Euclidean Voting

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1 Preliminaries

Let A denote the set of m alternatives, and $\mathcal{L}(A)$ and $\mathcal{O}$ denote the space of rankings of alternatives and outcomes, respectively. Thus, $|\mathcal{L}(A)| = m!$, and let $k = |\mathcal{O}|$. A profile $\pi \in \mathcal{L}(A)^n$ is a collection of votes (rankings). For any profile π, let $n(\pi, \sigma)$ denote the number of times σ appears in π. Let $\sigma_1, \ldots, \sigma_{m!}$ denote a fixed reference order of the rankings in $\mathcal{L}(A)$.

For any two profiles π_1 and π_2, let $\pi_1 + \pi_2$ be the union profile such that $n(\pi_1 + \pi_2, \sigma) = n(\pi_1, \sigma) + n(\pi_2, \sigma)$ for every $\sigma \in \mathcal{L}(A)$. Similarly, for any profile π, let $c\pi$ be the profile such that $n(c\pi, \sigma) = c \cdot n(\pi, \sigma)$ for every $\sigma \in \mathcal{L}(A)$.

Voting Rule A *voting rule* (more technically, a social welfare function - SWF) $r : \mathcal{L}(A)^n \to \mathcal{P}(\mathcal{L}(A)) \setminus \{\emptyset\}$ is a function that maps every profile of votes to a set of tied rankings.

Note that a voting rule can never output an empty set.

Anonymity A voting rule r is called *anonymous* if it only depends on the number of times each ranking appears in the profile: for every profiles π_1 and π_2 such that $n(\pi_1, \sigma) = n(\pi_2, \sigma)$ for every $\sigma \in \mathcal{L}(A)$, $r(\pi_1) = r(\pi_2)$.

Rank-Distinguishability A voting rule r is called *rank-distinguishing* if it can distinguish between any two rankings: for every two rankings $\sigma, \sigma' \in \mathcal{L}(A)$ ($\sigma \neq \sigma'$), there exists a profile π such that exactly one of σ or σ' is in $r(\pi)$.

In this paper, we only consider rank-distinguishing voting rules.

Unanimity A voting rule r is said to satisfy *unanimity* if on every profile that consists of copies of a single ranking, it uniquely outputs that ranking: for every profile π such that $n(\pi, \sigma) > 0$ for some $\sigma \in \mathcal{L}(A)$ and $n(\pi, \sigma') = 0$ for all $\sigma' \neq \sigma$, $r(\pi) = \{\sigma\}$.

Neutrality Given any profile $\pi = (\sigma_1, \dots, \sigma_n)$, let $\tau\pi = (\tau\sigma_1, \dots, \tau\sigma_n)$ be the profile where each vote is permuted according to τ. Similarly, given any set of rankings S, let $\tau S = \{\tau\sigma | \sigma \in S\}$. A voting rule r is called *neutral* if for every profile π and permutation τ, we have $r(\tau\pi) = \tau r(\pi)$.

Consistency A social welfare function r is called *consistent* (for rankings) if for every profiles π_1 and π_2 such that $r(\pi_1) \cap r(\pi_2) \neq \emptyset$, $r(\pi_1 + \pi_2) = r(\pi_1) \cap r(\pi_2)$.

Connectedness A voting rule r is called *connected* if for any two profiles π_1 and π_2 with $r(\pi_1) \cap r(\pi_2) \neq \emptyset$, there exist non-negative integers c and d such that $r(\pi_1) \cap r(c\pi_1 + d\pi_2) \neq \emptyset$ and $r(\pi_1) \neq r(c\pi_1 + d\pi_2)$.

[NS: Check if this is equivalent/related to continuity defined by Conitzer et al. [4]. Then we can add that to Proposition 3.]

Continuity Two profiles π_1 and π_2 satisfy $\pi_1 \approx \pi_2$ if they differ by one vote: for some σ and σ', $n(\pi_1, \sigma) = n(\pi_2, \sigma) - 1$, $n(\pi_1, \sigma') = n(\pi_2, \sigma') + 1$, and for every $\sigma \in \mathcal{L}(A) \setminus \{\sigma, \sigma'\}$, $n(\pi_1, \sigma) = n(\pi_2, \sigma)$. A voting rule r is called *continuous* if for every profile π and ranking σ, $\sigma \notin r(\pi)$ implies that there exists integer k such that for every profile $\pi' \approx k\pi$, $\sigma \notin r(\pi')$.

2 Background on Mean Proximity Rules and Generalized Scoring Rules

Mean Proximity Rules (Zwicker [9]) A voting rule is called a *mean proximity rule* if there exists an input embedding $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ and an output embedding $\psi : \mathcal{O} \to \mathbb{R}^k$ such that for any profile π with n votes, $r(\pi) = \arg\min_{o \in \mathcal{O}} \|\psi(o) - mean(\pi)\|$, where $mean(\pi) = (1/n) \cdot \sum_{\sigma \in \mathcal{L}(A)} n(\pi, \sigma) \cdot \phi(\sigma)$ is the mean of the input embeddings of the votes in π (along with multiplicity).

Generalized Scoring Rules (Zwicker [9]) A voting rule is called a *generalized scoring rule* if there exists a scoring function $s : \mathcal{L}(A) \times \mathcal{O} \to \mathbb{R}$ such that for any profile π, $r(\pi) = \arg\max_{o \in \mathcal{O}} \sum_{\sigma \in \mathcal{L}(A)} n(\pi, \sigma) \cdot s(\sigma, o)$.

For these two classes of voting rules, we have an elegant equivalence theorem by Zwicker [9]. The theorem uses an important result shown below.

Proposition 1 (Zwicker [9])**.** *For any embeddings $\phi, \psi : \mathcal{L}(A) \to \mathbb{R}^k$ and any profile π,*

$$\arg\min_{o \in \mathcal{O}} \|\psi(o) - mean(\pi)\| = \arg\min_{o \in \mathcal{O}} \sum_{\sigma \in \mathcal{L}(A)} n(\pi, \sigma) \cdot \|\psi(o) - \phi(\sigma)\|^2.$$

From the definition of generalized scoring rules and Proposition 1, it is clear that the scoring function $s(\sigma, o) = -\|\psi(o) - \phi(\sigma)\|^2$ represents the same mean proximity rule that has the embeddings (ψ, ϕ). Extending this to a two-way correspondence, Zwicker [9] showed the following.

Proposition 2 (Theorem 4.2.1, Zwicker [9])**.** *A voting rule is a mean proximity rule if and only if it is a generalized scoring rule.*

Due to this equivalence, any mean proximity rule has two representations: a pair of embeddings (ψ, ϕ), and a scoring function s, both of which may not be unique. Zwicker [8] also showed the following.

[NS: Check if continuity in [4] is same as continuity in [9]. If so, take the definition from [9], else from [4].]

Proposition 3 ([8])**.** *Any mean proximity rule is consistent, connected, continuous, and anonymous.*

This implies that any voting rule that is not consistent (in the SWF sense) is not a mean proximity rule. In fact, we have the following.

