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THE MEDIAN PROCEDURE IN CLUSTER ANALYSIS AND SOCIAL CHOICE THEORY

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Classical approachs for fitting and aggregation problems, specially in cluster analysis, social choice theory and paired comparisons methods, consist in the minimization of a remoteness function between relational data and a relational model. The notion of median, with its algebraic, metric, geometrical and statistical aspects, allow a unified treatment of many of these problems. Properties of median procedures are organized according to four directions: stabilities and axiomatic characterizations; Arrow-like properties; combinatorial properties; effective computational possibilities. Finally, interesting mathematical problems, related to the notion of median are surveyed.

Keywords: Cluster analysis; Aggregation; Median; Social choice theory; Distance.

1. Introduction

Mathematical models in social sciences use more and more binary relations or graphs, because these sciences need 'structural' (Harary, Norman, Cartwright, (1965)) or 'discrete' (Roberts (1976)) models. (It is not to say that more traditional continuous models have not been successfully used in these sciences.)

At a first level, binary relations modelize collected data or observed situations: preferences, scales, paired comparisons or similarity data, social networks, group structures, games, tournaments, codes, systems, hierarchies, etc. Then, properties or problems related to considered phenomena can be described by mathematical concepts of graph theory, or by 'transformations' of binary relations. This approach has the advantage of showing that problems occuring in very different situations can lead to the same mathematical problem (see, for example, the search for cliques in a graph). Here, we tackle such a very general problem: the problem of aggregation or fitting of binary relations, which appears, for example, in social choice theory (Arrow (1951)) in paired comparisons methods (Kendall (1948)), in balance theory (Flament (1963)), in qualitative or relational data analysis (Degenne (1972), Mirkin (1976), Hubert (1976)) etc. We present a general method, based on metric notions and we call it the median procedure. Many authors, in many situations, have independently proposed this method: Abelson–Rosenberg (1958), Kemeny (1959), Hays (1960), Slater (1961), Zahn (1965), Regnier (1966), Barbut (1966, 1967a), Mirkin–Chernyi (1970), etc. First, we give examples in cluster analysis and social choice theory where the median procedure was used. Then we give the latticial and metric definitions of the median relations, a unified treatment of the known properties of the related procedure, along with extensions and related or open problems. An aim of the paper is also to show that axiomatic, metric, statistical, geometrical,... approaches must be combined to obtain a satisfactory theory of aggregation and fitting of binary relations. Such a theory is not yet available: its development stretches from the pionneering work by Kendall (1948), Arrow (1950) or Guilbaud (1952), to the more recent work by many authors with especially Fishburn (1973) and Mirkin (1974, 1976). We hope this paper together with the more technical works refered to Monjardet (1978) and Barthelemy (1979) will contribute to further advances in this theory.

Finally we will point out that the study of median procedures shows that the mathematical tools needed are very diversified, even if they are mostly relevant to combinatorial mathematics (in the broader sense, including discrete optimization). That is perhaps an illustration of a more general fact: mathematicians working on problems arising from social sciences need at their disposal a panoply of techniques, even if the social scientist considers that these problems are of purely local interest.

Now, we give some basic definitions and notations used in this paper. $X = \{x, y, z, \ldots, n\}$ is a finite n-set; we name the elements of X objects (these 'objects' being either alternatives or candidates, motions, social states, etc ...). 2^X is the power set of X. Thus, 2^{X^2}, denoted $\mathscr{R}$, is the set of binary relations on X. We always denote a binary relation by a capital letter: R, for example. We write $(x,y) \in R$ $[(x,y) \notin R]$, if the two elements x and y are related [are not related] by R. A set of binary relations is always written as a capital cursive letter: $\mathscr{S}$, for example. To name special types of relations, we generally follow Fishburn's terminology (Fishburn, 1972). We give here only the principal definitions. A relation R is *complete* (or *connected*) if, for all $x, y \in X$, $(x,y) \in R$ or $(y,x) \in R$. A relation R is *antisymmetric* if, for all $x, y \in X$, $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$. A relation R is *symmetric*, if for all $x, y \in X$, $(x,y) \in R$ implies $(y,x) \in R$. A relation R is *transitive* if, for all $x, y, z \in X$, $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$. A *complete preorder* (or *ordering*) is a complete transitive relation. A *tournament* is a complete antisymmetric relation. A *partial order* is an antisymmetric transitive relation. A *linear order* ('complete ordering' (Fishburn)) is a complete partial order. An *equivalence* is a symmetric transitive relation. We denote by $\mathscr{C}$, $\mathscr{P}$, $\mathscr{A}$, $\mathscr{T}$, $\mathscr{O}$, $\mathscr{L}$, $\mathscr{S}$, $\mathscr{E}$, the set of all complete relations, complete preorders, antisymmetric relations, tournaments, partial orders, linear orders, symmetric relations, and equivalences on the set X, respectively. $V=\{1,2,\ldots,i,\ldots,v\}$ is a finite v set; we name the elements of V voters (these 'voters' being either judges or experts, criteria, variables, etc.). A *profile* of relations is a v-uple $\Pi=(R_1,\ldots,R_i,\ldots,R_v)$ of relations on X, where R_i is the relation associated to the voter i. Thus the set of all profiles is $\mathscr{R}^v$. A v-tuple in $\mathscr{S}^v$ will be called a $\mathscr{S}$-profile.

We denote d the distance in a metric space. In such a space, we need to measure the 'distance' between a point and a set of points. Many authors use the term distance in that case. Here, we use the term *remoteness* and we denote a remoteness by D or Δ. For example, $D(R,\Pi)$ denotes a remoteness between the relation R and the profile Π.

The symbol $\subset$ is the symbol of proper inclusion; the symbol $\subseteq$ is the symbol of inclusion. $R-S$ is the set of elements of R that are not in S. $|R|$ denotes the cardinality of the set R. $[x]$ denotes the integer part of the number x. $[x]^*$ denotes the least integer not less than the number x.

2. Examples

2.1. Cluster analysis

Cluster analysis sorts a set of objects X, described by several variables, into 'homogeneous' clusters. The usual approach is to compute a measure of association, or similarity $s(x,y)$, for every pair (x,y) of objects, by means of the values of the variables describing these objects. Then, a hierarchical or nonhierarchical clustering method is used (Sokal and Sneath (1963); Jardine and Sibson (1971); Benzecri (1973), etc.). Sometimes, the obtained similarity is dichotomous; for example, $s(x,y)\in\{0,1\}$. In the general case, it may be interesting to consider the 'truncated' simularity for a threshold s: $s(x,y)=1$ if $s(x,y)\geq s$, and $s(x,y)=0$ if $s(x,y)<s$. We obtain a symmetric relation on X, by considering the set of pairs (x,y) with $s(x,y)=1$. We can also obtain such a similarity relation directly by the method of paired comparisons: for every pair of objects, a subject must say if the two objects are similar or not. Now, the problem is to sort the objects of (X,R), R being a similarity relation on X. Matula (1970) or Hubert (1974) describe many possible answers, related to notions of graph theory. Zahn (1964) proposed the following method. Let $E\in\mathscr{E}$ be an equivalence relation on X. Let

$$\delta(R,E)=|R\cup E|-|R\cap E|=|R-E|+|E-R| \tag{1}$$

be a measure of distance between R and E. Then, Zahn suggests a classification of the objects according to the equivalence minimizing $\delta(R,E)$ in $\mathscr{E}$.

We will now return to the initial clustering problem where v variables describe the objects. We can associate an equivalence on X with each variable: x and y are in the same equivalence class if the values of this variable are the same for x and y. Thus, we obtain v equivalences on X: $R_1, \ldots, R_i, \ldots, R_v$. Regnier (1965), then Mirkin (1970, 1974) propose as a good clustering of X, a partition whose associated equivalence minimizes the quantity

$$\sum_{i=1}^{v} \delta(E, R_i) \tag{2}$$

among all the equivalences E in $\mathscr{E}$, δ being defined by (1). This partition is called a *central partition* by Regnier.

2.2. Social choice theory

It is well known that the mathematical approach to the problem of preferences aggregation begins with Borda (1781) and Condorcet (1785). Condorcet, criticizes the Borda method and proposes the following paired comparisons method, which we call the Condorcet procedure in this paper: in each comparison, the preferred object is that object preferred by a (simple) majority of voters. Formally, let $R_i(R)$ be the linear order expressing the preference of the voter i (the collectivity):

$$[(x,y) \in R] \Leftrightarrow [(x,y) \in R_i, \quad \text{for a majority of voters}]. \tag{3}$$

The relation R is a tournament if the number of voters is odd or when it is even with any tie-breaking rule. We call such a tournament the Condorcet tournament of the profile $(R_1, \ldots, R_i, \ldots, R_v)$. The fact that this tournament is not always transitive (i.e. a linear order), is the well-known paradox of voting (or 'effet Condorcet', Guilbaud, 1952). 'Solving' this paradox has been a major problem in social choice theory. Arrow's theorem (1951) showed that the paradox is not tied to the use of the majority rule, but is far more general. The attempts to escape to Arrow's theorem are numerous and all are more or less debatable: restricted domains, choice functions, nonindependant social choice functions. (See, for example, Fishburn 1973, Sen 1977, Kelly 1978, Bordes 1980, Salles 1980.) Here, we look at methods based on the Condorcet tournament, i.e., methods searching linear orders approximating at best this tournament. The general problem of fitting a tournament by a linear order has been tackled by Slater (1961) in connection with a paired comparisons method. Here the subject is required to give his preferred object for every pair and the problem is to best approximate the resulting tournament by a linear order (the obtained individual preference can reveal intransitivities). Slater proposed the following solution: take a linear order L minimizing the number of 'inversions' ('inconsistent responses') to the considered tournament T (there is an inversion between L and T if $(x,y) \in L$ and $(y,x) \in T$. Let us define a measure of distance between two arbitrary tournaments R and T by

