Homological Methods in Rewriting

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Equational Theories, Term Rewriting Systems (TRSs)

- Set of variables $V = \{x_1, x_2, x_3, \ldots\}$
- Signature (set of const/func symbols) $\Sigma = \{c, f, g, +, \ldots\}$
  - Terms: $f(x_1), f(c + x_1), g(x_2, f(x_1)), \ldots$
- Set of rules
  - $R = \{(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3), f(x_1 + x_2) = f(x_1) + f(x_2), \ldots\}$

Equational Theory (unordered)

or

  - $R = \{(x_1 + x_2) + x_3 \longrightarrow x_1 + (x_2 + x_3), f(x_1 + x_2) \longrightarrow f(x_1) + f(x_2), \ldots\}$

Term Rewriting System (ordered)

What This Talk is about

R : given an equational theory/TRS

Is there any smaller equational theory/TRS equivalent to R ?

How many rules are needed?

- find a lower bound using algebra.
- + brief intro & history of the algebra we are going to use.

Example: The Theory of Groups

$$
\begin{aligned}
(x_1 \cdot x_2) \cdot x_3 &= x_1 \cdot (x_2 \cdot x_3), & & \\
x_1 \cdot e &= x_1, & e \cdot x_1 &= x_1, \\
x_1 \cdot x_1^{-1} &= e, & x_1^{-1} \cdot x_1 &= e\,.
\end{aligned}
$$

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- Presentation with 2 axioms

$$x_1 \cdot (((x_2^{-1} \cdot (x_1^{-1} \cdot x_3))^{-1} \cdot x_4) \cdot (x_2 \cdot x_4)^{-1})^{-1} = x_3,$$
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- Presentation with 1 axiom is possible if we use division "/" instead of multiplication m.

$$x_1/((((x_1/x_1)/x_2)/x_3)/(((x_1/x_1)/x_1)/x_3)) = x_2$$

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Questions

- **Question 1.**

Is there a presentation with one axiom over signature $\{\ \cdot\ ,\ ^{-1}, e\ \}$?

- **Answer.**

No. [Tarski, Neumann, Kunen] We need at least 2 axioms.

- **Question 2.**

What about other equational theories/TRSs?

Is there a generic way to know how many rules are needed to present a given equational theory/TRS?

A lower bound by [Malbos-Mimram, FSCD'16]

(Σ, R) : complete (= terminating & confluent) TRS
∃ a computable number $MM(\Sigma, R)$ s.t.

$$MM(\Sigma, R) \leq \#R'$$

for any TRS (Σ', R') equivalent to (Σ, R).

- Not many TRSs are known to have $MM(\Sigma, R) > 1$

⇒ The inequality just tells "any equivalent TRS has at least 0 or 1 rule" for most examples. 😢

- "Equivalence" for TRSs with possibly different signatures

[Ikebuchi, FSCD '19]

Fix Σ. R : complete TRS over Σ. If $\deg(R)$ is 0 or prime,

$\exists\ e(R)$: (computable) nonnegative integer s.t.

$$\#R - e(R) \leq \#R'$$

for any R' over Σ equivalent to R. $(\overset{*}{\leftrightarrow}_R = \overset{*}{\leftrightarrow}_{R'})$

For a complete TRS R of the theory of groups over $\{ \cdot\ ,\ ^{-1}, e \}$, we get

$\deg(R) = 2$ and $\#R - e(R) = 2$.

"Any TRS presenting the theory of groups has at least 2 rules."

- Tarski's theorem is obtained **as a corollary**.

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Outline

- Definitions of $\deg$, $e(R)$
  - Examples
- Proof Overview
- More About Homology & History
- Conclusion

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Degree of a TRS

$\#_i t$: the number of occurrences of x_i in $t \in \mathrm{Term}(\Sigma, \{x_1, x_2, \ldots\})$,

$$\deg(R) = \gcd\{\#_i l - \#_i r \mid l \to r \in R, i = 1,2,\ldots\}$$

Example: $R = \{ f(x_1, x_2, x_2) \longrightarrow x_1, g(x_1, x_1, x_1) \longrightarrow e\}$

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$$\therefore \deg(R) = \gcd\{0,2,3\} = 1$$

$\deg(R) = 0$ iff $\to_R$ preserves the multiset of variables

E.g. $R = \{f(f(x_1, x_2), x_3) \to f(x_1, f(x_2, x_3)), \ \ g(f(x_1, x_1)) \to f(g(x_1), g(x_1))\}$

Matrix $D(R)$

$R = \{l_1 \to r_1, \ldots, l_n \to r_n\}$: complete TRS (n rules)

$$
\begin{tikzcd}[column sep=small]
 & t_1 \arrow[dl, "l_{a_1} \to r_{a_1}"'] \arrow[dr, "l_{b_1} \to r_{b_1}"] & \\
u_1 & & v_1
\end{tikzcd}
\quad , \cdots , \quad
\begin{tikzcd}[column sep=small]
 & t_m \arrow[dl, "l_{a_m} \to r_{a_m}"'] \arrow[dr, "l_{b_m} \to r_{b_m}"] & \\
u_m & & v_m
\end{tikzcd}
$$
: m critical pairs

Fix a rewriting strategy.

