Minesweeper Solver Analysis

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March 2, 2019

Introduction

This document discusses in detail my thought process in developing the algorithms. As a disclaimer I'd like to say that I understand that writing code doesn't usually include theoretical analysis like this, and in fact I wrote this document after coding up the algorithms I've described. The reason I wrote this is to explain my thought process in developing the solver algorithms, i.e. to demonstrate how I can see the essential, theoretical nature of a problem and from there develop effective and efficient algorithms. This theoretical clarity is useful both in terms of organizing code and communicating with teammates.

Henceforth I will mostly stick to the formal "we" is in mathematical writing.

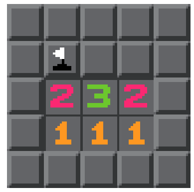
To begin with we consider the problem of beating a game of minesweeper. Simply put, we are trying to decide squares have mines and which do not. If we can find some such squares we flag the ones that have mines and reveal or "click" the squares that don't. In doing so, we have more information which helps us decide more squares to flag or reveal. As long as we can keep finding squares, eventually we'll cover the whole board and in this way we win the game.

Human-like Solving

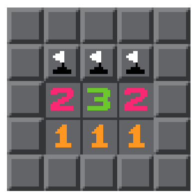
As a starting point we consider how a person plays minesweeper, as opposed to a computer executing an algorithm. Dropping the formal "we" momentarily, if I can formalize the strategies I use to play minesweeper (I'm assuming not everybody plays in exactly the way I do), I can turn them into an algorithm.

The Strategies

When I play minesweeper, my bread and butter strategy is to consider the information provided a by looking at a number on the board, or by looking at two numbers in conjunction and seeing if I can deduce where mines must be, or where mines must not be. Consider this example:



There is a 3 on the board with only one flag next to it and exactly two blank squares next to it (i.e. the other five squares adjacent to it have been revealed).The 3 requires two more flags to satisfy it, and since only two blank squares adjacent to 3 remain we know those must contain mines and so we flag them



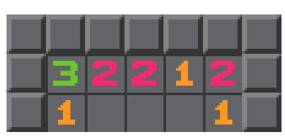
In doing so we are following a certain strategy: if the number of spaces around a number equals the number of flags that still need to be placed around it, then flag all those spaces. We now formalize that strategy. Let p be a point on the board, and let m be the number of mines around p that need to be flagged beyond those that have already been flagged, and let B be the blank squares adjacent to p (in the above example, $m = 2$ and $|B| = 2$). We state the strategy in a pseudocode snippet:

if $m = |B|$ **then** ▷ strategy 1
  flagAll(B)
end if

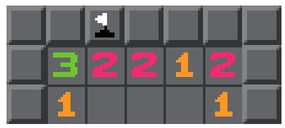
We also know that if there are no more flags required, then we can simply reveal all adjacent squares

if $m = 0$ **then** ▷ strategy 2
  revealAll(B)
end if

Those are the two strategies we follow when we look at one number. Now consider this example with two numbers.



Consider the 2 in the center and the 1 directly to the right of it. There are two squares adjacent to both of them, the space above the 2 and above the 1. Because these squares are adjacent to the 1, only one of them can contain a mine. But the 2 must have two mines adjacent to it, and so there must be another mine elsewhere to satisfy it. Outside of these two spaces shared by the one and two, the only blank square is above and to the left of the 2, so we flag this space:



To capture this idea in a strategy we define some variables: let p_1 and p_2 two points on the board, let m_1 and m_2 the number of mines needed to satisfy p_1 and p_2 respectively, and let B_1 and B_2 be the blank squares around p_1 and p_2 respectively. Set $I = B_1 \cap B_2$. Observe that since $I \subseteq B_1$ it can not contain any more mines than B_1 which contains m_1 mines, that is I can not contain more mines that B_1. We can apply the same logic with B_2 instead of B_1, and of course I can not have more mines than spaces, and so altogether we have a bound on the number of mines in I

$$i_{max} = \max(m_1, m_2, |I|)$$

With these definitions we state this strategy

if $m_1 - i_{max} = |B_1 - I|$ **then** ▷ Strategy 3
    flagAll($B_1 - I$)
end if

as well as the symmetric strategy

if $m_2 - i_{max} = |B_2 - I|$ **then** ▷ Strategy 4
    flagAll($B_2 - I$)
end if

If we call the 1 in our example p_1 and the 2 is p_2, then the strategy we used is Strategy 4.

