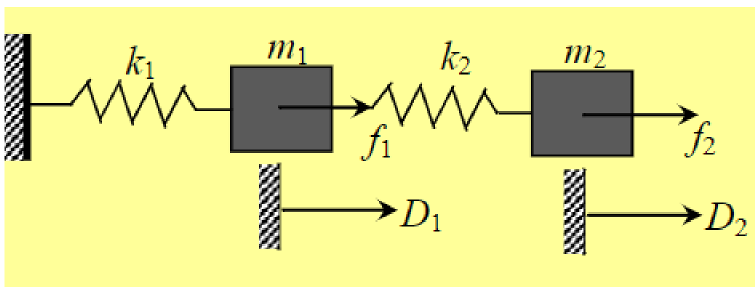
AEEM4058 - Homework 3

Slade Brooks

09.26.2023

Problem 1

Consider a 2 DOF system with two rigid blocks that can move only in the horizontal direction, as shown in the following figure.



Part 1

Give the expressions for the strain (potential) energy (Π), work done by external forces (W_f), and the total potential energy ($\Pi_t = \Pi - W_f$).

$\boxed{\Pi = \frac{1}{2}k_1D_1^2 + \frac{1}{2}k_2(D_2 - D_1)^2}$

$\boxed{W_f = D_1f_1 + D_2f_2}$

$\Pi_t = \Pi - W_f = \frac{1}{2}k_1D_1^2 + \frac{1}{2}k_2(D_2 - D_1)^2 - (D_1f_1 + D_2f_2)$

$\boxed{\Pi_t = \frac{1}{2}k_1D_1^2 + \frac{1}{2}k_2(D_2 - D_1)^2 - D_1f_1 - D_2f_2}$

Part 2

Derive the equilibrium equations for the system using the minimum potential energy principle.

$\Pi_t = \frac{1}{2}k_1D_1^2 + \frac{1}{2}k_2(D_2 - D_1)^2 - D_1f_1 - D_2f_2$

$\delta\Pi_t = 0 = \delta[\frac{1}{2}k_1D_1^2 + \frac{1}{2}k_2(D_2 - D_1)^2 - D_1f_1 - D_2f_2] = \delta[\frac{1}{2}k_1D_1^2 + \frac{1}{2}k_2(D_2^2 - 2D_1D_2 + D_1^2) - D_1f_1 - D_2f_2] = \delta[\frac{1}{2}k_1D_1^2 + \frac{1}{2}k_2D_2^2 - k_2D_1D_2 + \frac{1}{2}k_2D_1^2 - D_1f_1 - D_2f_2]$

vary D_1: $\delta D_1[k_1D_1 - k_2D_2 + k_2D_1 - f_1] = 0 \rightarrow D_1 = \frac{f_1+k_2D_2}{k_1+k_2}$

vary D_2: $\delta D_2[k_2D_2 - k_2D_1 - f_2] = 0 \rightarrow D_2 = \frac{f_2+k_2D_1}{k_2}$

$k_1D_1 - k_2D_2 + k_2D_1 - f_1 = 0 = k_1D_1 - k_2(\frac{f_2+k_2D_1}{k_2}) + k_2D_1 - f_1 = k_1D_1 - (f_2 + k_2D_1) + k_2D_1 - f_1 = 0$

$k_1D_1 = f_1 + f_2$

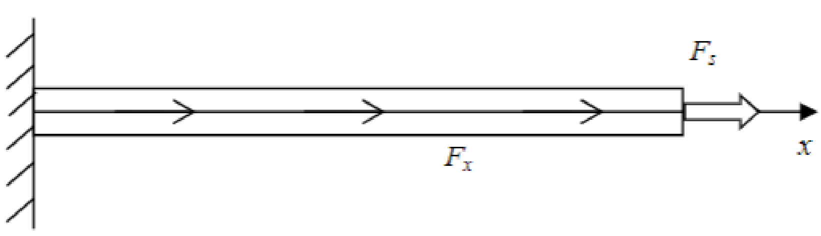
$\boxed{D_1 = \frac{f_1+f_2}{k_1}}$

$D_2 = \frac{f_2+k_2D_1}{k_2} = \frac{f_2+k_2(\frac{f_1+f_2}{k_1})}{k_2} = \frac{f_2}{k_2} + \frac{f_1+f_2}{k_1}$

$\boxed{D_2 = \frac{f_2}{k_2} + \frac{f_1+f_2}{k_1}}$

Problem 2

Consider again the simplest problem of a continuum 1D bar of uniform cross-section studied in Q1-HW1. The bar is fixed at the left-end and is of length l and section area A. It is subjected to a uniform body force f_x and a concentrated force F_s at the right end. The young's modulus of the material is E. Using the method of minimum potential energy, obtain the distribution of the displacement for the following cases.



