J. Geom. 82 (2005) 156 – 171
0047–2468/05/020156 – 16

DOI 10.1007/s00022-005-1752-8

Journal of Geometry

Cyclic polygons with given edge lengths: Existence and uniqueness

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Abstract. Let $a_1, \dots, a_n$ be positive numbers satisfying the condition that each of the a_i's is less than the sum of the rest of them; this condition is necessary for the a_i's to be the edge lengths of a (closed) polygon. It is proved that then there exists a unique (up to an isometry) convex cyclic polygon with edge lengths $a_1, \dots, a_n$. On the other hand, it is shown that, without the convexity condition, there is no uniqueness—even if the signs of all central angles and the winding number are fixed, in addition to the edge lengths.

Mathematics Subject Classification (2000): 51M04, 51M25, 52A25, 52A10.
Key words: Cyclic polygons, convex polygons, existence, uniqueness.

0. Introduction

A *polygon* or, more specifically, n-gon is any sequence $\mathcal{P} := (V_0, \dots, V_{n-1})$ of n points on the Euclidean plane $\mathbb{R}^2$. The smallest value that we shall allow here for the integer n is 3. The points $V_0, \dots, V_{n-1}$ are called the *vertices* of $\mathcal{P}$.

The segments, or closed intervals,

$$[V_{i-1}, V_i] := \operatorname{conv}\{V_{i-1}, V_i\} \quad \text{for} \quad i \in \{1, \dots, n\}$$

are called the *edges* of polygon $\mathcal{P}$, where

$$V_n := V_0.$$

The symbol conv denotes, as usual, the convex hull [16, page 12].

For a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$, let $\boldsymbol{a} = (a_1, \dots, a_n)$ be its edge lengths sequence, so that

$$a_i := |V_{i-1}V_i| \quad \text{for} \quad i \in \{1, \dots, n\},$$

where $|AB|$ stands for the Euclidean distance between points A and B.

A polygon is called *cyclic* if all its vertices lie on a circle.

For brevity, let us refer to the radius of the circumscribed circle of a c.p. simply as the *c.p.'s radius* and denote it by r; further, for $i = 1, \dots, n$, let α_i be the radian measure of the central angle corresponding to the chord of length a_i so that

$$\alpha_i = 2\arcsin\frac{a_i/2}{r} = 2\arcsin\frac{k(r)a_i}{2},$$

where

$$k(r) := 1/r$$

is the curvature of the circle.

For all polygons $\mathcal{P} = (V_0, \ldots, V_{n-1})$ considered in this paper, it will be assumed (unless specified otherwise) that their vertices are all distinct from one another: for all distinct i and j in the set $\{0, \ldots, n-1\}$, one has $V_i \neq V_j$. Such polygons were called *ordinary* in [11]. Note that the set of edges of any polygon can be represented as the union of the sets of edges of ordinary polygons.

Note also that any ordinary c.p. is *strict* in the sense that no three vertices V_i, V_j, and V_k with distinct i, j, and k in the set $\{0, \ldots, n-1\}$ are collinear (this follows because no straight line can have more than two distinct points in common with a circle).

Therefore, a necessary condition for the a_i's to be the edge lengths of an (ordinary) c.p. is that

$$\text{each of the } a_i\text{'s is less than the sum of the rest of them.} \tag{1}$$

Without loss of generality, let us assume that $a_n = \max\{a_i : 1 \leqslant i \leqslant n\}$. Then condition (1) simply means that $a_1 + \cdots + a_{n-1} > a_n$.

A c.p. can be determined by the circumscribed circle, a given vertex V_0 on it, the sequence of the central angles $(\alpha_1, \ldots, \alpha_n) \in (0, \pi]^n$ corresponding to the edges, and a sequence of signs $\boldsymbol{\varepsilon} := (\varepsilon_1, \ldots, \varepsilon_n) \in \{1, -1\}^n$. Indeed, vertex V_i can be obtained from V_{i-1} by the rotation through the central angle $\varepsilon_i\alpha_i$, for $i = 1, \ldots, n$. If $\alpha_i = \pi$, then the rotation through angle $(-\alpha_i)$ is the same as that through angle α_i; in such a case, let us avoid the indeterminacy by setting $\varepsilon_i := 1$. Let us refer to $\boldsymbol{\varepsilon}$ as the *signature* of the c.p.

For any given edge lengths sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$ and signature $\boldsymbol{\varepsilon} := (\varepsilon_1, \ldots, \varepsilon_n)$ of a c.p., introduce the function

$$f_{\boldsymbol{a},\boldsymbol{\varepsilon}}(k) := \frac{1}{\pi}\sum_{i=1}^{n} \varepsilon_i \arcsin\frac{ka_i}{2} \tag{2}$$

on the interval $[0, 2/a_n]$; recall that we assume that $a_n = \max\{a_i\colon 1 \leqslant i \leqslant n\}$. Let us refer to $f_{\boldsymbol{a},\boldsymbol{\varepsilon}}$ as the *winding function* (w.f.) corresponding to the given edge lengths $\boldsymbol{a}$ and signature $\boldsymbol{\varepsilon}$.

Note that the value

$$w := f_{\boldsymbol{a},\boldsymbol{\varepsilon}}(1/r) = \frac{1}{2\pi}\sum_{i=1}^{n}\varepsilon_i\alpha_i$$

must be an integer, which is known as the *winding number* of the polygon. (E.g., cf. [1], [2, page 178], [4], [8].)

Suppose that $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ are the edge lengths and signature of some c.p. $\mathcal{P}$. Then, applying the same permutation (say σ) to both $\boldsymbol{a}$ and $\boldsymbol{\varepsilon}$, one obtains another c.p. (say $\mathcal{P}^\sigma$), with the same radius; the central angles of c.p. $\mathcal{P}^\sigma$ can be obtained by applying the same permutation σ to the sequence $(\alpha_1, \dots, \alpha_n)$ of the central angles of c.p. $\mathcal{P}$. Let us say that $\mathcal{P}^\sigma$ is a permutation of $\mathcal{P}$. Also, switching from signature $\boldsymbol{\varepsilon}$ to $-\boldsymbol{\varepsilon}$ just changes the orientation of a c.p.

Let us say that two c.p.'s are *equivalent* to each other (*in the narrow sense*) if one of them can be obtained from the other by a permutation and/or the orientation switch; if at that the permutation is cyclical, let us say that two c.p.'s are *cyclically equivalent*; replacing here "c.p.'s" by "signatures", one defines the corresponding equivalence relation on the set $\{1, -1\}^n$ of signatures.

