REPRESENTATION THEORY OF THE SYMMETRIC GROUP

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1. Introduction

Representation Theory is a vast subject area with connections in Algebra, Geometry and Topology. In this course we mostly discuss the Algebraic aspects of Representation Theory by looking at finite groups. After creating a general theory for finite groups, we will create specific ideas for the Symmetric Group.

Definition: the set G is a *group*, if there is a binary operation from $G \times G \to G$; $((g,h) \mapsto g \circ h = gh)$, which satisfies the following axioms.

Associativity: $(g \circ h) \circ f = g \circ (h \circ f)$ for all $g, h, f \in G$.
Unit: there exists an element $e \in G$ such that $e \circ g = g \circ e = g$ for all $g \in G$.
Inverse: for all $g \in G$ there exists $g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$.

Example 1.1 (Basic Example). An example of a group is the *symmetric group* S_n where $n = 1, 2, \ldots$. This is the group of all permutations of $1, 2, \ldots, n$. For any $g \in \mathrm{S}_n$ ($g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$) we can describe g in two main ways. The first way we describe is a longer notation where we write down the image of each element of $1, 2, \ldots, n$

$$g = \begin{pmatrix} 1 & 2 & \ldots & n \\ g(1) & g(2) & \ldots & g(n) \end{pmatrix}.$$

The second way is a much shorter form of notation and is known as "cycle notation"

$$\underbrace{\big(1, g(1), g^2(1), \ldots, g^{k-1}(1)\big)}_{\text{a cycle}}\big(p, g(p), \ldots\big).$$

For this notation we note that $g^k(1) = 1$ and $g \neq g^\ell(1)$ for all ℓ.

Note. We will "read" the cycles from left to right.

Definition: Also, G is *commutative* if $g \circ h = h \circ g$ for all $g, h \in G$.

Claim. S_n is commutative if and only if $n = 1, 2$. $|\mathrm{S}_n| = n!$ (Proving these is an exercise for yourselves).

Definition: Choose a field, say $\mathbb{C}$ - the complex field. Take any vector space over $\mathbb{C}$, say V with $\dim V = 0, 1, \ldots, \infty$ (we note that if $\dim V = 0$ then $V = \{0\}$). Let $G = \mathrm{GL}_n(V)$, then G is the *general linear group* consisting of all invertible linear maps $g : V \to V$. G has composition of functions as its group operation, so for all $g, h \in G$ and $v \in V$ we have

$$g \circ h(v) = g\big(h(v)\big).$$

In G the unit element is the identity map $e : v \mapsto v$ for all $v \in V$. Also for all $g \in G$ we have that g^{-1} is the inverse transformation of the function.

Exercise: $\mathrm{GL}(V)$ is commutative only if $\dim V = 1, 0$ $(\mathrm{GL}\big(\{0\}\big) = \{e\})$.

Definition: A (linear) *representation* of a group G is a homomorphism $\rho : G \to \mathrm{GL}(V)$ for some vector space V. The representation is noted by writing (ρ, V).

The "homomorphism property" means that for all $g, h \in G$ we have

$$\mathrm{GL}(V) \ni \rho(g \circ h) = \rho(g)\rho(h)$$

Remark (Some Historical Remarks). George Frobenius is a German mathematician who spent most of his career in Berlin and started most of Representation Theory. His student Issai Schur ($\sim$ 1900) carried on his work. However, the most interesting work on the symmetric group was done by an English mathematician named Alfred Young (1901-1936). He wrote ten main papers on the subject and we will try and cover as much of this as possible.

Example 1.2. Let $G = (\mathbb{R}, +)$, so our group law is $t \circ s = t + s$ for all $t, s \in \mathbb{R}$. Now, let us look for homomorphisms $\rho : G \to \mathrm{GL}(\mathbb{R}^1)$ but we shall impose the extra restriction that for all $\rho(t)$ we have that $\rho'(t)$ exists. Now, for $\mathbb{R}$ we have the General Linear Group to be

$$\mathrm{GL}(\mathbb{R}^1) = \{\mathbb{R} \to \mathbb{R} : x \mapsto ax \mid a \in \mathbb{R}, a \neq 0\}.$$

Let us first exam the "homomorphism property" of ρ in this example. For all $t, s \in \mathbb{R} = G$ then

$$\rho(t + s) = \underbrace{\rho(t)}_{=a} \cdot \underbrace{\rho(s)}_{=b}$$

where $a, b \in \mathbb{R} \setminus \{0\}$. Now, take the derivative at $s = 0$. So,

$$\rho'(t) = \rho(t) \cdot \underbrace{\rho'(0)}_{=c}$$

which has a solution $\rho(t) = Ae^{ct}$ with $A \in \mathbb{R}$. By the general property of homomorphisms we know $\rho(0) = \mathrm{id} \in \mathrm{GL}(V)$ and thus

$$\rho(0) = A \cdot 1 = 1 \Rightarrow A = 1.$$

We also note that ρ satisfies the homomorphism property for all c because $e^{c(t+s)} = e^{ct} \cdot e^{cs}$.

Problem: describe all $\rho : \mathbb{R} \to \mathrm{GL}(\mathbb{R}^d)$, for $d = 1, 2, 3, \dots$ with $\rho(t+s) = \rho(t)\rho(s)$ such that $\rho'(t)$ exists.

2. Matrix Representations

Suppose $\dim V = n$ (i.e. finite dimensional), in $\mathrm{GL}(V)$. Now we can choose a basis in V, say $\{v_1, v_2, \dots, v_n\}$ and hence for all linear transformations $A : V \to V$ we can determine the matrix $M(A) = [a_{ij}]_{ij=1}^n$ by the standard rule.

$$V \ni Av_j = \sum_{i=1}^n a_{ij} v_i.$$

Then for all $A, B : V \to V$ we have $M(A \circ B) = M(A) \cdot M(B)$. We note that the group law for $\mathrm{GL}(V)$ was the composition of functions, i.e. $(A, B) \mapsto A \circ B$. A matrix representation is a map $\tilde{\rho} : G \to \mathrm{GL}_n(\mathbb{C})$ such that $\tilde{\rho}(gh) = \tilde{\rho}(g)\tilde{\rho}(h)$. Having ρ we can construct immediately $\tilde{\rho}$ by

$$\tilde{\rho}(g) = M\big(\rho(g)\big)$$

Example 2.1 (Continued from Example 1.4). Last time we examined the representations $\rho : \mathbb{R} \to \mathrm{GL}(\mathbb{R}^1) \cong \{a \in \mathbb{R} \mid a \neq 0\}$ and $A \mapsto a$ for all $A \in \mathrm{GL}(\mathbb{R}^1)$ with $At = at$ for all $t \in \mathbb{R}^1$.

2.1. Direct Sums of Representations

Let G be any group and (ρ, V), (σ, V) be two representations and thus

$$\rho : G \to \mathrm{GL}(V)$$
$$\sigma : G \to \mathrm{GL}(V)$$

are homomorphisms. Define a new representation $(\rho \oplus \sigma, V \oplus V)$ where $\rho \oplus \sigma$ is the homomorphism

$$\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus V).$$

We define the direct sum of two vector spaces V, U to be the set

$$V \oplus U = \big\{(v, u) \mid v \in V, u \in U\big\}$$

and then the mapping $\rho \oplus \sigma$ is defined as

$$\underbrace{(\rho \oplus \sigma)(v, u)}_{\in \mathrm{GL}(V \oplus U)} = \big(\rho(g)v, \sigma(g)u\big) \in V \oplus U.$$

However, we still need to check that $(\rho \oplus \sigma)(g)$ is an invertible transformation of $V \oplus U$.

Exercise: Check the homomorphism property, i.e. for all $g, h \in G$

$$(\rho \oplus \sigma)(gh) = \big[(\rho \oplus \sigma)(g)\big] \cdot \big[(\rho \oplus \sigma)(h)\big].$$

Suppose we have chosen a basis $\{v_1, \ldots, v_n\}$ in V and a basis $\{u_1, u_2, \ldots, u_m\}$ in U.

$$\mathrm{GL}(V) \ni A \mapsto M(A) = [a_{ij}]_{i,j=1}^{n}$$
$$\mathrm{GL}(V) \ni B \mapsto M(B) = [b_{kl}]_{k,l=1}^{m}.$$

Now condisder the basis $\{v_1, \ldots, v_n, u_1, \ldots, u_m\}$ in $V \oplus U$. with $A \oplus B : V \oplus U \to V \oplus U$ and $(A \oplus B)(v, u) = (Av, Bu)$. Then we have that

$$M(A \oplus B) = \begin{pmatrix} M(A) & 0 \\ 0 & M(B) \end{pmatrix}.$$

Check this as an exercise if required. If $A = \rho(g)$, $B = \sigma(g)$. Then

$$\widetilde{\rho \oplus \sigma}(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \sigma(g) \end{pmatrix}.$$

Question: Suppose we want to describe all representations of a given group G. How can we do this?

2.2. **Irreducible Representations**

Definition: Take any representation (ρ, V) of some group G. Let U be any vector subspace of V, i.e. $U \subseteq V$. Then U is called ρ-*invariant* (or just "invariant") if for all $u \in U$, $g \in G$ we have $\rho(g)u \in U$

Example 2.2. $U = \{0\}$ then $\rho(g) \cdot 0 = 0 \in U$ and hence $\{0\}$ is invariant for all representations ρ.

Example 2.3. $U = V$ then $\rho(g)u \in V$ and hence V is invariant for all representations ρ.

Definition: (ρ, V) is an *irreducible* representation, if the only ρ-invariant subspaces are $\{0\}$ and V itself.

Example 2.4. $\dim V = 1$. Then the only subspaces of V are $\{0\}$ and V. Then ρ is irreducible.

Definition: (ρ, V) is called *reducible*, if there is a ρ-invariant subspace $U \subset V$ such that $U \neq \{0\}, V$.

Definition: (ρ, V) is called *decomposible* if $V = U \oplus U'$ where U, U' are ρ-invariant subspaces of V such that $U, U' \neq \{0\}, V$. In this case

$$\sigma(g) = \rho(g)|_U \qquad \sigma(g)u = \rho(g)u \text{ for all } u \in U$$
$$\sigma'(g) = \rho(g)|_V \qquad \sigma'(g)u' = \rho(g)u' \text{ for all } u' \in U'.$$

Then $\rho = \sigma \oplus \sigma'$ by definition. Check this as an exercise.

A reducible representation is not necessarily decomposible.

Example 2.5. Let $G = \mathbb{R}$ and define the representations

$$\begin{aligned}\rho &: \mathbb{R} \to \mathrm{GL}(\mathbb{R}^2)\\ \tilde{\rho} &: \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})\\ t &\mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\end{aligned}$$

where we have

$$\tilde{\rho}(t)\tilde{\rho}(s) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \tilde{\rho}(t+s) = \begin{pmatrix} 1 & t+s \\ 0 & 1 \end{pmatrix}$$

Claim. this ρ is reducible but not decomposible!

Theorem 2.1 (Maschke). *Let G be any finite group and $\rho : G \to G$ be any representation of G, where V is a vector space over $\mathbb{C}$. Then any ρ-invariant subspace $U \subseteq V$ has a ρ-invariant complement $U' \subseteq V$ such that $V = U \oplus U'$.*

Proof. Consider any "projection" $P : V \to V$ (a linear operator), such that $Pu = u$ for all $u \in U$ and $P(V) = U$. Define $Q : V \to V$ such that

$$Qv = \frac{1}{|G|}\sum_{g\in G} \rho(g)P\rho(g)^{-1}v.$$

We claim that Q is another projection onto U. To show this we need to show it satisfies the above criteria.

- Does $Qu = u$? i.e.

  $$Qu = \frac{1}{|G|}\sum_{g\in G} \rho(g)P\rho(g)^{-1}u = u.$$

  We note that $u \in U \Rightarrow \rho(g)u \in U$ because U is ρ-invariant. Also, by the same logic $\rho(g)^{-1}u = \rho(g^{-1})u \in U$. This means that Qu becomes

  $$\begin{aligned}Qu &= \frac{1}{|G|}\sum_{g\in G} \rho(g)P\rho(g)^{-1}u \\ &= \frac{1}{|G|}\sum_{g\in G} \rho(g)\rho(g)^{-1}u && \text{because } P \text{ a projection} \\ &= \frac{1}{|G|}|G|u && \text{because } \rho(g)\rho(g)^{-1} = 1 \\ &= u.\end{aligned}$$

- Now we need to check that $Q(V) = U$ but what does this mean? Well we have
$$\begin{aligned} Q(V) &= \frac{1}{|G|}\sum_{g\in G}\rho(g)P\rho(g)^{-1}(V) \\ &= \frac{1}{|G|}\sum_{g\in G}\rho(g)P(V) && \big(\rho(g)^{-1}(V)=V\big) \\ &= \frac{1}{|G|}\sum_{g\in G}\rho(g)(U) && \big(P(V)=U\big) \\ &\subseteq U && \text{because } U \text{ is } \rho\text{-invariant.} \end{aligned}$$
Thus we have that $Q(V) \subseteq U$. We can see easily that $U \subseteq Q(V)$ because $Qu = u$ for all $u \in U$. Hence $Q(V) = U$.

