Orthogonal Matrices

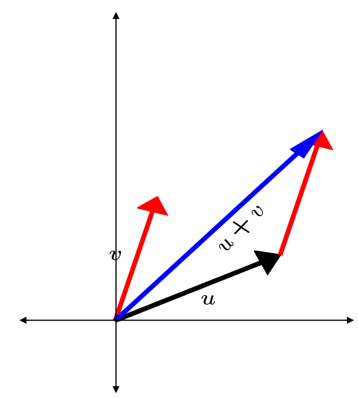
- If u and v are nonzero vectors then

$$u \cdot v = \|u\| \, \|v\| \cos(\theta)$$

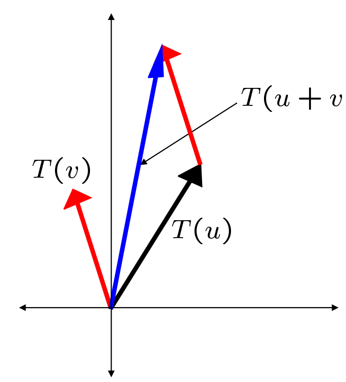
  is 0 if and only if $\cos(\theta) = 0$, i.e., $\theta = 90°$. Hence, we say that two vectors u and v are *perpendicular* or *orthogonal* (in symbols $u \perp v$) if $u \cdot v = 0$.

- A vector u is *a unit vector* if $\|u\| = 1$.

- Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be rotation around the origin by θ degrees. This is a linear transformation. To see this, consider the addition of u and v as in the following diagram.

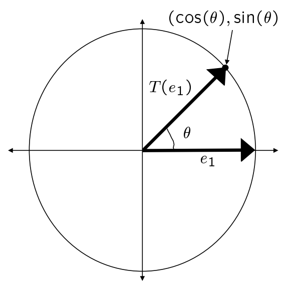


Now rotate the vectors, i.e., apply T.



This diagram shows that $T(u) + T(v) = T(u + v)$. It's even easier to check that $T(sv) = sT(v)$, so T is linear.

We need to find $T(e_1)$, $T(e_2)$. If we rotate e_1 by θ, we get $(\cos(\theta), \sin(\theta))$ by the unit circle definition of sin and cos.



The vector $w = (-\sin(\theta), \cos(\theta))$ is a unit vector and $w \cdot T(e_1) = 0$, so $T(e_2)$ must be either w or $-w$. Considering the sign of the first component shows that $T(e_2) = w$.

Thus, the matrix of T is

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

- Recall the addition laws for sin and cos.

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

  Recall also that

$$\cos(-\theta) = \cos(\theta)$$
$$\sin(-\theta) = -\sin(\theta)$$

- **Exercise:** Rotating by θ and then by φ should be the same as rotating $\theta + \varphi$. Thus, we have

$$R(\theta)R(\varphi) = R(\theta + \varphi) = R(\varphi)R(\theta).$$

  Use this and matrix multiplication to derive the addition laws for sin and cos. What is $R(\theta)^{-1}$?.

- **Exercise:** What is the matrix of reflection through the x-axis?

- A matrix A is *orthogonal* if $\|Av\| = \|v\|$ for all vectors v. If A and B are orthogonal, so is AB. If A is orthogonal, so is A^{-1}. Clearly I is orthogonal.. Rotation matrices are orthogonal.

  The set of orthogonal 2×2 matrices is denoted by $O(2)$.

Classification of Orthogonal Matrices

- If A is orthogonal, then $Au \cdot Av = u \cdot v$ for all vectors u and v. (It follows that A preserves the angles between vectors). To see this, recall that

$$2u \cdot v = \|u\|^2 + \|v^2\| - \|u - v\|^2.$$

Hence, if A is orthogonal, we have

$$\begin{aligned} 2Au \cdot Av &= \|Au\|^2 + \|Av\|^2 - \|Au - Av\|^2 \\ &= \|Au\|^2 + \|Av\|^2 - \|A(u - v)\|^2 \\ &= \|u\|^2 + \|v\|^2 - \|u - v\|^2 \\ &= 2u \cdot v \end{aligned}$$

so $Au \cdot Av = u \cdot v$.

- If A is orthogonal, we must have

$$\begin{aligned} &\|Ae_1\|^2 = Ae_1 \cdot Ae_1 = e_1 \cdot e_1 = 1 \\ (*) \qquad &\|Ae_2\|^2 = Ae_1 \cdot Ae_2 = e_2 \cdot e_2 = 1 \\ &Ae_1 \cdot Ae_2 = e_1 \cdot e_2 = 0. \end{aligned}$$

- **Exercise:** If A satisfies $(*)$, A is orthogonal.