Lemma 1 (Proposition 1,2,5 and Theorem 3 of [4])**.** *All positional scoring rules and the Kemeny rule are mean proximity rules. However, Bucklin's rule, Copeland's rule, the maximin rule, the ranked pairs method, and STV are not mean proximity rules since they do not satisfy consistency (under any tie-breaking scheme).*

3 Symmetric Mean Proximity Rules

In this paper, we are interested in social welfare functions (SWFs) that return a ranking (more formally, a set of tied rankings), so $\mathcal{O} = \mathcal{L}(A)$. Here, the scoring function $s : \mathcal{L}(A) \times \mathcal{L}(A) \to \mathbb{R}$ describes the *similarity* between two rankings. This special case was also defined and studied by Conitzer et al. [4] as *simple ranking scoring functions* (SRSFs). Under any fixed enumeration $\sigma_1, \ldots, \sigma_{m!}$, we can create an $m! \times m!$ matrix S such that $S_{ij} = s(\sigma_i, \sigma_j)$. This is called a *score matrix* of the generalized scoring rule (equivalently, mean proximity rule) corresponding to the scoring function s.

The class of mean proximity rules is very wide, even capturing rules that violate natural desiderata. For example, the mean proximity rule given by a scoring function s satisfying $s(\sigma, \sigma') > s(\sigma, \sigma)$ for some $\sigma, \sigma' \in \mathcal{L}(A)$ would not output σ even on the profile where all votes are σ, thus violating unanimity. To solve this problem, we propose a simple fix.

Symmetric Mean Proximity Rules A voting rule is called *symmetric mean proximity rule* if there exists a mean proximity representation with identical input and output embeddings, i.e., $\psi = \phi$.

Since the outcome space for SWFs is identical to the input space, $\psi = \phi$ is a natural restriction. It is easy to check that for any profile π where all votes are σ, $mean(\pi) = \phi(\sigma)$. Thus, the rule would output the singleton set $\{\sigma\}$.[1]

[1] Rank-distinguishability plays a key role here. Any embedding of a rank-distinguishable symmetric mean proximity rule must map all rankings to different Euclidean points.

Lemma 2. *While there exist mean proximity rules violating unanimity, all symmetric mean proximity rules satisfy unanimity.*

Symmetric mean proximity rules are not too restrictive, as they still capture all well-known mean proximity rules.

Lemma 3. *All positional scoring rules and the Kemeny rule are symmetric mean proximity rules.*

For this, note that the mean proximity representations of the positional scoring rules and the Kemeny rule constructed in [9] are indeed symmetric.

3.1 Characterization

Recall that mean proximity rules are equivalent to generalized scoring rules (Proposition 2). It is natural to ask: *What subclass of generalized scoring rules is equivalent to symmetric mean proximity rules?* For any embedding ϕ, define the scoring function s_ϕ by $s_\phi(\sigma, \sigma') = -\|\phi(\sigma) - \phi(\sigma')\|^2$ for all $\sigma, \sigma' \in \mathcal{L}(A)$. Then Proposition 1 implies that s_ϕ and ϕ are equivalent, i.e., they both represent the same symmetric mean proximity rule. Further, the score matrix generated by s_ϕ has a well-known structure.

Euclidean Distance Matrix (EDM) A $p \times p$ matrix $A = (a_{ij})$ is called a *Euclidean distance matrix (EDM)* if there exist $v_1, \ldots, v_p \in \mathbb{R}^k$ such that $a_{ij} = \|v_i - v_j\|^2$, $\forall i, j$.

Theorem 1. *A voting rule is a symmetric mean proximity rule if and only if it is a generalized scoring rule that has a score matrix whose negation is a Euclidean distance matrix.*

Proof. Take any symmetric mean proximity rule r. Let ϕ be arbitrary embedding of r, and let s_ϕ be its equivalent scoring function. Then it is easy to observe that the score matrix of s_ϕ is, by definition, negation of an EDM. Alternatively, given any score matrix that is negation of an EDM, by definition we can find an embedding ϕ such that the score matrix is generated by the scoring function s_ϕ. Hence, the rule is a symmetric mean proximity rule. $\square$

We provided two motivations for symmetric mean proximity rules: i) taking identical input and output embeddings is very natural, and ii) symmetric mean proximity rules achieve unanimity while still capturing all well-known mean proximity rules. We also identified the subclass of generalized scoring rules that is equivalent to symmetric mean proximity rules. In the next section, we analyze symmetric mean proximity rules satisfying another highly desired property – neutrality. While neutrality is also mild (all voting rules of interest are neutral), we show that it adds a lot of structure.

4 Neutral Symmetric Mean Proximity Rules

In this section, we study the restrictions neutrality imposes on the embeddings and on the scoring functions of symmetric mean proximity rules. We connect neutrality of symmetric mean proximity rule with neutrality of scoring functions, neutrality of embeddings, and positive semidefiniteness of score matrix. [NS: Consider switching from $\tau\sigma$ to $\tau(\sigma)$ to avoid formally dealing with interpretation of rankings as permutations. Or just say for a permutation τ, the permuted ranking $\tau(\sigma)$ will be denoted by $\tau\sigma$ for notational convenience. Also take care of all $\tau \in \mathcal{L}(A) \to \tau \in S_m$.]

4.1 Neutrality of Scoring Functions and Embeddings

We define neutral scoring functions as in [4], which are closely related to neutrality of generalized scoring rules.

Neutral Scoring Function A scoring function $s : \mathcal{L}(A) \times \mathcal{L}(A) \to \mathbb{R}$ is called *neutral* if $s(\tau\sigma, \tau\sigma') = s(\sigma, \sigma')$ for every $\sigma, \sigma', \tau \in \mathcal{L}(A)$. We say that a score matrix is neutral if the scoring function generating it is neutral.

In words, neutrality of a scoring function means that the similarity between two rankings does not change if we permute them in the same way. Conitzer et al. [4] showed that any scoring function s of a neutral GSR r can be *neutralized*, i.e., converted to an equivalent neutral scoring function.

Proposition 4 ([4])**.** *For any scoring function s of a neutral GSR r, there is an equivalent (representing r as well) neutral scoring function s^{NT} given by*

$$s^{NT}(\sigma, \sigma') = \sum_{\tau \in \mathcal{L}(A)} s(\tau\sigma, \tau\sigma'). \tag{1}$$

Using this, Conitzer et al. [4] translated neutrality of GSR (equivalently mean proximity rules) to neutrality of scoring functions.

Proposition 5 (Lemma 2, Conitzer et al. [4])**.** *A mean proximity rule is neutral if and only if it has a neutral scoring function.*

Proposition 5 and Theorem 1 together identify neutral symmetric mean proximity rules as those for which some scoring function is neutral and some scoring function generates a score matrix which is negation of an EDM. Next, we combine these two conditions using a natural definition of neutrality of embeddings using the connection between the embedding ϕ and the scoring function s_ϕ from Proposition 1.

