Mathematical Social Sciences 64 (2012) 11–27

Contents lists available at SciVerse ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/econbase

Voting with rubber bands, weights, and strings

Davide P. Cervone [a], Ronghua Dai [c], Daniel Gnoutcheff [b], Grant Lanterman [b], Andrew Mackenzie [d], Ari Morse [e], Nikhil Srivastava [f], William S. Zwicker [a,*]

[a] *Department of Mathematics, Union College, Schenectady, NY 12308, United States*
[b] *Union College, Schenectady, NY 12308, United States*
[c] *Gourmet Garden, Wyckoff, NJ 07481, United States*
[d] *Economics Department, University of Rochester, Rochester NY 14627, United States*
[e] *Skidmore College, Saratoga Springs, NY 12866, United States*
[f] *Institute for Advanced Study, Princeton, NJ 08540, United States*

ARTICLE INFO

Article history:
Received 11 May 2010
Received in revised form
16 August 2011
Accepted 23 August 2011
Available online 7 September 2011

ABSTRACT

We introduce some new voting rules based on a spatial version of the median known as the mediancentre, or Fermat-Weber point. Voting rules based on the mean include many that are familiar: the Borda Count, Kemeny rule, approval voting, etc. (see Zwicker (2008a,b)). These mean rules can be implemented by "voting machines" (interactive simulations of physical mechanisms) that use ideal rubber bands to achieve an equilibrium among the competing preferences of the voters. One consequence is that in any such rule, a voter who is further from consensus exerts a stronger tug on the election outcome, because her rubber band is more stretched.

While the $\mathbf{R}^1$ median has been studied in the context of voting, mediancentre-based rules are new. Voting machines for these rules require that the tug exerted by a voter be independent of his distance from consensus; replacing rubber bands with weights suspended from strings provides exactly this effect. We discuss some novel properties exhibited by these rules, as well as a broader question suggested by our investigations—What are the critical relationships among resistance to manipulation, decisiveness, and responsiveness for a voting rule? We argue that a distorted view may arise from an exclusive focus on the first, without due attention to the other two.

© 2011 Elsevier B.V. All rights reserved.

1. Overview

We begin, in Section 2, with a discussion of known voting rules (Kemeny, Borda, Approval, etc.) that can be implemented via "voting machines" that use ideal rubber bands. Each such rule is based on a form of mean, or average, of all ballots cast. In Section 2, we substitute strings and weights for the rubber bands to obtain a parallel class of (new) rules based on the *mediancentre*, one of several multivariate generalizations of the median. Our focus here is one such rule: *Mediancentre-Borda* or *M*^c*Borda* for short. Section 3 compares voting-theoretic properties of Borda and M^cBorda, tracing differences to the fundamental distinction between mean and mediancentre. The behavior of voting rules based on the mediancentre suggests some big-picture questions about how we should judge the relative merits of voting rules. We discuss these in Section 4.

The text occasionally suggests running the online *Evolver* program, a dynamic voting simulator. Observing the reactions of *Evolver*'s equilibrium position to changes in voter preference will give the reader additional insight into fundamental differences among voting rules. Alternately, with reference to the printed figures the article can be followed without running the simulation.

2. Rubber band voting machines

We will consider voting rules for which A is a finite set of *alternatives*, and a ballot consists of a strict ranking (linear ordering) of A's members. Let us start with $A = \{p, q, r\}$, so there are six possible ballots. Using any modern web browser (Safari, Firefox, Chrome, or Internet Explorer, for example), open the *Evolver* simulation at

http://www.math.union.edu/locate/voting-simulation.

Click on the link *Voting with rubber bands, weights, and strings*. You will find a regular hexagon, whose vertices are labeled with these six ballots, as in Fig. 1, along with links to other pages. Make sure

* Corresponding author.
E-mail address: zwickerw@union.edu (W.S. Zwicker).

0165-4896/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.mathsocsci.2011.08.003

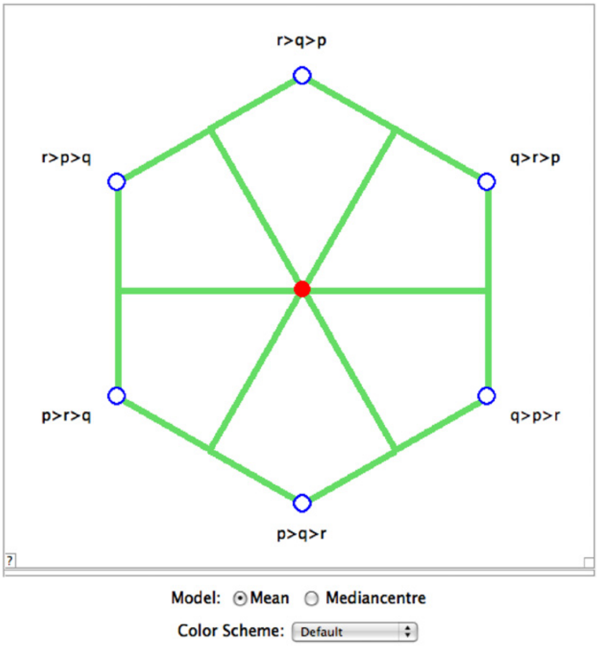

Fig. 1. The *Evolver* screen, with no ballots recorded. Each click on a vertex casts a single ballot for the corresponding preference ranking. To slide the control panel open, click once on the double line at the bottom of the square.

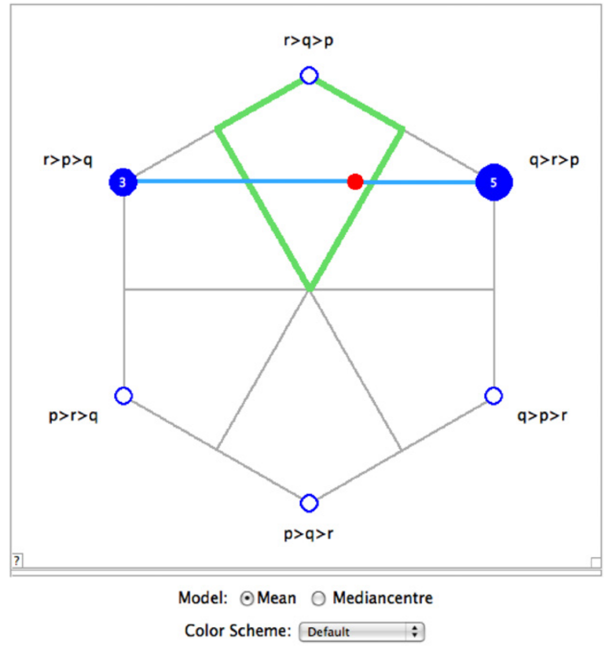

Fig. 2. The result of running the *Evolver* with 3 votes for $r > p > q$ and 5 for $q > r > p$ locates the mean point in the upper wedge of the hexagon.

that the *Model* button under the hexagon is toggled to the *Mean* position.

Click once on the vertex labeled "$r > p > q$". You have just cast one ballot expressing highest preference for alternative r, and lowest for q. As a result, there is now an *ideal rubber band* stretched between that vertex and the red point at the hexagon's center.[1] This band (represented by the blue line) exerts a force on the red point that is proportional to its length, and in the direction of the $r > p > q$ vertex. We will take the constant of proportionality to be 1; this means we can visualize that force as a vector with its tail on the red point and head on the vertex, with the length of the vector equal to the force's magnitude.

Click twice more on the $r > p > q$ vertex and then five times on the $q > r > p$ vertex, creating an *anonymous profile* for an 8-voter society (this configuration is available via the "$\Pi_1 = \langle 0, 0, 3, 0, 5, 0\rangle$" link). Voters can be removed from a vertex by holding down the *option* or *alt* key while clicking, in case you add too many. Now click once on the central, red point, releasing it, so that it is free to move in response to the 8 rubber bands tugging on it. It seeks a position M at which the forces are at equilibrium. The topmost "wedge" of the hexagon is now outlined in green, indicating that the vertex $r > q > p$ from this wedge is the closest vertex to M (see Fig. 2). We declare $r > q > p$ to be the *social ranking*, and its top alternative r to be the *social choice*, or *winner*.

Definition 2.1. An *anonymous profile* or *voting situation* for three alternatives and n voters is a vector

$$\Pi = \langle n_1, n_2, n_3, n_4, n_5, n_6\rangle,$$

of non-negative integers satisfying $n = \sum_{j=1}^{6} n_j$. We interpret n_j as the number of voters who cast ballots for the jth ranking on the following list:

$$n_1 : p > q > r$$
$$n_2 : p > r > q$$
$$n_3 : r > p > q$$
$$n_4 : r > q > p$$
$$n_5 : q > r > p$$
$$n_6 : q > p > r.$$

We will often refer to such a vector as a "profile", as all our profiles will be anonymous. The above list is generated by stepping around the *Evolver* hexagon in the clockwise direction, starting at the bottom vertex, and our initial 8-voter profile is

$$\Pi_1 = \langle 0, 0, 3, 0, 5, 0\rangle.$$

Clicking along the bottom boundary (of the square containing the hexagon, see Fig. 1) slides out the control drawer. You can hit the reset button and try different profiles, experiment with changing the weight or friction, or enable the *point trail* feature. The small square with a question mark (on your lower left) is the *Help* button.

How does this rather oddly defined voting rule behave? Is there any reason to believe it worthy of serious consideration? Before addressing these questions, let us consider a ninth voter who wishes to add her vote to Π_1, and whose sincere preferences are $q > r > p$. If she casts a sincere ballot, how does the red point respond? What happens if she instead votes insincerely, for $q > p > r$? The sincere ballot results in a 2-way tie between r and q (as indicated by the two green wedges with the red point on their common boundary), while the insincere ballot yields a clean victory for her top-ranked alternative q. If we assume that our voter would prefer the clean victory to the tie, then the insincere vote constitutes a *manipulation* by our ninth voter. Moreover, the rubber bands provide some mechanical insight; the insincere ballot is more distant from the previous equilibrium point M than is the sincere one, so it "yanks harder" at the red point, and moves it a greater distance.

[1] A real rubber band fails to be "ideal" in two respects. First, it has a rest length $L > 0$, with no force exerted when the band is at a length of L, while an ideal band would shrink to a rest length of zero if it were allowed to relax completely. Second, the force exerted by any real band stretched to a length $L + x$ is only approximately linear in the stretch x, and that approximation is increasingly bad as x increases.

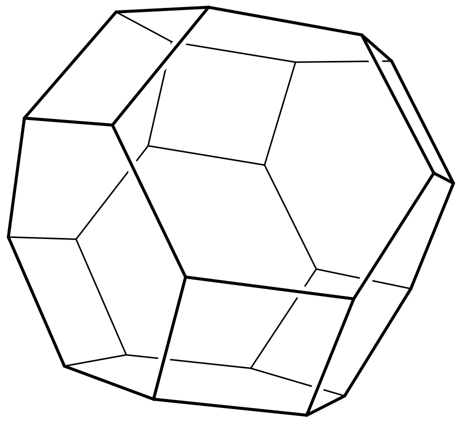

Fig. 3. The 4-permutahedron is a truncated octahedron.

2.1. Permutahedra and the Borda count

This approach is not limited to three alternatives. If A is a set of m alternatives, let $\{a_1, a_2, \ldots, a_m\}$ be any reference enumeration of A. For any ranking σ of A we write $x >_\sigma y$ if σ ranks alternative x above y, with $\geq_\sigma$ indicating the weak version. We define the *rank* $\rho(a_j)$ to be the number of elements a_k of A satisfying $a_j \geq_\sigma a_k$, and the *rank vector* $\rho(\sigma)$ to be the tuple $(\rho(a_1), \rho(a_2), \ldots, \rho(a_m))$. For example, with $m = 3$, $\rho(a_1 > a_2 > a_3) = (3, 2, 1)$, while $\rho(a_3 > a_1 > a_2) = (2, 1, 3)$. In this way, we associate to each ranking a point in $\mathbf{R}^m$; these comprise the vertices of the *m-permutahedron*, an $(m-1)$-dimensional polytope living in $\mathbf{R}^m$. Our function ρ labels each vertex with a ranking, in such a way that adjacent vertices are labeled by rankings that differ only in the reversal of a single pair of alternatives.

For example, the 3-permutahedron is a regular hexagon that lives in the plane $x_1 + x_2 + x_3 = 6$ within $\mathbf{R}^3$ (and is labeled as in Fig. 1), while the 4-permutahedron is a truncated octahedron—a semi-regular polyhedron known to Archimedes (see Fig. 3, an unlabeled representation). It lives in the hyperplane $x_1+x_2+x_3+x_4 = 10$ within $\mathbf{R}^4$.

Definition 2.2. The *Permutahedral Rubber Band rule* for m alternatives first finds the equilibrium position M under forces exerted via ideal rubber bands attached to the ballots of a profile, with the spatial locations of the ballots identified with the vertices of an m-permutahedron (as described earlier). If k voters cast a certain ballot, then there are k bands attached to that ballot. The social ranking σ is that whose vertex is closest in Euclidean distance to M, and the social choice is the alternative top-ranked by σ. When several vertices are equally close, the social choice is a tie among the alternatives that are top ranked by the rankings corresponding to these vertices; the social ranking is a tie among those rankings, of a kind that can be expressed as a single weak ordering (discussed below).