We now come to the main property of Q. For all $h \in G$ we have that $\rho(h)Q = Q\rho(h)$. We can see this as
$$\begin{aligned} \rho(h)Q &= \rho(h)\frac{1}{|G|}\sum_{g\in G}\rho(g)P\rho(g)^{-1} \\ &= \frac{1}{|G|}\sum_{g\in G}\rho(hg)P\rho(g)^{-1} \\ &= \frac{1}{|G|}\sum_{g'\in G}\rho(g')P\rho(h^{-1}g')^{-1} && \text{by letting } g'=hg \Rightarrow g=h^{-1}g' \\ &= \frac{1}{|G|}\sum_{g'\in G}\rho(g')P\rho(g')^{-1}\rho(h) && \text{by } \rho(h^{-1}g')^{-1} = [\rho(h^{-1})\rho(g')]^{-1} \\ &= Q\rho(h). \end{aligned}$$
Now define $U' = (1-Q)(V)$, we note that $U' \subseteq V$. Need to check $V = U \oplus U'$. In particular we need $U \cap U' = \{0\}$. Suppose $u \in U$ and $u \in U'$ then because Q is a projection onto U we have that
$$Qu = u. \tag{1}$$
We note that $1-Q$ is also a projection onto U'. We can check this by taking any element of U', say $(1-Q)v$ and applying $1-Q$. This gives us
$$(1-Q)\big[(1-Q)v\big] = (1-2Q+Q^2)v = (1-Q)v.$$
The last equality holds because $Q^2 = Q$ and hence $1-Q$ is a projection onto U'. Due to this and because $u \in U'$ we have that
$$(1-Q)u = u. \tag{2}$$

Taking the sum of (1) and (2) gives $u = 2u \Rightarrow u = 0$. Thus $U \cap U' = \{0\}$. Now we need to show that $V = U + U'$, i.e. that for all $v \in V$ we have $v = u + u'$ for some $u \in U$, $u' \in U'$. However we can break v down as

$$v = Qv + v - Qv = \underset{\in U}{Qv} + \underset{\in U'}{(1-Q)v} \in U + U'.$$

The final thing to check is that U' is ρ-invariant, i.e. that for all $u' \in U'$ we have $\rho(h)u' \in U'$. Writing $u' = (1-Q)v$ for some $v \in V$ we have, by definition,

$$\rho(h)(1-Q)v = (1-Q)\underbrace{\rho(h)v}_{=v'} = (1-Q)v' \in U'. \qquad \square$$

Definition: suppose (ρ, V) and (σ, U) are two representations of the same group G. Take any linear transformation $\varphi : U \to V$. It is called (ρ, σ)-*equivariant* (or (ρ, σ) - intertwining) if for all $g \in G$ $\rho(g)\varphi = \varphi\sigma(g)$

Theorem 2.2 (Schur's Lemma)**.** *Suppose that (ρ, V) and (σ, U) are irreducible representations of G over $\mathbb{C}$ and that $\varphi : U \to V$ is an intertwiner (as defined). Then φ is either invertible or is the zero map. Moreover if $(\rho, V) = (\sigma, U)$ then $\varphi = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$.*

Proof. Consider $\operatorname{Ker}\varphi = \{u \in U \mid \varphi u = 0\} \subseteq U$. We claim that $\operatorname{Ker}\varphi$ is σ-invariant. Let $k \in \operatorname{Ker}\varphi$ then

$$\varphi\big(\sigma(g)k\big) = \rho(g)(\varphi k) = \rho(g)0 = 0$$

which implies $\sigma(g)k \in \operatorname{Ker}\varphi$ and hence $\operatorname{Ker}\varphi$ is σ-invariant. However, U is an irreducible representation and hence $\operatorname{Ker}\varphi = \{0\}$ or $\operatorname{Ker}\varphi = U$, which implies $\operatorname{Ker}\varphi$ is the zero map or is 1 to 1.

Now, consider $\operatorname{Im}\varphi = \{v \in V \mid v = \varphi u \text{ for some } u \in U\} \subseteq V$. We claim that $\operatorname{Im}\varphi$ is ρ-invariant, i.e. for all $v \in \operatorname{Im}\varphi$ we have $\rho(g)v \in \operatorname{Im}\varphi$ (Exercise: check this). Again due to V being irreducible either $\operatorname{Im}\varphi = \{0\}$ or $\operatorname{Im}\varphi = V$, which implies that either φ is the zero map or onto. Hence φ is either invertible or the zero map.

Now suppose $(\rho, V) = (\sigma, U)$ then $\rho(g)\varphi = \varphi\rho(g)$, because φ is an intertwiner, for all $g \in G$. Now φ is a linear map $\varphi : V(=U) \to V(=U)$ but suppose $V \neq \{0\}$, then φ has an eigenvector in V. So, there exists $v \in V$, with $v \neq 0$, such that $\varphi v = \lambda v$ for some $\lambda \in \mathbb{C}$.

Consider the transformation $\psi : V \to V$ where $\psi = \varphi - \lambda \cdot I$. Then we have that for all $v \in V$, $v \in \operatorname{Ker}\psi$ because

$$\psi v = (\varphi - \lambda I)v = \lambda v - \lambda v = 0.$$

So, $\operatorname{Ker}\psi \subseteq V$ and $\operatorname{Ker}\psi \neq \{0\}$. We claim that $\operatorname{Ker}\psi$ is ρ-invariant. We know that φ is ρ-invariant and λI is ρ-invariant and also ψ is intertwining because

$$\psi\rho(g) = \varphi\rho(g) - \lambda\rho(g) = \rho(g)\varphi - \rho(g)\lambda = \rho(g)\psi$$

for all $g \in G$. Thus for any $k' \in \operatorname{Ker}\psi$ we have that

$$\psi\rho(g)k' = \rho(g)\psi k' = \rho(g)0 = 0.$$

Thus because V is irreducible we have $\operatorname{Ker}\psi = V$. Equivalently $(\varphi - \lambda I)u = 0 \Rightarrow \varphi u = \lambda u$ for all $u \in V$. □

Definition: Let (ρ, V) and (σ, U) be any two representations of (the same) group G. These two representations are called *equivalent* if there is a linear transformation, $\varphi : U \to V$ such that

(1) φ is (σ, ρ)-equivariant, i.e. $\varphi \circ \sigma(g) = \rho(g) \circ \varphi$ for all $g \in G$.
(2) φ has to be a bijection.

Note. Let us consider the matrix representations corresponding to σ and ρ and that U, V are finite dimensional with $\dim U = m$, $\dim V = n$. We choose a basis $\{u_1, \ldots, u_m\} \in U$ and a basis $\{v_1, \ldots, v_n\} \in V$.

- $\tilde{\sigma}(g)$ is an $m \times m$ matrix of $\sigma(g)$ relative to $\{u_1, \ldots, u_m\}$.
- $\tilde{\rho}(g)$ is an $n \times n$ matrix of $\rho(g)$ relative to $\{v_1, \ldots, v_n\}$.

Let $\tilde{\varphi}$ be the matrix corresponding to $\varphi : U \to V$ with n rows and m coloumns. Then by property (1) above we have

$$\tilde{\varphi}\tilde{\sigma}(g) = \tilde{\rho}(g)\tilde{\varphi}. \tag{3}$$

By the second property above (i.e. bijection) means that the dimensions must be equal, so $m = n$ and $\tilde{\varphi}$ is an invertible $n \times n$ matrix. Then (3) implies

$$\boxed{\tilde{\varphi}\tilde{\sigma}(g)\tilde{\varphi}^{-1} = \tilde{\rho}(g) \quad \text{for all } g \in G.}$$

Essentially that $\tilde{\sigma}(g)$ and $\tilde{\varphi}(g)$ differ by a change of basis.

Our Aim: is to study representations up to equivalence.

Definition: Two elements, say g and $g' \in G$, are called *conjugate* if there exists $h \in G$ such that $g' = hgh^{-1}$.

Definition: The *conjugacy class* of $g \in G$ is $\{hgh^{-1} \mid h \in G\}$

Then G is the union of different conjugacy classes.

Theorem 2.3. *Let G be a* ***finite group*** *then we establish the following relationships between two sets of G*

$$\{\textit{irreducible representations over } \mathbb{C} \textit{ of } G \textit{ up to equivalence}\} \cong \{\textit{conjugacy classes of } G\}$$

Corollary 2.1. *The first set is infinite.*

Proof. This will take a while and will be covered in the next few lectures. □

Note. The first set in Theorem 2.3 is dependent upon the choice of the field.

3. Conjugacy Classes in the Symmetric Group

Take any $g \in G$ such that

$$\begin{pmatrix} 1 & 2 & \dots & n \\ g(1) & g(2) & \dots & g(n) \end{pmatrix}.$$

Suppose $g = (k_1 k_2 \dots k_{r_1})(k_{r_1+1} k_{r_1+2} \dots k_{r_1+r_2} \dots)$ in the cycle notation. Here r_1, $r_2, \dots$ are the cycle lengths and so $r_1 + r_2 + \dots = n$.

Theorem 3.1. *we say $g \sim g'$ are conjugated in S_n if and only if they have the same* cycle type $\lambda = (r_1, r_2, \dots)$. *Note: we establish the convention from now on that the r_i are weakly decreasing.*

Proof. Suppose $i, j \in \{1, 2, \dots, n\}$ and $g(i) = j \Rightarrow h\big(g(i)\big) = h(j)$. Then

$$(hgh^{-1})\big(h(i)\big) = h(j),$$

we set $hgh^{-1} = g'$ and hence g and g' are conjugate. We have $g'\big(h(i)\big) = h(j)$ for all $i, j = 1, \dots, n$. A cycle representation for g' is:

$$g' = \big(h(k_1)h(k_2)\dots h(k_{r_1})\big)\big(h(k_{r_1+1})\dots h(k_{r_1+r_2})\big)\dots$$

We can see that $g \sim g'$ then they have the same cycle structure. Other way round, for g and g' with the same cycle structure we can find h. Then $hgh^{-1} = g'$. $\square$

This implies that irreducible representations of S_n over $\mathbb{C}$ (up to equivalence) are paramatrised by "cycle structures" $r_1 \geqslant r_2 \geqslant \dots$ such that $r_1 + r_2 + \dots = n$ with $r_k \in \mathbb{Z}$, $r_k > 0$ or by the partitions of n (into non-zero integral parts).

Example 3.1. Taking S_n with $n = 3$, then the possible cycle types are

$$\left.\begin{aligned} &3 = 3 \\ &3 = 2 + 1 \\ &3 = 1 + 1 + 1 \end{aligned}\right\} \text{all the partitions of } n = 3.$$

Example 3.2. If we take $n = 4$ we have

$$\left.\begin{aligned} &4 = 4 \\ &4 = 3 + 1 \\ &4 = 2 + 1 + 1 \\ &4 = 2 + 2 \\ &4 = 1 + 1 + 1 + 1 \end{aligned}\right\} \text{all the partitions of } n = 4.$$

We denote the number of partitions of n by $p(n)$ and we can easily write down the first couple of values.

n	1	2	3	4	5	...
$p(n)$	1	2	3	5	7	...

Ramanujan-Hardy-Rademache created a formula for $p(n)$, which is

$$p(n) \approx \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad n \to \infty.$$

We note that

$$\frac{p(n) - \text{RHS}}{n} \to 0 \qquad n \to \infty.$$

Using a computer we can calculate $p(200)$ using the above formula to be

$$p(200) = 3,972,999,029,288.$$

Definition: Let (ρ, V) be any representation of a group G where $\dim V < \infty$. Then the *character* of (ρ, V) is a function $\chi = \chi_\rho : G \to \mathbb{C}$ where $\chi(g) = \text{trace}\left(\tilde{\rho}(g)\right)$.

That is if $\tilde{\rho}$ is any matrix representation corresponding to ρ then $\chi(g) = \text{trace}\left(\tilde{\rho}(g)\right)$.

Remark. If $\tilde{\rho}$ and $\tilde{\tilde{\rho}}$ are two matrix representations corresponding to different bases in V then for all g

$$\tilde{\tilde{\rho}}(g) = C^{-1}\tilde{\rho}(g)C$$

where C is the coordinate change matrix between the two bases. Then

$$\text{trace}\left(\tilde{\tilde{\rho}}(g)\right) = \text{trace}\left(C^{-1}\tilde{\rho}(g)C\right) = \text{trace}(\tilde{\rho}(g)) = \chi(g).$$

Remark. If the representations (ρ, V) and (σ, V) are equivalent then $\chi_\rho(g) = \chi_\sigma(g)$.

Proof. we have a bijection $\varphi : U \to V$ such that $\rho(g)\varphi = \varphi\sigma(g)$ and so $\varphi^{-1}\rho(g)\varphi = \sigma(g)$. In terms of matrices we have $\tilde{\varphi}^{-1}\tilde{\rho}(g)\tilde{\varphi} = \tilde{\sigma}(g)$. Then we have

$$\chi_\sigma(g) = \text{trace}(\tilde{\varphi}^{-1}\tilde{\rho}(g)\tilde{\varphi}) = \text{trace}(\tilde{\rho}(g)) = \chi_\rho(g).$$ □

Definition: any function $\psi : G \to \mathbb{C}$ is called a *central function* (or *class-function*) if for all $g, h \in G$ we have $\psi(gh) = \psi(hg)$.

Proposition. For all (ρ, V) a representation of G, its character χ is a class function.

