AEEM4058 - Homework 6

Slade Brooks

11.07.2023

Problem 1

Show that the stiffness matrix of an isotropic linear triangular element whose thickness varies linearly in the element is:
$k_e = \bar{h}A_eB^TcB$
where B is the strain matrix, c is matrix of material constants, A_e is the area of the triangle and $\bar{h}$ the average thickness $(h_1+h_2+h_3)/3$, where h_1, h_2, and h_3 are the nodal thickness at the node.

$$N = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix}, \quad h = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$h(x,y) = Nh = L_1h_1 + L_2h_2 + L_3h_3$

$k_e = B^TcB\int_{A_e} h(x,y)dA = B^TcB\int_{A_e}(L_1h_1 + L_2h_2 + L_3h_3)dA$

from eisenberg and malvern: $\int_{A_e} L_1^m L_2^n L_3^p dA = \frac{m!n!p!}{(m+n+p+2)!}$
$h_1\int_{A_e} L_1 = h_1\frac{1!0!0!}{(1+0+0+2)!}2A_e = \frac{2h_1}{6}A_e = \frac{h_1A_e}{3}$
$h_2\int_{A_e} L_2 = h_2\frac{0!1!0!}{(0+1+0+2)!}2A_e = \frac{2h_2}{6}A_e = \frac{h_2A_e}{3}$
$h_3\int_{A_e} L_3 = h_3\frac{0!0!1!}{(0+0+1+2)!}2A_e = \frac{2h_3}{6}A_e = \frac{h_3A_e}{3}$

$k_e = B^TcB\frac{h_1+h_2+h_3}{3}A_e \rightarrow \boxed{k_e = \bar{h}A_eB^TcB}$

Problem 2

Show that the mass matrix of a linear triangular element whose thickness varies linearly within the plane of the element is:

$$m_e = \frac{\rho\bar{h}A_e}{60}\begin{bmatrix} 6+4\alpha_1 & 0 & 6-\alpha_3 & 0 & 6-\alpha_2 & 0 \\ & 6+4\alpha_1 & 0 & 6-\alpha_3 & 0 & 6-\alpha_2 \\ & & 6+4\alpha_2 & 0 & 6-\alpha_1 & 0 \\ & & & 6+4\alpha_2 & 0 & 6-\alpha_1 \\ & & & & 6+4\alpha_3 & 0 \\ sym. & & & & & 6+4\alpha_3 \end{bmatrix}$$

where ρ is the density, A_e is the area, $\bar{h}$ is the mean thickness and $\alpha_i = \frac{h_i}{\bar{h}}$ with i=1, 2, 3 for the three nodes.

$m_e = \rho\int_{A_e} h(x,y)N^TNdA$

$$m_e = \rho\int_{A_e}\begin{bmatrix} hN_1^2 & 0 & hN_1N_2 & 0 & hN_1N_3 & 0 \\ & hN_1^2 & 0 & hN_1N_2 & 0 & hN_1N_3 \\ & & hN_2^2 & 0 & hN_2N_3 & 0 \\ & & & hN_2^2 & 0 & hN_2N_3 \\ & & & & hN_3^2 & 0 \\ sym. & & & & & hN_3^2 \end{bmatrix}dA$$

$\int_{A_e} hN_1^2 = \int_{A_e}(h_1N_1^3 + h_2N_2N_1^2 + h_3N_3N_1^2)dA = \frac{12}{120}h_1A_e + \frac{4}{120}h_2A_e + \frac{4}{120}h_3A_e = \frac{60}{60}A_e(\frac{1}{10}h_1 + \frac{1}{30}(h_2+h_3))$
$\frac{A_e}{60}(4h_1 + 2(h_1+h_2+h_3)) = \frac{A_e}{60}(4h_1 + 6\bar{h}) = \frac{A_e\bar{h}}{60}(4\alpha_1 + 6)$, so $\int_{A_e} hN_i^2dA = \frac{A_e\bar{h}}{60}(6+4\alpha_i)$

$\int_{A_e} hN_1N_2dA = \int_{A_e}(h_1N_1^2N_2 + h_2N_1N_2^2 + h_3N_1N_2N_3)dA = \frac{4}{120}h_1A_e + \frac{4}{120}h_2A_e + \frac{2}{120}h_3A_e$
$\frac{60}{60}A_e(\frac{1}{30}h_1 + \frac{1}{30}h_2 + \frac{1}{60}h_3) = \frac{A_e}{60}(2h_1 + 2h_2 + h_3) = \frac{A_e}{60}(2(h_1+h_2+h_3) - h_3) = \frac{A_e}{60}(6\bar{h} - h_3) = \frac{A_e\bar{h}}{60}(6-\alpha_3)$,
so $\int_{A_e} hN_iN_jdA = \frac{A_e\bar{h}}{60}(6 - \alpha_{\text{not i or j}})$

then apply those definitions to the original m_e matrix to get the desired matrix:

$$
m_e = \frac{\rho \bar{h} A_e}{60}\begin{bmatrix} 6+4\alpha_1 & 0 & 6-\alpha_3 & 0 & 6-\alpha_2 & 0 \\ & 6+4\alpha_1 & 0 & 6-\alpha_3 & 0 & 6-\alpha_2 \\ & & 6+4\alpha_2 & 0 & 6-\alpha_1 & 0 \\ & & & 6+4\alpha_2 & 0 & 6-\alpha_1 \\ & & & & 6+4\alpha_3 & 0 \\ sym. & & & & & 6+4\alpha_3 \end{bmatrix}
$$

Problem 3

Figure 7-39 shows a plane strain problem to be solved using only one rectangular element. Determine the nodal displacements and element stresses.

