Neutrality and Geometry of Mean Voting

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[NS: ABSTRACT WOULD GO HERE]

Categories and Subject Descriptors: I.2.6 [**Computing Methodologies**]: Artificial Intelligence; J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*

General Terms: Algorithms, Economics

Additional Key Words and Phrases: Social choice, Mean proximity rules, Euclidean space, Neutrality

ACM Reference Format:
Nisarg Shah and Sébastien Lahaie, 2014. Neutrality and Geometry of Mean Voting. *ACM* X, X, Article X (February 2014), 19 pages.
DOI:http://dx.doi.org/10.1145/0000000.0000000

1. INTRODUCTION

Modern social choice literature has seen rising interest in studying families of voting rules for aspects like manipulability [**?**Xia 2013] and learnability [Procaccia et al. 2008, 2009]. Particularly, the framework of *generalied scoring rules* (GSRs) [**?**] has been a great success in this direction. More recently, Caragiannis et al. [2013] introduced the families of PM-c and PD-c rules as generalization of Condorcet consistenct rules and positional scoring rules, respectively. Their goal was to design rules that perform well in predicting an underlying ground truth from noisy data. These families of voting rules are extremely general, e.g., GSRs as well as the union of PM-c and PD-c rules capture almost all popular voting rules. However, this is a problem because not each rule in the family admits a theoretical justification, and some rules are arguably bad.

Another approach to social choice attempts to define families of voting rules where each rule in the family is justified. Two popular approaches are *distance rationalizability* [Elkind et al. 2009, 2010] and *maximum likelihood approach* [Conitzer and Sandholm 2005; Conitzer et al. 2009]. However, these approaches have been rather unfruitful because the definition of these families depend on the choices of a distance metric over permutations and a noise model for generating noisy estimates of a ground truth, respectively; these choices can be extremely general, thus both families end up capturing *almost all* voting rules.

The above discussion suggests the need for a family of voting rules that is not extremely general, and in which each rule has theoretical justification. The family of *mean proximity rules* introduced by Zwicker [2008b] is such a family; it is more restrictive than generalized scoring rules, distance rationalizable rules, and maximum likelihood estimators because it is a strict subset of all three of them (see Section **??**).

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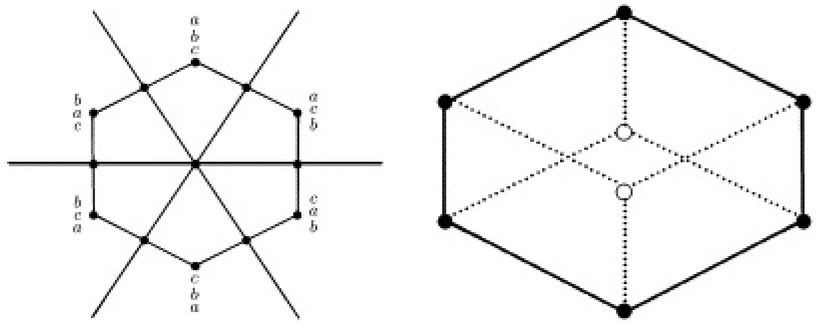
DOI:http://dx.doi.org/10.1145/0000000.0000000



(a) The Borda count for 3 alternatives (b) The Kemeny rule for 3 alternatives

Fig. 1: Geometric visualization of mean proximity rules

Additionally, each rule in the family has an intuitive theoretical justification: The goal of voting rules is to aggregate the input votes into an aggregate consensus. A plethora of voting rules have emerged due to the fact that there is no natural notion of *mean* of a set of rankings. Mean proximity rules embed rankings into a Euclidean space, and use the mean of the embeddings of the input votes to guide the consensus. When the mean does not represent embedding of any ranking, the ranking whose embedding is closest to the mean is chosen as the consensus. Different ways to embed the rankings into the Euclidean space gives rise to different mean proximity rules. In this sense, mean proximity rules use Euclidean spaces to achieve the mean representative of the input votes, which provides a theoretical justification for every mean proximity rule. Examples of mean proximity rules include the Kemeny rule and all positional scoring rules.

Mean proximity rules provide another important benefit — they are extremely easy to understand and sometimes even visualize. Consider, for example, the embedding given in Figure 1(a) that generates the Borda count for 3 alternatives. Each of the 6 rankings is embedded into a vertex of a regular hexagon. Each voter adds a unit weight to the vertex corresponding to his vote, and the ranking whose region contains the mean of all the votes is chosen as the output. For a simulation of this process using rubber bands, see http://omega.math.union.edu/research/2010-05-voting/. Similarly, Figure 1(b) shows the embedding generating the Kemeny rule for 3 alternatives. While the Kemeny rule is considered a black-box algebraic problem, this geometric visualization makes it extremely easy for a voter to understand how his vote affects the outcome of the rule. Saari [Saari 1995] argues that a clear understanding of voting rules is crucial for the rule to be accepted in the real-world, even more so for positional rules[1] that are arguably more complex than simple rules such as plurality or k-approval. He then makes a compelling case of using geometry in order to clarify and visualize voting rules. However, the geometry intuitions behind mean proximity rules are fundamentally different from those in Saari's work.

Despite the attractive outlook, mean proximity rules have received less attention in the literature. We tap the potential and uncover significant structures in mean proximity rules.

Our contributions. Mean proximity rules have two alternative representations, a geometric and an algebraic. We study mean proximity rules in a slightly limited con-

[1]By positional rules, we mean rules that pay attention to alternatives in later positions in the input votes as well, unlike plurality.

text of social welfare functions, and characterize the constraints the limited context imposes on both representations (Theorem 4.9). Next, we study the important axiom of neutrality in the framework of mean proximity rules. When introducing mean proximity rules, Zwicker [2008b] titled his paper "Consistency without neutrality in voting rules: When is a vote an average?". In this sense, our work fills the prime gap in the framework. We first give a characterization of constraints imposed by neutrality on both representations of mean proximity rules (Theorem 5.9). An important concept we introduce in this context is that of neutral embeddings.

Then, by drawing ideas from the representation theory of finite groups, we introduce *linear embeddings*, which are constructive in nature and therefore provide additional benefits over neutral embeddings. However, using a complicated algebraic proof, in Theorem 5.15 we establish that (almost) every neutral embedding is also a linear embedding, and therefore has the group theoretic structure that we can leverage. This leads to a constructive characterization of neutral mean proximity rules that has further implications such as characterization of mean proximity rules that admit low dimensional embeddings. We conclude by drawing connections between mean proximity rules and various lines of research in social choice literature, showing that mean proximity rules connect well with each line of research.

2. RELATED WORK

We build upon the work of Zwicker [2008b; 2008a] who introduced and studied mean proximity rules and their generalization called mean neat rules, and gave an axiomatic characterization of (rational) mean neat voting rules. The work of Cervone et al. [2012] builds upon mean proximity rules and utilizes another notion of geometric consensus known as mediancenter or the Fermat-Weber point. They use the intuition of a rubber band and each voter exerting force to bring the consensus close to himself which is similar to the intuition behind mean proximity rules.

The work of Saari [1995] uses a different geometric visualization where a profile is viewed in the space of $m!$ dimensions, each dimension encoding the number of times a particular ranking appears in a given profile. Daugherty et al. [2009] extend his work by using intricate results from group representation theory. Related lines of research include the work of Brams and Fishburn [2007] that argue in favor of approval voting for its simplicity and clarity, and the work of Crisman [Crisman] who connects the Borda count and the Kemeny rule using the geometry of permutahedron, the mean proximity embedding of the Borda count similar to Figure 1(a) but for 4 alternatives.

These lines of research must be separated from the models [] where the alternatives are embedded into a Euclidean space, and voters preferences are expressed as points in the same Euclidean space; the Euclidean space used in these models and the embeddings of the alternatives have specific meaning whereas the former lines of research use the Euclidean space merely as a tool to understand the voting rules.

