On the Role of Distances in Defining Voting Rules

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ABSTRACT

A voting rule is an algorithm for determining the winner in an election, and there are several approaches that have been used to justify the proposed rules. One justification is to show that a rule satisfies a set of desirable axioms that uniquely identify it. Another is to show that the calculation that it performs is actually maximum likelihood estimation relative to a certain model of noise that affects voters (MLE approach). The third approach, which has been recently actively investigated, is the so-called *distance rationalizability* framework. In it, a voting rule is defined via a class of consensus elections (i.e., a class of elections that have a clear winner) and a distance function. A candidate c is a winner of an election E if c wins in one of the consensus elections that are closest to E relative to the given distance. In this paper, we show that essentially any voting rule is distance-rationalizable if we do not restrict the two ingredients of the rule: the consensus class and the distance. Thus distance rationalizability of a rule does not by itself guarantee that the voting rule has any desirable properties. However, we demonstrate that the distance used to rationalize a given rule may provide useful information about this rule's behavior. Specifically, we identify a large class of distances, which we call *votewise* distances, and show that if a rule is rationalized via a distance from this class, many important properties of this rule can be easily expressed in terms of the underlying distance. This enables us to provide a new characterization of scoring rules and to establish a connection with the MLE framework. We also give bounds on the complexity of the winner determination problem for distance-rationalizable rules.

Categories and Subject Descriptors

I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems; I.2.4 [**Knowledge representation formalisms and methods**]

General Terms

Theory

Keywords

elections, voting, distance rationalizability, complexity

1. INTRODUCTION

Voting is an important tool that is used whenever a group of people—or, in general, a group of agents—needs to make a joint decision that in some way accommodates preferences and goals of all the participants. However, according to a famous result of Arrow there is no truly prefect voting system, and, throughout history, people have come up with a great number of very diverse election systems, each with its own advantages and flaws. The goal of this paper is to study a recently proposed framework for defining and analyzing voting rules, called distance rationalizability [21].

Curiously, the main idea behind this framework goes back to the early days of mathematical study of elections, namely, to the voting system of Dodgson. A winner of Dodgson election is a candidate that can be made a Condorcet winner (i.e., someone who is preferred to any other candidate by a majority of voters) via the least number of adjacent swaps in the voters' preference lists. In other words, Dodgson rule is defined via a notion of *consensus* (an election with a clear winner; here any election with a Condorcet winner is a consensus) and a distance function (here the distance function between voter profiles measures the number of adjacent swaps that transform one profile into the other). A candidate is a Dodgson winner of election E if she is a winner in a consensus elections that is closest to E.

Recently, Meskanen and Nurmi [21] (see also [1]) generalized Dodgson's approach by considering other consensus classes and distances, and showed that many other voting rules are distance-rationalizable, i.e., can be defined in terms of a class of consensus elections (such as, e.g., elections with Condorcet winners, elections where all voters rank the same candidate first, etc.) and appropriate distance functions. In particular, they gave distance-rationalizability proofs for Kemeny, plurality, Borda, Copeland, and some other rules. Subsequently, Elkind, Faliszewski and Slinko [9] expanded this list by showing distance-rationalizability of Maximin, Young, approval, and (almost) all scoring rules.

Now, the focus of papers [21, 9] was on proving distance rationalizability of specific voting rules. In this paper, we take a different approach: Instead of looking at specific voting rules we seek general results regarding *all* distance-rationalizable voting rules. However, this goal proves to be too ambitious: we show that essentially any voting rule is distance-rationalizable and thus distance rationalizability of a rule does not by itself guarantee that the voting rule has any desirable properties; this result holds (with small modifications) even if we restrict ourselves to a standard notion of consensus. We therefore rephrase our question by asking if we can derive useful conclusions about a voting rule based on the properties of a distance that is used to rationalize it.

We demonstrate that the rephrased question can be answered in positive. Specifically, we define a large class of distances that have a natural structure, and show that many important properties of voting rules that can be rationalized via distances from this class, such as anonymity, neutrality and consistency, can be expressed in terms of the underlying distances. This approach allows us to provide an

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alternative characterization of scoring rules in terms of rationalizability, and to establish interesting connections with the work on interpreting voting rules as maximum likelihood estimators [6, 5]. Further, for voting rules defined via certain type of distances from our class, we provide upper and lower bounds on the complexity of winner determination problem, thus demonstrating that, by constraining the allowable distances, we can turn distance rationalizability into a useful tool for reasoning about complexity-related issues.

Applications of voting theory play an important role in multiagent system design. Indeed, voting has been proposed as a tool to solve planning problems [10], to design recommender systems [13], and to build meta-search engines [8]. These applications have inspired research on several aspects of voting, such as the complexity of determining election winners (see, e.g., [14, 15, 4, 2]) or, if not all votes have been aggregated yet, possible winners (see, e.g., [18, 23, 3]), and the complexity of various types of attacks on elections (e.g., manipulation [7], bribery [11], and control [20, 12])[1]. So far, most of the research regarding these issues focused on specific voting rules. In contrast, distance rationalizability provides a unified framework for defining voting rules, and, as this paper illustrates, can also be used to reason about them. Thus, we believe that thinking about voting rules in terms of distances and consensuses may lead to more general results for various voting-related problems, and we view this paper as a first step in this direction.

2. PRELIMINARIES

Elections. An *election* is a pair $E = (C, V)$ where $C = \{c_1, \dots, c_m\}$ is the set of *candidates* and $V = (v_1, \dots, v_n)$ is an ordered list of *voters*. We denote the number of voters in a list V by $|V|$. We assume that each voter is represented by her *vote*, i.e., a preference order over the candidate set; we write $\succ_i$ to denote the i'th voter's preference order. We will refer to the list V as a *preference profile*. For example, if we have two voters, v_1 and v_2, and candidate set $C = \{c_1, c_2, c_3\}$ then we write $c_2 \succ_2 c_1 \succ_2 c_3$ to denote that the second voter prefers c_2 to c_1 to c_3. We assume the standard *rational voter model*, that is, preference orders are strict, total orders over C.

A *voting rule* $\mathcal{R}$ is a function that given an election $E = (C, V)$ returns a set of *election winners* $\mathcal{R}(E)$, i.e., the candidates that win this election. Note that it is legal for the set of winners to be empty or to contain more that one candidate. To simplify notation, we will sometimes write $\mathcal{R}(V)$ instead of $\mathcal{R}(E)$. In what follows, we sometimes consider voting rules defined for a particular number of candidates (or even a particular set of candidates) only.

Below we define several prominent voting rules. Let $E = (C, V)$ be an election where $C = \{c_1, \dots, c_m\}$ and $V = (v_1, \dots, v_n)$. For voting rules that assign points, the candidates with most points are winners.