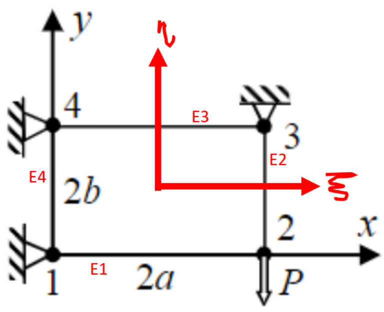

Figure 7-39

$$
\xi = \frac{x - \frac{x_2+x_1}{2}}{a}, \quad \eta = \frac{y - \frac{y_2+y_1}{2}}{b}
$$

$$
k_e = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ & K_{22} & K_{23} & K_{24} \\ & & K_{33} & K_{34} \\ sym. & & & K_{44} \end{bmatrix}, \quad k_e = \frac{abh}{16}\int_{-1}^{1}\int_{-1}^{1} B^T c B d\xi d\eta
$$

$$
B = \begin{array}{c} \begin{array}{cccc} \text{Node 1} & \text{Node 2} & \text{Node 3} & \text{Node 4} \end{array} \\ \left[\begin{array}{cc|cc|cc|cc} -\frac{1-\eta}{a} & 0 & \frac{1-\eta}{a} & 0 & \frac{1+\eta}{a} & 0 & -\frac{1+\eta}{a} & 0 \\ 0 & -\frac{1-\xi}{b} & 0 & -\frac{1+\xi}{b} & 0 & \frac{1+\xi}{b} & 0 & \frac{1-\xi}{b} \\ -\frac{1-\xi}{b} & -\frac{1-\eta}{a} & -\frac{1+\xi}{b} & \frac{1-\eta}{a} & \frac{1+\xi}{b} & \frac{1+\eta}{a} & \frac{1-\xi}{b} & -\frac{1+\eta}{a} \end{array}\right] \end{array}
$$

$$
C = \begin{bmatrix} C_{11} & C_{12} & 0 \\ & C_{22} & 0 \\ sym. & & C_{33} \end{bmatrix} = \frac{E(1-v)}{(1+v)(1-v)}\begin{bmatrix} 1 & \frac{v}{1-v} & 0 \\ & 1 & 0 \\ sym. & & \frac{1-2v}{2(1-v)} \end{bmatrix}
$$

$$
k_e d_e = f_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

matlab output:

X disp: $(72*P*a^2*b^2*(v-1)*(v+1))/(E*h*(64*a^4*v^2-96*a^4*v+32*a^4+128*a^2*b^2*v^2$ $-192*a^2*b^2*v+71*a^2*b^2+64*b^4*v^2-96*b^4*v+32*b^4))$
Y disp: $-(96*P*a*b*(v-1)*(v+1)*(2*a^2*v+2*b^2*v-a^2-2*b^2))/$ $(E*h*(64*a^4*v^2-96*a^4*v+32*a^4+128*a^2*b^2*v^2-192*a^2*b^2*v+71*a^2*b^2+$ $64*b^4*v^2-96*b^4*v+32*b^4))$

all other nodes have 0 disp b/c clamped

$$
\sigma_{xx} = E\varepsilon_{xx} = E\frac{dx}{l} \rightarrow \sigma_{e1} = \sigma_{xx} = \frac{Edx}{2a}
$$
$$
\sigma_{yy} = E\varepsilon_{yy} = E\frac{dy}{l} \rightarrow \sigma_{e2} = \sigma_{yy} = \frac{Edy}{2b}
$$

$$\boxed{\sigma_{e1} = \frac{E}{2a}\text{X disp}}$$

$$\boxed{\sigma_{e2} = \frac{E}{2b}\text{Y disp}}$$

no forces/disp in other elements so no stress in those

Matlab Code

```
clear; clc; close all;

syms n x a b v E C11 C12 C22 C33 P h
```

```
% B matrix
B = [-(1-n)/a 0 (1-n)/a 0 (1+n)/a 0 -(1+n)/a 0;
    0 -(1-x)/b 0 -(1+x)/b 0 (1+x)/b 0 (1-x)/b;
    -(1-x)/b -(1-n)/a -(1+x)/b (1-n)/a (1+x)/b (1+n)/a (1-x)/b -(1+n)/a];

% C matrix
C = (E*(1-v))/((1+v)*(1-v))*[1 v/(1-v) 0;
                            v/(1-v) 1 0;
                            0 0 (1-2*v)/(2*(1-v))];

% ke matrix
ke = (a*b*h/16)*int(int(transpose(B)*C*B, x, [-1, 1]), n, [-1, 1]);

% fe matrix
fe = [0; 0; 0; -P; 0; 0; 0; 0];

% calc disp at only node 2
disp2 = inv(ke(3:4, 3:4))*fe(3:4, 1);
fprintf("x disp @ node 2:")
simplify(disp2(1))
fprintf("y disp @ node 2:")
simplify(disp2(2))
```