When is a vote an average?

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June 30, 2006 version

Abstract Is there a precise sense in which the outcome of an election represents the preference of the average voter? While the original descriptions of the Borda Count, Kemeny, Plurality, and Approval voting systems make them seem quite different from each other, we show that each is a ***mean affine voting system***, in which the outcome of an election may be determined by the average spatial location of the votes cast. These systems are more closely related than we may have thought; in a sense, differences among them arise solely from the different choices we make when we locate, as points in Euclidean space, both the votes cast and the potential outcomes. We characterize the mean affine voting systems as precisely those that are consistent, in the sense of J. Smith [1973] and Young [1974, 1975], and ***connected***. Connectedness is a sort of discrete analogue of the Intermediate Value Theorem, guaranteeing the existence of ties in certain circumstances.

Key Words mean, voting system, convex, decomposition, consistent, connected, ties, hyperplane separation, Voronoi, thin convex, affine.

§1 Introduction

Suppose we were given a finite sequence of numbers, such as 3, 4, 11, 3, 9, and asked to name a single number that best represents the list. One possibility would be to take the average, or mean,[3] of the numbers (counting the 3 twice). On the other hand, if we were given a finite sequence of opinions (about who should be the next president, for example) and we were asked to make a decision that best represents the varying opinions, we might hold an election.

To what extent are these processes similar? That is, to what extent does the outcome of an election represent the opinion of a "mean" voter? Once we make this question precise (in one of several ways) we find that the answer depends on the choice of voting system.

[1] We gratefully acknowledge support from Grant SAB 2004-0264 of the Spanish Ministry of Education and Science. Much of the work was done while the author was visiting the Departamento Fundamentos del Análisis Económico at the University of Alicante and the Economics Department at the University of Caen.
[2] We thank Vincent Merlin and Hannu Nurmi for their assistance. Particular thanks are owed to Juan Enrique Martinez-Legaz for invaluable assistance with some key ideas in the proof.
[3] Alternatively, we might use some form of median. See §6.

In section 2 we point out that each of the following well-known voting systems – Borda count (or any scoring rule), Kemeny Rule, approval voting – can be given a ***Mean Proximity*** representation:[4]

- Each possible *vote* v is plotted as a *rational point* $R(v) \in \mathbf{Q}^n \subseteq \mathbf{R}^n$, via an *input plot function* R. These points are the vertices of some polytope $P \subseteq \mathbf{R}^n$.
- Each possible *election output* t is plotted as a rational point E(t) in P, via an *output plot function* E.
- The mean location q of the plotted votes (counting multiplicity) is found.
- The *election outcome* is the set of those outputs t for which E(t) is closest to q, with ties occurring when several points E(t) are equally closest to q.

Throughout this paper, except where we specify otherwise, "closest" refers to the Euclidean, ℓ_2 metric. Notice that this definition implies anonymity (symmetric treatment of voters), but not neutrality (symmetric treatment of alternatives).

A good example is the Borda count for three alternatives p, q, and r: each voter submits a strict preference ranking of these three, awarding +2 points, 0 points, and -2 points to their favorite, middle, and least favorite alternative, respectively.[5] The Borda election outcome is either the set of all *alternatives* who share the greatest point total (containing a single alternative when there is no tie), or is the *ranking* of the alternatives in descending order of point totals – a weak ranking, when there are ties. We consider both contexts.

Here is a mean proximity representation: plot the preference ranking $\sigma = p > q > r$ as the point $R(\sigma) = (2, 0, -2)$ in $\mathbf{R}^3$. Each other ranking τ is plotted as the corresponding vector of points awarded, by anyone voting τ, to p, q, and r respectively *taken in that order*. For example $\tau = q > r > p$ would yield $R(\tau) = (-2, 2, 0)$. The points $R(\tau)$ for the six possible rankings all lie on the plane $x_1 + x_2 + x_3 = 0$, and are the vertices of the regular hexagon appearing in Figure 1, which also shows the result of the sample election of Table 1.[6] The solid radial lines divide the hexagon into six "proximity regions." Each region contains the points of the (filled in) hexagon that are at least as close to the vertex of that region as they are to any other vertex. The 8 votes from Table 1 appear as labels on the corresponding vertices.

Adding the eight vectors corresponding to the coordinates of these vertices, and dividing by 8, we get the mean location of the plotted votes. It lies in the proximity region of the ranking $r > q > p$, which is thus the "winning" ranking. It is straightforward to show (see Zwicker [2005b]) that the result of this geometric calculation agrees with that of the Borda count for every possible profile.

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[4] A precise definition appears in section 5.

[5] It is more common to use Borda ***scoring weights*** of 3, 2, and 1 points (or of 2, 1, and 0), but these weights are equivalent; see the Observation in example 2.1.1.

[6] Although Fig 1 depicts the hexagon flat on the page, it is actually impossible to draw a regular hexagon in $\mathbf{R}^2$ so that every vertex gets rational coordinates. Given the important role of rational coordinates in the general theory, it is perhaps best to think of this hexagon as "living" on the plane $x + y + z = 0$ in $\mathbf{R}^3$ (see example 2.1.1).

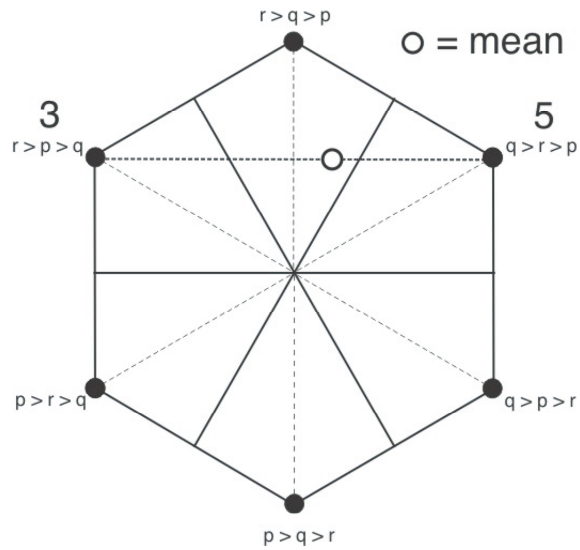

Figure 1 A mean proximity representation of the Borda count for three alternatives p, q, and r.

Profile #1	<u>0</u>	<u>0</u>	<u>3</u>	<u>0</u>	<u>5</u>	<u>0</u>
	p	p	r	r	q	q
	q	r	p	q	r	p
	r	q	q	p	p	r

Table 1 Each underlined number counts the votes for the ranking appearing beneath it.

Different profiles can place the mean at any point with rational coordinates that lies in the convex hull of the six vertices (the solid hexagon).

In this example each input (individual vote) and output (non-tied election result) is a ranking, and the functions R and E are the same. Voting systems that output rankings in this way are ***social welfare functions***.[7]

What about representing the Borda Count as a ***social choice function***, in which the possible outcomes are the sets of alternatives? The mean proximity representation for this context is very little different: the output p would be plotted at the midpoint of the two vertices corresponding to the rankings that place p on top. Thus E(p) = (2, -1, -1), E(q) = (-1, 2, -1) and E(r) = (-1, -1, 2). The proximity regions are subregions of the original hexagon, and each is the union of two bordering regions of Fig 1.

We do not know of a satisfactory characterization of the mean proximity voting systems. However, if we relax the requirement that outcome regions in our representations be

[7] Suppose the mean lies on a boundary between two (or six) proximity regions of the hexagon. Then the outcome given by the representation is the corresponding *set* T containing two (or six) strict rankings, and T is equivalent to a unique weak ranking (see Remark 2.1).

determined by proximity, while continuing to require that these regions share many of the qualitative features (see §4) of proximity regions, then the resulting, expanded class – the ***mean affine*** voting systems – is also quite natural, and seems more amenable to axiomatization. Specifically, the mean affine systems arise when we require that each pair of outcome regions be ***neatly separable*** – weakly separable by some hyperplane that has the same intersection with both regions. Section 2 provides examples that illustrate the distinction between these two classes.

Theorem 5, our principal result, characterizes the mean affine voting systems as those that are both ***consistent*** and ***connected***. Consistency was introduced in J. Smith [1973] and in Young [1974, 1975], and used by Myerson [1995], in their characterizations of scoring rules (see §6 for details). Young and Levenglick [1978] used it in their characterization of the Kemeny Rule (see §3.2).

Connectedness, a new property, is a discrete analogue of the Intermediate Value Theorem, and a natural partner to consistency, in that it guarantees the existence of exactly those situations – ties – governed by down consistency (one of the "components" of consistency). It is related to properties used in the three characterizations of scoring rules cited above.

Our results here build on two additional bodies of work in the literature. Barthélemy and Monjardet [1981] showed that several "median methods," such as the Kemeny voting rule, can be expressed via geometric representations and the mean.[8] Anyone familiar with the work of Saari [1994], and of Saari and Merlin [2000], will recognize the significant debt we owe to their geometric perspective. One example is our use, in examples 2.1.3 and 2.1.4, of Saari's *Representation* (hyper) *Cube*.

The rest of the paper is organized as follows. In section 2 we introduce several examples of voting systems and mean representations. Section 3.1 discusses the *abstract anonymous voting systems* of Myerson [1995] – the very broad class to which our main result applies. The definition deliberately blurs the distinctions between contexts that are typically treated separately in the literature. While this is essential for our purposes, a consequence is that comparisons between our results and others in the literature should be approached cautiously, with attention to the issue of context.

Section 3 continues with a discussion of the consistency and connectedness properties, including application to some of our examples. In section 4 we define ***mean representation*** and two associated classes of voting systems: the ***mean proximity*** and ***mean affine*** classes. Section 5 outlines, and begins, the proof of the main theorem. The remainder of the proof, which entails technical results in affine and convex geometry, appears in the appendix, and draws heavily on Cervone and Zwicker [2005]. In section 6

[8] Their representation of the Kemeny rule is different from ours. Note that our use of Barthélemy and Monjardet's term "median method" is likely to cause confusion in the current context. The methods they discuss are *medians* with respect to the Kendall metric, yet some prove to be *means* when expressed in terms of the Euclidean metric. The median methods in Zwicker [2005b] are quite different – they use a form of spatial median defined in terms of the Euclidean metric.

we discuss the relationship between our results and those in Myerson [1995], and point to several future lines of research.

§2 Examples of voting systems

We provide mean proximity representations for a first group of voting systems, followed by a single example of a system that has a mean affine representation, but lacks any mean proximity representation.[9] Our third group of examples lack representations of either type.

Remark 2.1 Social "welfare/preference" functions Several of our examples are both ***social choice functions*** – an election outcome is a non-empty set of alternatives, containing a unique alternative when there is no "tie" for first place – and ***social welfare functions*** – an election outcome is a single weak ordering of all alternatives, which is a strict ordering when there are no "ties" for first place, or for second, or (etc). Our framework, however, will require us[10] to represent a weak ordering σ in terms of the corresponding set $T(\sigma)$ of all strict orderings that refine σ (by breaking all ties in all possible ways). We'll use the term ***social welfare/preference function*** to refer to a ***converted*** social welfare function in which each election outcome is such a ***tie generated*** set $T(\sigma)$ of strict rankings. The term ***preference function*** was introduced in Young and Levenglick [1978] to describe the context for the Kemeny rule, in which an election outcome is a set T of strict rankings that is *not* always tie generated – T need not equal $T(\sigma)$ for any weak ordering σ.

2.1 Voting systems with mean proximity representations

It is relatively straightforward, with the exception of the Kemeny rule, to show that the particular representations given here indeed yield election outcomes identical to those of the original systems. The proofs appear in Zwicker [2005a, 2005b], and in footnote 20.

Example 2.1.1 – Rational Scoring Systems In a (simple)[11] rational[12] scoring system, each voter casts a ballot consisting of a strict ranking of the f alternatives. Each rank is

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[9] Precise definitions appear in §4.

[10] Smith [1973] and Young [1974] provide a definition of consistency for social welfare functions. Consistency (in their sense) of a social welfare function Y implies consistency (as we define it in §3) of the converted rule Y*, but the converse is false.

[11] We are using "simple" to distinguish these systems from the compound or general scoring systems of Smith [1973] and Young [1974], which allow a hierarchy of tie-breaking scoring vectors. Saari [1994] uses the term "positional voting" for simple scoring systems. However, there are voting methods that are not scoring systems, yet are defined solely in terms of positional information.

[12] It is equivalent to require that the scoring weights be integers (simply scale by a common denominator). Our rationality requirement here is not standard, but we will need it (see definitions 4.2.9, 4.2.10, and additional discussion in §6). Note that if we set an upper bound

pre-assigned a fixed rational number ***scoring weight***, and these weights form the ***scoring vector*** $(w_1, w_2, \ldots, w_f)$, which is required to satisfy $w_1 \geq w_2 \geq \ldots \geq w_f$ and $w_1 > w_f$. Each voter "awards" to their top-ranked alternative w_1 points, to their second ranked candidate w_2 points, etc. Each alternative receives the total of all points awarded. To obtain a social welfare/preference function we take the *weak ranking* of alternatives in descending order of point totals, and convert it as in Remark 2.1. For the social choice function version, we choose all alternatives that share the highest total score achieved. The Borda count and the plurality voting rule (with scoring vector $(1, 0, 0, \ldots, 0)$) are important special cases.[13] One useful fact is contained in the following, well-known observation.

Observation Scoring vectors $w = (w_1, w_2, \ldots, w_f)$ and $v = (v_1, v_2, \ldots, v_f)$ induce the same social choice (and social welfare) function if and only $v = \alpha w + (\beta, \beta, \ldots, \beta)$ for some real constants $\alpha > 0$ and β. In particular, the Borda count is induced by *any* scoring vector w for which $w_{i+1} - w_i$ is a fixed, strictly positive amount for all $i = 1, \ldots, f-1$.

To obtain a mean proximity representation, we begin by enumerating our set A of alternatives as $\{a_1, a_2, \ldots, a_f\}$. Let F denote $\{1, 2, \ldots, f\}$. A vote is then a bijective map $\sigma: F \rightarrow F$, corresponding to a voter who ranks alternative a_j in the "$\sigma(j)^{th}$ position" (where the first position is assigned to that voter's most favored alternative, and the f^{th} position is assigned to her least favored. The input plot function R takes any vote σ and positions it at the location $(w_{\sigma(1)}, w_{\sigma(2)}, \ldots, w_{\sigma(f)})$. The mean location q of the plotted votes is found, and a ranking is then in the set of winning rankings if its plotted position is among those closest to the mean. Thus the outcome plot function E is the same as the input plot function.

We need a different E, however, if we wish to represent a scoring rule as a social choice function. For each individual alternative a_j, $E(a_j)$ is located at the vector average of all points $R(\sigma)$ for which σ is a strict ranking in which a_j is top ranked.