Neutral Embedding An embedding $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ is *neutral* if $\|\phi(\sigma) - \phi(\sigma')\| = \|\phi(\tau\sigma) - \phi(\tau\sigma')\|$ for any $\tau, \sigma, \sigma' \in \mathcal{L}(A)$.

For any embedding ϕ, define the embedding ϕ^{NT}, called the *neutralization* of ϕ, as follows.

$$\phi^{NT}(\sigma) = [\phi(\tau_1\sigma)^T\phi(\tau_2\sigma)^T \dots \phi(\tau_{m!}\sigma)^T]^T, \tag{2}$$

where $\tau_1, \dots, \tau_{m!}$ is any fixed enumeration of all permutations. Note that $dim(\phi^{NT}) = m! \cdot dim(\phi)$. We show that ϕ^{NT} is neutral, and is equivalent to ϕ.

Lemma 4. *For any embedding ϕ of a neutral symmetric mean proximity rule r, the embedding ϕ^{NT} given in Equation (2) is neutral and also represents r. Further, $s_{\phi^{NT}} = s_\phi^{NT}$ is a scoring function of r that is neutral and generates a score matrix which is negation of an EDM.*

Proof. First, we show that ϕ^{NT} is neutral. For any $\tau \in \mathcal{L}(A)$,

$$\begin{aligned}\|\phi^{NT}(\tau\sigma) - \phi^{NT}(\tau\sigma)\|^2 &= \sum_{\tau' \in \mathcal{L}(A)} \|\phi(\tau'\tau\sigma) - \phi(\tau'\tau\sigma)\|^2 \\ &= \sum_{\tau' \in \mathcal{L}(A)} \|\phi(\tau'\sigma) - \phi(\tau'\sigma)\|^2 = \|\phi^{NT}(\sigma) - \phi^{NT}(\sigma)\|^2.\end{aligned}$$

Thus, ϕ^{NT} is neutral. Next, for any $\sigma, \sigma' \in \mathcal{L}(A)$,

$$\begin{aligned}s_{\phi^{NT}}(\sigma, \sigma') = -\|\phi^{NT}(\sigma) - \phi^{NT}(\sigma')\|^2 &= -\sum_{\tau \in \mathcal{L}(A)} \|\phi(\tau\sigma) - \phi(\tau\sigma')\|^2 \\ &= \sum_{\tau \in \mathcal{L}(A)} s_\phi(\tau\sigma, \tau\sigma') = s_\phi^{NT}(\sigma, \sigma').\end{aligned}$$

Thus, we have $s_{\phi^{NT}} = s_\phi^{NT} = s$ (say). The scoring function s_ϕ is equivalent to the embedding ϕ, and thus also represents rule r. From Proposition 4, the scoring function $s_\phi^{NT} = s_{\phi^{NT}}$ also represents r. Thus, the embedding ϕ^{NT} also represents r. Finally, $s = s_\phi^{NT}$ implies that s is neutral, and $s = s_{\phi^{NT}}$ implies that the score matrix of s is negation of an EDM. □

4.2 Positive Semidefiniteness of Score Matrix

Note that the embedding ϕ^{NT} constructed in Equation (2) satisfies an interesting property, in addition to being neutral: $\|\phi^{NT}(\sigma)\|^2 = \sum_{\sigma' \in \mathcal{L}(A)} \|\phi(\sigma')\|^2$, which is independent of σ. That is, ϕ^{NT} is an equal norm embedding.

Equal Norm Embedding An embedding $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ is said to have *equal norm* if $\|\phi(\sigma)\| = \|\phi(\sigma')\|$ for all $\sigma, \sigma' \in \mathcal{L}(A)$.

For equal norm embeddings, squares of Euclidean distances in Proposition 1 can be converted to inner products.

Lemma 5. *For any equal norm embedding ϕ of a symmetric mean proximity rule r, $s(\sigma, \sigma') = \langle\phi(\sigma), \phi(\sigma')\rangle$ is a scoring function of r.*

Proof. Let $c = \|\phi(\sigma)\|$, which is independent of σ since ϕ is an equal norm embedding. From Proposition 1, we know that on any profile π,

$$
\begin{aligned}
r(\pi) &= \arg\min_{\sigma \in \mathcal{L}(A)} \sum_{\sigma' \in \mathcal{L}(A)} n(\pi, \sigma') \cdot \|\phi(\sigma) - \phi(\tau\sigma')\|^2 \\
&= \arg\min_{\sigma \in \mathcal{L}(A)} \sum_{\sigma' \in \mathcal{L}(A)} n(\pi, \sigma') \cdot \big(c^2 + c^2 - 2 \cdot \langle \phi(\sigma), \phi(\sigma') \rangle\big) \\
&= \arg\min_{\sigma \in \mathcal{L}(A)} 2 \cdot n \cdot c^2 - 2 \cdot \sum_{\sigma' \in \mathcal{L}(A)} n(\pi, \sigma') \langle \phi(\sigma), \phi(\sigma') \rangle \\
&= \arg\max_{\sigma \in \mathcal{L}(A)} \sum_{\sigma' \in \mathcal{L}(A)} n(\pi, \sigma') \langle \phi(\sigma), \phi(\sigma') \rangle.
\end{aligned}
$$

It is now clear that $s(\sigma, \sigma') = \langle \phi(\sigma), \phi(\sigma') \rangle$ is indeed a scoring function of r. $\square$

We use this to generate score matrices of neutral symmetric mean proximity rules that are Gramian (or equivalently, positive semidefinite).

Gramian Matrix A $p \times p$ matrix $A = (a_{ij})$ is called *Gramian* if there exist vectors $v_1, \dots, v_p \in \mathbb{R}^k$ such that $a_{ij} = \langle v_i, v_j \rangle$ for all i, j. It is well-known that a matrix is Gramian if and only if it is positive semidefinite.

4.3 Final Characterization

Theorem 2. *For any mean proximity rule r, the following are equivalent.*

1. *r is neutral and symmetric.*
2. *r has a symmetric representation (identical input and output embeddings) where the embedding is neutral.*
3. *r has a score matrix which is neutral and negation of an EDM.*
4. *r has a score matrix that is neutral, positive semidefinite, and has equal diagonal entries.*

Proof. First, it is easy to show that the second and the third conditions imply the first condition. If r has a score matrix which is neutral and negation of an EDM, then by Proposition 5 and Theorem 1, r is a neutral mean proximity rule and r is a symmetric mean proximity rule, thus a neutral symmetric mean proximity rule. Similarly, if r is a symmetric mean proximity rule with a neutral embedding ϕ, then for any profile π, we have

$$
\begin{aligned}
r(\tau\pi) &= \arg\min_{\sigma \in \mathcal{L}(A)} \sum_{\sigma' \in \mathcal{L}(A)} n(\pi, \sigma') \cdot \|\phi(\sigma) - \phi(\tau\sigma')\|^2 \\
&= \tau \arg\min_{\sigma \in \mathcal{L}(A)} \sum_{\sigma' \in \mathcal{L}(A)} n(\pi, \sigma') \cdot \|\phi(\tau\sigma) - \phi(\tau\sigma')\|^2 \\
&= \tau \arg\min_{\sigma \in \mathcal{L}(A)} \sum_{\sigma' \in \mathcal{L}(A)} n(\pi, \sigma') \cdot \|\phi(\sigma) - \phi(\sigma')\|^2 = \tau r(\pi),
\end{aligned}
$$

where the first transition is due to Proposition 1, and the third transition is due to neutrality of ϕ. Thus, r is a neutral symmetric mean proximity rule.