Scoring rule $\mathcal{R}_{(\alpha_1,\dots,\alpha_m)}$. Each candidate receives α_j points for each vote that ranks her in the j'th position. A single scoring rule is defined for a fixed number of candidates, but many standard voting rules can be defined in terms of families of scoring protocols. For example, *plurality* is defined via the family of vectors $(1, 0, \dots, 0)$, *veto* is defined via the family of vectors $(1, \dots, 1, 0)$, and *Borda* is defined via the family of vectors $(m-1, m-2, \dots, 0)$; k-*approval* is the scoring rule with $\alpha_i = 1$ for $i \leq k$, $\alpha_i = 0$ for $i > k$.

Dodgson. A *Condorcet winner* is a candidate that is preferred to any other candidate by a majority of voters. The score of a candidate c is the smallest number of swaps of adjacent candidates that have to be performed on the votes to make c a Condorcet winner.

Kemeny. Let $\succ$ and $\succ'$ be two preference orders over C. The number of *disagreements* between $\succ$ and $\succ'$, denoted $t(\succ, \succ')$, is the number of pairs of candidates c_i, c_j such that either $c_i \succ c_j$ and $c_j \succ' c_i$ or $c_j \succ c_i$ and $c_i \succ' c_j$. A candidate c_i is a Kemeny winner if there exists a preference order $\succ$ such that c_i is ranked first in $\succ$ and $\succ$ minimizes the sum $\sum_{i=1}^{n} t(\succ, \succ_i)$.

Distances. Let X be a set. A function $d\colon X \to \mathbb{R} \cup \{\infty\}$ is a *distance* (or, a *metric*) if for each $x, y, z \in X$ it satisfies the following four conditions:

1. $d(x, y) \geq 0$ (nonnegativity),
2. $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles),
3. $d(x, y) = d(y, x)$ (symmetry),
4. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

If d satisfies all of the above conditions except the second one (identity of indiscernibles) then d is called a *pseudodistance*.

In the context of elections, it is useful to consider both distances over votes and over entire elections (that is, distances where the set X is the set of all linear orders over some given candidate set, and distances where X is the set of all possible elections); we remark that the former can be extended to the latter in a natural way (see Section 4).

Two particularly useful distances over votes are the discrete distance and the swap distance. Let C be a set of candidates and let u and v be two votes over C. The *discrete distance* $d_{\mathrm{discr}}(u, v)$ is defined to be 0 if $u = v$ and to be 1 otherwise. The *swap distance* $d_{\mathrm{swap}}(u, v)$ is the least number of swaps of adjacent candidates that transform vote u into vote v. The swap distance is sometimes called Dodgson distance.

Consensus classes. Intuitively, we say that an election $E = (C, V)$ is a consensus if it has an obvious, clear winner. Formally, a *consensus class* is a pair $(\mathcal{E}, \mathcal{W})$ where $\mathcal{E}$ is a set of elections and $\mathcal{W}$ is a voting rule such that for each election $E \in \mathcal{E}$ it holds that $\mathcal{W}(E)$ has exactly one member, which is called the *consensus winner*. We consider the following four natural consensus classes:

Strong unanimity. Denoted $\mathcal{S}$, contains elections $E = (C, V)$ where all voters report the same preference order. The consensus winner is the candidate ranked first by all the voters.

Unanimity. Denoted $\mathcal{U}$, contains all elections $E = (C, V)$ where all voters rank some candidate c first. The consensus winner is c.

Majority. Denoted $\mathcal{M}$, contains all elections $E = (C, V)$ where more than half of the voters rank some candidate c first. The consensus winner is c.

Condorcet. Denoted $\mathcal{C}$, contains all elections $E = (C, V)$ with a Condorcet winner (defined above). The Condorcet winner is the consensus winner.

One can certainly consider situations in which the voters reach a consensus that several candidates are equally well qualified to be elected. However, in this paper we limit ourselves to consensuses with unique winners.

Distance rationalizability. We are now ready to define what it means for a voting rule to be distance-rationalizable. The following two definitions are copied almost verbatim from [9].

[1]We should point out that references regarding these lines of work here are only examples of some of the recent works. It would be beyond the scope of this paper to even attempt an overview of last year's progress regarding manipulating elections.

DEFINITION 2.1. *Let d be a distance over elections and let $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ be a consensus class. We define the* $(\mathcal{K}, d)$-score *of a candidate c_i in an election E to be the distance (according to d) between E and a closest election $E' \in \mathcal{E}$ such that $c_i \in \mathcal{W}(E')$. The set of $(\mathcal{K}, d)$-winners of an election $E = (C, V)$ consists of those candidates in C whose $(\mathcal{K}, d)$-score is the smallest.*

DEFINITION 2.2. *A voting rule $\mathcal{R}$ is* distance-rationalizable *via a consensus class $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ and a distance d over elections (is $(\mathcal{K}, d)$-rationalizable), if for each election E, a candidate c is an $\mathcal{R}$-winner if and only if he or she is a $(\mathcal{K}, d)$-winner of E.*

Many of the common voting rules are distance-rationalizable in a very natural way, see [21] and [9].

Computational complexity. We assume that the reader is familiar with basic notions of computational complexity theory such as complexity classes P and NP. In this paper we also consider complexity classes one notch higher in the Polynomial Hierarchy, namely Θ_2^p and P^{NP}. A decision problem belongs to P^{NP} if it can be solved in polynomial time as long as one has access to an NP oracle, i.e., assuming one can solve NP decision problems at unit cost. Intuitively, a decision problem is in Θ_2^p if it is in P^{NP} with the additional restriction that all the queries to the NP oracle have to be prepared before any of the answers are received, i.e., the queries cannot be chosen adaptively. A catalog of reductions and complexity classes can be found in [16].

We will also consider our problems from *parameterized complexity* perspective. A problem is *fixed-parameter tractable (FPT)* with respect to some parameter if there is an algorithm that for each instance I of size n with parameter value j computes the solution to the problem in time $O(f(j)n^{O(1)})$, where f is a (computable) function of j.

3. UNIVERSAL DISTANCE-RATIONALIZABILITY RESULTS

Previous work on distance rationalizability of voting rules focuses on showing distance rationalizability (or impossibility thereof) of specific voting rules. In this section we take a different approach: we show that if we do not impose any additional constraints on allowable consensus classes and distance functions, then essentially all rules are distance-rationalizable.

We say that a voting rule $\mathcal{R}$ over a set of candidates C satisfies *nonimposition* if for every $c \in C$ there exists an election E_c with the set of candidates C in which c is the unique winner under $\mathcal{R}$. Clearly, nonimposition is a very weak condition that is satisfied by all common voting rules. Nevertheless, it turns out to be sufficient for distance-rationalizability.

THEOREM 3.1. *For any voting rule $\mathcal{R}$ over a set of candidates C that satisfies nonimposition, there is a consensus class $\mathcal{K}_\mathcal{R}$ and a distance $d_\mathcal{R}$ such that $\mathcal{R}$ is $(\mathcal{K}_\mathcal{R}, d_\mathcal{R})$-rationalizable.*

PROOF. Since $\mathcal{R}$ satisfied nonimposition, for each $c \in C$ there exists an election $E_c = (C, V_c)$ in which c is the unique winner. Define an undirected graph $G = (K, \mathcal{E})$ as follows. The set K consists of all tuples of votes over C (note that this set is infinite). The set $\mathcal{E}$ contains an edge between U and V if $|\mathcal{R}(U)| = 1$ and $\mathcal{R}(U) \subseteq \mathcal{R}(V)$ or $|\mathcal{R}(V)| = 1$ and $\mathcal{R}(V) \subseteq \mathcal{R}(U)$. For any two elections $E_U = (C, U)$ and $E_V = (C, V)$, we define $d_\mathcal{R}(E_U, E_V)$ to be the shortest path distance between U and V in G. It is easy to check that $d_\mathcal{R}$ is indeed a distance.