Example 2.1.2 – Approval voting In approval voting (see Brams and Fishburn [1983]), a vote is a subset L of the set A of alternatives (containing those alternatives of which the voter "approves"). Two voters may approve of different numbers of alternatives, and bullet votes are allowed. The social welfare/preference version ranks alternatives in descending order of the number of approving voters, and then applies Remark 2.1. To obtain a social choice function, choose the alternatives who share the highest number of approving voters.

In our representation, for each $L \subseteq A$ let $R_{approval}(L)$ equal the vector average of all points $R_{Borda}(\sigma)$ (defined in the previous example 2.1.1) for which σ is a strict ranking in which every alternative in L is ranked over every alternative not in L. Once the vector mean q

n on the number of voters then for every scoring vector there exists a rational vector that induces the same voting system.

[13] As with many natural ideas, the Borda count has been frequently rediscovered. At this time, the earliest known discoverer is Nicolaus Cusanus, (1401-1464) – see Hägele and Pukelsheim [2000, 2006], Pukelsheim's note in Garfunkel [2003], and McLean and Urkel [1995].

of the plotted votes is found, for the social welfare/preference version we choose the rankings σ for which $R_{Borda}(\sigma)$ is closest to q. For the social choice function representation, let $E(a) = R_{Approval}(\{a\})$, for each alternative $a \in A$.

Example 2.1.3 – The Kemeny Rule This system, proposed in Kemeny [1959], is closely related to several independent constructions that use a type of discrete median (see Barthélemy and Monjardet [1981]). An input vote is again a strict ranking σ of the set of alternatives. The rule employs the (non-Euclidean) ***Kendall metric*** d defined over rankings as follows: $d(\sigma, \tau)$ is equal to the number of pair-wise disagreements between the rankings – the number of pairs (a_i, a_j) for which $i < j$ and the rankings σ and τ differ as to which of these two alternatives a_i and a_j are ranked over the other.

This metric extends to a measure D of "total distance" between an individual ranking σ and a ***profile*** p – that is, an election in which each voter $v \in N$ chooses a strict ranking p(v) of the alternatives – by summing: $D(p,\sigma) = \sum_{v \in N} d(\sigma, p(v))$. The Kemeny rule election outcome is the set of rankings σ that minimizes this total distance $D(p, \sigma)$.[14] The rule is a "median" in that D is given as the sum of distances, not *squared* distances.

Before we give the details of the geometric representation for an arbitrary number of alternatives, we illustrate the special case for three alternatives, p, q, and r, in Figure 2. The 6 possible strict rankings are plotted as 6 of the 8 vertices of a 2×2×2 cube in $\mathbf{R}^3$. These 6 are the ***transitive*** vertices. The two "unused" ***intransitive*** vertices of this cube turn out to correspond to the two possible cycles: $p > q > r > p$, and $r > q > p > r$. Each ranking σ is plotted at the point $R(\sigma)$ indicated in the figure, the vector mean q of these points is determined, and the outcome is declared to be the set of rankings that label the vertices (from among the *transitive* vertices) closest to q.

Figure 2 also shows the 8-voter election from Table 1.1, and the mean location of the plotted votes (again shown as a hollow circle). It is easy to see that in this case the (sole) closest transitive vertex is (-1, +1, +1), corresponding to the ranking $q > r > p$, which is thus the Kemeny outcome. Recall that with the Borda count, the hexagon in Figure 1 showed that the outcome of the same election was different: $r > q > p$.

In fact, the 6 transitive vertices of Figure 2 do form a (non-planar) hexagon, and the order in which the rankings are plotted around this hexagon is the same as in Figure 1, but the Figure 2 hexagon is bent, so that the vertex angles are 90° rather than 120°; as a result, the vertex labeled $r > q > p$ in Figure 2 is pulled away from the line joining the two vertices that actually receive any votes. Thus, the $r > q > p$ vertex is now relatively farther from the mean than it was in Figure 1, and is no longer the closest.

For the general case, assume A is our set containing f alternatives. Choose any set $G \subseteq A \times A$ such that for each pair (x, y) of distinct alternatives of A, exactly one of (x, y)

[14] The outcome need not be tie generated (see Remark 2.1).

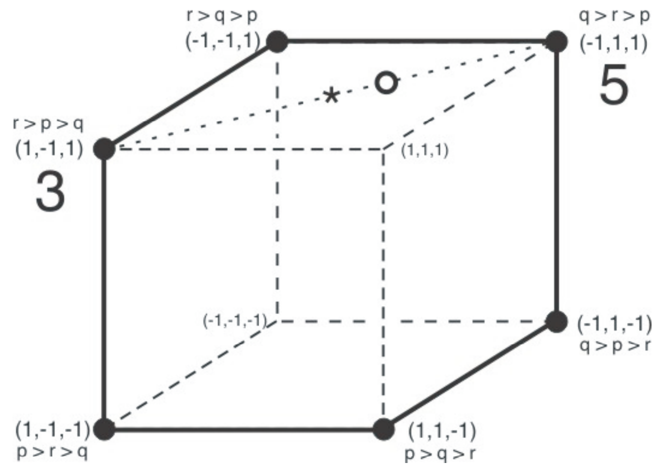

Figure 2 A mean proximity representation of the Kemeny rule for three alternatives.

or (y, x) is a member of G. Choose any enumeration $P = \left((b_1, c_1),\ (b_2, c_2),\ \ldots,\ (b_g, c_g)\right)$ of G, with $g = \binom{f}{2}$. Any ranking σ now generates a vector $R_{Kem}(\sigma)$ of +1s and -1s, according to whether σ ranks b_i over c_i:

$$R_{Kem}(\sigma)_i = \begin{cases} +1, & if \ \ \sigma(b_i) > \sigma(c_i) \\ -1, & if \ \ \sigma(b_i) < \sigma(c_i) \end{cases}.$$

We can recover each ranking σ from its image $R_{Kem}(\sigma)$, but some vectors of +1s and -1s are not ***transitive*** – that is, are not images under R_{Kem} of any transitive ranking σ. Thus our input plot function R_{Kem} places each ranking at a vertex of a certain hypercube, but only the transitive vertices are used. This same map R_{Kem} is the output plot function, and so the election outcome is a set containing one or more rankings.

We might consider the approach in J. Smith [1973] and construct a social choice version of the Kemeny rule as follows: the social choices are those alternatives atop at least one of the rankings that minimize D(p,σ). Unlike the original rule, however, this version is no longer consistent. As a consequence of our Theorem 5, then, this particular version of Kemeny as a social choice function has no mean affine representation.[15]

Example 2.1.4 – The "Extended" Condorcet Rule The Marquis de Condorcet [1785, 1994] suggested that alternatives be ranked by ***pairwise majority rule*** – i.e. that alternative a_i be ranked over alternative a_j when a strict majority of voters place a_i over a_j in the individual rankings that they submit as votes. Condorcet knew, of course, that this rule can lead to cycles and, more generally, to intransitive binary relations. The question of what to do when this happens – that is, of which "Condorcet extension" to use when

[15] But perhaps it is not so clear what the correct definition of the "Kemeny social choice function" should be, or even whether a good definition exists.

choosing election outcomes for those profiles yielding an intransitive majority preference relation – has been considered by many authors.

The method we are calling the "extended Condorcet rule" ducks this issue entirely. Loosely, it declares the election outcome to be the majority preference relation, whether or not it this relation proves to be transitive. The resulting rule is completely impractical, then, as a real voting system, but its geometric representation sheds some light on the issues raised in this paper, and on the role of the Kemeny rule as the unique consistent and neutral Condorcet extension (Young and Levenglick, [1978]).

In fact, we need to be a bit more careful in defining the extended Condorcet rule: the outcome of an extended Condorcet election is the set of all anti-symmetric and total (complete) binary relations B such that B contains the strict pairwise majority relation M. Thus, if there are exact ties in majority preference in one or more pairs of alternatives, our rule yields all possible ways to break all such ties.

For our mean proximity representation of this rule in the special case of 3 alternatives, we again refer the reader to Figure 2. The input plot function R for the extended Condorcet representation is that shown in the figure – exactly the same as for the Kemeny rule. The output plot function E for the extended Condorcet rule is an extension of this R; it employs R for the transitive rankings and maps the two possible cycles to the remaining two intransitive vertices of the cube, with the cycle $p > q > r > q$ mapped to (1, 1, 1) and the reverse cycle mapped to (-1, -1, -1).

In some ways the idea is more clear if we imagine starting with this mean proximity representation for the extended Condorcet rule. To obtain the representation of the Kemeny rule, discard the two cycles as potential outcomes, discarding as well as the corresponding pair of intransitive vertices. Now consider the 1×1×1 subcube consisting of those points of the bigger cube that are at least as close to the (1, 1, 1) vertex as to any other vertex. Once the intransitive vertices are discarded, the points in this subcube get divided up and parceled out to the proximity regions of 3 of the 6 remaining transitive vertices, while the points in the subcube corresponding to the other cycle are similarly reassigned to the proximity regions of the other 3.[16]

If we allow an arbitrary number f of alternatives, the mean proximity representation of the extended Condorcet rule should now be clear. The input plot function R is exactly the same as defined earlier, for the Kemeny rule. The output plot function E extends R by applying the definition given earlier for R to an arbitrary antisymmetric and total binary relation B, in the obvious way:

[16] Our debt to Saari [1994] and to Saari and Merlin [2000] is clear. There the reader will find related statements, and pictures. However, in their description of the Kemeny rule these authors do not describe these parceled out sections of subcubes in terms of proximity to vertices, but rather in terms of proximity to other subcubes. Consequently (see here Zwicker [2005a]) they must use the l_1 metric rather than the Euclidean (l_2) metric. It seems to us that the l_2 version yields additional information as to the geometric relationship between Condorcet and Kemeny, and insight into the connection between the Young and Levenglick characterization and this geometry.

$$E(B)_i = \begin{cases} +1, & if \ (b_i, c_i) \in B \\ -1, & if \ (c_i, b_i) \in B \end{cases}$$

Example 2.1.5 – A mean proximity grading system Suppose that in Mathematics 101 each student takes several exams during the semester, with each graded on a scale from 0 to 100 (and no fractional points awarded). These grades are then amalgamated to determine a letter grade for the course, with the possible letter grades being A (the highest course grade possible), B, C, D, and F (failing).

One way to amalgamate is to assign standard scores to each letter grade (say, A = 95, B = 85, C = 75, D = 65, F = 55), and to award as course grade the letter whose standard score is closest to a student's average test grade. This system is close to what is actually used in many colleges and universities in the United States.[17] The example may not look, at first, like a voting system, but it fits the broad definition we introduce in section 3; the "voters" are the exams, they "vote" on a particular student's level of knowledge, and the outcome of the "election" is that student's course grade. In fact, related grading systems provide important examples of voting with multiple levels of approval, as developed in Freixas and Zwicker [2003], [2005]; the weighted symmetric (j,k) games defined in the latter paper, when suitably modified to allow for a variable electorate, provide examples of mean affine voting systems if the weights and quotas are rational.

For this example, the one-dimensional mean proximity representation is implicit in the original description: the input plot function R is the identity function and the output function E is the assignment of standard scores to letter grades. Note that the set {0, 1, . . ., 100} of inputs is quite different from the set {A, B, C, D, F} of outputs, and that each endpoint dividing one proximity interval from the next must lie halfway between the two corresponding standard scores.

2.2 A mean affine system with no mean proximity representation

Example 2.2 Suppose that in Physics 101 there are frequent short quizzes given throughout the term. Each quiz lasts 5 minutes, consists of a single question, and receives one of two possible scores: 1 (for "correct") or 0 (for "incorrect"). A student's course grade is determined by the percentage M of quizzes that are correct, according to the following table:

Course grade of:	A	if $100\% \geq M \geq 80\%$
	B	if $80\% \geq M \geq 78\%$
	C	if $78\% \geq M \geq 58\%$
	D	if $58\% \geq M \geq 56\%$
	F	if $56\% \geq M \geq 0\%$

[17] Much of the rest of the world emplys a single, final examination. In one respect Example 2.1.5 is *not* very realistic – it yields "ties" when test averages fall exactly halfway between two standard scores. See discussion in §3.3.

We've chosen the five output intervals to alternate in length, with the longer intervals much longer than the shorter ones. With this variation in length, it is straightforward to show that these intervals cannot be proximity regions – it is impossible to assign standard scores to letter grades in such a way that each interval endpoint is the midpoint of adjacent standard scores.

It follows that this example does not have a mean proximity representation, but it is straightforward to provide a mean *affine* representation, in the sense of definition 4.2.10. This example is less realistic than the previous one, because it would be unusual to have a course grade be determined solely by this type of quiz, and even more unlikely to have such an extreme variation in grade intervals. However, we see later that the inductive nature of the proof of the main theorem forces us to consider similar examples.

In fact, the varying intervals of this example *could* be thought of proximity regions, if we allowed the points to which proximity is measured to lie above or below the number line that records percentages; the long intervals could use points located slightly above (or even at) their midpoints, with the short intervals using points that are located significantly below their midpoints. However, this would violate the definition of "mean proximity representation," which requires that these points E(t) lie in the polytope P spanned by the plotted input points R(v).[18] Also, note that points located off the number line would no longer correspond to "standard scores."

2.3 Voting systems with no mean affine representations

As discussed in sections 3.2 and 3.3, each of the following rules fails either to be consistent or to be connected. Theorem 5 then tells us that they have no mean affine representations.

Example 2.3.1 – The Copeland Rule An alternative a_i is ranked over a_j if the *number* of alternatives a_k for which a strict majority of voters rank a_i over a_k in their individual (strict) rankings is strictly greater than the number of alternatives a_k for which a strict majority of voters rank a_j over a_k. As in Remark 2.1, we obtain a social welfare/preference function by converting this single weak ranking into an equivalent set of strict rankings. Alternately, as a social choice function the Copeland rule chooses the alternatives who defeat the greatest number of other alternatives in the strict pair-wise majority sense. If there exists an alternative who defeats every other alternative in this sense (a ***Condorcet alternative***, or ***Condorcet winner***), then this alternative will be the sole Copeland social choice; the Copeland rule is a ***Condorcet extension***. The rule is particularly simple in the special case of three alternatives with an odd number of voters (so that the pair-wise majority rule relation M is total). In this case the Copeland *ranking*

[18] But is it *appropriate* to require E(t) ∈ P in the definition? We believe that the requirement is appropriate precisely because it is satisfied by so many natural examples, such as 2.1.1 - 2.1.5. However, this is largely a matter of taste. In Cervone and Zwicker [2005] we discuss the related distinction between affine , quasi-Voronoi, and Voronoi decompositions.

is either the Condorcet ranking or, in the case of a majority cycle, is a completely tied ranking (which gets converted to a six-way tie among all possible strict rankings). The Copeland social *choice* is either the Condorcet alternative or, in the case of a majority cycle, is the set containing all three alternatives. In both social choice and social welfare/preference forms, Copeland fails to be consistent.