Conversely, let r be any neutral symmetric mean proximity rule and ϕ be its arbitrary embedding. We already proved in Lemma 4 that the score matrix of the scoring function $s_{\phi}^{NT} = s_{\phi^{NT}}$ is both neutral and negation of an EDM, and that its equivalent embedding ϕ^{NT} is neutral and represents r. Thus, first three conditions are equivalent.

For equivalence with the fourth condition, take any neutral symmetric mean proximity rule r and its arbitrary embedding ϕ. Note that the embedding ϕ^{NT} has equal norm, thus its Gramian matrix S represents r (Lemma 5). Since S is Gramian, it is also positive semidefinite. Equal norm property of ϕ^{NT} implies that S has equal diagonal entries, and neutrality of ϕ^{NT} implies neutrality of S. Conversely, take any mean proximity rule r with a score matrix S that is neutral, positive semidefinite, and has equal diagonal entries. Then, we can find an embedding ϕ such that S is its Gramian matrix. Neutrality and equal diagonal entries of S imply neutrality and equal norm property of ϕ. Thus, due to Lemma 5, ϕ also represents r, showing that r is a neutral symmetric mean proximity rule. $\square$

5 Linear Embeddings

Theorem 2 allowed us to identify neutral symmetric mean proximity rules as those having a neutral embedding. Whilt it is straightforward to check if a given embedding is neutral, generating neutral embeddings is a non-trivial task. We show that neutrality of an embedding is equivalent an elegant structure that shows an easy way to generate neutral embeddings. In this section, we introduce *linear embeddings* by drawing ideas from representation theory of the symmetric group.

[NS: cite group theory literature on linear representations]

Linear Embeddings An embedding $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ is called *linear* if there exists a function $R : \mathcal{L}(A) \to \mathbb{R}^{k \times k}$ mapping each permutation to a $k \times k$ real matrix such that i) $\phi(\tau\sigma) = R(\tau)\phi(\sigma)$ for every $\tau, \sigma \in \mathcal{L}(A)$, and ii) $R(\tau_1\tau_2) = R(\tau_1)R(\tau_2)$ for all $\tau_1, \tau_2 \in \mathcal{L}(A)$. R is called the representation of ϕ.

Here, $\tau\sigma$ denotes the ranking obtained by permuting candidates in σ according to τ. It is known [NS: cite] that for representation R, $R(\tau^{-1}) = R(\tau)^{-1} = R(\tau)^T$ for every $\tau \in \mathcal{L}(A)$. In what follows, for notational convenience, we use R_τ instead of R_τ. Note that linear embeddings are quite structured: embedding of a ranking σ can be obtained from embedding of any other ranking σ' by the linear transformation corresponding to the permutation that converts σ' to σ.

Take any embedding ϕ of a neutral symmetric mean proximity rule r, and its equivalent neutral embedding ϕ^{NT} given in Equation (2). Observe that for any $\sigma, \sigma' \in \mathcal{L}(A)$, the coordinates of $\phi^{NT}(\sigma)$ and $\phi^{NT}(\sigma')$ are just permutations of each other. We use this to show the following.

Lemma 6. *For any embedding ϕ of a neutral symmetric mean proximity rule r, the equivalent neutral embedding ϕ^{NT} given in Equation (2) is linear.*

Proof. Let ϕ be a k-dimensional embedding. Hence, ϕ^{NT} has dimension $m! \cdot k$. We need to show that there exists a representation R such that for any $\tau, \sigma \in \mathcal{L}(A)$, $\phi^{NT}(\tau\sigma) = R_\tau \phi^{NT}(\sigma)$. Note that

$$\phi^{NT}(\tau\sigma) = [\phi(\tau_1\tau\sigma)^T \phi(\tau_2\tau\sigma)^T \dots \phi(\tau_{m!}\tau\sigma)^T]^T.$$

Let Π_τ be the $m! \times m!$ matrix such that $\Pi_{ij} = 1$ if and only if $\tau_j = \tau_i\tau$. It is easy to verify that Π_τ is a permutation matrix. Further, if the blocks of $\phi^{NT}(\sigma)$ were permuted according to Π_τ, then the i^{th} block in the resulting vector will be $\phi(\tau_j\sigma)$ where $\tau_j = \tau_i\tau$, which is the i^{th} block of $\phi^{NT}(\tau\sigma)$. Hence, applying Π_τ to the blocks of $\phi^{NT}(\sigma)$ results in $\phi^{NT}(\tau\sigma)$. Now construct an $(m! \cdot k) \times (m! \cdot k)$ matrix R_τ by replacing every value 1 in Π_τ by a $k \times k$ identity matrix, and every value 0 in Π_τ by a $k \times k$ zero matrix. Then, $R_\tau \phi^{NT}(\sigma) = \phi^{NT}(\tau\sigma)$. Finally, it is easy to verify that $R_{\tau_1\tau_2} = R_{\tau_1} R_{\tau_2}$ since $\Pi_{\tau_1\tau_2} = \Pi_{\tau_1}\Pi_{\tau_2}$ □

Combining the proof of Theorem 2 with Lemma 6, we know that every neutral symmetric mean proximity rule has a linear embedding. On the contrary, we show that every linear embedding represents a neutral symmetric mean proximity rule. For that, we use the following result which shows the real strength of linear embeddings.

Lemma 7. *For any linear embedding ϕ and any $\tau, \sigma, \sigma' \in \mathcal{L}(A)$,*

$$\langle \phi(\tau\sigma), \phi(\tau\sigma') \rangle = \langle \phi(\sigma), \phi(\sigma') \rangle.$$

Proof. Let ϕ be any linear embedding and let R be its representation. Then for any $\tau, \sigma, \sigma' \in \mathcal{L}(A)$,

$$\langle \phi(\tau\sigma), \phi(\tau\sigma') \rangle = \langle R_\tau\phi(\sigma), R_\tau\phi(\sigma') \rangle = \langle \phi(\sigma), R_\tau^T R_\tau \phi(\sigma') \rangle = \langle \phi(\sigma), \phi(\sigma') \rangle. \tag{3}$$

Here, the last transition holds because R_τ is an orthogonal matrix by definition. □

Taking $\sigma = \sigma'$, we get:

Corollary 1. *Every linear embedding has equal norm.*

Lemma 8. *Any linear embedding is neutral, and hence represents a neutral symmetric mean proximity rule.*

Proof. For any $\tau, \sigma, \sigma' \in \mathcal{L}(A)$,

$$\begin{aligned} \|\phi(\tau\sigma) - \phi(\tau\sigma')\|^2 &= \|\phi(\tau\sigma)\|^2 + \|\phi(\tau\sigma')\|^2 - 2 \cdot \langle \phi(\tau\sigma), \phi(\tau\sigma') \rangle \\ &= \|\phi(\sigma)\|^2 + \|\phi(\sigma')\|^2 - 2 \cdot \langle \phi(\sigma), \phi(\sigma') \rangle = \|\phi(\sigma) - \phi(\sigma')\|^2. \end{aligned}$$

Thus, ϕ is neutral. By Theorem 2, it represents a neutral symmetric mean proximity rule. □

Combining Lemmas 6 and 8 gives the following characterization.