The following theorem explains our particular interest in permutahedra, and in voting with rubber bands:

Theorem 2.1. *The Permutahedral Rubber Band rule for m alternatives is identical to the Borda count for m alternatives.*

Examples 2.3.1 and 2.3.2 in Zwicker (2008a) are closely related, but make no mention of rubber bands. Instead, M is defined there to be the mean position of the ballots (counting multiplicity, after each ballot is identified with the corresponding permutahedron vertex). Theorem 2.1 follows from the arguments presented with these examples, once we establish that the equilibrium position under ideal rubber band forces is equal to the mean position of the ballots.[2] One consequence of this result is that for any tied outcome arising from an M lying in the overlap of proximity regions for two or more vertices, there exists a single weak ordering ω of A such that the outcome is equal to the set of all strict rankings that can be generated by breaking ties in ω.

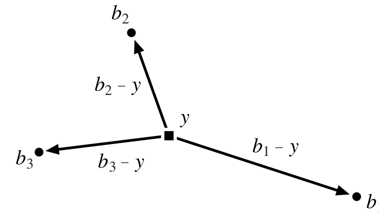

Fig. 4. The three vectors sum to zero when y is at the mean location of b_1, b_2, and b_3.

Although the application to voting appears to be new, the idea of using an equilibrium of physical forces to characterize the mean (as well as related concepts such as least squares line fit) is not; see, for example, Sall et al. (2007). The key idea is very simple:

Lemma 2.1. *Let $S = \{b_1, b_2, \ldots, b_n\}$ with $n \geq 1$ be a finite sequence of points of $\mathbf{R}^k$ (with repetitions in the sequence accounting for multiplicities). Then there is a unique point M of $\mathbf{R}^k$ at which the "ideal rubber band forces" are at an equilibrium, and it coincides with the mean $(b_1 + b_2 + \cdots + b_n)/n$.*

We will sketch one proof here, and another in the next subsection. Assume that there are $n = 3$ points in the sequence, arranged in a triangle as in Fig. 4. (It will be clear that that our argument applies just as well to the general case.) If the constant of proportionality (equivalent to a "spring constant") for our rubber bands is $C = 1$, then the net force acting on an arbitrary point y by ideal rubber bands anchored at b_1, b_2, and b_3 is $(b_1-y)+(b_2-y)+(b_3 - y)$. Thus y is at an equilibrium for these forces if and only if

$$(b_1 - y) + (b_2 - y) + (b_3 - y) = 0,$$

which is clearly equivalent to

$$y = (b_1 + b_2 + b_3)/3.$$

This line of reasoning applies for any positive constant C and can easily be extended to prove:

Lemma 2.2. *Let $b_1, b_2, \ldots, b_n$ be a sequence of points of $\mathbf{R}^k$, $M = \frac{1}{n}\sum_{j=1}^{n} b_j$ be their mean location, and y be any point in $\mathbf{R}^k$. Then $\sum_{j=1}^{n}(b_j - y) = n(M - y)$.*

Lemma 2.2 says that if we simultaneously translate all of the anchor points b_j from their initial positions to M, then there is no effect on the net force acting on an arbitrary point y. Thus ideal rubber band forces acting from multiple anchors give rise to a "central force field". This observation explains some of the dynamics observed on the *Evolver* simulation.

2.2. Potential energy $=$ discontent of the electorate

Alternately, consider the total potential energy stored by the rubber bands. For each individual band, this energy is equal to the work that would be done by the band if it were allowed to relax from its actual length to a length of zero. But the force exerted by a single band is linear in (actually, is equal to) its length x, so for each band, that P.E. is equal to half the square of its length:

$$\int_0^{\|y-b_j\|} x\,\mathrm{d}x = \frac{1}{2}\left\|y - b_j\right\|^2. \qquad (\dagger)$$

[2] Counting multiplicity, of course. We will not always remind you of this qualification; it applies throughout the rest of the paper.

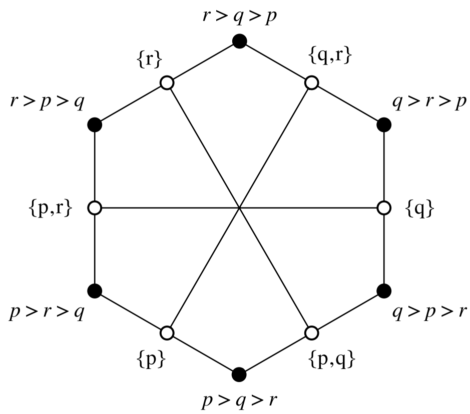

Fig. 5. In the rubber-band machine for approval voting, the bands are anchored at the ballots (white nodes), but the election's outcome is determined by proximity to a ranking (black node).

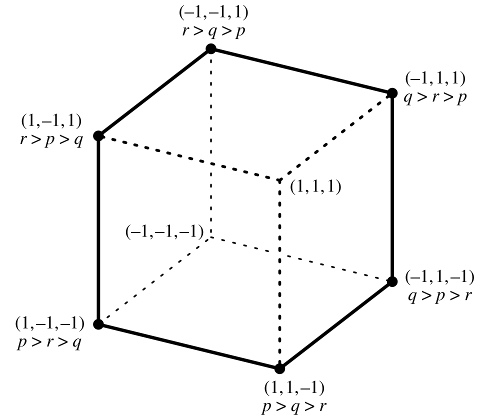

Fig. 6. In the rubber band machine for the Kemeny Rule, the hexagon's edges meet at 90°, forming part of a 3-dimensional cube.

If we assume that total P.E. is minimized at equilibrium,[3] then y is at equilibrium when the function

$$G(y) = \sum_{j=1}^{n} \left\| y - b_j \right\|^2 \qquad (\dagger\dagger)$$

is minimized. But the mean of the b_j is known to be the point minimizing the sum of squared distances to these points, so once again we may conclude that the rubber band equilibrium location is the mean.

The appeal of this second argument is that when we apply it to the voting context, we can interpret the quantity (†) as the *discontent* felt by voter j at the point y and the sum (††) as the *total discontent* of the electorate. The rubber band equilibrium thus minimizes this total discontent, and in this way constitutes a consensus position.

In particular, then, the Borda count for m alternatives has this interpretation: it first finds the point M minimizing the total discontent of the electorate, as measured by the sum (††), with b_j standing for the m-permutahedron vertex corresponding to voter j's ballot. Then it selects the ranking σ identified with the permutahedron vertex $b^\star$ closest to M.[4]

2.3. Kemeny's rule and approval voting

Moreover, the interpretation applies to quite a broad class of voting rules—any, in fact, that belong to the class of *generalized scoring rules* (equivalently, to the class of *mean proximity rules*, Zwicker, 2008a). Thus both the Kemeny rule, and approval voting, for example, correspond to suitable rubber band mechanisms, and can be expressed in terms of minimizing a sum of the form (††). Differences among these rules are due solely to differences in the way we identify ballots and election outcomes with spatial positions.

Fig. 5 shows a rubber band machine for approval voting with 3 alternatives. Approval ballots are now identified with the midpoints of the sides of the Fig. 1 hexagon; these serve as the "anchor points" for our rubber bands. The outcome, however, is determined by proximity to the hexagon's vertices, labeled with rankings as before. Thus the outcome of an approval election is a ranking, and one can show that this ranking is identical to that obtained via standard approval voting.[5]

The outcome of the Kemeny rule is usually described as the ranking σ that minimizes the sum, over all voters i, of the number of pairs of alternatives that are ordered oppositely by σ and by the ith voter. Fig. 6 depicts the rubber band machine for this rule when there are 3 alternatives; essentially this is the 3-permutahedron, bent in such a way that its vertices and edges form part of a 3-cube. More generally, any m-permutahedron is an orthogonal projection of an appropriate part of the hypercube of dimension $\binom{m}{2}$; this projection affects the question of which vertex is closest to M, and this effect represents the difference between Borda's rule, and Kemeny's.[6]

Barthélémy and Monjardet (1981) were the first to observe that the Kemeny rule can be described via the mean in this way, and pioneered the application of Huyghens' theorem in this context. Saari and Merlin (2000) have a related characterization of Kemeny, which uses the L_1 norm in place of Euclidean distance. For more on the Kemeny rule, and examples of other rules based on the mean, see Zwicker (2008a) and Zwicker (2008b).

3. Voting machines that use strings and weights

Open the *Evolver* control panel, hit "Reset", and then toggle from "Mean" to "Mediancentre" (using the buttons at the bottom of the window). With this change, each blue line now exerts a force on the red point whose magnitude is independent of distance and equal to the number of voters casting ballots for the corresponding ranking. Such forces can be realized, for example, via weights attached to strings. Fig. 7 shows three holes, with a unit weight suspended from each string; at the resting position of the ring the

3 This assumption can be justified, and the argument made rigorous, with an appeal to a theorem of Christiaan Huyghens (1629–1695), stating that $\sum_{j=1}^{n} \left\| y - b_j \right\|^2 = \sum_{j=1}^{n} \left\| M - b_j \right\|^2 + n \left\| M - y \right\|^2$; here M again denotes the mean of the b_j.

4 The ultimate outcome of the election is the ranking σ, not the point M, so one might argue in favor of directly minimizing the electorate's discontent with the vertex $b^\star$ corresponding to σ. However, when "discontent" is measured via the square of the distance, as in (††), Huyghens' formula (previous footnote) tells us that the two approaches agree: the vertex minimizing discontent (among vertices) is the one closest to the point M achieving the absolute minimum discontent.

5 Consider the ranking σ that labels the hexagon vertex closest to the equilibrium point M. Then the alternative top-ranked by σ is the one approved of by the greatest number of voters, the second-ranked alternative has the second greatest approval score, etc. A general version (for m alternatives) is proved in Zwicker (2008a), but with M defined as the mean. Once again, Lemma 2.1 translates that result into one in terms of equilibria under rubber band forces.

6 One might hope, then, that all distinctions between Borda and Kemeny could be explained purely in terms of the geometry. For example, Kemeny is Condorcet consistent while Borda is not, and in this case related geometric explanations can be found in Saari (1994) and Zwicker (1991). Also, Borda outcomes are easy to compute, while Kemeny outcomes are NP-hard (Bartholdi et al., 1989); a geometric explanation of this difference might be quite interesting.

unit forces in the directions of the holes will be at equilibrium. Forces whose magnitudes are independent of distance do not yield the same equilibria as those exerted by ideal rubber bands, so this resting position no longer represents the mean—instead, it is a point known as the *mediancentre*, which we will discuss shortly.

Now, imagine that the hexagon is painted on a similar tabletop, with six holes – one at each vertex – and with k unit weights hung from a string when there are k voters who cast the corresponding ballot. This version of the *Evolver* simulation computes a voting rule based on this different equilibrium notion, with the green outline again indicating which vertex is closest to the equilibrium position of the red point. Of course, this new "Mediancentre Borda" rule, or "M^cBorda" for short, is *different* from Borda. Can you anticipate any of the implications, in terms of the voting-theoretic properties of Borda vs. M^cBorda?

For example, what is the M^cBorda outcome for the 8-voter profile Π_1? What about the desire of our ninth voter to achieve a victory for alternative q—is a manipulation possible? What are the smallest profiles on which Borda and M^cBorda yield different outcomes? Enabling *Evolver's* "point trail" feature (available on the control panel) reveals that the dynamics are now quite different; in particular, the forces no longer form a central force field, as they did with rubber bands (see Lemma 2.2).

3.1. Voting in $\mathbf{R}^1$

The motivation for M^cBorda originates with the observation that certain specialized types of voting are best geometrized in $\mathbf{R}^1$, and in this setting the standard median has some attractive properties. Imagine, for example, that a group of n people are planning a hike together, and they have to decide on the length of the hike (in kilometers, for example). Suppose we assume that individuals' preferences over distances are *single peaked*: each voter i has a most-preferred "ideal hike length" d_i and whenever $d_i \leq d_i' < d_i''$ (or $d_i \geq d_i' > d_i''$) voter i prefers d_i' to d_i''. The group will hold an election to determine the actual hike length, with each ballot consisting of a non-negative real number (interpreted as a hike-length, in kilometers).

What decision rule should we use to determine the length of the hike? Suppose our last voter knows in advance that the actual hike length will be the mean of the ballots cast by the voters, and further knows (or has good estimates that suggest) that the first $n-1$ ballots will have a mean of around 5 km. If her own preferred length is 12 km, then by casting a ballot considerably larger than 12, she can make the mean come out very close to 12.

Perhaps it is unrealistic to presume that only one voter is willing to manipulate the outcome in this fashion. Suppose, instead, that they are all willing to do so, and each has perfect knowledge of the others' ideal lengths. Then there is a Nash equilibrium of the resulting voting game, in which all but one hiker casts 0 as their ballot, while a hiker whose ideal length is d_{i_0} is largest casts $n(d_{i_0})$ and the mean is d_{i_0}.[7]

Moreover, if our manipulating hikers' presumed knowledge of the others' preferences is actually quite faulty, then the outcome can be wildly inefficient. For example, if each overestimates the mean ideal length of the others, we might wind up with a mean of 0 km.

