AEEM 3062

HOMEWORK # 8

2022-2023 SPRING

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DUE: 11.59 PM on 3/30/2023

Problem 1

Figure 1 shows the geometry of the spar-strut system of a small, two-seat general aviation airplane. The spar ABC is clamped at one end (fuselage side) and is supported at point B by a strut DB that is pinned to the spar. The strut is also pinned at its other end. The spar is made of aluminum alloy AA 2024-T3 that has Young's modulus E =73.1 GPa and yield stress σ_Y =290 MPa. The strut is made of a different alloy AA 6061-T6 with E' =68.9 GPa and σ'_Y =276 MPa. Additionally, the spar has rectangular box section of 3 mm wall thickness while the strut is a circular tube with the same wall thickness. The geometrical properties of the cross sections of the spar and strut are shown in Fig. 2. The effect of aerodynamic forces during flight is modeled by considering a uniform lift distribution q = 700 N/m.

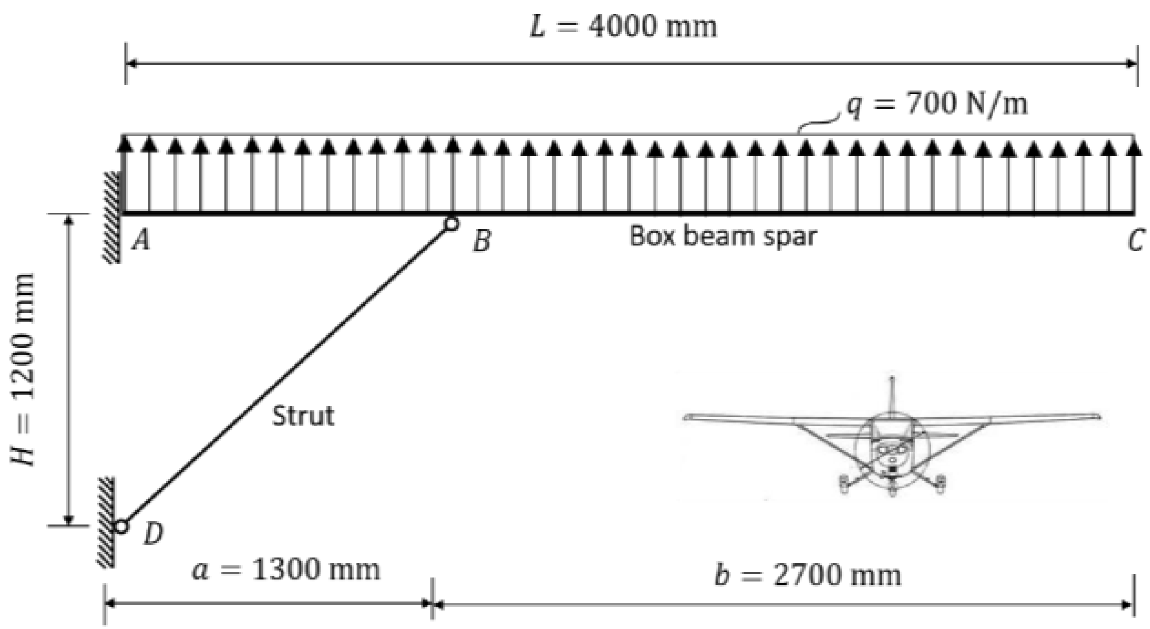


Figure 1. Diagram of the spar-strut wing structure

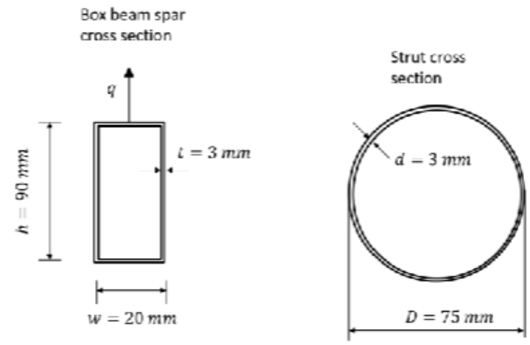
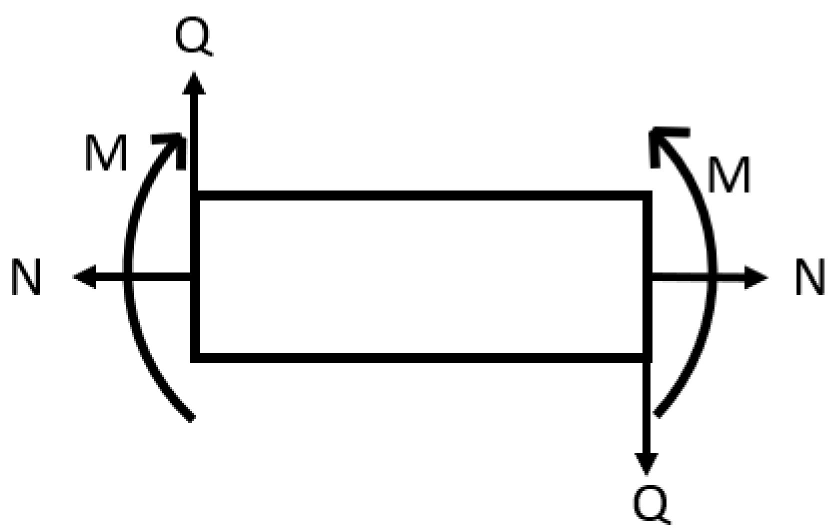