We can also establish a *lower* bound i_{min} on the number of flags in I. Namely, $B_1 - I$ can only contain $|B_1 - I|$ mines, and so I must contain the remaining $m_1 - |B_1 - I|$ mines required around p_1. Similarly I must contain at least $m_2 - |B_2 - I|$ mines, and so we say

$$i_{min} = \max(m_1 - |B_1 - I|, m_2 - |B_2 - I|)$$

Using this we derive three more strategies:

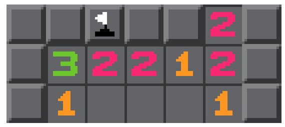
if $i_{min} = |I|$ **then** ▷ Strategy 5
    flagAll(I)
end if

if $i_{min} = m_1$ **then** ▷ Strategy 6
    revealAll($B_1 - I$)
end if

if $i_{min} = m_2$ **then** ▷ Strategy 7
    revealAll($B_2 - I$)

end if

In fact, if we apply Strategy 6 to our example (with p_1 as the 1 and p_2 as the 2, as before), we are able to reveal a square:



Human-like Algorithm

The basic idea for the algorithm is to just apply to the strategies to different points (or pairs of points) on the board until the game is won. It's just a matter of deciding when and where to apply the strategies. First off, observe that there is no use in applying Strategies 3-7 unless the two points have at least one blank neighbor in common. With this in mind, we write a subroutine that takes a point p and applies all the necessary strategies to it.

function ApplyStrategies(p)
  Apply Strategies 1 and 2 to p
  $p_1 \leftarrow p$
  for all points p_2 that share a blank neighbor with p_1 **do**
    Apply Strategies 3-7 to p_1 and p_2
  end for
end function

Now its just a question of which points to call ApplyStrategies on. Firstoff, we only need to check revealed points, i.e. squares with numbers. Secondly we only need to bother with points that have at least one blank neighbor- we will call the collection of all such points the **fringe** because it consists of those squares on the edge of the contiguous blobs of revealed squares. Finally, while we may meed to call ApplyStrategies repeatedly on a given point p, we only need to call it again *after* one of the blank neighbors of p has been flagged or revealed. To capture this we'll have a stack active which tracks which points need to be checked (which may have repeated values). Here is the pseudocode to solve a game G of minesweeper. We assume at least one move has been made so that there are revealed squares and a fringe to begin with

function HumanSolve(G) ▷ Human-Like algorithm
  active $\leftarrow$ fringe of G
  while *active* is not empty **do**
    $p \leftarrow$ *active*.pop()
    ApplyStrategies(p)
    for all revealed points q with a just revealed/flagged neighbor **do**
      active.push(q)
    end for
  end while
end function

Correctness and Completeness

The algorithm is correct in the sense that when it identifies a mine or a free square it does so correctly, and so can "correctly" win a game. However there is no guarantee that it will beat the game, and there are times

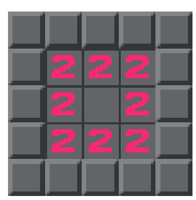
Space and Runtime

The only significant space usage is in the variable *active* which is of course bounded by the number of points b on the game board, so the algorithm uses $O(b)$ space.

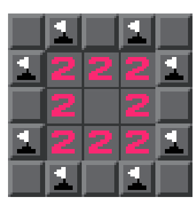
Observe that each strategy can be applied in constant time. Moreover all the points p_2 must be within two squares of p_2, and there are 25 such squares (within a five by five window centered around p_1). It follows that ApplyStrategies executes in constant time. Fix a point p. ApplyStrategies might be called on p once and then each time a point adjacent to it is revealed. There are only 8 adjacent points, so in total ApplyStrategies can be called up to 9 times on p. Thus this algorithm runs in $O(b)$ time, where again b is the number of points on the game board.