Proof. For all g, $h \in G$ we have

$$\begin{aligned}\chi_\rho(gh) &= \text{trace}\big(\rho(gh)\big)\\ &= \text{trace}\big(\rho(g)\rho(h)\big)\\ &= \text{trace}\big(\rho(h)\rho(g)\big)\\ &= \text{trace}\big(\rho(hg)\big)\\ &= \chi_\rho(hg)\end{aligned}$$

□

Proposition. let (ρ, V) and (σ, U) be any two representations of G. Take their direct sum $(\rho \oplus \sigma, V \oplus U)$. Then $\chi_{\rho\oplus\sigma}(g) = \chi_\rho(g) + \chi_\sigma(g)$ for all $g \in G$.

Proof. Revision exercise. □

Theorem 3.2. *for any representation (ρ, V) of G, where V is a vectorspace over $\mathbb{C}$ we have $\chi(g^{-1}) = \overline{\chi(g)}$*

4. Invariant inner product on V

Take any inner product $\langle u, v\rangle$ on V with u, $v \in V$ and $\langle u, v\rangle \in \mathbb{C}$. We say an inner product is *ρ-invariant* if for all $h \in G$ we have $\langle u, v\rangle = \langle \rho(h)u, \rho(h)v\rangle$ for all u, $v \in V$.

Question: how to get such an inner product?

Take any inner product $\langle u, v\rangle_\dagger$ on V. Then we define

$$\langle u, v\rangle = \frac{1}{|G|}\sum_{g\in G}\langle \rho(g)u, \rho(g)v\rangle_\dagger.$$

We claim that $\langle u, v\rangle$ is a ρ-invariant inner product. By definition

$$\begin{aligned}\langle \rho(h)u, \rho(h)v\rangle &= \frac{1}{|G|}\sum_{g\in G}\langle \rho(g)\rho(h)u, \rho(g)\rho(h)v\rangle_\dagger\\ &= \frac{1}{|G|}\sum_{g\in G}\langle \rho(gh)u, \rho(gh)v\rangle_\dagger\\ &= \frac{1}{|G|}\sum_{g'\in G}\langle \rho(g')u, \rho(g')v\rangle_\dagger\\ &= \langle u, v\rangle.\end{aligned}$$

We still, however, need to show that $\langle u, v\rangle$ is an inner product. The axioms of which are

(1) $\langle u, v\rangle = \overline{\langle v, u\rangle}$
(2) $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbb{C}$
(3) We also get from the above condition that $\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle$.
(4) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

(5) $\langle u, u\rangle \geqslant 0$
(6) $\langle u, u\rangle = 0$ then $u = 0$.

We need to show that $\langle u, v\rangle_\dagger$ imply similar properties of $\langle u, v\rangle$. It is left as an exercise to check this for (1)-(4) by using only the definition of $\langle u, v\rangle$.

$$\langle u, u\rangle = \frac{1}{|G|}\sum_g \underbrace{\langle \rho(g)u, \rho(g)u\rangle_\dagger}_{\geqslant 0 \text{ by (5) for } \langle\,,\rangle_\dagger} \geqslant 0.$$

Now, suppose $\langle u, u\rangle = 0$. Then for all $g \in G$ we have $\langle \rho(g)u, \rho(g)u\rangle_\dagger = 0 \Rightarrow \rho(g)u = 0$ by axiom (6) for $\langle\,,\rangle_\dagger$ and hence $u = 0$ as required.

Proof. (Theorem 3.2) Take any ρ-invariant inner product $\langle u, v\rangle$ on V, with $\dim V = d$ (we know this exists, as we have just shown how to construct one). Take an orthonormal basis in V, say $\{v_1, \ldots, v_d\}$, with respect to $\langle u, v\rangle$. The orthonormal property being that

$$\langle v_i, v_j\rangle = \delta_{ij}.$$

Let $\tilde{\rho}(g)$ be the matrix of $\rho(g)$ relative to this basis. Define $\tilde{\rho}(g) = A$. We know

$$Av_j = \sum_{i=1}^{d} a_{ij}v_i.$$

We also have that $\langle \rho(g)u, \rho(g)v\rangle = \langle u, v\rangle$, now putting $u = v_i$, $v = v_j$ then

$$\begin{aligned}
\langle \rho(g)v_i, \rho(g)v_j\rangle &= \langle v_i, v_j\rangle = \delta_{ij} \\
\Rightarrow \left\langle \sum_{k=1}^{d} a_{ki}v_k, \sum_{l=1}^{d} a_{lj}v_l \right\rangle &= \delta_{ij} \\
\Rightarrow \sum_{k,l=1}^{d} a_{ki}\overline{a_{lj}}\delta_{kl} &= \delta_{ij} \\
\Rightarrow \sum_{k=1}^{d} a_{ki}\overline{a_{kj}} &= \delta_{ij} \\
\Rightarrow &\boxed{A^t\overline{A} = I}
\end{aligned}$$

That is A is a *unitary* matrix. Now,

$$\begin{aligned}
\chi(g^{-1}) &= \operatorname{trace}\big(\rho(g^{-1})\big)\\
&= \operatorname{trace}\big(\rho^{-1}(g)\big)\\
&= \operatorname{trace}\big(A^{-1}\big)\\
&= \operatorname{trace}\big(\overline{A}^{t}\big)\\
&= \operatorname{trace}\big(\overline{A}\big)\\
&= \overline{\operatorname{trace}(A)}\\
&= \overline{\chi(g)}.
\end{aligned}$$ □

Aim: to prove that $\hat{G}$ is in bijective correspondence with the conjugacy classes of G for any finite group G. Then (ρ, V) is a representation of G and

$$\boxed{\chi(g) = \chi_\rho(g) = \operatorname{trace}\big(\rho(g)\big)}$$

Definition: the group algebra (group ring) of G consists of all finite linear combinations of the group elements with complex coefficients, i.e.

$$x_1 g_1 + \cdots + x_N g_N$$

where $\{g_1, \ldots, g_N\} = G$, $|G| = N$ with $x_1, \ldots, x_N \in \mathbb{C}$. The set, denoted by $\mathbb{C} \cdot G = \mathbb{C}G$, is a vectorspace under addition and scalar multiplication. We can clearly show that this satisfies closure and the other axioms for a vector space can be checked as well.

$$(x_1 g_1 + \cdots + x_n g_n) + (y_1 g_1 + \cdots + y_N g_N) = (x_1 + y_1) g_1 + \cdots + (x_N + y_N) g_N$$
$$\lambda(x_1 g_1 + \cdots + x_N g_N) = \lambda x_1 g_1 + \cdots + \lambda x_N g_N.$$

Defining multiplication to be

$$\left(\sum_{k=1}^{N} x_k g_k\right)\left(\sum_{\ell=1}^{N} y_\ell g_\ell\right) = \sum_{k,\ell=1}^{N} x_k y_\ell g_k g_\ell = \sum_{m=1}^{N}\left(\sum_{k,\ell=1}^{N} x_k y_\ell\right) g_m.$$

we can show that $\mathbb{C}G$ is a ring by checking the axioms.

Remark. $\mathbb{C}G$ can be identified with $\{f : G \to \mathbb{C}\}$ so that the linear combination $x_1 g_1 + \cdots + x_N g_N$ corresponds to the function f such that $f(g_k) = x_k$

Definition: Let $V = \mathbb{C}G$ with $\dim V = |G| = N$, then $\rho(g) : V \to V$ defined by

$$\rho(g)(x_1 g_1 + \cdots + x_N g_N) = x_1 g g_1 + \cdots + x_N g g_N$$

is the *left regular representation of* G.

We need to verify that this definition is in fact a representation of G. We can clearly see that $\rho(g)$ is a linear transformation and has an inverse defined to be $\rho(g)^{-1} = \rho(g^{-1})$, thus $\rho(g) \in \mathrm{GL}(V)$. We show the homomorphism property also holds. For all g, $h \in G$ we have

$$\rho(g)\rho(h)\sum_{k=1}^{N} x_k g_k = \rho(g)\sum_{k=1}^{N} x_k h g_k = \sum_{k=1}^{N} x_k g h g_k = \rho(gh)\sum_{k=1}^{N} x_k g_k.$$

Definition: Let $V' = \mathbb{C}G$, then $\rho'(g) : V' \to V'$ defined by

$$\rho'(g)(x_1 g_1 + \cdots + x_N g_N) = x_1 g_1 g^{-1} + \cdots + x_N g_N g^{-1}.$$

is the *right regular representation of* G.

As above we can easily check that the homomorphism property holds and we can clearly see $\rho'(g) \in GL(V')$.

Definition: any representation (σ, U) of G is *faithful* if

$$\sigma(g) = I \Rightarrow g = e.$$

We claim that the left regular representation of G is faithful. So, we require

$$\begin{aligned} \rho(g)(x_1 g_1 + \cdots + x_N g_N) &= x_1 g_1 + \cdots + x_N g_N \\ \Rightarrow \qquad x_1 g g_1 + \cdots + x_N g g_N &= x_1 g_1 + \cdots + x_N g_N \end{aligned}$$

for all $x_1, \ldots, x_N$. Set $x_1 = 1$, $x_2 = \cdots = x_N = 0$ then $gg_1 = g_1 \Rightarrow g = e$.

Task: to compute the character of the left regular representation of G. A basis in $V = \mathbb{C}G$ is $\{g_1, \ldots, g_N\}$. We have that the left regular representation is faithful by above, so $\chi(e) = |G|$ (because $\tilde{\rho}(e) = I$). Now, applying the left regular representation to a basis element g_k we obtain

$$\rho(g)g_k = \underbrace{gg_k}_{\text{basis vector}}.$$

Letting $\tilde{\rho} = A$ be the matrix representation of ρ then the above equation becomes

$$Ae_k = \sum_{\ell=1}^{N} a_{\ell k} e_\ell.$$

We are only interested in the diagonal elements, i.e. when $\ell = k$. However, if any element $a_{kk} = 1$ then we have $g_k = gg_k$ but this implies that $g = e$. So, if $g \neq e$ then we must have that all $a_{kk} = 0$. Thus the character of the left regular representation is

$$\chi_\rho(g) = \begin{cases} |G| & \text{for } g = e \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 4.1. We define the *trivial representation* of G, denoted as (τ, V), over $V = \mathbb{C}^1$ to be $\tau(g) = 1$. We can see that the homomorphism property holds trivially because

$$\tau(gh) = 1 = 1 \cdot 1 = \tau(g)\tau(h).$$

The matrix representation of τ is the 1×1 identity matrix and so $\chi_\tau(g) = 1$ for all $g \in G$.

Proposition. Let (ρ, V) be any finite-dimensional representations of a finite group G. Let χ be the character of ρ. Consider the direct sum decomposition $V = V_1 \oplus \cdots \oplus V_n$ with $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ where $(\rho_1, V_1), \ldots, (\rho_n, V_n)$ (which we know exists by Maschke's Theorem). Let m denote the multiplicity of the trivial representation in this direct sum. Then we have that

$$m = \frac{1}{|G|} \sum_{g \in G} \chi(g).$$

EXAMPLE 4.2. If ρ is the trivial representation then we have

$$1 = \frac{1}{|G|} \sum_{g \in G} 1 = 1.$$

EXAMPLE 4.3. Let ρ be the left regular representation then

$$m = \frac{1}{|G|} \sum_{g \in G} \chi(g) = \frac{1}{|G|} |G| = 1$$

A quick reminder about Tensor Products

Let U, V be finite-dimensional vector-spaces with $\dim U = m$ and $\dim V = n$. Now let $\{u_1, \ldots, u_m\}$ be any basis of U and $\{v_1, \ldots, v_n\}$ be any basis of V. By definition the tensor product $U \otimes V$ is the vector space of dimension mn with corresponding basis

$$\{u_i \otimes v_j \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}.$$

If we define $u \in U$, $v \in V$ to be

$$u = \sum_{k=1}^{m} a_k u_k \qquad\qquad v = \sum_{\ell=1}^{n} b_\ell v_\ell.$$

Then by definition we have that

$$\begin{aligned} u \otimes v &= \left(\sum_{k=1}^{m} a_k u_k \right) \otimes \left(\sum_{\ell=1}^{n} b_\ell u_\ell \right) \\ &= \sum_{k,l} a_k b_\ell (u_k \otimes v_\ell). \end{aligned}$$

Caution, though, as not every vector of $U \otimes V$ is of the form $u \otimes v$. Indeed, for $U = V = \mathbb{C}^2$ with standard basis $\{e_1, e_2\}$ we have that

$$\left. \begin{matrix} e_1 \otimes e_1 & e_2 \otimes e_2 \\ e_1 \otimes e_2 & e_2 \otimes e_1 \end{matrix} \right\} \text{form a basis in } \mathbb{C}^2 \otimes \mathbb{C}^2.$$

However we have that $e_1 \otimes e_1 + e_2 \otimes e_2 \neq u \otimes v$.

Definition: suppose $A : U \to U$ and $B : V \to V$ are linear transformations. Then $A \otimes B : U \otimes V \to U \otimes V$ and $U \otimes V = \operatorname{span}\{u \otimes v \mid u \in U, v \in V\}$. We define $A \otimes B(u \otimes v) = (Au) \otimes (Bv)$

Proof. (of previous Proposition) Let (ρ, V) be our representation with character χ and consider the linear operator

$$P = \frac{1}{|G|} \sum_{g \in G} \rho(g).$$

We claim that this P is a projection operator on the subspace in V, which consits of ρ-fixed vectors $V^G = \{v \in V \mid \rho(g)v = v \text{ for all } g \in G\}$. So, we check the axioms for a projection.