We identify the consensus profiles with the elections that have a unique winner, i.e., we set $\mathcal{K}_\mathcal{R} = \{E \mid |\mathcal{R}(E)| = 1\}$, and let the winner of the $\mathcal{K}_\mathcal{R}$-consensus election E be $\mathcal{R}(E)$.

Now, suppose that $E \in \mathcal{K}_\mathcal{R}$. Then $\{E' \mid d_\mathcal{R}(E, E') = 0\} = \{E\}$, and the unique winner in E under $\mathcal{R}$ is exactly the consensus winner in the nearest $\mathcal{K}_R$-consensus. On the other hand, if $E \notin \mathcal{K}_\mathcal{R}$, then $d_\mathcal{R}(E, E') \geq 1$ for any election E', and for any $c \in \mathcal{R}(E)$ we have $d_\mathcal{R}(E, E_c) = 1$. Moreover, for any $c \notin \mathcal{R}(E)$ and any $E' \in \mathcal{K}_\mathcal{R}$ such that $\mathcal{R}(E') = \{c\}$ we have $d_\mathcal{R}(E, E') \geq 2$. Thus, the set $\mathcal{R}(E)$ is exactly the set of consensus winners in the consensus profiles that are closest to E. $\square$

Arguably, the consensus class used in the proof of Theorem 3.1 includes many elections that would not normally be considered consensus elections. However, it turns out that we can use a similar idea to prove distance rationalizability with respect to a standard notion of consensus, albeit under a stronger condition.

DEFINITION 3.2. *Let $\mathcal{R}$ be a voting rule and let $(\mathcal{E}, \mathcal{W})$ be a consensus class. We say that $\mathcal{R}$ is* compatible *with $(\mathcal{E}, \mathcal{W})$, or* $(\mathcal{E}, \mathcal{W})$-compatible *if for each election $E = (C, V)$ in $\mathcal{E}$ it holds that $\mathcal{R}(E) = \mathcal{W}(E)$.*

The next theorem shows that compatibility with a particular consensus is equivalent to distance-rationalizability with respect to this consensus. In what follows we prove this result for the four consensus classes considered in this paper; however, it can be generalized to any consensus class that has the property that any candidate can be a consensus winner. Note also that any voting rule that is compatible with any such consensus class also satisfies nonimposition, so the compatibility condition is more restrictive than nonimposition.

THEOREM 3.3. *For any consensus class $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$, a voting rule $\mathcal{R}$ over a set of candidates C is $\mathcal{K}$-compatible if and only if there is a distance $d_\mathcal{R}^\mathcal{K}$ such that $\mathcal{R}$ is $(\mathcal{K}, d_\mathcal{R}^\mathcal{K})$-rationalizable.*

PROOF. The "if" direction is immediate: for any election $E \in \mathcal{K}$, there is only one election at distance 0 from it, so the $\mathcal{K}$-consensus winner of E is the only winner in E under $\mathcal{R}$.

For the "only if" direction, we use the same approach as in the proof of Theorem 3.1; however, we will modify it slightly to ensure that $d'_\mathcal{R}(E, E') = +\infty$ if E and E' have a different number of voters. Let $P(C)$ be the set of all possible votes over C. For each $n \in \mathbb{Z}^+$, we define an undirected graph $G(C, n) = (K, \mathcal{E})$, where the set of vertices K is $P(C)^n$ (i.e., K is the set of all possible profiles of n votes over C), and there is an edge between U and V if and only if $(C, U) \in \mathcal{K}$ and $\mathcal{R}(U) \subseteq \mathcal{R}(V)$ or $(C, V) \in \mathcal{K}$ and $\mathcal{R}(V) \subseteq \mathcal{R}(U)$.

We define the distance $d_\mathcal{R}^\mathcal{K}(E, E')$ between two elections $E = (C, U), E' = (C, V)$ to be the shortest path distance between U and V in $G(C, n)$ if U and V are both in $P(C)^n$ for some $n > 0$, and set $d_\mathcal{R}^\mathcal{K}(E, E') = +\infty$ if E and E' have a different number of voters. The argument that $\mathcal{R}$ is $(\mathcal{K}, d_\mathcal{R}^\mathcal{K})$-rationalizable follows the same lines as the proof of Theorem 3.1. $\square$

Observe that the compatibility requirement is more stringent for larger consensus classes: when the consensus class is $\mathcal{S}$, i.e., when all voters agree on the ordering of the candidates, it is very natural to demand that the voters' top choice should be the only winner. On the other hand, there are reasonable voting rules (e.g., scoring rules) that are not compatible with $\mathcal{C}$ (see, e.g., [22]). That is, using the approach of Theorem 3.3, it is easier to rationalize voting rules with respect to the strong consensus than with respect to any other consensus class. However, there are common voting rules such as, e.g., veto, and, more generally, k-approval for $k > 1$, that are not compatible with $\mathcal{S}$, and thus cannot be distance-rationalized with respect to any consensus class that contains $\mathcal{S}$. On the other hand, veto satisfies nonimposition and is therefore distance-rationalizable via Theorem 3.1.

We view Theorem 3.3 as a negative result: it shows that almost any voting rule is rationalizable with respect to the strong unanimity consensus. Thus, knowing that a rule is distance-rationalizable, even with respect to a standard notion of consensus, provides no further insights about the properties of the rule, and, moreover, the dichotomy between distance-rationalizable and non-distance-rationalizable rules becomes essentially meaningless.

However, observe that the distances employed in the proof of Theorem 3.3 are not particularly natural. For example, consider two elections with the set of candidates $\{a, b\}$. In the first election E_1, there are 100 voters with preference $a \succ b$ and 90 voters with preference $b \succ a$. In the second election E_2, there are 101 voters with preference $a \succ b$ and 89 voters with preference $b \succ a$. Suppose that we are trying to distance-rationalize some voting rule $\mathcal{R}$ with $\mathcal{R}(E_1) = \mathcal{R}(E_2) = \{a\}$ with respect to $\mathcal{S}$ using the approach of Theorem 3.3. By construction, the distance from both E_1 and E_2 to the nearest election in $\mathcal{S}$ is 1, while the distance between E_1 and E_2 is at least 2. However, intuitively, E_1 and E_2 are very similar to each other and very different from any strong consensus profile. Thus, the distance constructed in the proof of Theorem 3.3 does not respect our intuitive understanding of similarity between two preference profiles.