Example 2.3.2 – The Hare System This system, proposed by Thomas Hare (see Hare, [1859]) is also known as a special case of "single transferable vote" (or "STV"). It is most common in countries that once belonged to the British empire and is perhaps the most popular alternative voting system among amateur societies that favor voting reform (try Googling "electoral reform"). In Hare, an alternative is eliminated from each voter's strict ranking if it is among the alternatives top-ranked least often by the voters; imagine striking such alternatives from each voter's list and "closing up" the list, if need be, to fill the gap. Several alternatives, or even all the alternatives, may be eliminated simultaneously. These elimination rounds are repeated until all alternatives have been eliminated. The Hare ranking lists alternatives in terms of the number of rounds they survive, with the final batch eliminated ranked at the top. As before, we'll convert the resulting weak ranking to a corresponding set of strict rankings. The Hare social choices are the alternatives eliminated in the last round. As a social choice function, Hare is not consistent; as a social welfare/preference function, it is not connected.

Example 2.3.3 – The Omninominator rule Each voter ranks all alternatives. An alternative is among the winners if and only if she is top-ranked by at least one voter. This social choice function plays a role in the study of manipulation (see Taylor [2005]). It satisfies an important weak form of consistency, but is not fully consistent.

Example 2.3.4 – The Putin System (This example, along with the next, is a "toy" voting method, constructed to serve as an example that is consistent, but has no mean affine representation because it is not connected.) There are two alternatives, Putin and Bontilusic. Each voter casts a ballot for Putin, or for Bontilusic. Bontilusic wins if every voter, without exception, votes for Bontilusic, and Putin wins otherwise.

Example 2.3.5 – The Pythagoras System There are two alternatives, Pythagorus and Archytas. Each voter casts a ballot for one of the two. Archytas wins if the ratio

$$\frac{\textit{number of votes cast for Archytas}}{\textit{number of votes cast for Archytas} + \textit{number of votes cast for Pythagoras}}$$

exceeds $\sqrt{2}$, and Pythagoras wins otherwise.

§3 Consistency and connectedness in abstract anonymous voting systems

3.1 Abstract anonymous voting systems

The following definitions are due to Myerson [1995]:

Definition 3.1.1 Let I be a finite set (of ***input*** ballots) and O be a finite set (of election ***outputs***).[19] Let $\mathbf{Z}^+$ denote the set of non-negative integers. An ***anonymous profile***, henceforth ***profile***, is a function p: $I \rightarrow \mathbf{Z}^+$ satisfying N(p) > 0, where N(p) = $\sum_{j \in I} p(j)$ denotes the number of voters in the profile. We'll use $(\mathbf{Z}^+)^I$ to denote the set of all such profiles. An ***outcome*** is a non-empty subset of O, representing a potential election result; $C(O)$ denotes the collection of all such outcomes.

Definition 3.1.2 An ***abstract anonymous voting system*** $\mathcal{V}$ is a triple (I, O, $\mathcal{F}$), in which I and O are finite sets, and $\mathcal{F}$: $(\mathbf{Z}^+)^I \rightarrow C(O)$ is a function that specifies an election ***outcome*** for each possible profile.

The high level of abstraction encompasses the broad range of examples in the previous section, and is different from what we normally see in the theory of voting. In particular, Definition 3.1.1 imposes **no** structure on the set I of input ballots or on the set O of election outputs; a member of I may be anything whatsoever, including:

- an individual alternative (as in Plurality voting, wherein each voter may vote for a single candidate or alternative), or
- a set of alternatives (as in Approval voting), or
- a strict ranking of alternatives (as in the Borda count, or Kemeny rule), or
- an integer between 1 and 100 (as in some grading systems).

The distinction between outputs and outcomes is an important one; if $\mathcal{F}$(p) = {g}, then g is the sole winning output of the election, and when $|\mathcal{F}(p)| > 1$, the outcome is a tie among the output elements in $\mathcal{F}$(p). Our profiles are anonymous in that they do not keep track of who votes for each input, but only of how many do so, with p(j) representing the number of voters who vote for j.

Despite its breadth, Definition 3.1.2 has two implications that constitute limitations of a sort. First, in order for Definition 3.1.2 to work well with the definitions of consistency and connectedness, we are forced to represent social welfare functions as "social welfare/preference" functions – see Remark 2.1. Second, we are placing no upper bound

[19] Myerson [1995] calls members of O "alternatives," not "outputs." We've chosen to distinguish between the terms "alternative" and "election output," because it seems appropriate to do so for examples such as Kemeny and Extended Condorcet.

on the number N(p) of voters; the domain $(\mathbf{Z}^+)^{\mathcal{I}}$ of an abstract anonymous voting system is infinite, or "unbounded."

3.2 Consistency

In addition to its use of in characterizing scoring rules (see section 6), this property was employed by Young and Levengick [1978] to show that among preference functions the Kemeny rule is the unique neutral and consistent Condorcet extension.[20] The implications of consistency appear to be different in these results, because the contexts are not the same. Abstract anonymous voting systems are more or less context-free, and so the question of whether or not a given voting system is consistent according to the definition that follows may depend on how we view that system as an abstract system – that is, on what we declare to be the input set $\mathcal{I}$ and output set $\mathcal{O}$.

Definition 3.2.1 An abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is ***consistent*** if for every pair p_1 and p_2 of profiles satisfying $\mathcal{F}(p_1) \cap \mathcal{F}(p_2) \neq \emptyset$, $\mathcal{F}(p_1 + p_2) = \mathcal{F}(p_1) \cap \mathcal{F}(p_2)$.

Here $p_1 + p_2$ denotes the sum of p_1 and p_2 as functions: $(p_1 + p_2)(j) = p_1(j) + p_2(j)$ for each $j \in \mathcal{I}$. Intuitively, $p_1 + p_2$ represents the profile of the electorate obtained as the disjoint union of the separate electorates giving rise to p_1 and to p_2 respectively.

The following weak forms, or "components," of consistency, will be useful:

Definition 3.2.2 An abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is

- ***homogeneous*** if for every profile p and integer $k \geq 1$, $\mathcal{F}(kp) = \mathcal{F}(p)$,
- ***weakly consistent*** if for every pair p_1 and p_2 of profiles satisfying $\mathcal{F}(p_1) = \mathcal{F}(p_2)$, $\mathcal{F}(p_1 + p_2) = \mathcal{F}(p_1) = \mathcal{F}(p_2)$,
- ***up consistent*** if for every pair p_1 and p_2 of profiles, $\mathcal{F}(p_1 + p_2) \supseteq \mathcal{F}(p_1) \cap \mathcal{F}(p_2)$, and,
- ***down consistent*** if for every pair p_1 and p_2 of profiles satisfying $\mathcal{F}(p_1) \cap \mathcal{F}(p_2) \neq \emptyset$, $\mathcal{F}(p_1 + p_2) \subseteq \mathcal{F}(p_1) \cap \mathcal{F}(p_2)$.

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[20] As a corollary of this result, we obtain a short proof that the voting rule "K" given by the geometric representation in example 2.1.2 is the same as the Kemeny rule. It is clear from the general description that K is anonymous, and the symmetry inherent in the geometry guarantees that K is neutral. The easy direction of Theorem 5 tells us that, as K is mean affine, it is consistent as a preference function (see remark 2.1). It is easy to see that K is a Condorcet extension, as follows: divide the original $2\times2\times \ldots \times2$ cube into $1\times1\times \ldots \times 1$ subcubes, each containing a unique vertex of the larger cube. If the pairwise majority relation M is transitive, giving a weak ranking σ, then the mean location q will lie in all subcubes whose transitive vertices correspond to strict rankings extending σ (and in no other subcubes), so these vertices will collectively be closest (among transitive vertices) to q. We conclude that K is a neutral, anonymous Condorcet extension. By the theorem of Young and Levenglick [1978], K is the Kemeny rule.

Homogeneity is what allows the minimal sort of geometric representation to take place at all, because it tells us that the outcome of an election depends only on the *fraction of voters* (or relative proportions among the integers p(i)) voting for each possible input $i \in \mathcal{I}$, rather than on the absolute vote totals p(i). Smith [1973], page 1029, argues that it is an extremely natural property to expect. Surprisingly, however, in Fishburn [1977] we learn that the Dodgson and Young procedures can each fail to be homogeneous, depending on the some details in the precise formulation of these systems. Many other "natural" voting systems (including all examples in section 2) do satisfy homogeneity. Weak consistency appears in Saari [1994].

Proposition 3.2.3 The Hare system is not weakly consistent[21] as a social choice function.

Proof Consider the following profile of strict preferences for the four alternatives A, B, C, and D: ***Profile p_1***

8	3	2	4
A	B	C	D
B	C	B	A
C	D	D	C
D	A	A	B

Here, the underlined numbers give the number of voters who cast ballots for the corresponding strict rankings.

It is straightforward to check that the Hare procedure, applied to p_1, eliminates C first, then D, then B, then A, and the social choice is {A}. Let profile p_2 be obtained from profile p_1 by transposing alternatives B and C. Clearly, for p_2 Hare eliminates B first, then D, then C, then A, and the social choice is again {A}. If we set $p_3 = p_1 + p_2$, consistency demands that $\mathcal{F}(p_3) = \{A\}$, yet we find that for p_3, B and C are eliminated simultaneously in the first stage, then A, then D, and $\mathcal{F}(p_3) = \{D\}$. ∎

Together, "Up" and "Down" consistency imply full consistency, so we might think of them as the two halves of consistency. A related property called containment consistency appears in Merlin [1996a, b] (also see Chebatarev and Shamis [1998]). Suppose we consider the special case in which $\mathcal{O}$ consists of individual alternatives – candidates, for an elected presidency, for example. Up consistency then tells us that if two separate electorates elect Jill (with one or both possibly producing a tie among several candidates, including Jill) then the combined electorate should also elect Jill. A failure of up consistency would seem fairly outrageous, especially to the voters who supported Jill.

As pointed out in Saari [1994], the intuitive content of down consistency is quite different, telling us (mostly) that ties are unstable or "knife-edge."[22] For example, if p_1 and p_2 are profiles satisfying $\mathcal{F}(p_1) = \{a, b\}$ and $\mathcal{F}(p_2) = \{a\}$, then down consistency implies that $\mathcal{F}(p_1 + p_2) = \{a\}$. This holds even when $N(p_1) = 1{,}000{,}000$ and $N(p_2) = 1$, so

[21] Smith [1973] discusses the general inconsistency of scoring run-offs.

[22] The strong monotonicity used to characterize majority rule in May [1952] can be seen as a special case.

in this sense even one voter can break a tie in a very large election. Thus, down consistency loses much of its force when a voting system has no ties at all.

In fact, assume that some unspecified system has no ties whatsoever. While down consistency still has some effect under our assumption – it implies weak consistency – it carries no additional force beyond that of up consistency. Every up consistent system that has no ties is down consistent. In this sense, the down consistency of the Putin and Pythagoras systems is vacuous.

In contrast, the Copeland rule illustrates the setting in which the presence of ties makes down consistency "bite." With Copeland, ties are not at all knife-edge. For example, with three alternatives any sufficiently strong cycle (in which each pair-wise majority has a margin of 2 or more voters) results in a Copeland three-way tie. Any addition of a single voter fails to break the cycle, and so fails to break the three-way Copeland tie. The Omninomination rule satisfies $\mathcal{F}(p_1 + p_2) = \mathcal{F}(p_1) \cup \mathcal{F}(p_2)$ and so is a good example of a system that is up consistent and not down consistent. (That Copeland is not fully consistent is already implicit in Young [1975] and Young and Levenglick [1978]). We have just sketched the proof of the following:

Proposition 3.2.4 The Copeland rule is not down consistent, either as a social choice function or as a social welfare/preference function. The Omninomination rule is up, but not down, consistent.

It is not difficult to show that weak consistency and up consistency also fail for Copeland. Favardin et al [2002] compare Copeland to Borda, showing that Copeland is relatively resistant to manipulation. It seems possible that the failure of Copeland to be down consistent may be a partial explanation; this system may be relatively resistant to manipulation because the large number of ties makes it relatively resistant to any change at all. The Omninomination rule produces even more ties than Copeland, of course. Consequently, it is almost completely resistant to manipulation (as measured via the methods in Aleskerov and Kurbanov [1999], D. Smith [1999], and Favardin et al [2002]). With three alternatives and an arbitrary number of voters, there is in fact a unique (up to permutations of individuals and voters) manipulable profile.

3.3 Connectedness

Connectedness is designed to guarantee the existence of the ties that allow down consistency to bite.

Definition 3.3.1 A voting system $\mathcal{V}$ is ***connected*** if given any two anonymous profiles p_1 and p_2 with $\mathcal{F}(p_1) \cap \mathcal{F}(p_2) = \varnothing$, there exist non-negative integers c and d such that $\mathcal{F}(p_1) \cap \mathcal{F}(cp_1 + dp_2) \neq \varnothing$ and $\mathcal{F}(p_1) \neq \mathcal{F}(cp_1 + dp_2)$.

For example if we had an abstract anonymous voting system in which $O = \{a, b\}$, with $\mathcal{F}(p_1) = \{a\}$ and $\mathcal{F}(p_2) = \{b\}$, then the profile $cp_1 + dp_2$ provided by the definition would be forced to satisfy $\mathcal{F}(cp_1 + dp_2) = \{a, b\}$. Connectedness does not seem to be very interesting in the absence of consistency. In the presence of consistency, there are several equivalent ways to phrase definition 3.3.1.

We may think of the profile $cp_1 + dp_2$ as intermediate to p_1 and p_2, or as a sort of weighted average of the two. In any mean representation, profile $cp_1 + dp_2$ will be mapped to a point on the line segment joining the points to which p_1 and p_2 are mapped. The intuitive content of connectedness, then, is that as that as voters "gradually change their minds" from one profile to another, they will pass through one or more "connecting" profiles representing ties. In one sense, this cannot always be literally true, for with an odd number of voters there will never be a tie in majority rule. But there is a precise property – *commensurability* – that expresses this idea, and that holds of voting systems that are both consistent and connected (see Smaoui and Zwicker [2005]).

Many naturally occurring voting systems are connected, including all from section 2.1 and 2.2. Exceptions include grading systems that, while roughly similar to examples 2.1.5 and 2.2, are a bit more practical in that they rule out ties. In the normal order of business, an instructor is not allowed to assign a pair of tied grades for a course, so most instructors either adopt a rule that automatically assigns the higher grade for averages falling on a borderline, or break ties using a subjective judgment based on other aspects of a student's performance. In the former case, the system is not connected; in the latter case one might argue that the grading system itself *does* yield ties, and is connected.

