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A characterization of the rational mean neat voting rules

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ABSTRACT

A ***mean proximity rule*** is a voting rule having a mean proximity representation in Euclidean space. Legal ballots are represented as vectors that form the representing polytope. An output plot function determines a location for each possible election output in the same space, and these locations decompose the polytope into proximity regions according to which output is closest. The election outcome is then determined by which region(s) contain the mean position of all ballots cast. ***Mean neat rules*** are obtained by relaxing the requirement that the regions be determined by proximity, insisting only that they be neatly separable by a hyperplane. If each of these hyperplanes contains a dense set of rational points (vectors with all rational components), the mean neat voting rule is said to be ***rational***. The aim of this article is to prove that ***consistency*** and ***connectedness*** are necessary and sufficient conditions for mean neat rationality of any voting rule that is ***anonymous***. Connectedness can be viewed as a strong form of continuity, with an intuitive content related to the Intermediate Value Theorem (or to a discrete analogue of this theorem). The proof relies on a recent result in convexity theory [D. Cervone, W.S. Zwicker, Convex decompositions, J. Convex Anal. 2008 (in press)] and suggests a conjecture: if we relax connectedness to continuity, the class so characterized is that of the mean neat voting rules. This latter class properly contains *all* intuitive scoring rules.

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1. Introduction

Theorem 5.4 of Zwicker [2] (in this issue), states:

An abstract anonymous voting rule is a rational mean neat rule if and only if it is both consistent and connected.

Our goal here is to prove this result. The class of rational mean neat voting rules is broad, including all traditional scoring rules that employ rational scoring weights, as well as the Kemeny rule, approval voting, certain grading systems, etc. The theorem can be viewed as a generalization of the well-known characterizations of traditional scoring rules by Smith [3] and Young [4,5], and of their extension by Myerson [6]. For background on the theorem, and definitions of abstract anonymous voting system, homogeneity, consistency, and connectedness, see [2].

Here, we begin by reviewing the separation properties discussed in [1], and use them to develop a detailed definition of *rational mean neat voting rule*. When $u \neq v$ are distinct sets, the following separation properties are increasingly strong:

Definition 1.1. Two sets u and v of points of $\mathbf{R}^n$ are ***weakly separated by a hyperplane*** h of $\mathbf{R}^n$ if every point of u lies either on h or to one side of h, and every point of v lies either on h or to the other side of h. These sets are ***properly separated*** by h if they are weakly separated by h and, additionally, h does not contain both u and v as subsets; they are ***neatly separated*** by h if they are weakly separated and satisfy the requirement that $u \cap v = h \cap u = h \cap v$. Finally, they are ***strictly separated*** if they are weakly separated by some h disjoint from u and from v.

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Definition 1.2. An affine[1] subspace A of $\mathbf{R}^n$ of dimension k (or "codimension" $n-k$) is ***rational*** if any of the following equivalent conditions hold:

1. A is the solution space of a linear system of equations in which all coefficients and constants are integers.
2. A is the solution space of a linear system of $n-k$ independent equations in which all coefficients and constants are integers.
3. The set of rational points[2] of A is dense in A, with respect to the relative topology on A as induced by the Euclidean topology of $\mathbf{R}^n$.
4. There exist at least $k+1$ affinely independent rational points in A.
5. A is the affine span of a set of rational points.

We leave the proof of equivalence to the reader. A hyperplane h is an affine subspace of codimension 1. We immediately obtain the following:

Corollary 1.1. *For a hyperplane h of $\mathbf{R}^n$ the following are equivalent*

- *The set of rational points of h is dense in h.*
- *h is given by a single equation $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \beta$ in which both β and the components of the nonzero normal vector α to h are integers.*
- *h has a rational nonzero normal vector and contains at least one rational point.*

A straight line is an affine subspace of dimension 1. In this case we get:

Corollary 1.2. *For a line L of $\mathbf{R}^n$ the following are equivalent:*

- *The set of rational points of L is dense in L.*
- *L contains two rational points.*
- *L has a rational nonzero parallel vector and contains at least one rational point.*

Definition 1.3. An indexed ***decomposition*** of a subset $P \subseteq \mathbf{R}^n$ is a sequence $S = \{r_a\}_{a\in\Delta}$ of subsets of P called ***regions***, with finite index set Δ, satisfying $\bigcup_{a\in\Delta} r_a = P$. A ***decomposition*** is ***closed*** if its regions are closed sets.