Further, all distances considered in [21] and [9] are polynomial-time computable, even if they are used to distance-rationalize voting rules for which the winner determination problem is NP-hard (such as Dodgson, Young or Kemeny). In contrast, it is not difficult to see that $d^{\mathcal{K}}_{\mathcal{R}}$ is hard to compute for any such rule.

PROPOSITION 3.4. *For any $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$, given a polynomial-time algorithm for computing $d^{\mathcal{K}}_{\mathcal{R}}$, we can construct a polynomial-time algorithm that solves the winner determination problem for $\mathcal{R}$.*

PROOF. Suppose that we are given an election $E = (C, V)$. Clearly, we can check in polynomial-time if $E \in \mathcal{K}$, and if so, output the consensus winner; for all four consensus classes, this can be done in polynomial time.

Otherwise, for each candidate $c \in C$, we can construct an election $E_c = (C, V_c)$ with the set of candidates C in which all voters in V_c have the same preference v_c, c is ranked first in v_c, and all other candidates are ranked arbitrarily. Clearly, each E_c is a strong consensus with winner c. Now, $c \in \mathcal{R}(E)$ if and only if $d^{\mathcal{K}}_{\mathcal{R}}(E, E_c) = 1$; this follows from the fact that $\mathcal{S} \subseteq \mathcal{K}$ for any $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$. Thus we can query the distance oracle $d^{\mathcal{K}}_{\mathcal{R}}$ on $|C|$ inputs of the form (E, E_c) and output the set $\{c \mid d^{\mathcal{K}}_{\mathcal{R}}(E, E_c) = 1\}$. $\square$

Clearly, polynomial-time computability is a very important property. Thus, Proposition 3.4 provides yet another reason why the distances defined in the proof of Theorem 3.3 are not acceptable.

Therefore, we propose to restate the original question about distance-rationalizability of voting rules as follows:

Can a given voting rule be distance-rationalized via a natural *notion of consensus and a* natural *distance?*

Of course, the answer to this question depends on which notions of consensus and which distances are considered natural. Now, for the consensus classes, one can accept the classes used in [21], [9], and the current paper as a preliminary list[2]; while this list is probably incomplete, it is broad enough to allow us to distance-rationalize many common rules. However, so far no attempt was made to identify a suitable family of acceptable distances. In the next section, we will try to fill this gap.

4. IDENTIFYING GOOD DISTANCES

We have argued that to make productive use of the notion of distance rationalizability, we need to place restrictions on allowable distances. In this section, we propose a way of doing so, by describing a large family of distances that has a very intuitive interpretation and includes many distances that have been used so far to distance-rationalize well-known voting rules.

Specifically, many (though not all) distances used in [21, 9] are constructed by first defining a distance on individual votes and then extending it to distances over profiles of the same length by adding up distances between the corresponding votes. This technique can be interpreted as taking the direct product of the metric spaces that correspond to individual votes, and defining the distance on the resulting space via the ℓ_1-norm.

We can generalize this approach by allowing other types of product metrics. We start by recalling the necessary definitions.

DEFINITION 4.1. *Given a vector space S over $\mathbb{R}$, a* norm *on S is a mapping N from S to $\mathbb{R}$ that satisfies the following properties:*

(i) positive scalability: $N(\alpha u) = |\alpha| N(u)$ for any $u \in S$ and any $\alpha \in \mathbb{R}$;

(ii) positive semidefiniteness: $N(u) \geq 0$ for all $u \in S$, and $N(u) = 0$ if and only if u is the zero vector;

(iii) triangle inequality: $N(u + v) \leq N(u) + N(v)$.

A well-known class of norms on $\mathbb{R}^n$ are the *p-norms* ℓ_p given by

$$\ell_p(x_1, \dots, x_n) = \left(\sum_{i=1}^{n} (|x_i|^p) \right)^{\frac{1}{p}},$$

with the convention that $\ell_\infty(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$.

A norm N_n is said to be *symmetric* if it satisfies $N_n(x_1, \dots, x_n) = N_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for any permutation $\sigma : [1, n] \to [1, n]$; clearly, all p-norms are symmetric.

We can now define our family of "good" distances.

DEFINITION 4.2. *We say that a function d on pairs of preference profiles is* votewise *if the following conditions hold:*

1. *$d(E, E') = +\infty$ if E and E' have a different set of candidates or a different number of voters.*
2. *For any set of candidates C, there exists a distance $d_C(\cdot, \cdot)$ defined on votes over C;*
3. *For any $n \in \mathbb{N}$, there exists a norm N_n on $\mathbb{R}^n$ such that for any two preference profiles $E = (C, U)$, $E' = (C, V)$ with $U = (u_1, \dots, u_n)$ and $V = (v_1, \dots, v_n)$ we have $d(E, E') = N_n(d_C(u_1, v_1), \dots, d_C(u_n, v_n))$.*

It is well known that any function defined in this manner is a metric. Thus, in what follows, we refer to votewise functions as *votewise distances*; we will also use the term "N-votewise distance" to refer to a votewise distance defined via a norm N, and denote a votewise distance that is based on a distance d over votes by $\widehat{d}$. Similarly, we will use the term *N-votewise rules* to refer to voting rules that can

[2]Paper [21] does not mention the class $\mathcal{M}$, but makes use of a consensus class that consists of all elections that correspond to an acyclic tournament graph; paper [9] only uses the classes $\mathcal{S}, \mathcal{U}$ and $\mathcal{C}$ for the standard model of voting, but introduces a new consensus class to deal with approval voting.

be distance-rationalized via one of our four consensus classes and an N-votewise distance.

Clearly, any votewise distance is polynomial-time computable as long as both the underlying distance on votes and the norm N_n are.

Further, it can be argued that, unlike the distances constructed in the proof of Theorem 3.3, votewise distances respect similarity between preference profiles. For example, it can be shown that for any votewise distance $\widehat{d}$, any two profiles $V = (v_1, \dots, v_n)$ and $U = (u_1, \dots, u_n)$, and any $k \le n$ we have $\widehat{d}(V, (v_1, \dots, v_k, u_{k+1}, \dots, u_n)) \le \widehat{d}(V, U)$, in accordance with our intuition that the "hybrid" $(v_1, \dots, v_k, u_{k+1}, \dots, u_n)$ is more similar to V than U is. Note that the example provided in the previous section illustrates that the distances $d^{\mathcal{K}}_{\mathcal{R}}$ do not, in general, have this property: the election E_2 can be viewed as a "hybrid" of E_1 and the strong consensus in which all voters prefer a to b.

An important special case of our framework is when N_n is the ℓ_1-norm, i.e., $N_n(x_1, \dots, x_n) = x_1 + \dots + x_n$; we will call any such distance an *additively votewise* distance, or, in line with the notation introduced above, an *ℓ_1-votewise* distance. So far, ℓ_1-votewise distances were the only votewise distances used in distance rationalizability constructions: paper [21] uses them to distance-rationalize the Kemeny rule, Dodgson, Plurality and Borda, and [9] shows that the construction for Borda can be generalized to all scoring rules (also using an ℓ_1-votewise distance). However, N-votewise distances with $N \ne \ell_1$ are almost as easy to work with as ℓ_1-votewise distances and may be useful for rationalizing natural voting rules. We will now show that this is indeed the case for a simplified version of the Bucklin rule.