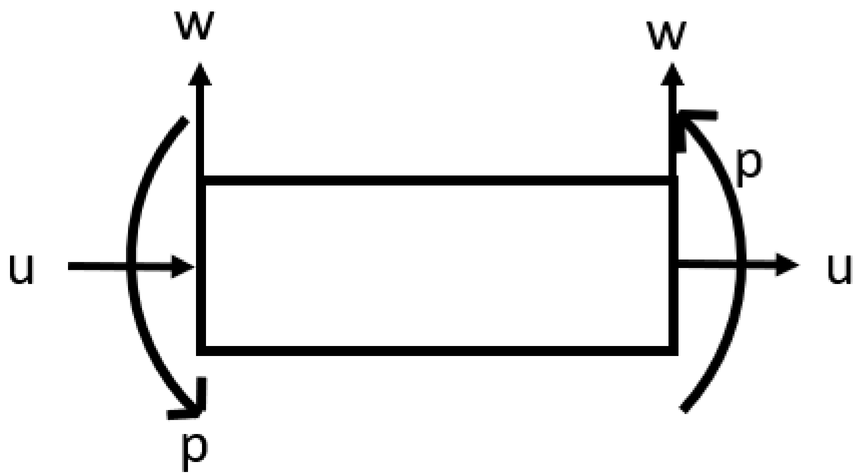
Figure 2. Cross sections of the spar and strut. Dimensions refer to the outer geometry of the sections

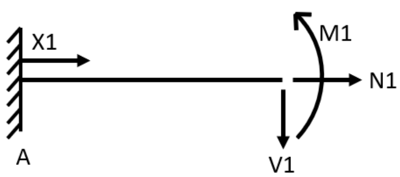
Part A

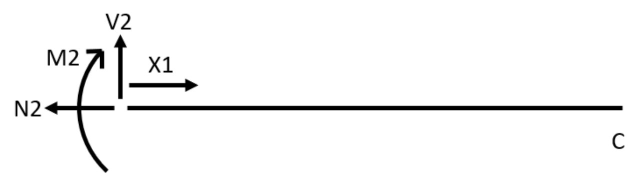
[70 pts] Select appropriate differential equations to describe the problem and write the system of algebraic equations needed to solve the problem. Your analysis must include:

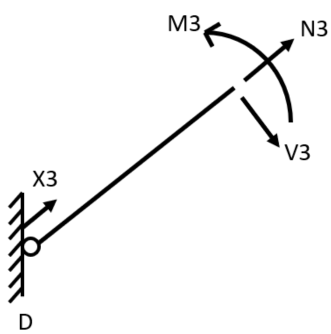
1. Statement of relevant boundary conditions
2. Free body diagrams on which the boundary conditions are based
3. Matrix representation of the system of algebraic equations – no symbolic solution of the system is required

