AEEM4058 - Homework 4

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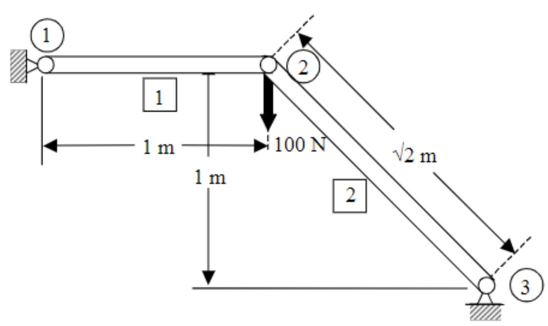
Problem 1

Figure 1 shows a truss structure with two uniform members made of same material. The truss structure is constrained at two ends. The cross-sectional area of all the truss members is 0.01 m^2, and the Young's modulus of the material is $2.0E10$ N/m^2. Using the finite element method, calculate
(a) all the nodal displacements;
(b) the internal forces in all the truss members; and
(c) the reaction forces at the supports.



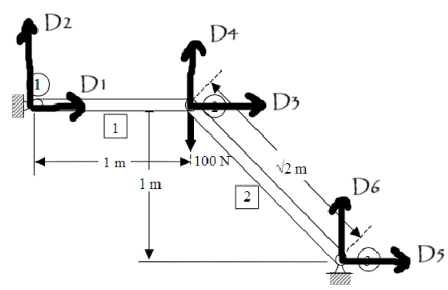


Figure 1

Part A

	Global node corresponding to		Coordinates in global		Direction cosines	
Element #	local node 1	local node 2	X_i, Y_i	X_j, Y_j	l_{ij}	m_{ij}
1	1	2	0, 0	1, 0	1	0
2	2	3	1, 0	2, -1	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$

Build Element Matrices

$$K_e = \frac{AE}{l_e}\begin{bmatrix} l_{ij}^2 & l_{ij}m_{ij} & -l_{ij}^2 & -l_{ij}m_{ij} \\ & m_{ij}^2 & -l_{ij}m_{ij} & -m_{ij}^2 \\ & & l_{ij}^2 & l_{ij}m_{ij} \\ sym. & & & m_{ij}^2 \end{bmatrix}$$

$$K_{e1} = \frac{0.01(2E10)}{1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ sym. & & & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 & 0 \\ & 0 & 0 & 0 \\ & & 2 & 0 \\ sym. & & & 0 \end{bmatrix} * 10^8 \text{ N/m}$$

$$K_{e2} = \frac{0.01(2E10)}{\sqrt{2}}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ & 0.5 & 0.5 & -0.5 \\ & & 0.5 & -0.5 \\ sym. & & & 0.5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \\ & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ & & 1/\sqrt{2} & -1/\sqrt{2} \\ sym. & & & 1/\sqrt{2} \end{bmatrix} * 10^8 \text{ N/m}$$

Build Global Matrices

combine K_{e1} and K_{e2}:

$$K = 10^8 * \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow K = 10^8 * \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2+1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

set up F and D matrices from boundary conditions:

$$F = \begin{bmatrix} R_{D1} \\ R_{D2} \\ 0 \\ -100 \\ R_{D5} \\ R_{D6} \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ D3 \\ D4 \\ 0 \\ 0 \end{bmatrix}$$

Solve for displacements

$KD = F$

$$10^8 * \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2+1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ D3 \\ D4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{D1} \\ R_{D2} \\ 0 \\ -100 \\ R_{D5} \\ R_{D6} \end{bmatrix}$$

use Matlab to solve for displacements at node 2 (D3 and D4):

$$\begin{bmatrix} -2 & 0 & 2+1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} D3 \\ D4 \end{bmatrix} = \begin{bmatrix} 0 \\ -100 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 0 \\ 0 \\ -5 * 10^{-7} \\ -1.9142 * 10^{-6} \\ 0 \\ 0 \end{bmatrix} \text{m}}$$