DEFINITION 4.3. *Given an election $E = (C, V)$ and a positive integer k, $1 \le k \le |C|$, we say that a candidate c is a k*-majority winner *if more than $\frac{|V|}{2}$ voters rank c among the top k candidates. Let k' be the smallest positive integer such that there is at least one k'-majority-winner for E. The* Bucklin score *of a candidate c is the number of voters that rank her in top k' positions. The* Bucklin winners *are the candidates with the highest Bucklin score; clearly, all of them are k'-majority winners. The* simplified Bucklin winners *are all k'-majority winners.*

It is easy to see that the Bucklin rule can be obtained from the simplified Bucklin rule by breaking ties via k'-approval.[3]

For any vote v, let $S_k(v)$ denote the set of all candidates ranked in top k positions in v. Now, for any two votes u and v over the same set of candidates C, set $d(u, v) = \min\{k \mid S_k(u) = S_k(v)\}$. It is easy to see that d is indeed a metric. We extend d to a distance d_{sb} over elections as follows. Let $E_U = (C, U)$ and $E_V = (C, V)$, where $U = (u_1, \dots, u_n)$ and $V = (v_1, \dots, v_n)$. Set

$$d_{sb}(E_U, E_V) = \max\{d(u_i, v_i) \mid 1 \le i \le n\}.$$

For elections with a different number of voters or over different candidate sets we set $d_{sb} = \infty$.

We have $d_{sb}(E, E') = \ell_\infty(d(u_1, v_1), \dots, d(u_n, v_n))$, i.e., $d_{sb}(E, E')$ is a votewise distance. We will now show that, together with the strict majority consensus, d_{sb} can be used to rationalize the simplified Bucklin rule.

THEOREM 4.4. *Simplified Bucklin is $(\mathcal{M}, d_{sb})$-rationalizable.*

PROOF. Let $E = (C, V)$ be an election with $V = (v_1, \dots, v_n)$, and let c be a candidate in C. Let k be the smallest integer such that c is a k-majority winner.

[3]We mention that sometimes the term "Bucklin rule" refers to a somewhat different voting rule. Our definition is, however, standard in computational social choice, and is used, e.g., in [6].

Now consider an arbitrary election $E_U = (C, U)$, $U = (u_1, \dots, u_n)$, in which c is a strict-majority winner. We have $d_{sb}(E, E_U) \ge k$. Indeed, for any $\ell < k$, it holds that $c \notin S_\ell(v)$ for at least $\frac{|V|}{2}$ voters $v \in V$ and $c \in S_\ell(u)$ for more than $\frac{|V|}{2}$ voters $u \in U$. Thus, there exists at least one value of i such that $S_\ell(u_i) \ne S_\ell(v_i)$.

On the other hand, there is a strict majority consensus $E_W = (C, W)$ with winner c such that $d_{sb}(E, E_W) = k$. Indeed, we can construct E_W from E by shifting c to the top in each vote that ranks c among the top k candidates (without changing anything else in those votes).

Thus, for each simplified Bucklin winner c of E there exists an election $E_W \in \mathcal{M}$ such that $E_W \in \arg\min_{E' \in \mathcal{M}} d_{sb}(E, E')$. That is, simplified Bucklin is $(\mathcal{M}, d_{sb})$-rationalizable. □

It turns out that the (regular) Bucklin rule can also be distance-rationalized via $\mathcal{M}$ and a distance that can be obtained from d_{sb} by a simple transformation (but is nevertheless not a votewise distance). We omit the proof of this fact due to space constraints.

We conclude that considering votewise distances that are not necessarily ℓ_1-votewise allows us to obtain a distance-based representation for a broader class of voting rules, while still combining distances between individual votes in a natural way.

In the rest of the paper, we will try to understand what can be said about a voting rule based on the fact that it can be defined via a votewise distance, or, more narrowly, an ℓ_1-votewise distance. We consider two issues. First, we analyze whether such rules satisfy the standard axioms such as anonymity, neutrality, and consistency. Second, we consider the complexity of winner determination under such rules.

4.1 Properties of Votewise Rules

In this section we consider three basic properties of voting rules. Specifically, given a consensus class $\mathcal{K}$ and a votewise distance $\widehat{d}$, we ask under what circumstances the voting rule that is distance-rationalizable via $(\mathcal{K}, \widehat{d})$ is anonymous, neutral, or consistent. To start, we recall the formal definitions of these properties.

Let $E = (C, V)$ be an election with $V = (v_1, \dots, v_n)$, and let σ and π be permutations of V and C, respectively. For any $C' \subseteq C$, set $\pi(C') = \{\pi(c) \mid c \in C'\}$. Let $\tilde{\pi}(v)$ be the vote obtained from v by replacing each occurrence of a candidate $c \in C$ by an occurrence of $\pi(c)$; we can extend this definition to preference profiles by setting $\tilde{\pi}(v_1, \dots, v_n) = (\tilde{\pi}(v_1), \dots, \tilde{\pi}(v_n))$.

Anonymity. A voting rule is *anonymous* if its result depends only on the number of voters reporting each preference order. Formally, a voting rule $\mathcal{R}$ is anonymous if for each election $E = (C, V)$ with $V = (v_1, \dots, v_n)$ and each permutation σ of V, the election $E' = (C, \sigma(V))$ satisfies $\mathcal{R}(E) = \mathcal{R}(E')$.

Neutrality. A voting rule is *neutral* if its result does not depend on the candidates' names. Formally, a voting rule $\mathcal{R}$ is neutral if for each election $E = (C, V)$, where $C = \{c_1, \dots, c_m\}$ and each permutation π of C, the election $E' = (C, \tilde{\pi}(V))$ satisfies $\mathcal{R}(E) = \pi^{-1}(\mathcal{R}(E'))$.

Consistency. A voting rule $\mathcal{R}$ is *consistent* if for any two elections $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$ such that $\mathcal{R}(E_1) \cap \mathcal{R}(E_2) \ne \emptyset$, the election $E = (C, V_1 + V_2)$ (i.e., the election where the collections of voters from E_1 and E_2 are concatenated) satisfies $\mathcal{R}(E) = \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$.

It turns out that for votewise distance-rationalizable rules a symmetric norm produces an anonymous rule.