(1) We need to show that $P(V) = V^G$. It is clear to see that $V^G \subseteq P(V)$, so for equality we only need to show $V^G \subseteq P(V)$. Take any $u \in V$ and consider $Pu \in V$, then for any $h \in G$ we have

$$\begin{aligned} \rho(h) P u &= \rho(h) \frac{1}{|G|} \sum_{g \in G} \rho(g) u \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(hg) u \\ &= \frac{1}{|G|} \sum_{g' \in G} \rho(g') u \\ &= Pu. \end{aligned}$$

Thus $Pu \in V^G$ and hence $P(V) = V^G$.

(2) Secondly we need to show that $Pv = v$ for all $v \in V^G$. This is shown easily as

$$\begin{aligned} Pv &= \frac{1}{|G|}\sum_{g\in G}\rho(g)v \\ &= \frac{1}{|G|}\sum_{g\in G} v \\ &= v. \end{aligned}$$

Let us now compute the trace of P in two different ways, the first being

$$\operatorname{trace}(P) = \frac{1}{|G|}\sum_{g\in G}\operatorname{trace}(\rho(g)) = \frac{1}{|G|}\sum_{g\in G}\chi(g).$$

The second way is to take V^G and choose a basis in this subspace, we note that $\dim V^G = m$ by definition of V^G. One can pick up a basis, say $v_1, \dots, v_m \in V^G$ and complete it to a basis $v_1, \dots, v_m, v_{m+1}, \dots, v_N$ in V so that $Pv_j = 0$, for $j > m$ (this is left as an exercise).

Now, $Pv_j = v_j$ for $j = 1, \dots, m$. Write the matrix of P in this basis of V

$$M = \left(\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \vdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{array}\right)$$

and $\operatorname{trace}(M) = m$ because the top left hand corner of the matrix is $m \times m$. $\square$

Definition: Let (ρ, V) be a representation of G and (σ, V) a representation of H. We have that the direct product of H and G is $H \times G = \{(h, g) \mid g \in G, h \in H\}$. We define the *representation of* $H \times G$ *on* $U \otimes V$ to be $(\sigma \otimes \rho, U \otimes V)$ by the formula

$$\sigma \otimes \rho(h, g) = \sigma(h) \otimes \rho(g).$$

Proposition. $\chi_{\sigma\otimes\rho}(h, g) = \chi_\sigma(h) \cdot \chi_\rho(g)$ for all $h \in H$, $g \in G$.

Proof. Let $\{u_1, \dots, u_m\}$ be a basis in U and $\{v_1, \dots, v_n\}$ a basis in V. Then $\{u_i \otimes v_j \mid i = 1, \dots, m \text{ and } j = 1, \dots, n\}$ is a basis in $U \otimes V$. We apply our representation $\sigma \otimes \rho$ to a generic basis element $u_i \otimes v_j \in U \otimes V$.

$$\begin{aligned}(\sigma \otimes \rho)(h,g)(u_i \otimes v_j) &= \sigma(h)u_i \otimes \rho(g)v_j \\ &= \sum_{k=1}^{m}\sum_{\ell=1}^{n} (\tilde{\sigma}(h)_{ki}u_k) \otimes (\tilde{\rho}(g)_{\ell j}v_\ell) \\ &= \sum_{k,\ell} \tilde{\sigma}(h)_{ki}\tilde{\rho}(g)_{\ell j} u_k \otimes v_\ell.\end{aligned}$$

Then we have

$$\chi_{\sigma\otimes\rho}(h,g) = \operatorname{trace}\big(\sigma \otimes \rho(h,g)\big) = \sum_{i,j} \tilde{\sigma}(h)_{ii}\tilde{\rho}(g)_{jj} = \chi_\sigma(h)\chi_\rho(g).$$ □

Definition:[Very Important] the *canonical* (Hermitian) inner product on $\mathbb{C}[G] = \{\varphi : G \to \mathbb{C}\}$ is

$$\langle \varphi_1, \varphi_2 \rangle = \frac{1}{|G|}\sum_{g\in G} \varphi_1(g)\overline{\varphi_2(g)}$$

Aim: to establish $\hat{G} \leftrightarrow$ the conjugacy classes of G.

Theorem 4.1. *the characters of irreducible representations of G are pairwise orthonormal ($\langle \chi_i, \chi_i \rangle = 1$ and $\langle \chi_i, \chi_j \rangle = 0$ for $i \neq j$) with respect to the $\langle\ ,\ \rangle$.*

Corollary 4.1. *now take any finite dimensional representation (V, ρ) of G such that*

$$(V,\rho) = \big(\underbrace{(V_1,\rho_1) \oplus \cdots \oplus (V_1,\rho_1)}_{m_1}\big) \oplus \big(\underbrace{(V_2,\rho_2) \oplus \cdots \oplus (V_2\rho_2)}_{m_2}\big) \oplus \dots$$

where (V_1, ρ_1), (V_2, ρ_2), etc. are irreducible and pairwise non-equivalent. Let χ, χ_1, χ_2, etc. be the corresponding characters, then we have

$$\chi = m_1\chi_1 + m_2\chi_2 + \dots$$

Consider for all $i = 1, 2, \dots$

$$\langle \chi, \chi_i \rangle = \langle m_1\chi_1 + m_2\chi_2 + \dots, \chi_i \rangle = m_i \underbrace{\langle \chi_i, \chi_i \rangle}_{=1} = m_i.$$

That is χ defines (V, ρ) uniquely up to equivalence.

Corollary 4.2. *take any representation (V, ρ) of G with character χ, then we claim that (V, ρ) is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

Proof. write $\chi = m_1\chi_1 + m_2\chi_2 + \dots$. Then

$$\begin{aligned}\langle \chi, \chi \rangle &= \langle m_1\chi_1 + m_2\chi_2 + \ldots, m_1\chi_1 + m_2\chi_2 + \ldots \rangle \\ &= m_1^2 + m_2^2 + \ldots\end{aligned}$$

Now, if χ is irreducible then $\chi = \chi_i$ and $\langle \chi_i, \chi_i \rangle = 1$. Conversely suppose $\langle \chi, \chi \rangle = 1$ then $m_1^2 + m_2^2 + \cdots = 1 \Rightarrow \exists j$ such that $m_i = 1$ if $i = j$ and $m_i = 0$ otherwise for $m_i \in \mathbb{Z}_{\geqslant 0} \Rightarrow \chi = \chi_j$. ☐

Proof. (Theorem 1). Let us take any two representations (ρ, V) and (σ, U) of G. Let χ_ρ and χ_σ be their characters. Now consider the vector span $\{\text{all linear maps } U \to V\} = W \ni \alpha$. Define a new representation of G on W, (τ, W) by the formula

$$\tau(g)\alpha = \rho(g) \circ \alpha \circ \sigma(g^{-1}).$$

We check this a representation by first checking the homomorphism property.

$$\begin{aligned}\tau(gh)(\alpha) &= \rho(gh) \circ \alpha \circ \sigma(h^{-1}g^{-1}) \\ &= \rho(g)\rho(h) \circ \alpha \circ \sigma(h^{-1})\sigma(g^{-1}) \\ &= \tau(g)\big(\tau(h)(\alpha)\big).\end{aligned}$$

Let us compute the character of this τ. Choose any bases $\{u_1, \ldots, u_m\}$ in U and $\{v_1, \ldots, v_n\}$ in V. Then

$$\rho(g)v_i = \sum_{k=1}^{n} \tilde{\rho}(g)_{ki} v_k \qquad \sigma(g)u_i = \sum_{k=1}^{m} \tilde{\sigma}(g)_{ki} u_k$$

consider linear transformations $w_{ij} : U \to V$ with

$$w_{ij}(u_k) = \begin{cases} v_i & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\{w_{ij} \mid i = 1, \ldots, n \text{ and } j = 1, \ldots, m\}$ form a basis in W.

$$\tau(g)(w_{ij}) = \rho(g) \circ w_{ij} \circ \sigma(g^{-1}) = ?$$

So, for $\sum_{i,j} \alpha_{ij} w_{ij} = \alpha \in W$ then

$$V \ni \alpha(w_k) = \left(\sum_{i,j} \alpha_{ij} w_{ij} \right)(w_k) = \sum_i \alpha_{ik} v_i.$$

So we apply

$$\begin{aligned}\rho(g)\circ w_{ij}\sigma(g^{-1})u_k &= \rho(g)w_{ij}\left(\sum_{\ell=1}^{m}\sigma(g^{-1})_{\ell k}u_\ell\right)\\ &= \rho(g)v_i\sigma(g^{-1})_{jk}\\ &= \sum_{p=1}^{n}\rho(g)_{pi}v_p\sigma(g^{-1})_{jk}.\end{aligned}$$

We need the coefficient $\rho(g)_{pi}\sigma(g^{-1})_{jk}$, with $p=i$, $j=k$. We compute the character

$$\sum_{i,j}\rho(g)_{ii}\sigma(g^{-1})_{jj} = \chi_\rho(g)\chi_\sigma(g^{-1}) = \chi_\rho(g)\overline{\chi_\sigma(g)}.$$

Now recall that ρ, σ are irreducible. Consider $W^G = \{\alpha \in W \mid \tau(g)\alpha = \alpha\}$ we have

$$\begin{aligned}\dim W^G &= \frac{1}{|G|}\sum_{g\in G}\chi_\tau(g) \qquad \text{by previous theorem.}\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)\overline{\chi_\sigma(g)} = \langle\chi_\rho,\chi_\sigma\rangle.\end{aligned}$$

But what is $\alpha \in W^G$? Well $\tau(g)(\alpha) = \alpha$ or $\rho(g)\circ\alpha\circ\sigma(g^{-1}) = \alpha \Leftrightarrow \rho(g)\circ\alpha = \alpha\circ\sigma(g)$. So, α is (ρ,σ)-inter twiner. Then

$$\dim W^G = \begin{cases}0 & \rho\not\sim\sigma\\ 1 & \rho\sim\sigma\end{cases}$$

by Schur Lemma. □

Theorem 4.2. *The characters of irreducible representations of a finite group G constitute a* complete family *of central functions on G with complex variables. For all central functions $\varphi : G \to \mathbb{C}$, i.e. $\varphi(gh) = \varphi(hg)$, we have that*

$$\langle\varphi,\chi\rangle = 0 \Rightarrow \varphi \equiv 0$$

for all χ.

Lemma 4.1. *Let $\varphi : G \to \mathbb{C}$ be central and let (ρ, V) be any representation of G. Consider the operator $A : V \to V$ defined to be*

$$A = \sum_{g\in G}\varphi(g)\rho(g).$$

Then A is a (ρ,ρ)-intertwiner, that is $A\rho(h) = \rho(h)A$ for all $h \in G$.

Proof. We have that

$$\begin{aligned}
A\rho(h) &= \sum_{g\in G} \varphi(g)\rho(g)\rho(h) \\
&= \sum_{g\in G} \varphi(g)\rho(gh) \\
&= \sum_{g'\in G} \varphi(hg'h^{-1})\rho(hg') \\
&= \sum_{g'\in G} \varphi(g')\rho(hg') && \text{because } \varphi \text{ a central function.} \\
&= \sum_{g'\in G} \varphi(g')\rho(h)\rho(g') \\
&= \rho(h)\sum_{g'\in G} \varphi(g')\rho(g') \\
&= \rho(h)A. && \square
\end{aligned}$$

Proof. (Theorem 2). Take any $\varphi : G \to \mathbb{C}$ such that φ is central and $\langle \varphi, \chi\rangle = 0$ for all χ. Then $\langle \chi, \varphi\rangle = \overline{\varphi, \chi\rangle} = 0$. Now, $\overline{\varphi}(g) = \overline{\varphi(g)}$ is also central.

Consider

$$A = \sum_{g} \overline{\varphi(g)}\rho(g) \tag{4}$$

for all ρ, a representation of G. Suppose that ρ is irreducible. Then A is a (ρ, ρ) intertwiner, by the Lemma. But by Schur's Lemma, we must have $A = \lambda I$, with $I : V \to V$ the identity and $\lambda \in \mathbb{C}$.
We have that

$$\begin{aligned}
\lambda &= \frac{1}{\dim V}\,\mathrm{trace}(A) \\
&= \frac{1}{\dim V}\sum_{g\in G} \overline{\varphi(g)}\,\mathrm{trace}\big(\rho(g)\big) \\
&= \frac{1}{\dim V}\sum_{g\in G} \overline{\varphi(g)}\chi(g) \\
&= \frac{|G|}{\dim V}\langle \chi, \varphi\rangle = 0.
\end{aligned}$$

Thus, A defined by (4) is $A = 0$ for all ρ-irreducible.