- Equations $(*)$ say that the columns of A are orthogonal unit vectors. Thus, if the first column of A is

$$\begin{bmatrix} a \\ b \end{bmatrix}, \qquad a^2 + b^2 = 1$$

the possibilities for the second column are

$$\begin{bmatrix} -b \\ a \end{bmatrix}, \quad \begin{bmatrix} b \\ -a \end{bmatrix}$$

So A must look like one of the matrices

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$

- Since $a^2 + b^2 = 1$, we can find θ so that $a = \cos(\theta)$ and $b = \sin(\theta)$. Thus, in one case, we get

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta),$$

so A is rotation by θ degrees. In the other case we have

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

Call this matrix $S(\theta)$. What does this represent geometrically?

- We claim that there is a line L through the origin so that $A = S(\theta)$ leaves every point on L fixed. To see this, it will suffice to find a unit vector

$$u = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix}$$

so that $Au = u$. If we do this, any other vector along L will have the form tu for some scalar t, and we will have $A(tu) = tAu = tu$. Since we only need to consider each line through the origin once, we can suppose $0 \leq \varphi < 180°$. We may as well suppose that $0 \leq \theta < 360°$. Thus, we are trying to find φ so that $Au = u$, i.e.

$$\begin{bmatrix} \cos\theta & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix} = \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix}$$

If we carry out the multiplication, this becomes

$$\begin{bmatrix}\cos(\theta)\cos(\varphi)+\sin(\theta)\sin(\varphi)\\ \sin(\theta)\cos(\varphi)-\cos(\theta)\sin(\varphi)\end{bmatrix}=\begin{bmatrix}\cos(\varphi)\\ \sin(\varphi)\end{bmatrix}$$

Using the addition laws, this is the same as

$$\begin{bmatrix}\cos(\theta-\varphi)\\ \sin(\theta-\varphi)\end{bmatrix}=\begin{bmatrix}\cos(\varphi)\\ \sin(\varphi)\end{bmatrix}$$

Because of our angle restrictions, we must have $\theta-\varphi=\varphi$, so $\varphi=\theta/2$. Thus, $Au=u$ for

$$u=\begin{bmatrix}\cos(\theta/2)\\ \sin(\theta/2).\end{bmatrix}$$

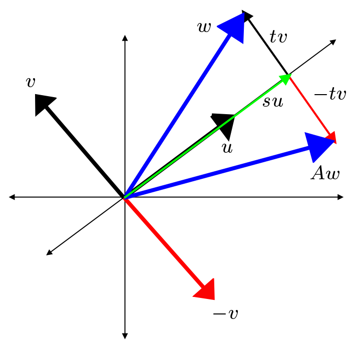
Consider the vector

$$v=\begin{bmatrix}-\sin(\theta/2)\\ \cos(\theta/2)\end{bmatrix},$$

which is a unit vector orthogonal to u. Calculate Av as follows

$$\begin{aligned}Av&=\begin{bmatrix}\cos\theta & \sin(\theta)\\ \sin(\theta) & -\cos(\theta)\end{bmatrix}\begin{bmatrix}-\sin(\theta/2)\\ \cos(\theta/2)\end{bmatrix}\\ &=\begin{bmatrix}-\cos(\theta)\sin(\theta/2)+\sin(\theta)\cos(\theta/2)\\ -\sin(\theta)\sin(\theta/2)-\cos(\theta)\cos(\theta/2)\end{bmatrix}\\ &=\begin{bmatrix}\sin(\theta-\theta/2)\\ -\cos(\theta-\theta/2)\end{bmatrix}\\ &=\begin{bmatrix}\sin(\theta/2)\\ -\cos(\theta/2)\end{bmatrix}\\ &=-v,\end{aligned}$$

So $Av=-v$. A little thought shows that A is reflection through the line L that passes through the origin and is parallel to u. Any vector w can be written as $w=su+tv$ for scalars s and t and $A(su+tv)=sAu+tAv=su-tv$.