Proposition 3.3.2 The Putin and Pythagoras systems, Examples 2.3.4 and 2.3.5, are consistent and are not connected.

The proof is an easy exercise. While these two examples may seem, at first, to be highly artificial, we show in the next proposition that the Hare social welfare/preference system fails to be connected for essentially the same reason as the Putin system:

- Among the possible profiles, there is a range corresponding to the rational points in the interval [0, 1],
- all profiles corresponding to points in (0, 1] have a common election outcome X, and
- the profile corresponding to 0 has an outcome disjoint from X.

Furthermore, it turns out that the *fundamentally irrational* scoring systems fail to be connected for essentially the same reason as the Pythagoras system (see Smaoui and Zwicker [2005]).

Proposition 3.3.3 The Hare social welfare/preference function is not connected.
Proof Recall profiles p_1, p_2, and $p_3 = p_1 + p_2$ from proposition 3.2.3. From the order of elimination of the alternatives, we see that, as a social welfare/preference function Hare

satisfies $\mathcal{F}(p_1) = \begin{Bmatrix} A \\ B \\ D \\ C \end{Bmatrix}$ and $\mathcal{F}(p_3) = \begin{Bmatrix} D & & D \\ A & & A \\ B & , & C \\ C & & B \end{Bmatrix}$. As $\mathcal{F}(p_3) \cap \mathcal{F}(p_1) = \varnothing$, connectedness demands integers j, k > 0 with $\mathcal{F}(jp_3 + kp_1) \cap \mathcal{F}(p_3) \neq \varnothing$ (and $\mathcal{F}(jp_3 + kp_1) \neq \mathcal{F}(p_3)$). But in this case, $jp_3 + kp_1 = (j+k)p_1 + jp_2$, and it is easy to see any profile $sp_1 + tp_2$ with $s > t$ eliminates alternatives in the same order as p_1 (for the same reasons). Thus

$\mathcal{F}((j+k)p_1 + kp_2) = \begin{Bmatrix} A \\ B \\ D \\ C \end{Bmatrix}$ for every two integers j, k > 0, and connectedness fails. ∎

§4 Mean representations of voting systems

4.1 Closed decompositions and mean representations

As suggested by the representations in section 2, after we plot votes as points in space and find their mean location, we use a subdivision of space to determine the outcome:

Definition 4.1.1 An indexed[23] ***decomposition*** of a subset $P \subseteq \mathbf{R}^n$ is a sequence $\{r_a\}_{a \in \Delta}$ of subsets of P called ***regions***, with finite index set Δ, satisfying $\bigcup_{a \in \Delta} r_a = P$.
A ***decomposition*** is ***closed*** if its regions are closed sets.

Our interest is in voting systems that can be represented geometrically via the mean:

Definition 4.1.2 A ***mean representation*** of an abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ consists of a triple (R, P, S), in which

1. $R: \mathcal{I} \to \mathbf{R}^n$ is a "plot" function that locates each possible vote as a point in space,
2. P is the convex hull $C(\{R(j): j \in \mathcal{I}\})$ of the finite set of vertices corresponding to the plotted votes,
3. $S = \{r_a\}_{a \in \mathcal{O}}$ is a closed decomposition of P indexed by $\mathcal{O}$, and
4. S ***represents*** $\mathcal{V}$: for each $p \in (\mathbf{Z}^+)^{\mathcal{I}}$, $\mathcal{F}(p) = \{a \in \mathcal{O}: \bar{p} \in r_a\}$, where $\bar{p}$ denotes the mean location $\dfrac{\sum_{j \in I} p(j)R(j)}{N(p)}$ of all plotted votes cast.

[23] Definition 4.1.2 explains why we need indices. We can have $r_a = r_b$ with $a \neq b$, or $r_a = \varnothing$, while these are impossible with the non-indexed decompositions of Cervone and Zwicker [2005]. The difference is unimportant, however, as properties of decompositions that we consider depend only on non-empty regions and distinct pairs of regions. We'll freely abuse notation by treating S as if it were a set, writing "$r \in S$" when $r = r_a$ for some $a \in \Delta$, and "$T \subseteq S$" when each $t \in T$ is equal to r_a for some $a \in \Delta$.

The map $\overline{}$: $(\mathbf{Z}^{+})^{I} \rightarrow \mathbf{R}^{n}$, defined via the mean, can be thought of as an extension, from "one-voter" profiles to arbitrary profiles, of map R. The existence of a mean representation, alone, tells us little about a voting system. However, further restrictions on the *type* of closed decomposition S employed in the representation are directly tied to the voting-theoretic properties of $\mathcal{V}$.

In what follows, we are more concerned with the decomposition S than with the plot function R. Clearly, however, R also has a role to play.[24] For example, neutrality is probably related to symmetries in R. Also, the minimal possible dimension of a representation – the minimal n for which there exists a mean representation (of some type) with a plot function R: $I \rightarrow \mathbf{R}^{n}$ – may be of interest. We conjecture that with three alternatives this minimal dimension is 2 for the Borda count, and 3 for the Kemeny rule.[25] More generally, for n alternatives the dimension suggested by the representations in §2 are n - 1 for the Borda count and $2^{\binom{n}{2}}$ for the Kemeny rule. But are these minimal?

4.2 Affine and Voronoi representations

We briefly review some key ideas, including ones from Cervone and Zwicker [2005], henceforth "CZ," which accompanies this paper.

Definition 4.2.1 Two sets u and v of points of $\mathbf{R}^{n}$ are ***weakly separated by a hyperplane*** h of $\mathbf{R}^{n}$ if every point of u lies either on h or to one side of h, and every point of v lies either on h or to the other side of h. These sets are ***properly separated*** by h if they are weakly separated by h and, additionally, h does not contain both u and v as subsets; they are ***neatly separated*** by h if they are weakly separated and satisfy the yet stronger requirement that $u \cap v = h \cap u = h \cap v$.

Definition 4.2.2 A closed decomposition S of P is ***Voronoi*** if the regions of S are determined by relative proximity to a finite selection F of points in P, and is ***affine*** if each pair of distinct regions of S is neatly separated by some hyperplane.

[24] The roles of S and R are somewhat interdependent. For example, from a given mean representation we can obtain an obtain a second, equivalent representation for the same voting system if we both compose a bijective affine transform L with R (replacing R with L∘R), and also replace the regions of S with their images under this L. This fact relies heavily on fundamental properties of the mean, and seems to be false for any reasonable alternative to the mean – see Zwicker [2005b].

[25] Here we are considering only mean *affine* representations. The hexagonal decomposition of Figure 1 sits on a tilted plane in $\mathbf{R}^{3}$. Its projection onto $\mathbf{R}^{2}$ is a distorted hexagon with rational vertices that serves as a mean affine representation (and not as a mean proximity representation) but lacks the symmetry corresponding to the neutrality of the Borda count.

Every Voronoi decomposition is affine: we can neatly separate a region u, of points most proximate to point f_u, from a region v, with proximity point $f_v \neq f_u$ via the hyperplane that perpendicularly bisects the line segment $[f_u, f_v]$. It is easy to construct decompositions that are affine but not Voronoi. Essentially, consistency of a voting system corresponds to neat separability of the regions in the representation, while connectedness corresponds to the requirement that the neatly separating hyperplanes be *rational* as affine subspaces[26]:

Definition 4.2.3 An affine subspace A of $\mathbf{R}^n$ of dimension k (or "codimension" n - k) is ***rational*** if any of the following equivalent conditions hold:

1. A is the solution space of a linear system of equations in which all coefficients and constants are integers
2. A is the solution space of a linear system of n - k independent equations in which all coefficients and constants are integers
3. The set of rational points of A is dense in A, with respect to the relative topology on A as induced by the Euclidean topology of $\mathbf{R}^n$
4. There exist at least k + 1 affinely independent rational points in A
5. A is the affine span of a set of rational points.

We leave the proof of equivalence to the reader. An hyperplane h is a special type of affine subspace – one of dimension n - 1 (codimension 1). We immediately obtain the following:

Corollary 4.2.4 For an hyperplane h of $\mathbf{R}^n$ the following are equivalent

- h is rational
- h is given by a single equation $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \beta$ in which both β and the components of the normal vector α to h are integers
- h has a rational normal vector and contains at least one rational point.

A straight line is another special case – an affine subspace of dimension 1 – and it is easy to see that in this case we get:

Corollary 4.2.5 For a line L of $\mathbf{R}^n$ the following are equivalent:

- L is rational
- L contains two rational points
- L has a rational parallel vector and contains at least one rational point.

Definition 4.2.6 A closed decomposition S of P is ***Q-affine*** if each pair of distinct regions of S is neatly separated by some rational hyperplane.

In our application, P is both closed and convex, as it is the convex hull of a finite point set. So, the following notion is central:

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[26] For background on affine subspaces and related material, see Rockafellar [1980].

Definition 4.2.7 A ***polytope*** P is the convex hull of a finite set of rational points of $\mathbf{R}^n$; equivalently, it is a bounded intersection of finitely many closed half-spaces of $\mathbf{R}^n$.

This equivalence is a basic result in convexity theory. It is straightforward to see that it extends, as follows:

Definition 4.2.8 A ***rational polytope*** P is the convex hull of a finite set of rational points of $\mathbf{R}^n$; equivalently, it is a bounded intersection of finitely many closed half-spaces of $\mathbf{R}^n$, each of whose bounding hyperplanes are rational.

Definition 4.2.9 A ***mean proximity representation*** of an abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ consists of a quadruple (R, P, F, S), in which

1. R: $\mathcal{I} \to \mathbf{Q}^n$ locates each possible vote as a *rational* point,
2. P is the convex hull C($\{R(j): j \in \mathcal{I}\}$), a rational polytope,
3. F = $\{f_a\}_{a\in\mathcal{O}}$ is a finite collection of *rational* points in P, indexed by $\mathcal{O}$, and
4. S = $\{r_a\}_{a\in\mathcal{O}}$ is the Voronoi decomposition of P into closed regions indexed by $\mathcal{O}$: each region r_a contains the points of P that are at least as close to f_a as they are to any f_b with $b \in \mathcal{O}$, and
5. S represents $\mathcal{V}$.

An abstract anonymous voting system $\mathcal{V}$ is a ***mean proximity voting system*** if it has such a representation.

Definition 4.2.10 A ***mean affine representation*** of an abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ consists of a triple (R, P, S) that satisfies parts 1, 2, and 5 of the previous definition, as well as

3′. S = $\{r_a: a \in \mathcal{O}\}$ is a closed, $\mathbf{Q}$-affine decomposition of P indexed by $\mathcal{O}$.

An abstract anonymous voting system $\mathcal{V}$ is a ***mean affine voting system*** if it has such a representation.

Our main result is that a voting system $\mathcal{V}$ is mean affine if and only if it is both consistent and connected. The official statement of this theorem (in section 5) includes other conditions based on the following notion from CZ:

Definition 4.2.11 If S = $\{r_a\}_{a\in\Delta}$ is a decomposition of a set P $\subseteq \mathbf{R}^n$, and A is a subset of $\mathbf{R}^n$, then the ***restriction*** of S to A is given by $S|A = \{r_a \cap A\}_{a\in\Delta}$. A closed decomposition S is ***regular*** if every restriction $S|L$ to a *line* of $\mathbf{R}^n$ is affine. Equivalently, for each line L, every region of $S|L$ is a closed interval of L[27], and every two such regions $r\cap L$ and $u\cap L$ are equal, or disjoint, or overlap only at a single point which is an endpoint of each.

Definition 4.2.12 A closed decomposition S of a set P $\subseteq \mathbf{R}^n$ is ***thin convex*** if whenever x and z lie in some common region of S and y lies in the open line segment (x, z), the

[27] It is convenient here to classify one-point sets, and $\varnothing$, as closed intervals.

regions of S containing y as a member are exactly those that contain both x and z as members.

The following theorem from CZ plays a key role in our proof of the main characterization theorem:

Theorem 4.2.13 Let S be a closed decomposition of a polytope. Then the following are equivalent:

(i) S is affine
(ii) S is regular
(iii) S is thin convex.

§5 The main characterization theorem and the barycentric representations

The added conditions, (ii) and (iii) below, are defined in sections 5.2 and 5.3:

Theorem 5 The Mean Affine Characterization Theorem Let $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ be an abstract anonymous voting system. Then the following are equivalent:

(i) $\mathcal{V}$ is consistent and connected.
(ii) The rational barycentric representation of $\mathcal{V}$ is $\mathbf{Q}$-regular.
(iii) The extended barycentric representation of $\mathcal{V}$ is a mean affine representation.
(iv) $\mathcal{V}$ is a mean affine voting system.

The proof of (i) ⇒ (ii) is in section 5.3. The difficult proof of (ii) ⇒ (iii) appears in the appendix. Clearly, (iii) ⇒ (iv) is a tautology.

5.1 (iv) ⇒ (i)

The following lemma, with an easy proof left to the reader, is a translation, into the context of abstract anonymous voting systems, of the statement that the mean satisfies the ***subcenter*** axiom. Subcenter (our term) is the principal ingredient in an attractive axiomatization of the mean (see p514-515 of Gleason et al [1980], and Zwicker [2005b]).

Lemma 5.1.1 Voting Subcenter Lemma Let p and q be anonymous profiles for an abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ and R: $\mathcal{I} \rightarrow \mathbf{R}^n$ be any plot function. Then

$$\overline{(p+q)} = \left(\frac{N(p)}{N(p)+N(q)}\right)\left(\overline{p}\right) + \left(\frac{N(q)}{N(p)+N(q)}\right)\left(\overline{q}\right).$$

Note that $\overline{(p+q)}$ is expressed as a convex combination of $\overline{p}$ and $\overline{q}$ using rational convex coefficients, so if R takes values in $\mathbf{Q}^n$ then $\overline{(p+q)}$ is a rational point in ($\overline{p}$, $\overline{q}$).

Proof of (iv) ⇒ (i) in Theorem 5 Assume $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is a mean affine voting system, with associated representation (R, P, S). To see that $\mathcal{V}$ is consistent, let p and q be anonymous profiles satisfying $\mathcal{F}(p) \cap \mathcal{F}(q) \neq \varnothing$. By Lemma 5.1.1, $\overline{(p+q)}$ lies on the line segment joining $\overline{p}$ and $\overline{q}$. By 4.2.13, S is thin convex, so as $\overline{p}$ and $\overline{q}$ belong to at least one common region of S, $\overline{(p+q)} \in r_a$ if and only if $\overline{p} \in r_a$ and $\overline{q} \in r_a$ for each $a \in \mathcal{O}$, whence $\mathcal{F}(p) \cap \mathcal{F}(q) = \mathcal{F}(p + q)$, as desired.