Take any ρ but then $\rho = \rho_1 \oplus \cdots \oplus \rho_N$ for irreducible $\rho_1, \ldots, \rho_N$ by Maschke's Thereom. So,

$$\tilde{\rho}(g) = \begin{pmatrix} \tilde{\rho_1}(g) & 0 & \cdots & 0 \\ 0 & \tilde{\rho_2}(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \tilde{\rho_N}(g) \end{pmatrix}.$$

We have that

$$A = \sum_{g \in G} \overline{\rho(g)}(\rho_1 \oplus \cdots \oplus \rho_N) = 0.$$

So, we have that $A = 0$ for all ρ, i.e. finite-dimensional representations of $G/$ In p[articular, we can choose ρ to be the left regular representation. But then all $\rho(g)$ are linearly independent in the space of operators $V \to V$. Then,

$$A = \sum \overline{\varphi(g)}\rho(g) = 0 \Rightarrow \overline{\varphi(g)} = 0$$

for all $g \in G$. $\square$

Theorem 4.3. *This is our main thereom. That the set $\hat{G}$ is in one-to-one correspondence with $\{$conjugacy classes of $G\}$.*

Proof. we know that the character of a representation defines the representation up to equivalence. So $\hat{G} = \{$characters of irreducible representations of $G\}$. Consider the vectorspace $\{\varphi : G \to \mathbb{C} \mid \varphi$ a central function$\} = \mathcal{P}$. By thereom 1, orthonormality implies linear independence of irreducible characters.

By thereom 2, the irreducible characters span $\mathcal{P}$. Suppose not. Then

$$\text{span}\{\text{irreducible characters}\} \subset \mathcal{P}.$$

Then there would be span$\{$irreducible characters$\}^{\perp} \ni \mathcal{P}$. So, $\{$irreducible characters$\}$ is a basis for $\mathcal{P}$. So,

$$\#\{\text{irreducible characters}\} = \dim \mathcal{P} = \#\{\text{conjugacy classes}\}.$$ $\square$

In the rest of this course, we will construct such a bijection explicitly for the symmetric group S_n with $n = 1, 2, \ldots$ for which $\{$conjugacy classes$\} = \{$partitions of $n\} = \{\alpha\}$. This will be the classical construction of Alfred Young.

Example 4.4. $\alpha = (n) \leftrightarrow$ the trivial representation of S_n. If $V = \mathbb{C}^1$ then $\rho(g) = 1$ for all $g \in \mathrm{S}_n$.

EXAMPLE 4.5. The next logical partition of n is to split it up into n parts, i.e.

$$\alpha = (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}).$$

The sign representation of S_n. Let $V = \mathbb{C}^1$ then

$$\rho(g) = \operatorname{sign}(g) = \begin{cases} 1 & \text{if } g \text{ is even.} \\ -1 & \text{if } g \text{ is odd.} \end{cases}$$

We know sign is a homomorphism because $\operatorname{sign}(gh) = \operatorname{sign}(g)\operatorname{sign}(h)$.

EXAMPLE 4.6. The case of S_3. We have that $|\,\mathrm{S}_3\,| = 6$

$$\left.\begin{aligned} 3 &= n = 3 \\ 3 &= n = 1+1+1 \\ 3 &= n = 2+1 \end{aligned}\right\} \text{partitions} \;\leftrightarrow\; \text{cycle structures}$$

The number of conjugacy classes of S_3 is 3. So,

$$\begin{aligned} (1,1,1) &\leftrightarrow \text{sign representation} \\ (3) &\leftrightarrow \text{trivial representation} \\ (2,1) &\leftrightarrow \ ? \end{aligned}$$

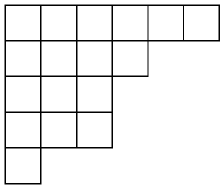
Going back to the first problem on the assignment sheet we have $n = 3$ and

$$V = \{(X_1, X_2, X_3) \mid X_1 + X_2 + X_3 = 0\}$$

with $\rho(g)(X_1, X_2, X_3) = (X_{g^{-1}(1)}, X_{g^{-1}(2)}, X_{g^{-1}(3)})$ is irreducible.

5. Representations of the Group S_n - in detail

Definition: Let λ be a partition of n so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ are the partitions; $\lambda_1 + \lambda_2 + \cdots = n$. Then the *Young diagram* of λ is



where we label the top row λ_1, the second λ_2, etc. (we note the total number of boxes is n). The tableau above is an example of $\lambda = (6, 4, 3, 3, 1)$ and $n = 17$. For example

Definition: a *Young tableau* Λ (of shape λ) is any bijective filling of the boxes of the Young diagram of λ with the numbers $1, 2, \ldots, n$.

Example 5.1. For the case $n = 8$ we could have

$$\begin{ytableau} 5 & 6 & 2 & 7 \\ 3 & 1 & 8 \\ 4 \end{ytableau} \qquad \qquad \begin{ytableau} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 \\ 8 \end{ytableau}$$

In the above cases we have $\lambda = (4, 3, 1)$.

Definition: A Young tableaux Λ is called *standard* if the numbers $1, 2, \ldots, n$ increase along the rows (from left to right) and down the columns.

Remark. For any given shape λ, there are exactly $n!$ tableaux of shape λ. In fact, for any λ we will construct an irreducible representation V_λ of S_n such that

$$\dim V_\lambda = \#\{\text{standard tableaux of shape } \lambda\}$$

and $V_\lambda \not\sim V_\mu$ for different partitions of n.

Example 5.2. Let our Young tableaux have shape $\lambda = (3, 2) \vdash 5$ then we have the standard tableaux to be

$$\begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \end{ytableau} \qquad \qquad \begin{ytableau} 1 & 3 & 4 \\ 2 & 5 \end{ytableau}$$

$$\begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \end{ytableau} \qquad \qquad \begin{ytableau} 1 & 3 & 5 \\ 2 & 4 \end{ytableau}$$

$$\begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \end{ytableau}$$

Fact (without proof): For all tableaux with shape $\lambda \vdash n$ then

$$\#\{\text{standard tableaux of shape } \lambda\} = \frac{n!}{h_1 \ldots h_n},$$

where $h_1, \ldots, h_n$ are the "hooks" of λ. We describe the hooks diagrammatically. In the case $n = 11$ and $\lambda = (5, 4, 2)$ we have the hooks to be

$$\begin{ytableau} 7 & 6 & 4 & 3 & 1 \\ 5 & 4 & 2 & 1 \\ 2 & 1 \end{ytableau} \; .$$

We obtain the numbers in the diagram by adding up the total number of boxes to the left, with the number of boxes underneath and finally add the box we're in. In our smaller example we have

$$\begin{ytableau} 4 & 3 & 1 \\ 2 & 1 \end{ytableau} \; .$$

Thus there are

$$\frac{5!}{4\times 3\times 2\times 1\times 1}=5$$

standard Young tableaux.

Definition: For any Young tableau Λ the *row subgroup* $R_\Lambda \subseteq \mathrm{S}_n$ consists of all permutations of $1,\ldots,n$ which preserve every row of Λ as a set. Similarly the *column subgroup* $C_\Lambda \subseteq \mathrm{S}_n$ consists of all permutations of $1,\ldots,n$ which preserve every column of Λ as a set.

EXAMPLE 5.3. Let $n=5$ and our tableaux be of shape $\lambda=(3,2)$ then we define

$$\Lambda = \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 4&2 \\ \cline{1-2}\end{array} \qquad\qquad \Lambda' = \begin{array}{|c|c|c|}\hline 5&3&2\\ \hline 4&1 \\ \cline{1-2}\end{array}.$$

Thus we have

$$\begin{aligned} R_\Lambda &= (\text{perm of } \{1,3,5\}) \times (\text{perm of } \{2,4\}) \\ C_\Lambda &= (\text{perm of } \{1,4\}) \times (\text{perm of } \{2,3\}) \times (\text{perm of } \{5\}) \end{aligned}$$

Consider the set of all tableaux Λ of a given shape λ (this set has $n!$ elements). The symmetric group S_n acts on this set, by applying permutations to the entries of the boxes. In the above example $\mathrm{S}_5 \ni g:\Lambda\to\Lambda'$ such that

$$g=\begin{pmatrix}1&3&5&4&2\\5&3&2&4&1\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\5&1&3&4&2\end{pmatrix}$$

Proposition. Suppose that $g(\Lambda)=\Lambda'$ for $g\in \mathrm{S}_n$ then $R_{\Lambda'}=gR_\Lambda g^{-1}$ and $C_{\Lambda'}=gC_\Lambda g^{-1}$.

Proof. Consider Λ and say its first row $\{k_1,\ldots,k_\ell\}$ is permuted by R_Λ. We have that $g(\Lambda)=\Lambda'$ and thus the first row of Λ' is $\{g(k_1),\ldots,g(k_\ell)\}$. Given any element $h\in R_\Lambda$ we have that h preserves $\{k_1,\ldots,k_\ell\} \Rightarrow ghg^{-1}$ preserves $\{g(k_1),\ldots,g(k_\ell)\}$. This is because

$$ghg^{-1}\big(g(k_i)\big)=gh(k_i)=g(k_j)\in \text{ first row of } \Lambda'.$$

This works for all rows and for all columns. $\square$

Definition: For any Young tableaux Λ of shape λ we define the following three operators:

$$P_\Lambda=\sum_{g\in R_\Lambda} g\in \mathbb{C}\,\mathrm{S}_n \qquad Q_\Lambda=\sum_{g\in C_\Lambda}(\operatorname{sign} g)g\in\mathbb{C}\,\mathrm{S}_n \qquad Y_\Lambda=P_\Lambda Q_\Lambda\in\mathbb{C}\,\mathrm{S}_n,$$

where P_Λ is the *row symmetrizer*, Q_Λ is the *column antisymmetrizer* and Y_Λ is the *Young symmetrizer*.

Theorem 5.1. *Consider the left regular representation of* S_n*, say* (ρ, V) *with* $V = \mathbb{C}\,\mathrm{S}_n$ *and* ρ *defined by*

$$\rho(g)\left(\sum_{h\in\mathrm{S}_n} a_n h\right) = \sum_h a_h gh.$$

Consider the left ideal $V_\Lambda \subseteq \mathbb{C}\,\mathrm{S}_n$*, generated by* Y_Λ*, such that*

$$V_\Lambda = \mathrm{span}\{gY_\Lambda \mid g \in \mathrm{S}_n\},$$

which is obviously a ρ*-invariant subspace. We claim that*

(1) *the restriction of* ρ *to* V_Λ *is an irreducible representation of* S_n*;*
(2) *for all* Λ *of the same shape* λ*, the* V_Λ*'s are equivalent to each other;*
(3) *for* Λ*,* Λ' *of different shapes then* V_Λ *is not equivalent to* $V_{\Lambda'}$*.*

Example 5.4. We define a tableaux of shape $\lambda = (n) \vdash n$ to be

$$\Lambda = \boxed{1}\boxed{\ldots}\boxed{n}$$

We have that $R_\Lambda = \mathrm{S}_n$ and $C_\Lambda = \{e\}$. The Row and Coloumn symmetrizers are therefore

$$P_\Lambda = \sum_{g\in\mathrm{S}_n} g \qquad\qquad Q_\Lambda = e,$$

which implies $Y_\Lambda = P_\Lambda$. Now, from the definition above

$$\begin{aligned} V_\Lambda &= \mathrm{span}\{gY_\Lambda \mid g \in \mathrm{S}_n\} \\ &= \mathrm{span}\{gP_\Lambda \mid g \in \mathrm{S}_n\} \\ &= \mathbb{C}P_\Lambda, \end{aligned}$$

which is irreducible. We notice that

$$\rho(h)P_\Lambda = hP_\Lambda = P_\Lambda \Rightarrow \rho(h) = 1 \text{ for all } h \in \mathrm{S}_n$$

and hence V_Λ is the trivial representation for any Λ.

Example 5.5. We define a tableaux of shape $\lambda = (1, \ldots, 1) \vdash n$ to be

$$\Lambda = \begin{array}{|c|}\hline 1 \\ \hline \\ \hline n \\ \hline \end{array}$$

We have that $R_\Lambda = \{e\}$ and $C_\Lambda = \mathrm{S}_n$, which is the opposite case to above. The Row and Coloumn symmetrizers are therefore

$$P_\Lambda = 1 \qquad\qquad Q_\Lambda = \sum_{g\in\mathrm{S}_n} (\mathrm{sign}\, g)g,$$

which implies $Y_\Lambda = Q_\Lambda$. Now, from the definition above

$$\begin{aligned} V_\Lambda &= \operatorname{span}\{gY_\Lambda \mid g \in \mathrm{S}_n\} \\ &= \operatorname{span}\{gQ_\Lambda \mid g \in \mathrm{S}_n\} \\ &= \mathbb{C}Q_\Lambda. \end{aligned}$$

We notice that

$$\rho(h)Q_\Lambda = \operatorname{sign}(h)Q_\Lambda \Rightarrow \rho(h) = \operatorname{sign}(h)$$

and hence V_Λ is the alternating representation.