On the other hand, if we use the median as our method for amalgamating the ballots and there are an odd number of hikers then it is easy to see that no voter has any incentive to cast a ballot for a length other than their true ideal (and for those whose

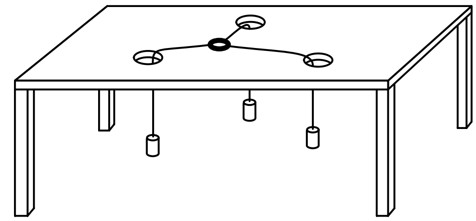

Fig. 7. Each weight exerts a force on the ring whose magnitude is independent of the distance from hole to ring. The equilibrium position of the ring is the Fermat-Weber point, or *mediancentre*—it minimizes the sum of distances from the ring to the holes. The device is called a *Varignon frame* (see Varignon, 1687).

knowledge of others' preferences is incomplete to the extent that there is a non-zero probability of being the median voter, there is a positive incentive favoring an honest vote).[8] Moreover, Moulin (1980) has shown that there exists a precise sense in which the median (along with certain slight variants) is the unique strategy-proof rule in this context.

Additionally, while knowledge of a voter's ideal point does not allow one to reconstruct their full preference ranking over non-negative real numbers, one can prove that it does suffice to identify the *Condorcet alternative* (see Section 4.5), which always exists in this setting and which coincides with the median. Thus, when voting on $\mathbf{R}^1$ with single-peaked preferences, strategy-proofness and Condorcet efficiency provide plausible grounds for choosing the median as our voting rule.

3.2. Voting in $\mathbf{R}^k$

Now let us move from $\mathbf{R}^1$ back to the context in which a ballot consists of a strict ranking of the alternatives in A, a finite set with 3 or more members. Can the advantages of the $\mathbf{R}^1$-median be transferred to this setting? Bassett and Persky (1999) suggest the use of the *Median Voting Rule* (MVR), in which the score of an alternative a_j is the median, taken over all voters i, of the rank $\rho_i(a_j)$ assigned to a_j by voter i, and the winning alternative is the one with highest score. Gehrlein and Lepelley (2003) fault this rule on a number of grounds. In particular, MVR is highly *indecisive* (tied outcomes are very common). For example, consider a three-way election in which no alternative is top-ranked by a strict majority of the voters, and none is bottom-ranked by a strict majority. For any such profile, MVR declares a three-way tie, with each alternative achieving a median rank of 2. Yet it is easy to cook up profiles that meet these rather minimal conditions and also make some single alternative x the winner by a landslide; for example, such an x can be both the scoring rule winner (under all scoring rules) and the Condorcet alternative, with lopsided margins in both senses.

So, are there alternatives to MVR—voting rules that exploit the median idea to achieve elevated resistance to manipulation, without suffering MVR's drawbacks? First, let us observe that MVR itself can be seen as a form of voting on the permutahedron, as follows. Define the *coordinate-wise median CWM* of a sequence of rank vectors in $\mathbf{R}^k$ to be the vector obtained by separately applying the $\mathbf{R}^1$-median in each coordinate. Identify each ballot with the corresponding vertex of the permutahedron, find the CWM of all ballots cast, and use the closest vertex to determine the election outcome (as before); the result will agree with MVR. In other contexts, however, statisticians have tended to see the coordinate-wise median as a badly behaved extension of the univariate median

7 When several hikers share this largest ideal length d_{i_0}, there are several equilibria, but each yields a mean equal to d_{i_0}.

8 With an even number, the situation is more complex, and depends on whether one uses the "high-school" definition of the median as the average of the middle two numbers, or a multi-valued definition that selects, as medians, all numbers in the closed interval between the two middle two.

to the multivariate context, and have considered a variety of alternative extensions in its stead.[9] For more on the general topic of multivariate medians, we suggest Small's nice survey article (Small, 1990) on the topic.

Perhaps the most venerable of the multivariate medians goes back to the work of Fermat, who considered the problem of finding the point y minimizing the sum of the distances to three given points in the plane. Notice that if we are given any odd number of points in $\mathbf{R}^1$, then the (standard, $\mathbf{R}^1$) median does indeed minimize the sum of the distances to the points given.

In the context of the facilities location problem, Weber (1909) extended Fermat's notion to an arbitrary number of points, and so the term "Fermat-Weber point" is sometimes used for the point in $\mathbf{R}^k$ that minimizes the sum of the Euclidean distances to a given finite selection of points of $\mathbf{R}^k$. But the same idea has been re-invented repeatedly under a variety of names, and there seems to be no consensus term; our use of "mediancentre" is taken from Gower (1974).[10]

Definition 3.1. The *mediancentre* of a finite set $\{b_j\}_{0\leq j\leq n}$ of points in Euclidean space is the point y minimizing the sum

$$F(y) = \sum_{j=1}^{n} \left\| y - b_j \right\| \qquad (\rhd\!\lhd\rhd\!\lhd)$$

of (unsquared) distances to the points.[11]

With an even number of points, both the median and mediancentre sometimes exhibit pathological behavior. Given an even number of points of $\mathbf{R}^1$, for which the middle two points are not at the same location, these two determine a closed interval, within which *every* point shares the distinction of minimizing the distance sum. Clearly, the mediancentre does the same, for an even number of collinear points with such a middle gap. However, it is possible to show (see footnote 11) that as long as a finite sequence $\{b_j\}_{0\leq j\leq n}$ of points of $\mathbf{R}^k$ are non-collinear, or are odd in number, there is a unique point that minimizes the sum of distances to the b_j—a unique mediancentre.

No three vertices of an m-permutahedron are collinear, however. So, if the b_j arise from an anonymous profile, and consist of such vertices, then the problematic configuration can only arise from a rather special type of profile:

Definition 3.2. An anonymous profile for $2i$ voters is 2-*split* if two of its coordinates (as a vector) are equal to i and the remaining coordinates are zeros.

Notice that when a 2-split profile is entered on *Evolver* with the mediancentre button engaged, the red point can be dragged to *any* location along the line segment connecting the corresponding vertices, and seen to be at equilibrium, (illustrating that these locations all minimize the distance sum). The 2-*Split Profile* link provides one example of this situation.

3.3. The Mediancentre-Borda Rule

Definition 3.3. The *Mediancentre-Borda* or *M*^{c}*Borda* voting rule for m alternatives first plots all ballots at corresponding vertices of the m-permutahedron and determines the mediancentre MC of the resulting sequence $\{b_j\}_{0\leq j\leq n}$ of points. The social ranking σ is that whose vertex is closest to MC, and the social choice is the alternative top-ranked by σ, with ties determined as in Definition 2.2. The M^{c}Borda outcome for a 2-split profile is considered undefined.

Thus the *Mediancentre-Borda Rule* (or *M*^{c}*Borda Rule*) can be seen as an attempt to implement the Bassett-and-Persky idea, using an alternative to their choice of the coordinate-wise version of spatial median, in the form of the mediancentre or Fermat-Weber point. Alternately (and this better describes the initial thinking), starting with the equivalent formulation of the Borda count as the proximate outcome to the mean position of the ballots in the m-permutahedron, M^{c}Borda is obtained by replacing the mean with the mediancentre.

Our decision to declare the M^{c}Borda outcome undefined for 2-split profiles is a temporary one, and not very satisfactory. It is tempting to declare the mediancentre for such a profile to be the entire line segment L joining the corresponding two vertices, and the M^{c}Borda outcome to be a tie that includes a ranking σ if and only if there exists *some* point $z \in L$ such that σ's vertex is at least as close to z as is any other vertex. But that yields some rather strange ties, so we have postponed any resolution of the issue.[12] This explains why our focus is on an odd number of voters, for much of the rest of the paper. Clearly, any serious attempt to promote M^{c}Borda for an actual application would require a better answer.[13]

The sum $(\rhd\!\lhd\rhd\!\lhd)$ minimized by the mediancentre can again be interpreted as the total discontent felt by the electorate at the point y, but of course our measure of individual discontent is now different. Each term in the sum can be obtained as an integral

$$\int_0^{\|y-b_j\|} 1\,\mathrm{d}x = \left\| y - b_j \right\| \qquad (\rhd\!\lhd)$$

and this integral can once again be interpreted as stored potential energy.

Imagine that each voter owns a unit weight resting on the floor beneath the table on which the hexagon is drawn (see earlier discussion). The weight for voter j is positioned directly under the vertex b_j corresponding to j's ballot, and is attached to one end of a vertical string, with the free end of the string located exactly at the height of the table-top (at the top of the hole drilled at b_j). As we draw the free end of the string from its initial position to the location y we lift the weight from the floor, and the integral $(\rhd\!\lhd)$ represents the work done.

Alternately, we can minimize the sum $F(y)$ by setting its gradient equal to 0. This gradient at y is equal to the sum

$$V(y) = \sum_{j=1}^{n} \frac{b_j - y}{\left\| b_j - y \right\|}$$

[9] There is only one viable "multivariate mean", of course, and it coincides with the coordinate-wise mean, but this can be put down to the fact that the coordinate-wise mean commutes with projection maps, while the coordinate-wise median does not.

[10] In Small (1990), Small suggests "L_1-median". However, while the distances to the given points are indeed summed, rather than square-summed, each individual distance is in the L_2 (Euclidean) norm, and so his suggestion came in for some criticism.

[11] If the b_j are non-collinear, then it is straightforward to show that F is *strictly convex*—F's value at any point z on the open line segment joining two other points x and y is strictly less than the height, at z, of the chord (secant line) joining the points $(x, F(x))$ and $(y, F(y))$. A strictly convex function can have at most one minimum, so the mediancentre of $\{b_j\}_{0\leq j\leq n}$ is uniquely defined in this case. If, on the other hand, the b_j all lie on some line L and are either odd in number, or are even but with the middle two sharing a common location, then the $\mathbf{R}^1$ median is unique; as the mediancentre coincides with the median for collinear points, it too is unique. Thus the mediancentre is defined and unique except for the case of an even number of collinear points with a middle gap.

[12] An alternate argument can be mounted in favor of declaring the mediancentre of a 2-split profile to be the midpoint of the line segment between the two vertices. This choice produces different election outcomes. Perhaps a closer examination of the axiomatic properties of M^{c}Borda will resolve the issue of which approach is more appropriate, in terms of preserving those axiomatic properties for 2-split profiles. At this moment, we lack the insight to resolve the matter.

[13] Keep in mind, however, that with a fixed number m of alternatives, the number of 2-split profiles is independent of the number $2i$ of voters. For 3 alternatives there are only 15 (6-choose-2) such profiles for each i, for example, which becomes a vanishingly small proportion of all profiles as i grows; in 5 of these 15, the voters are unanimous as to their top alternative.

of unit vectors at y (which point toward the vertices corresponding to the ballots cast). Thus, the mediancentre arises as the equilibrium position of unit forces. This establishes that our *Evolver* simulation indeed computes the M^{c}Borda rule, when the mediancentre button is engaged.[14]

3.4. *Calculating* M^{c}Borda*outcomes*

There is a significant literature concerned with algorithms that find the mediancentre of a selection of points in Euclidean space (see, for example, Kuhn, 1973; Eckhardt, 1980 and Brimberg, 1995). For 3 or 4 points, there are attractive geometric characterizations for the mediancentre, but with 5 points or more, we are left with approximation algorithms. In principle, we do not require the exact location of the mediancentre MC in order to calculate the M^{c}Borda outcome for a given profile—we only need to know which proximity zone contains MC. In practice, however, we have been using an iterative algorithm in order to locate MC with enough precision to be confident about identifying the proximity zone.

The classical approach of Weiszfeld (1937) uses an iteration involving a weighted sum of the b_j with weights inversely proportional to the distance to the current approximation. We take a slightly different approach that is motivated by the physical simulation. The algorithm starts at some location y_1, often taken to be the mean of the original sequence $b_1, b_2, \ldots, b_n$ of points. After calculating the unit vector sum $V(y_k)$, the algorithm sets

$$y_{k+1} = y_k + CV(y_k), \qquad (\star\star)$$

where the value of C is chosen dynamically as the simulation progresses in order to improve the rate at which our sequence $\{y_k\}$ converges to the mediancentre MC.[15] The algorithm halts when the length of $V(y_k)$ becomes sufficiently close to zero, and the final y_k is declared to be the approximate mediancentre. When C is made small enough, the points y_k turn out to be sufficiently close to make the path they trace resemble the smooth "descent curve" that would be traversed by a point whose velocity is everywhere tangent to the vector field $V(y)$. This is somewhat like the path that a ball would take as it rolled downhill on the graph surface $z = F(y)$ (from ($\bowtie\bowtie$)), except that a true ball has momentum that would cause it to veer off the descent curve.

Fig. 8 shows most of these paths for the case of 5 voters ranking 3 alternatives (the on-line version is in color—see the *Figures* links). Each white (or green) point represents the starting position (mean position) for some profile, while black (red) denotes the terminus (mediancentre) for that profile. Thus, paths that cross the gray radial lines indicate profiles for which M^{c}Borda and Borda yield different social rankings. Some mean points sprout several descent paths—can you guess the significance? Certain profiles result in a mean and mediancentre that are both at the center of the hexagon, other (unanimous) profiles place the mean and median both at the same vertex, and a third group yields a hexagon vertex as mediancentre; for these types, no descent paths appear in this figure.