$$\delta(R,T) = |R \cup T| - |R \cap T| = |R - T| + |T - R|. \tag{4}$$

Then a Slater linear order is obtained minimizing the quantity δ among all the linear orders. It is worth noticing that the first object of a Slater linear order is not necessarily an object with a maximum 'score' (Moon (1968)) in the considered tournament (Bermond (1972)). So, in social choice theory the winner of a Slater linear order of the Condorcet tournament is not always a 'Copeland' winner (Fishburn (1977)). Then, again in social choice theory, the use of the Slater method has an obvious drawback. It does not take into account the 'strength' of majorities: x can be preferred to y by every voter or by half plus one. The method suggested by Kemeny (1959), then Barbut (1967), remedies this drawback. Let $(R_1, \dots, R_i, \dots, R_v)$ be a profile of linear orders. The Kemeny method selects a linear order L in $\mathscr{L}$, that minimizes the quantity

$$\sum_{i=1}^{v} \delta(L, R_i), \tag{5}$$

where $\delta(L, R_i)$ is defined by (4). Kemeny calls such a linear order L a median of the profile. Kemeny justifies his method by giving an axiomatic characterization of the distance δ between linear orders defined by (4). Barbut (1967) gives a more convincing justification: it is easy to prove that a tournament T minimizing, among all the *tournaments* in $\mathscr{T}$, the quantity $\sum_{i=1}^{v} \delta(T, R_i)$ (δ defined by (4)), is a Condorcet tournament. Thus the Kemeny procedure is the Condorcet procedure, when there is no 'effet Condorcet', and it is a 'natural' extension, when the paradox of voting occurs. Moreover, Condorcet himself proposed this method, with a different formulation, to solve the paradox in the case of three objects (Condorcet's suggestions are not clear and can be differently interpreted, for more than three objects: Black, 1958). We shall see an other justification of the Kemeny procedure by Young and Levenglick in Section 3.1.2.

N.B. The above method has been often rediscovered under different formulations: Bowman–Colantoni (1973), Blin–Whinston (1975), Merchant–Rao (1976), Guenoche (1977); see Blin (1976b) or Adelsman–Whinston (1977) for the equivalence of the formulations. Many authors rediscovered also the obvious fact (following the Barbut result), that an unique linear order, equal to the majoritary relation, is obtained, when the paradox of voting does not occur.

3. Median relations

3.1. Definitions

In respect to the inclusion relation, $\subseteq$, $\mathscr{R}$ is a poset and a boolean lattice with respect to the set theoretic operations of union $\cup$ and intersection $\cap$. Let R and R' be two relations in $\mathscr{R}$, such that $R \subseteq R'$. We denote by $[R, R']$ the set of all relations S in $\mathscr{R}$ such that $R \subseteq S \subseteq R'$. This set is called the interval defined by R and R'. Let V be a finite v-set (the voters) and Π a profile: $\Pi = (R_i)_{i \in V}$. We denote by $R_\alpha(\Pi)$, or simply R_α, the relation defined by: $(x, y) \in R_\alpha$ if and only if there exists $\lfloor (v+2)/2 \rfloor$ voters i in V such that $(x, y) \in R_i$.

Equivalently, we can write, using the latticial operation of $\mathscr{R}$:

$$R_\alpha(\Pi)=\bigcup_{\substack{W\subset V\\|W|=[(v+2)/2]}}\left(\bigcap_{i\in W}R_i\right). \tag{6}$$

We denote R_β, the relation defined by: $(x,y)\in R_\beta$ if and only if there exists $[(v+1)/2]$ voters i in V such that $(x,y)\in R_i$. Thus

$$R_\beta(\Pi)=\bigcup_{\substack{W\subset V\\|W|=[(v+1)/2]}}\left(\bigcap_{i\in W}R_i\right). \tag{7}$$

It is easy to see that $R_\alpha\subseteq R_\beta$, for every profile.

Definition 1. The *median interval* of the profile Π is the interval $[R_\alpha(\Pi),R_\beta(\Pi)]$.

Definition 2. A relation R is a *median relation* of the profile Π if and only if R is in the median interval of Π. We denote a median relation by $M(\Pi)$, or simply M; we also say that M is a *median* of Π.

Thus, the set of medians of Π is the interval

$$\left[\bigcup_{\substack{W\subset V\\|W|=[(v+2)/2]}}\left(\bigcap_{i\in W}R_i\right),\ \bigcup_{\substack{W\subset V\\|W|=[(v+1)/2]}}\left(\bigcap_{i\in W}R_i\right)\right]. \tag{8}$$

A consequence of the above definitions is that a profile Π has an unique median if and only if $R_\alpha(\Pi)=R_\beta(\Pi)$, thus, if and only if for all x, y in X, $|\{i\in V$ such that $(x,y)\in R_i\}|$ is different from $v/2$.

For example, if $|V|=v=2p+1$ is an odd number, Π has an unique median.

$$M(\Pi)=\bigcup_{\substack{W\subset V\\|W|=p+1}}\left(\bigcap_{i\in W}R_i\right). \tag{9}$$

It is clear, that $|W|=p+1$ can be replaced by $|W|\geq p+1$ in eq. (9). Thus, in this case, $(x,y)\in M(\Pi)$ if and only if there exists a (simple) majority W of voters such that $(x,y)\in R_i$, for every i in W. For this reason, $M(\Pi)$ may also be called the majoritary relation.

Now, we give a metric characterization of the median relations of a profile Π. We denote by $R\mathbin{\Delta} S$, the symmetric difference (s.d.) of the two relations R and S:

$$\begin{aligned}R\mathbin{\Delta} S=(R-S)\cup(S-R)=\{&(x,y)\text{ such that }(x,y)\in R\text{ and}\\ &(x,y)\notin S,\text{ or }(x,y)\notin R\text{ and }(x,y)\in S\}.\end{aligned} \tag{10}$$

We denote by $\delta(R,S)$, the cardinality of the symmetric difference between R and S:

$$\delta(R,S)=|R\mathbin{\Delta} S|=|R-S|+|S-R|=|R\cup S|-|R\cap S|. \tag{11}$$

δ is a well known distance on $\mathscr{R}$ (see 3.2) which enumerates the 'disagreements' between R and S. Let R be any relation and Π a profile of arbitrary relations. We define a *remoteness* between R and the profile Π, by

$$\Delta(R,\Pi)=\sum_{i=1}^{\nu}\delta(R,R_i), \tag{12}$$

where $\delta(R,R_i)$ is the distance defined by (11).

Theorem 1. *A relation M is a median relation of a profile Π if and only if*

$$\Delta(M,\Pi)=\min_{R\in\mathscr{R}}\Delta(R,\Pi).$$

Thus, the medians of a profile Π are the relations minimizing the remoteness (12) from this profile.

Remark 1. The above definition of a median can be generalized to any lattice. In fact, this generalization is meaningful only for distributive lattices (see 5.12). The median classically used by statisticians occurs when the distributive lattice is a chain.

Let $\mathscr{S}\subseteq\mathscr{R}$ be a set of binary relations on X, and Π a $\mathscr{S}$-profile, i.e. a profile of relations in $\mathscr{S}$.

Definition 3. $\mathscr{S}$ is *median-stable* if and only if every median of any $\mathscr{S}$-profile Π is in $\mathscr{S}$.

Generally, an arbitrary set $\mathscr{S}$ is not median-stable, i.e. there are median intervals not included in $\mathscr{S}$. But, in many problems, we need to resume a $\mathscr{S}$-profile by a binary relation, which is in $\mathscr{S}$ or, more generally, in a prescribed set of relations $\mathscr{M}$. Thus, the 'algebraic' definition of medians, by eq. (8), is not a sufficient one. However, the above theorem suggests a metric definition of medians, extending the algebraic definition.

Definition 4. Let $\mathscr{S}$ and $\mathscr{M}$ be two sets of binary relations on X and Π a $\mathscr{S}$-profile. A relation M is a *$\mathscr{M}$-median* of Π if and only if

$$\Delta(M,\Pi)=\min_{R\in\mathscr{M}}\Delta(R,\Pi). \tag{13}$$

We denote $\mathrm{Med}_{\mathscr{M}}(\Pi)$, the set of $\mathscr{M}$-medians of Π; if $\mathscr{M}=\mathscr{S}$, we simply write $\mathrm{Med}(\Pi)$.

This above definition makes sense for any value of positive ν. If $\nu=1$, a $\mathscr{S}$-profile is any relation R in $\mathscr{S}$. Thus, a $\mathscr{M}$-median of R is only a relation in $\mathscr{M}$ minimizing the distance $\delta(M,R)$, among all the relations M in $\mathscr{M}$. We call such a relation a $\mathscr{M}$-*median fit* of R. Evidently, this fitting problem only occurs when $\mathscr{M}\neq\mathscr{S}$.

If $v=1$, and $\mathcal{M} \neq \mathcal{S}$, we are in the *fitting case*. If $v>1$, and $\mathcal{M}=\mathcal{S}$, we are in the *aggregation case*. If $v>1$, and $\mathcal{M} \neq \mathcal{S}$, we can call it the *fitting aggregation case*. In all this cases, the procedure that associates with any profile Π in $\mathcal{S}^v$, the set of $\mathcal{M}$-medians of Π is said the *median procedure*.

Let us now take another look at the examples of Section 2. All of them are examples of medians procedures. Let $\mathcal{S}=\mathcal{S}$ be the set of symmetric relations on X and $\mathcal{M}=\mathcal{E}$ the set of equivalence relations. An $\mathcal{E}$-median fitting of a symmetric relation R is the equivalence approximating R defined by Zahn. If $\mathcal{S}=\mathcal{M}=\mathcal{E}$, an $\mathcal{E}$-median of an $\mathcal{E}$-profile is a Regnier's central equivalence. Let $\mathcal{S}=\mathcal{T}$ be the set of tournaments on X and $\mathcal{M}=\mathcal{L}$ the set of linear orders. An $\mathcal{L}$-median fitting of a tournament R is the linear order approximating R defined by Slater. If $\mathcal{S}=\mathcal{L}$ and $\mathcal{M}=\mathcal{T}$, a $\mathcal{T}$-median of an $\mathcal{L}$-profile is a Condorcet tournament. If $\mathcal{S}=\mathcal{M}=\mathcal{L}$, an $\mathcal{L}$-median of an $\mathcal{L}$-profile is a Kemeny median. We will now point out some other examples. If $\mathcal{S}=\mathcal{M}=\mathcal{T}$, it is easy to determine the set of $\mathcal{T}$-medians of a $\mathcal{T}$-profile Π; it is $\mathcal{T} \cap \mathrm{Med}(\Pi)$. Thus, if v is odd, there is a unique $\mathcal{T}$-median equal to the majoritary relation. The case where $\mathcal{S}=\mathcal{T}$ and $\mathcal{M}=\mathcal{L}$ appears in the problems of paired comparisons (see Monjardet (1978), for an axiomatic study).