3. PRELIMINARIES

Let A be the set of alternatives; we denote $|A| = m$. Let $\mathcal{L}(A)$ denote the set of rankings over the alternatives in A. A profile $\pi \in \mathcal{L}(A)^n$ is a collection of votes (rankings). A *voting rule* (more technically, a social welfare function - SWF)[2] is a function that maps every profile to a ranking, or a set of tied rankings. Formally, we denote a voting rule by $r : \mathcal{L}(A)^n \to \mathcal{P}(\mathcal{L}(A)) \setminus \{\emptyset\}$, where $\mathcal{P}(\cdot)$ denotes the power set. Note that a voting rule must output at least one ranking.

[2]Another common definition of a voting rule is a social choice function (SCF), which maps every profile to a (set of tied) winning alternative(s).

Axiomatic properties of voting rules. Voting rules are often studied (and sometimes designed) from the viewpoint of axiomatic properties. Such properties define what a voting rule should intuitively do in certain special cases. Some of the widely studied properties are defined below.

Anonymity. A voting rule r is called *anonymous* if its output only depends on the collection of rankings in the profile (equivalently, the number of times each ranking appears), and not on the identities of the voters who voted for the different rankings.

Unanimity. A voting rule r is said to satisfy *unanimity* if on every profile π that consists of copies of a single ranking σ, the rule uniquely outputs that ranking, i.e., $r(\pi) = \{\sigma\}$.

Consistency. A voting rule r is called *consistent* if for every profiles π_1 and π_2 such that $r(\pi_1) \cap r(\pi_2) \neq \emptyset$, we have $r(\pi_1 + \pi_2) = r(\pi_1) \cap r(\pi_2)$.[3] Here, profile $\pi_1 + \pi_2$ denotes the union of the profiles π_1 and π_2.

Neutrality. A voting rule is called *neutral* if the labels of the alternatives do not matter, i.e., relabeling the alternatives would relabel the output of the rule (each ranking in the set returned) accordingly. Without loss of generality, let us choose the labels of the alternatives to be $1, 2, \ldots, m$. Then, a relabeling of the alternative is equivalent to a permutation of 1 through m, i.e., an element of the symmetric group S_m of size $m!$. For a permutation $\tau \in S_m$ and a ranking $\sigma \in \mathcal{L}(A)$, the ranking $\tau\sigma$ is obtained by relabeling the alternatives according to τ in σ.
Given a profile $\pi = (\sigma_1, \ldots, \sigma_n)$ and a permutation τ of the alternatives, let $\tau\pi = (\tau\sigma_1, \ldots, \tau\sigma_n)$ be the profile where each vote is permuted according to τ. Similarly, given any set of rankings S, let $\tau S = \{\tau\sigma | \sigma \in S\}$. Thus, formally, a voting rule r is neutral if for every profile π and permutation τ of the alternatives, we have $r(\tau\pi) = \tau r(\pi)$.

Rank-distinguishability. In the classical definition of social welfare functions that output a single ranking, non-imposition (also known as citizen sovereignty) of a voting rule dictates that every ranking of the alternatives should be achievable as the output of the voting rule on some profile. We define a mild generalization of non-imposition, which we call rank-distinguishability, to social welfare functions that output a set of rankings. We say that a voting rule r satisfies rank-distinguishability if it can distinguish between any two rankings: for every two distinct rankings $\sigma, \sigma' \in \mathcal{L}(A)$, there must exist a profile π such that exactly one of σ and σ' belongs to $r(\pi)$. The mildness of rank-distinguishability is evident from the simple observation that it follows from unanimity, which is itself considered very unrestrictive and almost always desirable.

Voting rules. Next, we define two prominent voting rules that play a key role in this paper.

The Kemeny rule. Given a profile π, define the weighted pairwise majority (PM) graph of a profile as the graph where the alternatives are the vertices and there is an edge from every alternative a to every other alternative b with weight equal to the number of voters that prefer a to b. The Kemeny score of a ranking is the total

[3]Consistency for social welfare functions that output a set of tied rankings is more general than consistency for social welfare functions that output a single ranking (i.e., a singleton set). It is also significantly different from consistency of the winning alternative for social choice functions.

weight of the edges of the weighted pairwise majority graph of π in its direction. The Kemeny rule selects the ranking or the set of tied rankings with the highest Kemeny score.

Positional scoring rules. A positional scoring rule is given by a scoring vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ where $\alpha_i \geq \alpha_{i+1}$ for all $i \in [m]$ and $\alpha_1 > \alpha_m$. Under this rule, for each vote σ in π and $i \in [m]$, α_i points are awarded to the i^{th} most preferred alternative in σ. The alternatives are then sorted in the descending order of their total points. The set of (tied) rankings where each group of alternatives with identical total points is sorted in all possible ways is returned. Some example of positional scoring rules include plurality, Borda count, veto, and k-approval.

4. MEAN PROXIMITY RULES

We begin by presenting important background on mean proximity rules that will be relevant for the rest of the paper. First, we formally define mean proximity rules. Mean proximity rules are defined in a scope greater than the scope of voting rules considered in this paper: While voting rules map every profile to a set of tied rankings, mean proximity rules in general map every profile to a set of tied outcomes, where the outcomes are chosen from the outcome space $\mathcal{O}$. In Section 4.1, we will fix $\mathcal{O} = \mathcal{L}(A)$ to analyze mean proximity rules in the context of voting rules.

Definition 4.1 (*Mean Proximity Rules ([Zwicker 2008b])*). A *mean proximity rule* is given by an input embedding $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ mapping input rankings to k-dimensional Euclidean space and an output embedding $\psi : \mathcal{O} \to \mathbb{R}^k$ mapping possible outcomes to the same Euclidean space. The outcome of the rule on a profile π is given by $\arg\min_{o \in \mathcal{O}} \|\psi(o) - \mu(\pi)\|$, where $\mu(\pi) = (1/n) \cdot \sum_{\sigma \in \pi} \phi(\sigma)$ is the mean of input embeddings of the votes in π.

In this paper, every summation of the form $\sum_{\sigma \in \pi}$ iterates over rankings in profile π; in particular, if a ranking appears multiple times in π, the summation contains one term for each appearance. Zwicker [2008b] also defined another family of rules, which he referred to as *generalized scoring rules*. However, this name has since been used primarily to represent a different family of voting rules that was introduced by Xia and Conitzer [2008]. We therefore refer to the family of Zwicker [2008b] as *scoring rules*. The naming is appropriate for another reason: In Section **??**, we observe that the family of scoring rules introduced by Zwicker [2008b] is a subset of the family of generalized scoring rules introduced by Xia and Conitzer [2008].

Definition 4.2 (*Scoring Rules ([Zwicker 2008b])*). A voting rule is called a *scoring rule* if there exists a scoring function $s : \mathcal{L}(A) \times \mathcal{O} \to \mathbb{R}$ such that for any profile π, we have $r(\pi) = \arg\max_{o \in \mathcal{O}} \sum_{\sigma \in \pi} s(\sigma, o)$.

That is, the scoring rule assigns a score to each outcome for each input vote, and the outcome or outcomes with the highest aggregate score are chosen. Zwicker [2008b] also showed equivalence between mean proximity rules and scoring rules using the following technical result.

PROPOSITION 4.3 ([ZWICKER 2008B]). *For embeddings $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ and $\psi : \mathcal{O} \to \mathbb{R}^k$, and profile π, we have*

$$\arg\min_{o \in \mathcal{O}} \|\psi(o) - \mu(\pi)\| = \arg\min_{o \in \mathcal{O}} \sum_{\sigma \in \pi} \|\psi(o) - \phi(\sigma)\|^2. \quad (1)$$

From the definition of scoring rules and Proposition 4.3, it is clear that the scoring function $s(\sigma, o) = -\|\psi(o) - \phi(\sigma)\|^2$ represents the same mean proximity rule that has the embeddings (ψ, ϕ). Extending this to a two-way correspondence, Zwicker [2008b] showed the following.

PROPOSITION 4.4 ([ZWICKER 2008B, THEOREM 4.2.1]). *The families of mean proximity rules and scoring rules coincide.*

Due to this equivalence, a mean proximity rule has two equivalent representations: a pair of embeddings (ψ, ϕ), and a scoring function s; it might be the case that neither of the two is unique. We conclude the background on mean proximity rules by mentioning a result due to Zwicker [2008a].