Definition 1.4. A closed decomposition S of P is ***neat*** if each pair of distinct regions of S is neatly separated by some hyperplane and is **Q**-***neat*** if each pair of distinct regions of S is neatly separated by some rational hyperplane.

In our application, P is both closed and convex, as it is the convex hull of a finite point set. So, the following notion is central:

Definition 1.5. A ***polytope*** P is the convex hull of a finite set of points of $\mathbf{R}^n$; equivalently, it is a bounded intersection of finitely many closed half-spaces of $\mathbf{R}^n$.

This equivalence is a basic result in convexity theory. It is straightforward to see that it extends, as follows:

Definition 1.6. A ***rational polytope*** P is the convex hull of a finite set of rational points of $\mathbf{R}^n$; equivalently, it is a bounded intersection of finitely many closed half-spaces of $\mathbf{R}^n$, each of whose bounding hyperplanes is rational.

Definition 1.7. A ***mean representation*** of an abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ consists of a quadruple (n, R, P, S), in which

1. $R : \mathcal{I} \to \mathbf{R}^n$ is an ***input plot function*** locating each possible ballot as a point in space,
2. P is a subset of the ***representing polytope***, which is the convex hull of the (finite) set $\{R(j) : j \in \mathcal{I}\}$,
3. $S = \{r_a\}_{a\in\mathcal{O}}$ is a decomposition of P indexed by $\mathcal{O}$, and
4. S ***represents*** $\mathcal{V}$: for each $p \in (Z^+)^{\mathcal{I}}$, $\mathcal{F}(p) = \{a \in \mathcal{O} : \overline{p} \in r_a\}$, where $\overline{p}$ denotes the mean location $\frac{\sum_{j\in\mathcal{I}} p(j)R(j)}{N(p)}$ of all plotted votes cast.

A ***mean neat representation*** satisfies two additional requirements: P is equal to the representing polytope and S is ***neat***. A ***rational mean neat representation*** is a mean neat representation for which P's vertices are rational points, and S is **Q**-***neat***. An abstract anonymous voting system $\mathcal{V}$ is a ***mean neat*** if it has a mean neat representation, and is ***rational mean neat*** if it has a rational mean neat representation.

[1] For background on affine and convex geometry see Rockafellar [7] or Webster [8].

[2] A *rational point* is a point whose coordinates are all rational numbers.

Our characterization of rational mean neat voting includes other conditions based on the following notion from [1]:

Definition 1.8. If $S = \{r_a\}_{a\in\Delta}$ is a decomposition of a set $P \subseteq \mathbf{R}^n$, and A is a subset of $\mathbf{R}^n$, then the ***restriction*** of S to A is given by $S|A = \{r_a \cap A\}_{a\in\Delta}$. A closed decomposition S is ***regular*** if every restriction $S|L$ to a *line* of $\mathbf{R}^n$ is neat. Equivalently, for each line L, every region of $S|L$ is a closed interval of L,[3] and every two such regions $r \cap L$ and $u \cap L$ are equal, or disjoint, or overlap only at a single point which is an endpoint of each.

Definition 1.9. A closed decomposition S of a set $P \subseteq \mathbf{R}^n$ is ***thin convex*** if whenever x and z lie in some common region of S and y lies in the open line segment (x, z), the regions of S containing y as a member are exactly those that contain both x and z as members.

The following theorem from [1] plays a key role in our main proof:

Theorem 1.3. *Let S be a closed decomposition of a polytope. Then the following are equivalent:*

1. *S is neat.*
2. *S is regular.*
3. *S is thin convex.*

Convention 1.1 (*The $*$-Convention*). When we write a point $x \in \mathbf{R}^n$ as x^* we indicate that x is a rational point. When we write a set $X \subseteq \mathbf{R}^n$ as X^* we indicate that X consists entirely of rational points; alternately, if X is a set that may also contain irrational points, X^* denotes $\{x^* \in X | x^* \text{ is rational}\}$. A ***Q-line*** L^* is the set of rational points on a *rational* line L of $\mathbf{R}^n$; L^* is the same as the "$\mathbf{Q}$-affine span" $\{\alpha p^* + \beta q^* | \alpha, \beta \in Q \text{ with } \alpha + \beta = 1\}$ of any two distinct rational points p^* and q^* of L.