Lemma 5.1. *For all $g \in R_1$ we have $gY_\Lambda = Y_\Lambda$.*

Proof. We have that

$$g \in R_\Lambda = \underbrace{\{\text{perm of first row of } \Lambda\}}_{\ni g_1} \times \underbrace{\{\text{perm of second row of } \Lambda\}}_{\ni g_2} \times \dots$$

Consider

$$gY_\Lambda = (g_1, \dots)P_\Lambda Q_\Lambda = P_\Lambda Q_\Lambda$$

using example 1 for each row. □

Lemma 5.2. *For all $h \in C_\Lambda$ we have $Y_\Lambda h = Y_\Lambda \operatorname{sign}(h)$.*

Proof. We have that

$$h \in C_\Lambda = \underbrace{\{\text{perm of first col of } \Lambda\}}_{\ni h_1} \times \underbrace{\{\text{perm of second col of } \Lambda\}}_{\ni h_2} \times \dots$$

Consider

$$\begin{aligned} Y_\Lambda h = P_\Lambda Q_\Lambda(h_1, h_2, \dots) &= P_\Lambda Q_\Lambda \operatorname{sign}(h_1)\operatorname{sign}(h_2)\dots \\ &= P_\Lambda Q_\Lambda \operatorname{sign}(h) \\ &= Y_\Lambda \operatorname{sign}(h). \end{aligned}$$

□

Lemma 5.3. *Up to a scalar factor, Y_Λ is a unique element of the symmetric group satisfying the systems of equations in Lemma 5.1 and 5.2 in S_n.*

Proof. Take any element $z \in \mathbb{C}\,\mathrm{S}_n$ satisfying Lemma 5.1, 5.2 instead of Y_Λ. Need to prove $z = \alpha Y_\Lambda$ for $\alpha \in \mathbb{C}$. We will show

$$z = \sum_{g \in \mathrm{S}_n} \underbrace{\alpha_g}_{\in \mathbb{C}} g.$$

We have

$$\alpha_g \neq 0 \Rightarrow g \in R_\Lambda C_\Lambda. \tag{5}$$

Then, if we prove the above we have

$$z = \sum_{g \in R_\Lambda C_\Lambda} \alpha_g g.$$

We now apply Lemma 5.1 (let $g' \in R_\Lambda$)

$$\begin{aligned} z = g'z &= \sum_{g \in R_\Lambda C_\Lambda} \alpha_g \underbrace{g'g}_{=g''} \\ &= \sum_{g'' \in R_\Lambda C_\Lambda} \alpha_{(g')^{-1}g''} g'' \\ &= \sum_{g \in R_\Lambda C_\Lambda} \alpha_{(g')^{-1}g} g. \end{aligned}$$

Note. Summing over $g'g$ no different as $g'R_\Lambda = R_\Lambda$

This implies $\alpha_g = \alpha_{(g')^{-1}g}$ for all $g \in R_\Lambda$. Similarly, by using Lemma 5.2

$$\alpha_g = \operatorname{sign}(h)\alpha_{gh} \text{ for all } h \in C_\Lambda,$$

which implies $\alpha_g = \operatorname{sign}(g)\alpha_e$.

Now let's show our condition (5) holds. Need to prove that if $g \notin R_\Lambda C_\Lambda$, then $\alpha_g = 0$. Consider another tableau, $\Lambda' = g\Lambda$. Suppose there exists a pair of numbers (a, b), with $a \neq b$, such that a, b appear in the same row of Λ and in the same column Λ'. Then $(ab) \in R_\Lambda$ and $(ab) \in C_{\Lambda'} = gC_\Lambda g^{-1}$. This implies

$$C_\Lambda \ni g^{-1}(ab)g = \big(g^{-1}(a), g^{-1}(b)\big).$$

Lemma 5.1 implies $\alpha_{(ab)g} = \alpha_g$. Then

$$\begin{aligned} \alpha_{g \cdot g^{-1}(ab)g} &= \operatorname{sign}\left[g^{-1}(ab)g'\right]\alpha_g \\ &= (-1) \cdot \alpha_g. \end{aligned}$$

So, $\alpha_{(ab)g} = -\alpha_g \Rightarrow \alpha_g = 0$. Now, let us derive (5) from here.

!! DIAGRAM !!

(1) There exists $h_1 \in C_{\Lambda'}$ such that the first row of $h_1\Lambda'$ has got the numbers $k_1, k_2, \ldots, k_\ell$ in it. There exists $g_1 \in R_\Lambda$ such that the first rows of $g_1\Lambda$ and $h_1\Lambda'$ coincide.

(2) Consider now the second row of $g_1\Lambda$ (the same as the second row of Λ) and so on. Continuing this process, we can find $g_1, \dots, r_r \in R_\Lambda$ and $h_1, \dots, h_r \in C_{\Lambda'}$ such that $g_1 \dots g_r\Lambda$ and $h_1 \dots h_r\Lambda'$ have the same first row, second row, etc. That is

$$\underbrace{g_1 \dots g_r}_{\in R_\Lambda}\Lambda = \underbrace{h_1 \dots h_r}_{\in C_{\Lambda'} = gC_\Lambda g^{-1}}\Lambda' = h_1 \dots h_r g\Lambda.$$

Now define $g_1 \dots g_r = g_0$ and $h_1 \dots h_r = h_0$. Note that $h_1 \dots h_r g \in gC_\Lambda$. Then we have that

$$g_0\Lambda = h_0 g\Lambda \Rightarrow g_0 = h_0 g = gh_\infty \Rightarrow g_0 h_\infty^{-1} = g.$$

Note that $h_0 g \in gC_\Lambda \Rightarrow h_0 g = gh_\infty$. □

Corollary 5.1. *We have that $Y_\Lambda^2 = Y_\Lambda Y_\Lambda = n_\Lambda Y_\Lambda$ for some $n_\Lambda \in \mathbb{C}$*

Proof. Y_Λ^2 and Y_Λ satisfy Lemma 5.1, 5.2 and so must be proportional to each other. □

Exercise: in fact $n_\Lambda \in \mathbb{Z}$.

Theorem 5.2. *the left ideal $(\mathbb{C}\,\mathrm{S}_n)Y_\Lambda = V_\Lambda \ni v$ is an irreducible representation of S_n where $g \in \mathrm{S}_n$ act by left multiplication; $\rho(g)v = gv$.*

Proof. there is a distinguished vector $v_0 \in V_\Lambda$ such that $v_0 = Y_\Lambda = 1 \cdot Y_\Lambda$. By our corollary

$$Y_\Lambda v_0 = Y_\Lambda Y_\Lambda = n_\Lambda Y_\Lambda \in \mathbb{C}Y_\Lambda = \mathbb{C}v_0.$$

So, now take any invariant subspace $U \subseteq V_\Lambda$. Then

$$Y_\Lambda(U) \subseteq Y_\Lambda \underbrace{(\mathbb{C}\,\mathrm{S}_n)Y_\Lambda}_{\in U} \subseteq \mathbb{C}Y_\Lambda = \mathbb{C}v_o$$

by our lemma. There our two cases to consider

(1) If $Y_\Lambda(U) \neq 0$ then $Y_\Lambda(U) = \mathbb{C}Y_\Lambda$. But we know that U is an invariant subspace and hence

$$\underbrace{Y_\Lambda(U)}_{Y_\Lambda} \subseteq U.$$

So, $U \ni Y_\Lambda$ and $gU \subseteq U$ for all $g \in \mathrm{S}_n$ by invariancy. Now, $gY_\Lambda \in U$ which spans V_Λ. So, $U = V_\Lambda$.

(2) Now suppose that $Y_\Lambda(U) = \{0\}$. Then

$$U \cdot U = \{XY \mid X, Y \in U\} \subset V_\Lambda U = U(\mathbb{C}\,\mathrm{S}_n)Y_\Lambda = \{0\}$$

So, $U \cdot U = \{0\}$. Now, $U \subset \mathbb{C}\,\mathrm{S}_n$. By Maschke's Theorem there is a direct sum decomposition of $\mathbb{C}\,\mathrm{S}_n = U \oplus U'$ where U, U' are preserved by left multiplications in $\mathbb{C}\,\mathrm{S}_n$ by all $g \in \mathrm{S}_n$. Consider the operator $P : \mathbb{C}\,\mathrm{S}_n \to \mathbb{C}\,\mathrm{S}_n; u + u' \mapsto u$ for all $u \in U$, $u' \in U'$. Then

$$P(gX) = gP(X) \tag{6}$$

for all $g \in \mathrm{S}_n$, $X \in \mathbb{C}S_n$ because U, U' are invariant.

Claim. ay operator P satisfying (6) should have the form $P(X) = XZ$ (for all $X \in \mathbb{C}\,\mathrm{S}_n$) for some fixed $Z \in \mathbb{C}\,\mathrm{S}_n$. So, $P^2 = P \Rightarrow Z^2 = Z$ but $P(1) = 1Z \in U \Rightarrow Z \in U$. But then $Z^2 = Z \in U$, however $Z^2 \in UU = \{0\} \Rightarrow Z = 0$. So, $P(X) = \{0\}$ for all X. By definition $\operatorname{Im} P = U = \{0\}$. So, $V_\Lambda$is irreducible. ☐

Theorem 5.3. *the representations V_Λ and V_Ω (where ω is a partition of n and Ω is a Young tableau of shape ω) of S_n are equivalent $\Leftrightarrow \lambda = \omega$ (that is, Λ and Ω have the same shape).*

Definition: *lexicographical ordering of paritions.* Suppose $\lambda, \omega \vdash n$. We say that $\lambda > \omega$ if and only if the first non-zero difference of $\lambda_1 - \omega_1$, $\lambda_2 - \omega_2, \dots$ is positive.

Lemma 5.4. *Suppose that $\lambda > \omega$. Then for all $A \in \mathbb{C}\,\mathrm{S}_n$ we have*

$$Y_\Omega A Y_\Lambda = 0, \tag{7}$$

where Λ and Ω are any tableaux of shapes λ and ω respectively.

Proof.

(1) $A = \sum_{g \in \mathrm{S}_n} \alpha_g g$ with $\alpha_g \in \mathbb{C}$. It is enough to prove (7) for $A = g$ for all $g \in \mathrm{S}_n$.

(2) $gR_\Lambda^{-1} g = R_g \Lambda$, $gC_\Lambda^{-1} g = C_{g\Lambda} \Rightarrow gP_\Lambda^{-1} g = P_{g\Lambda}$ and $gQ_\Lambda^{-1} g = Q_{g\Lambda}$. Therefore

$$Y_\Omega g Y_\Lambda = Y_\Omega g P_\Lambda g^{-1} g Q_\Lambda (g^{-1} g) = Y_\Omega P_{g\Lambda} Q_{g\Lambda} g = Y_\Omega Y_{g\Lambda} g.$$

This means that

$$Y_\Omega g Y_\Lambda = 0 \forall \Lambda \Leftrightarrow Y_\Omega e Y_{\Lambda'} = 0 \forall \Lambda' = g\Lambda.$$

So enough to consider the simplest case $g = e$.

(3) We will now show that there is a pair of numbers $a, b \in \{1, \dots, n\}$; $a \neq b$ such that a, b are in the same row of Λ and in the same column of Ω. Then

$$Y_\Omega Y_\Lambda = Y_\Omega \underbrace{(a,b)(a,b)}_{e''} Y_\Lambda = -Y_\Omega Y_\Lambda$$

(4) Let us prove the existence of such a pair a, b. Consider the first row of Λ. We know that $\lambda_1 \geqslant \omega_1$(otherwise $\lambda \not< \omega$). If all the numbers from the first row of Λ are in different columns of Ω then $\lambda_1 = \omega_1$. Moreover, then $g \in R_\Lambda$ and $h \in C_\Omega$ such that $g\Lambda$ and $h\Omega$ have the same first row. But then

$$Y_{h\Omega} Y_{g\Lambda} = h Y_\Omega h^{-1} g Y_\Lambda g^{-1} = h \operatorname{sign}(h^{-1}) Y_\Omega Y_\Lambda g^{-1}.$$

So $Y_\Omega Y_\Lambda = 0 \Leftrightarrow Y_{h\Omega} Y_{g\Lambda} = 0$. So, instead of proving $Y_\Omega Y_\Lambda = 0$ (7) we can prove $Y_{h\Omega} Y_{g\Lambda} = 0$. Then we can remove the first row from $g\Lambda$, $h\Omega$ and repeat the argument for the second row of $g\Lambda = \Lambda', h\Omega = \Omega'$. Where we would go in this way? Either at one of the steps k we find (a, b) in a same row of $\Lambda^{(k)}$ and some column of $\Omega^{(k)}$ so that $Y_{\Omega^{(k)}} Y_{\Lambda^{(k)}} = 0$ by (3). Or if not we have

$$\lambda_1 = \omega_1, \lambda_2 = \omega_2, \ldots, \lambda_k = \omega_k, \ldots$$

that is $\lambda = \omega$ but this is not possible.

□

Proof of Theorem 2. Suppose Λ and Ω are of the same shape $\lambda = \omega$. Then there exists $g \in \mathrm{S}_n$ such that $\Omega = g\Lambda$. Then $Y_\Omega = gY_\Lambda g^{-1}$. Define a linear map $V_\Lambda \to V_\Omega : A \mapsto Ag^{-1}$. If $A \in V_\Lambda$ then $A = BY_\Lambda$. Then

$$\varphi(A) = \varphi(BY_\Lambda) = BY_\Lambda g^{-1} = Bg^{-1}gY_\Lambda g^{-1} = Bg^{-1}Y_\Omega \in V_\Omega.$$

So φ is a bijection. Moreover, φ is S_n equivalent.

!! DIAGRAM !!