PROPOSITION 4.5 ([ZWICKER 2008A]). *All positional scoring rules and the Kemeny rule*[4] *are mean proximity rules. A mean proximity rule is consistent, connected, continuous, and anonymous.*

Consistency of mean proximity rules is evident from Equation (1) because if two profiles have the same output under a mean proximity rule, the same set of outcomes minimize the sum on the right hand side of Equation (1) for both profiles, and therefore in their union. Connectedness and continuity are two natural properties of voting rules defined in [Zwicker 2008b]. For an alternative, but related, definition of continuity, the reader may refer to [Conitzer et al. 2009]. Conitzer and Sandholm [2005] showed that other rules such as Bucklin's rule, Copeland's method, the maximin rule, and the ranked pairs method are not consistent as social welfare functions. Hence, these rules are not mean proximity rules.

4.1. Symmetric Mean Proximity Rules

We now begin our investigation of mean proximity rules in the context of social welfare functions that output rankings over alternatives. Thus, from here onwards, we fix $\mathcal{O} = \mathcal{L}(A)$. This has the following effect on different equivalent representations of mean proximity rules.

(1) The embeddings $\phi, \psi : \mathcal{L}(A) \to \mathbb{R}^k$ both embed rankings to a Euclidean space.
(2) The scoring function $s : \mathcal{L}(A) \times \mathcal{L}(A) \to \mathbb{R}$ describes *similarity* between two rankings. This special case was also defined and studied independently by Conitzer et al. [2009], who referred to such rules as *simple ranking scoring functions* (SRSFs).
(3) Under a fixed enumeration $\sigma_1, \ldots, \sigma_{m!}$ of $\mathcal{L}(A)$, we can represent a scoring function by an $m! \times m!$ *score matrix* S such that $S_{ij} = s(\sigma_i, \sigma_j)$ for $i, j \in [m!] \triangleq \{1, \ldots, m!\}$.

Mean proximity rules, even as social welfare functions, is a broad family of voting rules that also captures rules violating very natural desiderata. For example, a mean proximity rule given by a scoring function s that satisfies $s(\sigma, \sigma') > s(\sigma, \sigma)$ for distinct $\sigma, \sigma' \in \mathcal{L}(A)$ would not output σ even on the profile where all the votes are σ. Such a rule would violate unanimity. To solve this problem, we propose a simple fix.

Definition 4.6 (*Symmetric Mean Proximity Rules (SMPRs)*). We say that a voting rule is a *symmetric mean proximity rule* (SMPR) if there exists a mean proximity representation of the rule with identical input and output embeddings, i.e., with $\psi = \phi$.

Since the outcome space for SWFs is identical to the space of input votes (the set of rankings over the alternatives), $\psi = \phi$ is a natural restriction. Note that all embeddings representing an SMPR may not satisfy $\psi = \phi$. We call the embeddings that sat-

[4]With the outcome space being the set of rankings over the alternatives.

isfy this restriction *symmetric embeddings*. From here onwards, we refer to an SMPR using its symmetric embedding ϕ.

In what follows, we assume that the SMPRs additionally satisfy rank-distinguishability. For SMPRs, rank-distinguishability is equivalent to the very intuitive restriction that ϕ must map all rankings to different points of the Euclidean space. To see this, note that if ϕ mapped all rankings to distinct points, then the rule would output $\{\sigma\}$ on a profile with a single vote σ. Thus, such a rule would achieve rank-distinguishability. Conversely, if ϕ mapped two distinct rankings σ and σ' to the same point in the Euclidean space, then it is easy to see that on every profile either both rankings would belong to the output of the rule or neither.

OBSERVATION 4.7. *An SMPR satisfies rank-distinguishability if and only if every symmetric embedding ϕ representing the rule maps all rankings to distinct points of the Euclidean space.*

Under this assumption, it is easy to check that in a profile π where all the votes are σ, we have $\mu(\pi) = \phi(\sigma)$. Thus, the output of every symmetric mean proximity rule on π would be the singleton set $\{\sigma\}$. Hence, we remark that symmetry does not only impose intuitive structure on mean proximity rules, but also helps achieve natural desiderata. Further, this has no cost in terms of expressiveness: The embeddings constructed in [Zwicker 2008b] for positional scoring rules and the Kemeny rule in fact satisfy the restriction $\psi = \phi$, showing that both these well-known mean proximity rules are symmetric.

4.2. Characterization of symmetric mean proximity rules

Recall that mean proximity rules are equivalent to scoring rules (Proposition 4.4). We imposed symmetry on the embeddings of the mean proximity rules. It is natural to ask: *What restriction does symmetry place on other equivalent representations of mean proximity rules?* For an embedding ϕ, define the scoring function s_ϕ such that $s_\phi(\sigma, \sigma') = -\|\phi(\sigma) - \phi(\sigma')\|^2$ for all $\sigma, \sigma' \in \mathcal{L}(A)$. Then Equation (1) implies that s_ϕ and ϕ are equivalent, i.e., they both represent the same SMPR. Further, the score matrix generated by s_ϕ has a well-known structure.

Definition 4.8 (*Euclidean Distance Matrix (EDM)*). A $p \times p$ matrix $A = (a_{ij})$ is called a ***Euclidean distance matrix (EDM)*** if there exist $v_1, \ldots, v_p \in \mathbb{R}^k$ such that $a_{ij} = \|v_i - v_j\|^2$ for all $i, j \in [p]$.

Hence, the score matrix of s_ϕ is negation of an EDM, and represents the given SMPR. Conversely, given any score matrix that is negation of an EDM, by definition we can find a symmetric embedding ϕ such that the score matrix is generated by the scoring function s_ϕ. Hence, the rule represented by the score matrix must be an SMPR. This yields the following characterization.

THEOREM 4.9. *A voting rule is a symmetric mean proximity rule if and only if it is a scoring rule that has a score matrix whose negation is a Euclidean distance matrix.*

Theorem 4.9 gives an algebraic condition on the score matrix of a scoring rule (alternatively, simple ranking scoring function according to [Conitzer et al. 2009]) that translates to the geometric interpretation of existence of a symmetric embedding. To summarize, we provided two motivations for adding symmetry to mean proximity rules: (1) taking identical input and output embeddings is very natural when the outcome space coincides with the space of input votes, and (2) symmetric mean proximity rules achieve additional desiderata of unanimity while still capturing all well-known mean proximity rules. Then, we characterized symmetric mean proximity rules in all three equivalent representations. In the next section, characterize addition of another highly desired property, *neutrality*. While neutrality is also extremely mild (all voting rules of interest are neutral), we show that it adds significant structure to mean proximity rules.

5. NEUTRALITY AND SYMMETRIC MEAN PROXIMITY RULES

Recall that neutrality of a voting rule states that the labels of the alternatives do not matter, i.e., relabeling alternatives relabels the output in the same fashion. In this section, we study the restrictions imposed by neutrality on the different equivalent representations of *symmetric* mean proximity rules. We connect neutrality of SMPRs with a notion of neutrality in the embedding, a similar notion of neutrality in the scoring function [Conitzer et al. 2009], and positive semidefiniteness of the score matrix. In Section 5.1, we give a constructive characterization by drawing ideas from group representation theory.

We begin by defining neutrality of scoring functions as in [Conitzer et al. 2009].

Definition 5.1 (*Neutral Scoring Function [Conitzer et al. 2009]*). A scoring function $s : \mathcal{L}(A) \times \mathcal{L}(A) \to \mathbb{R}$ is called *neutral* if $s(\tau\sigma, \tau\sigma') = s(\sigma, \sigma')$ for all rankings $\sigma, \sigma' \in \mathcal{L}(A)$ and permutations $\tau \in S_m$. We say that a score matrix is neutral if its underlying scoring function is neutral.

In words, a scoring function is neutral if similarity between two rankings given by the function does not change if both rankings are permuted in the same way. Conitzer et al. [Conitzer et al. 2009] showed that any scoring function s of a neutral mean proximity rule r can be *neutralized*, i.e., converted to an equivalent neutral scoring function.