If two c.p.'s are equivalent to each other (in the narrow sense), then their w.f.'s are either the same or opposite to each other; any two c.p.'s with such w.f.'s may be called *equivalent in the broad sense*. These two kinds of equivalences coincide in the case when all the edge lengths a_i are distinct from one another. This follows from the linear independence of the functions $x \mapsto \arcsin a_i x$ for distinct positive a_i's, which in turn follows from the fact that $\arcsin'(1-) = \infty$.

In this paper, we shall consider the following questions.

(i) Given an n-tuple $\boldsymbol{a} := (a_1, \dots, a_n)$ of positive numbers, does there exist a cyclic polygon (c.p.) with edge lengths sequence $\boldsymbol{a}$?
(ii) Is such a c.p. essentially unique?

Clearly, one can express existence and uniqueness of c.p.'s in terms of the w.f.:

PROPOSITION 1. *A c.p. with a given lengths-signature-winding number (l.s.w.) triple* $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$ *exists iff the equation*

$$f_{\boldsymbol{a},\boldsymbol{\varepsilon}}(k) = w \tag{3}$$

has a solution k (*in the interval* $(0, 2/a_n]$)*; for the given triple* $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$*, such a c.p. is unique* (*up to an isometry*) *iff such a solution* k *of* (3) *is unique. At that, the radius of the c.p. is* $r = 1/k$ *and the central angles are*

$$\alpha_i = 2\arcsin\frac{ka_i}{2}, \quad \textit{for } i = 1, \dots, n.$$

(Note that Theorem 1 below establishes uniqueness—if only of convex c.p.'s—in a stronger sense, without specifying a signature or winding number in advance.)

1. Convex cyclic polygons

For convex c.p.'s, the answers to both the existence and uniqueness questions are "yes", provided the necessary condition (1).

Let us define the convex hull of a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ as the convex hull of the set of its vertices: conv $\mathcal{P} := \text{conv}\{V_0, \dots, V_{n-1}\}$.

Then a *convex polygon* can be defined as a polygon $\mathcal{P}$ such that the union of the edges of $\mathcal{P}$ coincides with the boundary of the convex hull of $\mathcal{P}$; cf. e.g. [17, page 5].

Let us emphasize that a polygon in this paper is a sequence and therefore ordered. In particular, even if all the vertices $V_0, \dots, V_{n-1}$ of a polygon $\mathcal{P} = (V_0, \dots, V_{n-1})$ are the extreme points of the convex hull of $\mathcal{P}$, it does not necessarily follow that $\mathcal{P}$ is convex. For example, if $V_0 = (0, 0)$, $V_1 = (1, 0)$, $V_2 = (1, 1)$, and $V_3 = (0, 1)$, then polygon (V_0, V_1, V_2, V_3) is convex, while polygon (V_0, V_2, V_1, V_3) is not.

THEOREM 1. *If condition* (1) *holds, then there exists a convex c.p. with edge lengths* $(a_1, \dots, a_n)$*. Such a polygon is unique up to an isometry.*

The proof of Theorem 1 is based in part on the following characterization of convex c.p.'s:

THEOREM A. *A c.p.* $\mathcal{P} = (V_0, \dots, V_{n-1})$ *is convex if and only if it is cyclically equivalent to a c.p. of one of the following two types, in terms of the signature* $\boldsymbol{\varepsilon}$ *and winding number* w*:*

(Type I) $w = 1$ *and* $\boldsymbol{\varepsilon} = (1, \dots, 1)$*;*

(Type II) $w = 0$ *and* $\boldsymbol{\varepsilon} = (1, \dots, 1, -1)$.

This characterization follows immediately from the main result in [11].

Now we are ready to turn to the

Proof of Theorem 1. Consider the following w.f., corresponding to signature $\boldsymbol{\varepsilon} = (1, \dots, 1)$:

$$f(k) := \frac{1}{\pi} \sum_{i=1}^{n} \arcsin \frac{k a_i}{2} \tag{4}$$

on the interval $[0, 2/a_n]$, on which f is clearly continuous and strictly increasing, from $f(0) = 0$ to $f(2/a_n) = \frac{1}{\pi} \sum_{i=1}^{n} \arcsin \frac{a_i}{a_n}$.

To prove the existence, we need to distinguish two cases, Case E1 and Case E2, where E stands for "existence".

CASE E1 ($f(2/a_n) \geqslant 1$). In this case, the equation $f(k) = 1$ has a (unique) solution $k = k_{\boldsymbol{a}}$ in the interval $(0, 2/a_n]$; by Theorem A, this implies the existence of a convex c.p. (of Type I) with edge lengths $a_1, \dots, a_n$, radius $r = 1/k_{\boldsymbol{a}}$, and central angles $\alpha_i = 2 \arcsin \frac{k_{\boldsymbol{a}} a_i}{2}$.

CASE E2 ($f(2/a_n) < 1$). In this case, consider the following w.f., corresponding to signature $\boldsymbol{\varepsilon} = (1, \dots, 1, -1)$:

$$g(k) := \frac{1}{\pi}\left(\sum_{i=1}^{n-1} \arcsin \frac{ka_i}{2} - \arcsin \frac{ka_n}{2}\right), \tag{5}$$

again on the interval $[0, 2/a_n]$. Note that $g(0) = 0$, $g'(0) = \frac{1}{2\pi}(\sum_{i=1}^{n-1} a_i - a_n) > 0$ (by (1)), and $g(2/a_n) = f(2/a_n) - 1 < 0$. Therefore, there exists a solution $k = k_{\boldsymbol{a}}$ of the equation $g(k) = 0$ in the interval $(0, 2/a_n)$. By Theorem A, this implies the existence of a convex c.p. (of Type II) with edge lengths $a_1, \dots, a_n$, radius $r = 1/k_{\boldsymbol{a}}$, and central angles $\alpha_i = 2 \arcsin \frac{k_{\boldsymbol{a}} a_i}{2}$.

Thus, we have proved the existence part of the theorem.

Consider now any convex c.p., say $\mathcal{P}_*$, with edge lengths $a_1, \dots, a_n$. In view of Theorem A, to prove that this c.p. is unique up to an isometry it suffices to consider two cases, corresponding to whether $\mathcal{P}_*$ is of Type I or II; in either case, it is enough to show that the radius, say r_*, of the c.p. with the given edge lengths is uniquely determined.

