AEEM 3062

HOMEWORK # 7

2022-2023 SPRING

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DUE: 11.59 PM on 2/25/2023

Problem 1

A beam AB is clamped at one end and joined to a bar, BC, at the other end by a hinge. The bar is pinned at end C. The length of the beam is L while the length of the bar is $L/2$. The beam and bar have the same circular cross section but are made of different materials. The beam is subject to a vertical load that varies linearly from q_0 at the clamp to zero at the hinge.

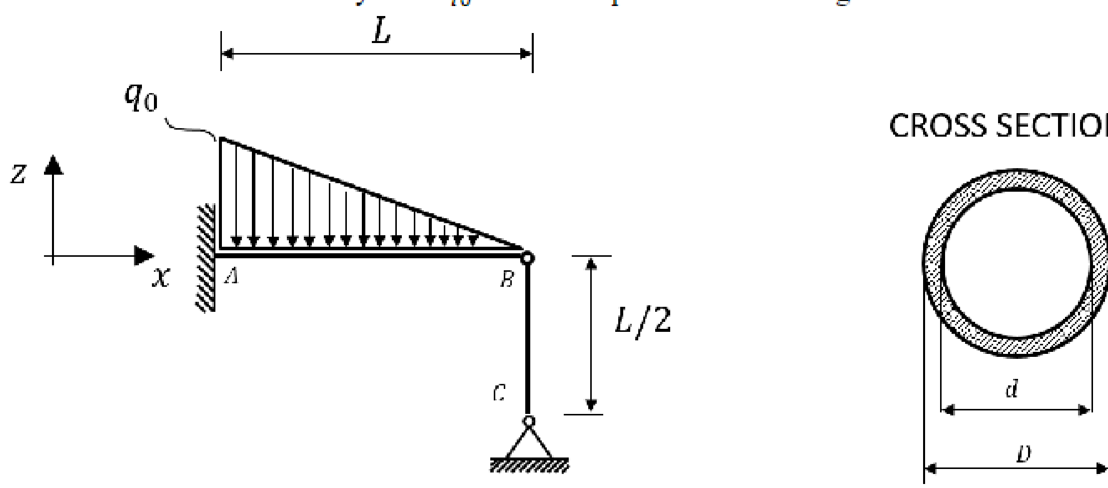
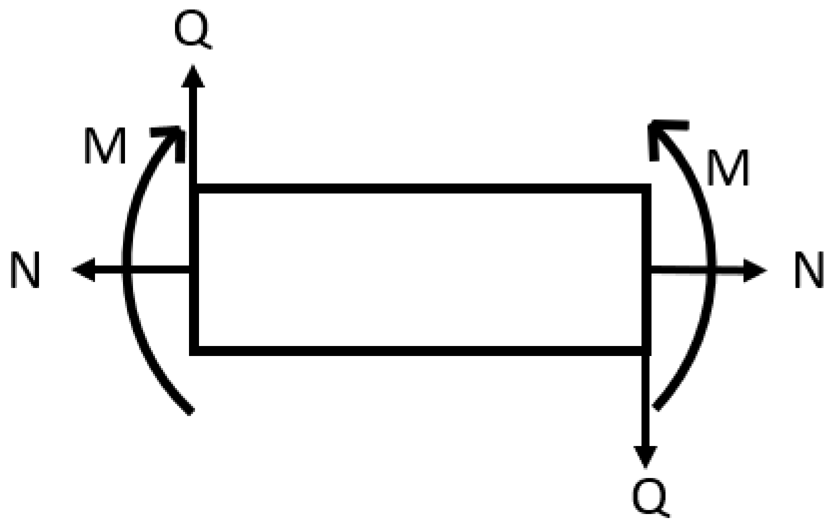