Remark 2. A by product of this approach is to show that the problem of 'restricted domains' for the majoritary rule in social choice theory, is a problem of median stability. E.g., if $\mathcal{L}' \subset \mathcal{L}$ is a set of linear orders satisfying an unimodality condition, $\mathcal{L}'$ is median-stable (with slight evident modification of the definition for v even). Reciprocally, every median-stable set $\mathcal{L}' \subset \mathcal{L}$ defines a restricted domain, where the paradox of voting does not occur. Similarly, it is easy to find median-stable sets of equivalences, i.e. cases where the Regnier minimization problem is trivial; e.g. distributive sublattices of the lattice of equivalences constituted by 'associable' equivalences (Ore, 1942) are such sets.

3.2. The symmetric difference distance: further considerations

The distance between relations used in the Sections 2 and 3.1 is the classical symmetric difference distance in a boolean lattice of sets: $\delta(A,B)=|A \,\Delta\, B|=|A-B|+|B-A|=|A \cup B|-|A \cap B|$. (See Birkhoff (1967) and Barbut–Monjardet (1970) for latticial aspects, Restle (1959) for a use in psychometry, Sokal–Michener (1958) for a use in taxonomy, Flament (1963) for a use in social networks, etc.). Let us recall an interesting betweenness property of δ:

$$[A \cap C \subseteq B \subseteq A \cup C] \Leftrightarrow [\delta(A,C)=\delta(A,B)+\delta(B,C)]. \tag{14}$$

Let R and S be two tournaments; we have $|R-S|=|S-R|=\frac{1}{2}\delta(R,S)$. Thus, this quantity is a distance enumerating the number of disagreeing pairs $\{x,y\}$ between R and S. This distance has been called Kendall's distance (see, for example, Degenne (1972)), because it was implicitly used by this author: the Kendall's τ (1938) measuring the rank correlation between two linear orders is an affine transformation of it:

$$\tau = 1 - \frac{4\delta}{n(n-1)}.$$

Now we may point out another interpretations of the distance δ between two arbitrary relations. Let $\boldsymbol{r}$ and $\boldsymbol{s}$ be the two characteristic vectors associated with R and S: X^2 being linearly ordered, $\boldsymbol{r}(x,y)=1$ (0) if $(x,y)\in R$ ($(x,y)\notin R$). We denote $\|\boldsymbol{x}\|_p=[\sum|x_i|^p]^{1/p}$ the Hölder–Minkowski norm of a vector space (p integer in $[1,\infty[$). It is easy to see, that for every p

$$\delta(R,S)=\|\boldsymbol{r}-\boldsymbol{s}\|_p^p. \tag{15}$$

Then, $\delta(R,S)$ is the familiar 'city block' or L_1 metric ($p=1$), the square of the canonical euclidean distance $d_E(\boldsymbol{r},\boldsymbol{s})$ ($p=2$), etc. It is also the Hamming distance between the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$, i.e. the number of different coordinates of these vectors.

The fact that the s.d. distance has been used in many problems shows that this distance is relevant in many cases. We can do this assertion more precise. Let $\mathcal{S}$ be a set of binary relations occurring in a problem, and let us suppose that we need to define a distance between two arbitrary relations of $\mathcal{S}$. We expect a distance to satisfy certain properties, related to the problem. Is it possible to induce such a distance from the desirable properties? At a general level the answer is not obvious. But it is possible to find certain sets $\mathcal{S}$ and certain properties for which there is an unique answer: the s.d. distance δ. Moreover, these properties are quite 'natural', for certain problems. The sets $\mathcal{S}$, for which such an axiomatic characterization of δ exists, are: $\mathcal{P}$ (Kemeny (1959)); $\mathcal{E}$ (Mirkin–Chernyi (1970)); $\mathcal{O}$ (Bogart (1973)); $\mathcal{A}$ (Bogart (1975)); $\mathcal{S}$, $\mathcal{C}$, $\mathcal{T}$ (Barthelemy (1979b)). This last reference also containing corrections and improvements of the previous axiomatics.

For example, the restriction of δ to $\mathcal{O}$ is characterized by the five independant axioms:

(1) $\delta(R,S)=\delta(S,R)$, for all $R,S\in\mathcal{O}$ with $R\subseteq S$;

(2) $\delta(R,S)=\delta(R,T)+\delta(T,S)$, for all $R,S,T\in\mathcal{O}$, with $R\subseteq T\subseteq S$;

(3) $\delta(R,S)=\delta(R,R\cap S)+\delta(R\cap S,S)$, for all $R,S\in\mathcal{O}$;

(4) $\delta(R,R\cup\{x,y\})=\delta(S,S\cup\{z,t\})$, for all $R,S\in\mathcal{O}$ and for all $x,y,z,t\in X$ with $(x,y)\notin R$, $(z,t)\notin S$, $R\cup\{x,y\}\in\mathcal{O}$, $S\cup\{x,y\}\in\mathcal{O}$;

(5) the minimum positive distance between partial orders is one.

It is worth noticing, that the triangle inequality, which is not necessarily a natural condition for certain problems, is very often implied by other properties. Notice also that conditions (2) and (3) are special cases of the betweenness property (14) of δ.

Among the properties of the distance δ on a set $\mathcal{S}$ of relations, there is a property which does not appear in the axiomatisation of δ, but which plays an important role in the proofs of the Pareto principle and the monotonicity of medians relations (see 4.2). We will describe this property.

The $\mathscr{S}$ *regularity* of δ means that if x, y are two distinct objects and R a relation in $\mathscr{S}$ such that $(x,y)\notin R$, one can find a relation R' such that:

(i) $(x,y)\in R'$,

(ii) $\delta(R',S)\le\delta(R,S)$ for every $S\in\mathscr{S}$ with $(x,y)\in S$.

For $\mathscr{S}=\mathscr{C}, \mathscr{T}, \mathscr{S}, \mathscr{O}$ R' is obtained by exchanging x and y in R.

For $\mathscr{S}=\mathscr{E}$ or $\mathscr{P}$ this notion of regularity needs some change: if $\mathscr{S}=\mathscr{E}$, R being an equivalence relation so that $(x,y)\notin R$, one can construct the two equivalence relations R'_1 and R'_2, R'_1 is obtained by putting x in the equivalence class of y, R'_2 is obtained by putting y in the class of x. One can prove (see Barthélemy (1976)) that if S is an equivalence relation with $(x,y)\in S$:

$$\delta(R',S)<\delta(R,S) \quad \text{with } R'=R'_1 \text{ or } R'=R'_2.$$

If $\mathscr{S}=\mathscr{P}$, two cases occur:

(a) S is so that $(x,y)\in S$ and $(y,x)\notin S$, then $\delta(R',S)>\delta(R,S)$, R' being obtained by exchanging x and y in R.

(b) S is so that both $(x,y)\in S$ and $(y,x)\in S$, we construct the complete pre-orders R'_1 and R'_2 (with the same way as for E), then $\delta(R',S)<\delta(R,S)$ with $R'=R'_1$ or $R'=R'_2$.

The above study of the distance δ justifies the use of this distance in many cases. But there are cases, where an other distance between relations is more appropriate (see for examples, 5.2.2, 5.3.1, Boorman and Arabie (1972), Arabie and Boorman (1973), Boorman and Oliver (1973), Cailliez and Pages (1975, Chapter 15), Barthélemy (1979), Marcotorchino–Michaud (1979)).

3.3. *The remoteness: further considerations*

Let us recall the definition of the remoteness between a relation R an a profile Π:

$$\Delta(R,\Pi)=\sum_{i=1}^{v}\delta(R,R_i). \tag{16}$$

We give other expressions for $\Delta(R,\Pi)$. Let (x,y) be an ordered pair of elements of X, and $\Pi=(R_1,\dots,R_i,\dots,R_v)$ a profile. We write:

$$V(x,y)=\{i\in V \text{ such that } (x,y)\in R_i\},$$

$$v(x,y)=|V(x,y)|, \qquad \bar v(x,y)=v-v(x,y),$$

$$w(x,y)=v(x,y)-\bar v(x,y)=2v(x,y)-v=v-2\bar v(x,y).$$

Let us point out the equality

$$\sum_{i=1}^{v}|R_i|=\sum_{(x,y)\in X^2}v(x,y).$$

Then, we have

$$\Delta(R,\Pi)=\sum_{(x,y)\in R}\bar v(x,y)+\sum_{(x,y)\notin R}v(x,y), \tag{17}$$

$$\Delta(R,\Pi) = \sum_{i=1}^{v} |R_i| - \sum_{(x,y)\in R} w(x,y). \tag{18}$$

Eq. (17) gives another intuitive interpretation of Δ, and permits an easy proof of Theorem 1.

Eq. (18) shows that minimizing Δ is equivalent to maximizing $\sum w(x,y)$ and gives the linear formulation of this problem. We can, for example, write that a $\mathscr{R}$-median of Π maximizes on $\mathscr{R}$ the linear form $\langle \boldsymbol{p}_\pi, \boldsymbol{r}\rangle$, where the coordinates of $\boldsymbol{p}_\pi$ are the numbers $[\{v(x,y)/v\} - \frac{1}{2}]$, and where $\boldsymbol{r}$ is the characteristic vector of R. (See consequences in Section 4.4.2).

