BME6013C HW#4

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Part 1

To prove linearity, we will first start with the proved version of the property and replace G_1 and G_2 with the definition of the Fourier transform:

$$\alpha G_1(f) + \beta G_2(f)$$

$$\alpha \int_{-\infty}^{\infty} e^{-j2\pi ft} g_1(t)dt + \beta \int_{-\infty}^{\infty} e^{-j2\pi ft} g_2(t)dt$$

$$\int_{-\infty}^{\infty} \alpha e^{-j2\pi ft} g_1(t)dt + \int_{-\infty}^{\infty} \beta e^{-j2\pi ft} g_2(t)dt$$

Now, we can combine the integrals into a single integral and factor like terms:

$$\int_{-\infty}^{\infty} \left(\alpha e^{-j2\pi ft} g_1(t) + \beta e^{-j2\pi ft} g_2(t)\right) dt$$

$$\int_{-\infty}^{\infty} e^{-j2\pi ft} \left(\alpha g_1(t) + \beta g_2(t)\right) dt$$

We can now clearly see from the result obtained in the previous step that the simplified version of the original term now resembles a Fourier transform where $g(t) = \alpha g_1(t) + \beta g_2(t)$. This would yield that our final integral expression is equal to $\mathcal{F}[g(t)] = \mathcal{F}[\alpha g_1(t) + \beta g_2(t)]$, and since we started at $\alpha G_1(f) + \beta G_2(f)$ to obtain that integral, it follows that $\mathcal{F}[\alpha g_1(t) + \beta g_2(t)] = \alpha G_1(f) + \beta G_2(f)$.

Part 2

First, we will start by taking the transform of $g(\alpha t)$:

$$\mathcal{F}[g(\alpha t)] = \int_{-\infty}^{\infty} e^{-j2\pi f t} g(\alpha t) dt$$

Now we can set a new variable t' and sub it in:

$$\begin{aligned} t' &= \alpha t \\ t &= t'/\alpha \\ dt &= dt'/\alpha \end{aligned}$$

$$\mathcal{F}[g(\alpha t)] = \int_{-\infty}^{\infty} e^{-j2\pi f t'/\alpha} g(t') dt'/\alpha = \frac{1}{\alpha} \int_{-\infty}^{\infty} e^{-j2\pi (f/\alpha) t'} g(t') dt'$$

Now we can see that the simplified integral now resembles the definition of the Fourier transform for $\mathcal{F}[g(t')]$ where f is replaced by f/α. The integral can then be replaced by the Fourier transform $G(f/\alpha)$ to get the final expression.

$$\mathcal{F}[g(\alpha t)] = \frac{G(f/\alpha)}{\alpha}$$

Part 3

First we will start by showing the Fourier transform $G(-f)$ from the definition for the Fourier transform and then take the conjugate of the result:

$$\begin{aligned} G(-f) = \int_{-\infty}^{\infty} e^{-j2\pi(-f)t} g(t) dt &= \int_{-\infty}^{\infty} e^{j2\pi f t} g(t) dt \\ [G(-f)]^* &= \left[\int_{-\infty}^{\infty} e^{j2\pi f t} g(t) dt \right]^* \end{aligned}$$

Since we know taking the conjugate is a linear operation, we can distribute it to the terms within the integral:

$$[G(-f)]^* = \int_{-\infty}^{\infty} (e^{j2\pi f t})^* g^*(t) dt$$

We know that the definition of a conjugate for a signal $(e^{j\phi})^* = e^{-j\phi}$. We also know that since $g(t)$ has no imaginary part, its conjugate is the same as $g(t)$.

$$[G(-f)]^* = \int_{-\infty}^{\infty} e^{-j2\pi ft} g(t)dt$$

The integral expression created is equal to the Fourier transform of $g(t)$, therefore:

$$\int_{-\infty}^{\infty} e^{-j2\pi ft} g(t)dt = \mathcal{F}[g(t)] = G(f) = [G(-f)]^*$$

Part 4

We can start by using the definition of the Fourier transform:

$$\mathcal{F}[g(t)] = \int_{-\infty}^{\infty} e^{-j2\pi ft} g(t)dt = \int_{-\infty}^{\infty} e^{-j2\pi ft} e^{-t/\tau} \mathcal{H}(t)dt$$

$$\mathcal{F}[g(t)] = \int_{-\infty}^{\infty} e^{-j2\pi ft - t/\tau} \mathcal{H}(t)dt = \int_{-\infty}^{\infty} e^{t(-j2\pi f - 1/\tau)} \mathcal{H}(t)dt$$

We can split out integral into two parts, one containing everything below 0 and one containing everything above.

$$\mathcal{F}[g(t)] = \int_{-\infty}^{0} e^{t(-j2\pi f - 1/\tau)} * 0dt + \int_{0}^{\infty} e^{t(-j2\pi f - 1/\tau)} * 1dt$$

The first integral will always be 0, so it goes away. We can evaluate the remaining integral:

$$\mathcal{F}[g(t)] = \frac{e^{t(-j2\pi f - 1/\tau)}}{-j2\pi f - 1/\tau}\Bigg|_0^\infty = \frac{e^{-t(j2\pi f + 1/\tau)}}{-j2\pi f - 1/\tau}\Bigg|_0^\infty$$

When we attempt to evaluate, we can see that at $t = \infty$, the term on top will be $e^{-\infty}$. e^x approaches 0 at $-\infty$, so this term will be 0. When $t = 0$, the term on top will be $e^0 = 1$.

$$\mathcal{F}[g(t)] = 0 - \frac{1}{-j2\pi f - 1/\tau} = \frac{1}{j2\pi f + 1/\tau} = \frac{\tau}{j2\pi f\tau + 1}$$

$$\boxed{\mathcal{F}[g(t)] = \frac{\tau}{j2\pi f\tau + 1}}$$

To sketch a plot, we will choose $\tau = 10$. We can see from the solution that at $f = 0$, the denomenator will be 1 so the mangitude is τ. As the frequency moves away from 0, the first term in the denominator will grow rapidly, meaning the magnitude will be τ divided by a large number. Since the magnitude is absolute value, this will be the same for negative frequencies. We should expect the plot of the magnitude with respect to frequency to be a sharp peak at $f = 0$, with a steep falloff that asymptotes to 0 on either side.