CASE U1 ($\mathcal{P}_*$ is of Type I). In this case, one has $f(1/r_*) = 1$, for the function f defined by (4). Since $1/r_* \leqslant 2/a_n$ and f is increasing on the interval $[0, 2/a_n]$, the root $1/r_*$ of f is unique on the interval $[0, 2/a_n]$. Therefore, polygon $\mathcal{P}_*$ is unique up to an isometry among all convex c.p.'s of Type I with edge lengths $a_1, \dots, a_n$. (So far, we have not excluded the possibility that there may also exist a convex c.p. of Type II with the same edge lengths $a_1, \dots, a_n$.) Moreover, since f is increasing on the interval $[0, 2/a_n]$, one has $f(2/a_n) \geqslant f(1/r_*) = 1$, so that Case E1 takes place here.

CASE U2 ($\mathcal{P}_*$ is of Type II). In this case, one has $g(1/r_*) = 0$, for the function g defined by (5). Now we want to show that this implies Case E2. Let here

$$\alpha_i := 2 \arcsin \frac{a_i/2}{r_*} \in (0, \pi), \text{ for } i = 1, \dots, n.$$

Observe that, for each $t \geqslant 1$, the function

$$\left[0, \arcsin \frac{1}{t}\right) \ni \alpha \longmapsto F(\alpha) := F_t(\alpha) := \frac{\sin \alpha}{\sqrt{1 - t^2 \sin^2 \alpha}}$$

is strictly convex, since $F''(\alpha) = \sin \alpha \cdot (t^2 - 1 + 2t^2 \cos^2 \alpha)(1 - t^2 \sin^2 \alpha)^{-5/2} > 0$ for all $\alpha \in [0, \arcsin 1/t)$. (We understand the term "convex/concave function" as, e.g., in [16] and [6].) But, for $k \in [1/r_*, 2/a_n)$,

$$\frac{\pi k}{t} g'(k) = \sum_{i=1}^{n-1} F_t\left(\frac{\alpha_i}{2}\right) - F_t\left(\frac{\alpha_n}{2}\right) < 0,$$

where $t := kr_* \geqslant 1$; this follows because F_t is strictly convex, $F_t(0) = 0$, $\alpha_i > 0\,\forall i$, and $\sum_{i=1}^{n-1} \alpha_i = \alpha_n$. Therefore, $g(2/a_n) < g(1/r_*) = 0$, whence $f(2/a_n) = g(2/a_n) + 1 < 1$; that is, one has Case E2 indeed.

Observe next that in Case E2 the function g is concave. Indeed, for $k \in (0, 2/a_n)$, one has

$$\frac{\pi g''(k)}{k} = \sum_{i=1}^{n-1} \frac{a_i^3}{(4 - a_i^2 k^2)^{3/2}} - \frac{a_n^3}{(4 - a_n^2 k^2)^{3/2}} < \frac{a_n^3(\sum_{i=1}^{n-1} y_i^3 - 1)}{(4 - a_n^2 k^2)^{3/2}},$$

where $y_i := a_i/a_n \in (0, 1]$, so that $\sum_{i=1}^{n-1} y_i^3 \leq \sum_{i=1}^{n-1} y_i^2$. Note also that $\frac{1}{\pi} \sum_{i=1}^{n-1} \arcsin y_i = g(2/a_n) + 1/2 < 1/2$.

Thus, to prove the concavity of g, it is enough to show that, for any natural m, the inequalities $0 \leqslant y_i \leqslant 1\,\forall i$ and $\sum_{i=1}^m \arcsin y_i \leqslant \pi/2$ imply $\sum_{i=1}^m y_i^2 \leqslant 1$. Let $(z_1, \dots, z_m)$ be a point of maximum of $\sum_{i=1}^m y_i^2$ over the set of all m-tuples $(y_1, \dots, y_m)$ such that $0 \leqslant y_i \leqslant 1\ \forall i$ and $\sum_{i=1}^m \arcsin y_i \leqslant \pi/2$. Then, without loss of generality, one may assume that $0 < z_i < 1$ for all i. Indeed, the zero z_i's can be simply discarded (and m correspondingly reduced); on the other hand, if one of the z_i's equals 1, then the rest of them must be zero (in order to have $0 \leqslant z_i \leqslant 1\,\forall i$ and $\sum_{i=1}^m \arcsin z_i \leqslant \pi/2$), so that $\sum_{i=1}^m z_i^2 = 1$. Hence, for some real number λ, all the z_i's satisfy the Lagrange condition $2z_i = \lambda/\sqrt{1 - z_i^2}$, so that $z_i^2(1 - z_i^2) = \lambda^2/4\,\forall i$. Thus, there may be only two subcases here: (i) the z_i's are all the same or (ii) the z_i^2's take on two distinct values, of the form z^2 and $1 - z^2$, for some $z \in (0, 1)$. In subcase (ii), one must have $m = 2$, because $z_i \in (0, 1)\,\forall i$ and $\arcsin z + \arcsin \sqrt{1 - z^2} = \pi/2$ for $z \in (0, 1)$; therefore, in this subcase, $\sum_{i=1}^m z_i^2 = 1$. In subcase (i), it suffices to prove the implication $0 \leqslant m\alpha \leqslant \pi/2 \Rightarrow m \sin^2 \alpha \leqslant 1$, for any natural m. For $m = 1$, this is obvious. Let us then assume that $m \geqslant 2$. To prove the implication, it is enough to show that $h(\alpha) := \pi \sin^2 \alpha - 2\alpha \leqslant 0$ for all $\alpha \in [0, \pi/4]$. But this follows because h is convex on $[0, \pi/4]$ and $h(0) = h(\pi/4) = 0$. This proves that function g is concave.

Since $g(0) = 0$, it follows from the concavity of g that the root $1/r_*$ of g is unique in the interval $(0, 2/a_n]$. Therefore, polygon $\mathcal{P}_*$ is unique up to an isometry among all convex c.p.'s of Type II with edge lengths $a_1, \dots, a_n$.

Finally, if there existed two convex c.p.'s, one of Type I and the other of Type II, with the same edge lengths $a_1, \dots, a_n$, then the above consideration of Cases U1 and U2 would imply that both Cases E1 and E2 take place at once, which would be a contradiction. This concludes the proof of the uniqueness part, and thereby that of entire Theorem 1.