Theorem 3. *A symmetric mean proximity rule is neutral if and only if it has a linear embedding.*

We showed that all linear embeddings are neutral, but we haven't shown that all neutral embeddings are linear as well. It turns out that this indeed holds, but requires a much deeper proof.

5.1 Connection between Neutrality and Linearity

In this section, we show that neutrality of an embedding is equivalent to linearity. For this, we first need some important results about embeddings of neutral symmetric mean proximity rules.

Lemma 9. *Let r be any neutral voting rule. Let π_{symm} be the profile containing each ranking exactly once, i.e., $n(\pi_{symm}, \sigma) = 1$ for all $\sigma \in \mathcal{L}(A)$. Then, $r(\pi_{symm}) = \mathcal{L}(A)$.*

Proof. Let $r(\pi_{symm}) = T \subseteq \mathcal{L}(A)$. Suppose $T \neq \mathcal{L}(A)$, so there exists a $\sigma' \notin T$. Further, by definition of a voting rule, $T \neq \emptyset$. Thus, there exists a $\sigma \in T$. Now, take τ to be the permutation that sends σ to σ'. It is easy to see that $\tau\pi_{symm} = \pi_{symm}$. Hence, $r(\tau\pi_{symm}) = r(\pi_{symm}) = T$. Thus, $\sigma' \notin r(\tau\pi_{symm})$. However, $\sigma' \in \tau r(\pi_{symm})$. Thus, $\tau r(\pi_{symm}) \neq r(\tau\pi_{symm})$. This implies that r violates neutrality, a contradiction. □

Therefore, for any embedding ϕ of a neutral symmetric mean proximity rule, $\|\phi(\sigma) - mean(\pi_{symm})\|$ must be a constant independent of σ so that all rankings are tied on π_{symm}. Let $\phi_{avg} = (1/m!) \cdot \sum_{\sigma \in \mathcal{L}(A)} \phi(\sigma)$. Define an embedding $\hat{\phi}$ such that $\hat{\phi}(\sigma) = \phi(\sigma) - \phi_{avg}$ for all $\sigma \in \mathcal{L}(A)$. We call $\hat{\phi}$ the *normalization* of ϕ. We argued that it is an equal norm embedding. Note that normalization of an embedding is just a translation. Intuitively, this does not change the geometry of the points to which the rankings are embedded, and hence should represent the same symmetric mean proximity rule. We show that in addition, normalization also preserves neutrality and linearity.

Lemma 10. *For any embedding ϕ of a neutral symmetric mean proximity rule r, its normalization $\hat{\phi}$ is an equivalent equal norm embedding. Further, an embedding is neutral (resp. linear) if and only if its normalization is neutral (resp. linear).*

Proof. Let $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ be any embedding of a neutral symmetric mean proximity rule r. Consider its normalization $\hat{\phi}$. From Lemma 9, it is clear that all $\phi(\sigma)$ must be at equal distance from $mean(\pi_{symm}) = \phi_{avg}$. Hence, $\hat{\phi} = \phi - \phi_{avg}$ must be an equal norm embedding. Further, for any rankings $\sigma, \sigma' \in \mathcal{L}(A)$, $\|\hat{\phi}(\sigma) - \hat{\phi}(\sigma')\| = \|\phi(\sigma) - \phi_{avg} - \phi(\sigma') + \phi_{avg}\| = \|\phi(\sigma) - \phi(\sigma')\|$. Now, Proposition 1 trivially implies that both ϕ and $\hat{\phi}$ represent the same rule.

If ϕ is neutral, then for any $\tau, \sigma, \sigma' \in \mathcal{L}(A)$, we have $\|\hat{\phi}(\tau\sigma) - \hat{\phi}(\tau\sigma')\| = \|\phi(\tau\sigma) - \phi(\tau\sigma')\| = \|\phi(\sigma) - \phi(\sigma')\| = \|\hat{\phi}(\sigma) - \hat{\phi}(\sigma')\|$. Hence, $\hat{\phi}$ is neutral too. The proof of the converse works exactly in the opposite direction.

Finally, assume ϕ is linear with representation R. First, we have that for any $\tau, \sigma \in \mathcal{L}(A)$, $\phi(\tau\sigma) = R_\tau \phi(\sigma)$. Averaging over all σ, we get that

$$\frac{1}{m!} \sum_{\sigma \in \mathcal{L}(A)} \phi(\tau\sigma) = \frac{1}{m!} \sum_{\sigma \in \mathcal{L}(A)} R_\tau \phi(\sigma) = R_\tau \left(\frac{1}{m!} \sum_{\sigma \in \mathcal{L}(A)} \phi(\sigma) \right).$$

However,

$$\frac{1}{m!} \sum_{\sigma \in \mathcal{L}(A)} \phi(\tau\sigma) = \frac{1}{m!} \sum_{\sigma \in \mathcal{L}(A)} \phi(\sigma) = \phi_{avg}.$$

Hence, we have $R_\tau \phi_{avg} = \phi_{avg}$ for all $\tau \in \mathcal{L}(A)$. Now, for any $\tau, \sigma \in \mathcal{L}(A)$,

$$\hat{\phi}(\tau\sigma) = \phi(\tau\sigma) - \phi_{avg} = R_\tau \phi(\sigma) - R_\tau \phi_{avg} = R_\tau \hat{\phi}(\sigma).$$

Thus, $\hat{\phi}$ is also linear with the same representation R. The proof of the converse works by observing that $\phi = \hat{\phi} + \phi_{avg}$. □

Lemma 10 shows that the normalization $\hat{\phi}$ of any neutral embedding ϕ of a neutral symmetric mean proximity is an equivalent equal norm and neutral embedding. We now show that it is also linear.

Theorem 4. *An embedding is neutral if and only if its normalization is linear.*

Proof. First, let ϕ be any embedding whose normalization $\hat{\phi}$ is linear. Then, $\hat{\phi}$ is also neutral (Lemma 8), implying that ϕ is neutral (Lemma 10).