A non-empty **Q**-***closed interval*** $[u^*, v^*]^*$, or **Q**-***open interval*** $(u^*, v^*)^*$, of a **Q**-line L^* consists of all (rational) points on L^* between points u^* and v^* of L^*, including (respectively, excluding) the endpoints u^* and v^*; $\varnothing$ is considered to be both a **Q**-closed interval and a **Q**-open interval and $\{x^*\}$ to be a **Q**-closed interval. Note that the endpoints of either type of interval are *required to have rational coordinates*. More generally, a **Q** ***interval*** is a subset I of a **Q**-line L^* satisfying that whenever $x^*, z^* \in I$ with $y^* \in (x^*, z^*)^*$, $y^* \in I$.

Definition 1.10. A decomposition S^* of a subset P^* of $\mathbf{Q}^n$ is **Q**-***regular*** if for every **Q**-line L^*:

1. $r^* \cap L^*$ is a **Q**-closed interval of L^* for each region $r^* \in S^*$, and
2. $r^* \cap L^*$ and $t^* \cap L^*$ are identical, or disjoint, or their intersection is an endpoint of each, for each pair of regions $r^*, t^* \in S^*$.

We are now ready to restate the main result in greater detail, as follows:

Theorem 1.4 (*The Rational Mean Neat Characterization Theorem*). *Let $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ be an abstract anonymous voting system. Then the following are equivalent:*

(i) *$\mathcal{V}$ is consistent and connected.*
(ii) *The rational barycentric representation of $\mathcal{V}$ is* **Q**-*regular.*
(iii) *The extended barycentric representation of $\mathcal{V}$ is a rational mean neat representation.*
(iv) *$\mathcal{V}$ is a rational mean neat voting system.*

In Section 2 we show (iv) ⇒ (i), and in Section 3 we introduce the barycentric representations. The proof of (i) ⇒ (ii) is in Section 4 and of (ii) ⇒ (iii) is in Section 5; (iii) ⇒ (iv) is clear.

2. Rational mean neat voting systems are consistent and connected

The following lemma is a translation, into the context of abstract anonymous voting systems, of the statement that the mean satisfies the ***subcenter*** axiom. Subcenter (our term) is the principal ingredient in an attractive axiomatization of the mean (see p 514–515 of Gleason et al. [9]). We leave the easy proof to the reader.

Lemma 2.1 (*Voting Subcenter Lemma*). *Let p and q be anonymous profiles for an abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ and $R : \mathcal{I} \to \mathbf{R}^n$ be any plot function. Then*

$$\overline{(p+q)} = \left(\frac{N(p)}{N(p)+N(q)}\right)(\overline{p}) + \left(\frac{N(q)}{N(p)+N(q)}\right)(\overline{q}).$$

Note that $\overline{(p+q)}$ is expressed as a convex combination of $\overline{p}$ and $\overline{q}$ using rational convex coefficients, so if R takes values in $\mathbf{Q}^n$ then $\overline{(p+q)}$ is a rational point in $(\overline{p}, \overline{q})$.

[3] It is convenient here to classify one-point sets, and $\varnothing$, as closed intervals.

Lemma 2.2 (*(iv) ⇒ (i) in* Theorem 1.4). *Let* $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ *be a rational mean neat voting system, with associated representation* (n, R, P, S). *Then* $\mathcal{V}$ *is consistent and connected.*

Proof. To show consistency, let p and q be anonymous profiles with $\mathcal{F}(p) \cap \mathcal{F}(q) \neq \varnothing$. By Lemma 2.1, $\overline{(p+q)}$ lies on the line segment joining $\overline{p}$ and $\overline{q}$. By Theorem 1.3, S is thin convex, so as $\overline{p}$ and $\overline{q}$ belong to at least one common region of S, $\overline{(p+q)} \in r_a$ if and only if $\overline{p} \in r_a$ and $\overline{q} \in r_a$ for each $a \in O$, whence $\mathcal{F}(p) \cap \mathcal{F}(q) = \mathcal{F}(p+q)$, as desired. ■

Notice that we never used the *rational* part of the hypothesis: mean neat voting systems are consistent.