Part B

$$F_e = \frac{AE}{l_e} \begin{bmatrix} -l_{ij} & -m_{ij} & l_{ij} & m_{ij} \end{bmatrix} \begin{bmatrix} D1 \\ D2 \\ D3 \\ D4 \end{bmatrix}$$

$$F_1 = \frac{0.01(2E10)}{1} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -5 * 10^{-7} \\ -1.9142 * 10^{-6} \end{bmatrix}$$

solve with matlab for F:

$$\boxed{F_1 = -100 \text{ N}}$$

$$F_2 = \frac{0.01(2E10)}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -5 * 10^{-7} \\ -1.9142 * 10^{-6} \\ 0 \\ 0 \end{bmatrix}$$

solve with matlab for F:

$$\boxed{F_2 = -141.42 \text{ N}}$$

Part C

plug in known values for displacements into KD=F:

$$10^8 * \begin{bmatrix} 2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2+1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -5 * 10^{-7} \\ -1.9142 * 10^{-6} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{D1} \\ R_{D2} \\ 0 \\ -100 \\ R_{D5} \\ R_{D6} \end{bmatrix}$$

solve for each missing force in F with matlab: $\boxed{\begin{bmatrix} R_{D1} \\ R_{D2} \\ R_{D5} \\ R_{D6} \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ -100 \\ 100 \end{bmatrix} \text{N}}$

Problem 2

Figure 2 shows a three-node truss element of length L and a constant cross-section area A. It is made of a material of Young's modulus E and density ρ. The truss is subjected to a uniformly distributed force b.
(a) Derive the stiffness matrix for the element.
(b) Write down the expression for the element mass matrix, and obtain m_{11} in terms of L, E, ρ, and A.
(c) Derive the external force vector.

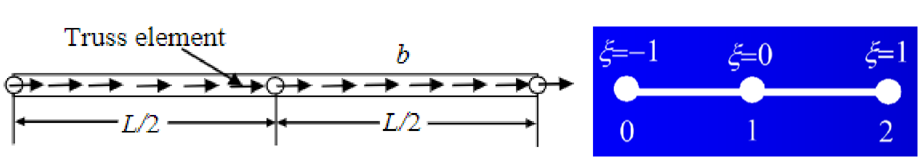


Figure 2

Part A

$N_0(\xi) = -\frac{1}{2}\xi(1-\xi) = -\frac{1}{2}\xi + \frac{1}{2}\xi^2$
$N_1(\xi) = (1+\xi)(1-\xi) = 1 - \xi^2$
$N_2(\xi) = \frac{1}{2}\xi(1+\xi) = \frac{1}{2}\xi + \frac{1}{2}\xi^2$

$a\xi = x \rightarrow d\xi = \frac{1}{a}dx \rightarrow \frac{d\xi}{dx} = \frac{1}{a}$

$K = \int_V B^T cB dV = A\int_{-L/2}^{L/2} B^T EB dx$
$B = \frac{dN}{dx} = \frac{dN}{d\xi}\frac{d\xi}{dx} = \frac{dN}{d\xi}\frac{1}{a}$
$B = \frac{1}{dx}\begin{bmatrix} dN_1 & dN_2 & dN_3 \end{bmatrix}$
$\frac{dN_1}{d\xi} = -\frac{1}{2} + \xi, \quad \frac{dN_2}{d\xi} = -2\xi, \quad \frac{dN_3}{d\xi} = \frac{1}{2} + \xi$
$B = \frac{1}{a}\begin{bmatrix} -\frac{1}{2}+\xi & -2\xi & \frac{1}{2}+\xi \end{bmatrix}$