What about completeness?

The algorithm outlined above works pretty well and is pretty efficient, however as we mentioned above it is not complete . Consider this example of a game in progress:



Our human-like algorithm can do nothing, but as it turns out there is a unique solution here:



Perhaps our algorithm would be able to make more progress if we added another strategy, maybe one that takes into account more than two squares at a time? Indeed by looking at, say the, the 2 in the top right corner as well as the 2 it's left and the 2 to directly below it we could conclude that the top right corner is blank (the 2 in the corner must share at least one mine with each of the other 2's and is satisfied in this way, leaving the corner free). By symmetry all the corners must be blank. So we were able to reveal four more squares in this way. In fact by looking at five squares at a time we can determing which of the remaining squares are also free and thus win the game (we spare the details).

Of course this begs the question do we really want to enumerate all the ways (strategies) to combine the information from five or even three numbers at a time? It seems like when the human-like algorithm hits a snag like this it might be worth it to just look at all the numbers at once. In the next section we develop an algorithm that does just that.

Exhaustive Solving

As a starting point we consider what exactly it means to beat a game of minesweeper. To win we must place flags on exactly those squares which contain mines and reveal the rest of the squares- in other words we must find the right "placement" of flags on the board. We say a **frame** of a game is a freeze frame of a game in progress, i.e. a frame is an array of squares representing the game board where each square is marked as flagged or blank, or is revealed and displying a number (0's are implicit and not shown). Because those numbers tell us how many mines there are adjacent to it, we call them **hints**. A **placement** is a subset of the blank squares on the board which we regard as a possible way of placing additional flags to designate where mines are. Since we aim to place flags on mines and only mines, these hints are constraints on the way flags can be placed (we already placed on the board are correctly placed). With an eye towards satisfying these constraints, we make the following definition:

Definition 1. A placement is **satisfactory** if after placing all flags in it we have this property: for every revealed square s, the number hint h in s is equal to the number of flags surrounding s.

Observe that the correct placement of flags on mines is itself a satisfactory placement, which we call the **actual placement**. It follows that if a square is flagged in *every* satisfactory placement, it must contain a mine, because in particular it is flagged in the actual placement. Therefore we say placing a flag on such a square is a **certain flag** (a flag placed with certainty). Likewise, if a square is *unflagged* in every mine placement, we know it must be free (i.e. it does not contain a mine). Thus we say revealing is a **certain reveal**. A **certain move** is a certain flag or a certain reveal. Consider the value of a certian move over an uncertain move: in playing Minesweeper, we do not know where the mines are but we can figure out what the satisfactory placements by looking at number hints, and so can deduce certain moves. Moreover we really *have* to resort to looking only at certain moves because if a reveal is uncertain, there are satisfactory placements in which the revealed square contains a mine, and if one of those satisfactory placements is the actual placement then revealing the square loses the game. Uncertain flags, while not immediately fatal, can lead to errors down the line. Hence we only want to make certain moves. This line of reasoning lends itself to an algorithm idea

Brute Force Algorithm

Note that this algorithm does not try to solve the game in its entirety, but rather tries to solve a "frame" of the game. To solve the game, we need to apply this version of solve repeatedly.