To see that $\mathcal{V}$ is connected, assume that $\mathcal{F}(p) \cap \mathcal{F}(q) = \varnothing$. Let L be the line through $\overline{p}$ and $\overline{q}$, and $S|L$ be the corresponding restriction of S. By Theorem 1.3, S is regular, so $P \cap L$ is a closed interval, equal to the union of finitely many closed intervals of the form $r_a \cap L$ that are ***proper*** (contain infinitely many points), each pair of which are disjoint, identical, or overlap only at a common endpoint. In particular, $\overline{p} \in r_a \cap L$ for one of these proper intervals. If we picture L with $\overline{p}$ to the left of $\overline{q}$, then the right endpoint x of $r_a \cap L$ lies strictly between $\overline{p}$ and $\overline{q}$, and x is thus the left endpoint of some proper interval $r_b \cap L$. As $x \in r_a \cap r_b$, if we can establish that $x = \overline{t}$ for some profile t of the form $cp + dq$, then $a \in \mathcal{F}(cp+dq) \cap \mathcal{F}(p)$, $\mathcal{F}(cp+dq) \neq \mathcal{F}(p)$, and connectedness follows. By assumption, there exists a rational hyperplane h neatly separating r_a and r_b. As x is the point of intersection of the rational line L and the rational hyperplane h, x is a rational point, and x is therefore a rational convex combination

$$x = \left(\frac{k}{m}\right)\overline{p} + \left(\frac{m-k}{m}\right)\overline{q} \tag{Ψ}$$

for some integers k and m with $k < m$. Let $c = kN(q)$ and $d = (m-k)N(p)$. We are done, then, if we establish the following:

Claim. $x = \overline{cp + dq}$.

Proof of claim. The definition of c and d imply $c(m-k)N(p) = dkN(q)$, so $cmN(p) = ckN(p) + dkN(q)$, whence

$$\frac{cN(p)}{cN(p) + dN(q)} = \frac{k}{m}, \quad \text{and} \quad \frac{dN(q)}{cN(p) + dN(q)} = \frac{m-k}{m}.$$

Substituting into (Ψ), we obtain

$$\begin{aligned} x &= \left[\frac{cN(p)}{cN(p)+dN(q)}\right](\overline{p}) + \left[\frac{dN(q)}{cN(p)+dN(q)}\right](\overline{q}) \\ &= \left[\frac{cN(p)}{cN(p)+dN(q)}\right](\overline{cp}) + \left[\frac{dN(q)}{cN(p)+dN(q)}\right](\overline{dq}) \\ &= \left[\frac{N(cp)}{N(cp)+N(dq)}\right](\overline{cp}) + \left[\frac{N(dq)}{N(cp)+N(dq)}\right](\overline{dq}) \\ &= \text{(by Lemma 2.1)}\ \overline{c \cdot p + d \cdot q}. \qquad ■ \end{aligned}$$