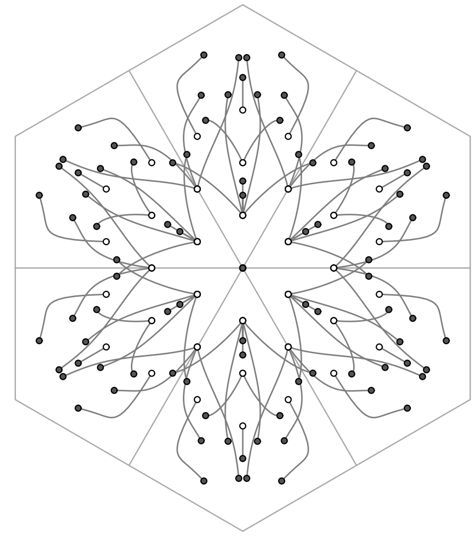

Fig. 8. Mediancentre descent paths for 5 voters. Each path starts at the mean location for a profile (white dot) and ends at the mediancentre location for that profile (black dot). Excluded are profiles whose mediancentre is a hexagon vertex.

These paths are related to the routes followed by the moving point in the *Evolver* simulation, but they are not quite the same, because *Evolver* builds in the momentum that is part of the actual physical system. The difference is minimized, however, if one ratchets up the friction while decreasing the weight (try setting weight = 0.1 and friction = 0.8). Note that you can click and drag the red point to an arbitrary starting position, or even "throw" it.

Fig. 9 shows descent paths for the single profile $\langle 0, 0, 2, 1, 1, 1\rangle$ of 5 voters with multiple starting points. This figure was originally developed as one of a number of checks on the behavior of the algorithm, but we include it here because it hints at the odd shape of the graph for the sum-of-distances function $z = F(y)$. So, what sort of descent paths would we see for the earlier sum-of-squared-distances function $G(y)$ from line (††)? A consequence of Lemma 2.2 is that such a picture would be boring; all descent paths are straight rays radiating from the mean, because the graph of $z = G(y)$ is a paraboloid of revolution, with minimum sitting directly above the mean position.

Our algorithm only approximates the mediancentre for a given profile, so how can we be certain that we have correctly identified the M^{c}Borda outcome? In particular, if the algorithm places the mediancentre near the boundary of two proximity regions, can we be confident that round-off error is not affecting our conclusion that there is (or is not) a tie for this profile? We have performed a number of checks designed to reveal errors due to approximation; we are confident that our most recent M^{c}Borda calculations are accurate for profiles having 25 or fewer voters. In particular, we have checked all near-ties in the calculated mediancentres through $n = 25$ and verified that none represent actual ties. (Given a specific profile, it is possible to write down algebraic conditions that would have to be satisfied for a tie to exist; these can be checked by hand for the relatively small number of mediancentres that fall near the borders.) We have also compared our results to those obtained by the Weiszfeld algorithm for $n = 5$, $n = 11$, and $n = 17$ and found that the solutions agree to within the tolerances used.

Because the *Evolver* simulation needs to operate in real time and respond in a realistic fashion, its determination of ties is not quite as accurate. Again, an exhaustive check of the near-tie profiles through $n = 25$ reveals that the first time the *Evolover* incorrectly

[14] We are glossing over a subtlety here, which we treat in greater detail in Cervone and Zwicker (2010). The function $F(y)$ given by ($\bowtie\bowtie$) fails to be differentiable at each of the original points b_j, so the gradient is undefined there. Thus, when the mediancentre is located at such a point b_j, we can no longer characterize the mediancentre as the equilibrium of our unit forces — at least not in quite the same way. A somewhat weaker notion of equilibrium does apply in this case, however.

[15] In fact, for reasons that may be apparent (see the previous footnote) the sequence typically fails to converge when MC is located at a vertex, and Eq. ($\star\star$) behaves badly if y_k coincides with one of the b_i. So, in practice, our algorithm first tests each vertex of the hexagon (among those voted for) to see whether it is equal to the mediancentre, only launching the iterative procedure if the answer is always *no*. (Let $V^*(v)$ denote the sum of the unit vectors from v to each of the ballots $b_i \neq v$. Then, except for 2-split profiles, the mediancentre coincides with the vertex v exactly when the number of voters j with $b_j = v$ is equal to, or exceeds, the magnitude $\|V^*(v)\|$. This condition expresses the weaker equilibrium notion alluded to in the previous footnote.)

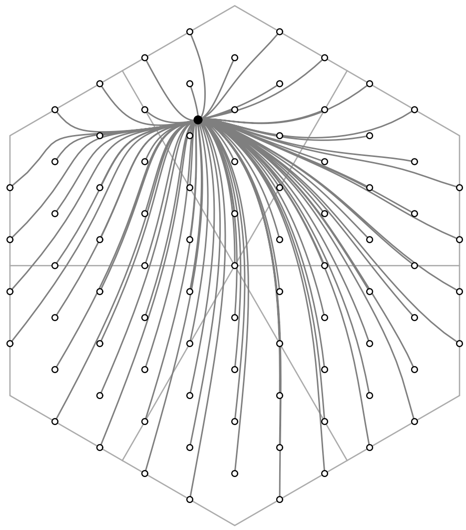

Fig. 9. Mediancentre descent paths for the profile $\langle 0, 0, 2, 1, 1, 1\rangle$, with many starting points.

marks a non-tie as a tie is for the profile $\langle 6, 0, 3, 5, 1, 0\rangle$ (and its rotations and reflections) when $n = 15$, so we are confident that the interactive program gives accurate results for $n < 15$. There is another such configuration when $n = 17$, one when $n = 18$, then two or three more in each of $n = 20$, 23, and 25, so the number of incorrectly reported ties is quite small over all.

4. Properties of the M^{c}Borda voting rule

With a few exceptions, we have found it difficult to prove analytical results for the M^{c}Borda rule, so our approach has been to explore its properties for the case of three and, more recently, four alternatives, by enumerating all possible *anonymous profiles* (also called *voting situations*) for n voters, for various values of n.

As each anonymous profile Π is enumerated, we use the algorithm described to compute the M^{c}Borda winner for Π and then to test Π for whichever voting-theoretic property we are exploring. For example, is Π *manipulable*—that is, does there exist at least one voter v who can, by changing her vote (from the ranking she expressed, as recorded by Π), obtain an outcome she prefers (by which we mean that she prefers the alternative that wins after she switches, to the one that won before she switched, where "prefers" is measured by her original ranking)?

The output is a fraction, representing the proportion of all anonymous profiles (for the given number of alternatives and voters) that have the property at hand. Alternatively, this fraction can be interpreted as the probability that a randomly drawn profile has the property, according to the probability distribution known as *Impartial Anonymous Culture* or *IAC*—see, for example, Berg (1985), Gehrlein (2002), or Stensholt (1999).

The properties we have tested M^{c}Borda for include manipulability, decisiveness, Condorcet efficiency, and one-way monotonicity, as explained below. Before we turn to those investigations, though, we consider several fundamental (and closely related) differences between the mean and mediancentre, and the resulting distinctions in voting theoretic properties of mean and mediancentre-based voting rules.

4.1. Majoritarian properties

Consider a sequence $(b_1, b_2, \ldots, b_n)$ of points in $\mathbf{R}^k$, with repetitions allowed. What can we say about the mediancentre when there are *lots* of repetitions?

1. If more than half of the b_i have the same spatial location y, then it is straightforward to show that the mediancentre of the sequence *is* y.
2. If exactly half the b_i have a common spatial location y while the other half collectively occupy at least two other locations not collinear with y, the mediancentre is again equal to y.
3. Suppose $k = 2j + 1$ with $j \geq 2$, j of the b_i share a common spatial location y, another j share a different common location w, a single b_i has a third location z, and z is closer to y than to w. Then as long as z is sufficiently far from the line segment $[y, w]$ and $\alpha = \angle zOy$ is sufficiently small (where O is the midpoint of segment $[y, w]$), the mediancentre of the sequence will be closer to y than to z or w.

The first two observations can be justified easily (using the V^* idea in footnote 15), and clearly do not apply to the mean.

We will say that a voting rule has the *Weak Majoritarian Property* if the election outcome is the ranking σ for every profile in which more than half of the voters cast the ballot σ, and has the *Strong Majoritarian Property* if the election outcome is the ranking σ for every profile in which half of the voters cast the ballot σ and the other half do not all cast ballots for a common alternative ranking $\tau \neq \sigma$. Then, as an immediate consequence of observations 1 and 2 above, together with the fact that no three vertices of a permutahedron are collinear, we obtain:

Proposition 4.1. *The* M^{c}Borda *rule has the strong and weak majoritarian properties.*

Versions of Proposition 4.1 hold for other rules that are similarly based on the mediancentre. (For example, one can define mediancentre versions of the Kemeny rule and of approval voting, the same way that M^{c}Borda is defined from Borda; we know little about these rules.) Borda, of course, violates both properties. Our profile Π_1 from Section 1 serves as a counterexample for the weak version, and explains why Borda has sometimes been described as favoring a "compromise" outcome, in contrast with the stronger majoritarian tendencies of *Condorcet extensions* (see definition in Section 4.5).

However, Borda treats large groups of voters with diametrically opposed views in a manner that is quite similar to that of many (if not most) Condorcet extensions, and that is arguably *not* majoritarian. For any $i, j \in A$, let a_{ij} be the number of voters who rank alternative i over alternative j minus the number of voters who rank alternative j over alternative i. Let $[a_{ij}]$ be the resulting antisymmetric matrix. We will say that a voting rule is *net pairwise* if outcomes under that rule depend only on the information in $[a_{ij}]$. Now, suppose that j voters cast ballots for ranking σ, while another j voters rank alternatives in the exact opposite order, with ballots of σ^{reverse}. Then any variable electorate rule that is net pairwise treats these voters as if the two groups canceled each other out. One consequence is that variable electorate net pairwise rules with the Pareto property (if all voters rank alternative x over y, then x is ranked over y in the outcome ranking) satisfy the following property:

Definition 4.1. A voting rule satisfies the *all-but-one cancellation property* if for any profile consisting of j voters casting some ballot σ, another j casting σ^{reverse}, and a single, $(2j + 1)$st voter casting τ, the election outcome is τ.

Borda fits the above assumptions, as do many Condorcet extensions, such as Copeland and Simpson (minimax), so these rules satisfy all-but-one cancellation. On the other hand, consider the following property, which represents a sort of antithesis:

Table 1
Counting manipulable anonymous profiles for three alternatives.

Voters	Anon. profiles	Borda			
		Strong		Weak	
		1st-choice manipulation	2nd-choice manipulation	1st-choice manipulation	2nd-choice manipulation
3	56	0	0	18	6
5	252	0	0	78	12
7	792	0	0	198	18
9	2002	0	0	438	42
		M^{c}Borda			
		Strong		Weak	
		1st-choice manipulation	2nd-choice manipulation	1st-choice manipulation	2nd-choice manipulation
3	56	0	0	6	0
5	252	6	12	18	0
7	792	36	36	24	0
9	2002	78	108	66	0

Voters	Anon. profiles	Weighted total		Ratio: W_B/W_{MC}	Un-weighted total		Ratio: T_B/T_{MC}
		Borda W_B	M^{c}BordaW_{MC}		Borda T_B	M^{c}BordaT_{MC}	
3	56	12	3	4.0000	24	6	4.0000
5	252	45	27	1.6667	90	36	2.5000
7	792	108	84	1.2857	216	96	2.2500
9	2002	240	219	1.0959	480	252	1.9048

Definition 4.2. Let $j \geq 2$ be an integer. A voting rule satisfies the *all-but-one tipping property at j* if, for each $k \geq j$, and each profile consisting of k voters casting some ballot σ, another k casting σ^{reverse}, and a single, $(2k+1)$st voter casting τ, the election outcome is either σ or σ^{reverse}.

M^{c}Borda for 3 alternatives satisfies the all-but-one tipping property at $j = 2$. M^{c}Borda for 4 alternatives fails to have the property, though it leans more towards tipping than cancellation: for each pair $(\sigma, \sigma^{\text{reverse}})$ there exist 2 rankings τ such that for all k the M^{c}Borda outcome is τ for the profile with k σ-voters, $k\,\sigma^{\text{reverse}}$-voters, and one τ-voter; for any $(2k+1)$st ranking λ other than these, the outcome is either σ or σ^{reverse} (when k is sufficiently large). We conjecture that when $m \equiv 2 (\text{mod } 4)$, or $m \equiv 3 (\text{mod } 4)$, M^{c}Borda for m alternatives satisfies tipping for a sufficiently large j. We know that when $m \equiv 0 (\text{mod } 4)$, or $m \equiv 1 (\text{mod } 4)$, the situation resembles that described for $m = 4$, but with more than two exceptional τ for large m.

These assertions are consequences of observation 3 together with the geometry of the 3- and 4-permutahedron. The observation and its consequences are discussed in greater detail in Cervone and Zwicker (2010), which also explores the ideas via some structured exercises using *Evolver*. The reader can use *Evolver* to test the tipping property for M^{c}Borda with $m = 3$ as j gradually increases.

4.2. Manipulability and decisiveness

Does M^{c}Borda fulfill its promise of increased resistance to manipulation, as compared to Borda? Table 1 suggests that the answer is *yes, to a certain extent, for small numbers of voters*. But there is a significant list of qualifications:

1. We have examined only manipulations by individual voters, not manipulations by coalitions, and not equilibrium behavior in the voting game. (Of course, it makes sense to consider changes by more than one voter, and we hope to do so in the future.)
2. With 3 alternatives, it is well known that Borda has no *single-winner* manipulations (see definition below). M^{c}Borda does have such manipulations, so in this sense M^{c}Borda is more manipulable than Borda. Corresponding tables for 4 or more alternatives might tell a different story, because they will count some single-winner Borda manipulations.
3. For election outcomes with ties, the matter of deciding what constitutes a manipulation is a delicate one; there is no standard, generally accepted definition.
4. The table shows the relative advantage of M^{c}Borda decreasing as the number of voters grows. It will be interesting to see where things go for larger numbers of voters.