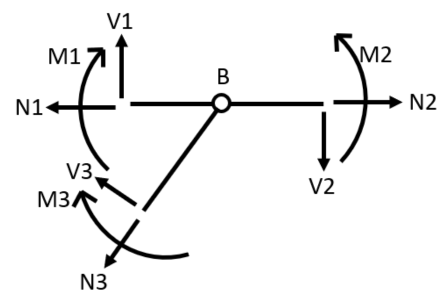


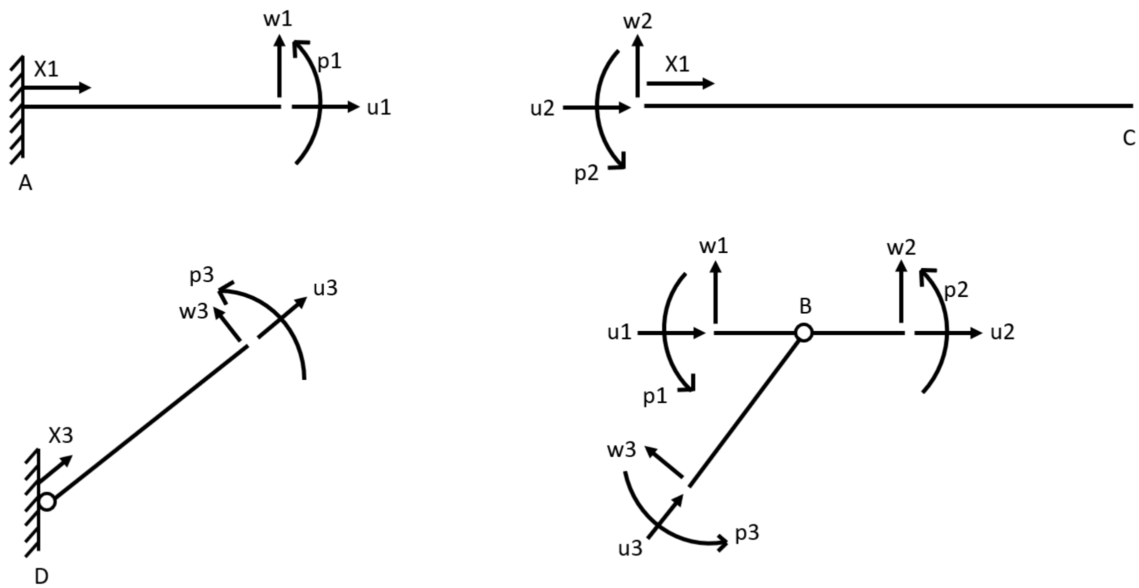

w
w
p
u
u
p


X1
M1
N1
A
V1


V2
M2
X1
N2
C


M3
N3
V3
X3
D


V1
M1
B
M2
N1
N2
V3
V2
M3
N3



Section 1:

$$\frac{\partial N_1}{\partial x_1} = 0,\ N_1 = C_1$$

$$\frac{\partial V_1}{\partial x_1} = q,\ V_1 = qx + C_2$$

$$\frac{\partial M_1}{\partial x_1} = V_1 = qx + C_2,\ M_1 = \frac{1}{2}qx^2 + C_2x + C_3a$$

$$\frac{\partial u_1}{\partial x_1} = \frac{N_1}{E\Sigma} = \frac{C_1}{E\Sigma},\ u_1 = \frac{1}{E\Sigma}(C_1x + C_4a)$$

$$\frac{\partial \varphi_1}{\partial x_1} = \frac{M_1}{EI} = \frac{\frac{1}{2}qx^2 + C_2x + C_3a}{EI},$$

$$\varphi_1 = \frac{1}{EI}(\frac{1}{6}qx^3 + \frac{1}{2}C_2x^2 + C_3ax + C_5a^2)$$

$$\frac{\partial w_1}{\partial x_1} = \varphi_1 = \frac{1}{EI}(\frac{1}{6}qx^3 + \frac{1}{2}C_2x^2 + C_3ax + C_5a^2),$$

$$w_1 = \frac{1}{EI}(\frac{1}{24}qx^4 + \frac{1}{6}C_2x^3 + \frac{1}{2}C_3ax^2 + C_5a^2x + C_6a^3)$$

Section 2:

$$\frac{\partial N_2}{\partial x_1} = 0,\ N_2 = C_7$$

$$\frac{\partial V_2}{\partial x_1} = q,\ V_2 = qx + C_8$$

$$\frac{\partial M_2}{\partial x_1} = V_2 = qx + C_8,\ M_2 = \frac{1}{2}qx^2 + C_8x + C_9b$$

$$\frac{\partial u_2}{\partial x_1} = \frac{N_2}{E\Sigma} = \frac{C_7}{E\Sigma},\ u_2 = \frac{1}{E\Sigma}(C_7x + C_{10}b)$$

$$\frac{\partial \varphi_2}{\partial x_1} = \frac{M_2}{EI} = \frac{\frac{1}{2}qx^2 + C_8x + C_9b}{EI},$$

$$\varphi_2 = \frac{1}{EI}(\frac{1}{6}qx^3 + \frac{1}{2}C_8x^2 + C_9bx + C_{11}b^2)$$

$$\frac{\partial w_2}{\partial x_1} = \varphi_2 = \frac{1}{EI}(\frac{1}{6}qx^3 + \frac{1}{2}C_8x^2 + C_9bx + C_{11}b^2),$$

$$w_2 = \frac{1}{EI}(\frac{1}{24}qx^4 + \frac{1}{6}C_8x^3 + \frac{1}{2}C_9bx^2 + C_{11}b^2x + C_{12}b^3)$$

Section 3:

$$\frac{\partial N_3}{\partial x_3} = 0,\ N_3 = C_{13}$$

$$\frac{\partial V_3}{\partial x_3} = 0,\ V_3 = C_{14}$$

$$\frac{\partial M_3}{\partial x_3} = V_3 = C_{14},\ M_3 = C_{14}x + C_{15}L_3$$

$$\frac{\partial u_3}{\partial x_3} = \frac{N_3}{E'\Sigma'} = \frac{C_{13}}{E'\Sigma'},\ u_3 = \frac{1}{E'\Sigma'}(C_{13}x + C_{16}L_3)$$

$$\frac{\partial \varphi_3}{\partial x_3} = \frac{M_3}{E'I'} = \frac{C_{14}x + C_{15}L_3}{E'I'},$$

$$\varphi_3 = \frac{1}{E'I'}(\frac{1}{2}C_{14}x^2 + C_{15}L_3x + C_{17}L_3^2)$$

$$\frac{\partial w_3}{\partial x_3} = \varphi_3 = \frac{1}{E'I'}(\frac{1}{2}C_{14}x^2 + C_{15}L_3x + C_{17}L_3^2),$$

$$w_3 = \frac{1}{E'I'}(\frac{1}{6}C_{14}x^3 + \frac{1}{2}C_{15}L_3x + C_{17}L_3^2x + C_{18}L_3^3)$$