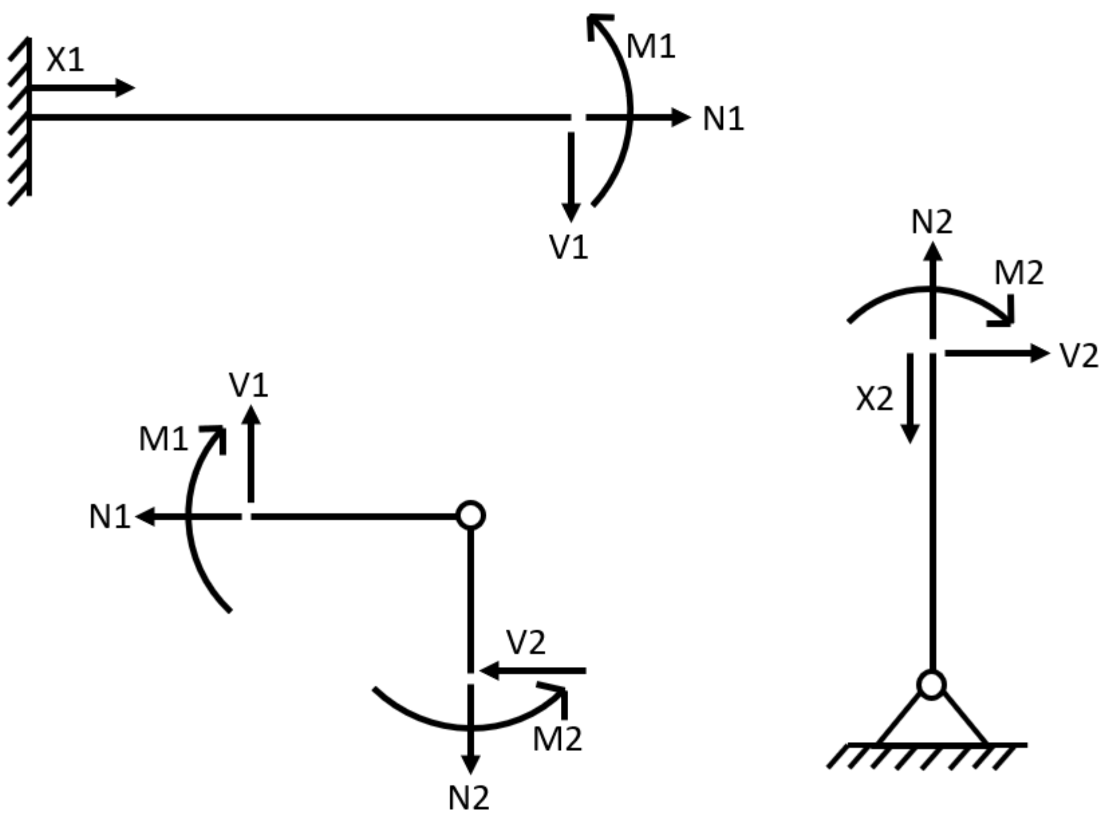
Figure 1

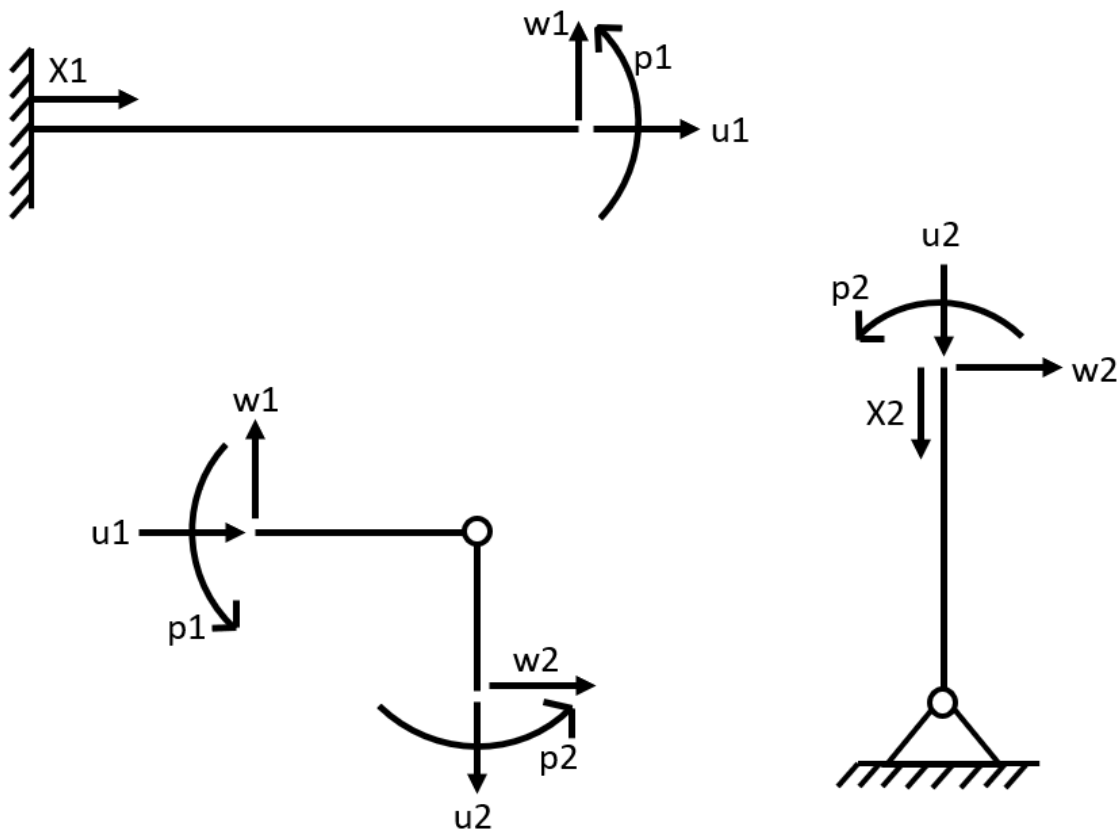
Part A

Write the system of equations needed to find the expressions of the functions describing how the forces, moments, rotation, and displacement components vary along the length of the beam and the bar. Assume that E_1 and E_2 are the Young's moduli of the beam and bar respectively, that the cross sectional area is Σ and that the second moment of area is I. Write the system of equations in matrix form [20 pts]

first create diagrams for each section and the corner


Q
M
M
N
N
Q


X1
M1
N1
V1
N2
M2
V2
X2
V1
M1
N1
V2
M2
N2



create differential equations for each section

Section 1:

$$\frac{\delta N_1}{\delta x_1} = 0,\ N_1 = C_1$$

$$\frac{\delta V_1}{\delta x_1} = -(-\frac{q_o}{L}x + q_o),\ V_1 = \frac{q_o}{2L}x^2 - q_o x + C_2$$

$$\frac{\delta M_1}{\delta x_1} = V_1,\ M_1 = \frac{q_o}{6L}x^3 - \frac{q_o}{2}x^2 + C_2 x + C_3 L$$

$$\frac{\delta u_1}{\delta x_1} = \frac{N_1}{E_1\Sigma} = \frac{C_1}{E_1\Sigma},\ u_1 = \frac{C_1}{E_1\Sigma}x + C_4$$

$$\frac{\delta \varphi_1}{\delta x_1} = \frac{M_1}{E_1 I},\ \varphi_1 = \frac{1}{E_1 I}(\frac{q_o}{24L}x^4 - \frac{q_o}{6}x^3 + \frac{C_2}{2}x^2 + C_3 L x + C_5 L^2)$$

$$\frac{\delta w_1}{\delta x_1} = \varphi_1,\ w_1 = \frac{1}{E_1 I}(\frac{q_o}{120L}x^5 - \frac{q_o}{24}x^4 + \frac{C_2}{6}x^3 + \frac{C_3 L}{2}x^2 + C_5 L^3 x + C_6 L^3)$$

Section 2:

$$\frac{\delta N_2}{\delta x_2} = 0,\ N_2 = C_7$$
$$\frac{\delta V_2}{\delta x_2} = 0,\ V_2 = C_8$$
$$\frac{\delta M_2}{\delta x_2} = V_2,\ M_2 = C_8 x + \frac{C_9 L}{2}$$
$$\frac{\delta u_2}{\delta x_2} = \frac{N_2}{E_2 \Sigma},\ u_2 = \frac{C_7}{E_2 \Sigma} x + C_{10}$$
$$\frac{\delta \varphi_2}{\delta x_2} = \frac{M_2}{E_2 I},\ \varphi_2 = \frac{1}{E_2 I}(\frac{C_8}{2} x^2 + \frac{C_9 L}{2} x + C_{11} L^2)$$
$$\frac{\delta w_2}{\delta x_2} = \varphi_2,\ w_2 = \frac{1}{E_2 I}(\frac{C_8}{6} x^3 + \frac{C_9 L}{4} x^2 + C_{11} L^2 x + C_{12} L^3)$$

Next we need to define the boundary conditions.