$D(R)$: $n \times m$ matrix, (i, j)-th entry $D(R)_{ij}$ is the difference between the numbers of $l_i \to r_i$ used in two normalizing paths

$$
\begin{tikzcd}
 & u_j \arrow[r] & u_{j,1} \arrow[r] & \cdots \arrow[r] & u_{j,k} \arrow[dr] & \\
t_j \arrow[ur] \arrow[dr] \arrow[rrrrr, bend left] \arrow[rrrrr, bend right] & & & & & \hat{t}_j \\
 & v_j \arrow[r] & v_{j,1} \arrow[r] & \cdots \arrow[r] & v_{j,k'} \arrow[ur] &
\end{tikzcd}
$$

Example:

$$\boxed{\deg(R) = 0}$$

$$R = \left\{ \begin{array}{ll} A_1 \,.\, -(-x_1) \to x_1, & A_2 \,.\, -f(x_1) \to f(-x_1), \\ A_3 \,.\, -(x_1 + x_2) \to (-x_1) \cdot (-x_2), & A_4 \,.\, -(x_1 \cdot x_2) \to (-x_1) + (-x_2) \,. \end{array} \right\}$$

$\boxed{C_1}$

$$\begin{tikzcd}
-(-f(x_1)) \arrow[d, "A_2"] \arrow[ddd, bend left, "A_1"] \\
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$$R = \left\{ \begin{array}{ll} A_1 \,.\, -(-x_1) \to x_1, & A_2 \,.\, -f(x_1) \to f(-x_1), \\ A_3 \,.\, -(x_1 + x_2) \to (-x_1) \cdot (-x_2), & A_4 \,.\, -(x_1 \cdot x_2) \to (-x_1) + (-x_2)\,. \end{array} \right\}$$

$\boxed{C_1}$

$$\begin{tikzcd} -(-f(x_1)) \arrow[d, "A_2"] \arrow[ddd, bend left, "A_1"] \\ -(f(-x_1)) \arrow[d, "A_2"] \\ f(-(-x_1)) \arrow[d, "A_1"] \\ f(x_1) \end{tikzcd}$$

$\boxed{C_2}$

$$\begin{tikzcd} -(-(-x_1)) \arrow[d, bend right, "A_1"'] \arrow[d, bend left, "A_1"] \\ -x_1 \end{tikzcd}$$

$\boxed{C_3}$

$$\begin{tikzcd} -(-(x_1 + x_2)) \arrow[d, "A_3"] \arrow[dddd, bend left, "A_1"] \\ -((-x_1) \cdot (-x_2)) \arrow[d, "A_4"] \\ (-(-x_1)) + (-(-x_2)) \arrow[d, "A_1"] \\ x_1 + (-(-x_2)) \arrow[d, "A_1"] \\ x_1 + x_2 \end{tikzcd}$$

$\boxed{C_4}$

$$\begin{tikzcd} -(-(x_1 \cdot x_2)) \arrow[d, "A_4"] \arrow[dddd, bend left, "A_1"] \\ -((-x_1) + (-x_2)) \arrow[d, "A_3"] \\ (-(-x_1)) \cdot (-(-x_2)) \arrow[d, "A_1"] \\ x_1 \cdot (-(-x_2)) \arrow[d, "A_1"] \\ x_1 \cdot x_2 \end{tikzcd}$$

$$D(R) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} 0 & & & \\ 2 & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

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$\boxed{C_2}$

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$\boxed{C_3}$

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$$D(R) = \begin{array}{c c} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Definition of $e(R)$

Let $d = \deg(R)$

Consider $D(R)$ as a matrix over $\mathbb{Z}/d\mathbb{Z}$

- $\simeq \mathbb{Z}$ if $d = 0$
- $\simeq \mathbb{F}_d$ (finite field) if d is prime

If d is prime: $e(R) = \operatorname{rank}(D(R))$

If $d = 0$: compute the "Smith normal form" of $D(R)$
by elementary row/column operations
$e(R) =$ (the number of ± 1s in the Smith n.f.)

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Definition of $e(R)$

$$\begin{pmatrix} e_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & e_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & 0 & e_r & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & 0 & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

e_i divides e_{i+1} $(1 \leq i < r)$

er $\mathbb{Z}/d\mathbb{Z}$

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Outline

- Definitions of $\deg$, $e(R)$
  - Examples
- Proof Overview
- More About Homology & History
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Example:

$$\boxed{\deg(R) = 0}$$

$$R = \left\{ \begin{array}{ll} A_1 \,.\, -(-x_1) \to x_1, & A_2 \,.\, -f(x_1) \to f(-x_1), \\ A_3 \,.\, -(x_1 + x_2) \to (-x_1)\cdot(-x_2), & A_4 \,.\, -(x_1 \cdot x_2) \to (-x_1) + (-x_2)\,. \end{array} \right\}$$

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x_1 + x_2
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-((-x_1)+(-x_2)) \arrow[d, "A_3"] \\
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Example:

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Example:

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$$\underline{e(R) = 1}$$

Example: (cont.)