Define boundary conditions for the external ends:

$$u_1(0) = 0$$
$$w_1(0) = 0$$
$$\varphi_1(0) = 0$$
$$N_2(L) = 0$$
$$V_2(L) = 0$$
$$M_2(L) = 0$$
$$M_3(0) = 0$$
$$u_3(0) = 0$$
$$w_3(0) = 0$$

Define boundary conditions @ pin B:

$$u_1(a) = u_2(a)$$
$$w_1(a) = w_2(a)$$
$$u_1(a)\cos(\theta) + w_1(a)\sin(\theta) - u_3(L_3) = 0$$
$$u_1(a)\sin(\theta) + w_1(a)\cos(\theta) - w_3(L_3) = 0$$
$$\varphi_1(a) = \varphi_2(a)$$
$$N_2(a)\cos(\theta) - N_1(a)\cos(\theta) - N_3(L_3) + V_1(a)\sin(\theta) - V_2(a)\sin(\theta) = 0$$
$$-N_2(a)\sin(\theta) + N_1(a)\sin(\theta) + V_3(L_3) + V_1(a)\cos(\theta) - V_2(a)\cos(\theta) = 0$$
$$M_1(a) = M_2(a)$$
$$M_3(L_3) = 0$$

Plug in and solve:

$$u_1(0) = 0 = \frac{1}{E\Sigma}(C_1 0 + C_4 a),\ C_4 a = 0,$$
$$C_4 = 0$$

$$w_1(0) = 0 = \frac{1}{EI}(\frac{1}{24}q0^4 + \frac{1}{6}C_2 0^3 + \frac{1}{2}C_3 a0^2 + C_5 a^2 0 + C_6 a^3),\ C_6 a^3 = 0,$$
$$C_6 = 0$$

$$\varphi_1(0) = 0 = \frac{1}{EI}(\frac{1}{6}q0^3 + \frac{1}{2}C_2 0^2 + C_3 a0 + C_5 a^2),\ C_5 a^2 = 0,$$
$$C_5 = 0$$

$$N_2(L) = 0 = C_7,$$
$$C_7 = 0$$

$$V_2(L) = 0 = qL + C_8,$$
$$C_8 = -qL$$

$$M_2(L) = 0 = \frac{1}{2}qL^2 - qL^2 + C_9 b,$$
$$C_9 = \frac{qL^2}{2b}$$

$$M_3(0) = 0 = C_{14}0 + C_{15}L_3,\ C_{15}L_3 = 0,$$
$$C_{15} = 0$$

$$u_3(0) = 0 = \frac{1}{E'\Sigma'}(C_{13}0 + C_{16}L_3),\ C_{16}L_3 = 0,$$
$$C_{16} = 0$$

$$w_3(0) = 0 = \frac{1}{E'I'}(\frac{1}{6}C_{14}0^3 + \frac{1}{2}C_{15}L_3 0 + C_{17}L_3^2 0 + C_{18}L_3^3),\ C_{18}L_3^3 = 0,$$
$$C_{18} = 0$$

$$u_1(a) = u_2(a),\ \frac{1}{E\Sigma}(C_1 a + 0a) = \frac{1}{E\Sigma}(0a + C_{10}b),\ C_1 a = C_{10}b,$$
$$(a)C_1 - (b)C_{10} = 0$$

$$w_1(a) = w_2(a),\ \frac{1}{EI}(\frac{1}{24}qa^4 + \frac{1}{6}C_2 a^3 + \frac{1}{2}C_3 a^3 + 0a^3 + 0a^3) =$$
$$\frac{1}{EI}(\frac{1}{24}qa^4 + \frac{1}{6}C_8 a^3 + \frac{1}{2}C_9 ba^2 + C_{11}b^2 a + C_{12}b^3),$$
$$(\frac{a^3}{6})C_2 + (\frac{a^3}{2})C_3 - (\frac{a^3}{6})C_8 - (\frac{ba^2}{2})C_9 - (b^2 a)C_{11} - (b^3)C_{12} = 0$$

$$u_1(a)\cos(\theta) + w_1(a)\sin(\theta) - u_3(L_3) = 0,$$
$$(\frac{1}{E\Sigma}(C_1 a + 0a))\cos(\theta) + (\frac{1}{EI}(\frac{1}{24}qa^4 + \frac{1}{6}C_2 a^3 + \frac{1}{2}C_3 a^3 + 0a^2 x + 0a^3))\sin(\theta)$$
$$-\frac{1}{E'\Sigma'}(C_{13}L_3 + 0L_3),$$
$$(\frac{a\cos(\theta)}{E\Sigma})C_1 + (\frac{a^3\sin(\theta)}{6EI})C_2 + (\frac{a^3\sin(\theta)}{2EI})C_3 - (\frac{L_3}{E'\Sigma'})C_{13} = -\frac{qa^4\sin(\theta)}{24EI}$$