3. The two barycentric representations

In the interesting direction of the main theorem we construct, for any consistent and connected system $\mathcal{V}$, a rational mean neat representation. Here, we define the construction, in two stages. The *rational barycentric representation* (k, R, P^*, S^*), in which S^* decomposes a set P^* of rational points, is our preliminary version. The *extended barycentric representation* (n, R, P^{Cl}, S^{Cl}) is then obtained by taking closures.[4] To define (k, R, P^*, S^*) we need only assume that $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is homogeneous as an abstract anonymous voting system. Identify the set $\mathcal{I}$ with some proper initial segment $K = \{1, 2, \ldots, k\}$ of $\mathbf{N}$, and let our plot function $R : K \to \mathbf{R}^k$ be given by $R(j) = e_j$. Here, $e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 appears in the jth coordinate. The convex hull of the e_j is the image of the $k-1$ simplex Δ_{k-1} under the standard embedding into $\mathbf{R}^k$ that identifies Δ_{k-1} with the intersection of the first orthant of $\mathbf{R}^k$ and the hyperplane of $\mathbf{R}^k$ having equation $x_1 + x_2 + \cdots + x_k = 1$. We will refer to this image as $\mathbf{B}^{k-1+}$, and to the hyperplane containing it as $\mathbf{B}^{k-1}$. Thus $\mathbf{B}^{k-1+}$ consists of the points of $\mathbf{B}^{k-1}$ lying in the first orthant of $\mathbf{R}^k$. For the rest of the proof, any mention of coordinates for a point in $\mathbf{B}^{k-1}$ should be understood to refer to Cartesian coordinates in $\mathbf{R}^k$. These are the so-called ***barycentric*** coordinates, which always sum to 1.[5]

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[4] A number of voting rules have "natural" rational mean neat representations that seem more interesting than these barycentric representations, and are of lower dimension; see Zwicker [2].

[5] Suppose that we identify $\mathbf{R}^{k-1}$ with the hyperplane of points of $\mathbf{R}^k$ whose last coordinate is 0. The reader may be tempted to identify $\mathbf{B}^{k-1}$ with this copy of $\mathbf{R}^{k-1}$, and indeed there is a rigid motion of $\mathbf{R}^k$-space carrying $\mathbf{B}^{k-1}$ to $\mathbf{R}^{k-1}$, consisting of a rotation of $\mathbf{B}^{k-1}$ about the intersection of $\mathbf{B}^{k-1}$ with $\mathbf{R}^{k-1}$. However, we will be concerned with the issue of ***rational points*** (those with rational coordinates), and no rigid motion of $\mathbf{R}^k$ can carry points with rational (barycentric) coordinates in $\mathbf{B}^{k-1}$ to points with rational coordinates in $\mathbf{R}^{k-1}$. The ***projection*** of $\mathbf{B}^{k-1}$ onto $\mathbf{R}^{k-1}$ given by $(x_1, \ldots, x_{k-1}, x_k) \mapsto (x_1, \ldots, x_{k-1}, 0)$ does identify these two spaces in such a way that identifies rational points with rational points. Furthermore, while this projection is not a rigid motion (it changes angles), it is a bijective linear transform (when its domain is taken to be $\mathbf{B}^{k-1}$), and it thus preserves most of the notions of actual concern in the proof. But it does not preserve the notion of "normal vector" for a hyperplane. In other words, the reader who thinks of $\mathbf{B}^{k-1}$ coordinatized as a "copy" of $\mathbf{R}^{k-1}$ will probably not be lead astray, but as the issue is somewhat delicate, we have chosen to stick with the coordinates of $\mathbf{R}^k$.

Due to our particular choice of plot function, R's extension to the mean map $^- : \mathcal{I}^{Z+} \rightarrow \mathbf{Q}^k$ (where $\mathbf{Q}^k$ consists of the rational points of $\mathbf{R}^k$) has a particularly simple form to which we will assign a special notation: $\overline{p}$ is just the "normalized version" $p^\Delta = \frac{p}{N(p)}$ of the profile p. Each component of p^Δ (a barycentric coordinate) is equal to the fraction of voters who vote for the corresponding element of $\mathcal{I}$. The image P^* of $\mathcal{I}^{Z+}$ under Δ is now the same as the set of rational points of $\mathbf{B}^{k-1+}$. We will denote this set as $\mathbf{QB}^{k-1+} = \mathbf{Q}^k \cap \mathbf{B}^{k-1+}$, and denote the set of rational points of the hyperplane $\mathbf{B}^{k-1}$ as $\mathbf{QB}^{k-1} = \mathbf{Q}^k \cap \mathbf{B}^{k-1}$.

Note that two profiles p and q satisfy $p^\Delta = q^\Delta$ if and only if some profile r is a common (integer) multiple, with $hp = gq = r$ for some integers h and g. Thus, each point in $\mathbf{QB}^{k-1+}$ corresponds to an *equivalence class* of profiles under a simple equivalence relation. Homogeneity is now equivalent to the statement that $\mathcal{F}$ is constant on each such equivalence class.