To see that $\mathcal{V}$ is connected, assume that $\mathcal{F}(p) \cap \mathcal{F}(q) = \varnothing$. Let L be the line through $\overline{p}$ and $\overline{q}$, and S|L be the corresponding restriction of S. By 4.2.13, S is regular, so P∩L is a closed interval, equal to the union of finitely many closed intervals of the form $r_a \cap L$ that are ***proper*** (contain infinitely many points), each pair of which are disjoint, identical, or overlap only at a common endpoint. In particular, $\overline{p} \in r_a \cap L$ for one of these proper intervals. If we picture L with $\overline{p}$ to the left of $\overline{q}$, then the right endpoint x of $r_a \cap L$ lies strictly between $\overline{p}$ and $\overline{q}$, and x is thus the left endpoint of some proper interval $r_b \cap L$. As $x \in r_a \cap r_b$, if we can establish that $x = \overline{t}$ for some profile t of the form cp + dq, then $a \in \mathcal{F}(cp + dq) \cap \mathcal{F}(p)$, $\mathcal{F}(cp + dq) \neq \mathcal{F}(p)$, and connectedness follows. By assumption, there exists a rational hyperplane h neatly separating r_a and r_b. As x is the point of intersection of the rational line L and the rational hyperplane h, x is a rational point, and x is therefore a rational convex combination

$$x = \left(\frac{k}{m}\right)\overline{p} + \left(\frac{m-k}{m}\right)\overline{q} \qquad (\Psi)$$

for some integers k and m with k < m. Let c = kN(q) and d = (m-k)N(p). We are done, then, if we establish the following:

Claim $x = \overline{cp + dq}$.

Proof of claim The definition of c and d imply (c)(m-k)(N(p)) = (d)(k)(N(q)), so (c)(m)(N(p)) = (c)(k)(N(p)) + (d)(k)(N(q)), whence

$$\frac{(c \cdot N(p))}{(c \cdot N(p) + d \cdot N(q))} = \frac{k}{m}, \text{ and } \quad \frac{(d \cdot N(q))}{(c \cdot N(p) + d \cdot N(q))} = \frac{m-k}{m}.$$

Substituting into (Ψ), we obtain

$$x = \left[\frac{(c \cdot N(p))}{(c \cdot N(p) + d \cdot N(q))}\right](\overline{p}) + \left[\frac{(d \cdot N(q))}{(c \cdot N(p) + d \cdot N(q))}\right](\overline{q})$$

$$= \left[\frac{(c \cdot N(p))}{(c \cdot N(p) + d \cdot N(q))}\right](\overline{c \cdot p}) + \left[\frac{(d \cdot N(q))}{(c \cdot N(p) + d \cdot N(q))}\right](\overline{d \cdot q})$$

$$= \left[\frac{(N(c \cdot p))}{(N(c \cdot p) + N(d \cdot q))}\right](\overline{c \cdot p}) + \left[\frac{(N(d \cdot q))}{(N(c \cdot p) + N(d \cdot q))}\right](\overline{d \cdot q}) = \text{(by 5.1.1) } \overline{c \cdot p + d \cdot q}. \blacksquare$$

5.2 The two barycentric representations

In the interesting direction of the main theorem we construct, for any consistent and connected system $\mathcal{V}$, a mean affine representation. Here, we define the construction, in two stages. The *rational barycentric representation* (R, P*, S*), in which S* decomposes a set P* of rational points, is our preliminary version. The *extended barycentric representation* (R, P^{Cl}, S^{Cl}) is then obtained by taking closures.

To define (R, P*, S*) we need only assume that $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is homogeneous as an abstract anonymous voting system. Identify the set $\mathcal{I}$ with some proper initial segment K = {1, 2, . . ., k} of $\mathbf{N}$, and let our plot function R: K → $\mathbf{R}^k$ be given by R(j) = e_j. Here, e_j = (0, 0, . . ., 0, 1, 0, . . ., 0), where the 1 appears in the j^{th} coordinate. The convex hull of the e_j is the image of the k-1 simplex Δ_{k-1} under the standard embedding into $\mathbf{R}^k$ that identifies Δ_{k-1} with the intersection of the first orthant of $\mathbf{R}^k$ and the hyperplane of $\mathbf{R}^k$ having equation $x_1 + x_2 + \ldots + x_k = 1$. We'll refer to this image as $\mathbf{B}^{k-1+}$, and to the hyperplane containing it as $\mathbf{B}^{k-1}$. Thus $\mathbf{B}^{k-1+}$ consists of the points of $\mathbf{B}^{k-1}$ lying in the first orthont of $\mathbf{R}^k$. For the rest of the proof, any mention of coordinates for a point in $\mathbf{B}^{k-1}$ should be understood to refer to Cartesian coordinates in $\mathbf{R}^k$. These are the so-called ***barycentric*** coordinates, which always sum to 1.[28]

[28] Suppose we identify $\mathbf{R}^{k-1}$ with the hyperplane of points of $\mathbf{R}^k$ whose last coordinate is 0. The reader may be tempted to identify $\mathbf{B}^{k-1}$ with this copy of $\mathbf{R}^{k-1}$, and indeed there is a rigid

Due to our particular choice of plot function, R's extension to the mean map $\overline{} : \mathcal{I}^{Z+} \to \mathbf{Q}^k$ (where $\mathbf{Q}^k$ consists of the rational points of $\mathbf{R}^k$) has a particularly simple form to which we will assign a special notation: $\overline{p}$ is just the "normalized version" $p^{\Delta} = \dfrac{p}{N(p)}$ of the profile p. Each component of p^{Δ} (a barycentric coordinate) is equal to the fraction of voters who vote for the corresponding element of $\mathcal{I}$. The image P* of $\mathcal{I}^{Z+}$ under Δ is now the same as the set of ***rational points*** (points whose barycentric coordinates are all rational) of $\mathbf{B}^{k-1+}$. We'll denote this set as $\mathbf{QB}^{k-1+} = \mathbf{Q}^k \cap \mathbf{B}^{k-1+}$, and denote the set of rational points of the hyperplane $\mathbf{B}^{k-1}$ as $\mathbf{QB}^{k-1} = \mathbf{Q}^k \cap \mathbf{B}^{k-1}$.

Note that two profiles p and q satisfy $p^{\Delta} = q^{\Delta}$ if and only if some profile r is a common (integer) multiple, with hp = gq = r for some integers h and g. Thus, each point in $\mathbf{QB}^{k-1+}$ corresponds to an *equivalence class* of profiles under a simple equivalence relation. Homogeneity is now equivalent to the statement that $\mathcal{F}$ is constant on each such equivalence class.

Definition 5.2.1 For any homogeneous abstract anonymous system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$, the ***rational extension*** $\mathcal{F}^*$ of $\mathcal{F}$ to the points x* of $\mathbf{QB}^{k-1+}$ is given by $\mathcal{F}^*(x^*) = \mathcal{F}(p^{\Delta})$ for any profile p satisfying $x^* = p^{\Delta}$.

Thus, it is homogeneity that makes this extension well defined, allowing even the most minimal type of geometric representation for a voting system. We are now ready for:

Definition 5.2.2 The ***rational barycentric representation*** (R, P*, S*) of any homogeneous abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is defined by:

- R: $\mathcal{I} \to \mathbf{R}^k$ by $R(j) = e_j$,
- P* is the set $\mathbf{QB}^{k-1+}$,
- Each rational region r^*_a is defined by $x^* \in r^*_a$ if and only if $a \in \mathcal{F}^*(x^*)$, and
- $S^* = \{r^*_a\}_{a \in \mathcal{O}}$.

---

motion of $\mathbf{R}^k$-space carrying $\mathbf{B}^{k-1}$ to $\mathbf{R}^{k-1}$, consisting of a rotation of $\mathbf{B}^{k-1}$ about the intersection of $\mathbf{B}^{k-1}$ with $\mathbf{R}^{k-1}$. However, we will be concerned with the issue of ***rational points*** (those with rational coordinates), and no rigid motion of $\mathbf{R}^k$ can carry points with rational (barycentric) coordinates in $\mathbf{B}^{k-1}$ to points with rational coordinates in $\mathbf{R}^{k-1}$. The ***projection*** of $\mathbf{B}^{k-1}$ onto $\mathbf{R}^{k-1}$ given by $(x_1, \ldots, x_{k-1}, x_k) \mapsto (x_1, \ldots, x_{k-1}, 0)$ does identify these two spaces in such a way that identifies rational points with rational points. Furthermore, while this projection is not a rigid motion (it changes angles), it is a bijective linear transform (when its domain is taken to be $\mathbf{B}^{k-1}$), and it thus preserves most of the notions of actual concern in the proof. But it does not preserve the notion of "normal vector" for a hyperplane. In other words, the reader who thinks of $\mathbf{B}^{k-1}$ coordinatized as a "copy" of $\mathbf{R}^{k-1}$ will probably not be lead astray, but as the issue is somewhat delicate, we have chosen to stick with the coordinates of $\mathbf{R}^k$.

Definition 5.2.3 The ***extended barycentric representation*** (R, P^{Cl}, S^{Cl}) of any homogeneous abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is defined by:

- R: $\mathcal{I} \to \mathbf{R}^k$ by $R(j) = e_j$
- P^{Cl} is the closure $\mathbf{B}^{k-1+}$ of $P^* = \mathbf{QB}^{k-1+}$,
- Each region ${r^*_a}^{Cl}$ is the closure of the corresponding region r^*_a of S*, and
- $S^{Cl} = \{{r^*_a}^{Cl}\}_{a \in \mathcal{O}}$.

5.3 Q-regularity and (ii) ⇒ (iii)

Here, we use consistency and connectedness to show that the rational barycentric representation satisfies **Q**-regularity, a modified version of regularity.

Convention 5.3.1 (the ***✳-convention***) When we write a point $x \in \mathbf{R}^n$ as x* we indicate that x is a rational point (with all coordinates in **Q**). When we write a set $X \subseteq \mathbf{R}^n$ as X* we indicate that X consists entirely of rational points; alternately, if X is a set that may also contain irrational points, X* denotes $\{x^* \in X \mid x^* \text{ is rational}\}$. A ***Q-line*** L* is the set of rational points on a *rational* line L of $\mathbf{R}^n$; L* is the same as the "**Q**-affine span" $\{\alpha p^* + \beta q^* \mid \alpha, \beta \in \mathbf{Q} \text{ with } \alpha + \beta = 1\}$ of any two rational points p* and q* of L.

A non-empty **Q**-***closed interval*** [u*, v*]*, or **Q**-***open interval*** (u*, v*)*, of a **Q**-line L* consists of all (rational) points on L* between points u* and v* of L*, including (respectively, excluding) the endpoints u* and v*, while ∅ is considered to be both a **Q**-closed interval and a **Q**-open interval and {x*} is a **Q**-closed interval. Note that the endpoints of either type of interval are *required to have rational coordinates*. More generally, a **Q** ***interval*** is a subset I of a **Q**-line L* satisfying that whenever x*, z* ∈ I with y* ∈ (x*, z*)*, y* ∈ I.

Definition 5.3.2 A decomposition S* of a subset P* of $\mathbf{Q}^n$ is **Q**-***regular*** if for every **Q**-line L*:

1. r*∩L* is a **Q**-closed interval of L* for each region r* ∈ S*, and
2. r*∩L* and t*∩L* are identical, or disjoint, or their intersection is an endpoint of each, for each pair of regions r*, t* ∈ S*.

The following lemma establishes (i) ⇒ (ii) in the main theorem:

Lemma 5.3.3 Let $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ be a homogeneous abstract anonymous voting system, and (R, P*, S*) be the rational barycentric representation. If $\mathcal{V}$ is consistent and connected then S* is a **Q**-regular decomposition.

Proof Assume $\mathcal{V}$ is consistent and connected. Clearly, (R, P*, S*) represents $\mathcal{V}$. If L is any rational line meeting $\mathbf{BQ}^{k-1+}$, then $L \cap \mathbf{BQ}^{k-1+}$ is a $\mathbf{Q}$-closed interval [p*, q*]*. It follows easily from the following three claims that $S^* | L^*$ satisfies the requirements of Definition 5.3.2.

Claim 1 $\mathcal{F}^*$ satisfies ***rational consistency***: for any three points x*, y*, and z* of $\mathbf{BQ}^{k-1+}$ with $y^* \in (x^*, z^*)^*$, if $\mathcal{F}^*(x^*) \cap \mathcal{F}^*(z^*) \neq \emptyset$, then $\mathcal{F}^*(y^*) = \mathcal{F}^*(x^*) \cap \mathcal{F}^*(z^*)$.

Claim 2 Every region r* of S* meets [p*, q*]* in a $\mathbf{Q}$-interval, any two such intervals that share more than one point are equal, and any two that share exactly one point w* each have w* as an endpoint.

Claim 3 Each $\mathbf{Q}$-interval in $S^* | L^*$ is $\mathbf{Q}$-closed.

proof of claim 1: Assume $x^*, z^* \in \mathbf{BQ}^{k-1+}$ with $y^* \in (x^*, z^*)^*$. Choose profiles p_x and p_z for which $x^* = p_x^{\Delta}$ and $z^* = p_z^{\Delta}$. It now suffices to show that there exist positive integers A and C such that $y^* = (Ap_x + Cp_z)^{\Delta}$, for then we have

$$\mathcal{F}^*(x^*) \cap \mathcal{F}^*(z^*) \neq \emptyset \Rightarrow \mathcal{F}(p_x) \cap \mathcal{F}(p_z) \neq \emptyset \Rightarrow \mathcal{F}(Ap_x) \cap \mathcal{F}(Cp_z) \neq \emptyset \Rightarrow$$
$$\mathcal{F}(Ap_x + Cp_z) = \mathcal{F}(Ap_x) \cap \mathcal{F}(Cp_z) \Rightarrow \mathcal{F}^*(y^*) = \mathcal{F}^*(x^*) \cap \mathcal{F}^*(z^*),$$

as desired. As $y^* \in (x^*, z^*)^*$ is rational, to find the integers A and B we can choose positive integers a < b for which $y^* = \frac{a}{b} x^* + \frac{b-a}{b} z^*$. Now if we set $A = aN(p_z)$ and $C = (b-a)N(p_x)$, we obtain (via the *subcenter formula* from Lemma 5.1.1):

$$\left(Ap_x + Bp_y\right)^{\Delta} = \left(\frac{N(Ap_x)}{N(Ap_x) + N(Bp_z)}\right)\left(Ap_x\right)^{\Delta} + \left(\frac{N(Bp_y)}{N(Ap_x) + N(Bp_z)}\right)\left(Bp_z\right)^{\Delta} =$$

$$\left(\frac{aN(p_z)N(p_x)}{aN(p_z)N(p_x) + (b-a)N(p_z)N(p_x)}\right)\left(p_x\right)^{\Delta} + \left(\frac{(b-a)N(p_z)N(p_x)}{aN(p_z)N(p_x) + (b-a)N(p_z)N(p_x)}\right)\left(p_z\right)^{\Delta} =$$

$\frac{a}{b} x^* + \frac{b-a}{b} z^* = y^*$, as desired.

proof of claim 2: Consider any three points x*, y*, and z* of $\mathbf{BQ}^{k-1+}$ with $y^* \in (x^*, z^*)^*$. From rational consistency, it follows that

- if x* and z* are both members of a region r* of S*, then $y^* \in r^*$, and
- if, additionally, $y^* \in v^* \in S^*$ for some $v^* \neq r^*$, then $x^* \in v^*$ and $z^* \in v^*$.