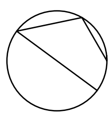
REMARK 1. The following statement, somewhat similar to Theorem 1, was made in [7]:

> *Given n positive real numbers $a_1, \ldots, a_n$, each of which is less than the sum of the others, there is a unique cyclic polygon $A_1A_2 \ldots A_n$ with $A_1A_2 = a_1, \ldots, A_{n-1}A_n = a_{n-1}$ and $A_nA_1 = a_n$.*

Apparently, in this quoted statement the polygon is tacitly assumed to be convex. As we shall see in the next section, without the polygon convexity condition the uniqueness statement is not true. The following heuristic proof was given in [7] to support the quoted claim:

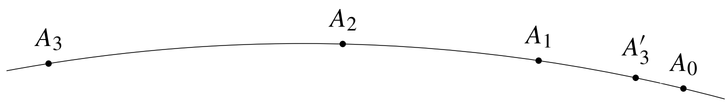
> Consider a circle of very large radius r. Take any point A_0 on it and construct chords of lengths $a_1, \ldots, a_n$ as shown [one next to another—I.P.]. For large enough r, the total arc $A_0A_1 \ldots A_n$ will be less than the circumference of the circle. Now reduce the radius. By a continuity argument, there will be a unique value of r such that A_n coincides with A_0 and $A_0A_1 \ldots A_n$ is one circumference of the circle.

Clearly, this reduction of the radius r from $r = \infty$ corresponds to the increase of the curvature $k = 1/r$ from $k = 0$. However, this radius-reduction process will work only in Case E1 considered in the proof of our Theorem 1. E.g., let $n := 3$ and $(a_1, a_2, a_3) := (3, 4, 6)$. Then, in the entire process of continuously reducing r from $r = \infty$ to the smallest feasible value $r = a_3/2 = 3$, the point A_3 will never coincide with A_0, because during this process the sum of the corresponding central angles will never exceed $f(2/a_3) = 2\arcsin\frac{3}{6} + 2\arcsin\frac{4}{6} + 2\arcsin\frac{6}{6} \approx 5.6 < 2\pi$.



Here is the situation that one has at $r = 3$, at the end of the radius-reduction process. After that, one needs to *increase* the value of r from 3 back to $\approx$3.3754, the radius of the circumscribed circle.

An anonymous referee suggested a modification of the above heuristics, whereby one can do without increasing the radius. Namely, the referee suggested that "the largest side be oriented in either the positive or negative direction, and then call on the ['shrinking'] continuity argument." For example, if, as above, $n = 3$ and $(a_1, a_2, a_3) = (3, 4, 6)$, then one can place points A_0, A_1, A_2, A_3, and A_3' on a large circle as shown below, so that $|A_0A_1| = 3$, $|A_1A_2| = 4$, and $|A_2A_3| = |A_2A_3'| = 6$.



Then let the circle shrink, keeping the four mentioned distances between the points, until either A_3 or A_3' coincides with A_0.

This illustrates the foregoing proof of Theorem 1. At that, the two different orientations of the longest edge correspond to the two cases in the proof of Theorem 1: E1 and E2 in the proof of existence, and U1 and U2 in the proof of uniqueness.

Here one may also recall that, according to the last paragraph of the proof of Theorem 1, there cannot exist two convex c.p.'s of the two different types, I and II, with the same edge lengths $a_1, \ldots, a_n$. That is, it is impossible that in the shrinking process there be two different times t and t' such that at time t point A_3 coincides with A_0 and at time t' point A_3' coincides with A_0.

REMARK 2. Another statement, also somewhat similar to Theorem 1, was made in [12]:

> *Let $\underline{B} = B_1 \ldots B_n$ be any given polygon (convex or not) and let $b_1, \ldots, b_n$ be the lengths of its sides. Then there exists the unique convex chordal polygon with sides of those lengths (and in this order).*

The term "chordal" here means "cyclic". The condition that $b_1, \ldots, b_n$ be the lengths of the sides of a polygon corresponds to our condition (1). However, no definition of polygon convexity is found in [12]. The uniqueness part of the quoted statement does not seem to be addressed in the corresponding proof in [12], which is given in a rather heuristic manner. The claim made in the first paragraph of that proof is somewhat similar to Theorem A of our paper but is presented in [12] without any substantiation. The reference to (13) made in [12] right after inequality (14) there should be apparently replaced by a reference to (11). Among the other comments that one can make about the proof, apparently the most important one is that the author of [12] seems to make the claim that, if ε_1 is small enough, then one has the implication

$$\sin \varepsilon_1 < \sin \varepsilon_2 + \cdots + \sin \varepsilon_n \Rightarrow \varepsilon_1 < \sin \varepsilon_2 + \cdots + \sin \varepsilon_n,$$

"[s]ince $\sin x \approx x$ for small x". At that, no explicit specifications on the signs or values of the ε_i's are stated in [12]. In fact, the values of $\varepsilon_2, \ldots, \varepsilon_n$ there seem to have to be not constant but vary simultaneously with ε_1 in a certain continuous manner. Therefore, to prove the above implication it is not enough just to refer to the fact that $\sin x \approx x$ for small x. Thus, one can see that the heuristic proof in [12] contains a few significant gaps.

Remarks 1 and 2 suggest that there likely are genuine and substantial difficulties with the proofs (and even formulations) of Theorems 1 and A that need to be overcome. It appears that at the root of these difficulties is the necessity to bridge the gap between such apparently distant notions as the polygon convexity and the central angles. Moreover, results and discussion in papers [9] and [10] suggest that the notion of polygon convexity is rather complex by itself, as it connects the notion of a polygon (and hence that of order) with the notion of convexity. Also, Remarks 1 and 2 suggest once again that heuristics needs to be complemented by rigorous definitions and proofs.

2. Non-convex cyclic polygons

Robbins [14, 15] presented 7 cyclic pentagons with edge lengths (29, 30, 31, 32, 33). The first two of the 7 c.p.'s, with radii ≈ 26.4 and 16.5, have the same signature $(1, 1, 1, 1, 1)$ but different winding numbers: 1 and 2.

It is therefore clear that, for possibly non-convex c.p.'s, one cannot guarantee uniqueness by specifying only edge lengths and signature (but not a winding number).

The other 5 of Robbins's cyclic pentagons, with radii $\approx 17.0, 17.6, 18.0, 18.3,$ and $18.7,$ have signatures $(1, 1, 1, 1, -1), (1, 1, 1, -1, 1), (1, 1, -1, 1, 1), (1, -1, 1, 1, 1),$ $(-1, 1, 1, 1, 1)$, respectively; each of these 5 c.p.'s has winding number 1.

