AEEM 3062

HOMEWORK # 2

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Problem 1

A reference frame of unit vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, is obtained by rotating a frame of unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, about the $\mathbf{e}_3$ axis by an angle $\varphi = 20°$ as shown in Fig. 1.

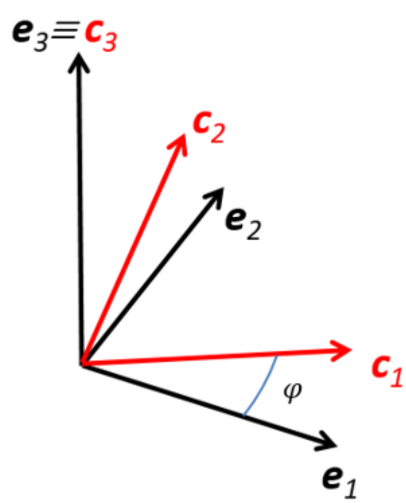

Figure 1

Part A

If vector $\mathbf{v}$ has coordinates (2,1,-3) in frame $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ find its coordinates in frame $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$. [20 pts]

equation to transform coordinates from one reference to another: $\hat{y} = R^T \hat{x}$

$$v_e = [\,2 \quad 1 \quad -3\,], \qquad R_{c_3} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \varphi = 20^\circ$$

Use the rotation vector for the third axis R_{c_3} from the notes.

Compute the transpose of R_{c_3} and v_e using Matlab's transpose() function and plug both values into the equation above. Solve using Matlab's mtimes() function.

$$v_c = R_{c_3}^T v_e^T = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos 20 & \sin 20 & 0 \\ -\sin 20 & \cos 20 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2\cos(20^\circ) + \sin(20^\circ) \\ \cos(20^\circ) - 2\sin(20^\circ) \\ -3 \end{bmatrix}$$

$$v_c = \begin{bmatrix} 2.221 \\ 0.256 \\ -3 \end{bmatrix}$$

Part B

Show that the length of **v** is the same in the two reference frames. [10 pts]

equation for length of a vector: $\quad |v_{e/c}| = \sqrt{v_{e1/c_1}^2 + v_{e2/c_2}^2 + v_{e3/c_3}^2}$

Plug in values and solve for the lengths.

$$|v_e| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4+1+9} = \sqrt{14} = 3.742$$

$$|v_c| = \sqrt{2.221^2 + 0.256^2 + (-3)^2} = \sqrt{4.933 + 0.0655 + 9} = \sqrt{14} = 3.742$$

$$|v_e| = 3.742 = |v_c|$$

Problem 2

Consider a reference frame of unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and an operator that in this frame is represented by matrix

$$A = \begin{bmatrix} 2 & 7 & 3 \\ 7 & -1 & -3 \\ 3 & -3 & 5 \end{bmatrix}$$

Part A

Find the eigenvectors and the corresponding eigenvalues of A. [10 pts]

$$A = \begin{bmatrix} 2 & 7 & 3 \\ 7 & -1 & -3 \\ 3 & -3 & 5 \end{bmatrix}, \quad [x_\gamma, \gamma] = eig(A)$$

Create a vector for A in Matlab and use the eig() function definition above to return the eigenvectors and values as two matrices.

$$x_\gamma = \begin{bmatrix} 0.6048 & -0.0357 & -0.7955 \\ -0.7342 & -0.4119 & -0.5397 \\ -0.3084 & 0.9105 & -0.2753 \end{bmatrix}, \quad \gamma = \begin{bmatrix} -8.0267 & 0 & 0 \\ 0 & 6.2393 & 0 \\ 0 & 0 & 7.7874 \end{bmatrix}$$

The eigenvectors in x_γ are each in their own column. The eigenvalues in γ are found along the diagonals. The first column corresponds to the first vector and value... and so on. Split them up into their individual forms.

$$x_{\gamma_1} = [0.6048 \quad -0.7342 \quad -0.3084], \quad x_{\gamma_2} = [-0.0357 \quad -0.4119 \quad 0.9105],$$
$$x_{\gamma_3} = [-0.7955 \quad -0.5397 \quad -0.2753]$$

$$\gamma_1 = -8.0267, \quad \gamma_2 = 6.2393, \quad \gamma_3 = 7.7874$$

Part B

Select two arbitrary eigenvectors and show that they are orthogonal. [10 pts]