The first bullet implies that every region r* of S* meets [p*, q*]* in a $\mathbf{Q}$-interval, and the second establishes the rest of the claim.

proof of claim 3: As degenerate (one-point) intervals of $S^*|L^*$ are clearly **Q**-closed, we consider only non-degenerate ones. Consider two distinct such intervals $r_a^*, r_b^* \in S^*|L^*$ that are adjacent on L^* (not separated by any intermediate, non-degenerate interval). To establish claim 3, it suffices to construct a common endpoint. Choose $x^* \in r_a^* - r_b^*$ and $z^* \in r_b^* - r_a^*$, as well as profiles with $p_x^\Delta = x^*$ and $p_z^\Delta = z^*$. Then $\mathcal{F}(p_x) \cap \mathcal{F}(p_y) = \emptyset$, or we contradict consistency. By connectedness we may choose positive integers c and d with the profile $p_y = cp_x + dp_z$ satisfying $\mathcal{F}(p_y) \neq \mathcal{F}(p_x)$ and $\mathcal{F}(p_y) \cap \mathcal{F}(p_x) \neq \emptyset$. Under these circumstances, consistency forces $\mathcal{F}(p_x)$ to be a proper subset of $\mathcal{F}(p_y)$. If we let y^* denote p_y^Δ, then rational consistency forces y^* to be an endpoint of r_a^*, and to lie weakly between any pair of points chosen one from r_a^* and the other from r_b^*. If we apply the same connectedness argument while reversing the roles of p_x and p_z we obtain a similar endpoint w^* of r_b^*, and it is straightforward to now show that $y^* = w^*$. ∎

§6 Conclusions

6.1 Results of Smith, Young, and Myerson

The results in the literature most closely related to ours include several of the mathematically deeper characterization theorems in the theory of voting:

Theorem 6.1 (J. Smith [1973]/Young [1974, 1975]) Among social welfare functions/social choice functions, a rule is neutral, anonymous, and consistent if and only if it is a "compound" scoring system.

Theorem 6.2 (J. Smith [1973]/ Young [1974, 1975]) Among social welfare functions/social choice functions, a rule is neutral, anonymous, consistent and Archimedean/continuous if and only if it is a (simple) scoring system.

Theorem 6.3 (Myerson [1995]) Among abstract anonymous voting systems, if a rule is *M-neutral*, anonymous, consistent, and has the *overwhelming majority* property then it is a (simple) *abstract scoring system.*[29]

Theorem 6.4 (Myerson [1995]) Among abstract anonymous voting systems, if a rule is anonymous, consistent, and has the *overwhelming majority* property then it is a *bilateral balance system.*

Theorem 6.3 takes up one direction of (both parts of) 6.2, and extends it to show that systems such as approval voting, in which a ballot need not consist of a ranking of

[29] We are using "abstract scoring system" and "M-neutral" in place of Myerson's "scoring rule" and "neutral." An abstract anonymous voting system is an abstract scoring system if for each $i \in O$ there exists a weight function $S_i: I \rightarrow \mathbf{R}$ such that the winning outputs for any profile are those with the greatest total weight, and is M-neutral if for every permutation π of O there exists a corresponding permutation π^* of I such that for every profile p, $\mathcal{F}(\pi^*(p)) = \pi[\mathcal{F}(p)]$.

alternatives, are nonetheless scoring systems, in the sense most natural to the abstract anonymous context. We have recently shown that among abstract anonymous systems the abstract scoring systems are identical to a variant form of our mean proximity rules, and there are other parallels between Myerson's results and ours. These will be the subject of a planned sequel (Zwicker [2006]) to this paper.

For the moment, we'll focus on some of the differences in between the two approaches. Myerson's neutrality hypothesis is quite strong, and is appropriate only for those abstract anonymous systems in which one can identify the set O of election outcomes with a completely symmetric set of alternatives. In particular, while the Kemeny and Extended Condorcet rules are clearly neutral in the sense most natural to their native context, they are not M-neutral; their outputs are binary relations over alternatives, which do not inherit the full symmetry of the underlying alternatives. Theorem 6.3 is applicable neither to these systems, nor to our grading system examples; "grades" are equipped with a linear ordering that breaks symmetry.

The bilateral balance systems of Theorem 6.4 are those in which each pair of alternatives can be separately compared via a vector of scoring weights that is specific to that particular pair. Loosely, this result can be expressed in our framework as follows: each pair of regions in the geometrical representation may be *weakly* separated by a hyperplane. Weak separability, as opposed to neat separability, does not carry the full force of consistency. Indeed, Myerson's proof of 6.4 only calls upon the application of weak consistency to single-winner outcomes, so the methods in this proof would not suffice to obtain results comparable to our Theorem 5. Consequently it is not surprising that the converse of Theorem 6.4 is false.

There is also an important distinction between Myerson's *overwhelming majority* property (the natural generalization of *Archimedean* and *continuity* to the abstract context) and our *connectedness*. In Smaoui and Zwicker [2005] we show that among abstract anonymous voting systems satisfying a weak form of consistency:

- overwheming majority is strictly weaker than connectedness, and
- a (simple) scoring system in the sense of Myerson satisfies connectedness if and only if it can be represented using integer (equivalently, rational) scoring weights.

Is our stronger assumption of connectedness in any way artificial? We would argue that it is not. It seems intuitively plausible that two profiles yielding disjoint election outcomes should be linked by some intermediate profile yielding a tie. In fact, it is difficult to think of any naturally occurring *consistent* voting systems that fail to be connected, unless one classifies as "natural" scoring rules with irrational scoring weights, or truly compound scoring rules. An additional argument for naturality appears in Smaoui and Zwicker [2005], where we define a precise sense in which, among consistent systems, the connected voting rules are those that are completely discrete.

6.2 Future research, continued

Our results suggest a few additional lines of inquiry:

1) Can differences among mean affine systems be explained by the differences in their representations? Can we construct new systems by finding new representations?
2) What common properties are shared by the mean affine voting systems?
3) Can we construct new classes, with different properties, by substituting some alternative for the mean?

Suppose we think of the computation of a mean affine system as taking place in two stages: in stage 1 the anonymous profile is converted into a mean location q, and in stage 2, q is converted into an election outcome. In general, each stage is a sort of projection, in which information is "lost" or compressed, and we might expect that for any particular voting system, the implications of employing the mean in stage 1 are greater if more information is compressed in this stage.

For example, with three alternatives the vertices of the hexagon used to represent the Borda count are the orthogonal projections (onto the plane x + y + z = 0) of the 6 transitive vertices of the cube used in the Kemeny representation. (This generalizes; the permutahedron for n alternatives is an orthogonal projection of the hypercube in $R^{\binom{n}{2}}$.) Because the orthogonal projection of the mean of a set of points is equal to the mean of the individually projected points, the mean point q in the Borda representation can be equivalently determined as follows: first find the mean location q′ according to the Kemeny representation, and then project q′ onto the plane x + y + z = 0. This shows that what we just called "stage 1" loses more information, in the case of Borda, than it does in the case of Kemeny. In this instance, it is possible to identify exactly what the additional lost information is: it is the "cyclic" or "spin" component of the pair-wise vote vector (the underlying tendency towards a majority cycle – see Zwicker [1991]; a similar analysis has been done by Saari [1994]).

The difference in representations does seem to tell us something interesting about the difference between these two systems, and it even suggests the possibility of systems intermediate to Borda and Kemeny, obtained via proximity representations that use partly compressed hypercubes – for example, if we move each of the six transitive vertices of the cube half-way along the line to their projections on the plane, and consider the resulting mean proximity voting systems, it seems that we might obtain a voting system in which the spin component plays some role, but less so than with the Kemeny rule.

In the case of the extended barycentric representation, stage 1 destroys almost no information; we can reconstruct the original anonymous profile from the mean point q simply by multiplying each coordinate by the original number n of voters. It is perhaps not surprising that the generic representation yielded by the proof of theorem 5 is thus one for which the mean seems less important (it destroys less information) than in a number of the representations of section 2. If we interpret the plurality voting system as one in which voters cast ballots for individual alternatives (rather than for rankings), then the extended barycentric representation is the same as the representation of plurality in Example 2.1.1 (for scoring rules), and seems to have lowest dimension possible among plurality representations.[30] With this interpretation, then, the role of the mean in plurality voting seems relatively less important than it is for some other systems.

This suggests that any properties common to all mean affine voting systems may nonetheless apply more strongly to some than to others. For example, the mean is notoriously sensitive to small numbers of "outliers" that are far from other points; if we imagine a "voting system" in which a vote is an unconstrained choice of a point in space, then a single voter who knows the others' votes can obtain any point whatsoever as the mean. Does this explain a common source of manipulability among mean affine voting systems? In (see Zwicker [2005b]) some of the simplest forms of manipulation of the Borda count an be explained, using the hexagon representation, by the fact that the mean is pulled more strongly by votes that are more distant. It seems possible that, because the mean plays a greater role in the Borda count than in plurality voting, the Borda count is correspondingly more subject to this particular form of manipulation.

Suppose we construct a new class of voting systems by replacing the mean, in any of our representations, by some substitute method for finding the "center" of a distribution of points. The resulting class might have very different properties, inherited from some fundamental axiomatic differences between the mean and our substitute. For example, the median is famously less sensitive to outliers than is the mean, so perhaps systems based on some spatial form of median will be less manipulable. Indeed, some preliminary computer enumerations (see Zwicker [2005b]) suggest that a system using the same representation hexagon as Borda, coupled with an appropriate spatial median[31], may be both less manipulable than the Borda count and more decisive (have fewer ties).

§A Appendix

Our goal is to prove (ii) ⇒ (iii) in Theorem 5.1. The following is a convenient package that carries the essential part of our assumption that the rational barycentric representation (R, P*, S*) of $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is $\mathbf{Q}$-regular:

Definition A.1 A ***standard situation*** is a vector D = (n, P, P*, S*, S^{Cl}) consisting of a non-negative integer n, a rational polytope $P \subseteq \mathbf{R}^n$, the set P* of rational points of P, a $\mathbf{Q}$-regular decomposition $S^* = \{r_a^*\}_{a\in\Delta}$ of P*, and $S^{Cl} = \{r_a^{*Cl}\}_{a\in\Delta}$.

[30] We have not proved this result.

[31] Our use of a *spatial* median distinguishes this system from that considered in Gehrlein and Lepelley [2003].

Assuming that D = (n, P, P*, S*, S^{Cf}) is a standard situation and $\mathcal{V}$ is an abstract anonymous voting system, we show, in sections A.1, A.2, and A.3 respectively, that:

1) if S* represents $\mathcal{V}$, then S^{Cf} represents $\mathcal{V}$.
2) S^{Cf} is "extended **Q**-regular" (defined below).
3) S^{Cf} is **Q**-affine.

It follows immediately that for consistent and connected systems the extended barycentric representation is a mean affine representation.

Definition A.2 A decomposition S of a set P ⊆ $\mathbf{R}^n$ is ***extended Q-regular*** if it is regular and for every rational line L the regions of S|L have rational endpoints.

A.0 Preliminaries

Basic notions A.0.1 Our proofs will use the following notions and notations of convexity theory: an ***affine combination*** of finitely many points of $\mathbf{R}^n$ with corresponding ***affine coefficients***; a ***convex combination*** of finitely many points of $\mathbf{R}^n$ with corresponding ***convex coefficients***; an ***affine subspace*** of $\mathbf{R}^n$; the ***affine span Aff(X)*** and ***convex hull C(X)*** of a set X ⊆ $\mathbf{R}^n$; an ***affinely independent*** set X ⊆ $\mathbf{R}^n$; an ***affine basis*** for an affine subspace of $\mathbf{R}^n$; the ***affine dimension, dim(X)*** of a set X ⊆ $\mathbf{R}^n$, which is the same as dim[Aff(X)]; the ***relative open ball*** $B_\delta{}^A(p) = \left\{q \in A \,\middle|\, \|p-q\| < \delta\right\}$ for a set A ⊆ $\mathbf{R}^n$ and corresponding ***relative*** (or ***induced***) ***topology*** on A; and the ***relative interior r.i.(X)*** of a set X ⊆ $\mathbf{R}^n$. We will also use ***r.i.$_A$(X)*** to denote the interior of X as calculated by the relative topology on A; thus, r.i.(X) = r.i.$_{\mathrm{Aff(X)}}$(X).

Lemma A.0.2 (rational convexity of regions and subregions) If D = (n, P, P*, S*, S^{Cf}) is a standard situation, {u_1*, . . ., u_j*} ⊆ S* with j ≥ 1, and $U^* = \bigcap_{i=1}^{j} u_i{}^*$ then C(U*)* ⊆ U*.
Proof For y* ∈ C(U*)*, choose {x_1*, . . ., x_m*} ⊆ U* with y* ∈ C({x_1*, . . ., x_m*}) and m minimal. Then there are unique convex coefficients c_i for which $y^* = \sum_{i=1}^{m} c_i x_i{}^*$ and it follows that the c_i are rational numbers. Using **Q**-regularity it is straightforward to prove by induction on m that $y^* = \sum_{i=1}^{m} c_i x_i{}^* \in U^*$. ▮

Lemma A.0.3 Let D = (n, P, P*, S*, S^{Cf}) be a standard situation, and u* be a region of S*. Then u^{*Cf} is convex.
Proof Straightforward. ▮

Lemma A.0.4 If X ⊆ $\mathbf{R}^n$ with |X| > 1, then r.i.(C(X)) is dense in C(X).