THEOREM 4.5. *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, \widehat{d})$-rationalizable, where $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$ and $\widehat{d}$ is an N-votewise distance, where N is a symmetric norm. Then $\mathcal{R}$ is anonymous.*

PROOF. Let $E_V = (C, V)$, where $|V| = n$, be an election, and let σ be a permutation of V. Fix a candidate $c \in \mathcal{R}(E)$, and let $E_U = (C, U)$ be a $\mathcal{K}$-consensus election with winner c that is closest to E. We form elections $E'_V = (C, V')$ and $E'_U = (C, U')$ by setting $V' = \sigma(V)$ and $U' = \sigma(U)$. Observe that for any $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$ the election E'_U is a $\mathcal{K}$-consensus, and, moreover, $\widehat{d}(E_V, E_U) = \widehat{d}(E'_V, E'_U)$. Now, suppose that there exists a $\mathcal{K}$-consensus election $E_W = (C, W)$ such that $\widehat{d}(E'_V, E_W) < \widehat{d}(E'_V, E'_U)$. Then for the election $E'_W = (C, \sigma^{-1}(W))$ we have

$$\widehat{d}(E_V, E'_W) = \widehat{d}(E'_V, E_W) < \widehat{d}(E'_V, E'_U) = \widehat{d}(E_V, E_U),$$

a contradiction with our choice of E_U. Thus, any winner of E_V is a winner of E'_V. By considering permutation σ^{-1}, we also obtain that any winner of E'_V is a winner of E_V. $\square$

We have shown that for votewise rules, anonymity is essentially a property of the underlying norm. In contrast, neutrality is inherited from the underlying distance over votes.

DEFINITION 4.6. *Let C be a set of candidates and let d be a distance on votes over C. We say that d is* neutral *if for each permutations π over C and any two votes u and v over C it holds that $d(u, v) = d(\tilde{\pi}(u), \tilde{\pi}(v))$. Further, we say that a votewise distance $\widehat{d}$ that corresponds to a distance d on votes is* neutral *if d is.*

Our next result shows that if a votewise rule is rationalized via a neutral distance then it itself is neutral.

THEOREM 4.7. *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, \widehat{d})$-rationalizable, where $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$ and $\widehat{d}$ is a neutral votewise distance. Then $\mathcal{R}$ is neutral.*

The proof of this theorem is similar to that of Theorem 4.5 and is omitted due to space constraints.

It is natural to ask if the converse is also true, i.e., if any neutral votewise rule can be rationalized via a neutral distance. Indeed, paper [5] provides a positive answer to a similar question in the context of representing voting rules as maximum likelihood estimators. However, the natural extension of the approach of [5] is not necessarily applicable in our setting. Nevertheless, all votewise distances that have so far arisen in the study of distance rationalizability of voting rules are neutral.

Our results for anonymity and neutrality are applicable to all consensus classes considered in this paper. In contrast, when discussing consistency, we need to limit ourselves to the unanimity consensus, and to ℓ_p-votewise rules.

THEOREM 4.8. *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{U}, \widehat{d})$-rationalizable, where $\widehat{d}$ is an ℓ_p-votewise distance. Then $\mathcal{R}$ is consistent.*

PROOF. We provide a proof for ℓ_1-votewise distances; the reader can easily verify that it generalizes to other values of p.

Let $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$ be two elections over the same candidate set such that $\mathcal{R}(E_1) \cap \mathcal{R}(E_2) \neq \emptyset$, and let $E = (C, V_1 + V_2)$. First, we will show that $\mathcal{R}(E_1) \cap \mathcal{R}(E_2) \subseteq \mathcal{R}(E)$. Fix a candidate $c \in \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$. By definition, there are two $\mathcal{U}$-consensuses, (C, U_1) and (C, U_2), such that for $i = 1, 2$, c is the unanimity winner of (C, U_i) and

$$U_i \in \arg \min_{(C,U) \in \mathcal{U}} \widehat{d}(V_i, U).$$

For the sake of contradiction, suppose that $c \notin \mathcal{R}(E)$. Clearly, $(C, U_1 + U_2)$ is a unanimity consensus with winner c. As $c \notin \mathcal{R}(E)$, there is another unanimity consensus $(C, W_1 + W_2)$ with $|W_1| = |V_1|$, $|W_2| = |V_2|$ such that

$$\widehat{d}(V_1 + V_2, U_1 + U_2) > \widehat{d}(V_1 + V_2, W_1 + W_2).$$

Since $\widehat{d}$ is an ℓ_1-votewise distance, this inequality is equivalent to

$$\widehat{d}(V_1, U_1) + \widehat{d}(V_2, U_2) > \widehat{d}(V_1, W_1) + \widehat{d}(V_2, W_2).$$

However, by the choice of U_1 and U_2, it holds that

$$\widehat{d}(V_1, U_1) \leq \widehat{d}(V_1, W_1), \quad \widehat{d}(V_2, U_2) \leq \widehat{d}(V_2, W_2),$$

which immediately yields a contradiction, and so $c \in \mathcal{R}(E)$.

To show that $\mathcal{R}(E) \subseteq \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$, consider a $c' \in \mathcal{R}(E)$. Since c and c' are both in $\mathcal{R}(E)$, there exists a unanimity consensus $(C, X_1 + X_2)$ with winner c' such that $|X_1| = |V_1|$, $|X_2| = |V_2|$ and

$$\widehat{d}(V_1 + V_2, X_1 + X_2) = \widehat{d}(V_1 + V_2, U_1 + U_2).$$

On the other hand, we have

$$\widehat{d}(V_1, U_1) \leq \widehat{d}(V_1, X_1), \quad \widehat{d}(V_2, U_2) \leq \widehat{d}(V_2, X_2).$$

It follows that both of the inequalities above are, in fact, equalities. Thus, by our choice of U_1 and U_2, for $i = 1, 2$ we obtain $X_i \in \arg\min_{(C,U) \in \mathcal{U}} \widehat{d}(V_i, U)$. Since c' is the unanimity winner in X_1 and X_2, it follows that $c' \in \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$. $\square$

While Theorem 4.8 may hold for some norms other than ℓ_p, we cannot hope to prove it for all votewise distances: fundamentally, consistency is a constraint on the relationship among N_s, N_t and N_{s+t} (i.e., the norms used for s voters, t voters, and $s + t$ voters), and our definition of votewise distances allows us to select norms N_n for different values of n independently of each other. Further, for our proof to work, the consensus class should be closed with respect to "splitting" and "merging" of the consensus profiles, and neither of the classes $\mathcal{S}$, $\mathcal{C}$, and $\mathcal{M}$ satisfies both of these conditions. Indeed, for $\mathcal{S}$ and $\mathcal{C}$ the conclusion of the theorem itself is not true: the counterexamples are provided by the Kemeny rule and the Dodgson rule, respectively (both are not consistent, yet rationalizable via the ℓ_1-votewise distance that is based on the swap distance).

By combining Theorems 4.5, 4.7 and 4.8, we conclude that any rule that is $(\mathcal{U}, \widehat{d})$-rationalizable, where $\widehat{d}$ is a neutral ℓ_1-votewise distance, is neutral, anonymous and consistent; it is not hard to check that the conclusion still holds if $\widehat{d}$ is a pseudodistance rather than a distance. Contrast this with Young's famous characterization result [24], which says that every voting rule that has all three of these properties is either a scoring rule or a composition of scoring rules (see [24] for an exact definition of composition of voting rules). It turns out that our framework allows us to refine Young's result by characterizing exactly the scoring rules themselves rather than their compositions. Moreover, we can actually "extract" the scoring rule from the corresponding distance, albeit not efficiently (see Section 4.2 for a discussion of the related complexity issues).