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By Main Theorem:

$$\#R - e(R) = 4 - 1 = 3 \leq \#R'$$

for any equivalent TRS R'

⇒There is no equivalent TRS with 2 rules

An equivalent TRS with 3 rules: $\{A_1, A_2, A_3\}$

Example (the theory of groups)

- Complete TRS

$$
\begin{array}{ll}
(x_1 \cdot x_2) \cdot x_3 \to x_1 \cdot (x_2 \cdot x_3) & e \cdot x_1 \to x_1 \\
x_1 \cdot e \to x_1 & x_1 \cdot x_1^{-1} \to e \\
x_1^{-1} \cdot x_1 \to e & x_1^{-1} \cdot (x_1 \cdot x_2) \to x_2 \\
e^{-1} \to e & (x^{-1})^{-1} \to x_1 \\
x_1 \cdot (x_1^{-1} \cdot x_2) \to x_2 & (x_1 \cdot x_2)^{-1} \to x_1^{-1} \cdot x_2^{-1}
\end{array}
$$

  with 48 critical pairs

- My program (https://github.com/mir-ikbch/homtrs) computes $MM(\Sigma, R)$, $\deg(R)$, $D(R)$, $e(R)$
  - $e(R) = 8$ ($\therefore \#R - e(R) = 2$), $MM(\Sigma, R) = 0$

Example (average and successors)

$A_1.\mathsf{ave}(0,0) \to 0,\quad A_2.\mathsf{ave}(x_1, \mathsf{s}(x_2)) \to \mathsf{ave}(\mathsf{s}(x_1), x_2),\quad A_3.\mathsf{ave}(\mathsf{s}(0), 0) \to 0,$
$A_4.\mathsf{ave}(\mathsf{s}(\mathsf{s}(0)), 0) \to s(0),\quad A_5.\mathsf{ave}(\mathsf{s}(\mathsf{s}(\mathsf{s}(x_1))), x_2) \to \mathsf{s}(\mathsf{ave}(\mathsf{s}(x_1), x_2)).$

$$\begin{tikzcd}[column sep=tiny]
& \mathsf{ave}(\mathsf{s}(\mathsf{s}(\mathsf{s}(x_1))), \mathsf{s}(x_2)) \arrow[dl, "A_2"'] \arrow[dr, "A_5"] & \\
\mathsf{ave}(\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(x_1))))), x_2) \arrow[dr, "A_5"'] & & \mathsf{s}(\mathsf{ave}(\mathsf{s}(x_1), \mathsf{s}(x_2))) \arrow[dl, "A_2"] \\
& \mathsf{s}(\mathsf{ave}(\mathsf{s}(\mathsf{s}(x_1)), x_2)) &
\end{tikzcd}$$

- $D(R)$ is the 5×1 zero matrix. $\Rightarrow e(R) = 0.\ \therefore \#R - e(R) = \#R = 5$
- Generally: Given a TRS, if any critical pair is of “this type”, then the TRS does not have any smaller equivalent TRSs.

Example (average and successors)

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\end{tikzcd}$$

the left path and the right path have the same multiset of rewrite rules

- $D(R)$ is the 5×1 zero matrix. ⇒ $e(R) = 0$. $\therefore$ $\#R - e(R) = \#R = 5$
- Generally: Given a TRS, if any critical pair is of "this type", then the TRS does not have any smaller equivalent TRSs.

Outline

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Assumption & Notation

- Assume $d = \deg(R)$ is prime for simplicity
  - $\mathbb{Z}/d\mathbb{Z} = \{0,1,\ldots,d-1\}$ forms a field
    - $\mathbb{Z}/d\mathbb{Z}^n = \underbrace{\mathbb{Z}/d\mathbb{Z} \times \ldots \times \mathbb{Z}/d\mathbb{Z}}_{n}$: n-dim. vector space
- (For $d = 0$, $\mathbb{Z}/d\mathbb{Z} \simeq \mathbb{Z}$ does not form a field, so the proof is more complicated.)

Main tools: linear algebra & Malbos-Mimram's results

Malbos-Mimram's Lower Bound

They introduced two linear maps

$$\tilde{\partial}_1 : \mathbb{Z}/d\mathbb{Z}^{\#R} \to \mathbb{Z}/d\mathbb{Z}^{\#\Sigma},$$

$$\tilde{\partial}_2 : \mathbb{Z}/d\mathbb{Z}^{\#\mathrm{CP}(R)} \to \mathbb{Z}/d\mathbb{Z}^{\#R}$$

$$MM(\Sigma, R) := \dim(\ker \tilde{\partial}_1 / \mathrm{im} \tilde{\partial}_2) \leq \#R$$

$MM(\Sigma, R) = MM(\Sigma', R')$ if (Σ, R) & (Σ', R') are equivalent.

(shown via homological algebra.