Conversely, take any neutral embedding $\phi : \mathcal{L}(A) \to \mathbb{R}^k$ and let r be the rule it represents. We want to show that $\hat{\phi}$ is linear. By Lemma 10, we know that $\hat{\phi}$ is an equal norm and neutral (also k-dimensional). Construct another equivalent embedding ϕ^f as follows. Let $k' \leq k$ be the dimension of the affine subspace of $\mathbb{R}^k$ spanned by the points $\hat{\phi}(\sigma_1), \ldots, \hat{\phi}(\sigma_{m!})$. If $k' = k$, then take $\phi^f = \hat{\phi}$. Otherwise, $k' < k$ and we can transform the points to $\mathbb{R}^{k'}$ to get ϕ^f that preserves all distances, and thus represents r as well.

Formally, note that $\hat{\phi}_{avg} = 0$. Hence, the affine space spanned by the embeddings under $\hat{\phi}$ form a linear subspace of $\mathbb{R}^k$ of dimension k'. Take any orthonormal basis $v_1, \ldots, v_{k'}$ of this linear subspace, write every point as a linear combination of the k' basis vectors, and let ϕ^f output the coefficients of the linear combination. Thus, if we construct a $k' \times k$ matrix P with $v_1, \ldots, v_{k'}$ as its rows, then $\phi^f(\sigma) = P\phi(\sigma)$ for all $\sigma \in \mathcal{L}(A)$. It is now standard to show that ϕ^f preserves distances between any two points in the affine subspace, hence the distance between any point in $\hat{\phi}(\sigma_1), \ldots, \hat{\phi}(\sigma_{m!})$ and any point in its convex hull. Note that by definition, preserving these distances is enough to represent the same rule r.

Thus, the embedding ϕ^f is a k'-dimensional equal norm neutral embedding of r. Further, ϕ^f is of *full dimension*, i.e., the points where the rankings are mapped span the whole of $\mathbb{R}^{k'}$. Note that this implies $k' \leq m!$. Consider the matrix $A = [\phi^f(\sigma_1), \ldots, \phi^f(\sigma_{m!})]$, the $k' \times m!$ matrix whose columns are the coordinate vectors of the points. This implies that A must have full rank k'. Fix any $\tau \in \mathcal{L}(A)$, and consider $B = [\phi^f(\tau\sigma_1), \ldots, \phi^f(\tau\sigma_{m!})]$. Then, $(B^T B)_{ij} = \langle \phi^f(\tau\sigma_i), \phi^f(\tau\sigma_j) \rangle = \langle \phi^f(\sigma_i), \phi^f(\sigma_j) \rangle = (A^T A)_{ij}$ for all i, j, where the second transition follows since $\hat{\phi}$ (thus, ϕ^f) is neutral (Lemma 10). Thus, $A^T A = B^T B$.

We now show that there exists a matrix R_τ such that $\phi^f(\tau\sigma) = R_\tau \phi^f(\sigma)$ for all $\sigma \in \mathcal{L}(A)$. Combining all the equations, it is equivalent to existence of R_τ such that $R_\tau A = B$, i.e., existence of an X such that $A^T X = B^T$. It is well-known that the system of equations $Ax = b$ has a solution if and only if $AA^+ b = b$, where A^+ is the Moore-Penrose pseudoinverse of A. Trivially extending this to multiple systems of linear equations, we can see that the necessary and sufficient condition for existence of the required R_τ is $A^T (A^T)^+ B^T = B^T$, or equivalently, $BA^+ A = B$. The last derivation uses the fact that $(A^T)^+ = (A^+)^T$. Now,

$$
\begin{aligned}
B &= BB^+B = B\left((B^T B)^+ B^T\right) B = B(B^T B)^+ \left(B^T B\right) \\
&= B(A^T A)^+ \left(A^T A\right) = B\left((A^T A)^+ A^T\right) A = BA^+ A.
\end{aligned}
$$

Refer [1] for the identities $X = XX^+X$ (used in the first transition) and $X^+ = (X^T X)^+ X^T$ (used in the second and the fifth transitions) regarding Moore-Penrose pseudoinverses. The fourth transition follows since $A^T A = B^T B$. Hence, we have shown that for every $\tau \in \mathcal{L}(A)$, there exists a matrix R_τ such that $\phi^f(\tau\sigma) = R_\tau \phi^f(\sigma)$ for all $\sigma \in \mathcal{L}(A)$. Further, one solution of $Ax = b$ is $x = A^+ b$. Extending this, we obtain that one solution of $A^T X = B^T$ is $X = (A^T)^+ B^T$. Hence, $R_\tau = X^T = BA^+$. Choose this solution for every $\tau \in \mathcal{L}(A)$.

Recall that the rank of product of two matrices is at most the minimum of the rank of the two matrices, and $R_\tau A = B$. Also, both A and B are rank k' matrices. Hence, $rank(R_\tau) \geq k'$. However, R_τ is a $k' \times k'$ matrix. Hence, we conclude that R_τ is invertible for every $\tau \in \mathcal{L}(A)$. We now show that $R_{\tau_1\tau_2} = R_{\tau_1} R_{\tau_2}$ for all $\tau_1, \tau_2 \in \mathcal{L}(A)$.

Fix any $\tau_1, \tau_2 \in \mathcal{L}(A)$. For any $\sigma \in \mathcal{L}(A)$, $\phi^f(\tau_1\tau_2\sigma) = R_{\tau_1\tau_2}\phi^f(\sigma)$. Also, $\phi^f(\tau_1\tau_2\sigma) = R_{\tau_1}\phi^f(\tau_2\sigma) = R_{\tau_1}R_{\tau_2}\phi^f(\sigma)$. Thus, we have that $R_{\tau_1\tau_2}\phi^f(\sigma) = R_{\tau_1}R_{\tau_2}\phi^f(\sigma)$ for all $\sigma \in \mathcal{L}(A)$. We now show that this implies $R_{\tau_1\tau_2} = R_{\tau_1}R_{\tau_2}$. Form the partial A matrix, call it A_p, by only taking k' linearly independent columns of A. Then, A_p is an invertible square matrix. Also, $R_{\tau_1\tau_2}A_p = R_{\tau_1}R_{\tau_2}A_p$. Multiplying by A_p^{-1} on both sides, we get the desired result. Thus, ϕ^f is a linear embedding. We now show that $\hat{\phi}$ is also linear.