Proof Given $x_1 \in C(X)$, expand $\{x_1\}$ to an affine basis $X' = \{x_1, x_2, \ldots x_{k+1}\} \subseteq C(X)$ for $Aff(X) = Aff(C(X))$. Each convex combination $y = \sum_{i=1}^{k+1} c_i x_i$ using strictly positive convex coefficients $c_i > 0$ satisfies $y \in$ r.i.(C(X)), and it is clear that we can construct such points y arbitrarily close to x_1. ∎

Lemma A.0.5 If $X \subseteq \mathbf{R}^n$ and Aff(X) is a rational affine subspace of $\mathbf{R}^n$ then (r.i.(C(X)))* is dense in C(X).
Proof Rational points are dense in Aff(X), so via A.0.4 we can find a rational point $w^* \in$ r.i.(C(X)) as close as we desire to any $x \in C(X)$. ∎

Lemma A.0.6 Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation, with $u^*, t^* \in S^*$. Assume that $P \subseteq Aff(u^*)$ and $u^* - t^* \neq \emptyset$. Then $\dim(u^* \cap t^*) < Aff(u^*)$.
Proof If not, let $X^* = \{x_1, \ldots, x_{j+1}\} \subseteq u^* \cap t^*$ be an affine basis for $Aff(u^*)$. As $X^* \subseteq u^* \cap t^*$, $C(X^*)^* \subseteq u^* \cap t^*$. Choose rational points $b^* \in \text{r.i.}_{Aff(u^*)}(C(X^*))$ and $d^* \in u^* - t^*$. Clearly, $[b^*, d^*]^* \cap t^* \cap u^* = [b^*, c^*]$, with $c^* \in (b^*, d^*)$. Thus $b^*, d^* \in u^*$, $c^* \in t^*$, and $d^* \notin t^*$, contradicting $\mathbf{Q}$-regularity. ∎

A.1 Representation

The definition of the rational barycentric representation guarantees that it *represents* $\mathcal{V}$: for every $p \in (\mathbf{Z}^+)^I$ and $a \in O$, $\bar{p} \in r^*_a \Leftrightarrow a \in \mathcal{F}(p)$. Here we show that the same holds when $(r^*_a)^{Cl}$ replaces r^*_a.

Proposition A.1.1 ("No new rational points") Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation. Then $S^* = S^{Cl}|P^*$.

Proof Clearly, each region r^* of S^* is contained in the corresponding region $r^{*Cl} \in S^{Cl}$, whence $r^* \subseteq (r^{*Cl} \cap P^*)$. The reverse containment establishes the proposition, and follows immediately from the following two claims.

Claim 1 Let y^* be a rational point in r.i.(r^{*Cl}). Then $y^* \in r^*$.
Proof of claim 1 Choose $\delta > 0$ with $B_\delta^{Aff(r^*)}(y^*) \subseteq r^{*Cl}$. Choose points $a_1, \ldots, a_k$ in $B_\delta^{Aff(r^*)}(y^*)$ so that

- $\{a_1, \ldots, a_k\}$ is an affine basis for A, and
- $y^* \in$ r.i.$(C(\{a_1, \ldots, a_k\}))$.

As each a_j lies in r^{*Cl}, we can replace each a_j with a point $a_j^* \in r^*$ that is as close to a_j as we like. By choosing them close enough, it follows that

- $\{a_1^*, \ldots, a_k^*\}$ is an affine basis for A, and
- $y^* \in$ r.i.$(C(\{a_1^*, \ldots, a_k^*\})$.

Now as $y^* \in C(r^*)^*$, $y \in r^*$ by Lemma A.0.2.

Claim 2 Let y^* be a rational point in r^{*Cl}. Then $y^* \in r^*$.
Proof of claim 2 Construct an affine basis $\{y^*, a_2^*, \ldots, a_k^*\}$ for A with $\{a_2^*, \ldots, a_k^*\} \subseteq r^*$. For any point z of A, let C_z denote the convex hull of $\{z, a_2^*, \ldots, a_k^*\}$.

Sub-claim 2.1 Any member z^* of r^* sufficiently close to y^* satisfies

- $\{z^*, a_2^*, \ldots, a_k^*\}$ is an affine basis of A, and
- $\text{r.i.}(C_{z^*}) \subseteq \text{r.i.}(r^{*Cl})$.

We leave the proof of this sub-claim to the reader.
Now let $w^* = \sum_{i=2}^{k}\left(\frac{1}{k-1}\right)a_i^*$ so that w^* is a convex combination of $a_2^*, \ldots, a_k^*$ using strictly positive convex coefficients, and consider the line L through w^* and y^*.

Sub-claim 2.2 The open interval (w^*, y^*) of L lies entirely inside $\text{r.i.}(r^{*Cl})$.
Proof of sub-claim As $y^* \in r^{*Cl}$, we can find points z^* of r^* as close as desired to y^*. For each such z^*, there is a point v_z^* such that $\text{r.i.}(C_{z^*}) \cap (w^*, y^*) = (w^*, v_z^*)$, and so by sub-claim 2.1, $(w^*, v_z^*) \subseteq \text{r.i.}(r^{*Cl})$. By choosing z^* sufficiently close to y^* we can force v_z^* to be as close as we wish to y^*, so $(w^*, y^*) \subseteq \text{r.i.}(r^{*Cl})$.

Thus, by claim 1 $(w^*, y^*)^* \subseteq r^*$. By $\mathbf{Q}$-regularity, $L \cap r^*$ is a $\mathbf{Q}$-closed interval $[a^*, b^*]^* \supseteq (w^*, y^*)^*$. So $y^* \in [a^*, b^*]^* \subseteq r^*$, and $y^* \in r^*$ as desired. ∎

Lemma A.1.2 Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation, u^* be a region of S^*, and A be any rational affine subspace such that $A = \text{Aff}(u^{*Cl} \cap A)$. Then $u^* \cap \text{r.i.}(u^{*Cl} \cap A)$ is dense in $u^{*Cl} \cap A$.
Proof By A.0.3, $u^{*Cl} \cap A = C(u^{*Cl} \cap A)$. Also, $\text{Aff}[(u^*)^{Cl} \cap A] = A$ so $\text{Aff}[(u^*)^{Cl} \cap A]$ is a rational affine subspace. Hence by A.0.5, $[\text{r.i.}(u^{*Cl} \cap A)]^*$ is dense in $u^{*Cl} \cap A$. But by A.1.1, $[\text{r.i.}(u^{*Cl} \cap A)]^* \subseteq u^*$. ∎

Corollary A.1.3 Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation, and u^* be a region of S^*. Then $u^* \cap \text{r.i.}(u^{*Cl})$ is dense in u^{*Cl}.
Proof Take A to be $\text{Aff}(u^{*Cl}) = \text{Aff}(u^*)$ in Lemma A.1.2. ∎

Proposition A.1.4 Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation, u^* be a region of S^*, and A be a rational affine subspace of $\mathbf{R}^n$. Then $(u^* \cap A)^{Cl} = (u^*)^{Cl} \cap A$.

Proof Suppose that (P^*, S^*, u^*, A) represents a failure of this lemma. Then both $(P^* \cap \text{Aff}(u^*), S^*|\text{Aff}(u^*), u^*, A)$, and $(P^*, S^*, u^*, A \cap \text{Aff}(u^*))$ also represent failures, so any failure triggers a ***smooth*** failure: one satisfying $\text{Aff}(u^*) = \text{Aff}(P^*)$ and $A \subseteq \text{Aff}(u^*)$. Clearly, any smooth failure satisfies $\dim(A) < \dim(u^*)$.

If (P^*, S^*, u^*, A) is a smooth failure satisfying $\dim(A) < \dim(u^*) - 1$, let B be any rational affine subspace with $A \subseteq B \subseteq \mathrm{Aff}(u^*)$ and $\dim(A) < \dim(B) < \dim(u^*)$. Then either (P^*, S^*, u^*, B) or $(P^*\cap B, S^*|B, u^*\cap B, A)$ must be a (smooth) failure, else (P^*, S^*, u^*, A) is not itself a failure. So any smooth failure triggers a smooth and ***sequential*** failure: one satisfying $\dim(u^*) = 1 + \dim(A)$.

Now assume that $G = (P^*, S^*, u^*, A)$ is a smooth and sequential failure that is ***minimal***: $k = \dim(A)$ is as small as possible among such failures. We will obtain a contradiction by constructing a failure with a lower k. Note that $k > 0$ by Lemma A.1.1. Clearly $(u^*)^{cl} \cap A \supseteq (u^*\cap A)^{cl}$, so choose any point $y \in (u^*)^{cl} \cap A - (u^*\cap A)^{cl}$.

Claim $\dim[(u^*)^{cl} \cap A] < k$.
Proof of claim If not, then $A = \mathrm{Aff}[(u^*)^{cl} \cap A]$, so by A.1.2, $u^*\cap \mathrm{r.i.}(u^{*cl}\cap A)$ is dense in $u^{*cl}\cap A$. Thus $y \in (u^*\cap A)^{cl}$, contradicting our assumption.

As $y \in (u^*)^{cl} \cap A$, we know by A.1.3 that there are points x^* in $u^*\cap \mathrm{r.i.}(u^{*cl})$ arbitrarily close to y, but by assumption this is false if we additionally require $x^* \in A$. Let x^* be any element of $u^*\cap \mathrm{r.i.}(u^{*cl}) - A$.

Our failure was sequential, so $\mathrm{Aff}(u^*) = \mathrm{Aff}(A\cup\{x^*\})$. Thus, there are points of $\mathrm{Aff}(A\cup\{x^*\})\cap u^*\cap \mathrm{r.i.}(u^{*cl})$ arbitrarily close to y. For each natural number $n > 0$, choose a (rational) point $w_n^* \in \mathrm{Aff}(A\cup\{x^*\})\cap u^*\cap \mathrm{r.i.}(u^{*cl})$ with $\|w_n^* - y\| < 1/n$, and choose a rational point v_n^* on A with $\|v_n^* - y\| < 1/n$. For all sufficiently large n, $v_n^* \notin u^*$ and we consider the rational line segment $[v_n^*, w_n^*]^*$. As $w_n^* \in u^*$, $u^*\cap[v_n^*, w_n^*]^* = [p_n^*, w_n^*]^*$ for some point p_n^* lying strictly between v_n^* and w_n^*. Note that $\|p_n^* - y\| < 1/n$. As P^* contains $[v_n^*, p_n^*]^*$, by $\mathbf{Q}$-regularity there exists some region t_n^* of S^* such that $[v_n^*, w_n^*]^*\cap t_n^* = [q_n^*, p_n^*]^*$ where $q_n^* \in [v_n^*, p_n^*)^*$. Thus $w_n^* \notin t_n^*$.

Via the pigeonhole principle choose a region t^* of S^* with $t^* = t_n^*$ for each integer $n \in K$, where K is infinite. Let $V = \mathrm{Aff}(\{p_n^* | n \in K\})$, so that V contains points of $t^*\cap u^*$ arbitrarily close to y.

By Lemma A.0.6, $\dim(V) < k + 1$. The rational affine subspace $W = A \cap V$ satisfies $\dim(W) < k$, and $y \in (u^*\cap V)^{cl} \cap W - (u^*\cap V\cap W)^{cl}$, so $G' = (P^*\cap V, S^*|V, u^*\cap V, W)$ is a smooth failure with $\dim(W) < k$. The process by which G' triggers a smooth sequential failure G'' does not increase the dimension of the fourth coordinate of the failure, so the dimension of the fourth coordinate of G'' is $< k$, contradicting G's minimality. ∎

A.2 Extended Q-Regularity

Proposition A.2.1 Let $D = (n, P, P^*, S^*, S^{cl})$ be a standard situation. Then S^{cl} is regular.

Corollary A.2.2 Let D = (n, P, P*, S*, S^{Cl}) be a standard situation. Then S^{Cl} is extended **Q**-regular.

Proof of A.2.2 Via A.1.1, regularity implies extended Q-regularity. ∎

Proof of A.2.1 A ***failure*** of A.2.1 is a tuple H = (n, P, P*, S*, S^{Cl}, r*, u*, x, y, z, u*, L) in which D = (n, P, P*, S*, S^{Cl}) is a standard situation; r*, u* ∈ S*; x, z ∈ r^{*Cl}; L is the line through x and z; y ∈ (x, z); and y ∈ u^{*Cl} while x and z are not both members of u^{*Cl}. Given A.0.3, **Q**-regularity follows from showing that no such failures exist. A failure H is ***level*** if P* ⊆ Aff(r^{*Cl}) = Aff(r*) and the ***dimension*** of a level failure is the common value of dim(P*) = dim(Aff(r^{*Cl}). If H = (n, P, P*, S*, S^{Cl}, r*, u*, x, y, z, u*, L) is a failure then H′ = (n, P∩Aff(r*), P*∩Aff(r*), S*|Aff(r*), S^{Cl}|Aff(r*), r*, u*∩Aff(r*), x, y, z, L) is a level failure , so it suffices to rule out level failures.

By way of contradiction, let H be level failure that is ***minimal***: its dimension k is the smallest possible among level failures. We will show that k ≠ 1, and also that if k > 1 then H triggers a level failure H″ with dim(H″) < dim(H).

<u>k ≠ 1</u> If k = 1, then P* is a closed Q-segment [a*, b*]* of a rational line M. From **Q**-regularity, the regions r* of S* are **Q** -closed intervals [c*, d*]* and any two distinct regions that intersect have, as their sole point of intersection, a (rational) endpoint of each. As the rational points are dense in M, P = [a*, b*], and each r^{*Cl} ∈ S^{Cl} is a closed segment [c*, d*] of M with rational endpoints. Thus every two distinct regions of S^{Cl} that intersect have, as their sole point of intersection, a (rational) endpoint of each. Finally, L = M as they share the points x ≠ z, so H is not a failure.

<u>Assume k > 1</u>

<u>Case 1</u> Assume that for each ε > 0, there exists a z° ∈ r.i.(r^{*Cl}) ∩ (y, z) with $\left\|z^o - z\right\| < \frac{\varepsilon}{2}$. Then for each such ε, choose δ in $\left(0, \frac{\varepsilon}{2}\right)$ with $B_\delta^{\mathrm{Aff}(r^*)}$(z°) ⊆ r^{*Cl}. Choose η > 0 with

$$\eta < \delta\left(\frac{\frac{1}{2}\|x-y\|}{\|z-y\| + \frac{1}{2}\|x-y\|}\right).$$

Choose y* ∈ u* with ||y* - y|| < η and x* ∈ r* with ||x* - x|| < η, and let M be the (rational) line through y* and x*. By proportional triangles (see Figure 4), the closest point w of M to z° is in $B_\delta^{\mathrm{Aff}(r^*)}$(z°). Thus M contains a rational point z* ∈ r*∩$B_\delta^{\mathrm{Aff}(r^*)}$(z°). By **Q**-rationality, as x*, z* ∈ r* and y* ∈ u*∩(x*, z*), both x* and z* lie in u*. As $\left\|z - z^o\right\| < \frac{\varepsilon}{2}$ and $\left\|z^o - z^*\right\| < \frac{\varepsilon}{2}$, $\left\|z - z^*\right\| < \varepsilon$. As both x and z are arbitrarily close to points in u*, both x and z are in u^{*Cl}, contradicting our choice of H as a failure.