Using the w.f. tool, one can see that these 7 cyclic pentagons are the only ones with edge lengths $\boldsymbol{a} = (29, 30, 31, 32, 33)$, up to the equivalence. The cause of this is that these 5 lengths are close to one another, and so, for any signature $\boldsymbol{\varepsilon}$ equivalent to $(1, 1, 1, -1, -1)$, the value $f_{\boldsymbol{a},\boldsymbol{\varepsilon}}(k)$ of the w.f. is close to one of the summands $\frac{1}{\pi}\varepsilon_i \arcsin\frac{ka_i}{2}$ in (2) and hence stays within the open interval $(0, 1)$ or $(-1, 0)$ for all $k \in (0, 2/a_n]$, never reaching any $w \in \mathbb{Z}$.

Thus, for non-convex c.p.'s, there is no universal existence, and we have not discerned any simple general condition—except that given by Proposition 1—for the existence of possibly non-convex c.p.'s with given edge lengths and signatures.

Moreover, Robbins's example of the 7 cyclic pentagons shows that, without the convexity condition, a c.p. with given edge lengths does not have to be unique. However, one can notice that any two distinct cyclic pentagons of Robbins's 7 ones do differ either in the signature or in the winding number.

What may appear as more surprising is that there exist, as we shall show, two or more distinct c.p.'s with the same l.s.w. triple $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$, that is, with the same edge lengths, signatures, and winding numbers!

Before we state our general result on non-uniqueness, let us make a few preliminary observations.

REMARK 3. All 3-gons are obviously convex and thus covered by Theorem 1. Therefore, we may further assume that $n \geqslant 4$ (recall that n stands for the number of edges).

REMARK 4. For $n = 4$, there is a simple example of non-uniqueness. Indeed, let $\boldsymbol{a} = (a, b, a, b)$, where $0 < a < b$, and let $\boldsymbol{\varepsilon}$ be $(1, 1, -1, -1)$ (or $(1, -1, -1, 1)$). Then $f_{\boldsymbol{a},\boldsymbol{\varepsilon}}(k) = 0$ for all $k \in [0, 2/b]$. This means that, for *any* $r \geqslant b/2$, there exists a (necessarily non-convex) cyclic 4-gon with radius r and the l.s.w. triple $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$ with $w = 0$. The set of vertices of any such cyclic 4-gon coincides with that of a symmetric trapezoid. Similar examples can be produced for any even n, with $\boldsymbol{a} = (a, b, c, \dots, a, b, c, \dots)$ and $\boldsymbol{\varepsilon} = (1, 1, 1, \dots, -1, -1, -1, \dots)$. However, all such c.p.'s are in a sense trivial, and will be referred here to as *degenerate*, for these c.p.'s are equivalent (even in the narrow sense) to the trivial c.p.'s with $\boldsymbol{a} = (a, a, b, b, c, c, \dots)$ and $\boldsymbol{\varepsilon} = (1, -1, 1, -1, 1, -1, \dots)$, with each edge traversed twice, in the two opposite directions. (Note that the trivial c.p.'s are not ordinary; however, degenerate c.p.'s in general may be ordinary.) Moreover, any such example of non-uniqueness is unstable: if, for instance, one changes here the value of only one of the edge lengths a_i to any other value $\tilde{a}_i > 0$, then the non-uniqueness phenomenon disappears; indeed, after such a change the w.f. takes on the values $\frac{1}{\pi}\varepsilon_i(\arcsin\frac{k\tilde{a}_i}{2} - \arcsin\frac{ka_i}{2})$ in the interval $(0, 1/2)$ or $(-1/2, 0)$ for all $k \in (0, 2/a_n]$. E.g., there is no c.p. with signature $(1, 1, -1, -1)$ and edge lengths $(\tilde{a}, b, a, b)$ for any $\tilde{a} \neq a$.

We thus come to

DEFINITION 1. Let us say that an l.s.w. triple $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$ is an *instance of stable non-uniqueness* if, for every $\tilde{\boldsymbol{a}}$ in some neighborhood of $\boldsymbol{a}$, there exist (at least) two non-isometric c.p.'s with the same l.s.w. triple $(\tilde{\boldsymbol{a}}, \boldsymbol{\varepsilon}, w)$.

We shall also show that there exist instances of multiple non-uniqueness with any given number (say m) of distinct values of the radius of circumscribed circle. Moreover, at that the values of the m radii can be chosen to be arbitrarily close to any m prescribed positive distinct values.

DEFINITION 2. Let us say that an m-tuple $\boldsymbol{r} := (r_1, \dots, r_m)$ of distinct positive real numbers *possesses an instance of non-uniqueness* if there exist m c.p.'s with the distinct radii $r_1, \dots, r_m$ but the same l.s.w. triple $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$.

Now we are prepared to state our main result on non-uniqueness of c.p.'s with given edge lengths.

THEOREM 2. *Let n be the number of edges, as before.*

1. *For $n = 3$, there is no non-uniqueness.*
2. *For $n = 4$, the only cases of non-uniqueness are those of degenerate cyclic 4-gons described in Remark 4, and in all these cases the non-uniqueness is not stable.*
3. (a) *For* every winding number $w \in \mathbb{Z}$, *there exist some $n = n(w)$ (necessarily $\geqslant 5$), edge lengths $\boldsymbol{a}$, and signature $\boldsymbol{\varepsilon}$ such that $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$ is an instance of stable non-uniqueness; at that, one may choose $n(0) = n(1) = n(-1) = 5$.*

(b) *For every* $n \geqslant 5$ *and* every signature $\boldsymbol{\varepsilon}$ *except for the signatures equivalent to* $(1, \ldots, 1)$*, there exist edge lengths* $\boldsymbol{a}$ *and a winding number* $w \in \{1, 0, -1\}$ *such that* $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$ *is an instance of stable non-uniqueness.*

(c) *For* every signature $\boldsymbol{\varepsilon}$ *containing at least four* 1*'s and at least four* (-1)*'s* and *for* every winding number $w \in \{1, 0, -1\}$*, there exist edge lengths* $\boldsymbol{a}$ *such that* $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$ *is an instance of stable non-uniqueness.*

4. *For any natural* m *and any* $w \in \mathbb{Z}$*, in any neighborhood of any* m*-tuple* $\boldsymbol{r} := (r_1, \ldots, r_m)$ *of distinct positive real numbers there exists an* m*-tuple* $\tilde{\boldsymbol{r}} := (\tilde{r}_1, \ldots, \tilde{r}_m)$ *which possesses an instance* $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$ *of non-uniqueness.* (*Recall Definition* 2.)

Proof. 1. Part 1 of Theorem 2 immediately follows from Remark 3.

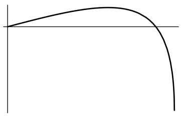
2. If the signature is $(1, 1, 1, 1)$ (or $(-1, -1, -1, -1)$), then the c.p. is either convex (with winding number $w = \pm 1$ and no non-uniqueness possible, by Theorem 1) or degenerate (with $w = \pm 2$ and $a_i = 2r$ for all i).