$$u_1(a)\sin(\theta) + w_1(a)\cos(\theta) - w_3(L_3) = 0,$$
$$(\frac{1}{E\Sigma}(C_1 a + 0a))\sin(\theta) + (\frac{1}{EI}(\frac{1}{24}qa^4 + \frac{1}{6}C_2 a^3 + \frac{1}{2}C_3 a^3 + 0a^2 x + 0a^3))\cos(\theta)$$
$$-\frac{1}{E'I'}(\frac{1}{6}0x^3 + \frac{1}{2}0L_3 x + C_{17}L_3^3 + 0L_3^3) = 0,$$
$$(\frac{a\sin(\theta)}{E\Sigma})C_1 + (\frac{a^3\cos(\theta)}{6EI})C_2 + (\frac{a^3\cos(\theta)}{2EI})C_3 - (\frac{L_3^3}{E'I'})C_{17} = -\frac{qa^4\cos(\theta)}{24EI}$$

$$\varphi_1(a) = \varphi_2(a),\ \frac{1}{EI}(\frac{1}{6}qa^3 + \frac{1}{2}C_2 a^2 + C_3 a^2 + 0a^2) =$$
$$\frac{1}{EI}(\frac{1}{6}qa^3 + \frac{1}{2}C_8 a^2 + C_9 ba + C_{11}b^2),$$
$$(\frac{a^2}{2})C_2 + (a^2)C_3 - (\frac{a^2}{2})C_8 - (ba)C_9 - (b^2)C_{11} = 0$$

$$N_2(a)\cos(\theta) - N_1(a)\cos(\theta) - N_3(L_3) + V_1(a)\sin(\theta) - V_2(a)\sin(\theta) = 0,$$
$$0\cos(\theta) - C_1\cos(\theta) - C_{13} + (qa + C_2)\sin(\theta) - (qa + C_8)\sin(\theta) = 0,$$
$$-(\cos(\theta))C_1 + (\sin(\theta))C_2 - (\sin(\theta))C_8 - C_{13} = 0$$

$$-N_2(a)\sin(\theta) + N_1(a)\sin(\theta) + V_3(L_3) + V_1(a)\cos(\theta) - V_2(a)\cos(\theta) = 0,$$
$$-0\sin(\theta) + C_1\sin(\theta) + 0 + (qa + C_2)\cos(\theta) - (qa + C_8)\cos(\theta) = 0,$$
$$(\sin(\theta))C_1 + (\cos(\theta))C_2 - (\cos(\theta))C_8 = 0$$

$$M_1(a) = M_2(a),\ \frac{1}{2}qa^2 + C_2 a + C_3 a = \frac{1}{2}qx^2 + C_8 a + C_9 b,$$
$$(a)C_2 + (a)C_3 - (a)C_8 - (b)C_9 = 0$$

$$M_3(L_3) = 0,\ C_{14}L_3 + 0L_3 = 0,\ C_{14}L_3 = 0,$$
$$C_{14} = 0$$

Now create a matrix:

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{a^3}{6} & \frac{a^3}{2} & 0 & 0 & 0 & 0 & -\frac{a^3}{6} & -\frac{ba^2}{2} & 0 & -b^2a & -b^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{a\cos(\theta)}{E\Sigma} & \frac{a^3\sin(\theta)}{6EI} & \frac{a^3\sin(\theta)}{2EI} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{L_3^3}{E'\Sigma'} & 0 & 0 & 0 & 0 & 0 \\
\frac{a\sin(\theta)}{E\Sigma} & \frac{a^3\cos(\theta)}{6EI} & \frac{a^3\cos(\theta)}{2EI} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{L_3^3}{E'I'} & 0 \\
0 & \frac{a^2}{2} & a^2 & 0 & 0 & 0 & 0 & -\frac{a^2}{2} & -ba & 0 & -b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0 & 0 & -\sin(\theta) & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 & 0 & -\cos(\theta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & a & 0 & 0 & 0 & 0 & -a & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \\ C_{13} \\ C_{14} \\ C_{15} \\ C_{16} \\ C_{17} \\ C_{18} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -qL \\ \frac{qL^2}{2b} \\ 0 \\ 0 \\ -\frac{qa^4\sin(\theta)}{24EI} \\ -\frac{qa^4\cos(\theta)}{24EI} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Part B

[30 pts] Using the parameters of the problem solve the system of equations numerically and produce the following diagrams for the spar and the strut (where applicable)

a. Normal force
b. Shear force
c. Bending moment
d. Rotation
e. Deformation – it is sufficient to show the vertical component of displacement of the spar alone
f. Reactions – the reaction forces at A and D should be provided in terms of horizontal and vertical components.