Definition 3.1. For any homogeneous abstract anonymous system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$, the ***rational extension*** $\mathcal{F}^*$ of $\mathcal{F}$ to the points x^* of $\mathbf{QB}^{k-1+}$ is given by $\mathcal{F}^*(x^*) = \mathcal{F}(p^\Delta)$ for any profile p satisfying $x^* = p^\Delta$.

Thus, it is homogeneity that makes this extension well defined, allowing even the most minimal type of geometric representation for a voting system. We are now ready for:

Definition 3.2. The ***rational barycentric representation*** (k, R, P^*, S^*) of any homogeneous abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is defined by:

- $R : \mathcal{I} \rightarrow \mathbf{R}^k$ by $R(j) = e_j$,
- P^* is the set $\mathbf{QB}^{k-1+}$,
- Each rational region r_a^* is defined by $x^* \in r_a^*$ if and only if a $\in \mathcal{F}^*(x^*)$, and
- $S^* = \{r_a^*\}_{a \in \mathcal{O}}$.

Definition 3.3. The ***extended barycentric representation*** (k, R, P^{Cl}, S^{Cl}) of any homogeneous abstract anonymous voting system $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is defined by:

- $R : \mathcal{I} \rightarrow \mathbf{R}^k$ by $R(j) = e_j$,
- P^{Cl} is the closure $\mathbf{B}^{k-1+}$ of $P^* = \mathbf{QB}^{k-1+}$,
- Each region r_a^{*Cl} is the closure of the corresponding region r_a^* of S^*, and
- $S^{Cl} = \{r_a^{*Cl}\}_{a \in \mathcal{O}}$.

4. Q-regularity from consistency and connectedness

Lemma 4.1 (*(i) ⇒ (ii) in* Theorem 1.4). *Let $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ be a homogeneous abstract anonymous voting system, and (k, R, P^*, S^*) be the rational barycentric representation. If $\mathcal{V}$ is consistent and connected then S^* is a $\mathbf{Q}$-regular decomposition.*

Proof. Assume that $\mathcal{V}$ is consistent and connected. Clearly, (k, R, P^*, S^*) represents $\mathcal{V}$. If L is any rational line meeting $\mathbf{BQ}^{k-1+}$, then $L \cap \mathbf{BQ}^{k-1+}$ is a $\mathbf{Q}$-closed interval $[p^*, q^*]^*$. It follows easily from the following three claims that $S^*|L^*$ satisfies the requirements of Definition 1.10.

Claim 1. $\mathcal{F}^*$ satisfies ***rational consistency***: for any three points x^*, y^*, and z^* of $\mathbf{BQ}^{k-1+}$ with $y^* \in (x^*, z^*)^*$, if $\mathcal{F}^*(x^*) \cap \mathcal{F}^*(z^*) \neq \varnothing$, then $\mathcal{F}^*(y^*) = \mathcal{F}^*(x^*) \cap \mathcal{F}^*(z^*)$.

Claim 2. Every region r^* of S^* meets $[p^*, q^*]^*$ in a $\mathbf{Q}$-interval, any two such intervals that share more than one point are equal, and any two that share exactly one point w^* each have w^* as an endpoint.

Claim 3. Each $\mathbf{Q}$-interval in $S^*|L^*$ is $\mathbf{Q}$-closed.

Proof of claim 1. Assume that $x^*, z^* \in \mathbf{BQ}^{k-1+}$ with $y^* \in (x^*, z^*)^*$. Choose profiles p_x and p_z for which $x^* = p_x^\Delta$ and $z^* = p_z^\Delta$. It now suffices to show that there exist positive integers A and C such that $y^* = (Ap_x + Cp_z)^\Delta$, for then we have