External Boundaries:

$$u_1(0) = 0$$
$$w_1(0) = 0$$
$$\varphi_1(0) = 0$$
$$u_2(L/2) = 0$$
$$w_2(L/2) = 0$$
$$M_2(L/2) = 0$$

Corner:

$$w_1(L) = -u_2(0)$$
$$u_1(L) = w_2(0)$$
$$N_1(L) + V_2(0) = 0$$
$$V_1(L) = N_2(0)$$
$$M_1(L) = 0$$
$$M_2(0) = 0$$

Then we can apply the boundary conditions to the equations:

$$
\begin{aligned}
u_1(0) &= 0 = C_4 \\
w_1(0) &= 0 = \frac{C_6 L^3}{E_1 I} \\
\varphi_1(0) &= 0 = \frac{C_5 L^2}{E_1 I} \\
u_2(L/2) &= 0 = \frac{C_2 L}{2E_2 I} + C_{10} \\
w_2(L/2) &= 0 = \frac{1}{E_2 I}\left(\frac{L^3}{48}C_8 + \frac{L^3}{16}C_9 + \frac{L^3}{2}C_{11} + L^3 C_{12}\right),\ \frac{1}{48}C_8 + \frac{1}{16}C_9 + \frac{1}{2}C_{11} + C_{12} = \\
M_2(L/2) &= 0 = \frac{L}{2}(C_8 + C_9),\ C_8 + C_9 = 0 \\
w_1(L) + u_2(0) &= 0 = C_{10} + \frac{1}{E_1 I}\left(\frac{q_o}{120}L^4 - \frac{q_o}{24}L^4 + \frac{C_2}{6}L^3 + \frac{C_3}{2}L^3 + C_5 L^3 x + C_6 L^3\right), \\
C_{10} &+ \frac{L^3}{E_1 I}\left(\frac{C_2}{6} + \frac{C_3}{2} + C_5 + C_6\right) = \frac{L^4 q_o}{30 E_1 I} \\
u_1(L) - w_2(0) &= \frac{L C_1}{E_1 \Sigma} + C_4 - \frac{L^3 C_{12}}{E_2 I} \\
N_1(L) + V_2(0) &= 0 = C_1 + C_8 \\
V_1(L) - N_2(0) &= 0 = \frac{q_0 L}{2} - q_o L + C_2 - C_7,\ C_2 - C_7 = \frac{q_0 L}{2} \\
M_1(L) &= 0 = \frac{L^2 q_o}{6} - \frac{L^2 q_o}{2} + L(C_2 + C_3),\ C_2 + C_3 = \frac{L q_o}{3} \\
M_2(0) &= 0 = \frac{L C_9}{2}
\end{aligned}
$$

Lastly we can create a matrix from the equations:

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{L^3}{E_1 I} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{L^2}{E_1 I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{L}{2E_2 I} & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{48} & \frac{1}{16} & 0 & \frac{1}{2} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & \frac{L^3}{6E_1 I} & \frac{L^3}{2E_1 I} & 0 & \frac{L^3}{E_1 I} & \frac{L^3}{E_1 I} & 0 & 0 & 0 & 1 & 0 & 0 \\
\frac{L}{E_1 \Sigma} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{L^3}{E_2 I} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{L}{2} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{L^4 q_o}{30 E_1 I} \\ 0 \\ 0 \\ \frac{q_o L}{2} \\ \frac{q_o L}{3} \\ 0
\end{bmatrix}
$$

Part B

Solve the system of equations assuming that: L = 500 mm, D = 25 mm, d = 22 mm, E_1=68.9 GPa, E_2=3 GPa, and $q_0 = 10$ N/mm. Provide the expressions describing how the shear force and bending moment vary along beam AB using the appropriate units [20 pts]

We can convert all values to meters and Pa, then plug in to the matrix above and solve for the constants. I created the matrix in matlab with variables for each value and used matlab to solve.

$$C_1 = 0$$
$$C_2 = 2496.4$$
$$C_3 = -829.73$$
$$C_4 = 0$$
$$C_5 = 0$$
$$C_6 = 3.583E - 14$$
$$C_7 = -3.6023$$
$$C_8 = 0$$
$$C_9 = 0$$
$$C_{10} = 0.03911$$
$$C_{11} = 0$$
$$C_{12} = 0$$

Shear Force:

$$V_1(x) = \frac{q_o}{2L}x^2 - q_o x + C_2 = \frac{10000\mathrm{N/m}}{2(0.5)}x^2 + (10000\mathrm{N/m})x + 2496.4$$

$$\boxed{V_1(x) = (10000\mathrm{N/m})x^2 + (10000\mathrm{N/m})x + 2496.4\mathrm{N}}$$

Since x is in meters, the units for shear force are Newtons.

Moment:

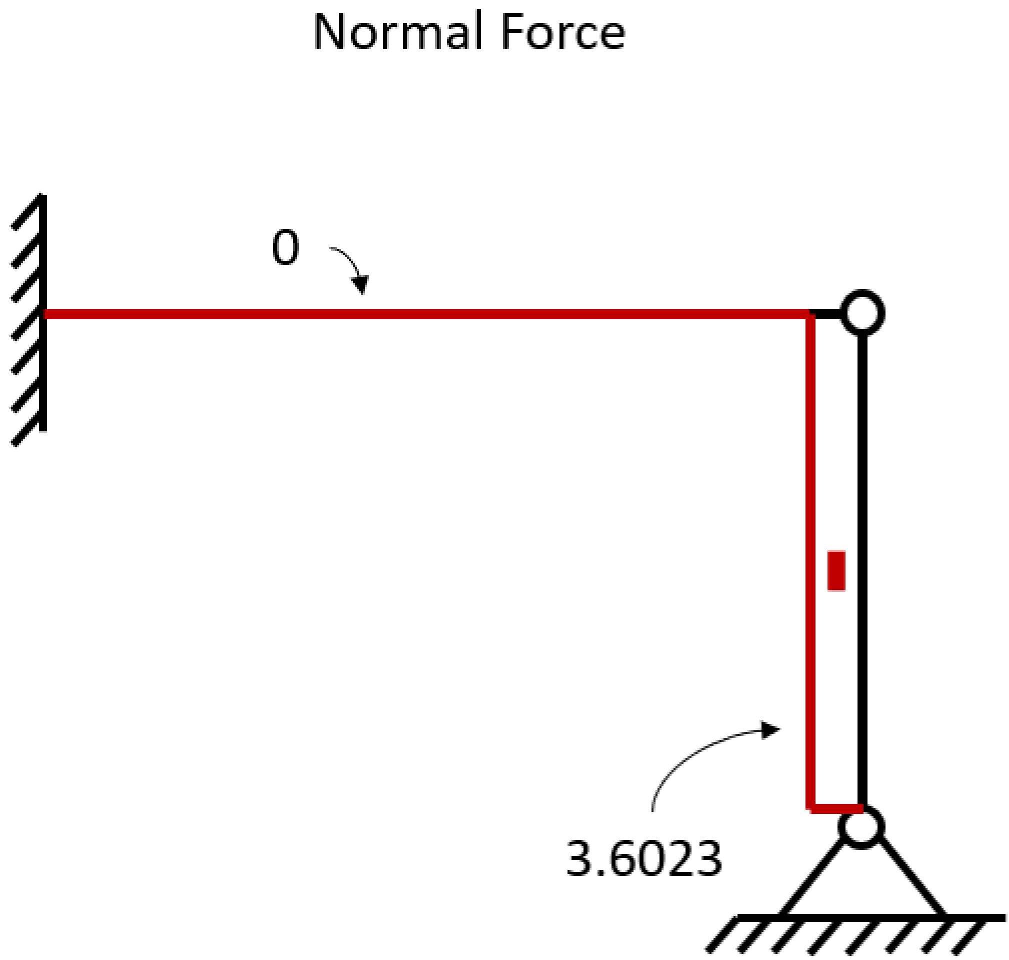
$$M_1(x) = \frac{q_o}{6L}x^3 - \frac{q_o}{2}x^2 + C_2 x + C_3 L = \frac{(10000\mathrm{N/m})}{6(0.5\mathrm{m})}x^3 - \frac{(10000\mathrm{N/m})}{2}x^2 + (2496.4\mathrm{N})x -$$