$\ker \tilde{\partial}_1 / \mathrm{im} \tilde{\partial}_2$ is called the "second homology")

$\therefore MM(\Sigma, R) \leq \#R'$ for any R' equivalent to R

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If R is complete, the matrix representation is $D(R)$

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Malbos-Mimram's Lower Bound

They introduced two linear maps

$$\tilde{\partial}_1 \colon \mathbb{Z}/d\mathbb{Z}^{\#R} \to \mathbb{Z}/d\mathbb{Z}^{\#\Sigma},$$
$$\tilde{\partial}_2 \colon \mathbb{Z}/d\mathbb{Z}^{\#\mathrm{CP}(R)} \to \mathbb{Z}/d\mathbb{Z}^{\#R}$$

If R is complete, the matrix representation is $D(R)$

$\ker \tilde{\partial}_1 = \{x \mid \tilde{\partial}_1(x) = 0\}$
$\mathrm{im}\tilde{\partial}_2 = \{\tilde{\partial}_2(x) \mid x \in \mathbb{Z}/d\mathbb{Z}^{\#\mathrm{CP}(R)}\}$
subspaces of $\mathbb{Z}/d\mathbb{Z}^{\#R}$

$$MM(\Sigma, R) := \dim(\ker \tilde{\partial}_1 / \mathrm{im}\tilde{\partial}_2) \leq \#R$$

$MM(\Sigma, R) = MM(\Sigma', R')$ if (Σ, R) & (Σ', R') are equivalent.
(shown via homological algebra.
$\ker \tilde{\partial}_1 / \mathrm{im}\tilde{\partial}_2$ is called the "second homology")

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$$MM(\Sigma, R) := \dim(\ker \tilde{\partial}_1 / \mathrm{im}\tilde{\partial}_2) \leq \#R$$
$$\because \dim(\ker \tilde{\partial}_1 / \mathrm{im}\tilde{\partial}_2) \leq \dim(\ker \tilde{\partial}_1) \leq \dim(\mathbb{Z}/d\mathbb{Z}^{\#R}) = \#R$$

$MM(\Sigma, R) = MM(\Sigma', R')$ if (Σ, R) & (Σ', R') are equivalent.

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$\ker \tilde{\partial}_1 / \mathrm{im}\tilde{\partial}_2$ is called the "second homology")

$\therefore MM(\Sigma, R) \leq \#R'$ for any R' equivalent to R

Proof Overview

- $\#R - e(R)$ equals the dimension of $V := (\mathbb{Z}/d\mathbb{Z}^{\#R})/\mathrm{im}\tilde{\partial}_2$

$$\left(\begin{aligned} \because \dim((\mathbb{Z}/d\mathbb{Z}^{\#R})/\mathrm{im}\tilde{\partial}_2) &= \dim(\mathbb{Z}/d\mathbb{Z}^{\#R}) - \dim(\mathrm{im}\tilde{\partial}_2) \\ &= \#R - \mathrm{rank}(D(R)) = \#R - e(R) \end{aligned} \right)$$

- By more theorems from linear algebra,

$$\dim(V) = \dim(\ker \tilde{\partial}_1/\mathrm{im}\tilde{\partial}_2) + \dim(\mathrm{im}\tilde{\partial}_1) \leq \#R$$

Any equivalent R, R' give the same $\dim(\mathrm{im}\tilde{\partial}_1)$

and the same $\dim(\ker \tilde{\partial}_1/\mathrm{im}\tilde{\partial}_2) = MM(\Sigma, R)$

$\#R - e(R) = \dim(V) = \dim(\ker \tilde{\partial}_1/\mathrm{im}\tilde{\partial}_2) + \dim(\mathrm{im}\tilde{\partial}_1)$: invariant

$$\therefore \#R - e(R) \leq \#R' \qquad \square$$

Main Theorem

Fix Σ. R : complete TRS over Σ. If $\deg(R)$ is 0 or prime,

$\exists\ e(R)$: (computable) nonnegative integer s.t.

$$\#R - e(R) \leq \#R'$$

for any R' over Σ equivalent to R.

What if $d = \deg(R)$ is not either 0 or prime?

- $\mathbb{Z}/d\mathbb{Z}$ has zero divisors.
  - e.g., for $d = 4$, $2 \times 2 = 4 \equiv 0 \bmod 4$.

$\Rightarrow$ Many useful theorems don't work.

  - e.g., "Smith normal form" is no longer well defined.

Outline

- Definitions of $\deg$, $e(R)$
  - Examples
- Proof Overview
- More About Homology & History
- Conclusion

String Rewriting Systems

- String Rewriting Systems (SRSs)
  - Alphabet Σ
  - Rules $R = \{ s_1 \longrightarrow t_1, s_2 \longrightarrow t_2, \dots \}$ $s_i, t_i \in \Sigma^*$ (strings over Σ)
- Example
  - $\Sigma = \{a, b\}, R = \{ ba \longrightarrow ab, abb \longrightarrow \varepsilon \}$