Let now the signature be $(1, 1, 1, -1)$; recall the convention $a_n = \max_i a_i$. Because $\alpha_i \in (0, \pi]$ and $\alpha_4 \geqslant \alpha_i \; \forall i$, here $2\pi w = \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 \leqslant \alpha_1 + \alpha_2 < 2\pi$, so that $w < 1$—unless $\alpha_1 = \alpha_2 = \pi = \alpha_4 = \alpha_3$, in which case the c.p. is degenerate. On the other hand, one has $2\pi w > -\alpha_4 \geqslant -\pi$, so that the winding number $w = 0$, and so, by Theorem A, the c.p. must be convex, except for the degeneracy case. Hence, no stable non-uniqueness is possible in this case. The same holds for signature $(-1, -1, -1, 1)$.

Next, let the signature be $(1, 1, -1, 1)$. Then the sum of the third and fourth summands in the w.f. is nondecreasing, because the function arcsin is convex on $[0, 1]$. Hence, here the w.f. is strictly increasing, and no non-uniqueness is possible. Signatures $(-1, -1, 1, -1)$, $\pm(1, -1, 1, 1)$, and $\pm(-1, 1, 1, 1)$ are similar.

It remains to consider the signature $(1, 1, -1, -1)$ (or its equivalent), for which the winding number of a c.p. is necessarily 0. Therefore, in any case of non-uniqueness here, there would exist positive roots, say k and ℓ with $k < \ell$, of the w.f. $f_{a,\varepsilon}$ in the interval $(0, 2/a_4]$. For all $i \in \{1, 2, 3, 4\}$, let $\gamma_i := \arcsin(a_i k/2)$ and $\beta_i := \arcsin(a_i \ell/2)$, so that $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$ and $\beta_1 + \beta_2 = \beta_3 + \beta_4$; one also has $\beta_i = H(\gamma_i)$ for all i, where $H(\gamma) := \arcsin(t \sin \gamma)$ and $t := \ell/k > 1$. Next, $H''(\gamma) = (t^2 - 1)t(1 - t^2 \sin^2 \gamma)^{-3/2} \sin \gamma$, which implies that H is a strictly convex function. Now it is easy to see directly (or refer, e.g., to the general Theorem 108 in [3]) that the sets $\{\gamma_1, \gamma_2\}$ and $\{\gamma_3, \gamma_4\}$ must be the same. Therefore, the sets $\{a_1, a_2\}$ and $\{a_3, a_4\}$ must be the same. This implies that the c.p. must be degenerate.

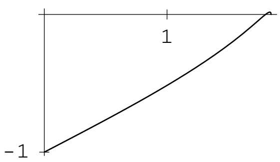
3a. That n here must be necessarily $\geqslant 5$ immediately follows from parts 1 and 2 of Theorem 2. Next, as was noted, the function g given by (5) is a w.f. Moreover, in the consideration of Case U2 in the proof of Theorem 1, it was shown that in Case E2 and under condition (1) one has that $g(0) = 0$, $g'(0) > 0$, g is concave, and $g(2/a_n) < 0$.

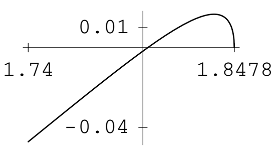


Here is the graph of function g for $\boldsymbol{a} = (3, 4, 6)$. It follows that there exists some $\nu_0 > 0$ such that for every $\nu \in (0, \nu_0)$ there exist two distinct positive roots k_1 and k_2 of the equation $g(k) = \nu$.

Observe now that any integer multiple $mf_{\boldsymbol{a},\boldsymbol{\varepsilon}}$ of any w.f. $f_{\boldsymbol{a},\boldsymbol{\varepsilon}}$ is a w.f. as well; one only has to repeat $|m|$ times each of the a_i's and each of the ε_i's, and also change $\boldsymbol{\varepsilon}$ to $-\boldsymbol{\varepsilon}$ if $m < 0$. It follows that for any *nonzero* integer w, there is an integer m such that the equation (3) with w.f. mg in place of $f_{\boldsymbol{a},\boldsymbol{\varepsilon}}$ has two distinct positive roots, say k_1 and k_2. Thus, one has two c.p.'s with the same l.s.w. triple $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$ but different radii, $1/k_1$ and $1/k_2$. Moreover, then the triple $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$ is an instance of stable non-uniqueness, because the w.f. mg is strictly concave or convex (depending on the sign of m), and $f_{\boldsymbol{a},\boldsymbol{\varepsilon}}$ continuously depends on $\boldsymbol{a}$.

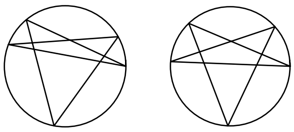
A suitable modification of this approach leads to an instance of stable non-uniqueness with as few as 5 edges (and winding number 1). Indeed, let $\varphi(b) := f_{\boldsymbol{a}(b),\boldsymbol{\varepsilon}}(2/b) = \frac{1}{\pi}(4 \arcsin \frac{1}{b} - \frac{\pi}{2})$, where $b \geq 1$, $\boldsymbol{a}(b) := (1, 1, 1, 1, b)$, and $\boldsymbol{\varepsilon} = (1, 1, 1, 1, -1)$. Then $\varphi(b)$ decreases from 3/2 to $-1/2$ as b increases from 1 to ∞. It follows that there is a unique value $b_0 > 1$ such that $f_{\boldsymbol{a}(b_0),\boldsymbol{\varepsilon}}(2/b_0) = 1$; in fact, $b_0 = 1/\sin\frac{3\pi}{8} \approx 1.0824$. One also has $f_{\boldsymbol{a}(b_0),\boldsymbol{\varepsilon}}(0) = 0$ and $\lim_{k\uparrow(2/b_0)} \frac{d}{dk} f_{\boldsymbol{a}(b_0),\boldsymbol{\varepsilon}}(k) = -\infty$. Hence, there are (at least) two distinct roots k of the equation $f_{\boldsymbol{a}(b_0),\boldsymbol{\varepsilon}}(k) = 1$ in the interval $(0, 2/b_0]$, one of the two roots being the endpoint $2/b_0$.