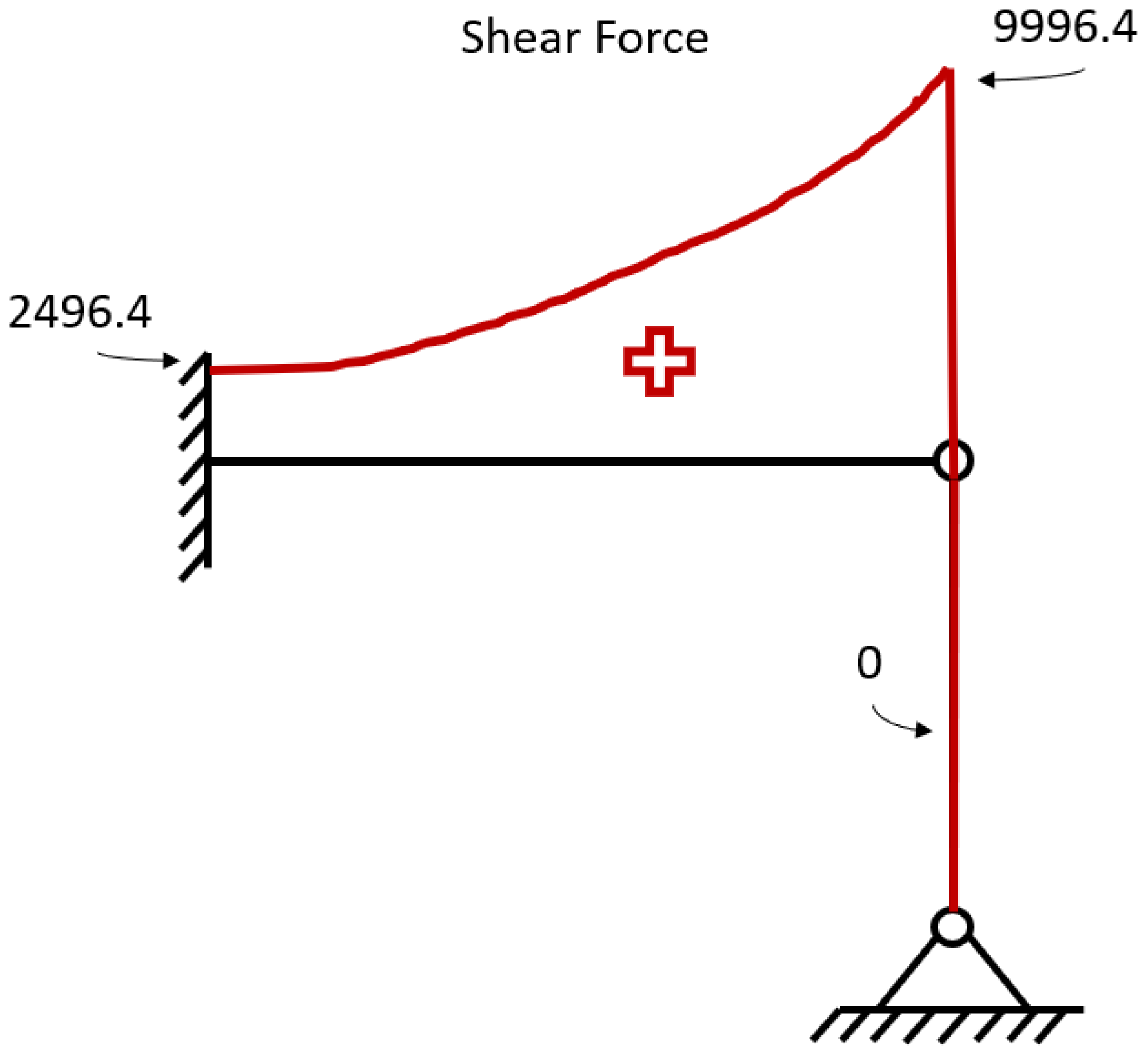
$$\boxed{M_1(x) = (3333.3\mathrm{N/m^2})x^3 - (5000\mathrm{N/m})x^2 + (2496.4\mathrm{N})x - 414.864\mathrm{Nm}}$$

Since x is in meters, the units for bending moment are Newton-meters.