$K = EA\int_{-1}^{1} B^T B a d\xi$

$$B^T B = \frac{1}{a^2}\begin{bmatrix} -\frac{1}{2}+\xi & -2\xi & \frac{1}{2}+\xi \end{bmatrix}\begin{bmatrix} -\frac{1}{2}+\xi \\ -2\xi \\ \frac{1}{2}+\xi \end{bmatrix} = \frac{1}{a^2}\begin{bmatrix} \frac{1}{4}-\xi+\xi^2 & \xi-2\xi^2 & -\frac{1}{4}+\xi^2 \\ & 4\xi^2 & -\xi-2\xi^2 \\ sym. & & \frac{1}{4}+\xi+\xi^2 \end{bmatrix}$$

integrate and ignore the odd power terms of ξ since they will cancel:

$$K = \frac{EA}{a}\int_{-1}^{1}\begin{bmatrix} \frac{1}{4}-\xi+\xi^2 & \xi-2\xi^2 & -\frac{1}{4}+\xi^2 \\ & 4\xi^2 & -\xi-2\xi^2 \\ sym. & & \frac{1}{4}+\xi+\xi^2 \end{bmatrix} d\xi = \frac{EA}{a}\begin{bmatrix} \frac{1}{4}\xi+\frac{1}{3}\xi^3 & -\frac{2}{3}\xi^3 & -\frac{1}{4}\xi+\frac{1}{3}\xi^3 \\ & \frac{4}{3}\xi^3 & -\frac{2}{3}\xi^3 \\ sym. & & \frac{1}{4}\xi+\frac{1}{3}\xi^3 \end{bmatrix}_{-1}^{1}$$

then evaluate from 0 to 1 and multiply by 2 since those terms are gone:

$$K = \frac{2EA}{a}\begin{bmatrix} \frac{1}{4}\xi+\frac{1}{3}\xi^3 & -\frac{2}{3}\xi^3 & -\frac{1}{4}\xi+\frac{1}{3}\xi^3 \\ & \frac{4}{3}\xi^3 & -\frac{2}{3}\xi^3 \\ sym. & & \frac{1}{4}\xi+\frac{1}{3}\xi^3 \end{bmatrix}_{0}^{1} = \frac{2EA}{a}\begin{bmatrix} \frac{1}{4}+\frac{1}{3} & -\frac{2}{3} & -\frac{1}{4}+\frac{1}{3} \\ & \frac{4}{3} & -\frac{2}{3} \\ sym. & & \frac{1}{4}+\frac{1}{3} \end{bmatrix}$$

then plug in $a = L/2$:

$$\boxed{K = \frac{EA}{L}\begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ & \frac{16}{3} & -\frac{8}{3} \\ sym. & & \frac{7}{3} \end{bmatrix}}$$

Part B

$m_e = \int_{V_e} \rho N^T N dV = A\rho a\int_{-1}^{1} N^T N d\xi$

$$N^T N = \begin{bmatrix} -\frac{1}{2}\xi+\frac{1}{2}\xi^2 \\ 1-\xi^2 \\ \frac{1}{2}\xi+\frac{1}{2}\xi^2 \end{bmatrix}\begin{bmatrix} -\frac{1}{2}\xi+\frac{1}{2}\xi^2 & 1-\xi^2 & \frac{1}{2}\xi+\frac{1}{2}\xi^2 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{4}\xi^2-\frac{1}{2}\xi^3+\frac{1}{4}\xi^4 & -\frac{1}{2}\xi+\frac{1}{2}\xi^2+\frac{1}{3}\xi^3-\frac{1}{2}\xi^4 & -\frac{1}{4}\xi^2+\frac{1}{4}\xi^4 \\ & 1-2\xi^2+\xi^4 & \frac{1}{2}\xi+\frac{1}{2}\xi^2-\frac{1}{2}\xi^3-\frac{1}{2}\xi^4 \\ sym. & & \frac{1}{4}\xi^2+\frac{1}{2}\xi^3+\frac{1}{4}\xi^4 \end{bmatrix}$$

integrate and ignore the odd power terms of ξ since they will cancel:

$$m_e = A\rho a\int_{-1}^{1}\begin{bmatrix} \frac{1}{4}\xi^2-\frac{1}{2}\xi^3+\frac{1}{4}\xi^4 & -\frac{1}{2}\xi+\frac{1}{2}\xi^2+\frac{1}{3}\xi^3-\frac{1}{2}\xi^4 & -\frac{1}{4}\xi^2+\frac{1}{4}\xi^4 \\ & 1-2\xi^2+\xi^4 & \frac{1}{2}\xi+\frac{1}{2}\xi^2-\frac{1}{2}\xi^3-\frac{1}{2}\xi^4 \\ sym. & & \frac{1}{4}\xi^2+\frac{1}{2}\xi^3+\frac{1}{4}\xi^4 \end{bmatrix} d\xi =$$

$$A\rho a\begin{bmatrix} \frac{1}{12}\xi^3+\frac{1}{20}\xi^5 & \frac{1}{6}\xi^3-\frac{1}{10}\xi^5 & -\frac{1}{12}\xi^3+\frac{1}{20}\xi^5 \\ & \xi-\frac{2}{3}\xi^3+\frac{1}{5}\xi^5 & \frac{1}{6}\xi^3-\frac{1}{10}\xi^5 \\ sym. & & \frac{1}{12}\xi^3+\frac{1}{20}\xi^5 \end{bmatrix}_{-1}^{1}$$

then evaluate from 0 to 1 and multiply by 2 since those terms are gone:

$$m_e = 2A\rho a\begin{bmatrix} \frac{1}{12}\xi^3+\frac{1}{20}\xi^5 & \frac{1}{6}\xi^3-\frac{1}{10}\xi^5 & -\frac{1}{12}\xi^3+\frac{1}{20}\xi^5 \\ & \xi-\frac{2}{3}\xi^3+\frac{1}{5}\xi^5 & \frac{1}{6}\xi^3-\frac{1}{10}\xi^5 \\ sym. & & \frac{1}{12}\xi^3+\frac{1}{20}\xi^5 \end{bmatrix}_{0}^{1} = 2A\rho a\begin{bmatrix} \frac{1}{12}+\frac{1}{20} & \frac{1}{6}-\frac{1}{10} & -\frac{1}{12}+\frac{1}{20} \\ & 1-\frac{2}{3}+\frac{1}{5} & \frac{1}{6}-\frac{1}{10} \\ sym. & & \frac{1}{12}+\frac{1}{20} \end{bmatrix}$$

then plug in $a = L/2$:

$$\boxed{m_e = A\rho L\begin{bmatrix} \frac{2}{15} & \frac{1}{15} & -\frac{1}{30} \\ & \frac{8}{15} & \frac{1}{15} \\ sym. & & \frac{2}{15} \end{bmatrix}}$$

find m_{11}:

$$\boxed{m_{11} = \frac{2A\rho L}{15}}$$

Part C

$f_e = \int_{V_e} N^T f_b dV + \int_{S_e} N^T f_s dS = A \int_{-1}^{1} N^T f_b a d\xi + L \int_{-1}^{1} N^T f_s a d\xi$

$f_e = (Aaf_b + Laf_s) \int_{-1}^{1} N^T d\xi = (Aaf_b + Laf_s) \int_{-1}^{1} \begin{bmatrix} -\frac{1}{2}\xi + \frac{1}{2}\xi^2 \\ 1 - \xi^2 \\ \frac{1}{2}\xi + \frac{1}{2}\xi^2 \end{bmatrix} d\xi =$

$(Aaf_b + Laf_s) \begin{bmatrix} -\frac{1}{4}\xi^2 + \frac{1}{6}\xi^3 \\ \xi + \frac{1}{3}\xi^3 \\ \frac{1}{4}\xi^2 + \frac{1}{6}\xi^3 \end{bmatrix}_{-1}^{1} = (Aaf_b + Laf_s) \begin{bmatrix} -\frac{1}{12} + \frac{5}{12} \\ \frac{2}{3} + \frac{2}{3} \\ \frac{5}{12} - \frac{1}{12} \end{bmatrix}$

$$\boxed{f_e = (Aaf_b + Laf_s) \begin{bmatrix} \frac{1}{3} \\ \\ \frac{4}{3} \\ \\ \frac{1}{3} \end{bmatrix}}$$