Here are the graphs of the function $k \mapsto f_{\boldsymbol{a}(b_0),\boldsymbol{\varepsilon}}(k) - 1$ on intervals $[0, 2/b_0]$ and $[1.74, 2/b_0] \approx [1.74, 1.8478]$. However, this non-uniqueness is not stable: it disappears if one replaces b_0 by any $b \in (1, b_0)$, because $f_{\boldsymbol{a}(b),\boldsymbol{\varepsilon}}(2/b) = \varphi(b) > 1$ if $b \in (1, b_0)$ (since $\varphi(b_0) = 1$ and φ is decreasing). To obtain an instance of stable non-uniqueness here, it suffices to replace b_0 by a slightly greater value of b, so that $f_{\boldsymbol{a}(b),\boldsymbol{\varepsilon}}(2/b) < 1$. On the other hand, b should not exceed another value, b_1, where (b_1, k_1) is the only solution of the system of equations $f_{\boldsymbol{a}(b_1),\boldsymbol{\varepsilon}}(k_1) = 1$ and $\frac{d}{dk} f_{\boldsymbol{a}(b_1),\boldsymbol{\varepsilon}}(k)|_{k=k_1} = 0$ with $b_1 > 1$ and $k_1 \in (0, 2/b_1)$, because $\max\{f_{\boldsymbol{a}(b),\boldsymbol{\varepsilon}}(k) \colon 0 \leqslant k \leqslant 2/b\} < 1$ if $b > b_1$.

Hence, one has to choose b in the rather narrow interval $(b_0, b_1) \approx (1.0824, 1.0886)$. Thus, one may choose $b = 1.0860$. Then equation (3) with $w = 1$, $\boldsymbol{a} = \boldsymbol{a}(b)$, and $\boldsymbol{\varepsilon} = (1, 1, 1, 1, -1)$ has roots $k_{1,2} \approx 1.8107, 1.8402$, which correspond to radii $r_{1,2} = 1/k_{1,2} \approx 0.5523, 0.5434$, which are close but different. The two corresponding c.p.'s, each with $\boldsymbol{a} = (1, 1, 1, 1, b)$ and $\boldsymbol{\varepsilon} = (1, 1, 1, 1, -1)$, are shown next.

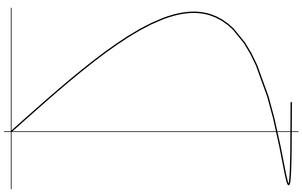


It is seen that the left-hand side c.p. has the greater radius, as its longest edge (of length $b = 1.0860$)—which is of course the one closest to the center—is farther away from the center than the corresponding longest edge of the right-hand side c.p.

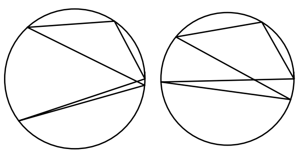
Here, the cumulative values of the central angles $\sum_{i=1}^{k} \varepsilon_i \alpha_i \, (k = 1, \ldots, 5)$ for the two c.p.'s are $\approx$ 2.26, 4.53, 6.79, 9.09, 2π and 2.34, 4.67, 7.01, 9.35, 2π, respectively. The initial vertex V_0 of each c.p. shown, here and below, is assumed to be at the rightmost point of the circumscribed circle.

To obtain an instance of stable non-uniqueness with winding number $w = 0$, one can modify the above approach as follows. Note that, for any given $a > 0$ and $q > 0$, the function $k \mapsto h(k) := \frac{1}{\pi q} \frac{\partial \arcsin(ka/2)}{\partial a}$ is continuous and increasing from 0 to ∞ on the interval $(0, 2/a)$. Hence, for every $\delta_1 \in (0, 2/a)$ there is some $\delta_2 > 0$ such that $0 < \delta h(k) \le \delta_1$ for all $\delta \in (0, \delta_2)$ and all $k \in (0, 2/a - \delta_1)$, while still $\delta h(k) \uparrow \infty$ as $k \uparrow 2/a$. On the other hand, the above function g (say again with $\boldsymbol{a} = (3, 4, 6)$) is continuous and changes sign from $+$ to $-$ in the interval $(0, 2/6)$. Hence, for $a = 6$ and every small enough $\delta > 0$, the continuous function $g + \delta h$ will have a sign change in the interval $(0, 2/6)$ from $+$ to $-$, while $g(k) + \delta h(k) \uparrow \infty$ as $k \uparrow 2/6$. It follows that the function $g + \delta h$ will have at least two roots in the interval $(0, 2/6)$, for every small enough $\delta > 0$. Hence, the same will hold for the function $mg + h$ if m is a large enough positive constant.

Moreover, for large enough q, the function h can be approximated by the w.f. $k \mapsto \frac{1}{\pi} \cdot (\arcsin \frac{k(a+1/q)}{2} - \arcsin \frac{ka}{2})$. Hence, adding the latter w.f. (again with $a = 6$) to a large enough integer multiple mg of the function g, one obtains a w.f. with at least two different roots in the interval $(0, 2/(a + 1/q))$.



Thus (using the multiplier $m = 1$) one obtains the w.f. $f_{\boldsymbol{a},\boldsymbol{\varepsilon}}$ shown here, with $\boldsymbol{a} = (3, 4, 6, 6, 6.1)$ and $\boldsymbol{\varepsilon} = (1, 1, -1, -1, 1)$. The nonzero roots of this w.f. are $k_{1,2} \approx$ 0.3110, 0.3278, which correspond to radii $r_{1,2} = 1/k_{1,2} \approx$ 3.2157, 3.0509. The two corresponding c.p.'s are shown next.



Here, the cumulative values of the central angles $\sum_{i=1}^{k} \varepsilon_i \alpha_i$ $(k = 1, \ldots, 5)$ for the two c.p.'s are, respectively, $\approx$ 0.97, 2.31, -0.09, -2.50, 0 and 1.03, 2.46, -0.32, -3.09, 0.

3b. Because of continuity in the a_i's, one can modify the two instances of stable non-uniqueness described above, with signatures $(1, 1, 1, 1, -1)$ and $(1, 1, -1, -1, 1)$, by introducing an additional number of sufficiently small edges (and also making permutations and the orientation switch, if necessary), to obtain instances of stable non-uniqueness for any signatures $\boldsymbol{\varepsilon}$ with $n \geqslant 5$ except for the signatures equivalent to those of the form $(1, \ldots, 1)$. At that, the winding number can always be made to be either 0 or 1 or -1, possibly depending on $\boldsymbol{\varepsilon}$.