PROPOSITION 5.2 ([CONITZER ET AL. 2009]). *Let s be a scoring function representing a mean proximity rule r that is neutral. Then, an equivalent neutral scoring function s^{NT} representing the same rule r can be defined as follows. For all rankings $\sigma, \sigma' \in \mathcal{L}(A)$ and permutations $\tau \in S_m$,*

$$s^{NT}(\sigma, \sigma') = \sum_{\tau \in S_m} s(\tau\sigma, \tau\sigma'). \tag{2}$$

Additionally, it is easy to check that by definition every neutral scoring function represents a neutral mean proximity rule. It immediately follows that:

PROPOSITION 5.3 ([CONITZER ET AL. 2009, LEMMA 2]). *A mean proximity rule is neutral if and only if there exists a neutral scoring function representing it.*

What does this mean for us? Proposition 5.3 characterizes the condition on the scoring function (and thus on the score matrix) that neutrality imposes on mean proximity rules. However, we are interested in addition of neutrality to *symmetric* mean proximity rules. Together with Theorem 4.9, Proposition 5.3 implies that a voting rule is a symmetric mean proximity rules if and only if it has a neutral score matrix and a score matrix that is negation of an EDM. *Is there one score matrix satisfying both conditions? What conditions does neutrality impose on the embedding of an SMPR?* To answer this, we first introduce a notion of neutrality for an embedding.

Definition 5.4 (*Neutral Embedding*). We say that an embedding $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ is *neutral* if for all rankings $\sigma, \sigma' \in \mathcal{L}(A)$ and permutations $\tau \in S_m$, we have $\|\phi(\sigma) - \phi(\sigma')\| = \|\phi(\tau\sigma) - \phi(\tau\sigma')\|$.

It can be checked that if the embedding ϕ is neutral, then its corresponding score function s_ϕ is also neutral. Further, similarly to scoring functions, an embedding can also be neutralized.

LEMMA 5.5. *Let $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ be an embedding of an SMPR r. Then, ϕ^{NT} defined in Equation* (3) *below is an equivalent neutral embedding of the same SMPR r.*

$$\phi^{NT}(\sigma) = [\phi(\tau_1\sigma)^T \phi(\tau_2\sigma)^T \dots \phi(\tau_{m!}\sigma)^T]^T, \tag{3}$$

where $\tau_1, \dots, \tau_{m!}$ is a fixed enumeration of the permutations in S_m. Further, the neutralization of ϕ and the neutralization of its scoring function s_ϕ are connected via the equation $s_{\phi^{NT}} = s_\phi^{NT}$, where the neutral scoring function s_ϕ^{NT} defined in Equation (2) *represents r.*

PROOF. First, we show that ϕ^{NT} is neutral. For all rankings $\sigma, \sigma' \in \mathcal{L}(A)$ and permutations $\tau \in S_m$,

$$\begin{aligned}\|\phi^{NT}(\tau\sigma) - \phi^{NT}(\tau\sigma')\|^2 &= \sum_{\tau' \in S_m} \|\phi(\tau'\tau\sigma) - \phi(\tau'\tau\sigma)\|^2 \\ &= \sum_{\tau' \in S_m} \|\phi(\tau'\sigma) - \phi(\tau'\sigma)\|^2 = \|\phi^{NT}(\sigma) - \phi^{NT}(\sigma)\|^2,\end{aligned}$$

where the second equality holds because for $\tau \in S_m$, we have $\{\tau'\tau | \tau' \in S_m\} = S_m$, which is a property of any group. Hence, ϕ^{NT} is neutral. Next, for all rankings $\sigma, \sigma' \in \mathcal{L}(A)$,

$$\begin{aligned}s_{\phi^{NT}}(\sigma, \sigma') = -\|\phi^{NT}(\sigma) - \phi^{NT}(\sigma')\|^2 &= -\sum_{\tau \in \mathcal{L}(A)} \|\phi(\tau\sigma) - \phi(\tau\sigma')\|^2 \\ &= \sum_{\tau \in \mathcal{L}(A)} s_\phi(\tau\sigma, \tau\sigma') = s_\phi^{NT}(\sigma, \sigma').\end{aligned}$$

Thus, we have $s_{\phi^{NT}} = s_\phi^{NT}$. The scoring function s_ϕ is equivalent to the embedding ϕ, and thus also represents the rule r. From Proposition 5.2, the scoring function $s_\phi^{NT} = s_{\phi^{NT}}$ also represents r. Hence, the embedding ϕ^{NT} also represents r. □

We remark that the neutralized embedding ϕ^{NT} is not very satisfactory because it maps rankings to a Euclidean space with dimension $m!$ times the dimension used by ϕ. However, it has significant structure. For example, in addition to neutrality we also have

$$\|\phi^{NT}(\sigma)\|^2 = \sum_{\tau \in S_m} \|\phi(\tau\sigma)\|^2 = \sum_{\sigma' \in \mathcal{L}(A)} \|\phi(\sigma')\|^2.$$

Hence, $\|\phi^{NT}(\sigma)\|$ is independent of σ. That is, ϕ^{NT} is an *equal norm embedding*.

Definition 5.6 (*Equal Norm Embedding*). We say that an embedding $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ has *equal norm* if $\|\phi(\sigma)\| = \|\phi(\sigma')\|$ for all rankings $\sigma, \sigma' \in \mathcal{L}(A)$.

For equal norm embeddings, squares of Euclidean distances in Proposition 4.3 can be replaced by inner products as follows.

LEMMA 5.7. *For an equal norm embedding ϕ of an SMPR r, the scoring function s given by $s(\sigma, \sigma') = \langle \phi(\sigma), \phi(\sigma') \rangle$ for all rankings $\sigma, \sigma' \in \mathcal{L}(A)$ represents r.*

PROOF. Let the equal norm embedding ϕ have $\|\phi(\sigma)\| = c$ for all $\sigma \in \mathcal{L}(A)$. From Proposition 4.3, we know that on a profile π,

$$
\begin{aligned}
r(\pi) &= \operatorname*{arg\,min}_{\sigma\in\mathcal{L}(A)} \sum_{\sigma'\in\pi} \|\phi(\sigma)-\phi(\sigma')\|^2 \\
&= \operatorname*{arg\,min}_{\sigma\in\mathcal{L}(A)} \sum_{\sigma'\in\pi} \left(c^2 + c^2 - 2\langle\phi(\sigma),\phi(\sigma')\rangle\right) \\
&= \operatorname*{arg\,min}_{\sigma\in\mathcal{L}(A)} 2nc^2 - 2\sum_{\sigma'\in\pi} \langle\phi(\sigma),\phi(\sigma')\rangle = \operatorname*{arg\,max}_{\sigma\in\mathcal{L}(A)} \sum_{\sigma'\in\pi} \langle\phi(\sigma),\phi(\sigma')\rangle.
\end{aligned}
$$

Hence, by definition s is a scoring function representing r. □

The score matrix generated by the inner product scoring function of Lemma 5.7 has a well-known structure.

Definition 5.8 (*Gramian Matrix*). A $p\times p$ matrix $A = (a_{ij})$ is called *Gramian* if there exist vectors $v_1,\ldots,v_p \in \mathbb{R}^k$ such that $a_{ij} = \langle v_i, v_j\rangle$ for all $i,j \in [k]$. It is well-known that a matrix is Gramian if and only if it is positive semidefinite.

With this machinery, we are ready to answer the questions we posed regarding conditions on various representations of an SMPR imposed by neutrality. We present the answer in the form of the following characterization.