Now suppose that $\lambda \neq \omega$. Need to show that $V_\Lambda \not\sim V_\Omega$. Can assume $\lambda > \omega$ otherwise (if $\lambda < \omega$) swap λ and ω. We have the left regular representation of S_n, $h \in \mathrm{S}_n$ acts via left multiplication and we can extend this action to $\mathbb{C}\,\mathrm{S}_n \ni \sum_h \alpha_h h = A$ then

$$\begin{aligned} A : B \mapsto AB &= \left(\sum \alpha_h h\right) B \\ &= \sum \alpha_h \rho(h) B. \end{aligned}$$

Compare the action of $Y_\Omega \in \mathbb{C}\,\mathrm{S}_n$ in both V_Λ and V_Ω. In $V_\Lambda \ni B = CY_\Lambda$ (where $C \in \mathbb{C}\,\mathrm{S}_n$) we have

$$Y_\Omega B = Y_\Omega C Y_\Lambda = 0$$

by Lemma. So the action of Y_Ω in Y_Λ is the zero operator. In $V_\Omega \ni 1 \cdot Y_\Omega$ we have

$$Y_\Omega \cdot 1 \cdot Y_\Omega = Y_\Omega^2 = h_\Omega Y_\Omega \neq 0$$

because $h \neq 0$. No equivalence. □

$\mathbb{C}\,\mathrm{S}_n$ the Young symmetrizer

$$\boxed{Y_\Lambda^2 = Y_\Lambda n_\Lambda}$$

with $n_\Lambda \in \mathbb{C}$.

Lemma 5.5. *We have*

$$n_\Lambda = \frac{n!}{\dim V_\Lambda}$$

where $V_\Lambda = (\mathbb{C}\,\mathrm{S}_n)Y_\Lambda$

Proof. Consider the operator M of the right multiplication in $\mathbb{C}\,\mathrm{S}_n$ by Y_Λ. The image of M is V_Λ. Compute the trace of this operator.

(1) $M : \mathbb{C}\,\mathrm{S}_n \to \mathbb{C}\,\mathrm{S}_n; g \mapsto gY_\Lambda = \sum_{h\in \mathrm{S}_n} \alpha_h(g)h$ with $\alpha_h \in \mathbb{C}$. Want to know only the coefficient $\alpha_g(g)$ (for every $g \in \mathrm{S}_n$) that is $\alpha_h(g)$ with $h = g$. Then

$$Y_\Lambda = P_\Lambda Q_\Lambda = \left(\sum_{r\in R_\Lambda} r\right)\left(\sum_{c\in C_\Lambda} c\,\mathrm{sign}(c)\right).$$

but $R_\Lambda \cap C_\Lambda = \{e\}$

$$= 1\cdot e + \dots$$

This implies however that

$$\boxed{rc = e \Rightarrow r = e \quad c = e}$$

Thus we have

$$gY_\Lambda = gP_\Lambda Q_\Lambda = g\left(1\cdot e + \sum_{f\in \mathrm{S}_n} \beta_f f\right) = 1\cdot ge + \sum_{f\neq e} \beta_f \underbrace{gf}_{\neq g}.$$

Relative to the basis $\{g \mid g \in \mathrm{S}_n\}$ the matrix of M has the diagonal

$$\begin{pmatrix} 1 & \dots & \dots & \star \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \star & \dots & \dots & 1 \end{pmatrix}$$

Thus we have $\mathrm{trace}(M) = n!$.

(2) Consider the possibility of $n_\Lambda = 0$. This would mean that $Y_\Lambda^2 = 0$ or $M^2 = 0$. Consider the Jordan normal form of M. If $v \in \mathbb{C}\,\mathrm{S}_n$ is an eigenvector with eigenvalue λ, then $Mv = \lambda v$ or $M^2 v = \lambda^2 v \Rightarrow \lambda = 0$. So, in the Jordan normal form, the matrix of M would be of the form

$$\begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$$

so that $\mathrm{trace}\, M = 0 \neq n!$ but this is not possible.

(3) So we now know that $Y_\Lambda^2 = Y_\Lambda n_\Lambda$ and $n_\Lambda \neq 0$ with $\bar{Y}_\Lambda = \frac{Y_\Lambda}{n_\Lambda}$. Then

$$\overline{Y}_\Lambda^2 = \left(\frac{Y_\Lambda}{n_\Lambda}\right)^2 = \frac{n_\Lambda Y_\Lambda}{n_\Lambda^2} = \frac{Y_\Lambda}{n_\Lambda} = \overline{Y}_\Lambda.$$

Right multiplication by $\overline{Y}_\Lambda$ in $\mathbb{C}\,\mathrm{S}_n$ is a projection onto V_Λ. In some basis, the matrix of the operator $\overline{M}_\Lambda$ of right multiplication. Then $\overline{Y}_\Lambda$ would be

$$\left(\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & \star & \cdots & \star \\ 0 & 1 & \cdots & 0 & \vdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 1 & \star & \cdots & \star \\ \hline \star & \cdots & \cdots & \star & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \cdots & \cdots & \star & 0 & \cdots & 0 \end{array}\right)$$

Then trace $\overline{M}_\Lambda = \dim V_\Lambda$ but

$$\frac{\mathrm{trace}(M_\Lambda)}{n_\Lambda} = \frac{n!}{n_\Lambda} \Rightarrow n_\Lambda = \frac{n!}{\dim V_\Lambda}$$ □

6. Towards the proofs of Theorem 3 and 4

Notation: for any two Young tableaux Λ, Λ' of the same shape $\lambda \vdash n$ denote $E_{\Lambda\Lambda'} = gP_\Lambda Q_\Lambda$ where $g \in \mathrm{S}_n$ such that $g\Lambda = \Lambda'$

Note. $E_{\Lambda\Lambda'}$ where Λ is standard but fixed, while Λ' ranges over all standard tableaux of shape λ, form a basis in V_Λ – Theorem 4 to be proved.

Lemma 6.1. *$P_{\Lambda'}Q_\Lambda = c_{\Lambda\Lambda'}E_{\Lambda\Lambda'}$ for some $c_{\Lambda\Lambda'} \in \mathbb{C}$*

Proof. $g\Lambda = \Lambda' \Rightarrow gP_\Lambda g^{-1} = P_{\Lambda'}$ or $g^{-1}P_\Lambda = P_\Lambda g^{-1} \Rightarrow g^{-1}P_{\Lambda'}Q_\Lambda = P_\Lambda g^{-1}Q_\Lambda$

Note. $hZ = Z$ for all $h \in R_\Lambda$and $Zf = Z(\mathrm{sign}\, f)$ for all $f \in C_\Lambda \Rightarrow Z = cY_\Lambda = cP_\Lambda Q_\Lambda$.

Then we have $g^{-1}P_{\Lambda'}Q_\Lambda = cP_\Lambda Q_\Lambda$ by our main Lemma, which implies

$$\begin{aligned} P_{\Lambda'}Q_\Lambda &= cgP_\Lambda Q_\Lambda \\ &= cE_{\Lambda\Lambda'} \end{aligned}$$

y definition. Hence $c = c_{\Lambda\Lambda'}$. □

Corollary 6.1. $Q_\Lambda P_{\Lambda'} = c_{\Lambda\Lambda'}Q_\Lambda g^{-1}P_{\Lambda'}$

Proof.

$$\begin{aligned} P_{\Lambda'}Q_\Lambda &= c_{\Lambda\Lambda'}gP_\Lambda Q_\Lambda \\ &= c_{\Lambda\Lambda'}P_{\Lambda'}gQ_\Lambda. \end{aligned} \tag{$\star$}$$

Consider the linear map $\alpha : \mathbb{C}\,\mathrm{S}_n \to \mathbb{C}\,\mathrm{S}_n; h \mapsto h^{-1}$. This is injective. Then $\alpha(AB) = \alpha(B)\alpha(A)$ for all $A, B \in \mathbb{C}\,\mathrm{S}_n$ this is because

$$\alpha(hf) = (hf)^{-1} = f^{-1}h^{-1} = \alpha(f)\alpha(h).$$

This is true for A, B by linearity of α. Applying α to both sides of $(\star)$ we get

$$(\star\star) \qquad \alpha(Q_\Lambda)\alpha(P_{\Lambda'}) = c_{\Lambda\Lambda'}\alpha(Q_\Lambda)g^{-1}\alpha(P_{\Lambda'})$$

However we note that

$$\alpha(Q_\Lambda) = \alpha\left(\sum_{h\in C_\Lambda} \mathrm{sign}(h)h\right) = \sum_{h\in C_\Lambda} \underbrace{\mathrm{sign}(h)}_{\mathrm{sign}(h^{-1})} h^{-1} = Q_\Lambda.$$

Thus we have $(\star\star)$ if and only if

$$Q_\Lambda P_{\Lambda'} = c_{\Lambda\Lambda'}Q_\Lambda g^{-1}P_{\Lambda'} \qquad \square$$

Proposition. Let Λ, Λ' be tableaux of shape λ and Ω, Ω' be tableaux of shape ω where $\lambda, \omega \vdash n$. Then

$$E_{\Omega\Omega'}E_{\Lambda\Lambda'} = \begin{cases} 0 & \text{if } \lambda \neq \omega \\ \frac{n!}{\dim V_\Lambda} c_{\Lambda\Lambda'}E_{\Lambda\Omega'} & \text{if } \lambda = \omega. \end{cases}$$

Proof. we want to compute $(\mathrm{S}_n \ni h : \Omega \to \Omega')$

$$E_{\Omega\Omega'}E_{\Lambda\Lambda'} = P_{\Omega'}hQ_\Omega P_{\Lambda'}gQ_\Lambda$$

by definition. Consider three cases.

(1) $\lambda > \omega$ on our lexicographical ordering of partitions. Then $Q_\Omega P_{\Lambda'} = 0$ by the proof of our Lemma (before Theorem 1) [can find a pair $a \neq b$ appearing in the same column of Ω and the same row of Λ']. So $E_{\Omega\Omega'}E_{\Lambda\Lambda'} = 0$ as needed.

(2) $\lambda < \omega$. Let us write

$$\mathbb{C}\,\mathrm{S}_n \ni hQ_\Omega P_{\Lambda'}g = \sum_{f\in \mathrm{S}_n} \alpha_f f$$

with $\alpha_f \in \mathbb{C}$. We will show that

$$P_{\Omega'}fQ_\Lambda = 0$$

for all $f \in \mathrm{S}_n$. This implies that

$$(8) \qquad E_{\Omega\Omega'}E_{\Lambda\Lambda'} = 0$$

as required. Put $\Lambda'' = f\Lambda$. Then $P_{\Omega'}fQ_\Lambda = P_{\Omega'}Q_{\Lambda''}f$. Now I claim that here $P_{\Omega'}Q_{\Lambda''} = 0$ ($\Rightarrow$ (8)). Consider the produce $Q_{\Lambda''}P_{\Omega'} = 0$ because $\lambda < \omega$ (as in the proof of the already mentioned Lemma can find $c \neq d$ such that they appear in the same column of Λ'' and the same row of Ω'). Let us apply to $Q_{\Lambda''}P_{\Omega'} = 0$ our anti-automorphism of $\mathbb{C}\,\mathrm{S}_n$, $\beta : x \mapsto x^{-1}$. Then we have

$$\beta(P_{\Omega'})\beta(Q_{\Lambda''}) = P_{\Omega'}Q_{\Lambda''} = 0.$$

(3) $\lambda = \omega$. But then there exists $f \in \mathrm{S}_n$ such that $f\lambda' = \Omega$. Then $\Lambda' = g\Lambda$, so $(fg)\Lambda = f\Lambda' = \Omega$ or $(fg)^{-1}\Omega = \Lambda$. Then

$$E_{\Omega\Omega'}E_{\Lambda\Lambda'} = c_{\Lambda\Lambda'}P_{\Omega'}h(Q_\Omega fP_{\Lambda'})gQ_\Lambda$$

by the Corollary of last lecture.

$$\begin{aligned} &= c_{\Lambda\Lambda'}P_{\Omega'}hQ_\Omega fgP_\Lambda Q_\Lambda \\ &= c_{\Lambda\Lambda'}P_{\Omega'}(hfg)Q_\Omega P_\Lambda Q_\Lambda. \end{aligned}$$

We note that our permutations look like

$$\Lambda \underset{g}{\to} \Lambda' \underset{f}{\to} \Omega \underset{h}{\to} \Omega'.$$

Thus $(hfg)\Lambda = \Omega'$. So

$$\begin{aligned} E_{\Omega\Omega'}E_{\Lambda\Lambda'} &= c_{\Lambda\Lambda'}(hfg)(P_\Lambda Q_\Lambda)(P_\Lambda Q_\Lambda). \\ &= c_{\Lambda\Lambda'}(hfg)P_\Lambda Q_\Lambda n_\Lambda \end{aligned}$$

where $n_\Lambda \in \mathbb{C}$. Finally, by definition

$$= c_{\Lambda\Lambda'}E_{\Lambda\Omega'}n_\Lambda.$$

□

Definition: Suppose that Λ, Λ' are *standard* tableaux of the same shape $\lambda \vdash n$. Then we say that $\Lambda' < \Lambda$ in the *first letter order* if the first (of $1, 2, \ldots, n$) letter of disagreement between Λ and Λ' is *higher* in Λ' than in Λ.

EXAMPLE 6.1. $\lambda = (3, 2)$ then we have

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} <_3 \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} <_4 \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} <_2 \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} <_4 \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}.$$

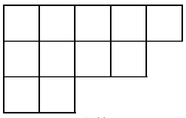
We note in the first case they disagree on where the number 3 is placed. However in the first diagram 3 is in the first row and in the second 3 is in the second row. Thus 3 is *higher* in the first than in the second.

Observations on this ordering

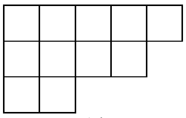
(a) for any two different standard tableaux Λ, Λ' $(\Lambda \neq \Lambda')$ we have either $\Lambda > \Lambda'$ or $\Lambda < \Lambda'$.
(b) this is a *transitive ordering*. This means that if $\Lambda'' < \Lambda'$ and $\Lambda' < \Lambda$ then $\Lambda'' < \Lambda$.