Consider the orthonormal basis $v_1, \ldots, v_{k'}$ that was used in constructing ϕ^f from $\hat{\phi}$ and the matrix P that has $v_i' s$ as its rows. Complete this to a basis $v_1, \ldots, v_k$ of $\mathbb{R}^k$, and consider the matrix Q that has all the $v_i' s$ as its rows. Then, $Q\hat{\phi}(\sigma) = [\phi^f(\sigma) 0 \ldots 0]^T$ where the number of zeros is $k - k'$. For any

$\tau \in \mathcal{L}(A)$, construct the $k \times k$ matrix

$$R'_\tau = Q^T \left[\begin{smallmatrix} R_\tau & 0 \\ 0 & I \end{smallmatrix}\right] Q.$$

Here, R_τ is a $k' \times k'$ matrix and I is the $(k-k') \times (k-k')$ identity matrix. Note that

$$R'_\tau \hat{\phi}(\sigma) = Q^T \left[\begin{smallmatrix} R_\tau & 0 \\ 0 & I \end{smallmatrix}\right] Q\hat{\phi}(\sigma) = Q^T \left[\begin{smallmatrix} R_\tau & 0 \\ 0 & I \end{smallmatrix}\right] [\phi^f(\sigma)0\ldots 0]^T = Q^T[\phi^f(\tau\sigma)0\ldots 0]^T = \hat{\phi}(\tau\sigma).$$

Thus, $R'_\tau \hat{\phi}(\sigma) = \hat{\phi}(\tau\sigma)$ for all τ and σ. Further,

$$\begin{aligned} R'_{\tau_1} R'_{\tau_2} &= Q^T \begin{bmatrix} R_{\tau_1} & 0 \\ 0 & I \end{bmatrix} QQ^T \begin{bmatrix} R_{\tau_2} & 0 \\ 0 & I \end{bmatrix} Q = Q^T \begin{bmatrix} R_{\tau_1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} R_{\tau_2} & 0 \\ 0 & I \end{bmatrix} Q \\ &= Q^T \begin{bmatrix} R_{\tau_1} R_{\tau_2} & 0 \\ 0 & I \end{bmatrix} Q = Q^T \begin{bmatrix} R_{\tau_1\tau_2} & 0 \\ 0 & I \end{bmatrix} Q = R'_{\tau_1\tau_2}. \end{aligned}$$

Here, the second transition follows since QQ^T is identity matrix as rows of Q form an orthonormal basis. Thus, $\hat{\phi}$ is a linear embedding and R' is its representation. $\square$

6 Summary of Results

The following table summarizes all the equivalence results presented above.

Mean proximity rule is	iff $\exists$ score matrix S	iff $\exists$ embeddings (ψ, ϕ)
symmetric	S = -EDM	$\psi = \phi$
symmetric+neutral	i) S = neutral, -EDM ii) S = neutral + PSD + has equal diagonal	i) $\psi = \phi$ = neutral ii) $\psi = \phi$ = linear

In addition, we showed that an embedding ϕ is neutral if and only if its normalization $\hat{\phi} = \phi - \phi_{avg}$ is linear.

7 Low Dimensional Embeddings

In this section, we analyze the linear embeddings in small dimensional Euclidean spaces. Low dimensional representations have beem emphasized in the group representation theory [2, 6]. Low dimensional embeddings are interesting, because they are very structured and are easy to visualize.

Lemma 11 (1-Dimension)**.** *The only linear embeddings in dimension* 1 *are obtained as follows.*

1. *Partition the set of rankings $\mathcal{L}(A)$ into two sets $\mathcal{L}(A)_1$ and $\mathcal{L}(A)_2$, both of equal sizes, such that all even permutations take rankings of each partition within itself, and all odd permutations take rankings of one partition to the other.*

2. *Any linear embedding of dimension* 1 *maps all rankings in* $\mathcal{L}(A)_1$ *to some point* $\lambda \in \mathbb{R}$*, and either maps all of* $\mathcal{L}(A)_2$ *also to* λ*, or all of* $\mathcal{L}(A)_2$ *to* $-\lambda$*.*

The only rank-distinguishable rule with a 1*-dimensional embedding is the majority rule over* 2 *alternatives.*

Lemma 12 (2-Dimension)**.** *The only linear embeddings in dimension* 2 *are obtained as follows.*

1. *Take the generator set* $\tau_r = (12 \ldots m)$ *(cycle) and* $\tau_t = (12)$ *(transposition).*

2. *Fix* R_{τ_r} *to be the orthogonal transformation that rotates by* $2\pi/m$ *clockwise, take* R_{τ_t} *to be any flip, and generate all other* R_τ*'s by their combinations.*

For 3 *alternatives, this means having two equilateral triangles on a circle centered at the origin. Among these, the Borda rule and all positional scoring rules are given in Figure* **??***.*

PUT THE FIGURE HERE

Lemma 13. *All* 2*-dimensional embeddings of any* 3*-alternative neutral symmetric mean proximity rule are linear.*

Lemma 14. *All* 2*-dimensional embeddings of any neutral symmetric mean proximity rule (with any number of alternatives) are equivalent, up to translation, scaling, rotation and reflection.*

8 Unique Representation Theorem

Theorem 5. *Any two embeddings of a neutral SMPR are similar, i.e., equivalent up to translation, rotation, reflection, scaling (dilation), and dimension-lifting.*

Proof. Lemma [**?**] showed that any embedding of a neutral SMPR maps all rankings to points on a sphere centered at the mean of all the points. Take any two embeddings ϕ and ϕ' of a neutral SMPR r. For any subset of rankings P, let $S_\phi)(P) = \{\phi(\sigma)\}_{\sigma \in P}$ and $S_{\phi'}(P) = \{\phi'(\sigma)\}_{\sigma \in P}$.

With slight abuse of notation, let $S_\phi = S_\phi(\mathcal{L}(A))$ and $S_{\phi'} = S_{\phi'}(\mathcal{L}(A))$. Let D_ϕ and $D_{\phi'}$ denote the Delaunay tessellations of S_ϕ and $S_{\phi'}$ respectively. Note that D_ϕ and $D_{\phi'}$ are convex polyhedra since the points in S_ϕ lie on a sphere, and so do the points in $S_{\phi'}$.

We use Cauchy's rigidity theorem to establish that D_ϕ and $D_{\phi'}$ are similar to each other. That means, when viewed in their respective minimum Euclidean dimension, one "solid" can be obtained from another by translation, rotation, reflection, and scaling. For this, we need to establish two conditions:

1. The combinatorial structures of D_ϕ and $D_{\phi'}$ are identical. That is, for a subset P of rankings, $S_\phi(P)$ forms a Delaunay facet of D_ϕ if and only if $S_{\phi'}(P)$ forms a Delaunay facet of $D_{\phi'}$.

2. There exists a constant λ such that all the edge lengths of D_ϕ are λ times their counterparts in $D_{\phi'}$. That is, for all σ and σ',

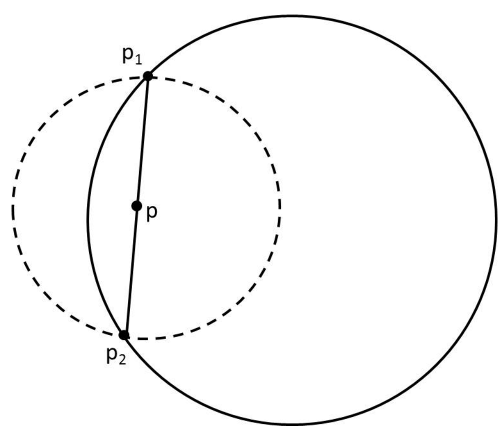
$$\|\phi(\sigma) - \phi(\sigma')\| = \lambda \cdot \|\phi'(\sigma) - \phi'(\sigma')\|.$$

The first condition is rather easy to establish. Note that $S_\phi(P)$ forms a Delaunay facet of D_ϕ if and only if the convex hull of $S_\phi(P)$ has empty intersection with the Voronoi cell of every $\sigma' \notin P$ in the Voronoi diagram of S_ϕ. In other words, every convex combination of $S_\phi(P)$ is closer to some point in $S_\phi(P)$ than every point in $S_\phi(\mathcal{L}(A) \setminus P)$. This happens if and only if for every profile π containing only rankings from P, $r(\pi) \subseteq P$.[2]

Note that the last condition is a property of the voting rule r itself. Hence, for every $P \subseteq \mathcal{L}(A)$, $S_\phi(P)$ forms a Delaunay facet of D_ϕ if and only if $S_{\phi'}(P)$ forms a Delaunay facet of $D_{\phi'}$.