First we can calulate all of the values (and convert everything to m and Pa):

$$\theta = \tan^{-1}\left(\frac{H}{a}\right) = 42.71^\circ = 0.7454\text{rad}$$

$$L_3 = \sqrt{H^2 + a^2} = 1.769\text{m}$$

$$\Sigma = hw - (h - 2t)(w - 2t) = 6.24E - 4\text{m}^2$$

$$\Sigma' = \pi\left(\frac{D}{2}\right)^2 - \pi\left(\frac{d}{2}\right)^2 = 6.786E - 4\text{m}^2$$

$$I = \frac{bh^3 - b_1h_1^3}{12} = 5.235E - 7\text{m}^4$$

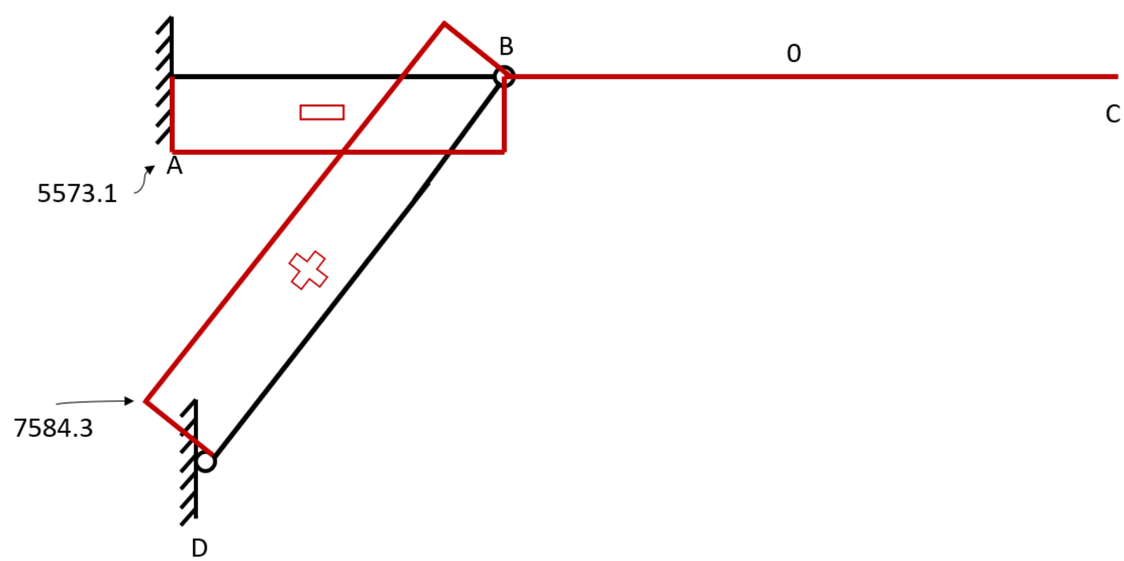
$$I' = \frac{\pi(D^4 - d^4)}{64} = 4.405E - 7\text{m}^4$$

Then plug in to the matrix in Matlab and solve for the coefficients:

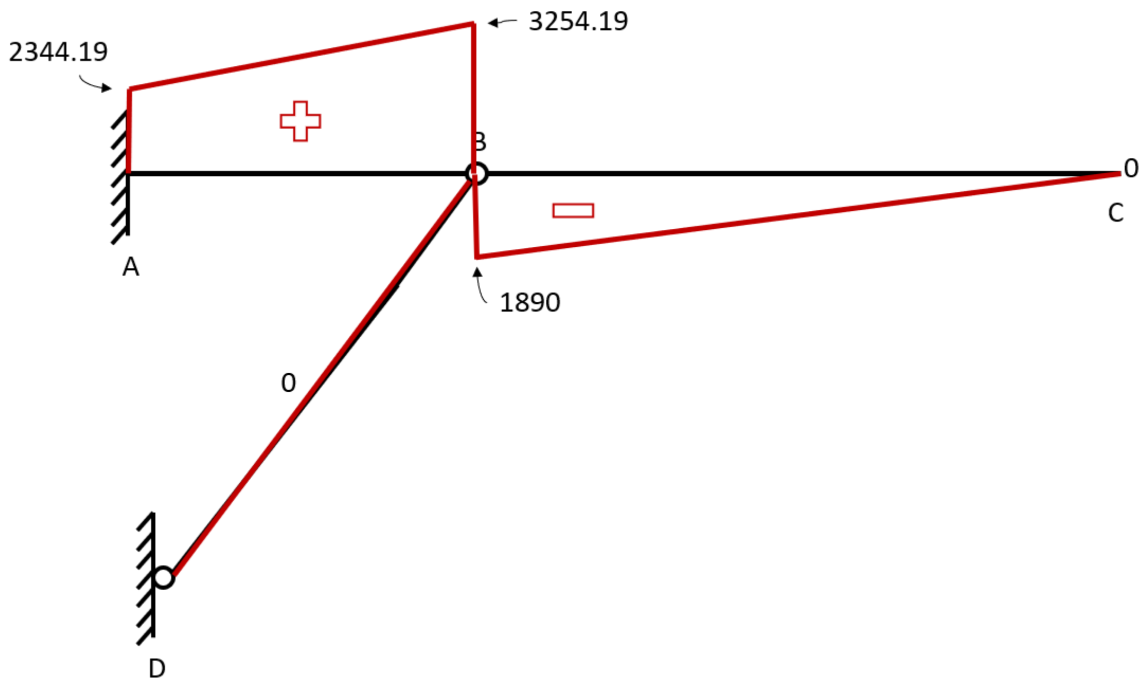
$$C_1 = -55731.1$$
$$C_2 = 2344.2$$
$$C_3 = -836.4971$$
$$C_4 = 0$$
$$C_5 = 0$$
$$C_6 = 0$$
$$C_7 = 0$$
$$C_8 = -2800$$
$$C_9 = 2074.1$$
$$C_{10} = -2683.3$$
$$C_{11} = -596.2744$$
$$C_{12} = 95.6984$$
$$C_{13} = 7584.3$$
$$C_{14} = 0$$
$$C_{15} = 0$$
$$C_{16} = 0$$
$$C_{17} = 1.807$$
$$C_{18} = 0$$