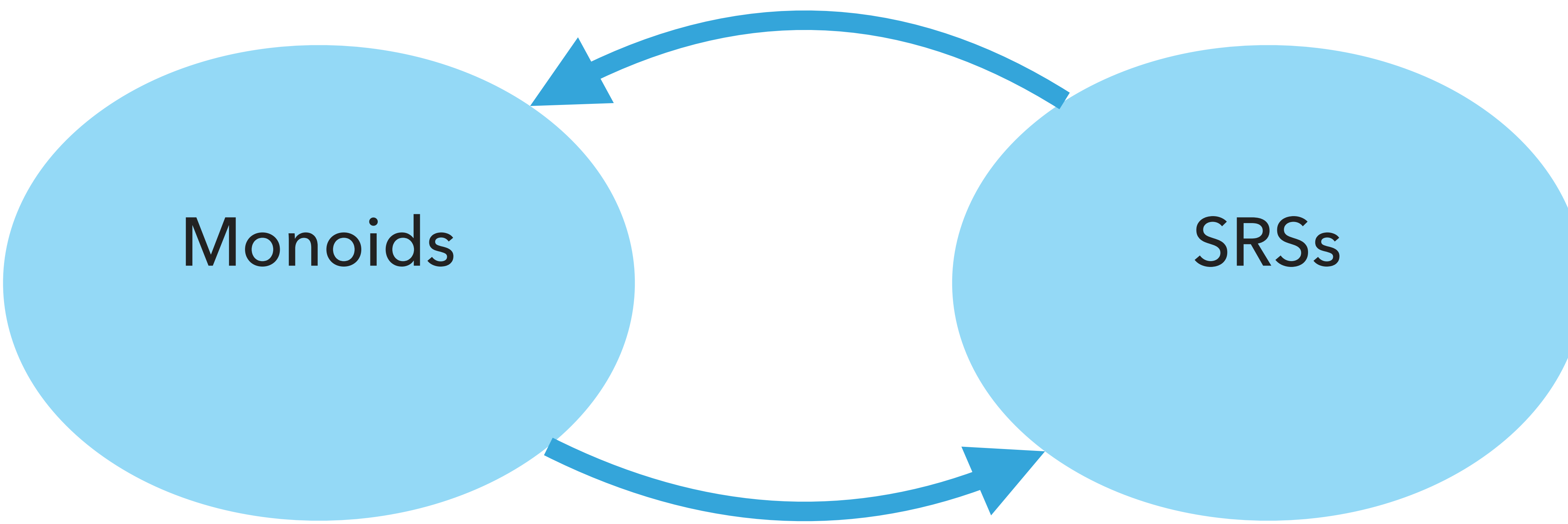
$$abab \longrightarrow aabb \longrightarrow a$$

How SRSs relate to algebra? — Monoids Presentation

- Any SRS (Σ, R) presents a monoid $M = \Sigma^* / \leftrightarrow_R^*$ (multiplication: string concatenation)
- Example:
  - $\Sigma = \{a\}, R = \{aa \to \varepsilon\} \Rightarrow \Sigma^* = \{a^n\},$
    $$M = \{[\varepsilon], [a]\},\ [aa] = [\varepsilon]$$
  - $\Sigma = \{a, b\}, R = \{ba \to ab\} \Rightarrow \Sigma^* = \{\varepsilon, a, b, aa, ab, ba, \ldots\},$
    $$M = \{[a^n b^m]\},\ [ba] = [ab], [bba] = [abb], \ldots$$

Monoids vs SRSs

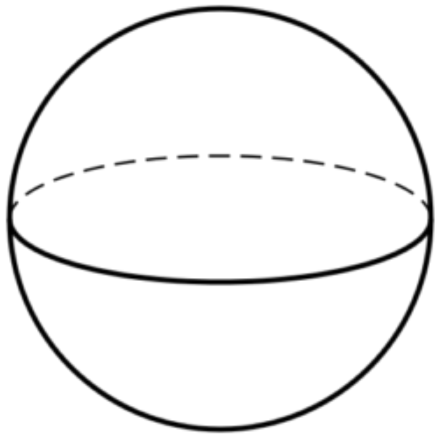
- Equivalent SRSs present isomorphic monoids
- Any monoid can be presented by an SRS (possibly with an infinite alphabet & rules)



Homology Groups in General

- There are many types of homology groups
  - Homology groups of a topological space
  - Homology groups of a group
  - ...
  - Homology groups of a general algebraic system (Quillen)
- Corresponds an "object" to a sequence of abelian groups that extracts some information from the object

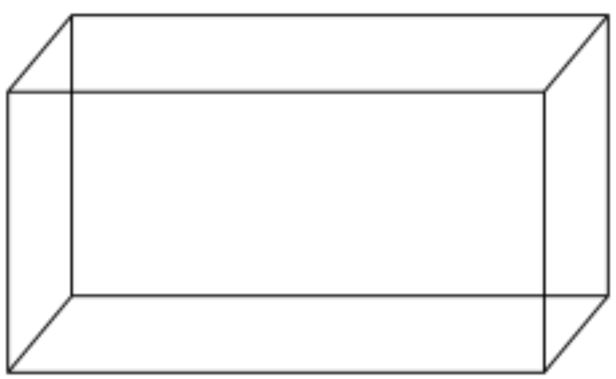
For topological spaces:

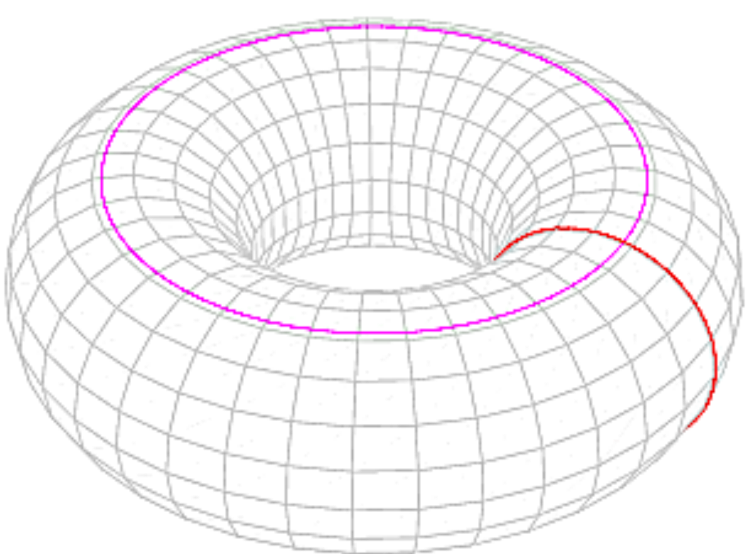


(topologically) isomorphic

$\mathbb{Z}, \{0\}, \mathbb{Z}, \{0\}, \{0\}, \ldots$

0th 1st 2nd ...



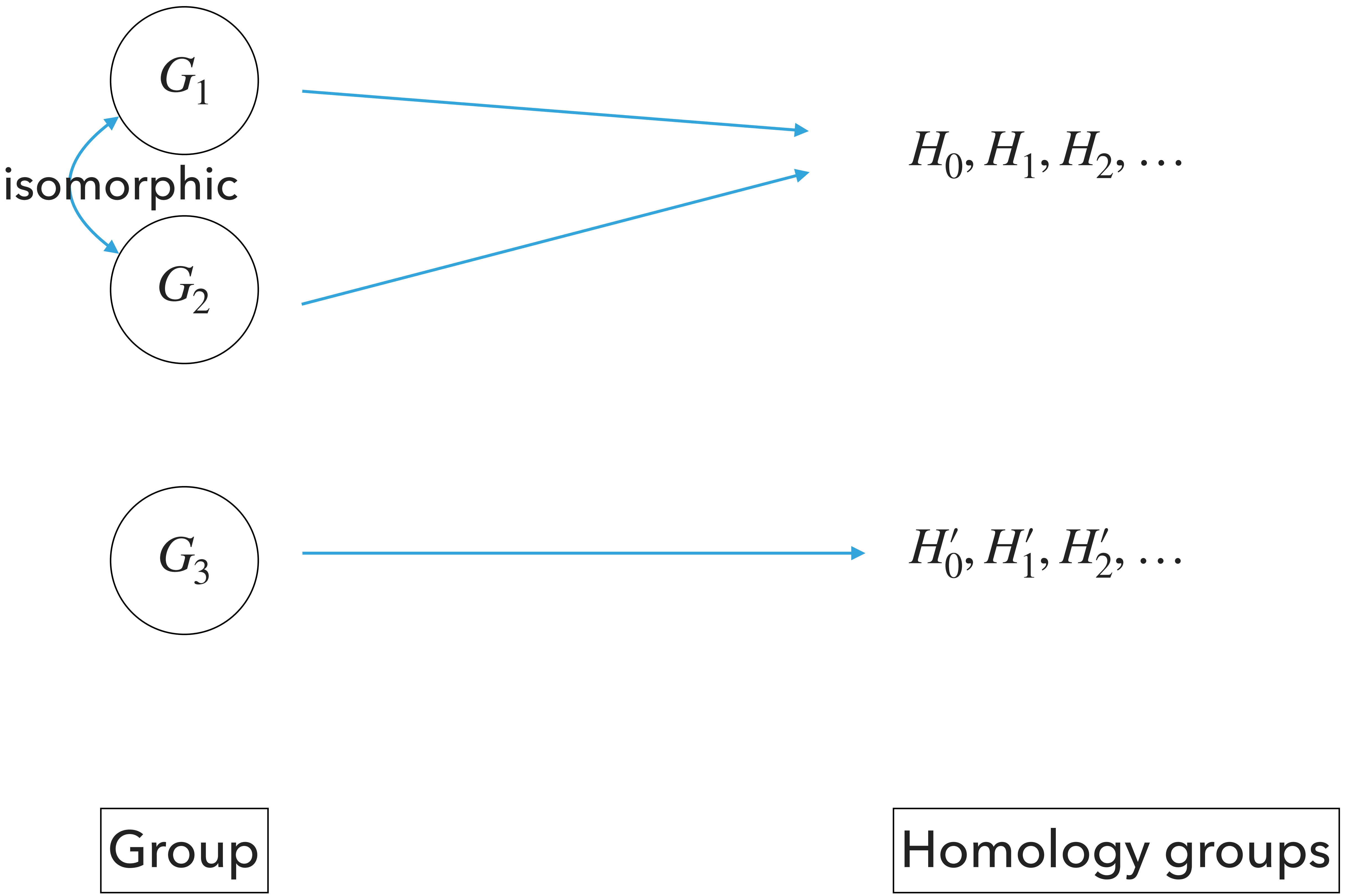


$\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}, \mathbb{Z}, \{0\}, \{0\}, \ldots$