Simple put, we need to generate all satisfactory placements, check which squares are consistently flagged or unflagged across each placement and make certain moves accordingly. Let's assume for the moment we have a function satisfactoryPlacements(G) which takes a game of minesweeper G and generates all satisfactory placements. Here is the algorithm in pseudocode:

```
function BruteSolve(G)                                ▷ Brute force algorithm
    Mines ← game board
    Free ← game board
    for P in satisfactoryPlacements(G) do
        Mines ← Mines ∩ P
        Free ← Free − P
    end for
    flagAll(Mines)
    revealAll(Free)
end function
```

We just need to implement satisfactoryPlacements(). We start with a brute force approach: go over all mine placements and yield the ones that are satisfactory. Mine placements are just subsets of the game board B, i.e. elements of the powerset $\mathcal{P}(B)$.

```
function satisfactoryPlacements(G)
    for P in 𝒫(B) do
        if P is satisfactory then
            yield P
        end if
    end for
end function
```

Correctness and Completeness

Regarding correctness, we reiterate our discussion above.The algorithm only flags squares in the intersection of all satisfactory placements, which is a subset of the actual placement. It only reveals squares that are not a part of any satisfactory placement, in particular they are not a part of the actual placement.

In the discussion of our human like algorithm we alluded to the fact that in a certain sense it was not complete, and we can now state more precisely what we meant. Our notion of completeness is nuanced, because it does not mean the algorithm always beats the game. In general completeness means that when there is a solution the algorithm finds it. For our purposes we will say there is a solution when there is at least one certain move to be made. (This definition puts solution in terms of frames, not in terms of the game as a whole). We gave an example above where there are certain moves to be made (where in fact the game could be beaten) which the human-like algorithm did not find. In this sense the human-like algorithm is complete. On the other hand our brute algorithm was created explicitly to find all certain moves, and so is complete.

Space and Runtime

This method checks $|\mathcal{P}(B)| = 2^b$ placements, where $b = |B|$ is the size of the board. We can check if a placement is satisfactory in linear time, and since the placements are bound by the board size, we have that satisfactoryPlacements() has an $O(b2^b)$ runtime. The other work in solve is relatively insignificant and so this also the runtime for solve(). We must apply solve() repeated to actually beat the game. We can't assume that solve() makes move on more than one square for each call, so the beat the game takes up to b applications of solve(), or $O(b^2 2^b)$.

As far as space, if we regard SatisfactoryPlacements() as a generator a la Python (which is in fact how it is implemented in my code), then all variables are bounded in size by the game board, and so the algorithm uses $O(b)$ space.

Idea: Reducing Scope

An $O(b^2 2^b)$ runtime is pretty bad, but there's a reason we call this the brute force algorithm. Here we consider one way to make it more efficient.

If you've ever played minesweeper though it might occur to you that it's unecessary to look at the entire board, since in a sense we only have information about the blank squares which are adjacent to revealed squares (hints). With this in mind we make the following definitions: a square is **in-play** if (in the frame in question) it is unrevealed and unflagged and adjacent to a revealed square. We will refer to all in-play squares collectively as the **perimiter** because they circumscribe the revealed part of the board (different from the fringe, defined above). We will make the algorithm more efficient by reducing the scope of the algorithm to the perimiter.

Lemma. *If S is a satisfactory placement and P is the perimiter, then $S' = S \cap P$ is also a satisfactory placement.*

Proof. Let r be a revealed square. All the flags in S adjacent to r are also in $S' = P \cap S$ because the perimiter P contains all the in-play squares in the current frame. In particular r has the same number of flags adjacent to it in P' as in P. This goes for each revealed square r, so S' like S is a satisfactory placement. ☐

Claim 1. *1. A flag is certain if and only if it occurs in each satisfactory placement contained within the perimiter. Additionally, certain flags always occur within the perimiter*

2. *A reveal is certain if and only if it is in the perimiter and the space to be revealed is unflagged in each satisfactory placement contained within the perimiter.*

3. *Certain moves occur only in the perimiter of the board.*

Proof. 1. If a flag is certain if it is contained in every satisfactory placement, in particular it is a part of the satisfactory placements contained within the perimiter. Conversely, suppose s is a space that is flagged in every satisfactory placement that is contained within the perimiter. Let S be an arbitrary satisfactory placement. Set S' be the intersection of S and the perimiter. By the lemma, S' is also satisfactory, and so by our hypothesis, $s \in S'$. Of course $S' \subset S$, and so $s \in S$. Thus s is in every satisfactory placement S, meaning s is a certain flag. This proves the iff statement.