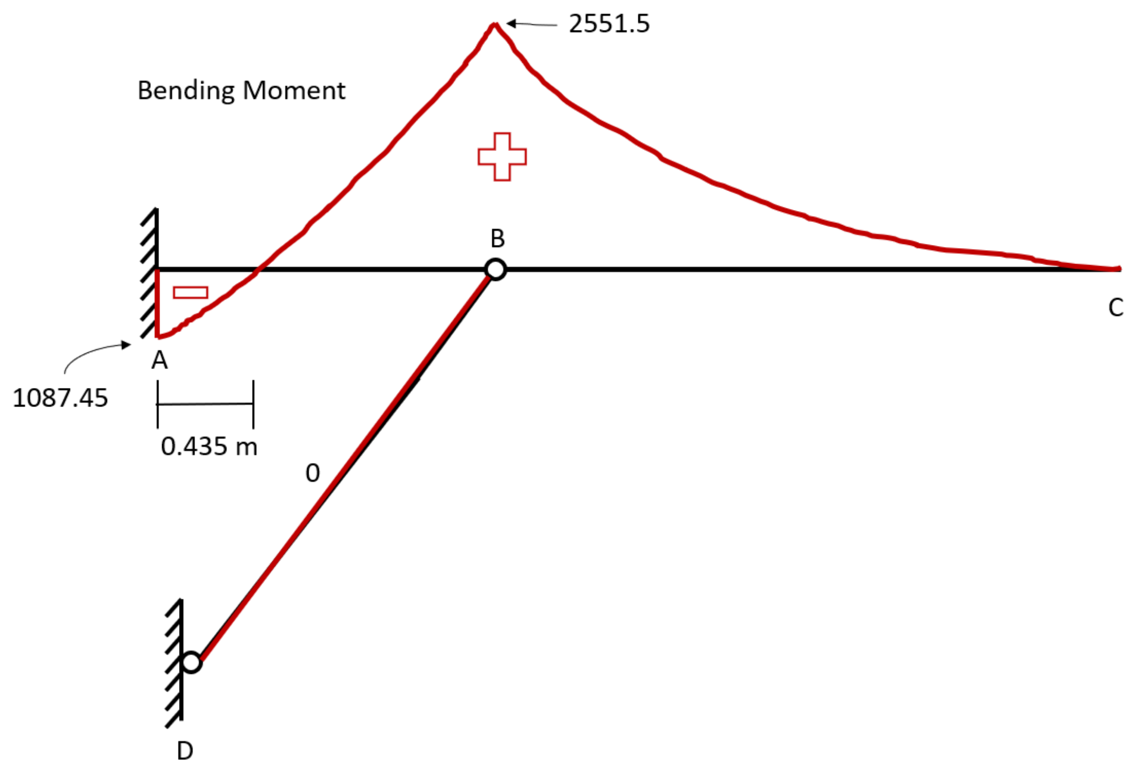
Then I programmed the equations with their coefficients into matlab to get the plots (I am not showing all of that here because i programmed the equations from above into matlab and used the constants directly from the matrix that solved for them):