- **Exercise:** Verify the following equations.

$$
\begin{aligned}
R(\theta)S(\varphi) &= S(\theta+\varphi) \\
S(\varphi)R(\theta) &= S(\varphi-\theta) \\
S(\theta)S(\varphi) &= R(\theta-\varphi).
\end{aligned}
$$

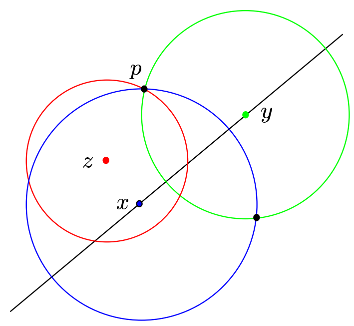
Isometries of the Plane

- If a point has coordinates (x, y), the vector with components (x, y) is the vector from the origin to the point. This is called the position vector of the point. Generally, we use the same notation for the point and its position vector.
- The distance between two points p and q in $\mathbb{R}^2$ is given by $d(p, q) = \|p - q\|$.
- A transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called an *isometry* if it is 1-1 and onto and $d(T(p), T(q)) = d(p, q)$ for all points p and q.
- One kind of isometry we have discovered is $T(p) = Ap$, where A is an orthogonal matrix.
- Another kind of isometry is translation. Let v be a fixed vector and define T_v by

$T_v(p) = p + v$. Translations are *not* linear.

- The identity mapping $\mathrm{id}\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathrm{id}(p) = p$ is obviously an isometry. The composition of two isometries is an isometry. The inverse of an isometry is an isometry.

- Let x, y and z be noncollinear points. Then a point p is uniquely determined by the three numbers $d(x,p)$, $d(y,p)$ and $d(z,p)$.



- If x, y and z are three noncollinear points and T is an isometry such that $T(x) = x$, $T(y) = y$ and $T(z) = z$, then $T = \mathrm{id}$.

  To see this, suppose that p is any point. Then p is determined by the numbers $d(x,p)$, $d(y,p)$ and $d(z,p)$. Since T is an isometry $d(x,p) = d(T(x),T(p)) = d(x,T(p))$. Similarly, $d(y,T(p)) = d(y,p)$ and $d(z,T(p)) = d(z,p)$. Thus, we must have $T(p) = p$.

- **Problem:** If points x,y and z form a right triangle with the right angle at x, then $T(x)$, $T(y)$ and $T(z)$ form a right triangle with the right angle at $T(x)$.

- **Theorem** If T is an isometry, there is a vector v and an orthogonal matrix A so that $T(p) = Ap + v$ for all p, i.e., T is the composition of a rotation or reflection at the origin and a translation.

  Proof: The points 0, e_1 and e_2 are noncollinear and form a right triangle, so the points $T(0)$, $T(e_1)$ and $T(e_2)$ form a right triangle.

  Let $v = -T(0)$. Let T_v be translation by v. Then $T' = T_v T$ is an isometry and $T'(0) = 0$. We must have $d(0, T'(e_1)) = d(0, e_1) = 1$, so we can rotate $T'(e_1)$ around the origin so it coincides with e_1. Let R be the rotation matrix for this rotation. Then $T'' = RT_v T$ is an isometry with $T''(0) = 0$ and $T''(e_1) = e_1$. Since $d(T''(e_2), 0) = 1$ and the points $0, e_1$ and $T''(e_2)$ form a right triangle, $T''(e_2)$ must be either e_2 or $-e_2$. If it's e_2 let $M = I$ and if it's $-e_2$ let M be the matrix of reflection through the x axis. In either case M is an orthogonal matrix and $MT''(e2) = e_2$. Let $T''' = MRT_v T$. Then $T'''(0) = 0$, $T'''(e_1) = e_1$ and $T'''(e_2) = e_2$, so T''' must be the identity. If we set $A = MR$, A is orthogonal and we have $T''' = AT_v T = \mathrm{id}$.

  Thus, for any point point p, $AT_v T(p) = p$. Multiplying by A^{-1} we get $T_v T(p) = A^{-1}p$. Apply T_{-v} to both sides to get $T(p) = T_{-v}A^{-1}p = A^{-1}p - v$. Since A^{-1} is orthogonal, the proof is complete.

- Every isometry can be written in the form $T(x) = Ax + u$. To abbreviate this, we'll simply denote this isometry by $(A \mid u)$, so the rule is

  $$(A \mid u)(x) = Ax + u$$

  In this notation, $(I \mid 0) = \mathrm{id}$, $(I \mid u)$ is translation by u, and $(A \mid 0)$ is transformation $x \mapsto Ax$.