Surface

Homology groups

For groups:


G_1
isomorphic
G_2
G_3
$H_0, H_1, H_2, \ldots$
$H'_0, H'_1, H'_2, \ldots$
Group
Homology groups

Homology groups of a group (= group homology)

- Group presentation – Σ : alphabet, R : set of strings on $\Sigma \cup \Sigma^{-1}$ ($\Sigma^{-1} = \{a^{-1} \mid a \in \Sigma\}$, a^{-1} is the formal inverse of a)
- Monoid presented by alphabet $\Sigma \cup \Sigma^{-1}$ and rules $\{w \to \varepsilon \mid w \in R \cup \{xx^{-1}, x^{-1}x \mid x \in \Sigma\}\}$ forms a group
- Any group can be presented in this way.
- [Epstein, Q. J. Math., 1961] If G is presented by finite Σ, R,

$$\#R - \#\Sigma \geq s(H_2(G)) - \mathrm{rank} H_1(G)$$

2nd & 1st homology groups of G

- We can construct homology groups for monoids/SRSs

$$\begin{tikzcd}
\{aba \to \varepsilon\} \arrow[r] \arrow[d, bend right, leftrightarrow, "\text{equivalent}"'] & H_0, H_1, H_2, \dots \\
\{ba \to ab, abb \to \varepsilon\} \arrow[ur] & \\
\{aa \to a\} \arrow[r] & H'_0, H'_1, H'_2, \dots
\end{tikzcd}$$

SRS/Monoid

Homology groups

but no application to rewriting known <u>until 1987</u>

[Squier, J. Pure Appl. Algebra, 1987]

- Solved an open problem at the time: "Does there exist a monoid with a solvable word problem that cannot be presented by any finite complete SRS?" - Yes
  - Word problem is solvable = equality is decidable
  - If a finite complete SRS presents a monoid, the word problem of the monoid is solvable
- Squier discovered that if the 3rd homology group constructed from a complete SRS is not finitely generated, then the SRS is infinite. (His main theorem is even stronger)

Monoids vs SRSs

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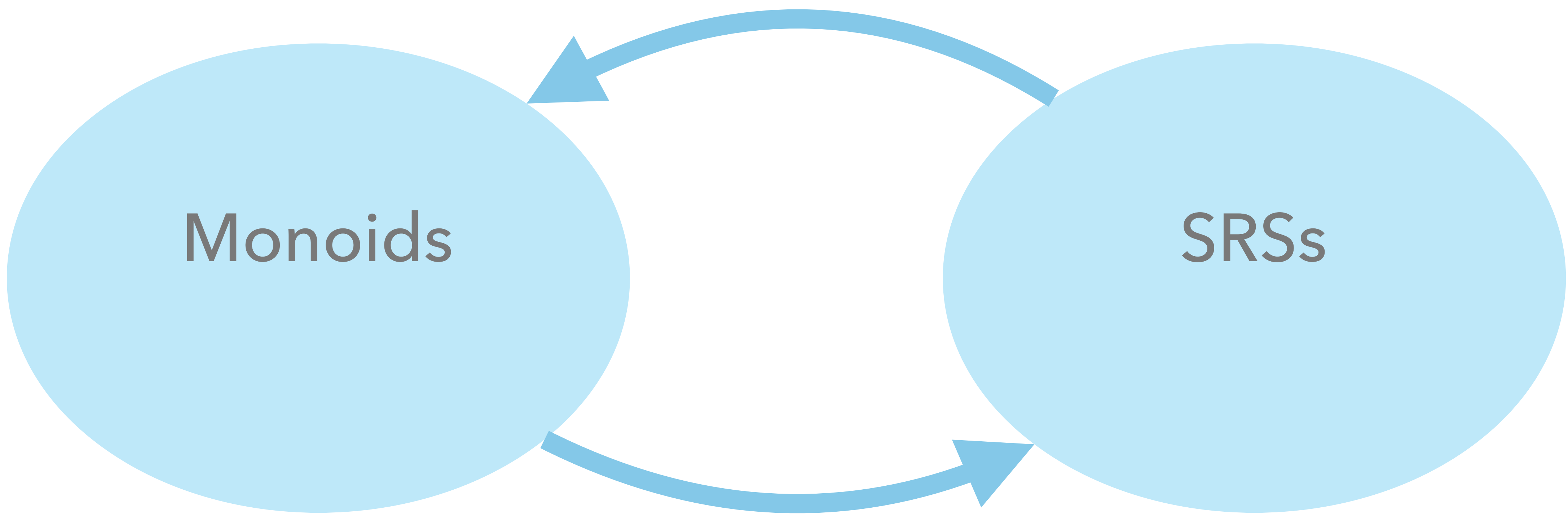