THEOREM 5.9. *For a mean proximity rule r, the following are equivalent.*

(*1*) *r is neutral and symmetric.*
(*2*) *There exists a neutral symmetric embedding representing r.*
(*3*) *There exists a score matrix representing r which is neutral and negation of an EDM.*
(*4*) *There exists a score matrix representing r which is neutral, positive semidefinite, and has equal diagonal entries.*

PROOF. It is easy to show that the second and the third conditions imply the first condition. If r has a score matrix which is neutral and negation of an EDM, then by Proposition 5.3 and Theorem 4.9, r is a neutral SMPR. Similarly, if r is a symmetric mean proximity rule with a neutral embedding ϕ, then for a profile π, we have

$$
\begin{aligned}
r(\tau\pi) &= \operatorname*{arg\,min}_{\sigma\in\mathcal{L}(A)} \sum_{\sigma'\in\pi} \|\phi(\sigma)-\phi(\tau\sigma')\|^2 \\
&= \tau \ \operatorname*{arg\,min}_{\sigma\in\mathcal{L}(A)} \sum_{\sigma'\in\pi} \|\phi(\tau\sigma)-\phi(\tau\sigma')\|^2 \\
&= \tau \ \operatorname*{arg\,min}_{\sigma\in\mathcal{L}(A)} \sum_{\sigma'\in\pi} \|\phi(\sigma)-\phi(\sigma')\|^2 = \tau r(\pi),
\end{aligned}
$$

where the first transition follows from Proposition 4.3, and the third transition follows from neutrality of ϕ. Thus, r is a neutral SMPR.

Conversely, let r be a neutral SMPR and ϕ be its embedding. We proved in Lemma 5.5 that $s_\phi^{NT} = s_{\phi^{NT}} = s$ (say). Then, $s = s_\phi^{NT}$ implies that the score matrix corresponding to the scoring function s is neutral. Further, $s = s_{\phi^{NT}}$ implies that the score matrix is also negation of an EDM. Hence, the score matrix corresponding to the scoring function s is both neutral and negation of an EDM, implying the third condition. Further, the equivalent embedding ϕ^{NT} is neutral and represents r, implying the second condition. Thus, we have shown that the first three conditions are equivalent.

For equivalence with the fourth condition, take a neutral SMPR r and its embedding ϕ. We saw that the embedding ϕ^{NT} has equal norm, and hence the score matrix S corresponding to its scoring function s given by $s(\sigma, \sigma') = \langle \phi^{NT}(\sigma), \phi^{NT}(\sigma') \rangle$ for all $\sigma, \sigma' \in \mathcal{L}(A)$ is a Gramian score matrix representing r (Lemma 5.7). Since S is Gramian, it is also positive semidefinite. Equal norm property of ϕ^{NT} implies that S has equal diagonal entries. Finally, neutrality of ϕ^{NT} implies neutrality of S. Hence, S satisfies the requirements of the fourth condition.

Conversely, take a mean proximity rule r with a score matrix S that is neutral, positive semidefinite, and has equal diagonal entries. Then, we can find an embedding ϕ such that S is its Gramian matrix. Neutrality and equal diagonal entries of S imply neutrality and equal norm property of ϕ. The latter implies that ϕ also represents r (Lemma 5.7). Hence, r must be a neutral SMPR. □

Theorem 5.9 translates neutrality of an SMPR to neutrality of its embedding. While it is straightforward to check if a given embedding is neutral, generating neutral embeddings is a non-trivial task. This limits the constructiveness of the characterization. Next, we improve the characterization by showing that neutrality of an embedding is equivalent to a more elegant structure that gives an easy way to generate neutral embeddings.

5.1. Linear Embeddings

Group representation theory has played a key role in an extremely diverse set of fields that includes but is not limited to[5] coding theory [MacWilliams and Sloane 1977], quantum mechanics [Barut and Raczka 1986], and crystallography (in chemistry) [Kovalev et al. 1993]. Linear representation of a group essentially maps every group element to a linear transformation of a vector space (or from a vector space to another vector space), i.e., to an element of the general linear group on the vector space. When a basis is chosen for the vector space (or vector spaces), the linear transformation can be represented via a matrix. This is known as matrix representation of a group.

In our case, we use representation theory of the symmetric group, which has gained much attention and for which many interesting structures have been discovered leveraging the extreme symmetry of the group [James et al. 1984]. We fix the standard Euclidean basis for simplicity. Now, we are ready to introduce *linear embeddings* using ideas from linear representations of the symmetric group.

Definition 5.10 (*Linear Embeddings*). We say that an embedding $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ is *linear* if there exists a representation function $R : S_m \to \mathbb{R}^{k \times k}$, called the representation of ϕ, mapping each permutation to a $k \times k$ real matrix such that

(1) the identity permutation τ_e is mapped to the identity matrix, i.e., $R(\tau_e) = I_k$,
(2) R respects the multiplication operator of the symmetric group; $R(\tau_1 \tau_2) = R(\tau_1) R(\tau_2)$ for all permutations $\tau_1, \tau_2 \in S_m$ (where $\tau_1 \tau_2$ represents the multiplication of τ_1 and τ_2 within S_m),
(3) R maps permutations to orthogonal matrices, i.e., $R(\tau)^T = R(\tau)^{-1}$ for all permutations $\tau \in S_m$, and
(4) permuting a ranking is equivalent to rotating—because R uses orthogonal matrices—its embedding appropriately; $\phi(\tau\sigma) = R(\tau)\phi(\sigma)$ for all rankings $\sigma \in \mathcal{L}(A)$ and permutations $\tau \in S_m$.

It can be seen very easily that the first and the second conditions together imply that $R_{\tau^{-1}} = R(\tau)^{-1}$ for all permutations $\tau \in S_m$, where τ^{-1} is the inverse of τ in

[5] See http://wdjoyner.com/repn_thry_appl.html for a wider range of applications.

S_m. Further, condition (3) is completely unrestrictive: A striking result from the group representation theory states that for a representation R of any finite group (not just the symmetric group), there exists a matrix P such that the equivalent representation R' defined by $R'(g) = PR(g)P^{-1}$ for each group element g uses orthogonal matrices (see, e.g., [Boerner et al. 1963, Theorem 6.3]). That is, any representation could be converted to a representation that uses orthogonal matrices. In what follows, we use R_τ instead of $R(\tau)$ for notational convenience.

Before we dive into the equivalence between neutrality and linearity, we build its pillars by presenting a few results about linear embeddings.

LEMMA 5.11. *For a linear embedding ϕ, rankings $\sigma, \sigma' \in \mathcal{L}(A)$, and permutation $\tau \in S_m$,*

$$\langle \phi(\tau\sigma), \phi(\tau\sigma') \rangle = \langle \phi(\sigma), \phi(\sigma') \rangle.$$

PROOF. Let R be the representation of ϕ. Then, we have

$$\langle \phi(\tau\sigma), \phi(\tau\sigma') \rangle = \langle R_\tau \phi(\sigma), R_\tau \phi(\sigma') \rangle = \langle \phi(\sigma), R_\tau^T R_\tau \phi(\sigma') \rangle = \langle \phi(\sigma), \phi(\sigma') \rangle, \tag{4}$$

where, the last transition holds because R_τ is an orthogonal matrix by definition. □

Taking $\sigma = \sigma'$ in Lemma 5.11, we get the following corollary.

COROLLARY 5.12. *Every linear embedding has equal norm.*

While every linear embedding has equal norm, every neutral embedding may not have equal norm. However, the next result shows that every neutral embedding has a similar equal distance property.

LEMMA 5.13. *Let ϕ be a neutral embedding, and $\phi_{avg} = (1/m!) \cdot \sum_{\sigma \in \mathcal{L}(A)} \phi(\sigma)$ denote the mean of embeddings of all rankings under ϕ. Then, $\|\phi(\sigma) - \phi_{avg}\|$ is a constant independent of the ranking σ.*

PROOF. Let r be the mean proximity rule represented by ϕ. By Theorem 5.9, r must be neutral. Let π_{symm} denote the profile containing each ranking from $\mathcal{L}(A)$ exactly once. Let $r(\pi_{symm}) = T \subseteq \mathcal{L}(A)$, and suppose that $T \neq \mathcal{L}(A)$. Thus, there must exist a ranking $\sigma' \notin T$. Further, by definition of a voting rule, $T \neq \emptyset$. Thus, there must also exist a ranking $\sigma \in T$. Now, take $\tau \in S_m$ to be the permutation that converts σ to σ'.

