BME6013C HW#3

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For a square wave defined as:

$$\begin{aligned} u(t) &= -A, -T/2 \leq t < 0, \\ &= A, 0 \leq t < T/2, \\ &= u(t+T) \end{aligned}$$

and the integral fourier series definition:

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} e^{-j2\pi nt/T} u(t) dt$$

First we will solve for C_n. The first step is to split the integral into 2 pieces so we can sub in the value of $u(t)$ that is known for each interval.

$$C_n = \frac{1}{T} \left[\int_{-T/2}^{0} e^{-j2\pi nt/T} (-A) dt + \int_{0}^{T/2} e^{-j2\pi nt/T} (A) dt \right]$$

$$C_n = \frac{A}{T} \left[\int_{0}^{T/2} e^{-j2\pi nt/T} dt - \int_{-T/2}^{0} e^{-j2\pi nt/T} dt \right]$$

Now, we can evaluate the integrals.

$$C_n = \frac{A}{T} \left[-\frac{T}{j2\pi n} \left(e^{-j2\pi nt/T} \right) \Big|_{0}^{T/2} - \left(-\frac{T}{j2\pi n} \right) \left(e^{-j2\pi nt/T} \right) \Big|_{-T/2}^{0} \right]$$

$$C_n = \frac{A}{j2\pi n} \left[- \left(e^{-j2\pi n(T/2)/T} - e^{-j2\pi n(0)/T} \right) + \left(e^{-j2\pi n(0)/T} - e^{-j2\pi n(-T/2)/T} \right) \right]$$

Next we will simplify the result:

$$C_n = \frac{A}{j2\pi n}\left[-\left(e^{-j\pi n}-1\right)+\left(1-e^{-j\pi n}\right)\right]$$
$$C_n = \frac{A}{j2\pi n}\left(2-2e^{-j\pi n}\right)$$
$$C_n = \frac{A}{j\pi n}\left(1-e^{-j\pi n}\right)$$

Convert the e to sin notation using the Euler identity:

$$C_n = \frac{A}{j\pi n}\left(1-\cos\left(\pi n\right)-j\sin\left(\pi n\right)\right)$$

Since the sin of any integer multiple of π is 0, that term disappears. The cos of any integer multiple of π is -1^n, which is a known relationship. This can be substituted in to simplify further.

$$C_n = \frac{A}{j\pi n}\left(1-(-1)^n\right)$$

This expression can be evaluated with some thinking. For any even value of n, the term in the parentheses will be $1-1=0$, so the coefficient $C_n = 0$ for any even value of n. If n is odd, it will be $1--1=2$. Thus:

$$C_n = 0, n \text{ is even}$$
$$C_n = \frac{2A}{j\pi n}, n \text{ is odd}$$

Then, this can be plugged in to the series:

$$u(t) = \sum_{-\infty}^{\infty} C_n e^{j2\pi nt/T}$$

Since $C_n = 0$ for any even value of n, we can replace the summation limits. For simplicity, instead of going from -infinity to +infinity, we can create a second constant C_{-n} that is the negative pair to go with each n and only use positive, odd values of n starting at 1.

$$u(t) = \sum_{n=1,3,5,\cdots}^{\infty} C_n e^{j2\pi nt/T} + C_{-n} e^{-j2\pi nt/T}$$

$$u(t) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{2A}{j\pi n} e^{j2\pi nt/T} + \frac{2A}{-j\pi n} e^{-j2\pi nt/T}$$

$$u(t) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{2A}{j\pi n} \left(e^{j2\pi nt/T} - e^{-j2\pi nt/T}\right)$$

Now, the function is starting to look like the real sinusoid definition. We will multiply both the numerator and denomenator by 2 to make it match:

$$u(t) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{4A}{j2\pi n} \left(e^{j2\pi nt/T} - e^{-j2\pi nt/T}\right)$$

$$u(t) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{4A}{\pi n} \left(\frac{e^{j2\pi nt/T} - e^{-j2\pi nt/T}}{j2}\right)$$

And finally, we can use the definition mentioned in the previous step to plug in and reach the final answer:

$$u(t) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{4A}{\pi n} \sin\left(\frac{2\pi nt}{T}\right)$$