The above formulas can be simplified in some cases. For example, if π is a $\mathscr{L}$-profile and R a tournament, we obtain

$$\Delta(R,\Pi) = 2 \sum_{\substack{(x,y)\in R\\ x\neq y}} v(y,x) = \frac{n(n-1)v}{2} - \sum_{\substack{(x,y)\in R\\ x\neq y}} w(x,y).$$

Now, we give some general properties of the remoteness we called Δ. Mathematically, Δ is a function on the product set $\mathscr{R} \times \mathscr{R}^V$ to the set of nonnegative real numbers $\mathbb{R}^+$. Δ satisfies the following properties.

Property 0. δ-Extension. *For all $R, R' \in \mathscr{R}$, and for every $\Pi \in \mathscr{R}^V$*

$$[\delta(R,R_i) = \delta(R',R_i), \textit{ for every } i\in V] \Rightarrow [\Delta(R,\Pi) = \Delta(R',\Pi)].$$

Property 1. Positivity. *For every $R \in \mathscr{R}$*, and for every $\Pi \in \mathscr{R}^V$,

$$\Delta(R,\Pi) \geq 0.$$

Property 2. Definiteness. *For every $R \in \mathscr{R}$, and for every $\Pi \in \mathscr{R}^V$*

$$[\Delta(R,\Pi) = 0] \Leftrightarrow [R_1 = \cdots R_i = \cdots R_v = R].$$

Property 3. Isotonicity. *For all $R, R' \in \mathscr{R}$ and for every $\Pi \in \mathscr{R}^V$*

$$[\delta(R,R_i) \leq \delta(R',R_i), \textit{ for every } i\in V] \Rightarrow [\Delta(R,\Pi) \leq \Delta(R',\Pi)].$$

Property 4. Monotonicity. *For all $R, R' \in \mathscr{R}$ and for every $\Pi \in \mathscr{R}^V$*

$$[\delta(R,R_i) < \delta(R',R_i), \textit{ for every } i\in V] \Rightarrow [\Delta(R,\Pi) < \Delta(R',\Pi)].$$

Property 5. Strict Monotonicity. *For all $R, R' \in \mathscr{R}$ and for every $\Pi \in \mathscr{R}^V$*

$$[\delta(R,R_i) \leq \delta(R',R_i), \textit{ for every } i\in V \textit{ and there exists } i \textit{ such that } \delta(R,R_i) < \delta(R',R_i)] \Rightarrow [\Delta(R,\Pi) < \Delta(R',\Pi)].$$

These properties, along with the $\mathscr{S}$-regularity of δ, are important to prove many properties of median relations (see Section 4). They can also be considered abstractly and leads to the definitions of a δ-remoteness or a general remoteness used in Section 5.2. Here, let us only point out that other definitions of a remoteness, using δ, should been given. For example:

$$\Delta_p(R,\Pi)=\left[\sum_{i=1}^{v}\delta(R,R_i)^p\right]^{1/p}. \tag{19}$$

These are 'Minkowski-δ-remoteness' they satisfy all the above properties; $\Delta_1=\Delta$ may be said to be an additive remoteness.

$$\Delta_\infty(R,\Pi)=\sup_{i\in V}[\delta(R,R_i)]. \tag{20}$$

Δ_∞ is said a uniform δ-remoteness. It does not satisfy $P5$.

We have always $P5$ implies $P4$, and $\{P5,P0\}$ imply $P3$.

3.4. Geometrical and statistical aspects

There exists many possible embeddings (codings) of binary relations in an euclidean space. In such an embedding, a relation R is represented (or coded) by a vector (or point) in the space, denoted, for example, by $\boldsymbol{r}$. On the other hand, we will denote in the same way a set or a profile of relations, and the set or the family of associated vectors. We will now consider the embedding in the usual Euclidean space $\mathbb{R}^{n^2}$, obtained by the use of the characteristic vectors defined in Section 3.2. Let $\Pi=(\boldsymbol{r}_1,\dots,\boldsymbol{r}_i,\dots,\boldsymbol{r}_v)$ and $\boldsymbol{g}=\sum_{i=1}^{v}\boldsymbol{r}_i/v$ the mean vector (or gravity center) of Π; the coordinates of $\boldsymbol{g}$ are the (relative) frequencies $v(x,y)/v$ of voters relating x to y. The remoteness Δ between an arbitrary relation R and Π is identical to the second moment of Π about the point $\boldsymbol{r}$:

$$\Delta(R,\Pi)=\sum_{i=1}^{v}\delta(R,R_i)=\sum_{i=1}^{v}d_{\mathrm{E}}^2(\boldsymbol{r},\boldsymbol{r}_i), \tag{21}$$

where d_{E} is the Euclidean distance in $\mathbb{R}^n$. Then, (Huyghens' theorem):

$$\Delta(R,\Pi)=\sum_{i=1}^{v}d_{\mathrm{E}}^2(\boldsymbol{g},\boldsymbol{r}_i)+vd_{\mathrm{E}}^2(\boldsymbol{r},\boldsymbol{g}) \tag{22}$$

and a relation M is a $\mathscr{R}$-median of Π if and only if

$$d_{\mathrm{E}}^2(\boldsymbol{m},\boldsymbol{g})=\min_{r\in\mathscr{R}}d_{\mathrm{E}}^2(\boldsymbol{r},\boldsymbol{g}). \tag{23}$$

Let us denote by $\mathrm{cl}(\mathscr{R})$ the convex closure of $\mathscr{R}$ in the Euclidean space. It is a convex polytope whose vertices are the points in $\mathscr{R}$ (a relation cannot be a convex combination of different relations). Thus, in geometrical terms, a median of Π is a vertice of the polytope $\mathrm{cl}(\mathscr{R})$, closest to the gravity center.

Remark. The study of polytopes $\mathrm{cl}(\mathscr{R})$ is important, for example in view to solve related binary linear programming problems, but it is generally difficult. There are results for the following sets of binary relations

- $\mathcal{T}$: Moon–Pullman (1972)
- $\mathcal{L}$: Guilbaud (1970), Bowman (1972), Meggido (1977), Young (1978), Dridi (1980) (the first two references contain false assertions on cl($\mathcal{L}$))
- $\mathcal{E}$: Regnier, de la Vega (1976)
- $\mathcal{O}$: Hausmann, Korte (1978).

In the case of linear orderings there exist other polyhedral representations of $\mathcal{L}$. For example, the convex closure in $\mathbb{R}^{n-1}$ of the score vectors of linear orderings (see 5.2.2) has been defined and used by many French authors, under the name of 'permutoedre': Guilbaud–Rosenstiehl (1963, 1970), Feldman–Hogasen (1969, 1970), Benzecri (1970), Kreweras (1970) who define a bijection between the set of all faces of the permutoedre and the set $\mathcal{P}$ of all complete preorders on X, Barbut–Monjardet (1970), Frey (1971), Jacquet Lagreze (1971, 1977), Degenne (1972), Romero (1978), Marcotorchino–Michaud (1979), Terrier (1980). Notice that the obtained uniform polytope has been also considered in the study of the group of permutations S_n (see Coxeter, Moser (1972) p. 65) and that the shortest path distance in the adjacency graph of this polytope is nothing else that the Kendall distance $\delta/2$.

An other representation of $\mathcal{L}$ is the well-known polytope of doubly bistochastic matrices (see Brualdi–Gibson (1977), for a recent investigation, and Blin (1976a) for an application in social choice theory).

Let $\Pi=(\boldsymbol{r}_1,\dots,\boldsymbol{r}_i,\dots,\boldsymbol{r}_v)$ be a $\mathcal{L}$-profile. The usual dispersion measure of Π in $\mathbb{R}^{n^2}$ is the minimum value of the second moment of Π about an arbitrary point i.e. the value obtained about the mean vector $\boldsymbol{g}$. Dividing by v, we obtain the variance formula: $(1/v)\sum_{i=1}^{v} d_{\mathrm{E}}^2(\boldsymbol{g},\boldsymbol{r}_i)$. By analogy (and remarking that $\boldsymbol{g}$ is the characteristic vector of a relation, only if the profile Π is unanimous), we will define the $\mathcal{M}$-dispersion $D(\mathcal{M},\Pi)$ of Π:

$$D(\mathcal{M},\Pi)=\sum_{i=1}^{v} d_{\mathrm{E}}^2(\boldsymbol{m},\boldsymbol{r}_i)=\Delta(M,\Pi)=\min_{\mathcal{M}}\Delta(R,\Pi), \tag{24}$$

where M is a $\mathcal{M}$-median of Π, and $\boldsymbol{m}$ its characteristic vector. So, the $\mathcal{M}$-dispersion is the second moment of Π about a $\mathcal{M}$-median point. Furthermore, in order to compute a normalized $\mathcal{M}$-dispersion coefficient, we must compute the maximum $\mathcal{M}$-dispersion in $\mathcal{L}^v$. So, we write

$$f(\mathcal{M},\mathcal{L}^v,n)=\max_{\Pi\in\mathcal{L}^v}\left[\min_{R\in\mathcal{M}}\Delta(R,\Pi)\right]. \tag{25}$$

Unfortunately, the expressions (24) or (25) are generally difficult to compute. Nevertheless, some results have been obtained (see Section 4.3).