It is easy to see that $\tau\pi_{symm} = \pi_{symm}$. Hence, $r(\tau\pi_{symm}) = r(\pi_{symm}) = T$. Thus, $\sigma' \notin r(\tau\pi_{symm})$. However, $\sigma' \in \tau T = \tau r(\pi_{symm})$. Thus, $\tau r(\pi_{symm}) \neq r(\tau\pi_{symm})$, which is a contradiction because r is neutral. Hence, we must have $r(\pi_{symm}) = \mathcal{L}(A)$. Observing that $\mu(\pi_{symm}) = \phi_{avg}$ and $r(\pi_{symm}) = \arg\min_{\sigma \in \mathcal{L}(A)} \|\phi(\sigma) - \phi_{avg}\|$ yields the required result. □

Given Lemma 5.13, we can easily derive an equal norm embedding given a neutral embedding: Given a neutral embedding ϕ, we say that the embedding $\hat{\phi}$ defined by $\hat{\phi}(\sigma) = \phi(\sigma) - \phi_{avg}$ for all $\sigma \in \mathcal{L}(A)$ is the *normalization* of ϕ. Note that $\hat{\phi}$ is simply a translation applied to ϕ, which is distance-preserving. Hence, $\hat{\phi}$ represents the same neutral SMPR as ϕ does. This normalization achieves the equal norm property (Lemma 5.13), and also preserves neutrality and linearity of the original embedding.

LEMMA 5.14. *An embedding is neutral (resp. linear) if and only if its normalization is neutral (resp. linear).*

PROOF. If ϕ is neutral, then for all $\sigma, \sigma' \in \mathcal{L}(A)$ and $\tau \in S_m$, we have

$$\|\hat{\phi}(\tau\sigma) - \hat{\phi}(\tau\sigma')\| = \|\phi(\tau\sigma) - \phi(\tau\sigma')\| = \|\phi(\sigma) - \phi(\sigma')\| = \|\hat{\phi}(\sigma) - \hat{\phi}(\sigma')\|.$$

Hence, $\hat{\phi}$ is neutral too. The proof of the converse works in the same way by observing that $\phi(\sigma) = \hat{\phi}(\sigma) + \phi_{avg}$ for all $\sigma \in \mathcal{L}(A)$.

Similarly, if ϕ is linear with representation R, then for all $\sigma \in \mathcal{L}(A)$ and $\tau \in S_m$, we have $\phi(\tau\sigma) = R_\tau \phi(\sigma)$. Averaging over all $\sigma \in \mathcal{L}(A)$, we get

$$\frac{1}{m!} \sum_{\sigma \in \mathcal{L}(A)} \phi(\tau\sigma) = \frac{1}{m!} \sum_{\sigma \in \mathcal{L}(A)} R_\tau \phi(\sigma) = R_\tau \left(\frac{1}{m!} \sum_{\sigma \in \mathcal{L}(A)} \phi(\sigma) \right).$$

However,

$$\frac{1}{m!} \sum_{\sigma \in \mathcal{L}(A)} \phi(\tau\sigma) = \frac{1}{m!} \sum_{\sigma \in \mathcal{L}(A)} \phi(\sigma) = \phi_{avg}.$$

Hence, we have $R_\tau \phi_{avg} = \phi_{avg}$ for all $\tau \in S_m$. Now, for all $\sigma \in \mathcal{L}(A)$ and $\tau \in S_m$,

$$\hat{\phi}(\tau\sigma) = \phi(\tau\sigma) - \phi_{avg} = R_\tau \phi(\sigma) - R_\tau \phi_{avg} = R_\tau \hat{\phi}(\sigma).$$

Thus, $\hat{\phi}$ is also linear with the same representation R. Again, the proof of the converse works by observing that $\phi(\sigma) = \hat{\phi}(\sigma) + \phi_{avg}$ for all $\sigma \in \mathcal{L}(A)$. $\square$

Lemma 5.14 shows that the normalization of a neutral embedding is also neutral, in addition to having the equal norm property. We now show that this normalization is in fact linear — but this requires a much deeper proof.

THEOREM 5.15. *An embedding is neutral if and only if its normalization is linear.*

PROOF. We first show that a linear embedding is also a neutral embedding. Let ϕ be a linear embedding. Then for all $\sigma, \sigma' \in \mathcal{L}(A)$ and $\tau \in S_m$, we have

$$\begin{aligned} \|\phi(\tau\sigma) - \phi(\tau\sigma')\|^2 &= \|\phi(\tau\sigma)\|^2 + \|\phi(\tau\sigma')\|^2 - 2 \cdot \langle \phi(\tau\sigma), \phi(\tau\sigma') \rangle \\ &= \|\phi(\sigma)\|^2 + \|\phi(\sigma')\|^2 - 2 \cdot \langle \phi(\sigma), \phi(\sigma') \rangle = \|\phi(\sigma) - \phi(\sigma')\|^2, \end{aligned}$$

where the second transition holds due to Lemma 5.11 and its Corollary 5.12. Thus, ϕ is also neutral.

Now, for our main result, let ϕ be an embedding whose normalization $\hat{\phi}$ is linear. Then, $\hat{\phi}$ is also neutral as shown above. Due to Lemma 5.14, this implies that ϕ is also neutral.

The converse is much trickier, therefore we provide the high-level overview of the proof. Take a neutral embedding $\phi : \mathcal{L}(A) \to \mathbb{R}^k$, and let r be the rule it represents. We want to show that $\hat{\phi}$ is linear. By Corollary 5.12 and Lemma 5.14, we know that $\hat{\phi}$ is a (k-dimensional) equal norm neutral embedding. We first construct an equivalent embedding ϕ^f as follows. Let $k' \le k$ be the dimension of the affine subspace of $\mathbb{R}^k$ spanned by the points $\{\hat{\phi}(\sigma)\}_{\sigma \in \mathcal{L}(A)}$. Then, we obtain ϕ^f by taking a distance-preserving mapping to $\mathbb{R}^{k'}$. Next, we show that ϕ^f is linear by explicitly constructing its representation using Moore-Penrose pseudoinverses. Finally, we use this to construct an explicit representation for $\hat{\phi}$ itself.

Now we provide the formal proof. First, it is easy to check that $\hat{\phi}_{avg} = 0$. Hence, the affine space spanned by the embeddings under $\hat{\phi}$ form a linear subspace of $\mathbb{R}^k$ of dimension k'. Take an orthonormal basis $v_1, \ldots, v_{k'}$ of this linear subspace, write $\hat{\phi}(\sigma)$ as a linear combination of the k' basis vectors, and let $\phi^f(\sigma)$ encode the coefficients of the linear combination. Equivalently, if P is the $k' \times k$ matrix with $v_1, \ldots, v_{k'}$ as its rows, then $\phi^f(\sigma) = P\phi(\sigma)$ for all $\sigma \in \mathcal{L}(A)$. It is now standard to show that the transformation

P preserves all distances. This has two implications: First, ϕ^f represents the same neutral SMPR r as $\hat{\phi}$ and ϕ do. Second, ϕ^f is also an equal norm neutral embedding like $\hat{\phi}$. These hold because the definitions of mean proximity rules, equal norm property, and neutrality of embeddings only use distances between pairs of embeddings.

By construction, ϕ^f is of *full dimension*, i.e., the points where the rankings are mapped span the whole of $\mathbb{R}^{k'}$.[6] Next, consider the matrix $A = [\phi^f(\sigma_1), \ldots, \phi^f(\sigma_{m!})]$, the $k' \times m!$ matrix whose columns are the embeddings under ϕ^f. Here, $\sigma_1, \ldots, \sigma_{m!}$ is an enumeration of $\mathcal{L}(A)$. This implies that A must have full row rank k'. Fix a permutation $\tau \in S_m$, and consider $B = [\phi^f(\tau\sigma_1), \ldots, \phi^f(\tau\sigma_{m!})]$. Then for all $i, j \in [m!]$,

$$(B^T B)_{ij} = \langle \phi^f(\tau\sigma_i), \phi^f(\tau\sigma_j) \rangle = \langle \phi^f(\sigma_i), \phi^f(\sigma_j) \rangle = (A^T A)_{ij}$$

where the second transition follows because ϕ^f is neutral. Thus, $A^T A = B^T B$.