   To prove that all certain flags are in the perimiter let M be the actual mine placement, which is satisfactory (for the purposes of this proof M could be any satisfactory mine placement). By the lemma $M' = M \cap perimiter$, the intersection of M and the perimiter, is satisfactory. By the definition of certain, all certain flags are in M' which is in the perimiter.

2. Suppose s is a certain reveal. By definition then, s is not contained in any satisfactory mine placement, in particular those contained within the perimiter. Suppose for contradiction s was not in the perimiter. Let M be the actual mine placement (or any satisfactory mine placement for that matter), and set $M' = M \cup \{s\}$. Since s is not in the perimiter it is not in play meaning that adding it to a placement would not change the number of flags around any revealed square. In particular adding it to M would yield a placement $M' = M \cup \{s\}$ which had the same flag counts around each revealed square, i.e. another satisfactory placement. Thus s is in a satisfactory placement M', which is a contradiction because s is a certain reveal.

   Conversely, suppose s is in the perimiter and is unflagged in each satisfactory placemnt contained within the perimiter. Let S be a satisfactory placement, and let S' be it's intersection with the perimiter. By the lemma, S' is a satisfactory placement, and so by out hypothesis $s \notin S'$. But since s is in the perimiter and S' is the intersection of S with the perimiter, the only way $s \notin S'$ is if $s \notin S$. This shows that s is contained in no satisfactory placement and so by definition s is a certain reveal.

3. This statement follows immediately from the first two.

□

This claim shows us that our algorithm need only focus on the perimiter as opposed to the whole board. That is, we can initialize mines and free to be just the perimiter and we only need to generate satisfactory placements that are contained within the perimiter. Here are the revised versions of BruteSolve() and satisfactoryPlacements():

```
function BruteSolve(G)                                          ▷ Version 2
    Mines ← perimiter of G
    Free ← perimiter of G
    for P in satisfactoryPlacements(G) do
        Mines ← Mines ∩ P
        Free ← Free − P
    end for
    flagAll(Mines)
    revealAll(Free)
end function
function SatisfactoryPlacements(G)                              ▷ Version 2
    for P in 𝒫(perimiter of G) do
        if P is satisfactory then
            yield P
        end if
    end for
end function
```

Correctness and Completeness

The claims establish that correctness and completness are preserved after we reduce the scope of the algorithm.

Space and Runtime

This reduces the runtime of BruteSolve() to $O(p2^p)$ where p is the size of the perimiter. While this is still exponential, the perimiter tends to be much smaller than the board. To wave my hands for a moment: the board has 2 dimensions while the perimiter is more like a 1-dimensional curve embedded in it. In the worst case we still need to apply solve b times, so we would say that it takes $O(bp2^p)$ time to beat the game except that this is an abuse of notation: p is a constant for each *frame* of the game, but the perimter changes as more moves are made. Techinically we can still only say that the runtime is $O(b^2 2^b)$.

Space used per frame is $O(p)$, and over the course of the game is $O(b)$.

Idea: Building Satisfactory Placements

Even when we restrict our attention to the perimiter, it seems like we're doing exta work when we go back and check if a placement is satisfactory- after all this is where the linear factor p in $O(p2^p)$ comes from. What if rather than looking at every placement, we "built up" satisfactory placements according to the hints on the board. If we do it correctly we don't have to check that it we are getting exactly the satisfactory placements. Recall this term: the **fringe** consists of all revealed squares (numbers) with at least one adjacent blank square. Note the distinction between the fringe and the perimiter: the fringe consists of the edge (or edges), of the revealed part(s) of the board, while the perimiter is unrevealed and wraps around the fringe. Functionally speaking the fringe contains all the hints on the board that have yet to be satisfied, and as such is our focus if we are building up satisfactory placements.