THEOREM 4.9. *Let $\mathcal{R}$ be a voting rule. There exists a neutral ℓ_1-votewise pseudodistance $\widehat{d}$ such that $\mathcal{R}$ is $(\mathcal{U}, \widehat{d})$-rationalizable if and only if $\mathcal{R}$ can be defined via a family of scoring rules.*

PROOF. The "if" direction was essentially shown in [9]; it is not hard to see that the distance used in that proof is a neutral ℓ_1-votewise pseudodistance.

For the "only if" direction, let $\mathcal{R}$ be a $(\mathcal{U}, \widehat{d})$-rationalizable voting rule, where $\widehat{d}$ is a neutral, ℓ_1-votewise (pseudo)distance based on a (pseudo)distance d. We will now show how to derive a scoring rule $\mathcal{R}_{(\alpha_1,\dots,\alpha_m)}$ that corresponds to $\mathcal{R}$ for m candidates.

Let $C = \{c_1, \dots, c_m\}$, and consider an arbitrary preference profile $V = (v_1, \dots, v_n)$ over C. Fix any vote $v \in V$ and let the corresponding preference order be $c_{j_1} \succ \dots \succ c_{j_m}$. For any $k = 1, \dots, m$, let β_k be the distance from v to the nearest vote u_k that ranks c_{j_k} first. Note that by neutrality the value of β_k is independent of our choice of v. Now, consider a candidate c that is ranked in position t_i in v_i for $i = 1, \dots, n$. Clearly, the distance from V to the nearest profile in $\mathcal{U}$ in which c wins is given by $\beta_{t_1} + \dots + \beta_{t_n}$. Thus, to transform the vector $\beta = (\beta_1, \dots, \beta_m)$ into a scoring rule, we need to "reverse" it by setting $\alpha_j = B - \beta_j$ for $j = 1, \dots, m$ where B is large enough, e.g., $B = \max_{j=1}^{m} \beta_j$. It is immediate that $R_{(\alpha_1,\dots,\alpha_m)}$ is exactly $\mathcal{R}$ for m candidates. □

REMARK 4.10. Note that in this paper, following Young [24], we do not require $(\alpha_1, \dots, \alpha_m)$ to be nondecreasing or integer. Indeed, the distance rationalizability framework does not impose any ordering over different positions in a vote, so it works equally well for a scoring rule with, e.g., $\alpha_1 < \alpha_2$.

The above theorem allows us to partially resolve a question from [9] regarding the relation between voting rules that are distance-rationalizable and so-called MLEWIV voting rules. Briefly put, Conitzer and Sandholm [6] defined a voting rule to be MLEWIV if it can be interpreted as a maximum likelihood estimator for the winner of the election.

COROLLARY 4.11. *A neutral voting rule is MLEWIV if and only if it is distance-rationalizable via a neutral ℓ_1-votewise distance and unanimity consensus.*

Due to space constraints we cannot formally define MLEWIV rules here and so we skip the proof. However, the idea is to show that all neutral MLEWIV voting rules are in fact families of scoring protocols. The proof, in essence, combines the ideas from our proof of Theorem 4.9 and from a proof from [5].

Papers [6, 5] also consider maximum-likelihood estimation of entire rankings produced by voting rules. It would be interesting to understand how to translate results between this form of maximum-likelihood estimation and distance-rationalizability.

4.2 The Winner Determination Problem

In this section, we focus on the computational complexity of the winner determination problem for distance-rationalizable rules. Clearly, to prove upper bounds on the complexity of this problem. we need to impose restrictions on the complexity of the distance itself. Thus, in what follows, we focus on distances that take values in $\mathbb{Z} \cup \{\infty\}$ and are polynomial-time computable; we will call a distance *normal* if it has both of these properties.

The winner determination problem can be formally stated as follows.

DEFINITION 4.12. *Let $\mathcal{R}$ be a voting rule. In the $\mathcal{R}$-*winner problem *we are given an election $E = (C, V)$ and a candidate $c \in C$ and we ask whether $c \in \mathcal{R}(E)$.*

This problem can be hard even for ℓ_1-votewise rules: for Dodgson and Kemeny, its is known to be Θ_2^p-complete [14, 15]. On the positive side, for both of these rules it can be solved in polynomial time if the number of candidates is fixed. In fact, a stronger statement is true: the winner determination problem for both Dodgson and Kemeny is fixed parameter tractable with respect to the number of candidates.

We will now show that, in a sense, from the complexity perspective, Dodgson and Kemeny exhibit the worst-case behavior.

Our next theorem provides an upper bound on the complexity of the winner determination problem for rules that are rationalizable via a large subclass of normal distances. In particular, this bound applies to all normal votewise distances.

THEOREM 4.13. *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, d)$-rationalizable, where $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$, and d is a normal distance that satisfies $d((C_1, V_1), (C_2, V_2)) = +\infty$ whenever $C_1 \neq C_2$ or $|V_1| \neq |V_2|$. Then the $\mathcal{R}$-winner problem is in P^{NP}. Moreover, if, in addition, for any two elections $E_1 = (C, V_1)$, $E_2 = (C, V_2)$, the distance $d(E_1, E_2)$ is either $+\infty$ or at most polynomial in $|C| + |V_1| + |V_2|$, then the $\mathcal{R}$-winner problem is in Θ_2^p.*

PROOF (SKETCH). Due to space constraints, we only provide a brief outline of the proof. We can use the NP-oracle to guess a consensus election with a given winner whose distance from the input election does not exceed a certain (integer) value k. By using binary search (or simply submitting an independent query for each value of k if d is polynomially bounded), for each candidate c we can determine the distance to the nearest consensus where c is the winner. This allows us to identify the winner of the original election. The additional restrictions on d are needed to ensure that the nearest consensus profile is not too "large" relative to the input. □

Note that the distance used to rationalize Dodgson and Kemeny is polynomially bounded. On the other hand, there are natural distances that are not polynomially bounded: an example is provided by the distances used in [9] to rationalize scoring rules.

If, in addition to being normal, the distance in question is an ℓ_1-votewise distance, the winner determination problem is fixed-parameter tractable with respect to the number of candidates.

THEOREM 4.14. *Suppose that a voting rule $\mathcal{R}$ is $(\mathcal{K}, d)$-rationalizable, where $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$. and d is a normal ℓ_1-votewise distance. Then the $\mathcal{R}$-winner problem is FPT with respect to the number of candidates.*

Due to limited space we skip the proof. The main idea is that our problem can be reformulated as an integer linear program with $O(|C|!)$ variables, where $|C|$ is the number of candidates. We can then use Lenstra's algorithm [19]. For consensus classes $\mathcal{S}, \mathcal{U}$, and $\mathcal{M}$ we can also derive algorithms that do not rely on Lenstra's result. (This is quite useful, as Lenstra's algorithm has a prohibitively large multiplicative constant in its running time.)

In the previous section we have seen that neutral ℓ_1-votewise rules that use unanimity consensus correspond to families of scoring rules. Thus, one would expect their winner problems to be in P. Note, however, that in our setting we are given the distance, but not the scoring vector and computing the latter from the former might be hard. Nevertheless, we can easily determine the winner if we are allowed to use polynomial-size *advice*.