We now show that there exists a matrix R_τ such that $\phi^f(\tau\sigma) = R_\tau \phi^f(\sigma)$ for all $\sigma \in \mathcal{L}(A)$. Combining the equations for all $\sigma \in \mathcal{L}(A)$, it is equivalent to show the existence of an R_τ such that $R_\tau A = B$, i.e., existence of a solution X of the linear system $A^T X = B^T$. It is well-known that the system of equations $Ax = b$ has a solution if and only if $AA^+ b = b$, where A^+ is the Moore-Penrose pseudoinverse of A (see, e.g., [Barata and Hussein 2012]). Trivially extending this to multiple systems of linear equations, we can see that the necessary and sufficient condition for existence of the required R_τ is $A^T (A^T)^+ B^T = B^T$, or equivalently, $BA^+ A = B$. The last derivation uses the fact that $(A^T)^+ = (A^+)^T$. Now,

$$\begin{aligned} B &= BB^+ B = B\left((B^T B)^+ B^T\right) B = B(B^T B)^+ \left(B^T B\right) \\ &= B(A^T A)^+ \left(A^T A\right) = B\left((A^T A)^+ A^T\right) A = BA^+ A. \end{aligned}$$

Refer [Barata and Hussein 2012] for the identities $X = XX^+ X$ (used in the first transition) and $X^+ = (X^T X)^+ X^T$ (used in the second and the fifth transitions) regarding Moore-Penrose pseudoinverses. The fourth transition holds because $A^T A = B^T B$. Hence, we have shown that for every permutation $\tau \in S_m$, there exists a matrix R_τ such that $\phi^f(\tau\sigma) = R_\tau \phi^f(\sigma)$ for all $\sigma \in \mathcal{L}(A)$. Further, one solution of $Ax = b$ is $x = A^+ b$. Extending this, we can see that one solution of $A^T X = B^T$ is $X = (A^T)^+ B^T$. Hence, $R_\tau = X^T = BA^+$. We choose this particular solution for every $\tau \in S_m$ and show that it satisfies the conditions of linearity.

Recall that the rank of the product of two matrices is at most the minimum of the rank of the two matrices, and $R_\tau A = B$. Also, both A and B are rank k' matrices. Hence, $rank(R_\tau) \geq k'$. However, R_τ is a $k' \times k'$ matrix. Hence, we conclude that R_τ is invertible for every $\tau \in S_m$. We now show that $R_{\tau_1\tau_2} = R_{\tau_1} R_{\tau_2}$ for all $\tau_1, \tau_2 \in S_m$.

Fix $\tau_1, \tau_2 \in S_m$. For every $\sigma \in \mathcal{L}(A)$, we have $\phi^f(\tau_1\tau_2\sigma) = R_{\tau_1\tau_2}\phi^f(\sigma)$. Also, $\phi^f(\tau_1\tau_2\sigma) = R_{\tau_1}\phi^f(\tau_2\sigma) = R_{\tau_1} R_{\tau_2}\phi^f(\sigma)$. Thus, we have that $R_{\tau_1\tau_2}\phi^f(\sigma) = R_{\tau_1} R_{\tau_2}\phi^f(\sigma)$ for all $\sigma \in \mathcal{L}(A)$. It is easy to show that this implies $R_{\tau_1\tau_2} = R_{\tau_1} R_{\tau_2}$. To see this, consider the submatrix of A—let us call it A_p—by taking k' linearly independent columns of A. Then, A_p is an invertible square matrix. Also, $R_{\tau_1\tau_2} A_p = R_{\tau_1} R_{\tau_2} A_p$. Multiplying by A_p^{-1} on both sides, we get the desired result. To complete the proof, we need to show that R_τ is an orthogonal matrix for all $\tau \in S_m$, i.e., $R_\tau^{-1} = R_\tau^T$. Note that

$$R_\tau^T B = (A^+)^T B^T B = (A^+)^T A^T A = (AA^+)^T A = IA = A,$$

where the second transition holds because $B^T B = A^T A$, and the fourth transition holds because $AA^+ = I$ for any full row-rank matrix A (see, e.g., [Barata and Hussein 2012]). Thus, we have that $R_\tau^T B = A$. Applying R_τ on both sides, we get $R_\tau R_\tau^T B =$

[6]While it is not relevant for the proof, we remark that the fact that the $m!$ points span $\mathbb{R}^{k'}$ implies $k' \leq m!$.

$R_\tau A = B$. Similarly to A_p, we can construct an invertible submatrix B_p of B, and get $R_\tau R_\tau^T B_p = B_p$, which, upon right-multiplication by B_p^{-1}, yields the required equation $R_\tau^{-1} = R_\tau^T$. Thus, ϕ^f is a linear embedding. We now show that $\hat{\phi}$ is also linear.

Consider the orthonormal basis $v_1, \ldots, v_{k'}$ that was used for constructing ϕ^f from $\hat{\phi}$, and the corresponding matrix P with v_i's as its rows. Complete this to a basis $v_1, \ldots, v_k$ of $\mathbb{R}^k$, and consider the matrix Q that has all the v_i's as its rows. Then, $Q\hat{\phi}(\sigma) = [\phi^f(\sigma)^T, 0, \ldots, 0]^T$, where the number of zeros is $k - k'$. For a permutation $\tau \in S_m$, construct the $k \times k$ matrix

$$R'_\tau = Q^T \begin{bmatrix} R_\tau & 0 \\ 0 & I \end{bmatrix} Q.$$

Here, R_τ is a $k' \times k'$ matrix and I is the $(k - k') \times (k - k')$ identity matrix. Note that

$$R'_\tau \hat{\phi}(\sigma) = Q^T \begin{bmatrix} R_\tau & 0 \\ 0 & I \end{bmatrix} Q\hat{\phi}(\sigma) = Q^T \begin{bmatrix} R_\tau & 0 \\ 0 & I \end{bmatrix} [\phi^f(\sigma) 0 \ldots 0]^T = Q^T [\phi^f(\tau\sigma)^T, 0, \ldots, 0]^T = \hat{\phi}(\tau\sigma).$$

Thus, $R'_\tau \hat{\phi}(\sigma) = \hat{\phi}(\tau\sigma)$ for all $\tau \in S_m$ and $\sigma \in \mathcal{L}(A)$. Further,

$$\begin{aligned} R'_{\tau_1} R'_{\tau_2} &= Q^T \begin{bmatrix} R_{\tau_1} & 0 \\ 0 & I \end{bmatrix} QQ^T \begin{bmatrix} R_{\tau_2} & 0 \\ 0 & I \end{bmatrix} Q = Q^T \begin{bmatrix} R_{\tau_1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} R_{\tau_2} & 0 \\ 0 & I \end{bmatrix} Q \\ &= Q^T \begin{bmatrix} R_{\tau_1} R_{\tau_2} & 0 \\ 0 & I \end{bmatrix} Q = Q^T \begin{bmatrix} R_{\tau_1 \tau_2} & 0 \\ 0 & I \end{bmatrix} Q = R'_{\tau_1 \tau_2}. \end{aligned}$$

Here, the second transition follows since QQ^T is identity matrix as rows of Q form an orthonormal basis. Because R'_τ is product of three invertible matrices, R'_τ itself is invertible. Further,

$$(R'_\tau)^T = Q^T \begin{bmatrix} R_\tau & 0 \\ 0 & I \end{bmatrix}^T Q = Q^T \begin{bmatrix} R_\tau^T & 0 \\ 0 & I \end{bmatrix} Q = Q^T \begin{bmatrix} R_{\tau^{-1}} & 0 \\ 0 & I \end{bmatrix} Q = R'_{\tau^{-1}}.$$

Thus, R'_τ is also an orthogonal matrix. This completes the proof that $\hat{\phi}$ is a linear embedding and R' is its representation. □

Theorem 5.9 showed that an SMPR is neutral if and only if it has a symmetric embedding that is neutral. The latter happens if and only if the rule also has a symmetric embedding that is normalization of a neutral embedding. Together with Theorem 5.15, this provides another characterization of neutral SMPR.