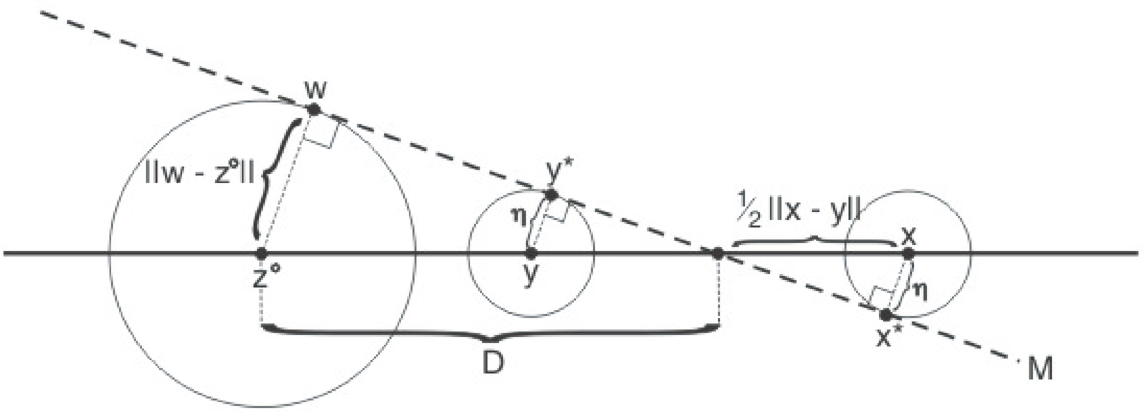

Figure 4 Let $D = \left\|z^\circ - y\right\| + \frac{1}{2}\left\|x - y\right\|$. Then, in the limiting worse case shown, $\eta = \left\|x - x^\circ\right\| = \left\|y - y^\circ\right\|$, and $\frac{\left\|w - z^\circ\right\|}{D} = \frac{\eta}{\frac{1}{2}\left\|x - y\right\|} < \frac{\delta}{D}$, so $\left\|w - z^\circ\right\| < \delta$.

As P* is a rational polytope, we can list rational hyperplanes $h_1, \ldots, h_m$ of $\mathbf{R}^n$ such that P is the intersection of Aff(r*) with m closed half spaces corresponding to the h_i, and such that m is minimal. Each h_i

- contains Aff(r*), or
- is disjoint from Aff(r*), or
- intersects Aff(r*) in a rational affine subspace g_i of dimension k - 1 or less,

but each h_i on our list is of the last type, or it could safely have been struck from the list, so we obtain a new list T = $g_1, \ldots, g_m$ of these subspaces.

Choosing an $\varepsilon > 0$ for which no point z° (with the Case 1 property) exists, we obtain a non-empty interval (z, z′) of L contained in r^{*Cl} - r.i.(r^{*Cl}). For each w in this interval and each positive integer n, choose rational points $w_n^* \in B_{\frac{1}{n}}^{Aff(r^*)}(w) - r^*$ and (using A.1.3) $v_n^* \in B_{\frac{1}{n}}^{Aff(r^*)}(w) \cap r^* \cap$ r.i.(r^{*Cl}); note that $[w_n^*, v_n^*] \subseteq B_{\frac{1}{n}}^{Aff(r^*)}(w)$. If $w^*_n \notin P^*$, choose t_n^* to be a subspace $g_i \in T$ that separates v_n^* from w_n^* in Aff(r*), with $\{p_n^*\} = [v_n^*, w_n^*] \cap g_i$. If $w_n^* \in P^*$, $[v_n^*, w_n^*] \cap r^* = [v_n^*, p_n^*]^*$ for some $p_n^* \in (v_n^*, w_n^*) \cap r^*$. Choose $t_n^* \in S^*$ with $p_n^* \in r^* \cap t_n^*$ and $v_n^* \notin t_n^*$. As there are finitely many possible choices for t_n^*, choose some value t(w) that is equal to t_n^* for infinitely many n. If t(w) is one of the rational subspaces g_i, then $w \in g_i$, and if $t(w) = t^* \in S^*$, then $w \in Aff(r^* \cap t^*)$. As t(w) takes on only finitely many possible values, there exist $w_1 \neq w_2$ in (z, z′) with $t(w_1) = t(w_2) = t_0$.

If $t_0 = g_i$ then $L \subseteq g_i$, $\dim(g_i) < \dim(Aff(r^*))$, and g_i serves as the desired affine subspace A of Aff(r*). If $t_0 = t^*$ then $L \subseteq Aff(r^* \cap t^*)$. By A.0.6, $\dim(r^* \cap t^*) < \dim(r^*)$, with any of the points v_n^* witnessing $r^* - t^* \neq \emptyset$. So $Aff(r^* \cap t^*)$ serves as the desired affine subspace A of Aff(r*). ∎

A.3 Neat separation via rational hyperplanes

From A.2.2, we know that in any standard situation, S^{Cl} is extended **Q**-regular.
From S^{Cl}'s regularity the first three claims of Lemma 2.2 of [CZ] then immediately yield the following:

Corollary A.3.1 Let D = (n, P, P*, S*, S^{Cl}) be a standard situation. Then the relative interiors of the regions of S^{Cl} are pairwise disjoint, every pair of distinct regions of S^{Cl} can be properly separated by a hyperplane, and each region of S^{Cl} is a polytope.

The fourth claim of Lemma 2.2 of [CZ] then shows that S^{Cl} is affine. Here we modify that proof to show, from extended **Q**-regularity, that S^{Cl} is **Q**-affine.

Definition A.3.2 A set $F \subseteq r$ is a ***face*** of the polytope r if for every $x, z \in r$ and $y \in (x, z) \cap F$, $x \in F$ and $z \in F$.

Definition A.3.3 A ***facet*** of a dimension-j polytope r is a face of r with dimension j - 1.

Definition A.3.4 The ***algebraic difference*** $r -_{ALG} u$ of two subsets of $\mathbf{R}^n$ is the set

$\{x - y \mid x \in r \text{ and } y \in u\}$ of all algebraic differences of their respective elements.

Lemma A.3.5 Let D = (n, P, P*, S*, S^{Cl}) be a standard situation, r* ∈ S, and F be a facet of r^{*Cl}. Then Aff(F) is rational.

Proof As F = $[\text{r.i.}(F)]^{Cl}$, it suffices to show that there are points of F* arbitrarily close to any point y ∈ r.i.(F). Let ε > 0 be arbitrary, and without loss of generality assume that ε is small enough so that no point in $B_\varepsilon(y)$ lies on any face of r other than F. Now choose rational points z* and w* in $B_\varepsilon^{\text{Aff}(r^*)}(y)$ with z* ∈ r.i.(r^{*Cl}) and w* ∉ r^{*Cl}. Then the entire line segment [z, w] lies in $B_\varepsilon^{\text{Aff}(r^*)}(y)$ with one endpoint in r^{*Cl}'s interior. It follows from extended **Q**-regularity that $r^{*Cl} \cap [z^*, w^*] = [z^*, v^*]$ with v* rational. Now v* is the desired rational point in $F \cap B_\varepsilon[y]$. ∎

Lemma A.3.6 Let D = (n, P, P*, S*, S^{Cl}) be a standard situation and r* ∈ S*. Then the vertices of the polytope r^{*Cl} are rational.

Proof Each vertex v is an intersection of the finitely many rational affine subspaces Aff(F) for the facets F containing v. So v is the unique solution of a system of linear equations, each equation having integer coefficients and constants. ∎

Lemma A.3.7 Let D = (n, P, P*, S*, S^{Cl}) be a standard situation and r*, u* ∈ S*. Then $r^{*Cl} -_{\text{ALG}} u^{*Cl}$ is a polytope with rational vertices.

Proof Each extreme point of $r^{*Cl} -_{\text{ALG}} u^{*Cl}$ is the difference $x_r - x_u$ of extreme points of r^{*Cl} and u^{*Cl} respectively. But the extreme points of r^{*Cl} (or of u^{*Cl}) are its finitely many rational vertices. Thus $r^{*Cl} -_{\text{ALG}} u^{*Cl}$ has finitely many extreme points, each of which is rational. By the Krein-Milman theorem, as $r^{*Cl} -_{\text{ALG}} u^{*Cl}$ is bounded it is the convex hull of its extreme points, each of which is a rational differences $x_r^* - x_u^*$. ∎

Lemma A.3.8 Let h and A be rational affine subspaces with h ⊆ A and dim(A) = dim(h) + 1. Then there is a rational hyperplane h′ of $\mathbf{R}^n$ with A∩h′ = h.

Proof Choose any rational point v* on A - h. Let w* be the orthogonal projection of v* onto h. Then w* is rational with v* - w* normal to h, and h consists of all points x of A with (x - w*) ⊥ (v* - w*). So we can take h′ to be the (rational) hyperplane through w* normal to v* - w*. ∎

Lemma A.3.9 Let D = (n, P, P*, S*, S^{Cl}) be a standard situation with r*, u* ∈ S* and r* ≠ u*. Then r^{*Cl} and u^{*Cl} can be properly separated by a rational hyperplane h.

<u>Case 1</u> Assume $0 \notin r^{*Cl} -_{\text{ALG}} u^{*Cl}$. Then as r^{*Cl} and u^{*Cl} are disjoint closed polytopes, choose any hyperplane g separating them strictly. Obtain the desired h by perturbing the

constants in the equation of g so that they have rational values so close to the original values that h still separates r and u strictly.

Case 2 Assume $0 \in r^{*cl} -_{ALG} u^{*cl}$. Via Corollary A.3.1, choose a hyperplane h′ that properly separates r^{*cl} and u^{*cl}. Choose m and c with $m \bullet x \geq c$ for all $x \in r^{*cl}$, $m \bullet y \leq c$ for all $y \in u^{*cl}$, and $m \bullet z \neq c$ for some $z \in r^{*cl} \cup u^{*cl}$. Let f be the hyperplane through 0 with normal m. Then $r^{*cl} -_{ALG} u^{*cl}$ is not contained in f, but is contained in one of the two closed half spaces formed by f. Thus 0 lies on the "relative boundary" of $r^{*cl} -_{ALG} u^{*cl}$ (the topological boundary in the relative topology of $\text{Aff}(r^{*cl} -_{ALG} u^{*cl})$), whence 0 lies on some facet F of $r^{*cl} -_{ALG} u^{*cl}$. By A.3.5, Aff(F) is a rational affine subspace. Using A.3.8**,** extend F to a rational hyperplane g through 0, with $g \cap \text{Aff}(r^{*cl} -_{ALG} u^{*cl}) = \text{Aff}(F)$. The rational normal α of g satisfies $\alpha \bullet x \geq \alpha \bullet y$ for all $x \in r^{*cl}$ and $y \in u^{*cl}$. At some (rational) vertex v_0^* of r^{*cl}, $x \mapsto \alpha \bullet x$ achieves its minimum value on r^{*cl}. Let $\beta = \alpha \bullet v_0$. Then β is rational, $\alpha \bullet x \geq \beta$ for all $x \in r^{*cl}$, and $\alpha \bullet y \leq \beta$ for all $y \in u^{*cl}$. Equation $\alpha \bullet x = \beta$ provides our rational properly separating hyperplane h. ▮

Theorem A.3.10 Let $D = (n, P, P^*, S^*, S^{cl})$ be a standard situation Then S^{cl} is $\mathbf{Q}$-affine.

Proof We'll use ***dim(D)*** to denote dim(P). The argument is by induction on j = dim(D), with the dimension $n \geq j$ of the ambient space held fixed. If j = 1, then Aff(P) is a rational line L, $S^{cl} = S^{cl}|L$ is affine by Proposition 1.12 [CZ] and Definition 1.13 [CZ] of regularity. The neatly separating hyperplanes $h_{a,L}$ in the 1.12 proof of are rational, as follows. Each endpoint a* of an interval of S^{cl} is rational. As L is rational we may take $b^* - a^*$ as the normal to $h_{a,L}$, where $b^* \in L^*$ with $b^* \neq a^*$. The resulting equation $(b^* - a^*) \cdot x = (b^* - a^*) \cdot a^*$ for $h_{a,L}$ has a rational constant, and all coefficients rational. So S^{cl} is $\mathbf{Q}$-affine.

Assume S^{cl} is $\mathbf{Q}$-affine for every standard situation of dimension j or less. Let $D = (n, P, P^*, S^*, S^{cl})$ be a standard situation with dim(P) = j + 1, and $r = r^{*cl}$ and $u = u^{*cl}$ be distinct regions of S^{cl}. We need to neatly separate r and u with a rational hyperplane. Via A.3.9, choose a rational, properly separating hyperplane h′ with equation $\alpha \bullet x = \beta$ and assume $\alpha \bullet x \geq \beta$ on r.

If h′ neatly separates r and u we are done. If $r \cap h' = \varnothing$ then any sufficiently small rational increase in β yields a parallel rational hyperplane h″ that strictly separates r and u, while if $u \cap h = \varnothing$ a small rational decrease in β achieves strict separation.

So, assume $\varnothing \neq r \cap h' \neq u \cap h' \neq \varnothing$ (see Figure 3 of [CZ]). The restriction $S^{cl}|h'$ is an extended $\mathbf{Q}$-regular closed decomposition of the rational polytope $P \cap h'$, and $\dim(P \cap h') < \dim(P)$. By induction, choose a rational hyperplane k′ of $\mathbf{R}^n$ that neatly separates the distinct regions $r \cap h'$ and $u \cap h'$ of $S^{cl}|h'$, and let $k = k' \cap h'$. As in [CZ], k is a codimension 2 affine subspace of $\mathbf{R}^n$. Clearly, k is rational. Define v and w as in [CZ], and let h_ε be the hyperplane obtained by rotating h' about k, taking v toward w through an angle of ε. As in [CZ], for any sufficiently small value of $\varepsilon > 0$, h_ε neatly separates r and u.

It remains only to show that there are arbitrarily small values of ε for which h_ε is rational. Choose an affine basis $X^* = \{x_1^*, \ldots, x_{n-1}^*\}$ of k. We'll show that there are arbitrarily small values of ε for which h_ε contains a rational point $x_n^* \notin k$, for then $X^* \cup \{x_n^*\}$ must be an affine basis for h_ε, whence h_ε is rational. But this is clear – no matter how small is $\varepsilon > 0$, as the angle increases from 0 to ε, the rotating h' sweeps out a region of $\mathbf{R}^n$ whose non-empty interior is disjoint from k, and contains rational points of $\mathbf{R}^n$ - k. ∎

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Note

There is a second version of this paper that adds a number of comments filling in details. Most of these comments appear in the appendix. Contact the author to request a copy.