We turn now to connections between the median procedure and statistical coefficients of concordance between relations, as defined by Kendall and Smith (1939, 1940) for linear orders or tournaments. Here we consider only this case, but generalizations would be possible. Every tournament R has a dual tournament R^c $[(x,y)\in R^c$ if and only if $(x,y)\notin R]$, and we find more convenient to imbed $\mathcal{T}$ in an

Euclidian space, in such a way that the vectors associated with R and R^c are opposite. Hence, we define an arbitrary linear order on the set of the n^2 ordered pairs (x,y) of X, and we code the tournament R by the following vector ε in $\mathbb{R}^{n^2}$:

$$\begin{aligned} x=y:&\quad \varepsilon(x,x)=0,\\ x\neq y:&\quad \varepsilon(x,y)=+1 \quad \text{if and only if } (x,y)\in R\\ &\qquad\qquad\; =-1 \quad \text{if and only if } (x,y)\in R^c. \end{aligned}$$

In the Euclidean space $\mathbb{R}^{n^2}$, we have

$$d_E^2(\varepsilon,\varepsilon')=4\delta(R,R').$$

The convex closure $\mathrm{cl}(\mathcal{T})$ is a convex polytope inscribed in a sphere with radius $\varrho=\sqrt{n(n-1)}$ and situated in a hyperplane $\sum_{(x,y)}\varepsilon(x,y)=0$. Let now $\Pi=(\varepsilon_1,\ldots,\varepsilon_i,\ldots,\varepsilon_v)$ be a $\mathcal{T}$-profile. The n^2 coordinates of $g=\sum_{i=1}^{v}\varepsilon_i/v$ are, for $x\neq y$, the frequencies $w(x,y)/v$, and for $x=y$, zero. A natural measure of the 'homogeneity' of Π in $\mathbb{R}^{n^2}$ is the ratio, or square ratio, between the Euclidean norm of the vector g and the radius of the sphere. Then, we define

$$K(\Pi)=\frac{\|g\|^2}{\varrho^2}=\sum_{(x,y)}\frac{w^2(x,y)}{v^2n(n-1)}. \tag{26}$$

In fact, it is easy to see that K is an affine transformation of the Π-variance:

$$K=1-\sum_{i=1}^{v}\frac{d_E^2(\varepsilon_i,g)}{v\varrho^2}. \tag{27}$$

Thus, $K=1$ if and only if the Π-variance is null (all the tournaments in Π are the same) and K is minimum if and only if the Π-variance is maximum ($\min K=0$ for v even, $\min K=1/v^2$ for v odd). Moreover, we easily get

$$K(\Pi)=\sum_{(i,j)\in V^2}\frac{\tau(\varepsilon_i,\varepsilon_j)}{v^2}, \tag{28}$$

where

$$\tau(\varepsilon_i,\varepsilon_j)=1-\frac{d_E^2(\varepsilon_i,\varepsilon_j)}{\varrho^2}=1-4\frac{\delta(R_i,R_j)}{n(n-1)} \tag{29}$$

is an obvious generalization of the Kendall τ, in the case of arbitrary tournaments. We now emphasize three facts:

(1) The coefficient K, defined by Degenne (1972) with the formula (28) is equal to $\{U(v-1)+1\}/v$ where U is the Kendall–Smith–Ehrenberg (1940, 1952) coefficient of agreement. In fact, U is the average of the $\tau(\varepsilon_i,\varepsilon_j)$ on all the pairs $\{i,j\}$ of voters. These two coefficients can be used for tournaments or linear orderings with the advantage of an easy computation.

(2) Let Π be a $\mathcal{T}$-profile (a $\mathcal{L}$-profile) and R be an arbitrary tournament (linear order). Let us denote $\Pi+R$ the profile obtained by adding R to Π. Then, we have:

$$[K(\Pi+R)\text{ maximum}]\Leftrightarrow[R\text{ is a }\mathcal{T}\text{-median }(\mathcal{L}\text{-median})].$$

This is an obvious consequence of the formulas (28), (29) and the median definition. The same result is clearly valid with U, as noticed by Marcotorchino–Michaud (1979), and with $\sum_{i=1}^{v} \tau(R, R_i)$ (Hays (1960)).

(3) The same geometrical ideas results are valid when the score vectors are used. In the case of linear orders, for example, one obtain the classical Spearman ϱ and Kendall W.

A detailed study of concordance coefficients is in Kendall (1970) (see also the fifth chapter in Marcotorchino–Michaud (1979)). Related geometrical aspects may be found in Degenne (1972) and Monjardet (1981).

Finally, we mention some statistical studies of median relations. The median linear order of a $\mathcal{L}$-profile may be obtained as a maximum likelihood ranking in a probabilistic model: Thompson–Remage (1964), Remage–Thompson (1966), Astie (1970, 1971), Hubert–Schultz (1975), etc. Regnier (1977) studies the statistical stability of clustering procedures including the median equivalence procedure.

4. Properties of the median procedure

Most of the properties of the medians are established in the aggregation case ($\mathcal{Y} = \mathcal{M}$, $v > 1$).

4.1. The two stabilities by restrictions

We are here concerned by the following question. Is the aggregation procedure defined by medians stable in subsets of voters and in subsets of objects?

4.1.1. Stability for the voters

This is the so-called Young consistency condition (Young–Levenglick (1978)), a choice theoretic version of this condition is given in Young (1974)). If V is divided into two parts i.e. $V = W \cup \overline{W}$ and $W \cap \overline{W} = \emptyset$, we get the restricted profiles Π_W and $\Pi_{\overline{W}}$: $\Pi_W = (R_i \mid i \in W)$, $\Pi_{\overline{W}} = (R_j \mid j \in \overline{W})$. Applying the median procedure to the sets $\mathcal{Y}^w$ and $\mathcal{Y}^{v-w}$ (with $w = |W|$). We obtain the sets $\mathrm{Med}(\Pi_W)$ and $\mathrm{Med}(\Pi_{\overline{W}})$. It is easy to prove that if $\mathrm{Med}(\Pi_W) \cap \mathrm{Med}(\Pi_{\overline{W}}) \neq \emptyset$, then $\mathrm{med}(\Pi) = \mathrm{med}(\Pi_W) \cap \mathrm{Med}(\Pi_{\overline{W}})$.

As a consequence of this: if a relation, say R_1, occuring in the profile Π is a median for the others (i.e. for the profile $\Pi_{V-\{1\}}$) then R_1 is *the* median for Π.

4.1.2. Stability for the objects

Monjardet (1973) established that, in the case of tournament aggregation, the median procedure is stable by restriction to any subset of X (applying the median procedure to a profile, then restricting the obtained tournament to a subset of X is just the same as restricting the relations of the profile to that subset, then applying, on those restricted relations the median procedure). Unfortunately, this remains not true in the case of linear orders (that is a consequence of Arrow's theorem). However, Jacquet-Lagrèze (1969) established that if M is a median order for the profile $\Pi \in \mathcal{L}^v$, the median procedure (on Π) restricted to the intervals of M is stable (it will be remembered that the interval $[a, b]$, for M, is the set: $\{x \mid (a,x) \in M, (x,b) \in M\}$). (See also, Younger (1963), for the case $v = 1$).

Formally, $\mathcal{S}$ being a set of binary relations on X and Z a subset of X, let $\mathcal{S}^Z$ denote the set $\{R \cap (Z \times Z) \mid R \in \mathcal{S}\}$ (so $\mathcal{S}^Z$ is the *restriction* of $\mathcal{S}$ to Z). For a profile $\Pi = (R_1, \ldots, R_v)$, Π yields the restricted profile $\Pi^Z = R_1^Z, \ldots, R_v^Z)$. We note $\mathrm{Med}^Z(\Pi)$, the set, in $\mathcal{S}^Z$, of medians for Π^Z. Using this notation, we shall call a *median stability*, a map s from $\mathcal{S}$ to 2^{2^X}, so that: for every profile $\Pi \in \mathcal{S}^v$, $M \in \mathrm{Med}(\Pi)$ if and only if, for every $Z \in s(M)$, $M^Z \in \mathrm{Med}^Z(\Pi)$. $s(M)$ will be called a *stability family* for M.

The following stability families have been obtained:

$s(M) = 2^X$, if M in $\mathcal{S} = \mathcal{P}$ or $\mathcal{T}$ (Monjardet (1973));

$s(M) = \{\text{intervals of } M\}$ for M in $\mathcal{L}$ (Jacquet-Lagrèze (1969)) and M in $\mathcal{W}$ (Barthélemy (1976));

$s(M) = \{\text{unions of } M' \text{ classes}\}$ for M in $\mathcal{E}$ (Barthélemy (1976)).

4.1.3. A consequence of the two stabilities

In the case of linear orders, the stability by intervals implies an extension of the Condorcet condition: for every pair $\{x, y\}$, the Condorcet winner cannot be placed just before the loser. Hence, if in the Condorcet tournament one observes the cyclical majorities: a preferred to c, c preferred to b, b preferred to a, the following configurations are excluded in a median order:

c preferred to a,	a preferred to b;
b preferred to c,	c preferred to a;
a preferred to b,	b preferred to c.

From a more formal point of view, an aggregation procedure φ on linear orders (i.e. a map from $\mathcal{L}^v$ to $2^{\mathcal{L}}$ is *Condorcet* (Young-Levenglick (1978)) if:

– When x, y are such that $v(x,y) < v(y,x)$ for the profile Π, there is no $M \in \varphi(\Pi)$ in which y succeeds to x.

– When x, y are such that $v(x,y) = v(y,x)$ for any linear order M, if M' denotes the order obtained from M by exchanging x and y, then $M \in \varphi(\Pi)$ if and only if $M' \in \varphi(\Pi)$.

Another property of the median procedure is its neutrality: an aggregation procedure φ is *neutral* when, for every permutation σ on X and every profile Π: $\varphi(\Pi^\sigma) = \varphi^\sigma(\Pi)$; Π^σ being the profile $(R_1^\sigma, \ldots, R_v^\sigma)$ (with $R^\sigma = \{(x,y)/(\sigma(x), \sigma(y)) \in R\}$) if $\Pi = (R_1, \ldots, R_v)$ and φ^σ being the procedure which associates, to the profile Π the set $\{R^\sigma / R \in \varphi(\Pi)\}$.

In the previously quoted paper, Young-Levenglick proved that:

The median procedure on $\mathcal{L}$ is the only aggregation procedure which is neutral, consistent and Condorcet.

Hence the two stabilities (voters and objects) together with the neutrality, do in fact characterize median orders entirely.

4.2. *Arrow-like conditions*

Let us now look at some properties arising from the axiomatic approach in Social Choice Theory (Arrow (1951), Sen (1968), Murakami (1970), Fishburn (1973), Arkhipoff (1975), etc.).