THEOREM 5.16. *A symmetric mean proximity rule is neutral if and only if there is a linear embedding representing it.*

Theorem 5.16 should be viewed as a fifth equivalent condition of Theorem 5.9. Next, we show that Theorem 5.16 could also be derived using a much simpler proof, but later we describe the advantages of our equivalence result between neutrality and linearity over the alternative proof.

LEMMA 5.17. *For an embedding ϕ of a neutral SMPR r, its neutralization ϕ^{NT} given in Equation* (3) *is a linear embedding.*

PROOF. Let ϕ be a k-dimensional embedding. Hence, ϕ^{NT} has dimension $m! \cdot k$. Observe that for all $\sigma, \sigma' \in \mathcal{L}(A)$, the coordinates of $\phi^{NT}(\sigma)$ and $\phi^{NT}(\sigma')$ are simply permutations of each other. First, we show that there exists a representation R such that for all $\sigma \in \mathcal{L}(A)$ and $\tau \in S_m$, we have $\phi^{NT}(\tau\sigma) = R_\tau \phi^{NT}(\sigma)$. Note that

$$\phi^{NT}(\tau\sigma) = [\phi(\tau_1\tau\sigma)^T \phi(\tau_2\tau\sigma)^T \ldots \phi(\tau_{m!}\tau\sigma)^T]^T.$$

Let Π_τ be the $m! \times m!$ matrix such that $\Pi_{ij} = 1$ if and only if $\tau_j = \tau_i\tau$. It is easy to verify that Π_τ is a permutation matrix. Further, if the blocks ($\phi(\tau_i\tau\sigma)$ for $i \in [m!]$) of

$\phi^{NT}(\sigma)$ were permuted according to Π_τ, then the i^{th} block in the resulting vector would be $\phi(\tau_j\sigma)$, where $\tau_j = \tau_i\tau$, the i^{th} block of $\phi^{NT}(\tau\sigma)$.

Hence, applying Π_τ to the blocks of $\phi^{NT}(\sigma)$ results in $\phi^{NT}(\tau\sigma)$. Now construct an $(m! \cdot k) \times (m! \cdot k)$ matrix R_τ by replacing every 1 in Π_τ by a $k \times k$ identity matrix, and every 0 in Π_τ by a $k \times k$ zero matrix. Then, it is easy to see that $R_\tau\phi^{NT}(\sigma) = \phi^{NT}(\tau\sigma)$. Finally, note that each R_τ is a permutation matrix originating from the permutation matrix Π_τ of the permutation τ. Also, we have $\Pi_{\tau_1\tau_2} = \Pi_{\tau_1}\Pi_{\tau_2}$ for all $\tau_1, \tau_2 \in S_m$. Hence, it can be verified that $R_{\tau_1\tau_2} = R_{\tau_1}R_{\tau_2}$ for all $\tau_1, \tau_2 \in S_m$, and every R_τ for $\tau \in S_m$ is an orthogonal matrix. $\square$

Theorem 5.9 and Lemma 5.17 together provide another proof of Theorem 5.16. However, our original proof that showed a stronger connection (equivalence) between normalization of neutral embeddings and linear embeddings bears certain advantages described below.

The proof of Theorem 5.15 is constructive, meaning that one can use it to explicitly find the representation of the normalization of any given neutral embedding of an SMPR. The multiplicative property $R_{\tau_1\tau_2} = R_{\tau_1}R_{\tau_2}$ implies that one only needs to store the representation matrices corresponding to a generating set of S_m, and can generate the rest of the matrices by appropriate multiplications and inverses.[7]

It is well-known that the symmetric group S_m has a generating set of size two, irrespective of the value of m: the set containing a cycle which shifts the labels of the alternatives in a cyclic fashion, and a transposition which exchanges the labels of two alternatives adjacent in the previous cycle. The matrices corresponding to these two elements along with the value of $\phi(\sigma)$ for a single $\sigma \in \mathcal{L}(A)$ is enough to generate $\phi(\sigma)$ for all $\sigma \in \mathcal{L}(A)$. Hence, if the dimension of ϕ is k, then the corresponding SMPR can be expressed in terms of $2k^2 + k$ paramters, which is the *first* representation to the best of our knowledge that is not direclty exponential in m.[8]

This succint representation has many advantages. First, it reduces the storage space required to encode a neutral SMPR. Second, we believe that this succint representation would also inform the design of efficient algorithms based on mean proximity rules. Finally, one of the motivations for studying families of voting rules is to be able to learn the most effective voting rule from the family, for which less number of parameters implies more efficient learning.

Further, Lemma 5.17 gives a linear embedding that has dimension $m!$ times the dimension of the input embedding, whereas Theorem 5.15 produces a linear embedding of the same dimension as the input embedding. Together with the added benefits of the succint representation presented above, this provides significant additional benefit.

This discussion inspires a very interesting basic question: *Which neutral SMPRs admit low dimensional embeddings?* Low dimensional embeddings are important because they provide simple intuition behind the working of the rules they represent. Turns out, the structure of linear embeddings allow us to characterize all neutral SMPRs which admit one dimensional and two dimensional embeddings. One dimensional embeddings partition the set of rankings into two — the sets of rankings obtained by applying all odd and even permutations[9] to any single ranking. The embedding then maps all rankings in one set to the same point, thus mapping all possible rankings to one of two distinct points. The set of two dimensional embeddings can be mapped to

[7]A generating set of a group is a subset of the group such that every element of the group can be expressed as the product (under the group operation) of finitely many elements of the subset and their inverses.

[8]It might be the case that k is still exponential in m, but as we mention later, there exist interesting low dimensional embeddings.

[9]The parity (oddness or evenness) of a permutation σ is the parity of the number of inversions in σ, i.e., the number of pairs x, y such that $x < y$ but $\sigma(x) > \sigma(y)$.

points on a circle as follows. In the matrices corresponding to the generating set of S_m of size 2, it can be shown that the matrix corresponding to the cycle must be a rotation by $4\pi/m!$, where as the matrix corresponding to the transposition must be a reflection around a line passing through the origin at angle θ, where $\theta \in [0, 2\pi)$.[10] Further, we were able to show that for 3 alternatives, imposing Pareto efficiency in the winner and in the loser[11] to the set of two dimensional embeddings yields a characterization of positional scoring rules over 3 alternatives. We omit these proofs due to lack of space and because they are fairly simple.

Summary of the results till now. Table I summarizes the characterizations presented in the paper till now. In case of multiple conditions within a table cell, each condition is equivalent to the required restriction on the mean proximity rule. A prime result that is omitted from the table is that an embedding ϕ is neutral if and only if its normalization $\hat{\phi} = \phi - \phi_{avg}$ is linear.

A mean proximity rule is	**iff $\exists$ a score matrix S such that**	**iff $\exists$ embeddings (ψ, ϕ) such that**
symmetric	S is negation of a Euclidean distance matrix	$\psi = \phi$
symmetric and neutral	i) S is neutral and negation of a Euclidean distance matrix ii) S is neutral, positive semidefinite, and has equal diagonal entries	i) $\psi = \phi$ is neutral ii) $\psi = \phi$ is linear

Table I: Summary of the characterization results.

6. CONNECTIONS TO OTHER APPROACHES

[NS: TO BE WRITTEN]

Voting rules have been analyzed from three viewpoints in the literature [NS: cite].

GSR — MPRs are strict subset of GSRs. Also connected axiomatically: mean neat rules (a slight generalization) characterized by anonymity, consistency, and two technical conditions of connectedness and continuity. GSR characterized by anonymity and finite local consistency. Also, both families have both algebraic and geometric versions.

DR — subset with a specific choice of intuitive distance function.

MLE — again very direct connection.

Aximoatic — weak. Essentially characterization of mean neat voting, and positional scoring rules (a subset of MPR).

7. DISCUSSION

[NS: TO BE WRITTEN]

[NS: See the image attached]

[10]Orientation of the $4\pi/m!$ rotation as well as the initial embedding can be fixed without loss of generality

[11]When all voters rank the same alternative first or the same alternative last, that alternative should be first or last respectively in the output.

Also: after leveraging algorithmic etc insights, and deeper understanding — also low dimensional representations (group theory citations)

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