The methodology for Table 1 is as follows. In an *individual* manipulation, one voter v switches ballots, resulting in a change in the social choice that represents an improvement according to v's original ballot (which we interpret as representing her sincere preference ranking). For such a manipulation to be *single-winner*, there must be no ties involved—the change in outcome is from a single winning alternative t to some other single winner s, and the meaning of "improvement" is clear: v ranks s over t according to her sincere ballot. But what about ties?

In a *strong first-choice* manipulation, a voter v's top-ranked candidate x is not among the winning alternatives when she votes sincerely, but there exists an insincere vote by v that changes the outcome so that x becomes the unique winner. A *strong second-choice* manipulation is defined similarly: neither v's top-ranked candidate nor her second-ranked candidate y are among the winning alternatives when she votes sincerely, but there exists an insincere vote by v that makes y the unique winner. Any single-winner manipulation is thus a strong first- or second-choice manipulation—in fact (see later discussion) it seems likely that with 3 alternatives and an odd number of voters, all strong first- or second-choice M^{c}Borda manipulations are actually single-winner.

Notice that the gray lines radiating from the center of our hexagon (in the *Evolver* simulation) represent boundaries between proximity regions; in a strong manipulation, the red point crosses at least one such boundary.

Suppose a voter v's top-ranked candidate x is not among the winning alternatives when she votes sincerely, but there exists an insincere vote by v that makes x become one among several winners, or suppose x *is* among two or more winners, and there exists an insincere vote by v that makes x become the unique winner. In either situation, we declare this to be a *weak first-choice* manipulation. Similarly, *weak second-choice* manipulations are defined via two scenarios. In the first, v's top-ranked candidate x as well as her second-ranked candidate y are both excluded from the winning alternatives when she votes sincerely, but there exists an insincere vote by v that leaves y (but not x) among at least two

Table 2
Types of manipulations (assuming our manipulating voter has preference $x > y > z$).

		Winning alternatives *before* attempted manipulation						
		x	y	z	xy	xz	yz	xyz
	x	–	Strong first	Strong first	Weak first	Weak first	Strong first	Weak first
	y		–	Strong second			Weak second	
	z			–				
Winning alternatives *after* attempted manipulation	xy		Weak first	Weak first	–		Weak first	
	xz		Weak first	Weak first		–	Weak first	
	yz			Weak second			–	
	xyz		Weak first	Weak first			Weak first	–

winners. In the second scenario, y (but not x) is initially among at least two winners, and there is an insincere vote by v that leaves y as the unique winner.

Thus, in a weak manipulation our voter moves the red point in *Evolver* onto a certain boundary line, or off such a line, without strictly crossing it. Table 2 shows exactly which changes constitute manipulations according to the definitions above. It considers a voter with sincere preference ranking $x > y > z$. There are 7 possible outcomes (non-empty subsets of $\{x, y, z\}$), hence 49 possible changes in outcome. The table classifies these 49 changes according to type of manipulation in our classification scheme; a blank entry indicates "no manipulation".

The literature contains alternative methods for defining "manipulation" in the context of ties. One method modifies the original voting rule by employing a fixed ordering of the alternatives – a *tie-breaking agenda* – to resolve all tied outcomes; if there are several winners according to the original rule, the modified rule selects the highest-ranked (according to the agenda) from among them. As long as the original rule is *neutral* the choice of agenda has no effect in terms of counting the number of manipulations for the modified rule. This is the most common approach used in papers that compare voting rules in terms of their *relative* degree of vulnerability to manipulation; see Kelly (1993), Aleskerov and Kurbanov (1999), Smith (1999) and Favardin et al. (2002).

Another approach is to use a *set extension* principle, to extend an individual's preferences over individual alternatives to a partial ordering over sets of alternatives. There are a variety of such extensions (see Gärdenfors, 1979 and Barberà et al., 2004). This is the more common approach for defining an *absolute* sense in which an irresolute voting rule (ones with ties) is or is not manipulable. Quite recently, however, the set extension method has been employed to measure relative degree of vulnerability, as well; see Aleskerov et al. (2011) and Aleskerov et al. (in press).

After choosing some method for defining "manipulation" in the context of ties, one still must decide on a method for counting manipulations. For example, suppose a certain profile can be manipulated by several voters (whose ballots may differ), or can be manipulated several different ways by a single voter. When we measure the degree or extent of manipulability for the rule at hand, should such a profile be weighed more heavily than a profile having an essentially unique manipulation? Again a variety of approaches are possible (see Kelly, 1993; Smith, 1999 and Aleskerov and Kurbanov, 1999). Essentially, our choice was to look at the most serious manipulation possible for a given profile P, and classify P as manipulable in that sense only. Our priority list, with more serious manipulations listed above less serious ones, is as follows:

1. Strong first-choice manipulations.
2. Strong second-choice manipulations.
3. Weak first-choice manipulations.
4. Weak second-choice manipulations.

In particular, each profile P contributes at most once to the numbers in Table 1; when it does contribute, it is assigned to exactly one of the four manipulation types.

This counting method does not address the important matter of how to weigh weak vs. strong manipulations. For example, with 3 alternatives, M^{c}Borda has some individual, single-winner manipulations while the Borda count has none. Does this distinction alone represent a sense in which Borda is *more* resistant than M^{c}Borda to manipulation?

We would argue against that interpretation, on two grounds. First, suppose we take some social choice function F – one that *does* have some single-winner manipulations – and modify it by introducing some additional ties, as follows: define the outcome of $\mathrm{Blur}_k(F)$ applied to some profile P to be a tie among all outcomes of the form $F(P^\star)$, where $P^\star$ ranges over all profiles that may be obtained from P by changing between 0 and k of the votes. For *any* such F, $\mathrm{Blur}_1(F)$ will have no strong single-winner manipulations. So, by blurring boundary lines in this way we have bought complete resistance to single-winner manipulations, at the expense of heightened indecisiveness. It would seem odd to declare $\mathrm{Blur}_1(F)$ more resistant to manipulation than F—at least, not without taking decisiveness into account at the same time.

Second, when we compare the single-winner M^{c}Borda manipulations having the fewest voters – hence, in a sense, the simplest examples – to what is happening with Borda for the same profiles, what we find seems somewhat similar to the F versus $\mathrm{Blur}_1(F)$ story above. From the *Voting with rubber bands* $\cdots$ page, click on the *Smallest M^c Borda Manipulation* link, where you will find the following profile Π_{SMc} displayed:

$\Pi_{\mathrm{SMc}} = \langle 2, 0, 1, 0, 1, 1\rangle.$

That is:

- One voter* ranks $q > p > r$.
- One voter ranks $q > r > p$.
- One voter ranks $r > p > q$.
- Two voters rank $p > q > r$.

The M^{c}Borda winner for Π_{SMc} is p, while Borda yields a two-way tie between p and q. (With the "Mean" button engaged, you should see the red point sitting on a boundary, with two proximity regions highlighted in green.) When the first voter above (with the asterisk) switches her vote to $q > r > p$ the single winner of the resulting profile $\langle 2, 0, 1, 0, 0, 2\rangle$ is q via both M^{c}Borda and Borda. Thus, Borda exhibits a weak first-choice manipulation whereas M^{c}Borda has a strong first-choice manipulation (and all 6 examples of strong first-choice M^{c}Borda manipulations for 5 voters can be obtained via symmetries from Π_{SMc}).

Speaking loosely, as the red point moves from the $p > q > r$ proximity region to the $q > p > r$ region, in the case of M^{c}Borda it crosses the boundary between these regions by "skipping" from one side to the other. With Borda, however, there is a profile that catches it exactly on the boundary. To make this idea more precise, consider what happens for the profile $3\Pi_{\mathrm{SMc}} = \langle 6, 0, 3, 0, 3, 3\rangle$, obtained by tripling the number of voters that back each ranking according to Π_{SMc}. The M^{c}Borda and Borda outcomes for $3\Pi_{\mathrm{SMc}}$ are the same as for Π_{SMc}.[16]

If you click on the *Crossing Boundaries* link (also on the *Voting with rubber bands* ... page) you will see a modified version $3\Pi^{\dagger}_{\mathrm{SMc}} = \langle 6, 0, 3, 0, 0, 6\rangle$, obtained by taking the three $q > r > p$ voters of $3\Pi_{\mathrm{SMc}}$ and temporarily moving them down to the $q > p > r$ vertex. Now, with the "Mean" button engaged, transfer the six $q > p > r$ voters, one-at-a-time, up to the $q > r > p$ vertex, creating the sequence:

$\langle 6, 0, 3, 0, 0, 6\rangle$

$\langle 6, 0, 3, 0, 1, 5\rangle$

$\vdots$

$\langle 6, 0, 3, 0, 6, 0\rangle$.

You will see the red point pause on the boundary after 3 of the 6 have moved, and then move in to the $q > r > p$ region. If you make the same one-at-a-time transfer with the "Mediancentre" button engaged, the point will skip from one side to the other of this same boundary (when the last of the six voters moves up).

Thus, as the red point gradually moves across the boundary line, it gets counted as two weakly manipulable profiles for Borda, and as one strongly manipulable profile for M^{c}Borda. In the column of Table 1 called "Weighted total" we add the corresponding 4 columns while weighting each weakly manipulable profile as half of a strongly manipulable profile. This method, applied to the sequence of profiles in the previous paragraph, will count a total of 1 manipulation for Borda, and 1 for M^{c}Borda, neither rewarding nor punishing Borda for its pause on the boundary. On the other hand, returning to the original 5-voter profile Π_{SMc}, this method has the effect of rewarding Borda for its indecisiveness. For this reason we also include an "Unweighted Total" column that counts a weakly manipulable profile the same as a strongly manipulable one.[17]

4.3. Decisiveness: 3-way ties

Our early computer enumerations suggested that M^{c}Borda has fewer ties than Borda. Indeed, as shown by Table 3, the difference is both significant and growing as a function of the number n of voters; with 13 voters, the number of anonymous profiles with Borda ties is over 18 times as great as the number with M^{c}Borda ties.

Separating ties into 3-way and 2-way versions explains much of what is going on. In particular, for 3-way ties, a complete analysis is possible.

Definition 4.3. An *elementary reversal* for 3 alternatives is an anonymous profile consisting of one vote cast for a ranking σ together with one vote cast for the reverse ranking σ^{reverse}. An *elementary cycle* for 3 alternatives is an anonymous profile consisting of one vote each cast for the rankings $p > q > r$, $q > r > p$, and $r > p > q$, or one vote each cast for the rankings $q > p > r$, $p > r > q$, and $r > q > p$. A profile P is *elementary central* if it is an elementary reversal or an elementary cycle, and is *central* if it can be written as a sum of positive-integer-multiples of elementary central profiles.

The link *Central Profile* shows the profile

$$\begin{aligned}\langle 1, 2, 4, 0, 3, 3\rangle &= 1\langle 1, 0, 1, 0, 1, 0\rangle + 2\langle 0, 1, 0, 0, 1, 0\rangle \\ &\quad + 3\langle 0, 0, 1, 0, 0, 1\rangle.\end{aligned}$$

Notice that any elementary central profile consists of vertices located symmetrically about the hexagon, so that at the center of the hexagon, the net force exerted by ideal rubber bands, as well as that exerted by weights and strings, is 0. It follows immediately that the same is true for any linear combination of central profiles. As 0 net force at the center is both necessary and sufficient for a 3-way Borda or (for non-2-split profiles) for a 3-way M^{c}Borda tie, we conclude that for an odd number of voters every central profile yields a 3-way tie according to both rules—establishing half the proof of the following result:

Proposition 4.2. *Let Π be a profile for 3 alternatives and an odd number of voters. Then the following are equivalent:*

- *Π is central.*
- *Π yields a 3-way M^{c}Borda tie.*
- *Π yields a 3-way Borda tie.*

In fact, for Borda the result holds for any even number of voters, as well, and for M^{c}Borda it holds for any even number that do not form a 2-split profile.

Proof. It remains only to show that if a profile yields 0 net force at the center, then it must be central. Let us say that a profile P *contains* a (non-zero) profile Q if we can write P as the (vector) sum $P = Q + P'$, in which case it follows that P has more voters than P'. Suppose, by way of contradiction, that there exist non-central profiles yielding 0 net force at the center. Let P be a minimal (with respect to the number of voters) such profile.

If $P = Q + P'$ with Q elementary central, then it is easy to see that P' is also central, and is smaller than P. Hence P contains no central elementary profiles. This places severe restrictions on P—in particular, the votes can be distributed among at most 3 "active" vertices of the hexagon, all of which must be on the same "side" of the center: we can draw a line L with the center strictly to one side, and all active vertices strictly to the other. But then if F is the net force at the center, the component of F orthogonal to L cannot be 0, so $F \neq 0$, our desired contradiction. □

We have used two independent methods to determine the value of the function $T^{\mathrm{Borda}}_{3\text{-way}}(n)$ giving the number of three-way Borda ties as a function of the number n of voters.[18] First we applied elementary methods to count the number of (positive, integer) linear combinations of basic central profiles for n voters, while subtracting terms to compensate for the over-count (because central profiles can typically be represented as such linear combinations in multiple ways).