The core of the algorithm is to look at a point in the fringe and add the appopriate number of flags around it to satisfy it. There are usually multiple ways to add the appropriate number of flags, so we choose one way and proceed to the next point, doing the same with this point. We continue in this manner until we have a satisfactory placement, which we then yield. We then backtrack an consider other ways of adding flags around points.

Recursive Backtracking Algorithm

For this change we are only updating the SatisfactoryPlacements function, which we can simply plug into Version 2 of the BruteSolve() function. It is a recursive backtracking algorithm.

function SatisfactoryPlacements(G) ▷ Version 3
    SPHelper($G, 0, \emptyset$)
end function
function SPHelper($G, i, placement$)
    if $i =$ length($G.fringe$) **then** ▷ recursion base case
        yield *placement*
    end if
    $point \leftarrow G.fringe[i]$ ▷ Get the ith point in the fringe
    if *point* has too many flags around it when including those in *placement* **then**
        return ▷ Go back up one level of the call stack
    else if *point* already has the right number of flags around it **then**
        SPHelper($G, i + 1, placement$)
    else
        for all sets of flags *newFlags* that can be added to satisfy *point* without adding flags around $G.fringe[0], G.fringe[1],...,G.fringe[i - 1]$ **do**
            $placement \leftarrow placement \cup newFlags$
            SPHelper($G, i + 1, placement$)
            $placement \leftarrow placement - newFlags$
        end for
    end if
end function

Correctness and Completeness

As we have shown already, to prove the correctness and completeness of the algorithm it suffices to show that SatisfactoryPlacements() generates all satisfactory placements within the perimiter.

Claim 2. *The above version of SatisfactoryPlacements() yields exactly those satisfactory placements contained within the perimiter.*

Proof. The perimiter is just the union of all in-play squares adjacent to points in the fringe, so we might rephrase the claim as SatisfactoryPlacements() yields exactly those satisfactory placements that contain only flags adjacent to points in the fringe. Now consider the following propositions which depends on the variable k.

OnlySatisfactory(k): Whenever a call of the form SPHelper($G, k, placement$) is made, which is to say a call with $i = k$, then the variable *placement* is a placement that satisfies the points $G.fringe[0], ..., G.fringe[k-1]$

AllSatisfactory(k): Each placement which (a) satisfies the points $G.fringe[0]$, ...,$G.fringe[k-1]$ and (b) contains only flags adjacent to $G.fringe[0], ..., G.fringe[k-1]$ is stored in the variable *placement* at the beginning of one of the function calls of the form SPHelper($G, k, placement$)

The statement OnlySatisfactory(length($G.fringe$)) should be interpreted as follows: when the base case of the recursion is entered (because i equals length($G.fringe$)) the *placement* variable contains a satisfactory placement because all points in $G.fringe$ are satisfied, and thus the placement that's yielded is satisfactory. The statement AllSatisfactory(length($G.fringe$)) guarantees that all satisfactory placements contained within the perimiter are yielded. Together these statements tell us that SatisfactoryPlacements() yields exactly those satisfactory placements contained within the perimiter, which is our goal. We prove OnlySatisfactory(length($G.fringe$)) and AllSatisfactory(length($G.fringe$)) by induction on k.

There are no points to be satisfied in base case ($k = 0$), so OnlySatisfactory(0) is trivial. Recursive calls are made with strictly ascending values of i, so SPHelper() is only called once with $i = 0$ once and in this case $placement = \emptyset$. Indeed $\emptyset$ is the only placement which satisfies conditions (a) and (b) in AllSatisfactory(0)

