BME6013C HW#5

Slade Brooks
M13801712

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Part 1

First we will split up the function as much as possible:

$$v(t) = \frac{d}{dt}\left(\frac{d}{dt}\left(e^{-t^2/2\sigma^2}e^{j2\pi f_0 t}\right)\right)$$

Now we will create a function $w(t)$ to define everything inside of the first d/dt.

$$w(t) = \frac{d}{dt}\left(e^{-t^2/2\sigma^2}e^{j2\pi f_0 t}\right) = \frac{d}{dt}g(t)$$
$$v(t) = \frac{d}{dt}w(t)$$

Now, from our F.T. sheet we can see that the F.T. of the derivative of a function is:

$$\mathcal{F}\left[\frac{d}{dt}w(t)\right] = V(f) = j2\pi f W(f)$$
$$\mathcal{F}\left[\frac{d}{dt}g(t)\right] = W(f) = j2\pi f G(f)$$
$$V(f) = (j2\pi f)^2 G(f)$$

We can see that the final expression for the F.T. is an expression times the F.T. of the original function inside the derivative $g(t)$. We can see from the F.T. sheet that the tranform of a function times $e^{j2\pi f_0 t}$ is a frequency shift on the transform of the function by f_0. We can see that the function resembles a gaussian pulse, so we can multiply by 1 to get it into pure gaussian form, then take the F.T. and apply the frequency shift the get the transform $G(f)$.

$$g(t) = e^{j2\pi f_0 t} e^{-t^2/2\sigma^2} = \frac{\sqrt{2\pi}\sigma}{\sqrt{2\pi}\sigma} e^{j2\pi f_0 t} e^{-t^2/2\sigma^2}$$

$$g(t) = \sqrt{2\pi}\sigma e^{j2\pi f_0 t} \frac{e^{-t^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$

$$G(f) = \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(f-f_0)^2}$$

$$\boxed{V(f) = -(2\pi f)^2 \sigma \sqrt{2\pi} e^{-2\pi^2\sigma^2(f-f_0)^2}}$$

Part 2

We will start by taking the Fourier transform of $g(x-\zeta, y-\eta)$:

$$\mathcal{F}\left[g(x-\zeta, y-\eta)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(ux+vy)} g(x-\zeta, y-\eta)dxdy$$

Now we will create a new variable to sub in:

$$x' = x - \zeta$$
$$x = x' + \zeta$$
$$dx = dx'$$
$$y' = y - \eta$$
$$y = y' + \eta$$
$$dy = dy'$$

$$\mathcal{F}\left[g(x-\zeta, y-\eta)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(u(x'+\zeta)+v(y'+\eta))} g(x', y')dx'dy'$$

$$\mathcal{F}\left[g(x-\zeta, y-\eta)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(ux'+u\zeta+vy'+v\eta)} g(x', y')dx'dy'$$

$$\mathcal{F}\left[g(x-\zeta, y-\eta)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(ux'+vy')} e^{-j2\pi(u\zeta+v\eta)} g(x', y')dx'dy'$$

$$\mathcal{F}\left[g(x-\zeta, y-\eta)\right] = e^{-j2\pi(u\zeta+v\eta)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(ux'+vy')} g(x', y')dx'dy'$$

We can see that after simplifying that we are left with the definition of a 2D Fourier transform of $g(x', y')$ and the other exponential term is a constant that can be removed from the integral. This means we can finally show that the shift theorem is correct and:

$$\mathcal{F}\left[g(x-\zeta, y-\eta)\right] = e^{-j2\pi(u\zeta+v\eta)} G(u, v)$$

Part 3

First we will separate the two "parts" of the given function. This will be a more convenient way to analyze it.

$$f(x,y) = \frac{e^{-x^2/(2\sigma_x^2)}}{\sqrt{2\pi}\sigma_x} \frac{\mathcal{H}(L_y/2 - |y|)}{L_y}$$

This form clearly shows that these are separable since $f(x,y) = g(x) \cdot h(y)$. We know from the definition of separable functions that this means that $\mathcal{F}[f(x,y)] = G(u) \cdot H(v)$. The first term is a Gaussian pulse that exactly follows the Fourier transform sheet definition for the F.T. of a Gaussian pulse. We can simply replace t with x and σ with σ_x. The second term is a Heaviside function divided by L_y. The F.T. sheet shows that the F.T. of a Heaviside is almost a sinc function:

$$\begin{aligned} H(f) &= \frac{\sin \pi f L_y}{\pi f} \\ sinc(L_y f) = \frac{\sin \pi f L_y}{\pi f L_y} &= \frac{1}{L_y} \frac{\sin \pi f L_y}{\pi f} \end{aligned}$$

We can see from this that since L_y is a constant, the F.T. of a Heaviside divided by its period is a sinc function. Since we found the function is separable and now know the F.T. of both parts of it, we can write the final equation:

$$\boxed{F(u,v) = e^{-2\pi^2\sigma_x^2 u^2} sinc(L_y v)}$$