Next, we establish the harder second condition required by Cauchy's theorem. Consider a two-dimensional facet f of D_ϕ and the corresponding facet f' of $D_{\phi'}$ (recall that both D_ϕ and $D_{\phi'}$ have identical combinatorial structures). Let $P \subseteq \mathcal{L}(A)$ be the set of rankings whose points form both facets. Let $|P| = k$, where $k \geq 3$. Hence, the facet must be a k-gon in both D_ϕ and $D_{\phi'}$. Also, note that the k-gon must be cyclic, otherwise it would be triangulated further in the Delaunay triangulations.

First, we show that the vertices of P must be arranged in the same order in both f and f' (up to mirror image). Let us consider f. Consider two points $p_1, p_2 \in P$ as shown in Figure 8. Let p be the midpoint of p_1 and p_2. Let $N(p)$ denote the set of points in P that are closest to p.



[2]While the profiles are only *rational* convex combinations of the points, the equivalence can still be observed due to the fact that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$ for every k.

1. If $N(p) = \{p_1, p_2\}$, then no point in P can lie on the arc joining p_1 and p_2 that is opposite to the center of the circle. Hence, p_1 and p_2 must be adjacent in the k-gon.

2. If $N(p) \neq \{p_1, p_2\}$, then there are two possibilities.

   (a) Either p_1 and p_2 are not adjacent in the k-gon, or

   (b) The line joining p_1 and p_2 either passes through the center of the circle, or separates the rest of the polygon from the center of the circle.

   In both cases, there can be at most one such pair (p_1, p_2) in the polygon.

Hence, we see that there would either be exactly $k-1$ or k pairs (p_1, p_2) such that $N(p) = \{p_1, p_2\}$, where p is the midpoint of p_1 and p_2, and all such pairs are adjacent in the k-gon. Thus, they completely specify the order of the points in the k-gon.[3]

Further, note that whether $N(p)$ is exactly the output of the voting rule r on profile that consists of one occurrence of p_1 and one occurrence of p_2. Thus, it must be the same in both f and f'. Hence, the order of points of P in both f and f' must be identical, up to mirror image. Without loss of generality, assume that the order of the vertices is the same. If it is not, we take reflection, which does not change edge-lengths.

Let the vertices be $\sigma_1, \sigma_2, \ldots, \sigma_k$ in order. We show that

$$\frac{\|\phi(\sigma_1) - \phi(\sigma_2)\|}{\|\phi'(\sigma_1) - \phi'(\sigma_2)\|} = \frac{\|\phi(\sigma_2) - \phi(\sigma_3)\|}{\|\phi'(\sigma_2) - \phi'(\sigma_3)\|} = \ldots = \frac{\|\phi(\sigma_k) - \phi(\sigma_1)\|}{\|\phi'(\sigma_k) - \phi'(\sigma_1)\|} \tag{4}$$

Take the triangle $\sigma_1, \sigma_2, \sigma_3$. In both ϕ and ϕ', the barycentric coordinates (linear combination of the vertices) of the circumcenter of the triangle must be equal, since the profile described by the barycentric coordinates must have output $\{\sigma_1, \ldots, \sigma_k\}$, which is only possible at the circumcenter of the k-gon. However, for triangle of side lengths a, b, and c, the barycentric coordinates are proportional to

$$(a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)).$$

It can be easily shown that two triangles with identical barycentric coordinates of the circumcenter must have proportional side-lengths. Hence, we get

$$\frac{\|\phi(\sigma_1) - \phi(\sigma_2)\|}{\|\phi'(\sigma_1) - \phi'(\sigma_2)\|} = \frac{\|\phi(\sigma_2) - \phi(\sigma_3)\|}{\|\phi'(\sigma_2) - \phi'(\sigma_3)\|}$$

Applying this repeatedly to triangles formed by all triplets of consecutive vertices gives Equation (4). Finally, note that all the two-dimensional facets of the Delaunay triangulations D_ϕ and $D_{\phi'}$ are connected via shared edges. Thus,

[3]In case of $k-1$ adjacent pairs, there is only one possibility left for the remaining edge.

joining the Equation (4) of all polygons via shared edges, we get that for some $\lambda > 0$,

$$\|\phi(\sigma) - \phi(\sigma')\| = \lambda \cdot \|\phi'(\sigma) - \phi'(\sigma')\|,$$

for all σ, σ', which is the final condition required. $\square$

9 Main Research Questions

1. Proving that if d is the minimum dimension required, then there exists a d-dimensional embedding.

2. Proving a generalization of the fact that all embeddings of a neutral SMPR are equivalent up to some sort of transformations and dimension uplifting.

3. How to find the minimum dimension required for any given voting rule. What is the minimum dimension for PSR and the Kemeny rule?

4. Exploring MLE connection - efficient sampling.

5. Conditions imposed on the embeddings by other properties like monotonicity.

10 Other Research Questions

1. Alternative proof of the characterization of positional scoring rules by observing that they are obtained by the same representation R where R_τ is the permutation matrix generated by τ (which permutes rankings according to the mapping $\sigma \to \tau\sigma$), and different initial embeddings.

2. Conjecture by Conitzer et al. [4]: Consistent + continuous + neutral $\Leftrightarrow$ Neutral SRSF.

3. What about notions of consensus in Euclidean spaces other than mean $\to$ e.g., minimize maximum distance (everyone "lets go" equally)?

11 Scribbled Notes

These are just high level intuitions. Need further investigations.

1. Neutral $\Rightarrow$ all rows permutations of each other $\Rightarrow$ sum = constant $\Leftrightarrow$ $[1, \ldots, 1]$ is an eigenvector $\Leftrightarrow$ first coordinate of the embedding constant $\Leftrightarrow$ there is an embedding of rank-1

2. What is the permutation of all rankings generated by applying τ to each of them? What are all these $m!$ permutations?

3. Anything interesting about inner product maximizing rules that do not have embeddings of equal norm? Even symmetric ones may not be unanimous.

4. An illustration: the *Least Frequent Ranking Rule* is not SMPR - satisfies consistency, anonymity and neutrality. But violates unanimity.

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