THEOREM 4.15. *Suppose that a voting rule $\mathcal{R}$ is distance-rationalizable via a normal neutral ℓ_1-votewise distance and unanimity consensus. Then $\mathcal{R}$-winner is in* P/poly.

P/poly is a complexity class that captures the power of polynomial computation "with advice." Due to space constraints, we omit its formal definition (see, e.g., [16]) as well as the proof of Theorem 4.15. However, the intuition behind the proof is very simple: we just use the appropriate scoring rule as the advice.

Karp–Lipton theorem [17] says that if there is an NP-hard problem in P/poly then the Polynomial Hierarchy collapses. Thus, for voting rules that are distance rationalizable via a normal neutral ℓ_1-votewise distance and the consensus class $\mathcal{U}$ the winner determination problem is unlikely to be NP-hard. In contrast, this problem is hard for both Dodgson and Kemeny, even though they are both rationalizable via a normal neutral ℓ_1-votewise distance (and consensus classes $\mathcal{C}$ and $\mathcal{S}$, respectively). Thus, from computational perspective, the unanimity consensus appears to be easier to work with than the strong consensus and the Condorcet consensus. Indeed, both $\mathcal{S}$ and $\mathcal{C}$ impose a "global" constraint on the closest consensus. On the other hand, $\mathcal{U}$ only imposes a "local" constraint: For each vote in an election we simply seek the closest vote with a particular candidate ranked first. We conjecture that the winner determination problem is hard for all rules that are rationalizable via an ℓ_1-votewise distance and $\mathcal{C}$. The situation in the case of $\mathcal{S}$ is more interesting, because the (rather silly) voting rule that is rationalized by $(\mathcal{S}, \widehat{d}_{\rm discr})$ has a polynomial-time winner determination procedure. We leave establishing the complexity of winner determination for all normal ℓ_1-votewise rules as an open problem.

5. CONCLUSIONS

We have shown that essentially any voting rule is distance-rationalizable, even with respect to a standard notion of consensus. Thus, distance-rationalizability of a rule does not by itself indicate that the rule has any desirable properties. However, we have demonstrated that if the class of allowable distances is restricted, we can derive conclusions about certain features of a voting rule based on the properties of the distance that was used to rationalize it. This enables us to give a new characterization of scoring rules in terms of distances and consensuses. We have also demonstrated connections between our framework and the MLE approach.

We believe that there are many other applications of the distance rationalizability framework to the analysis of voting rules: indeed, our paper illustrates that this framework can be useful for proving results not just for particular voting rules, but for entire families of rules. Thus, a promising topic of future research is to find distances and consensus classes that are both intuitively appealing and lead to voting rules with attractive properties. Further, while this paper took a constructive approach to defining acceptable distances, developing a normative approach, i.e., identifying axioms that should be satisfied by all distances or all consensus classes is an exciting research direction as well.

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6. REFERENCES

[1] N. Baigent. Metric rationalisation of social choice functions according to principles of social choice. *Mathematical Social Sciences*, 13(1):59–65, 1987.

[2] N. Betzler, J. Guo, and R. Niedermeier. Parameterized computational complexity of Dodgson and Young elections. In *Proceedings of SWAT-08*, pages 402–413, July 2008.

[3] N. Betzler, S. Hemmann, and R. Niedermeier. A multivariate complexity analysis of determining possible winners given incomplete votes. In *Proceedings of IJCAI-09*, July 2009.

[4] I. Caragiannis, J. Covey, M. Feldman, C. Homan, C. Kaklamanis, N. Karanikolas, A. Procaccia, and J. Rosenschein. On the approximability of Dodgson and Young elections. In *Proceedings of SODA-09*, pages 1058–1067, Jan. 2009.

[5] V. Conitzer, M. Rognlie, and L. Xia. Preference functions that score rankings and maximum likelihood estimation. In *Proceedings of IJCAI-09*, July 2009.

[6] V. Conitzer and T. Sandholm. Common voting rules as maximum likelihood estimators. In *Proceedings of UAI-05*, pages 145–152, July 2005.

[7] V. Conitzer, T. Sandholm, and J. Lang. When are elections with few candidates hard to manipulate? *Journal of the ACM*, 54(3):Article 14, 2007.

[8] C. Dwork, R. Kumar, M. Naor, and D. Sivakumar. Rank aggregation methods for the web. In *Proceedings of WWW-01*, pages 613–622, Mar. 2001.

[9] E. Elkind, P. Faliszewski, and A. Slinko. On distance rationalizability of some voting rules. In *Proceedings of TARK-09*, pages 108–117, July 2009.

[10] E. Ephrati and J. Rosenschein. A heuristic technique for multi-agent planning. *Annals of Mathematics and Artificial Intelligence*, 20(1–4):13–67, 1997.

[11] P. Faliszewski, E. Hemaspaandra, and L. Hemaspaandra. How hard is bribery in elections? *JAIR*, 35:485–532, 2009.

[12] P. Faliszewski, E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Llull and Copeland voting computationally resist bribery and constructive control. *JAIR*, 35:275–341, 2009.

[13] S. Ghosh, M. Mundhe, K. Hernandez, and S. Sen. Voting for movies: The anatomy of recommender systems. In *Proceedings of Agents-99*, pages 434–435, 1999.

[14] E. Hemaspaandra, L. Hemaspaandra, and J. Rothe. Exact analysis of Dodgson elections: Lewis Carroll's 1876 voting system is complete for parallel access to NP. *Journal of the ACM*, 44(6):806–825, 1997.

[15] E. Hemaspaandra, H. Spakowski, and J. Vogel. The complexity of Kemeny elections. *Theor. Comp. Sci.*, 349(3):382–391, 2005.

[16] L. Hemaspaandra and M. Ogihara. *The Complexity Theory Companion*. Springer-Verlag, 2002.

[17] R. Karp and R. Lipton. Some connections between nonuniform and uniform complexity classes. In *Proceedings of STOC-80*, pages 302–309, Apr. 1980.

[18] K. Konczak and J. Lang. Voting procedures with incomplete preferences. In *IJCAI-05 Workshop on Advances in Preference Handling*, pages 124–129, 2005.

[19] H. Lenstra, Jr. Integer programming with a fixed number of variables. *Math. of Op. Res.*, 8(4):538–548, 1983.

[20] R. Meir, A. Procaccia, J. Rosenschein, and A. Zohar. The complexity of strategic behavior in multi-winner elections. *JAIR*, 33:149–178, 2008.

[21] T. Meskanen and H. Nurmi. Closeness counts in social choice. In M. Braham and F. Steffen, editors, *Power, Freedom, and Voting*. 2008.

[22] H. Moulin. *Axioms of Cooperative Decision Making*. Cambridge University Press, 1991.

[23] L. Xia and V. Conitzer. Determining possible and necessary winners under common voting rules given partial orders. In *Proceedings of AAAI-08*, pages 196–201, July 2008.

[24] H. Young. Social choice scoring functions. *SIAM Journal on Applied Mathematics*, 28(4):824–838, 1975.