4.2.1. *Conditions obviously verified by the median procedure*

We will mention:
- *Neutrality:* pointed out in Section 4.1.3.
- *The Non Restricted Domain Condition:* every profile in $\mathcal{D}$ admits $\mathcal{M}$-medians.
- *Sovereignty:* for every $x, y \in X$ there exists a profile $\Pi \in \mathcal{D}^V$ so that for one (at least) $M \in \mathrm{Med}(\Pi)$, $(x, y) \in M$.
- *Surjectivity of the procedure:* every $M \in \mathcal{M}^V$ is a median for some profile (when $\mathcal{M} \subseteq \mathcal{D}$).
- *Non-dictatoriality:* there is no voter $i \in V$ so that, for every profile $\Pi = (R_1, \dots, R_v)$, $R_i \subseteq M$, with $M \in \mathrm{Med}(\Pi)$.

Notice also that the restricted stabilities of Section 4.1.2 are weakened versions of Arrow's independance condition.

4.2.2. *The Pareto principle*

The *Pareto relation* of the profile $\Pi = (R_1, \dots, R_v)$ is the intersection $P(\Pi) = \bigcap_{i=1}^{v} R_i$, $P(\Pi)$ is the unanimous part of Π. The median procedure, in $\mathcal{D}$, will satisfy the *Pareto principle* when, for every $\Pi \in \mathcal{D}^V$ and every $M \in \mathrm{Med}(\Pi)$, $P(\Pi) \subseteq R$ (all unanimous pairs are in the medians).

In $\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$, $\mathcal{L}$, $\mathcal{O}$, $\mathcal{E}$ the median procedure satisfies the Pareto principle. The proofs can be found in:
- Monjardet (1973): $\mathcal{T}$, $\mathcal{L}$;
- Feldman (1973): $\mathcal{L}$;
- Barthélemy (1976): $\mathcal{C}$, $\mathcal{T}$, $\mathcal{L}$, $\mathcal{P}$, $\mathcal{E}$;
- Michaud–Marcotorchino (1979) who make a distinction between $\mathcal{D}$ and $\mathcal{M}$: $\mathcal{D}$ is the set of partial preorders on X, $\mathcal{M} = \mathcal{P}$.

Notice also that the Pareto principle for $\mathcal{E}$ was first pointed out by Regnier (1965), with a sketched proof.

We will now examine a dualization of the Pareto principle. The scope of such a dualization is the hope that all the pairs related by a median of a profile Π are also related by (at least) one relation occuring in Π.

The *co-Pareto relation* of $\Pi = (R_1, \dots, R_v)$ is the union $P^*(\Pi) = \bigcup_{i=1}^{v} R_i$: $P^*(\Pi)$ is the 'non exterior part' of Π. The median procedure, in $\mathcal{D}$, satisfies the *co-Pareto principle* if, for every $\Pi \in \mathcal{D}^V$ and for every $M \in \mathrm{Med}(\Pi)$, $R \subseteq P^*(\Pi)$.

Arguments about valuations on posets (cf. Section 5.3.1) show that this principle is verified in $\mathscr{T}$, $\mathscr{P}$, $\mathscr{L}$ and $\mathscr{C}$ (Barthélemy 1979a, 1979c). One can also prove it for $\mathscr{S}$ and $\mathscr{A}$.

Notice that the co-Pareto principle has been introduced by Mirkin (1975) with the aim of providing an axiomatic approach to the problem of partitions aggregation. Mirkin (1979) gives also axiomatic characterizations of 'oligarchic' procedures, i.e. procedures that associate with every $\mathscr{S}$-profile Π, the intersection of a prescribed subset of relations in Π.

4.2.3. *Monotonicity*

We will now study a condition not far removed from Sen's positive responsiveness (Sen (1970)). This condition was first pointed out by Feldman (1973) in the case of median linear orders and extended by Barthélemy (1979) to complete preorders.

The obtained result indicates that if y is preferred to x by every median of a profile Π and if the profile Π is modified by exchanging x and y in a relation where x is strictly preferred to y, then y is always preferred to x in every median of the new profile. Technically:

Suppose $\mathscr{M} = \mathscr{S} = \mathscr{L}$ or $\mathscr{P}$. If $\Pi = (R_1, \ldots, R_v) \in \mathscr{S}^v$ let x, y be objects so that $x \neq y$ and $(x, y) \in M$ for every $M \in \mathrm{Med}(\Pi)$. Let k be a voter such that $(x, y) \notin R_k$. If $\Pi' = (R'_1, \ldots, R'_v)$ is the profile defined by:

- *$R'_i = R_i$, for $i \neq k$,*
- *R'_k is obtained by exchanging x and y in R_k.*

Then, if $M' \in \mathrm{Med}(\Pi')$, $(x, y) \in M'$.

4.3. *Combinatorial properties*

Medians lead to interesting combinatorial problems, mainly investigated in the fitting case ($v = 1$, $\mathscr{S} \neq \mathscr{M}$).

Let us first consider the case of the median linear orders of a tournament. It is easy to see that finding such an order is a case of a well known problem in graph theory: finding a 'minimum feedback arc set' for a directed graph (Seshu and Ree (1961), Younger (1963)), i.e., a minimum set of arcs to delete in order to obtain a graph without circuit (or more generally, finding a minimum transversal set of a hypergraph). Using this connection, Bermond (1973), then Hardouin du Parc (1975) and Bermond–Kodratoff (1976), give several old and new results on median-linear orders. Let us give some examples:

- The median orders are Hamilton paths of the tournament.
- Let, generally $F(\mathscr{M}, \mathscr{S}, n) = \max_{D \in \mathscr{S}} \min_{M \in \mathscr{M}} \delta(d, M)$:

$$\left[\frac{n}{3}\left[\frac{n-1}{2}\right]\right] \leq F(\mathscr{L}, \mathscr{T}, n) \leq \left[\frac{n}{2}\right]\left[\frac{n-3}{2}\right].$$

$-F(\mathcal{T}, \mathcal{L}, n)$ is equivalent to $(n(n-1))/4$ for n infinite.
$-F(\mathcal{T}, \mathcal{L}, n)$ is known for $n \leq 13$.

Let us now consider the case of a median equivalence relation of a symmetric relation. This has been investigated by Zahn (1964), Moon (1966, 1971), Tomescu (1974) and Defays (1975). One interesting result, due to Tomescu, is worth quoting:

$$F(\mathcal{E}, \mathcal{S}, n) = [(n(n-2))/4]^*.$$

Moreover, the extremal symmetric relations are determined, and the same result is valid if $\mathcal{E}$ is replaced by the set of equivalence relations with at most k classes ($k \geq 2$).

The aggregation case ($v > 1$), mainly investigated by Monjardet (1973) for tournaments and linear orders, leads to some results. For example:

(a) The median orders are Hamilton paths of the Condorcet tournament of a $\mathcal{L}$-profile (notice that the converse is false). An easy consequence is that a median order contains the weak order induced by the transitive closure of the Condorcet tournament.

Generally let $F(\mathcal{M}, \mathcal{D}^v, n) = \text{Max}_{D \in \mathcal{D}^v} \text{Min}_{M \in \mathcal{M}} \Delta(D, M)$,

$$F(\mathcal{T}, \mathcal{T}^{2p}, n) = F(\mathcal{T}, \mathcal{L}^{2p}, n) = p\binom{n}{2}.$$

(b) $$p\binom{n}{2} + F(\mathcal{T}, \mathcal{L}, n) \leq F(\mathcal{T}, \mathcal{L}^{2p+1}, n) \leq \binom{n}{2}\frac{v}{2} - \frac{1}{2}\left[\frac{n}{2}\right].$$

(c) $$p\binom{n}{2} \leq F(\mathcal{T}, \mathcal{T}^{2p+1}, n) \leq \binom{n}{2}\frac{v}{2} - \frac{1}{2}\left[\frac{n}{2}\right],$$

with the upper bound obtained for $n \leq 3$.

(d) $F(\mathcal{T}, \mathcal{L}^v, n)$ is equivalent to $\binom{n}{2} v/2$ for n infinite.

Notice that no result is known for $F(\mathcal{E}, \mathcal{S}^v, n)$, $F(\mathcal{E}, \mathcal{E}^v, n)$.

4.4. Computation of medians

To compute the $\mathcal{M}$-medians of a profile is generally a difficult problem. First we review a lot of proposed methods that are unsatisfactory because either they give the optimal solutions only for small values of n, or they give sub-optimal solutions (heuristics). Then, we emphasize the linear programming formulation which has recently allowed significant advances.

4.4.1. Combinatorial methods

Many methods have been proposed for the computation of *median* or *near-median linear orders*. Let us mention first two reformulations of this problem leading to a host of techniques: the minimum cover (or weight cover) interpretation: Barbut (1966), Lempel and Cederbaum (1966), Glover *et al.* (1974), Merchant and

Rao (1976); the quadratic assignment interpretation: Lawler (1964), Blin and Whinston (1974, 1975), Hubert and Schultz (1975).

The exhaustive method to obtain all the optimal solutions is to compute the remoteness for all the $n!$ linear orders on X. The 'easier' way is an inductive computing using the factorial tree enumerating all these linear orders. Many 'backtrack' techniques (Reingold et al., 1977) have been proposed to obtain the optimal orders without construct the complete tree: Bellman's dynamic programming: Lawler (1964), Remage and Thompson (1966), Bermond and Kodratoff (1976), Guenoche (1977); Branch and Bound: De Cani (1972), Burkov and Groppen (1972), Cook and Saipe (1976); Branch search: Flueck and Korsh (1974). These methods give all the median linear orders, but only for small values of n ($n<20$).

Many heuristics (sub-optimal algorithms) have been proposed to tackle with greater values of n. Their general strategy is to improve a given linear order 'locally' until such improvements are impossible; then a 'local optimum' is obtained but it is generally a sub-optimal solution. The tested local transformations can be induced by various local necessary conditions for a median linear order. Such heuristics can be found, for example, in Younger (1963), Jacquet–Lagreze (1969), Baker–Hubert (1977), Marcotorchino and Michaud (1979); the last two references seek to appraise the efficiency of their techniques and give reasons supporting their claim that it is 'good'.