Given isometries $(A \mid u)$ and $(B \mid v)$, we have

$$\begin{aligned}(A \mid u)(B \mid v)(x) &= (A \mid u)(Bx + v)\\ &= A(Bx + v) + u\\ &= ABx + Av + u\\ &= (AB \mid u + Av)(x),\end{aligned}$$

so the rule for combining these symbols is

$$(A \mid u)(B \mid v) = (AB \mid u + Av).$$

- **Exercise:** Verify that $(A \mid u)^{-1} = (A^{-1} \mid -A^{-1}u)$
- A *Glide Reflection* is reflection in some line followed by translation in a direction parallel to the line.
- **Theorem** Every isometry falls into one of the following mutually exclusive cases.

  I. The identity.

  II. A translation.

  III. Rotation about some point (i.e., not necessarily the origin).

  IV. Reflection about some line.

  V. A glide reflection.

- We already know that $(I \mid 0)$ is the identity (Case I), $(0 \mid u)$ is translation (Case II) and that $(A \mid 0)$ is either a rotation about the origin or a reflection about a line through the origin.
- Consider $(A \mid w)$ where A is a rotation matrix, $A = R(\theta)$. A Rotation leaves fixed the rotation center, so we expect $(A \mid w)$ to have a fixed point, i.e., a point p so that $Ap + w = p$. Rearranging this equation gives $w = p - Ap = (I - A)p$, so p is the

solution, if any, of the equation

$$(*) \qquad w = (I - A)p.$$

The matrix $I - A$ is given by

$$I - A = \begin{bmatrix} 1 - \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & 1 - \cos(\theta) \end{bmatrix}.$$

so

$$\begin{aligned} \det(A) &= (1 - \cos(\theta))^2 + \sin^2(\theta) \\ &= 1 - 2\cos\theta + \cos^2(\theta) + \sin^2(\theta) \\ &= 2 - 2\cos\theta. \end{aligned}$$

This is nonzero as long as $\cos(\theta) \neq 1$. If $\cos(\theta) = 1$, then $A = I$, and we've already dealt with that case. So, we can assume $\det(I - A) \neq 0$.

This means that $I - A$ is invertible, so equation $(*)$ has the unique solution $p = (I - A)^{-1}w$.

Now that we know the fixed point p exists, we can write $(A \mid w) = (A \mid p - Ap)$. Then we have

$$(A \mid p - Ap)(x) = Ax + p - Ap = A(x - p) + p.$$

If we think about this geometrically, this shows that $(A \mid p - Ap)$ is rotation about the point p: We take the vector $x - p$ that points from p to x, take the arrow at the origin, rotate the arrow by A and move the arrow back to p. This gives us the rotation of x about p.

- Consider the case $(A \mid w)$ where A is a reflection matrix. Recall that for a reflection matrix, there are orthogonal unit vectors u and v so that $Au = u$ and $Av = -v$. Consider first the case where $w = sv$. Let $p = sv/2$. We claim that p is fixed.

$$\begin{aligned}(A \mid sv)(p) &= Ap + sv \\ &= A(sv/2) + sv \\ &= (s/2)Av + sv \\ &= (s/2)(-v) + sv \\ &= (s - s/2)v \\ &= sv/2 \\ &= p.\end{aligned}$$

Thus, $sv = 2p$ and we can write $(A \mid w) = (A \mid 2p)$. Note $Ap = -p$.

As t runs over all real numbers, the point $p + tu$ run along the line, call it L, that passes through p and is parallel to u. For points on L, we have

$$\begin{aligned}(A \mid 2p)(p + tu) &= A(p + tu) + 2p \\ &= Ap + tu + 2p \\ &= -p + tu + 2p \\ &= p + tu,\end{aligned}$$

so all the points on L are fixed. We can also write

$$\begin{aligned}(A \mid 2p)(x) &= Ax + 2p \\ &= Ax + p + p \\ &= Ax - (-p) + p \\ &= Ax - Ap + p \\ &= A(x - p) + p.\end{aligned}$$

By the same reasoning as before, this shows that $(A \mid 2p)$ is reflection through L.

- **Exercise:** Show that $(A \mid w)$ has fixed points if and only if $w = sv$.
- Consider finally the case $(A \mid w)$ where w is not a multiple of v. Then we can write $w = \alpha u + \beta v$. We then have

$$\begin{aligned}(A \mid w) &= (A \mid \alpha u + \beta v) \\ &= (I \mid \alpha u)(A \mid \beta v)\end{aligned}$$

We know that $(A \mid \beta v)$ is reflection about some line parallel to u and $(I \mid \alpha u)$ is translation in a direction parallel to u, so $(A \mid w)$ is a glide reflection. A glide reflection has no fixed points.