As a check, we wrote the necessary and sufficient conditions for a 3-way Borda count tie in terms of a system of linear inequalities, and then applied the increasingly popular method of polynomial interpolation via Barvinok's algorithm, to obtain an

16 It is easy to check that both Borda and M^{c}Borda are *homogeneous* voting rules—the outcome for a profile are unchanged when that profile is scaled by a positive integer.

17 As far as we know, every method for measuring a "degree" or amount of manipulation for a voting rule can be criticized as being *ad hoc*, and our weighted or unweighted totals are no exception.

18 Proposition 4.2 tells us that when n is odd $T^{\mathrm{Borda}}_{3\text{-way}}(n)$ also gives the number of three-way M^{c}Borda ties. When n is even, $T^{\mathrm{Borda}}_{3\text{-way}}(n)$ includes in the count those 2-split profiles involving a ranking and its reversal. Thus, if we adopt either of the two resolutions of 2-split profiles discussed in Section 3.3, then $T^{\mathrm{Borda}}_{3\text{-way}}(n) = T^{\mathrm{M^cBorda}}_{3\text{-way}}(n)$ holds for all n.

Table 3
Decisiveness: Borda vs. M^{c}Borda.

Voters	Anon. Profiles	Profiles with Borda ties	Profiles with M^{c}Borda ties	% of profiles with Borda ties	% of profiles with M^{c}Borda ties	Ratio of Borda to M^{c}Borda ties
3	56	14	2	25.00	3.57	7
5	252	54	6	21.43	2.38	9
7	792	132	12	16.67	1.52	11
9	2002	298	22	14.89	1.10	13.55
11	4368	576	36	13.19	0.82	16
13	8568	990	54	11.55	0.63	18.33

Ehrhart quasipolynomial for $T^{\text{Borda}}_{\text{3-way}}(n)$, which agreed with our first approach:

$$T^{\text{Borda}}_{\text{3-way}}(n) = \frac{n^3 + 9n^2 + \langle\langle 42, 15\rangle\rangle n + \langle\langle 72, -25, 88, -9, 57, 7\rangle\rangle}{72}.$$ [19]

In a quasi-polynomial, coefficients are modular. For example, the expression given by $\langle\langle 72, -25, 88, -9, 57, 7\rangle\rangle$ indicates that the constant term is 72 when $n \equiv 0 \pmod 6$, is -25 when $n \equiv 1 \pmod 6$, etc., while the coefficient $\langle\langle 42, 15\rangle\rangle$ is 42 or 15 according to n's value (mod 2).

4.4. Decisiveness: 2-way ties

For 3 alternatives and an odd number of voters, we have seen that Borda and M^{c}Borda agree exactly on 3-way ties. In terms of 2-way ties their behavior could not be more different, however. Borda has a significant number of 2-way ties. To our surprise, our enumerations reveal that M^{c}Borda has *no* such ties for odd $n \leq 25$, and we conjecture that this pattern continues. What explains the difference?

Fig. 10 (available in color through the on-line link) shows the mean position, on the hexagon, for all profiles of 9 voters. Different profiles can yield the same mean position, so each marked point has a multiplicity, which is keyed to the color of the point in the on-line version of the figure, and to its size in the version here. Points closer to the center have higher multiplicities.

Notice that the geometry of the point grid, together with that of the boundary lines for proximity regions, conspire to place a number of these points on single boundary lines, reflecting two-way Borda ties. In fact, the Ehrhart polynomial $T^{\text{Borda}}_{\text{2-way}}(n)$ counting 2-way Borda ties begins:

$$T^{\text{Borda}}_{\text{2-way}}(n) = \frac{15}{864}n^4 + \cdots.$$

Thus, the number of 2-way ties grows roughly by the fourth power of the number n of voters, in contrast with the third power of n for 3-way ties (previous section).

Perhaps this should not be surprising, given the dimension of various regions within the representing simplex. The 5-simplex Δ^5 consists of all points in $\mathbf{R}^6$ that have non-negative coordinates and lie on the hyperplane

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1.$$

If we normalize a profile P by dividing by the number of voters we obtain a point

$$\Pi^\star = \left\langle \frac{n_1}{n}, \frac{n_2}{n}, \frac{n_3}{n}, \frac{n_4}{n}, \frac{n_5}{n}, \frac{n_6}{n} \right\rangle$$

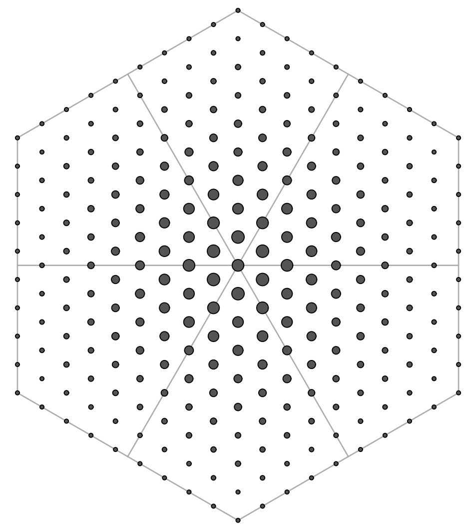

Fig. 10. Mean locations, all profiles for 9 voters. Larger nodes indicate a greater number of profiles sharing that location, with nodes on the perimeter corresponding to exactly one profile.

in this simplex, with points of the form $\Pi^\star$ for some fixed value of n forming a regular grid in Δ^5. As n increases, the spacing between these grid points decreases, and so the number of anonymous profiles increases. Moreover, decreasing this spacing is equivalent to dilating the simplex while leaving the grid-spacing fixed. As the simplex dilates we expect its volume to increase according to the fifth power of the dilation factor. With a regular enough grid and appropriate bounding hyperplanes, we expect the number of grid points in the dilated simplex to reflect that simplex's 5-volume, and thus we expect the number of anonymous profiles for 3 alternatives and n voters to grow as the fifth power of n, as indeed it does.[20]

There is a hyperplane (affine subspace of co-dimension 1 relative to the simplex), $H_{x,y}$, which separates the normalized profiles for which alternative x amasses a higher Borda score than y from those where y does better than x, with two-way ties between x and y holding for normalized profiles exactly on $H_{x,y}$, and three-way ties holding along the intersection of two such hyperplanes. Reasoning as above, the portion of $H_{x,y}$ lying inside Δ^5 constitutes a polytope of dimension 4, so we expect the number of grid-points in this intersection – that is, the number of 2-way Borda ties – to grow as the fourth power of n.[21] By the same token, the region of

[19] This approach is discussed, for example, in Lepelley et al. (2008), and in Wilson and Pritchard (2007). The same basic idea was applied earlier to social choice theory by Gehrlein (2002) and by Huang and Chua (2000).

[20] To make the argument (as well as the definition of "appropriate") precise, we would apply Ehrhart's theory, which requires that the simplex be a convex hull of finitely many rational points—which it is, of course.

[21] If one replaces the Borda count with any scoring rule employing irrational scoring weights, this line of reasoning falls apart; $H_{x,y}$'s equation has irrational coefficients, and so the polytope we are describing is no longer the convex hull of finitely many rational points. Nonetheless, it is easy to show that the example $\langle 2, 0, 1, 0, 1, 1\rangle$ shown on the link *A Stubborn Two-Way Tie* yields the same 2-way tie for *all* scoring rules (but not for M^{c}Borda).

3-way Borda ties has dimension 3, and so we expect the number of 3-way Borda ties to grow as the cube of n.

In fact, Fig. 10 represents a projection of the situation described above, with Δ^5 projecting to the (filled-in) hexagon and the hyperplanes of form $H_{x,y}$ projecting to the three lines that divide the hexagon into its proximity regions; all the nodes falling on these lines arise from normalized profiles lying on the original hyperplanes, but the projection sends many such profiles to a single node (and sends the entire 3-dimensional intersection of the three hyperplanes to the central point of the hexagon).

Nonetheless, not every social choice function for 3 alternatives has 3-way ties growing as the 3rd power of the number n of voters, or has 2-way ties growing as the 4th power, because the assumptions in the preceding argument do not always apply. For example, with 3 alternatives the *Copeland* voting rule (see, for example, Taylor, 2005) declares a 3-way tie for every profile that yields a strict majority cycle (also called a *Condorcet* cycle). It is known that in the limit as $n \to \infty$ the fraction of anonymous profiles that yield such a cycle approaches $\frac{1}{16}$; a detailed explanation can be found in Gehrlein (2006). It follows that for Copeland, the leading term in the Ehrhart polynomial is of degree 5—that is, the growth rate of 3-way Copeland ties is greater than that of 3-way Borda (or M^{c}Borda) ties, by two full orders of magnitude.[22]

On the other hand, 2-way M^{c}Borda ties grow very slowly indeed—for odd n there seem to be none at all. It must be that the mediancentres of all profiles for a fixed number n of voters do not form the sort of regular grid that we see in Fig. 10, or one would expect some of them to land on boundaries. We had two guesses: perhaps mediancentres are staying far away from the boundary lines, or perhaps they were coming quite close while just missing.

Fig. 11 shows all mediancentres for profiles of $n = 11$ voters (shown in color on-line, along with versions for other values of n). As with the earlier, mean version, the story is not complete without knowing the multiplicities, which surprised us, as did the overall appearance. The center of the hexagon has multiplicity $T^{\text{Borda}}_{\text{3-way}}(11) = 36$, and the multiplicity of each vertex is 359, thanks partly to the strong and weak majoritarian properties. But each other node in Fig. 11 has multiplicity 1—it represents the mediancentre of a unique profile for 11 voters.

The symmetries about 6 lines through the hexagon's center can easily be explained. Our original motivation for creating these figures was to resolve which (if either) of our original guesses were correct. Our interpretation is that *both* guesses were partly correct. On the one hand, there is some concentration of mediancentres on or near the rays from the center to the vertices, which tends to keep these points away from the proximity boundaries. On the other hand, there are a number of nodes that appear to touch these boundaries only because they are so close that the resolution of the figure fails to show the separation; blown-up versions of the figure show these points miss the boundaries, and their actual coordinates confirm the miss. It is tempting to speculate that there be some connection between the absence of 2-way ties and the wealth of mediancentres with multiplicity 1, but at the moment speculation is all we have.

When n is even, 2-way M^{c}Borda ties do exist, of course. Take any line L that bisects a pair of opposite hexagon sides. Then any profile of votes located symmetrically about L will yield a 2-way *bilateral* tie (unless it constitutes a 2-split or a 3-way tie). Of course all bilateral profiles yield 2-way or 3-way Borda ties, as well.

More interesting is that there is a type of 2-way M^{c}Borda tie that is not also a Borda tie. An example of such a *cross tie* is:

$$Q = \langle 5, 2, 5, 2, 0, 0\rangle$$

[22] There are various ways to define exactly what constitutes a 3-way Copeland tie, but each can be captured via a system of linear inequalities (with rational coefficients), and thus can be counted via an Ehrhart quasi-polynomial.

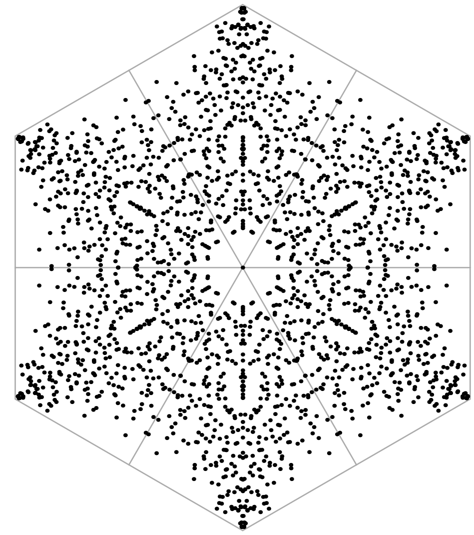

Fig. 11. Mediancentre locations, all profiles for 11 voters. The central node is the mediancentre of 36 profiles, each hexagon vertex is the mediancentre for 359 profiles, and each other node corresponds to a unique profile.

(shown on the *Cross Tie* link), which yields a two-way tie between p and r. A moment's thought about the unit vectors will explain why the equilibrium lies on the boundary line, as it would for any profile of form $P_{k,j} = \langle k, j, k, j, 0, 0\rangle$ (with $k \neq j$, to distinguish cross ties from bilateral ones) or any rotation of $P_{k,j}$ about the hexagon. However, if we add a cross tie to a bilateral tie the result is a bit of a surprise. For example, if we add the above pr cross tie Q to the bilateral tie $\langle 0, 3, 3, 0, 1, 1\rangle$ (also a pr tie) then r is the sole winning alternative. This peculiar phenomenon represents a failure of the *consistency* axiom.

Conjecture 4.1. *Every* 2-*way* M^{c}Borda *tie for* 3 *alternatives is either bilateral or cross.*

Conjecture 4.2 (*This is a Corollary of* Conjecture 4.1)**.** *There are no* 2-*way* M^{c}Borda *ties for* 3 *alternatives and an odd number of voters.*

We have some evidence for these conjectures. The version of the M^{c}Borda algorithm used to check these conjectures used an error tolerance ϵ, declaring a tie whenever the calculated approximate mediancentre is within ϵ of a boundary line on the hexagon. With our value of ϵ, for each $n < 15$, our algorithm correctly declares every bilateral, cross, and central profile to be a M^{c}Borda tie, and declares no other ties.