The problem to compute median equivalences has been far less studied. For the case of aggregation, Regnier (1975) and Mirkin (1976, 1977) have given heuristics similar to those above. The fitting case has been tackled by Heuchenne (1970) and Defays (1975).

Remark. The computation of median linear orders can be seen as a particular case of the general problem of ordering 'at best' a set of objects based on an asymmetric proximity function (see Section 5.1.4). A fairly complete review of this last problem is in Hubert (1976) where the reader will find many additional references.

4.4.2. Median procedure and linear programming

For any sets of binary relations $\mathcal{S}$ and $\mathcal{M}$, $\Delta(M,\Pi)$ is obviously a linear function of the coefficients of the $\{0,1\}$-matrix associated to M (see Section 3.3). Hence integer linear programming (De Cani (1969), Bowman and Colantoni (1973), Marcotorchino and Michaud (1978)), leads to an exact solution for $\mathrm{Min}_{S\in\mathcal{M}}\,\Delta(S,\Pi)$ when the definition axioms are linearizable, (for example, if S is a transitive relation). Moreover although the zero-one linear program and the associated 'relaxed' linear program are not generally equivalent, it seems that very often (always in the Marcotorchino–Michaud examples) the relaxed program gives a zero-one optimal solution, thus an optimal solution of the initial program. A drawback is the high order of the constraints in the relaxed linear program (for instance, associativity is in n^3). The remedy is the use of the dual program: (Michaud and Marcotorchino, 1978). Another drawback, unfortunately unsolvable as yet, is the fact that just one solution is computed and no answer can easily be given to the questions: is the solution the only one? Are the other solutions (if any) far from the computed one?

However, this method allows the computation of an exact solution up to $n = 80$. Let us remark, that the heuristics in Section 4.4.1 are interesting to provide a 'good' starting solution for the linear programming algorithm, allowing to shorten the computing time.

5. Extensions, related matters, open problems

5.1. Extensions of the notion of median

5.1.1. An infinite case

The median procedure can be extended when X is infinite with the following manner:

Let $(X, \mathscr{A}, \mu)$ be a measured space, the measure μ being positive and finite (in particular μ can be a probability measure). When R, S are measurable relations ($R, S \in \mathscr{A} \times \mathscr{A}$: for the notations see Halmos (1968)) we write:

$$\delta(R,S) = \mu \times \mu(R \Delta S)$$

(previously we used the cardinality as a measure).

The main problem is the existence of medians, it has been solved in the two cases hereunder:

– Measurable equivalence relations with a finite number of classes (Barthélemy (1977a)).

– Measurable complete preorders (Barthélemy (1977b), results given without proof).

For these two cases, the existence of medians (which is non-trivial from a measure theoretic point of view) is strongly related to the Pareto principle. Notice also that those medians can be computed as in the finite case.

5.1.2. Theory of median in finite distributive lattices

The first noteworthy occurrence of this median is in Birkhoff and Kiss (1947) in the case of three elements in a lattice. This ternary case has led to many developments subsumed under the name of 'median algebras' (see Bandelt and Hedlikova (1979) for a review). In a (finite) lattice, the notion of median has been extended by Barbut (1961) to an odd numbered family of elements.

In a lattice the median interval of such a family $(e_1, \ldots, e_{2p+1})$ is the interval $[\alpha, \beta] = \{x \mid \alpha \le x \le \beta\}$ with

$$\alpha = \bigvee_{\substack{|K| = p+1 \\ K \subseteq \{1, \ldots, 2p+1\}}} \left(\bigwedge_{i \in K} e_i \right), \quad \beta = \bigwedge_{\substack{|K| = p+1 \\ K \subseteq \{1, \ldots, 2p+1\}}} \left(\bigvee_{i \in K} e_i \right).$$

When this interval admits just one element (this is the case for the lattice $\mathscr{R}$ cf. Section 3.1) we say that the family $(e_1, \ldots, e_{2p+1})$ *admits a median*.

The following result (Barbut, 1961) establishes a strong connection between the concepts of distributivity, median and existence of a δ like distance for a lattice.

Let L be a finite lattice, the following conditions are equivalent:

(i) *L is distributive,*

(ii) *every odd numbered family of L's elements admits a median,*

(iii) *one can construct a function d from $L \times L$ to the set $\mathbb{R}$ of real numbers so that:*

– *d is positive and $d(x,y)=0$ implies $x=y$*

– *$d(x,y)=d(x,t)+d(t,y)$ if $t \in [x \wedge y, x \vee y]$ and $d(x,y)=d(y,x)$ if $x \leq y$.*

The two following remarks are essential:

(a) d is, in fact a distance function and the second condition of (ii) is not other than a weakened form of the betweenness condition (14) occurring in the axiomatic characterization of δ (cf. Section 3.2).

(b) The median of the family $(e_1, \ldots, e_{2p+1})$ is the (only) solution of:

$$\min_{x \in L} \sum_{i=1}^{2p+1} d(x, e_i). \tag{30}$$

Notice that this can be easily generalised in the even case, only the unicity is lost and the medians form an interval in L. So, distributive lattices appear as a good 'explaining structure' for median theory and axiomatic of δ (for the first point see Monjardet (1980), for the second Barthélemy (1979)).

Notice also that the set of 'quantiles' of a family in L (with a suitable generalization) forms a chain strongly related to the nearest chain of a subset of L (Barbut, 1967).

5.1.3. Extensions to graphs

The distance d in a distributive lattice mentioned in the previous section, is not other than a minimal path length in the covering graph of L.

More generally, the *covering graph* $G(L)$ of a poset L is the graph (undirected) whose vertices are L's elements, u_{xy} being an edge if and only if y covers x ($x<y$ and there is no z such that $x<z<y$) or x covers y. It is easy to show that, in a distributive lattice, the function:

$$d(x,y) = \text{length of a minimal path, in } G(L), \text{ between } x \text{ and } y \tag{31}$$

verifies the condition (iii) of the previous section.

Thus, in any connected graph G, taking (31) as the definition of a distance d, we get a notion of median (extending the median in distributive lattices) by means of (30), for every family (not necessarily odd) $(e_1, \ldots, e_p)$ of vertices. Remark that the medians of all G's vertices is sometimes called the *centroid* of G and that finding this centroid is a classical problem in graph theory.

The particular case of a tree has been studied by Slater (1978). Among his results we will point out the two following facts:

– The medians of a family of vertices of a tree T consist in the vertices of a path in T.

– If a set S of T's vertices has an odd cardinality, S admits just one median.

Indeed the case of a tree is of great interest because it can be shown (Barthélemy, 1980) that for any set S of vertices of a graph G and any median m of S in G, there exists a subtree T of G, whose set of vertices contains S, such that m is a median of S in T.

Notice also that the graphs where three elements admit only one median, lead to interesting problems (related to median algebras and hypergraphs (see Sholander (1954) and Mulder and Schrijver (1979)).

5.1.4. Extensions to other relations

We will now return to the lattice $\mathscr{R}$. Median procedures can be used for other relations than those previously studied. The median complete preorders and the median linear orders of a semiorder have been studied by Jacquet-Lagrèze (1978) and Vincke (1978). (Notice that semiorders have recently been studied in a social choice context by Blau (1979) and Blair and Pollack (1979), who establish the analogue of Arrow's impossibility theorem). Then, we will mention the work of Bogart and Meek (1979), using a median procedure to find a 'consensus' for signed digraphs.

Finally, we will emphasize the following facts. The computation of a median relation for a profile is only based on the n^2 values $v(x,y)$ or $w(x,y)$ defined in Section 3.3. These values define a particular, non necessarily symmetric, proximity (or similarity) measure on X. We can equivalently speak of a 'fuzzy', 'probabilistic' or 'weighted' binary relation. We can search an $\mathscr{U}$-median-like relation for such an arbitrary weighted relation. For example, to find a median – like linear order of a weighted relation, i.e. a permutation that maximizes the sum of the above-diagonal elements in the associated matrix, is a well known problem in archaeological or psychological seriation, or in economics (see, Hubert (1976) for a review). Furthermore, if we allow the median-like relation to be itself a weighted relation of a specified type, we get a paradigm covering a considerable amount of structural models in social sciences (see, for example, Jardine and Sibson (1971), and Mirkin (1976, 1977, 1979)).

5.2. Extensions of the notion of remoteness

5.2.1. A slight extension: weighted voters

Suppose that each voter i, $1 \leq i \leq v$, has a weight p_i ($p_i > 0$, $\sum_{i=1}^{v} p_i = 1$). Then we can define the remoteness Δ by:

$$\Delta(S,\Pi) = \sum_{i=1}^{v} p_i \delta(S,R_i).$$

It is clear that all the properties of the median procedure, exposed in this paper, remain true in this case.

5.2.2. *General remoteness*

The properties of Δ pointed out in Section 3.3 may be used for the definition of a remoteness on a subset $\mathcal{P}$ of $\mathcal{R}$. Thus a *remoteness* on $\mathcal{P}$ will be a function D from $\mathcal{P} \times \mathcal{P}^v$ to the set $\mathbb{R}$ of real numbers satisfying the conditions of positivity (P_1), definiteness (P_2), isotonicity (P_3), monotonicity (P_4). If (P_4) is replaced by (P_5) we say that D is strictly monotonous.

Let $\mathcal{S}$, $\mathcal{M}$ be two subsets of $\mathcal{P}$; for a profile $\Pi \in \mathcal{S}^v$, a relation $M \in \mathcal{M}$ so that:

$$D(M,\Pi) = \min_{S \in \mathcal{M}} D(S,\Pi) \tag{32}$$

will be called an $(\mathcal{M}, D)$ *central relation* for Π.

A few results have been obtained in the case of general central relations. They are essentially given in Barthélemy (1979) and (1980) (they concern the Pareto principle). We shall just mention here that the well-known *Borda procedure* leads to a special kind of central relations.