Arbitrarily select 2 eigenvectors.

$$x_{\gamma_2} = [-0.0357 \quad -0.4119 \quad 0.9105], \quad x_{\gamma_3} = [-0.7955 \quad -0.5397 \quad -0.2753]$$

Rearrange the dot product equation to solve for θ, the angle between the vectors.

$$x_{\gamma_2} * x_{\gamma_3} = |x_{\gamma_2}||x_{\gamma_3}|\cos\theta, \quad \theta = \arccos\frac{x_{\gamma_2} * x_{\gamma_3}}{|x_{\gamma_2}||x_{\gamma_3}|}$$

Use Matlab's norm() function to compute the lengths of the eigenvectors.

$$|x_{\gamma_2}| = norm(x_{\gamma_2}) = 1, \quad |x_{\gamma_3}| = norm(x_{\gamma_3}) = 0.9999$$

Use Matlab's dot() function to compute the dot product of the eigenvectors.

$$x_{\gamma_2} * x_{\gamma_3} = dot(x_{\gamma_2}, x_{\gamma_3}) = 0$$

Plug the dot product and the lengths into the rearranged dot product equation to find θ.

$$\theta = \arccos\frac{0}{0.9999} = \arccos 0$$

$$\theta = 90^\circ$$

θ is 90° meaning that the eigenvectors are orthogonal.

Part C

Find the expression of operator A in the reference frame formed by the eigenvectors [15 pts]

To find the expression of a matrix in a new reference frame we need to compute the rotation vector. The rotation vector R is found by multiplying the eigenvectors by the vectors of the original reference frame e. This was computed using matlab to give the rotation vector R. Then we compute its transpose. By using the first equation shown below, we can calculate the new matrix A' by multiplying the transpose of R by A and then that result by R using Matlab's mtimes() command.

equation for the expression of a matrix in a new reference frame: $A' = R^T AR$

$$R = \begin{bmatrix} x_{\gamma_1} * \hat{e}_1 & x_{\gamma_2} * \hat{e}_1 & x_{\gamma_3} * \hat{e}_1 \\ x_{\gamma_1} * \hat{e}_2 & x_{\gamma_2} * \hat{e}_2 & x_{\gamma_3} * \hat{e}_2 \\ x_{\gamma_1} * \hat{e}_3 & x_{\gamma_2} * \hat{e}_3 & x_{\gamma_3} * \hat{e}_3 \end{bmatrix}, \quad \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The rotation matrix R is given above, as well as the original reference frame unit vectors $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$.

Using the Matlab function mtimes() for each index of the matrix R gives us the rotational matrix R. Its transpose is also calculated using Matlab's transpose() function.

$$R = \begin{bmatrix} 0.6048 & -0.0357 & -0.7955 \\ -0.7342 & -0.4119 & -0.5397 \\ -0.3084 & 0.9105 & -0.2753 \end{bmatrix}, \quad R^T = \begin{bmatrix} 0.6048 & -0.7342 & -0.3084 \\ -0.0357 & -0.4119 & 0.9105 \\ -0.7955 & -0.5397 & -0.2753 \end{bmatrix}$$

Plug R, R^T, and A into the A' equation.

$$A' = \begin{bmatrix} 0.6048 & -0.7342 & -0.3084 \\ -0.0357 & -0.4119 & 0.9105 \\ -0.7955 & -0.5397 & -0.2753 \end{bmatrix} \begin{bmatrix} 2 & 7 & 3 \\ 7 & -1 & -3 \\ 3 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0.6048 & -0.0357 & -0.7955 \\ -0.7342 & -0.4119 & -0.5397 \\ -0.3084 & 0.9105 & -0.2753 \end{bmatrix}$$

Solve using Matlab's mtimes() function. First do mtimes(R^T,A) and then mtimes() with the resultant of the previous operation and R.

$$A' = \begin{bmatrix} -8.0262 & 0 & 0 \\ 0 & 6.2390 & 0 \\ 0 & 0 & 7.7865 \end{bmatrix}$$

Part D

What can be said about the form of A in the reference frame formed by the eigenvectors [5 pts]

A in the reference form of the eigenvectors takes the same form as the matrix defining the eigenvalues. It is a diagonal matrix with the eigenvalues along the diagonal.