4.5. Condorcet efficiency

A (strong) *Condorcet alternative* x for a profile P is one preferred to y by (strictly more than) half of the voters, for each alternative $y \neq x$. We will consider only profiles for an odd number n of voters; all Condorcet alternatives are then strong. Of course, not every profile has a Condorcet alternative.

A voting rule is a *Condorcet extension* if it declares the sole election winner to be the Condorcet alternative, for each profile that *has* a Condorcet alternative. Some authors find the Condorcet alternative to be so compelling that they argue in favor of using Condorcet extensions for real world elections. However, Condorcet extensions have been shown to suffer from some pathologies. For example, Moulin (1986) shows that with 4 or more alternatives and sufficiently many voters, every *resolute* Condorcet extension suffers from the *No-Show* paradox of Brams and Fishburn (1983): by not voting, a voter can sometimes obtain an outcome that he strictly prefers to that obtained when he votes sincerely. Here, a social choice correspondence is *resolute* if it yields a unique winning alternative for each profile. A resolute social choice correspondence is also called a social choice *function*.

More recently, Sanver and Zwicker (2009) show, under the same assumptions as Moulin, that each such Condorcet extension fails to satisfy *half-way monotonicity*—it can be manipulated by a voter who tips her ranking upside down. By misrepresenting her relative preference over *every* pair $\{x, y\}$ of alternatives, that voter is manipulating via a "total lie".

For voting rules that are not Condorcet extensions, we can measure the extent to which they fall short of the ideal in the form of their *Condorcet efficiency*—the probability, conditional on the assumption that a profile Π has a Condorcet alternative, that the Condorcet alternative is the sole election winner. As always, such a measure depends in part on the choice of probability distribution, and we will once again be using the IAC distribution. Thus the IAC Condorcet efficiency of a rule is the same as the fraction (or percentage) formed by dividing the number of anonymous profiles for which the rule chooses the Condorcet alternative by the number for which a Condorcet alternative exists.

The majoritarian properties of M^cBorda made us suspect that it might have superior Condorcet efficiency to that of Borda—the sample profile Π_1 from the introduction is a suggestive example in this respect. Once again, one must decide how to count ties: if there are $k \geq 2$ Borda winners for a given profile, and one of these is the Condorcet winner, does that profile count wholly against Borda's efficiency, or partly, or not at all? Our method here was to use alphabetical order to break all ties with a tie-breaking agenda, and work with the resulting resolute versions of Borda and M^cBorda.[23]

Table 4 displays some of our results, in the form of *Condorcet inefficiency*, which is the deficit

$$100 - \text{Condorcet efficiency}$$

in percentage terms.

Our interpretation of Table 4 is that M^cBorda has a significant advantage over Borda in terms of Condorcet efficiency.[24]

4.6. Participation, one-way monotonicity, and half-way monotonicity

Every scoring rule occasionally fails to elect the Condorcet alternative. Resolute Condorcet extensions, on the other hand, exhibit the no-show paradox, and can be manipulated by a voter who reverses his ranking—pathologies that never vex scoring rules. One can argue, then, that the history of social choice theory continues to play out the fundamental nature of the dichotomy between the approaches of Borda and Condorcet.

We expect the M^cBorda rule to behave somewhat similarly to Borda, thanks to the common geometric basis for these two rules, and examples of profiles such as Π_{SMc} seem to bear this out. On the other hand, the majoritarian tendencies of M^cBorda suggest that mediancentre rules also share some properties with Condorcet extensions. It is natural, then, to ask whether these majoritarian tendencies incur some of the same costs for M^cBorda as they do for resolute Condorcet extensions, in the form of common pathologies. Before addressing the issue, though, some precise definitions are in order.

Definition 4.4. Let S be a set of voters, P be any profile for the voters in S, v be a voter not in S, σ and τ be individual ballots, and $P \wedge \sigma$ be the profile obtained by adding, to P, one additional ballot of σ, cast by v. Let F be a social choice function. Then *participation* requires of F that

$$F(P \wedge \sigma) \geq_\sigma F(P)$$

hold in all cases. *Half-way monotonicity* (HWM) requires that

$$F(P \wedge \sigma) \geq_\sigma F(P \wedge \sigma^{\text{reverse}})$$

hold in all cases. *One-way monotonicity* (OWM) requires that

$$F(P \wedge \sigma) \geq_\sigma F(P \wedge \tau) \quad \text{or} \quad F(P \wedge \tau) \geq_\tau F(P \wedge \sigma)$$

hold in all cases. *Reversal cancellation* requires that

$$F(P \wedge \sigma \wedge \sigma^{\text{reverse}}) = F(P)$$

hold in all cases.

Notice that *participation* (Moulin, 1986) asserts the absence of no-show paradoxes (equivalently, it asserts that the rule cannot be manipulated via abstention), while *half-way monotonicity* asserts that the rule cannot be manipulated by a voter who reverses her ranking. In Sanver and Zwicker (2009) introduced half-way monotonicity, showing that it is implied by participation, and that the converse holds for rules that satisfy both homogeneity and reversal cancellation. In this connection it is worth pointing out that M^cBorda violates reversal cancellation (a property that is satisfied by many Condorcet extensions and by Borda, though not by any other scoring rules for 3 alternatives).

One-way monotonicity, also introduced in Sanver and Zwicker (2009), is a strong form of half-way monotonicity and so it is similarly violated by resolute Condorcet extensions for 4 or more alternatives. *OWM*, as well, has an interesting interpretation in terms of strategy-proofness (discussed in Sanver and Zwicker, 2009, and briefly mentioned here in Section 5) and is satisfied by all scoring rules.

We used a computer enumeration to look for all M^cBorda *OWM* violations. Our initial approach was to ignore all profiles with ties; that way we could apply the above definitions directly, to look for "resolute" failures of *OWM*. We found no such resolute failures for 12 or fewer voters. (To be clear, if $F(P \wedge \sigma)$ or $F(P \wedge \tau)$ yields a tie, we never classify it as a failure of resolute *OWM*.)

We have earlier argued, however, that this sort of approach can reward a rule for having ties, which seems perverse. The matter of how best to adapt Definition 4.4 to the irresolute realm is a subtle one however. Our approach here is the one most common in the literature—we use a fixed agenda (ordering of the alternatives, such as alphabetical order) to break all ties in outcomes (by choosing the alphabetically earliest of the winners), thus rendering the rule resolute, and then we apply Definition 4.4 to this resolute version of the rule.[25]

23 In this case the agenda method has a very tame effect on the results. Three-way ties play no role at all; we have seen that each such tie arises from a central profile, and it is clear that central profiles do not have strong Condorcet alternatives. Neutrality guarantees that any agenda will count exactly half of the 2-way ties (that include the Condorcet alternative) towards the Condorcet inefficiency total. The effect, then, is to weight each such 2-way tie half as much as a profile having a unique winner unequal to the Condorcet alternative for that profile. (In fact, we have seen that there are no 2-way M^cBorda ties arising for an odd number of voters less than 25.)

24 We have mentioned that other mean rules (Kemeny, for example) can be paired with mediancentre versions. There are some reasons to suspect that these alternative pairings would yield comparisons somewhat similar to that between Borda and M^cBorda, but we do not know that the Borda–M^cBorda lessons all carry over. In particular, under *IAC* Borda does not maximize Condorcet efficiency among all scoring rules in the 3-alternative case. (It *is* the maximizer under the *IC* distribution, for the limiting case as $n \to \infty$, as shown in Gehrlein and Fishburn, 1978.) The maximally efficient scoring rule under *IAC* varies according to n (see Gehrlein, 2003 and Lepelley et al., 2000), and has been found for the limiting case as $n \to \infty$ in Cervone et al. (2005), where it is shown to be different from Borda. So it might be interesting to compare the Condorcet efficiencies of some of the maximally efficient scoring rules with those of their mediancentre analogues.

25 We were originally quite skeptical of this methodology. However, in Sanver and Zwicker (in press), we show that for neutral rules the tie-breaking agenda method gains credibility through its equivalence (when applied to participation, *HWM*, and *OWM*) to the *two-at-a-time* approach as well its equivalence (when applied to participation and *HWM*) to the set extension method via the extension that Gärdenfors calls $\mathcal{R}_F$ – for Fishburn – in Gärdenfors (1979).

Table 4
Condorcet inefficiency: Borda vs. M^{c}Borda.

Voters	Anon. Profiles	Anon. profiles with Condorcet winners	Anon. profiles for which M^{c}Borda selects Condorcet winner	Anon. profiles for which Borda selects Condorcet winner	M^{c}Borda Condorcet inefficiency (%)	Borda Condorcet inefficiency (%)	Ratio McBorda-to-Borda inefficiency
3	56	54	54	48	0.00	11.11	0.00
5	252	240	234	210	2.50	12.50	0.20
7	792	750	726	660	3.20	12.00	0.27
9	2002	1890	1824	1668	3.49	11.75	0.30

Using this methodology, we have found M^{c}Borda violations of "irresolute *OWM*". The smallest such has 7 voters, and uses the profile change

$$\Pi_3 = \langle 1, 1, 3, 0, 0, 2\rangle \longmapsto \Pi_3' = \langle 1, 0, 3, 0, 1, 2\rangle.$$

The link *Irresolute OWM Failure* shows Π_3. Notice that the Π_3M^{c}Borda outcome is r as sole winner, but when the single $p > r > q$ voter switches her vote to $q > r > p$, the result is a 3-way pqr tie, which is broken in favor of p via alphabetical order, our tie-breaking agenda. As the switch is a complete reversal, this example shows a failure of irresolute *HWM*, as well. Moreover, the profile change

$$\Pi_3 = \langle 1, 1, 3, 0, 0, 2\rangle \longmapsto \Pi_3'' = \langle 1, 1, 3, 0, 1, 2\rangle$$

shows a M^{c}Borda violation of participation, in its resolute form.[26]

We have not yet performed a computer search for all M^{c}Borda violations of participation—the above example was found by "playing" with the irresolute *OWM* violation. Consequently, we cannot say how common such violations might be.

Violations of (irresolute) *OWM* seem to be quite rare, however. With 7 voters, all 12 violations of irresolute *OWM* arise from the example above via symmetries of the hexagon; thus, only one "basic" violation exists for 7 voters.[27] The same is true for 8, 9, 10, or 11 voters. With 12 voters there are two basic violations, and with 13 there are 3.

5. Decisiveness, responsiveness, and resistance to manipulation

Imagine that a single *focal voter* v is about to cast one of two ballots: σ or τ. Meanwhile, the other voters have already made their choices, as recorded by the profile Π. Some social choice correspondence F is at hand, and so our voter v is contemplating two possible outcomes:

$$F(\Pi \wedge \sigma) = \{x\} \quad \text{or} \quad F(\Pi \wedge \tau) = \{y\}.$$

Here F has declared a unique winning alternative for each profiles, so if $x \neq y$, then it is possible that the change in outcome due to a switch by v will constitute a manipulation. Now suppose we are comparing F to some less decisive voting rule G for which

$$G(\Pi \wedge \sigma) = \{x, y\} = G(\Pi \wedge \tau).$$

The same situation, then, does not count as a manipulation for G. If this sort of thing happens often, we will conclude, correctly, that F is more frequently manipulable than G. Yet if G's relatively high score on resistance to manipulation is earned largely by declaring many ties, surely our conclusion should mention that fact—otherwise we would not be comparing these rules on a level playing field.

Of course, it is possible that G's double tie at $\Pi \wedge \sigma$ and at $\Pi \wedge \tau$ merely shifts the profile at which a transition in election outcome is manifested. Nonetheless, it seems that if a voting rule declares *enough* ties, it can completely dodge some transitions that count as manipulations for other rules. The *omninominator* social choice correspondence $\mathcal{OMN}$ declares an alternative x to be a social choice if at least one voter top-ranks x. Of course $\mathcal{OMN}$ declares an enormous number of ties, and for that reason would be disqualified for many voting applications. By the same token, it can serve as an interesting test case. For example, if we apply $\mathcal{OMN}$ to the seven-profile sequence of Section 4.2, we see that it declares a three-way pqr tie for each of the profiles in the sequence.

How does $\mathcal{OMN}$ fare when we measure manipulability using the methodology of Table 1? Under the first-or-second choice, strong-or-weak approach, $\mathcal{OMN}$ has no manipulations whatsoever.

Suppose, instead, we use a tie-breaking agenda (such as alphabetical order, $p > q > r$) to break ties in all outcomes. On the one hand, the resulting resolute variant $\mathcal{OMN}^\star$ of $\mathcal{OMN}$ *is* manipulable in the following circumstance (and in no other): a voter has sincere preference $r > p > q$, and none of the other ballots top-ranks p. Few profiles fit those conditions, but we can no longer argue that $\mathcal{OMN}^\star$ achieves its high resistance to manipulation via indecisiveness. On the other hand, $\mathcal{OMN}^\star$ rarely responds to a change in profile with a change in outcome; p is the winner for that vast majority of profiles for which at least one voter top-ranks p. This example suggests that there is an additional characteristic – let us call it *responsiveness* – that should also be considered whenever we compare rules in terms of resistance to manipulation. It seems that a voting rule can achieve a high resistance to manipulation at the expense of low decisiveness or low responsiveness.

Is there some particular way in which measures of decisiveness and responsiveness should be weighted against measures of resistance to manipulation, in order to get a more fair or meaningful overview, when comparing voting rules? And what, exactly, is "responsiveness"? At this stage our understanding is inadequate to tackle the first question. In the remainder of this section, we offer some tentative thoughts in connection with the second.