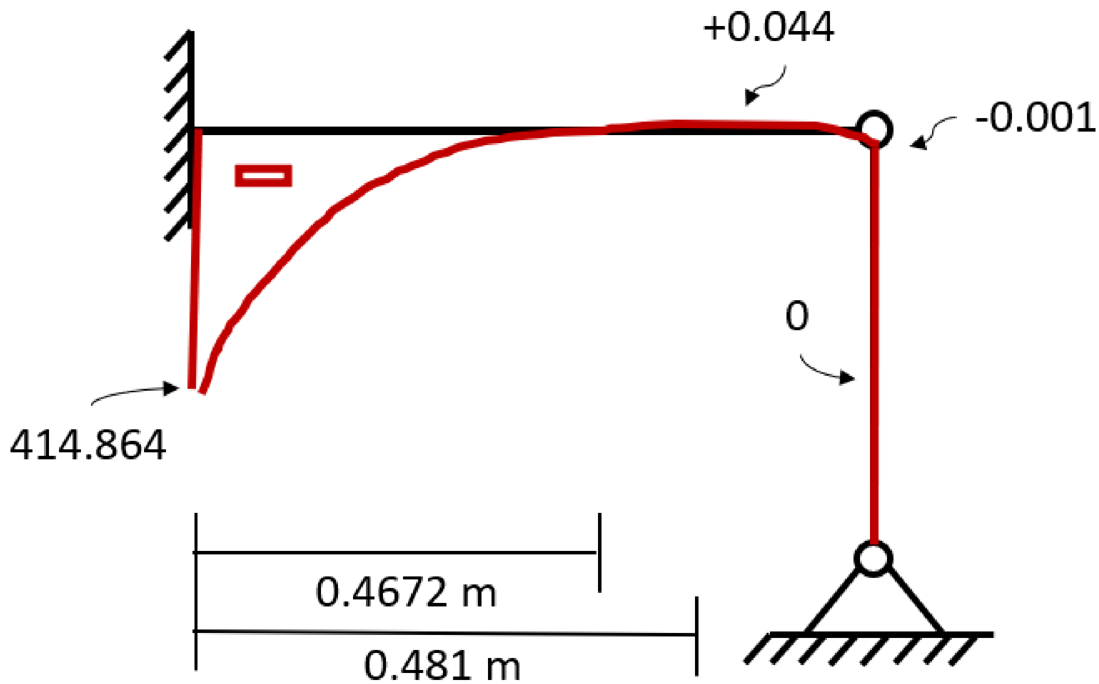
Part C

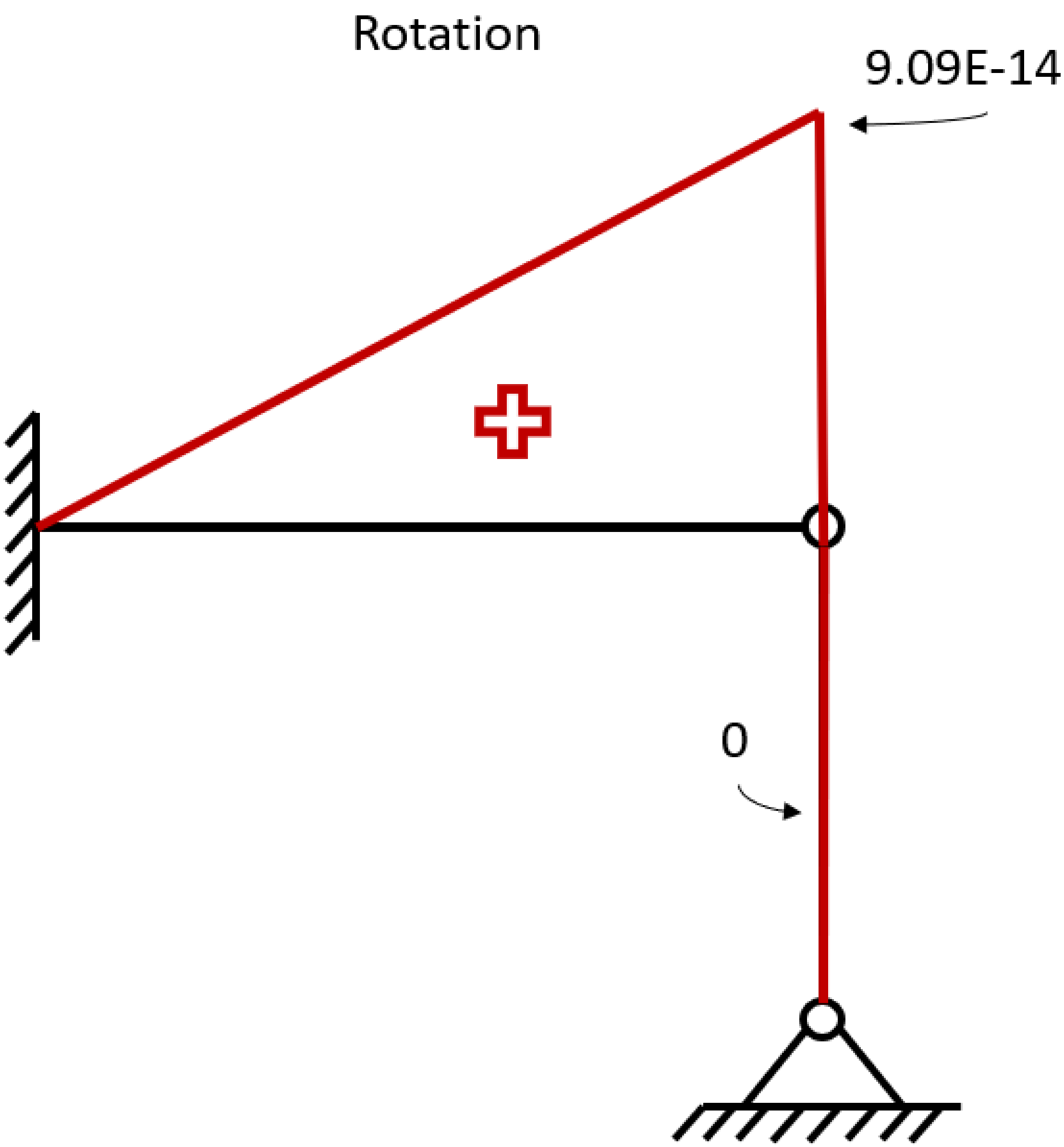
Plot the relevant force, moment, rotation, and displacement diagrams for the beam and bar. Moreover, show the reaction diagram for the entire structure (no explanation needed) [20 pts]