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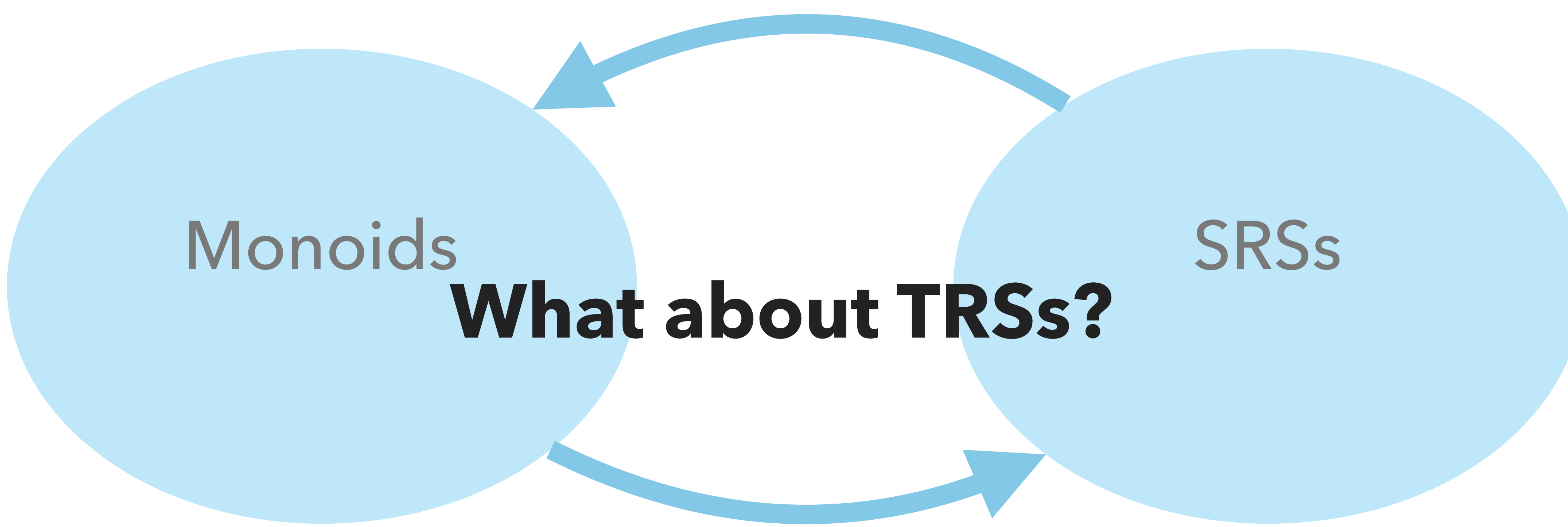

Algebraic Structure on Terms

- Multiplication? – substitution of tuples of terms:
- $f(g(x_1), x_2) \cdot \langle c, f(x_2, x_1) \rangle = f(g(c), f(x_2, x_1))$
- $\langle g(x_1), f(x_2, x_3) \rangle \cdot \langle c, f(x_2, x_1), g(c) \rangle = \langle g(c), f(f(x_2, x_1), g(c)) \rangle$
- (n-tuple with k kinds of vars) • (k-tuple with m kinds of vars) → (n-tuple with m kinds of vars)
- Monoids with typed (sorted) multiplication

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- Monoids with typed (sorted) multiplication = Category

Category of Terms

- Objects: natural numbers $0, 1, 2, 3, \ldots$
- Morphisms $k \longrightarrow n$: n-tuples of terms with vars in $\{x_1, \ldots, x_k\}$
- Composition (multiplication): $(k \to n) \cdot (m \to k) : (m \to n)$

$\langle t_1, \ldots, t_n \rangle \cdot \langle s_1, \ldots, s_k \rangle = \langle t_1[s_1/x_1, \ldots, s_k/x_k], \ldots, t_n[s_1/x_1, \ldots, s_k/x_k] \rangle$

- Identity: $\langle x_1, \ldots, x_n \rangle : n \to n$

Term version of the free monoid Σ^*.

Lawvere Theories

- A Lawvere theory is a category whose objects are $0,1,2,\ldots$ where n equals the nth categorical power of 1

  (Any morphism $n \to k$ is a n-tuple of $1 \to k$)

- (SRS vs Monoid) = (TRS vs Lawvere theory)

- The Lawvere theory presented by a TRS R: Any term t is identified with s iff $t \leftrightarrow_R^* s$

Homology Groups for Lawvere theories/TRSs

- [Jibladze & Pirashvili, J. of Algebra, 1991] defined cohomology groups of Lawvere theories
- [Malbos & Mimram, FSCD 2016] figured out how to compute the 2nd homology H_2 when the given TRS is complete and # of rules is bounded below by # of generators of H_2.
- [Ikebuchi, FSCD 2019] better lower bound I showed today

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Conclusion

- We obtained a lower bound of the number of rewrite rules to present a TRS over a fixed signature.
- Relationship between rewriting and abstract algebra
- New algebraic tools & more research directions of TRSs/ equational theories