Problem 3

A wall is formed by bending a rectangular strip of width t to form a circular arc of radius R as show in Fig. 2. Note that the wall is not closed and its free ends are distance from the y axis $\pm\alpha$ The wall is subject to a uniform internal pressure P. Find the components F_y and F_z of the force resultant from the pressure [30 pts].

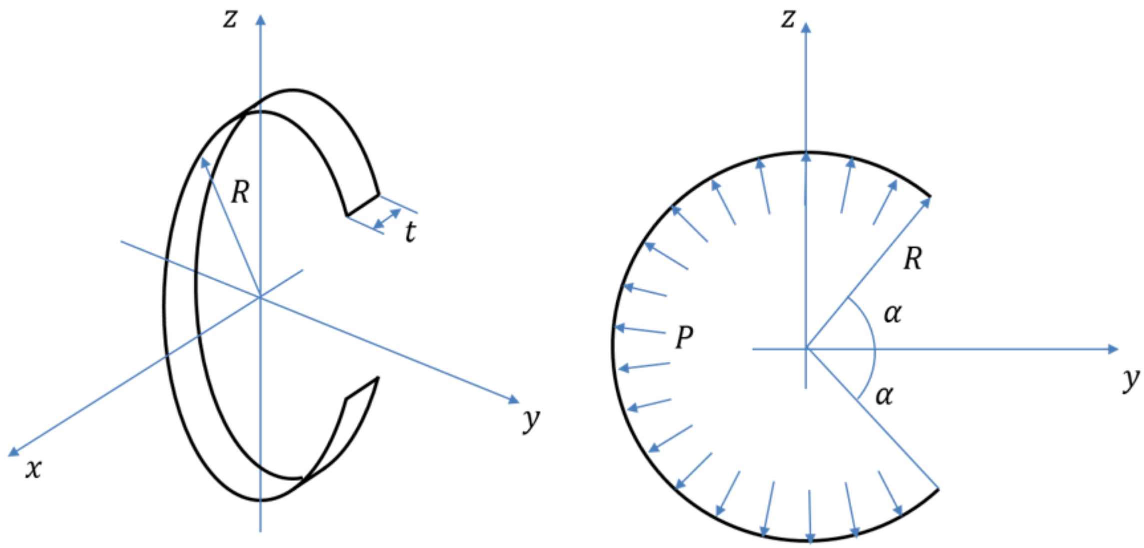

Figure 2

Define the force in the y direction as a surface integral using polar coordinates. We know that pressure is force per area, so force is pressure times area. This allows us to set up and integral describing the force as a function of pressure, radius, and width t.

Pressure will be uniform across the surface. Since the force will be the area times the pressure, we can keep the pressure term outside of the integral as it is a scalar. t can also be outside of the integral as it is constant. We integrate from 0 to R and from α to $-\alpha$ to cover the entire arc length of the circle. The angle integral is of $\cos\theta$ so that we capture the y component of the force.

$$F_y = Pt\int_0^R dr\int_\alpha^{-\alpha}\cos\theta d\theta$$

Integrate each term and solve.

$$F_y = Pt[r]_0^R[\sin\theta]_\alpha^{-\alpha} = PtR(\sin(-\alpha) - \sin(\alpha)) =$$
$$PtR(-\sin(\alpha) - \sin(\alpha)) = PtR(-2\sin\alpha)$$

$$F_y = -2PRt\sin\alpha$$

To find the z component, we simply switch the $\cos\theta$ for a $\sin\theta$ to reference the z component instead of the y.

$$F_z = Pt\int_0^R dr\int_\alpha^{-\alpha}\sin\theta d\theta$$

Integrate each term and solve.

$$F_z = Pt[r]_0^R[-\cos\theta]_\alpha^{-\alpha} = PtR(-\cos(-\alpha) + \cos(\alpha)) = PtR(-\cos(\alpha) + \cos(\alpha)) = PtR(0)$$

$$F_z = 0$$