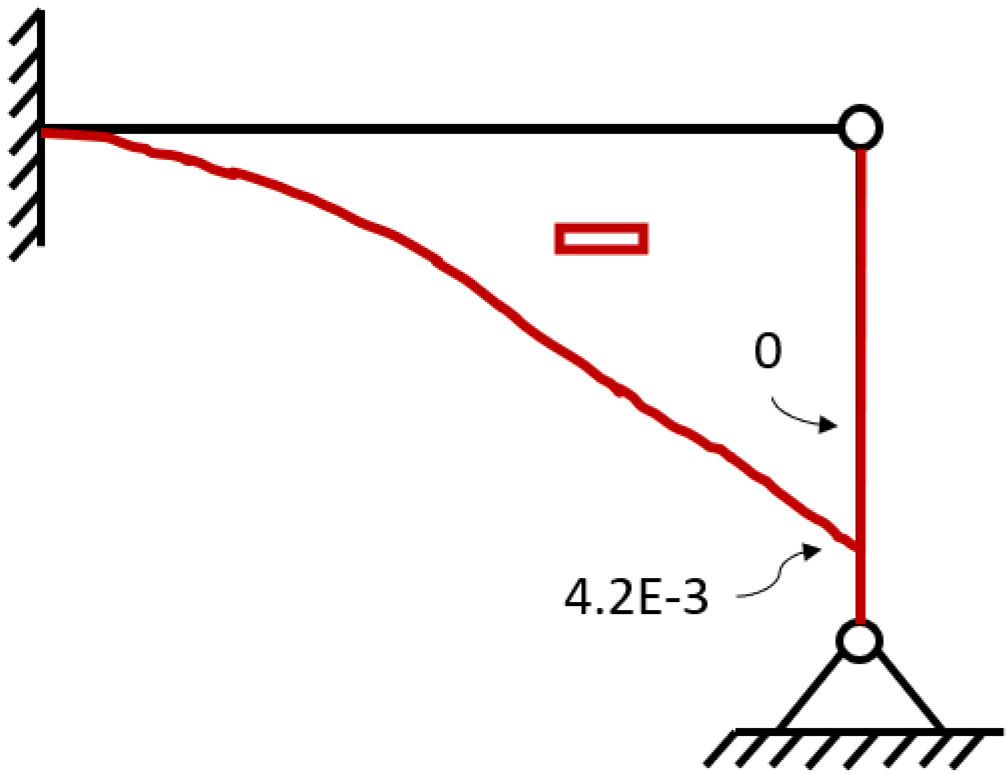

Shear Force
9996.4
2496.4
0

Bending Moment

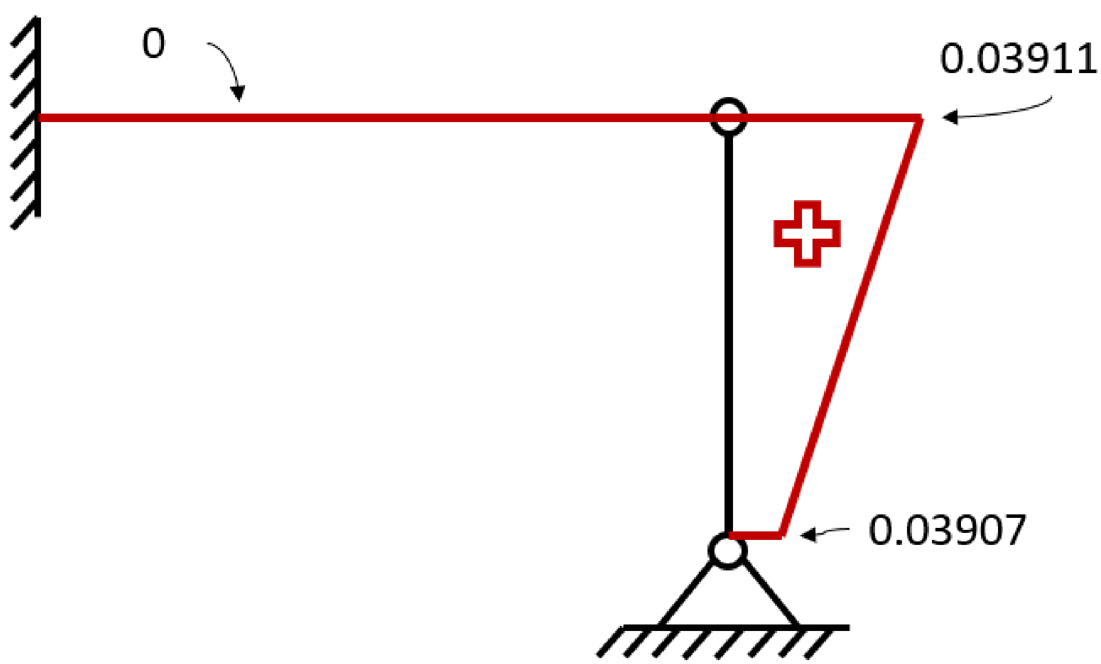



Rotation
9.09E-14
0

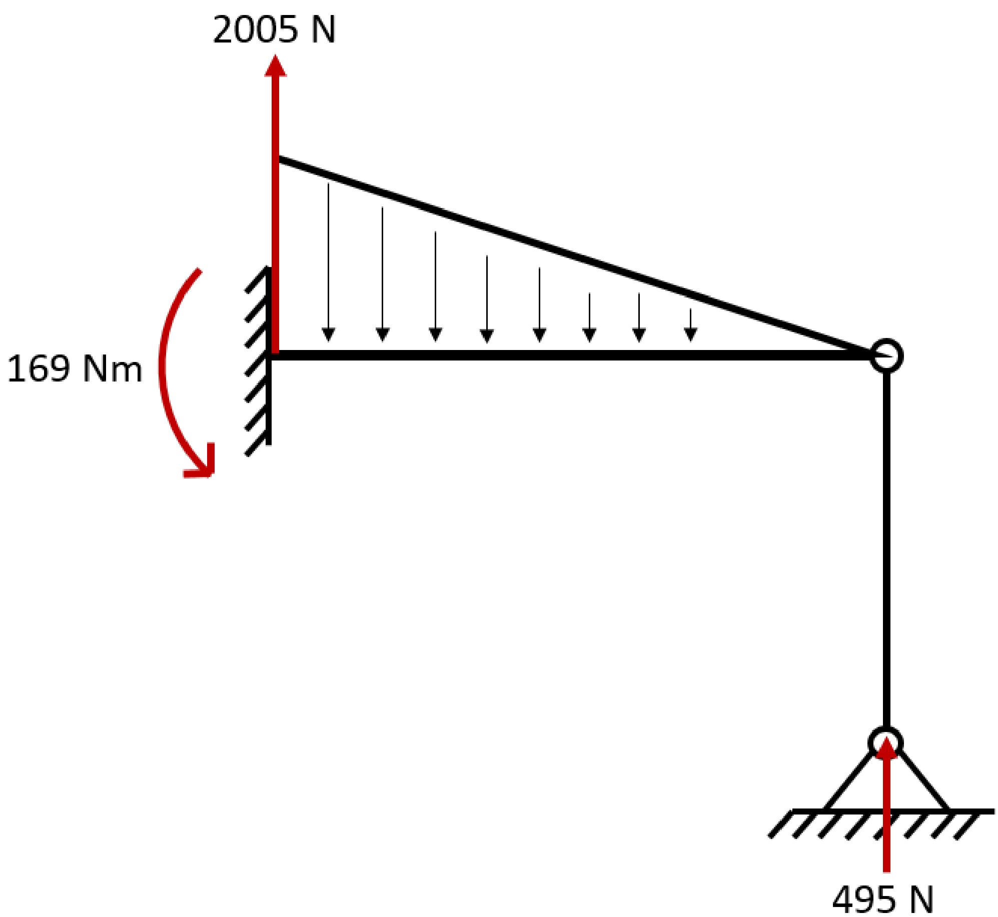
Displacement (z)



Displacement (x)



Reactions



Part D

Determine the maximum compressive and tensile stresses and indicate where in the structure they occur [10 pts]

The max compressive stress will be throughout x2 (since N2 is constant along it). There is no tensile because there is no positive N in either bar.

$$\sigma_{max} = \frac{3.6023}{A_2} = \frac{3.6023}{\pi(D^2 - d^2)} = \boxed{8132.25\text{N/m} = \sigma_{max}}$$