Now suppose OnlySatisfactory(k) is true. Calls of the form SPHelper($G, k+1, placement$) are only made within calls of the form SPHelper($G, k, placement$). When such a call is made from the else if clause, then *placement* already satisfies the hint at point $G.fringe[k]$ and OnlySatisfactory(k) tells us that it also satisfies $G.fringe[0], ..., G.fringe[k-1]$. This establishes the statment OnlySatisfactory($k+1$) for the if else clause. Otherwise the call SPHelper($G, k+1, placement$) is made within the for each loop. Adding $newFlags$ to *placement* guarantees that *placement* satisfies $G.fringe[k]$, but by the condition in the for each loop $newFlags$ doesn't contain any flags around $G.fringe[0], ..., G.fringe[k-1]$. OnlySatisfactory($k$) guarantees that *placement* satisfied $G.fringe[0], ..., G.fringe[k-1]$ to begin with, and after adding $newFlags$ which doesn't contain any flags around these points, they continue to be satisfied. Thus OnlySatisfactory($k+1$) is true.

Now suppose AllSatisfactory(k) is true. Let P be a placement as described in AllSatisfactory($k+1$). Let N be the flags from P which are adjacent to $G.fringe[k]$ but are not adjacent to $G.fringe[0], ..., G.fringe[k-1]$($N$ may be empty), and set P' equal to $P - N$. P' is one of the placements described in AllSatisfactory(k), and so there is a call of the form SPHelper($G, k, placement$) with *placement* equal to P'. If $N = \emptyset$ then $placement = P' = P$ satisfies $G.fringe[0], ..., G.fringe[k]$ already and the else if clause makes the call SPHelper($G, k + 1, placement$) as required. Otherwise if $N \neq \emptyset$, one of the iterations of the for each loop sets $newMines$ equal to N and adds $newMines$ to *placement*. Since *placement* equals P' before this it now equals $P' \cup N = P$. Thus SPHelper($G, k + 1, placement$) is called with $placement = P$ as desired. □

Space and Runtime

Regarding space, SPHelper proceeds depth first and so the call stack is only ever as deep as the index i, Thus there are a maximum of f calls on the call stack where f is the size of the fringe, i.e. length($G.fringe$). Moreover, each call only stores a reference to the variable *placement*, so the overall space usage is $O(f)$. We note here that since each square has 8 adjacents squares, $f \leq 8p$ and $p \leq 8f$, so $O(f) = O(p)$. Thus for the sake of comparison we can say the algorithm still uses $O(p)$ space per frame.

Now onto runtime. First observe that each iteration of the for each loop in SPHelper() includes a recursive call to, so we can associate the work done in this iteration as a a prt of the recursive call (i.e. we associate the work with the call SPHelper($G, i + 1, placement$, not with the current SPHelper($G, i, placement$ call). All other work done in a call to SPHelper is constant, thus the runtime is proportional to the number of calls to SPHelper. Now observe that the beginning of each call to SPHelper *placement* stores a different subset of the perimiter (we will not bother to, but we could prove this by induction), and so the number of calls is bounded by the number of subsets of the perimiter, i.e. 2^p where p is the size of the perimiter. Thus the runtime is $O(2^p)$.

As desired we've dropped the factor of p from the previous version of SatisfactoryPlacements(). Beyond a better asymptotic runtime though, there's an additional efficiency to this version that's not clear from asymptotic bound $O(2^p)$. We can think of SatisfactoryPlacements as "searching" a tree where each node is a subset of the perimiter, that subset being stored in *placement* as the algorithm proceeds. Each node at level i of the tree is a subset that satisfies points $G.fringe[0], ..., G.fringe[i-1]$. Adding flags to a node at level i to satisfy $G.fringe[i]$ yields a child node at level $i+1$. If we were not careful we would search the entire tree, which consists of 2^p nodes/subsets. However the algorithm specifically cuts off at the points where its unable to satisfy the next point $G.fringe[i]$ (the else if statement) and also never satisfies $G.fringe[i]$ in a way that would put to many flags around $G.fringe[0], ..., G.fringe[i-1]$ (the condition in the for all statement). In this way the algorithm "prunes" the tree as it searches, so in some cases the algorithm runs much faster that $O(2^p)$. This is in contrast to our original implementation of SatisfactoryPlacements() which *always* considered all 2^p subsets of the perimiter, on top of checking if it was satisfactory.