3c. Suppose indeed that the signature $\boldsymbol{\varepsilon}$ contains at least four 1's and at least four (-1)'s and that the winding number w is required to be in the set $\{1, 0, -1\}$. Consider first the case $w = 0$. Since $\boldsymbol{\varepsilon}$ contains at least three 1's and at least two (-1)'s, one can start with the example of stable non-uniqueness described in the above proof of part 3a for signature $(1, 1, -1, -1, 1)$ and $w = 0$, and then add $n - 5$ sufficiently small edges (also making permutations, if necessary), to obtain an instance of stable non-uniqueness for the given signature $\boldsymbol{\varepsilon}$ and $w = 0$. The case $w = 1$ is similar; only here one should start with signature $(1, 1, 1, 1, -1)$ and $w = 1$ instead. The case $w = -1$ can be reduced to case $w = 1$ by the orientation switch.

4. One can further modify the w.f. $f_{\boldsymbol{a},\boldsymbol{\varepsilon}}$ constructed at the end of the proof of part 3a, with $\boldsymbol{a} = (3, 4, 6, 6, 6.1)$ and $\boldsymbol{\varepsilon} = (1, 1, -1, -1, 1)$, in a manner similar to the manner in which that w.f. $f_{\boldsymbol{a},\boldsymbol{\varepsilon}}$ was obtained from the function g, in order to obtain a w.f. with three nonzero roots. Continuing thus, one can obtain a w.f. with any given number (say m) of nonzero roots, which results in m c.p.'s with distinct radii but with the same l.s.w. triple $(\boldsymbol{a}, \boldsymbol{\varepsilon}, w)$ (with $w = 0$). This approach can be modified in order to deal with an arbitrary winding number $w \in \mathbb{Z}$. Alternatively, one can use the following approach.

Let $s(b) := s(b, k) := \arcsin(k\sqrt{b}/2)$ and $S(b) := 2\sqrt{b}\, s'(b)$, where $b > 0$ and $0 \le k\sqrt{b}/2 < 1$. Observe that, for $n = 0, 1, \ldots$, the nth derivative of S is given by

$$S^{(n)}(b) = \frac{(2n-1)!!}{2^n}\left(\frac{k^2}{4 - bk^2}\right)^{n+1/2}.$$

Hence, by the Leibniz formula for the nth derivative of a product,

$$\left(\frac{k^2}{4 - bk^2}\right)^{n+1/2} = \frac{2^{n+1}}{(2n-1)!!}\sum_{j=0}^{n}\binom{n}{j}(-1)^{j-1}\frac{(2j-3)!!}{2^j}b^{-j+1/2}s^{(1+n-j)}(b).$$

Approximating here the derivatives $s^{(p)}(b)$ by the pth order difference quotients

$$\delta^{-p}\sum_{i=0}^{p}\binom{p}{i}(-1)^{p-i}s(b + (i - p/2)\delta),$$

where δ is a small positive number, one obtains an approximation of the function $k \mapsto \frac{1}{\sqrt{b}}(\frac{k^2}{4-bk^2})^{n+1/2}$ by a rational multiple of a w.f.—provided that b and δ are both rational numbers. Thus, the product of the positive function $k \mapsto \frac{1}{\sqrt{b}}(\frac{k^2}{4-bk^2})^{1/2}$ and any function of k which is a polynomial in $\frac{k^2}{4-bk^2}$ with rational coefficients can be approximated arbitrarily closely by a rational multiple of a w.f. Therefore and because the continuous function $k \mapsto \frac{k^2}{4-bk^2}$ maps the interval $(0, 2/\sqrt{b})$ monotonically onto $(0, \infty)$, for any given integer w one can obtain a w.f. $f_{\boldsymbol{a},\boldsymbol{\varepsilon}}$ such that equation (3) has m roots which are arbitrarily close to $1/r_1, \dots, 1/r_m$. Thus, for any $w \in \mathbb{Z}$, in any neighborhood of any m-tuple $\boldsymbol{r} := (r_1, \dots, r_m)$ of distinct positive real numbers there exists an m-tuple $\widetilde{\boldsymbol{r}} := (\tilde{r}_1, \dots, \tilde{r}_m)$ which possesses an instance of non-uniqueness.

E.g., letting $m = 3$, $b = 1$, $\delta = \frac{1}{11}$, and $\boldsymbol{r} = (r_1, r_2, r_3) = (\sqrt{\frac{12}{24}}, \sqrt{\frac{9}{24}}, \sqrt{\frac{8}{24}}) \approx (0.707, 0.612, 0.577)$ (so that $\frac{k_i^2}{4-bk_i^2} = i$, where $k_i := 1/r_i$, $i = 1, 2, 3$), one obtains an instance of triple non-uniqueness with $w = 0$ and $\tilde{\boldsymbol{r}} \approx (0.701, 0.609, 0.597)$ (which is somewhat close to $\boldsymbol{r}$), where the edge lengths and signature can be symbolically given as

$$\left((2662)\sqrt{\frac{18}{22}},\ (3267)\sqrt{\frac{19}{22}},\ (-10087)\sqrt{\frac{20}{22}},\ (-9828)\sqrt{\frac{21}{22}},\ (14850)\sqrt{\frac{22}{22}},\right.$$
$$\left.(9828)\sqrt{\frac{23}{22}},\ (-10087)\sqrt{\frac{24}{22}},\ (-3267)\sqrt{\frac{25}{22}},\ (2662)\sqrt{\frac{26}{22}}\right).$$

Here, e.g., the "multiplier" (-10087) at $\sqrt{\frac{20}{22}}$ means that 10087 of the edge lengths a_i equal $\sqrt{\frac{20}{22}}$ and, given the minus sign, the corresponding ε_i's equal -1. Note that the number of edges here is $66538 (= 2^{16} + 2)$ and $\varepsilon_1 + \cdots + \varepsilon_{66538} = 0$. (Is there an instance of triple non-uniqueness with a number of edges less than 66538?) □

Concerning especially part 3a of Theorem 2, note also that large absolute values of the winding number w require large enough values of n.

It would be interesting to consider similar questions of existence and uniqueness of polyhedra with given edge lengths which are inscribed into a sphere.

REMARK 5. In [12, Theorem 3], a trigonometric polynomial equation of degree $n-1$ was offered, apparently for the radius of a *convex* c.p. An extension to certain non-convex c.p.'s was offered in [13, Theorem 4]. Those equations seem much more complicated and difficult to analyze than equation (3) above. In fact, no systematic analysis of those equations was given.

Acknowledgments

The author became interested in the problem of existence and uniqueness of cyclic polydons with given edge lengths after reading the front-page article [5] (which, however, concerned the rather different problem considered in [14, 15]). The author is grateful to an anonymous referee for useful and stimulating comments.

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Received 10 September 2003; revised 4 June 2004.