Problem 2

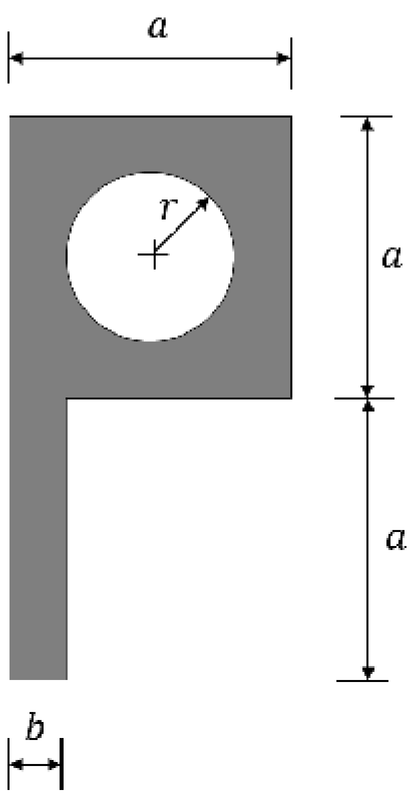


Figure 2

The cross section shown in Fig. 2 consists of a solid square of size a = 50 mm attached to a rectangle of width b = 10 mm. The square has a hole of radius r = 15 mm at its center.

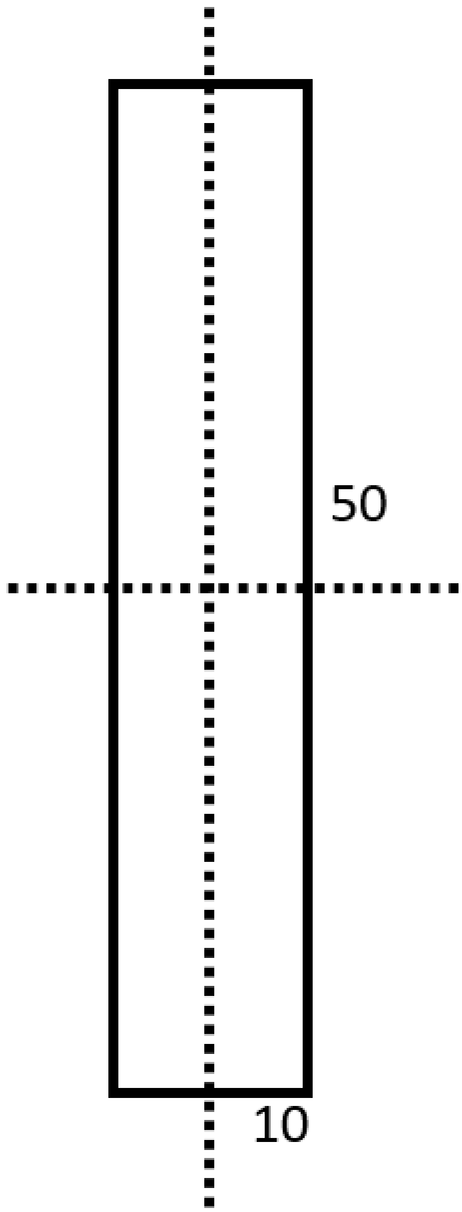
Part A

Find the position of the centroid of the cross section [10 pts]

To find the centroid we will split the cross section into 2 shapes. One will be a 50x10 rectangle and the other will be a 50x50 square with a 15 radius hole in the center.

First find the centroid and area of each component individually.

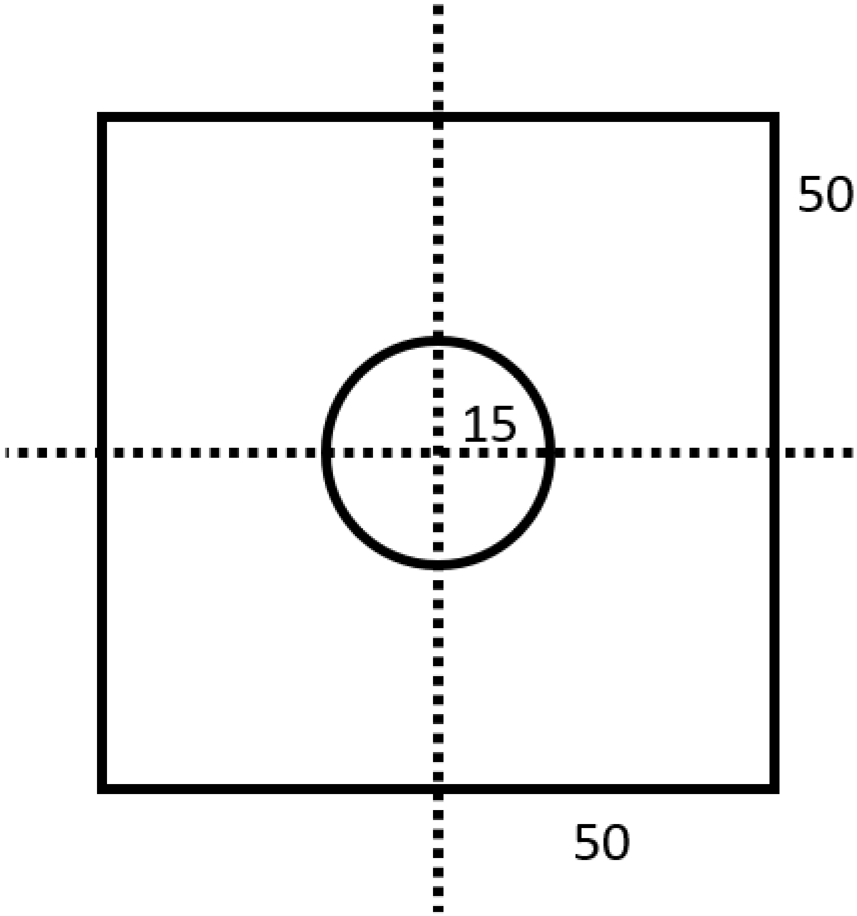
Rectangle



The centroid of the rectangle will be at the intersection of its two axes of symmetry, which is the midpoint of both edges.

Its centroid is at $(5, 25)$ and its area is $50 * 10 = 500mm^2$.

Square



The centroid of the square will be at the intersection of its two axes of symmetry, which is the midpoint of both edges.

Since we defined the origin at the bottom left corner of the rectangle, the centroid of the square is $(25, 25+50) = (25, 75)$ and its area is $50 * 50 - \pi * (15)^2 = 1793.14165mm^2$.

Combined

The combined centroid is defined as the sum of each component's centroid times its area divided by the total area of the shape.

For y direction:

$$y_C = \frac{5 * 500 + 25 * 1793.14165}{500 + 1793.14165} = 20.64mm$$

For z direction:

$$z_C = \frac{25 * 500 + 75 * 1793.14165}{500 + 1793.14165} = 64.1mm$$

Therefore the centroid of the cross section is $(20.64, 64.1)$ mm.