Proof of transitivity. suppose $\Lambda'' <_k \Lambda$ and $\Lambda' <_\ell \Lambda$. If $k = \ell$ then Λ'' disagrees with Λ in $k = \ell$ and k is higher in Λ'' than in Λ' and k is higher in Λ' than in Λ. So $\Lambda'' <_k \Lambda$

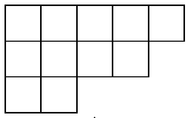
We could have $k > \ell$



Λ''



Λ'



Λ

So ℓ is the first letter of disagreement between Λ, Λ'' and ℓ is higher in Λ'' than in Λ. So, $\Lambda'' <_\ell \Lambda$.

$k < \ell$. Then, similarly, Λ'' and Λ argree in $1, \ldots, k-1$ and disagree in k, and k is higher in Λ'' than in Λ, Λ'. So $\Lambda'' <_k < \Lambda$. □

Proposition. Our observations mean that all standard tableaux can be organised in a chain.

Proposition. (a) $\Lambda' = \Lambda \Rightarrow c_{\Lambda\Lambda'} = 1$, $g : \Lambda \to \Lambda'$
(b) $\Lambda' < \Lambda \Rightarrow c_{\Lambda\Lambda'} = 0$, where $P_{\Lambda'} Q_\Lambda = c_{\Lambda\Lambda'} E_{\Lambda\Lambda'}$

Proof. We've take each case separately.

(a) $\Lambda = \Lambda' \Rightarrow g = e$. Then

$$P_\Lambda Q_\Lambda = c_{\Lambda\Lambda} E_{\Lambda\Lambda} = c_{\Lambda\Lambda} P_\Lambda Q_\Lambda$$

$\Rightarrow c_{\Lambda\Lambda} = 1$.

(b) Suppose $\Lambda' < \Lambda$ in first letter ordering. This means that for $a \in \{1, \ldots, n\}$ the numbers $1, \ldots, a-1$ appear in Λ, Λ' in the same positions and a appears in Λ' higher than in Λ.

!! DIAGRAM !!

But b appears in Λ and Λ' in the same position and as b is standard $\Rightarrow$ b is to the left of a. Conclusion is that b, a appear in the same row of Λ' and column of $\Lambda \Rightarrow P_\Lambda Q_{\Lambda'} = 0 \Rightarrow c_{\Lambda\Lambda'} = 0$.

□

Notation: $\mathbb{C}\,\mathrm{S}_n \ni X = \sum_{f \in \mathrm{S}_n} \alpha_f f$ where $\alpha_e = x|_e$.

Proposition. We suggest that

$$c_{\Lambda\Lambda} = E_{\Lambda\Lambda'}|_e$$

Proof. When proving the existence of $c_{\Lambda\Lambda'}$ we showed that

$$P_\Lambda g^{-1} Q_\Lambda = c_{\Lambda\Lambda'} P_\Lambda Q_\Lambda$$

but $P_\Lambda Q_\Lambda|_e = 1$. Therefore

$$E_{\Lambda\Lambda'}|_e = (g P_\Lambda Q_\Lambda)|_e$$

Apply our favourite antihomomorphism

$$\varphi(E_{\Lambda\Lambda'}|_e) = E_{\Lambda\Lambda'}|_e = (\underbrace{Q_\Lambda}_{=X}\underbrace{P_\Lambda g^{-1}}_{=Y})|_e$$

Also $X, Y \in \mathbb{C}\,\mathrm{S}_n$ then $(XY)|_e = (YX)|_e$. Therefore

$$(XY)|_e = (YX)|_e = (P_\Lambda g^{-1} Q_\Lambda)|_e = (c_{\Lambda\Lambda'} P_\Lambda Q_\Lambda)|_e = c_{\Lambda\Lambda'}.$$

□

Corollary 6.2. *Let λ range over all partitions of n and Λ, Λ' range over all standard tableaux of shape λ. Then $E_{\Lambda\Lambda'}$ are linearly independent.*

Proof. Consider any linear combination

$$A = \sum_{\lambda \vdash n} \sum_{\Lambda,\Lambda'} a_{\Lambda\Lambda'} E_{\Lambda\Lambda'}$$

where Λ, Λ' are standard tableaux of shape λ. By proposition 1 we have

$$\begin{aligned} E_{\Omega\Omega'} A &= E_{\Omega\Omega'} \left(\sum_{\lambda,\Lambda,\Lambda'} a_{\Lambda\Lambda'} E_{\Lambda\Lambda'} \right) \\ &= \sum_{\lambda,\Lambda,\Lambda'} \delta_{\lambda\omega} a_{\Lambda\Lambda'} c_{\Omega\Lambda'} E_{\Lambda\Omega'} \underbrace{\frac{n!}{\dim V_\Lambda}}_{n_\Lambda \neq 0} \\ \Rightarrow (E_{\Omega\Omega'} A)|_e &= \sum_{\lambda,\Lambda,\Lambda'} \delta_{\lambda\omega} a_{\Lambda\Lambda'} c_{\Omega\Lambda'} c_{\Lambda\Omega'}. \end{aligned}$$

Consider now the transformation

$$A \mapsto (E_{\Omega\Omega'} A)|_e \in \mathbb{C}.$$

Regard this as a transformation.

$$(\text{collection of } a_{\Lambda\Lambda'}) \mapsto (\text{collection of } (E_{\Omega\Omega'} A)|_e)$$

The left hand side of this is a matrix, which we wish to make into a column vector. This transformation is linear

$$\#(E_{\Omega'\Omega} A)|_e = \sum_{\lambda,\Lambda,\Lambda'} \delta_{\lambda\omega} a_{\Lambda\Lambda'} \underbrace{c_{\Omega'\Lambda'} c_{\Lambda\Omega}}_{=b^{\Lambda\Lambda'}_{\Omega\Omega'}}$$

If $b^{\Lambda\Lambda'}_{\Omega\Omega'} \neq 0 \Rightarrow \Omega' \leqslant \Lambda'$, $\Lambda \leqslant \Omega$ and $b^{\Lambda\Lambda'}_{\Omega\Omega'} = 1$ if $\Omega' = \Lambda'$ and $\Omega = \Lambda$. We're going to order the pairs Λ, Λ' like this:

Suppose for given λ, in the first letter ordering we have: $M_1, M_2, \ldots, M_{N(\lambda)}$ all standard tableaux of shape λ.Then order the pairs like this:

$$(\Lambda, \Lambda') = (M_1, M_1)(M_1, M_2)\ldots(M_1, M_n)(M_2, M_1)(M_2, M_2)\ldots$$

Further order the pairs (Ω, Ω') like this:

$$(M_N, M_1)(M_N, M_2)\ldots(M_N, M_N)(M_{N-1}, M_1)(M_{N-2}, M_2)\ldots$$

Then the matrix $b^{\Lambda\Lambda'}_{\Omega\Omega'}$ is upper triangular with only units on the diagonal. My linear transform

$$(a_{\Lambda\Lambda'}) \mapsto (E_{\Omega\Omega'}A)|_e$$

is a matrix transformation with an invertible matrix

$$b^{\Lambda\Lambda'}_{\Omega\Omega'} = \begin{pmatrix} 1 & \cdots & \star \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

So when $a_{\Lambda\Lambda'}$ vary $(E_{\Omega\Omega'}A)|_e$ must be linearly independent. So $E_{\Omega\Omega'}$ are linearly independent as the transformation is invertible. $\square$

Comment on the proof of the last Corollary

$$A = \sum_{\lambda \vdash n} \sum_{\Lambda, \Lambda'} a_{\Lambda\Lambda'} E_{\Lambda\Lambda'}$$

where Λ, Λ' are standard tableaux of shape λ. We proved

$$\mathbb{C} \ni E_{\Omega\Omega'} A\big|_e$$

is linearly independent as (linear) scalar functions of $\{a_{\Lambda\Lambda'}\}$ $\Rightarrow$ the elements $E_{\Lambda\Lambda'}$ are linearly independent in $\mathbb{C}\,\mathrm{S}_n$. Indeed, consider any linear combination

$$\sum_{\omega, \Omega, \Omega'} x_{\Omega\Omega'} E_{\Omega\Omega'} = 0.$$

Then consider

$$\left(\sum_{\omega, \Omega, \Omega'} x_{\Omega\Omega'} E_{\Omega\Omega'} \right) A \Bigg|_e (a_{\Lambda\Lambda'}) \equiv 0$$

linear combination of functions $E_{\Omega\Omega'}A|_e$ is the zero function $\Rightarrow x_{\Omega\Omega'} = 0$ for all Ω, Ω'.

Proposition. let λ run through partitions of n. Denote by f_λ the number of standard tableaux of shape λ

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

Theorem 6.1. *We have that*

$$\mathbb{C}\,\mathrm{S}_n = \bigoplus_{\lambda \vdash n} \bigoplus_{\Lambda} V_\Lambda$$

where Λ are standard tableaux of shape λ.

Theorem 6.2. $V_\Lambda = \oplus_{\Lambda'} \mathbb{C} E_{\Lambda\Lambda'}$ *where* $E_{\Lambda\Lambda'} = gP_\Lambda Q_\Lambda = gY_\Lambda$ *and* $g : \Lambda \to \Lambda'$.

Proof. Need to show that $E_{\Lambda\Lambda'}$ span V_Λ where Λ' varies (theorem 4) that when Λ, λ also vary (as well as Λ) then $E_{\Lambda\Lambda'}$ span $\mathbb{C}\,\mathrm{S}_n$ (theorem 3). Consider $\widetilde{V}_\Lambda$ = span of $E_{\Lambda\Lambda'} \subseteq V_\Lambda$ where Λ' varies. Need to show that $\widetilde{V}_\Lambda = V_\Lambda$ and $\dim \widetilde{V}_\Lambda = f_\lambda$.

Suppose for some Λ we have $\dim V_\Lambda > f_\Lambda$ we have the equality

$$\sum_\lambda f_\lambda^2 = n!$$

(prop). Also by our Corollary

$$\begin{aligned} \#\{E_{\Lambda\Lambda'}\} = \sum_\lambda (\dim V_\Lambda) f_\lambda \leqslant \dim \mathbb{C}\,\mathrm{S}_n = n! \\ < \sum_\lambda f_\lambda f_\lambda = n! \end{aligned}$$

but this is a contradiction! So, $\dim V_\Lambda = f_\Lambda \Rightarrow V_\Lambda = \widetilde{V}_\Lambda$. $\square$

7. Robinson-Schensted-Knuth Algorithm

Establishes a bijection between

$$\{g \in \mathrm{S}_n\} = \mathrm{S}_n \to \{(\Lambda, \Lambda') \text{ where } \Lambda, \Lambda' \text{ are standard of the same shape } \lambda, \lambda \text{ vary}\}$$

We notice that we must have equality between the cardinality of the sets. This is

$$n! = \sum_{\lambda \vdash n} f_\lambda^2.$$

Example 7.1. A permutation of S_8 is

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 1 & 3 & 5 & 8 & 2 & 6 \end{pmatrix}.$$

Our algorithm gives us

$$\Lambda = \begin{ytableau} 1 & 2 & 5 & 6 \\ 3 & 8 \\ 4 \\ 7 \end{ytableau} \qquad \Lambda' = \begin{ytableau} 1 & 4 & 5 & 6 \\ 2 & 8 \\ 3 \\ 7 \end{ytableau}$$

The right hand side tableau is called the log. It records the order in which boxes are created in the tableau Λ.

Observations:

- Λ, Λ' are standard
- Λ, Λ' of the same shape
- Need to show that RSK is reversible.

Example 7.2. The reverse RSK.

$$\Lambda = \begin{ytableau} 1 & 3 & 6 & 8 \\ 2 & 5 \\ 4 & 7 \end{ytableau} \qquad \Lambda' = \begin{ytableau} 1 & 2 & 4 & 6 \\ 3 & 5 \\ 7 & 8 \end{ytableau}$$

$$\Lambda = \begin{ytableau} 1 & 5 & 6 & 8 \\ 2 & 7 \\ 4 \end{ytableau} \qquad \Lambda' = \begin{ytableau} 1 & 2 & 4 & 6 \\ 3 & 5 \\ 7 \end{ytableau}$$

$$\Lambda = \begin{ytableau} 2 & 5 & 6 & 8 \\ 4 & 7 \end{ytableau} \qquad \Lambda' = \begin{ytableau} 1 & 2 & 4 & 6 \\ 3 & 5 \end{ytableau}$$

$$\Lambda = \begin{ytableau} 2 & 5 & 6 \\ 4 & 7 \end{ytableau} \qquad \Lambda' = \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \end{ytableau}$$

$$\Lambda = \begin{ytableau} 2 & 5 & 7 \\ 4 \end{ytableau} \qquad \Lambda' = \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}$$

$$\Lambda = \begin{ytableau} 2 & 5 & 7 \\ 4 \end{ytableau} \qquad \Lambda' = \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}$$

$$\Lambda = \begin{ytableau} 2 & 5 & 7 \\ 4 \end{ytableau} \qquad \Lambda' = \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}$$

$$\Lambda = \begin{ytableau} 2 & 5 \\ 4 \end{ytableau} \qquad \Lambda' = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$$

This gives us the permutation

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 2 & 7 & 6 & 8 & 1 & 3 \end{pmatrix}.$$