For a relation $R \in \mathcal{R}$ the score $\hat{R}(x)$ of $x \in X$ is defined by:

$$\hat{R}(x) = \frac{|\{y \in X \mid (y,x) \in R, (x,y) \notin R\}| + |\{y \in X \mid (y,x) \in R\}| - 1}{2}$$

(if R is a linear order $\hat{R}(x)$ is the x' rank; in the tournament case, one recognizes the classical notion of score (cf. Moon (1968))).

If $\Pi = (R_1, \dots, R_v)$ is a profile, we write:

$$\hat{\Pi}(x) = \sum_{i=1}^{v} \hat{R}(x),$$

then $\beta(\Pi)$ is the complete preorder defined by:

$$(x,y) \in \beta(\Pi) \quad \text{if and only if } \hat{\Pi}(x) \leq \hat{\Pi}(y).$$

Let $\beta'(\Pi)$ be the set of linear orders included in $\beta(\Pi)$. We have defined the function β' from $\mathcal{R}^v$ to $2^{\mathcal{L}}$ (it is the so-called *Borda procedure*).

Suppose now that X admits an arbitrary linear order, then $R \in \mathcal{R}$ can be associated with its *score-vector* $\hat{R}$ whose coordinate on x is $\hat{R}(x)$. Restricting Borda procedure on $\mathcal{P}$ we can consider the remoteness (on $\mathcal{P}$):

$$B(S,\Pi) = \sum_{i=1}^{v} \|\hat{S} - \hat{R}_i\|^2 \tag{33}$$

(the norm being the classical one in $\mathbb{R}^n$, and $\Pi \in \mathcal{P}^v$).

One can show (cf. Kendall (1962)) that linear orders in $\beta'(\Pi)$ are $(\mathcal{L}, \mathcal{B})$-central relations for Π. Let us notice that Cook and Seiford (1978) study the central complete preorder obtained in replacing in (32) the classical norm L_2 by the norm L_1.

5.2.3. The special case of δ-remoteness

A remoteness on $\mathscr{R}$ verifying the condition P_0 (δ-extension) of Section 3.3 will be called a *δ-remoteness*. In Sections 2, 3 and 4 we considered the special case of additive δ-remoteness but many results can be extended in more general situations. These questions were studied first by Feldman (1973) who pointed out, in the special case of linear orders, the central part of the monotonicity (P_4) and afterwards by Barthélemy (1978) in the case of other relations (complete preorders, equivalence relations, tournaments and complete relations).

For central relations, relative to any δ-remoteness, the properties of Section 4.2.1 can be extended without difficulty. The Pareto principle remains true for $\mathscr{C}$ and $\mathscr{L}$. In the case of linear orders one must assume, moreover, that the δ-remoteness is strictly monotonous. For complete preorders, only a weakened version of the Pareto principle is obtained: if D is a strictly monotonous δ-remoteness every $M \in \mathscr{P}$, central for $\Pi = (R_1, \dots, R_v) \in \mathscr{P}^v$ contains every pair (x, y) so that $(x, y) \in R_i$, for every voter i and, at least for one voter j, $(y,x) \notin R_j$ (in other words every pair (x, y), y being unanimously prefered — and non unanimously indifferent — to x, is in M). For equivalence relations, proofs of the Pareto principle need the additivity of the δ-remoteness.

The property of Section 4.2.3 (monotonicity of the procedure) may be extended to any strictly monotonous δ-remoteness.

Young's consistency (stability in the set of voters) depends only upon the strict monotonicity of a (general) remoteness. No stability in subsets of objects is known in the non additive case.

5.3. Related matters

5.3.1. Metric properties of posets

We have already pointed out that δ can be interpretated as a minimal path length in the covering graph of the distributive lattice $\mathscr{R}$ (see Section 5.1.2). More generally we may consider a real-valued function θ on a poset E so that $\theta(x) < \theta(y)$ if $x < y$. θ induces a valuation on edges of the covering graph $G(E)$ ($\theta(u_{xy}) = |\theta(y) - \theta(x)|$). When E is connected, the minimal length, with respect to θ, of a path between x and y is a distance function d_θ on E. The study of d_θ is very closely related to the study of 'valuations' on the poset E. In particular, in some cases, d_θ can be computed with the help of a formula, e.g. in a lattice $d_\theta(x, y) = 2\theta(x \vee y) - \theta(x) - \theta(y) = d_\theta(x \vee y, x) + d_\theta(x \vee y, y)$, if θ is so that: $\theta(x) + \theta(y) \geq \theta(x \vee y) + \theta(x \wedge y)$ for every x, y.

In the case, for example, of the lattice $\mathscr{E}$, the following functions θ are of special interest ($A_1, \dots, A_r$ being the equivalence classes of R):

$$\theta(R) = \sum_{i=1}^{p} |A_i|^2 \quad (d_\theta = \delta),$$

$\theta(R) = n$-number of R's classes (d_θ = length of a minimal path in $\mathscr{E}$),

$$\theta(R)=\sum_{i=1}^{p}|A_i|\log|A_i| \quad \text{(it is the so-called } \textit{entropy function} \text{ and } d_\theta(R,S)=\theta(R)+\theta(S)-2\theta(R\cap S)).$$

The subject of metrics associated with valuations on posets is studied in: Birkhoff (1967), Boorman and Oliver (1973), Bordes (1976). The subject of valuations and minimal connecting path has been investigated in: Haskins and Gudder (1972) and Monjardet (1976) in the semi-modular case, Arabie and Boorman (1972) in the case of equivalence relations; Comyn and Van Dorpe (1976) and Grimonprez and Van Dorpe (1976) in the latticial and semi-latticial cases; Barthélemy (1978) in the general case. For a recent connexion with information theory cf. Abdi, Barthélemy and Luong (1979).

5.3.2. *Typological aggregation*

As we saw, for a profile $\Pi=(R_1,\ldots,R_v)$ of linear orders, the quantity $\min_{M\in\mathscr{L}}\Delta(M,\Pi)$ (with a suitable normalization) can measure the dispersion of Π. When this dispersion is too important, an alternative to aggregation, combining cluster analysis and median orders, can be proposed. One forms clusters of orders (occuring in Π) with small dispersion together with an aggregate of the elements of each cluster.

The necessity, in some situations, of such a typological aggregation has been clearly seen by Anderson (1973). This subject has been studied by Lemaire (1976, 1977) who uses Diday's Dynamic Cluster Method (cf. Diday (1970)) together with median and Borda linear orders. Other approaches are due to Jacquet–Lagréze (1971, 1972). A review of all these methods can be found in Lemaire (1980).

5.3.3. *Ultradissimilarity*

Constructing a hierarchy on a set Y of objects, from a dissimilarity relation on Y (often induced by a similarity measure) is a current problem in classification theory. Such a dissimilarity relation is not other than a complete preorder on $X=Y\times Y$ and a hierarchy on Y is equivalent to an ultradissimilarity (i.e. a dissimilarity R such that, for $x,y,z\in Y$, $((x,z),(x,y))\in R$ or $((x,z),(y,z))\in R$.

Starting from this idea, Schader (1980) solves the problem

$$\min_{S\in\mathscr{U}} d(S,R),$$

$\mathscr{U}$ being the set on ultradissimilarity on Y, $R\in\mathscr{P}$ and d being the latticial distance on $\mathscr{P}$ (the distance of a minimal path in the covering graph of $\mathscr{P}$ cf. Section 5.3.1). The special case where S admits only two classes, i.e. where the hierarchy induced by S is not other than a partition, is also solved by Schader (1979).

5.4. *A list of open problems*

5.4.1. *Algorithmical problems*

(1) What is the algorithmical classification of a median search? (for the

definitions, see Reingold et al. (1977)). In fact, it is known that the minimum feedback edge set problem is NP hard (Reingold et al. (1977)). Is the same result generely valid for a $\mathcal{M}$-median computation problem?

(2) The computation of a median can be formulated as a zero-one linear program (cf. Section 4.4.2). Define 'cases' where the associated relaxed linear program always gives a zero-one optimal solution.

(3) Improve significantly the values of n for which *all* the medians of a profile can be computed (see Section 4.4).

5.4.2. Combinatorial problems

(4) Evaluate the analogue of the function F (cf. Section 4.3) for other sets of binary relations ($\mathcal{C}$, $\mathcal{T}$, $\mathcal{P}$, $\mathcal{E}$ in the case $v>1,\ldots$).

(5) Can we deduce, from the profile Π, some informations about the number of medians of Π (see also problem 3)?

(6) In particular, find conditions on Π assuring the unicity of the -median? (a trivial condition is that the Condorcet relation associated with Π is in $\mathcal{M}$ and unique).

(7) Evaluate the proportion of profiles admitting a unique median.

(8) Can we get some informations about the dispersion $\min_{M\in\mathcal{M}} D(M,\Pi)$ without the computation of a median?

(9) What about medians in graphs in the general case (only covering graphs of distributive lattices and trees have been studied, cf. Sections 5.1.2 and 5.1.3).

5.4.3. Axiomatic characterizations

Beside the Young–Levenglick theorem (Section 4.1.1), the only axiomatic characterizations of median procedure, known as yet are:

– Axiomatic characterization of median for a tournaments' profile (i.e., the Condorcet procedure (Monjardet, 1978)).

– Axiomatic characterization of the Borda rule as a choice function (Young (1974)) and for profiles of tournaments as a social welfare function (Nitzan and Rubinstein, 1979).

(10) Give an axiomatic characterization of medians in $\mathcal{T}$, $\mathcal{E}$ and $\mathcal{P}$.

5.4.4. Statistical properties

(11) What is the statistical meaning of median fitting for the pairs ($\mathcal{C}$, $\mathcal{P}$) and ($\mathcal{T}$, $\mathcal{E}$)? (Only ($\mathcal{T}$, $\mathcal{L}$) have been studied.)

5.4.5. Extensions to other relations

(12) What can we say concerning the questions here studied (closed or open) in the case of partial orders and semi-orders? And in particular the following:

(13) How is the Pareto principle to be applied in the case of partial orders and semi-orders?

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