Part B

Find the principal axes of the cross section and the corresponding second moments of area. Sketch and label the axes on the cross section. [20 pts]

First we will find the moment of intertia about each axis of each component.

Square:

$$I_{sz} = \frac{a^4}{12} - \frac{\pi r^2}{4}$$
$$I_{sy} = \frac{a^4}{12} - \frac{\pi r^2}{4}$$

Rectangle:

$$I_{rz} = \frac{ab^3}{12}$$
$$I_{ry} = \frac{a^3 b}{12}$$

We will define y_s, y_r, z_s, and z_r as the distances between the centroid of the total shape and the individual centroids of each component. These will be used in the parallel axis theorem in the next step.

Square:

$$z = A_s z_s^2$$
$$y = A_s y_s^2$$

Rectangle:

$$z = A_r z_r^2$$
$$y = A_r y_r^2$$

Then we can define the arbitrary axis moments of inertia (about the centroid). I_η is the sum of the moments of intertia in the z direction corrected with the parallel axis them (and I_ζ is the same but in the y direction).

$$I_\eta = \frac{a^4}{12} - \frac{\pi r^4}{4} + A_s z_s^2 + \frac{ab^3}{12} + A_r z_r^2$$
$$I_\zeta = \frac{a^4}{12} - \frac{\pi r^4}{4} + A_s y_s^2 + \frac{a^3 b}{12} + A_r y_r^2$$
$$I_{\eta\zeta} = A_s z_s y_s + A_r z_r y_r$$

Then we can evaluate these with the given values:

$$I_\eta = \frac{50^4}{12} - \frac{\pi(15)^2}{4} + (1793.14165)(64.1-75)^2 + \frac{50*10^3}{12} + (500)(64.1-25)^2 = 1.5E6\mathrm{m}$$

$$I_\zeta = \frac{50^4}{12} - \frac{\pi(15)^2}{4} + (1793.14165)(20.64-25)^2 + \frac{50^3*10}{12} + (500)(20.64-5)^2 = 7.8E5\mathrm{r}$$

$$I_{\eta\zeta} = (1793.14165)(64.1-75)(20.64-25) + (500)(64.1-25)(20.64-5) = 3.91E5\mathrm{mm}^4$$

Now we can determine the rotation of the principle axes through the centroid using the equation:

$$2\theta' = \tan^{-1}\left(\frac{2I_{\eta\zeta}}{I_\zeta - I_\eta}\right)$$

$$\theta' = \frac{1}{2}\tan^{-1}\frac{2(3.91E5)}{7.8E5 - 1.5E6}$$

$$\theta' = -23.7^\circ$$

Then we can determine the second moments of area:

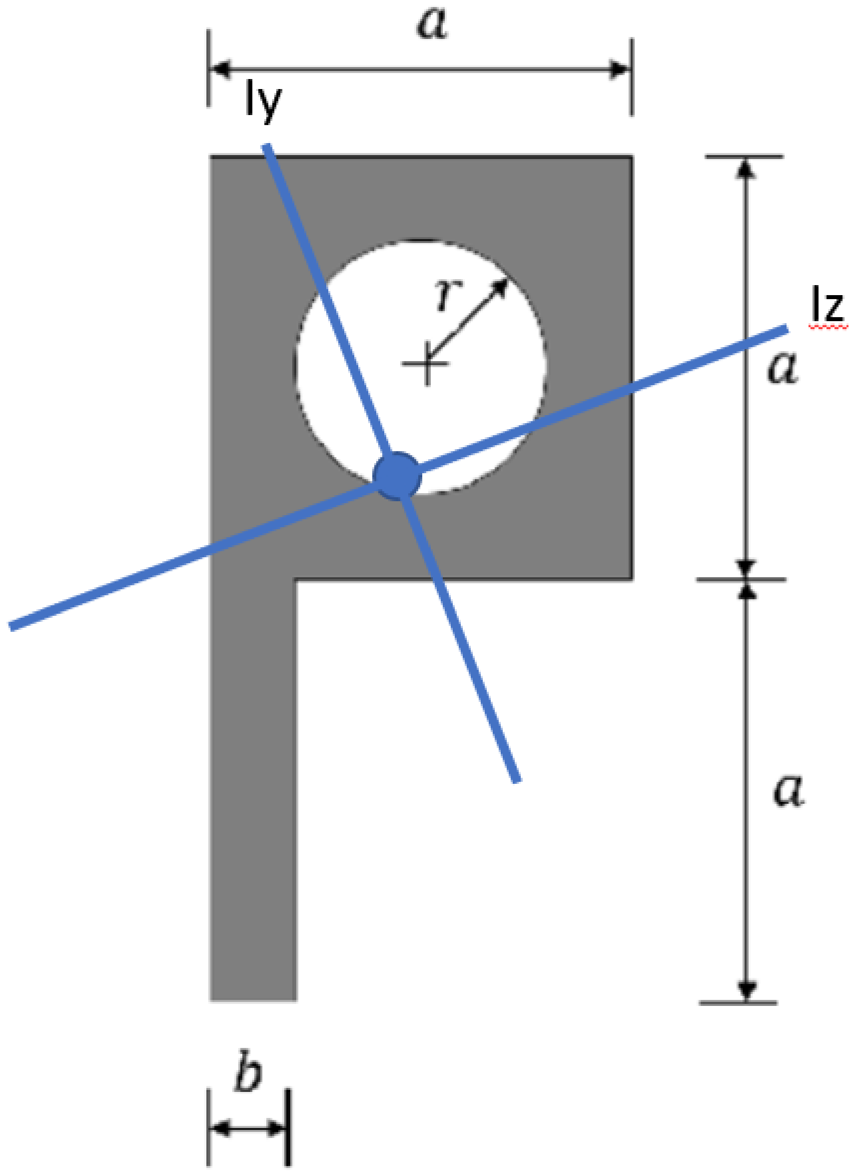
$$\begin{matrix} I_y \\ I_z \end{matrix} = \frac{I_\eta + I_\zeta}{2} \pm \sqrt{\left(\frac{I_\eta + I_\zeta}{2}\right)^2 + I_{\eta\zeta}^2}$$

We can substitute in the values we calculated:

$$\begin{matrix} I_y \\ I_z \end{matrix} = \frac{1.5E6 + 7.8E5}{2} \pm \sqrt{\left(\frac{1.5E6 + 7.8E5}{2}\right)^2 + (3.91E5)^2}$$

$$\boxed{I_y = 2.345E6\mathrm{mm}^4}$$

$$\boxed{I_z = -6.52E4\mathrm{mm}^4}$$


a
Iy
r
Iz
a
a
b