To simplify the analysis, however, we will move decisiveness temporarily off the table by limiting ourselves to consideration of a resolute voting rule F. With that in mind, we return to the scene set at the start of this section: our focal voter v is considering the possible ballots: σ and τ, leading to two possible outcomes:

$$F(\Pi \wedge \sigma) = \{x\} \quad \text{and} \quad F(\Pi \wedge \tau) = \{y\}.$$

We consider three possible *interpretations* of the choice faced by v:

- *Interpretation* I: v has not yet decided which of σ, τ represents her true preference ranking.

26 That there seems to be no corresponding M^{c}Borda violation of resolute *HWM* – at least, not for $n \leq 12$ – suggests that the assumption of reversal cancellation is necessary, in the proof that *HWM*$\Longrightarrow$ participation (from Sanver and Zwicker, 2009).

27 There are 6 permutations of the alternatives, each of which induces a permutation of rankings that is bound to generate a "new" violation from a given violation, for any neutral rule. Each of these can be composed with the antipodal map (sending each ranking to its reversal, and corresponding to rotation of the hexagon by π radians), which is *not* induced by a permutation of alternatives.

- *Interpretation* II: v's sincere preference ranking is σ, but she is considering casting the insincere vote τ.
- *Interpretation* III: v's sincere preference ranking is τ, but she is considering casting the insincere vote σ.

We consider the implications of these interpretations, in each of the following five scenarios:

- Scenario A: $x >_\sigma y$ and $y >_\tau x$.
- Scenario B: $y >_\sigma x$ and $x >_\tau y$.
- Scenario C: $x >_\sigma y$ and $x >_\tau y$.
- Scenario D: $y >_\sigma x$ and $y >_\tau x$.
- Scenario E: $x = y$.

Under Interpretation I, Scenario A looks quite appealing. If v expresses preference for x over y by casting the ballot σ, she obtains her preferred alternative (of the two) x, but if she changes her mind and expresses a preference for y over x by casting τ instead, the rule F reacts appropriately by handing her y. Scenario A looks good under Interpretations II and III, as well. In neither case would the insincere ranking under consideration succeed as a manipulation—in fact, it would yield a strictly *worse* outcome from v's point of view (regardless of which of σ or τ represents that point of view). Suppose v's knowledge of other voters' ballots is imperfect. Then if Scenario A were known to occur for many values of Π, this frequency might serve as a general disincentive to attempts at manipulation.

By the same token, Scenario B stands for ''Bad'' under any of the three interpretations. Under Interpretation I, our voter would feel that F was responding perversely to her shift in sincere preference. Under Interpretations II and III this scenario presents the opportunity to manipulate successfully.

Under typical circumstances, and with a reasonably large number of voters, we expect Scenario E to be the most common of the five. Still, there is a problem if it is too common, because voters would have a decreased incentive to vote. Thus, while E looks just as good as A does if we are counting instances of manipulation, Interpretation I suggests that E is less desirable than A.

Scenario C is puzzling. Under Interpretation I, someone who changed their mind from preferring τ to preferring σ might see F as responding appropriately, but the opposite change of heart would be discouraging. Under Interpretation III, C represents an opportunity to manipulate successfully, but under II this scenario presents a disincentive against manipulating. Of course D is C's mirror image.

For a given resolute rule F, each triple (Π, σ, τ) fits exactly one of the five scenarios; if a different rule G causes one of these scenarios to occur more frequently than it does with F, then of course some of the other scenarios must occur correspondingly less often. The literature that compares rules based on degree of manipulability is typically measuring something closely related to the combined frequencies of Scenarios B, C, and D.

For example, Favardin et al. (2002) compare Copeland and Borda in terms of degree of manipulability, using the tie-breaking agenda method to create resolute versions Copeland* and Borda*. They find Copeland* to have fewer manipulations than Borda*. But we have seen that Copeland has many more ties than Borda; as a result we suspect that Scenario E occurs more often for Copeland* than for Borda*.

Now, if Scenarios B, C, and D occur less often for Copeland* only because Scenario E occurs correspondingly more often, perhaps that should not be interpreted as a relative advantage for Copeland over Borda. In fact, that is not the only trade-off between these two rules. We know that ''bad'' Scenario B occurs for Copeland (and for Copeland*) but not for Borda (or Borda*). A failure of half-way monotonicity implies (and a failure of one-way monotonicity is equivalent to) an occurrence of Scenario B; from Sanver and Zwicker (2009) we know resolute Condorcet extensions (for 4 or more alternatives) violate half-way monotonicity, while scoring rules satisfy the stronger property of one-way monotonicity.

As far as we know, no one has systematically measured rules in terms of the frequency of all five scenarios (and of course there are additional scenarios for irresolute rules). We have almost no understanding of any fundamental trade-offs among or limitations on the frequency of the various scenarios. That understanding would seem to be a prerequisite for concocting any measure of manipulability that takes decisiveness and responsiveness into account.

6. Concluding remarks

In our investigation of manipulability of the M^{c}Borda rule we accidentally noticed its unusually high level of decisiveness. In turn, that suggested linkage between decisiveness and manipulability; considerations of responsiveness were then a natural outgrowth. It was these developments that initially suggested the one-way monotonicity and half-way monotonicity properties studied in Sanver and Zwicker (2009).

In this way, our study of M^{c}Borda has led to new and general insights. Consider, as well, the dichotomy between rules satisfying the all-but-one cancellation property and those satisfying its antithesis, the all-but-one tipping property; this distinction would not have suggested itself without M^{c}Borda's role as an example.

Compared to some of the better-known social choice rules, M^{c}Borda seems quite an odd beast. (But we have seen examples of different mechanisms that lead to the same rule, so some caution is appropriate—we have not compared M^{c}Borda to every rule in the literature, and the possibility that it duplicates some lesser-known rule has not been definitively ruled out.) As a result, it has prompted some very different ideas, and it may be that this role outweighs any actual value M^{c}Borda may have as a practical rule. More broadly, it suggests that Social Choice Theory might benefit from more such strange examples, because of their potential to expand the universe of interesting voting properties beyond those we now know.

Acknowledgments

We owe a debt to Karl-Dieter Crisman, James Henle, and Christopher Small for their advice and insight. Extensive comments of two anonymous referees significantly improved the final manuscript. R. Dai, D. Gnoutcheff, and A. Mackenzie acknowledge support from the Union College Summer Research Fund; A. Morse acknowledges support from the Skidmore College Summer Research Fund. These five, along with N. Srivastava, were undergraduates when they worked on this project.

References

Aleskerov, F., Karabekyan, D., Sanver, M.R., Yakuba, V., 2011. On the degree of manipulability of multi-valued social choice rules. In: Essays in Honor of Hannu Nurmi. In: Homo Oeconomicus, 28. pp. 1–12.

Aleskerov, F., Karabekyan, D., Sanver, M.R., Yakuba, V., 2011. On manipulability of voting rules in the case of multiple choice. Mathematical Social Sciences (this issue).

Aleskerov, F., Kurbanov, E., 1999. A degree of manipulability of known social choice procedures. In: Aliprantis, S., Alkan, A., Yannelis, N. (Eds.), Current Trends in Economics: Theory and Applications. Springer Verlag, Berlin, pp. 13–28.

Barberà, S., Bossert, W., Pattanaik, P.K., 2004. Ranking sets of objects. In: Barberà, S., Hammond, P.J., Seidl, C. (Eds.), Handbook of Utility Theory, Volume II: Extensions. Kluwer, Dordrecht, pp. 893–977.

Barthélémy, J.P., Monjardet, B., 1981. The median procedure in cluster analysis and social choice theory. Mathematical Social Sciences 1, 235–267.

Bartholdi, J., Tovey, C.A., Trick, M.A., 1989. Voting schemes for which it can be difficult to tell who won the election. Social Choice and Welfare 6, 157–165.

Bassett Jr., G.W., Persky, R., 1999. Robust voting. Public Choice 99, 299–310.

Berg, S., 1985. Paradox of voting under an urn model: the effect of homogeneity. Public Choice 47, 377–387.
Brams, S.J., Fishburn, P., 1983. Paradoxes of preferential voting. Mathematics Magazine 56, 207–214.
Brimberg, J., 1995. The Fermat Weber location problem revisited. Mathematical Programming 71, 71–76.
Cervone, D.P., Gehrlein, W.V., Zwicker, W.S., 2005. Which scoring rule maximizes Condorcet efficiency? Theory and Decision 58, 145–185.
Cervone, D.P., Zwicker, W.S., 2010. The mean and median as equilibria of physical systems: a web-based simulation. Working Paper.
Eckhardt, U., 1980. Weber's problem and Weiszfeld's algorithm in general spaces. Mathematical Programming 18, 186–196.
Favardin, P., Lepelley, D., Serais, J., 2002. Borda rule, Copeland method and strategic manipulation. Review of Economic Design 7, 213–228.
Gärdenfors, P., 1979. On definitions of manipulation of social choice functions. In: Laffont, J.-J. (Ed.), Aggregation and Revelation of Preferences. North-Holland, Amsterdam, pp. 29–36.
Gehrlein, W.V., 2002. Obtaining representations for probabilities of voting outcomes with effectively unlimited precision integer arithmetic. Social Choice and Welfare 19, 503–512.
Gehrlein, W.V., 2003. Weighted scoring rules that maximize Condorcet efficiency. In: Sertel, M.R., Koray, S. (Eds.), Advances in Economic Design. Springer-Verlag, pp. 53–64.
Gehrlein, W.V., 2006. Condorcet's Paradox. Springer.
Gehrlein, W.V., Fishburn, P., 1978. Coincidence probabilities for simple majority and positional voting rules. Social Science Research 7, 272–283.
Gehrlein, W.V., Lepelley, D., 2003. On some limitations of the median voting rule. Public Choice 117, 177–190.
Gower, J.C., 1974. Algorithm AS 78: the mediancentre. Journal of the Royal Statistical Society, Series C (Applied Statistics) 23, 466–470.
Huang, H.C., Chua, V.C.H., 2000. Analytical representation of probabilities under the IAC condition. Social Choice and Welfare 17, 143–155.
Kelly, J., 1993. Almost all social choice rules are highly manipulable, but a few aren't. Social Choice and Welfare 10, 161–175.
Kuhn, H.W., 1973. A note on Fermat's problem. Mathematical Programming 4, 98–107.
Lepelley, D., Louichi, A., Smaoui, H., 2008. On Ehrhart polynomials and probability calculations in voting theory. Social Choice and Welfare 30, 363–383.
Lepelley, D., Pierron, P., Valognes, F., 2000. Scoring rules, Condorcet efficiency, and social homogeneity. Theory and Decision 49, 175–196.
Moulin, H., 1980. On strategy-proofness and single peakedness. Public Choice 35, 437–455.
Moulin, H., 1986. Choosing from a tournament. Social Choice and Welfare 3, 271–291.
Saari, D., 1994. Geometry of Voting. Springer Verlag.
Saari, D.G., Merlin, V.R., 2000. A geometric examination of Kemeny's rule. Social Choice and Welfare 17, 403–438.
Sall, J., Creighton, L., Lehman, A., 2007. JMP Start Statistics: A Guide to Statistics and Data Analysis Using JMP, fourth ed. SAS Institute, Cary, North Carolina.
Sanver, M.R., Zwicker, W.S., 2009. One-way monotonicity as a form of strategy-proofness. International Journal of Game Theory 38, 553–574.
Sanver, M.R., Zwicker, W.S., 2011. Monotonicity properties and their adaptation to irresolute social choice rules. Social Choice and Welfare (in press).
Small, C.G., 1990. A survey of spatial medians. International Statistical Review 58, 263–277.
Smith, D.A., 1999. Manipulability measures of common social choice functions. Social Choice and Welfare 16, 639–661.
Stensholt, E., 1999. Beta distribution in a simplex and impartial anonymous cultures. Mathematical Social Sciences 37, 45–57.
Taylor, A.D., 2005. Social Choice and the Mathematics of Manipulation. Cambridge University Press.
Varignon, P., 1687. Projet d'une nouvelle méchanique. Chez la Veuve d'Edme Martin, J. Boudot & E. Martin, Paris.
Weber, A., 1909. Über den Standort der Industrien, Erster Teil: Reine Theorie des Standortes. Mohr, Töbingen.
Weiszfeld, E., 1937. Sur le point pour lequel la somme des distances de n points donnés est minimum. Tôhoku Mathematics Journal 43, 355–386.
Wilson, M.C., Pritchard, G., 2007. Probability calculations under the IAC hypothesis. Mathematical Social Sciences 54, 244–256.
Zwicker, W.S., 1991. The voters' paradox, spin, and the Borda count. Mathematical Social Sciences 22, 187–227.
Zwicker, W.S., 2008a. Consistency without neutrality in voting rules: when is a vote an average? In: Belenky, A. (Ed.), Mathematical Modeling of Voting Systems and Elections: Theory and Applications. In: Mathematical and Computer Modelling, vol. 48. pp. 1357–1373. (special issue on).
Zwicker, W.S., 2008b. A characterization of the rational mean neat voting rules. In: Belenky, A. (Ed.), Mathematical Modeling of Voting Systems and Elections: Theory and Applications. In: Mathematical and Computer Modelling, vol. 48. pp. 1374–1384. (special issue on).