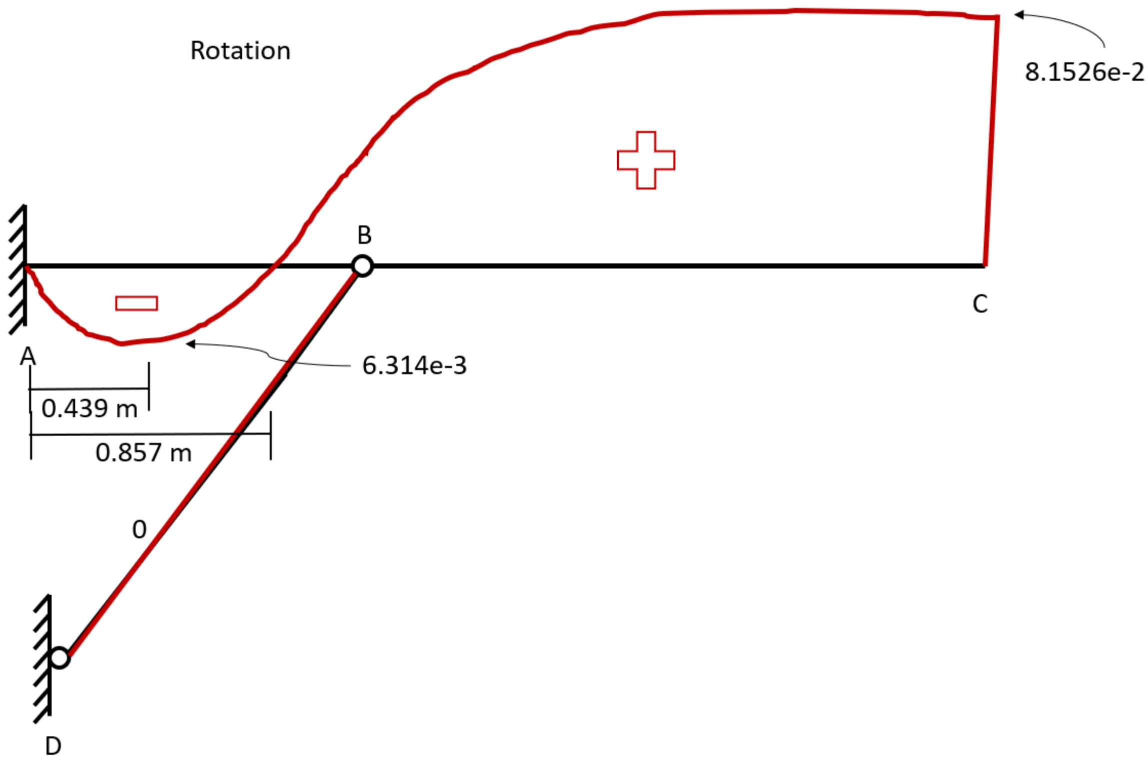
Normal Force

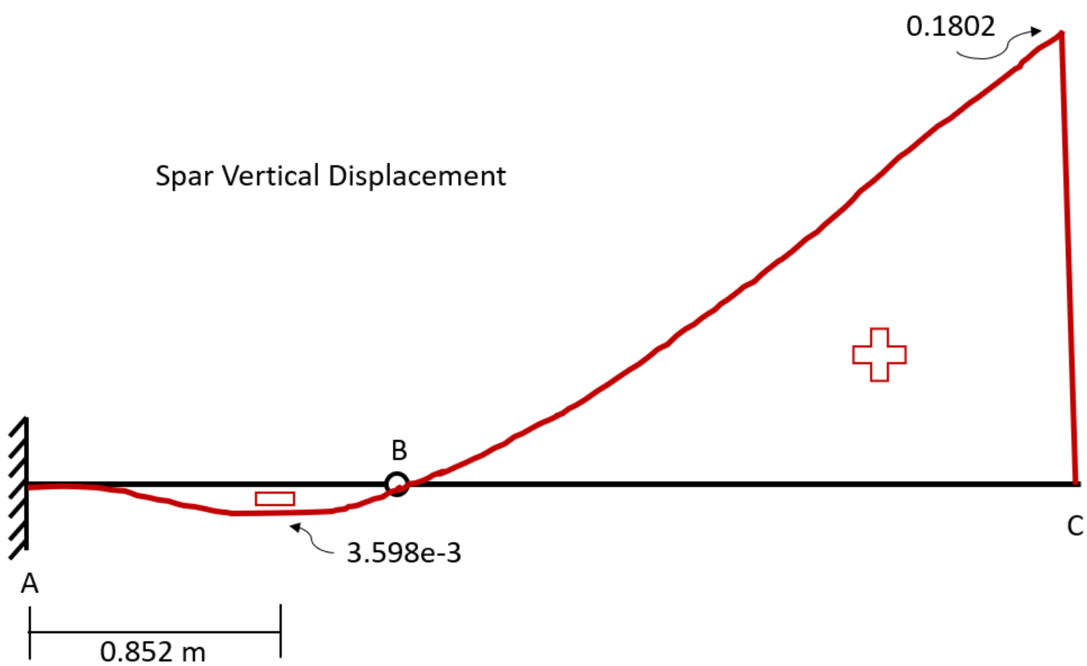


Shear Force


3254.19
2344.19
B
0
A
C
1890
0
D


2551.5
Bending Moment
B
C
A
1087.45
0.435 m
0
D


Rotation
8.1526e-2
B
C
A
6.314e-3
0.439 m
0.857 m
0
D


0.1802
Spar Vertical Displacement
B
C
A
3.598e-3
0.852 m

Reactions