Part 1

$f_x = 0$ and $F_s =$ constant

$\Pi = \frac{1}{2}\int_V \varepsilon^T \sigma dV$, $\sigma = E\varepsilon$, $\Pi = \frac{E}{2}A\int_0^l \varepsilon^2 dx$

assume $u(x) = C_0 + C_1 x \rightarrow \varepsilon = \frac{du}{dx} = C_1$

$\Pi = \frac{EA}{2}\int_0^l C_1^2 dx = \frac{EA}{2} l C_1^2$
$W_f = u(l)F = C_1 l F_s$
$\Pi_t = \Pi - W_f = \frac{EA}{2} l C_1^2 - l F_s C_1$
$\delta\Pi_t = 0 = \delta C_1[EAlC_1 - lF_s] = \delta C_1[EAC_1 - F_s] = 0$
$C_1 = \frac{F_s}{EA}$
enforce boundary condition: $u(0) = 0 = C_0$
$\boxed{u(x) = \frac{F}{EA}x}$

Part 2

$f_x =$ constant and $F_s =$ constant

$\Pi = \frac{E}{2}A\int_0^l \varepsilon^2 dx$

assume $u(x) = C_0 + C_1 x + C_2 x^2 \rightarrow \varepsilon = \frac{du}{dx} = C_1 + 2C_2 x$

$\Pi = \frac{EA}{2}\int_0^l (C_1 + 2C_2 x)^2 dx = \frac{EA}{2}\int_0^l (4C_2^2 x^2 + 4C_1 C_2 x + C_1^2) dx = \frac{EA}{2}(\frac{4C_2}{3} l^3 + 2C_1 C_2 l^2 + C_1^2 l)$
enforce boundary condition: $u(0) = 0 = C_0$
$W_f = u(l)F + \int_V u^T f_b dV = (C_1 l + C_2 l^2)F_s + A\int_0^l (C_1 x + C_2 x^2) f_x dx = (C_1 l + C_2 l^2)F_s + A(\frac{C_1}{2}l^2 + \frac{C_2}{3}l^3)f_x$

$\Pi_t = \Pi - W_f = \frac{EA}{2}(\frac{4C_2^2}{3}l^3 + 2C_1 C_2 l^2 + C_1^2 l) - (C_1 l + C_2 l^2)F_s - A(\frac{C_1}{2}l^2 + \frac{C_2}{3}l^3)f_x$
$\delta\Pi_t = 0 = \delta C_1(\frac{EA}{2}(2C_2 l^2 + 2C_1 l) - (l)F_s - A(\frac{l^2}{2})f_x)$
$\delta\Pi_t = 0 = \delta C_2(\frac{EA}{2}(\frac{8C_2}{3}l^3 + 2C_1 l^2) - (l^2)F_s - A(\frac{l^3}{3})f_x)$

find C_1: $\frac{EA}{2}(2C_2 l^2 + 2C_1 l) - (l)F_s - A(\frac{l^2}{2})f_x = 0$
$(2C_2 l^2 + 2C_1 l) = \frac{A\frac{l^2}{2}f_x + lF_s}{\frac{EA}{2}}$
$C_1 = \frac{\frac{A}{2}lf_x + F_s}{EA} - C_2 l$
$C_1 = \frac{lf_x}{2E} + \frac{F_s}{EA} - C_2 l$

find C_2: $\frac{EA}{2}(\frac{8C_2}{3}l^3 + 2C_1 l^2) - (l^2)F_s - A(\frac{l^3}{3})f_x$
$(\frac{8}{3}C_2 l^3 + 2C_1 l^2) = \frac{A\frac{l^3}{3}f_x + l^2 F_s}{\frac{EA}{2}} = \frac{2l^3 f_x}{3E} + \frac{2l^2 F_s}{EA}$
$\frac{8}{3}C_2 l^3 = \frac{2l^3 f_x}{3E} + \frac{2l^2 F_s}{EA} - 2C_1 l^2$
$C_2 = \frac{f_x}{4E} + \frac{3F_s}{4EAl} - \frac{3}{4l}C_1$

$C_2 = \frac{f_x}{4E} + \frac{3F_s}{4EAl} - \frac{3}{4l}(\frac{lf_x}{2E} + \frac{F_s}{EA} - C_2 l)$
$C_2 = \frac{f_x}{4E} + \frac{3F_s}{4EAl} - \frac{3f_x}{8E} - \frac{3F_s}{4EAl} + \frac{3}{4}C_2$
$\frac{1}{4}C_2 = \frac{-f_x}{8E}$
$C_2 = \frac{-f_x}{2E}$

$C_1 = \frac{lf_x}{2E} + \frac{F_s}{EA} - l(\frac{-f_x}{2E})$
$C_1 = \frac{lf_x}{E} + \frac{F_s}{EA}$
$C_1 = \frac{Alf_x + F_s}{EA}$

$\boxed{u(x) = \frac{Alf_x + F_s}{EA}x - \frac{f_x}{2E}x^2}$