$$\begin{aligned}\mathcal{F}^*(x^*) \cap \mathcal{F}^*(z^*) \neq \varnothing &\Rightarrow \mathcal{F}(p_x) \cap \mathcal{F}(p_z) \neq \varnothing \\ &\Rightarrow \mathcal{F}(Ap_x) \cap \mathcal{F}(Cp_z) \neq \varnothing \\ &\Rightarrow \mathcal{F}(Ap_x + Cp_z) = \mathcal{F}(Ap_x) \cap \mathcal{F}(Cp_z) \\ &\Rightarrow \mathcal{F}^*(y^*) = \mathcal{F}^*(x^*) \cap \mathcal{F}^*(z^*),\end{aligned}$$

as desired. As $y^* \in (x^*, z^*)^*$ is rational, to find the integers A and B we can choose positive integers $a < b$ for which $y^* = \frac{a}{b}x^* + \frac{b-a}{b}z^*$. Now if we set $A = aN(p_z)$ and $C = (b-a)N(p_x)$, we obtain (via the *subcenter formula* from Lemma 2.1):

$$\begin{aligned}\left(Ap_x + Bp_y\right)^\Delta &= \left(\frac{N(Ap_x)}{N(Ap_x) + N(Bp_z)}\right)(Ap_x)^\Delta + \left(\frac{N(Bp_y)}{N(Ap_x) + N(Bp_z)}\right)(Bp_z)^\Delta \\ &= \left(\frac{aN(p_z)N(p_x)}{aN(p_z)N(p_x) + (b-a)N(p_z)N(p_x)}\right)(p_x)^\Delta + \left(\frac{(b-a)N(p_z)N(p_x)}{aN(p_z)N(p_x) + (b-a)N(p_z)N(p_x)}\right)(p_z)^\Delta \\ &= \frac{a}{b}x^* + \frac{b-a}{b}z^* = y^*, \quad \text{as desired.}\end{aligned}$$

Proof of claim 2. Consider any three points x^*, y^*, and z^* of $\mathbf{BQ}^{k-1+}$ with $y^* \in (x^*, z^*)^*$. From rational consistency, it follows that

- if x^* and z^* are both members of a region r^* of S^*, then $y^* \in r^*$, and
- if, additionally, $y^* \in v^* \in S^*$ for some $v^* \neq r^*$, then $x^* \in v^*$ and $z^* \in v^*$.

The first bullet implies that every region r^* of S^* meets $[p^*, q^*]^*$ in a $\mathbf{Q}$-interval, and the second establishes the rest of the claim.

Proof of claim 3. As degenerate (one-point) intervals of $S^*|L^*$ are clearly $\mathbf{Q}$-closed, we consider only non-degenerate ones. Consider two distinct such intervals $r_a^*, r_b^* \in S^*|L^*$ that are adjacent on L^* (not separated by any intermediate, non-degenerate interval). We now construct a common endpoint. Choose $x^* \in r_a^* - r_b^*$ and $z^* \in r_b^* - r_a^*$, as well as profiles with $p_x^\Delta = x^*$ and $p_z^\Delta = z^*$. Then $\mathcal{F}(p_x) \cap \mathcal{F}(p_y) = \varnothing$, or we contradict consistency. By connectedness we may choose positive integers c and d with the profile $p_y = cp_x + dp_z$ satisfying $\mathcal{F}(p_y) \neq \mathcal{F}(p_x)$ and $\mathcal{F}(p_y) \cap \mathcal{F}(p_x) \neq \varnothing$. Under these circumstances, consistency forces $\mathcal{F}(p_x)$ to be a proper subset of $\mathcal{F}(p_y)$. If we let y^* denote p_y^Δ, then rational consistency forces y^* to be an endpoint of r_a^*, and to lie weakly between any pair of points chosen one from r_a^* and the other from r_b^*. If we apply the same connectedness argument while reversing the roles of p_x and p_z we obtain a similar endpoint w^* of r_b^*, and it is straightforward to now show that $y^* = w^*$. To establish claim 3, it remains only to produce two additional endpoints, where L^* intersects the walls of $\mathbf{BQ}^{k-1+}$. But these walls are rational affine subspaces defined by linear systems with integer coefficients and constants, so they intersect L^* at rational points. ■

5. Taking closures

Our goal is to prove (ii) ⇒ (iii) in Theorem 1.4.

5.0. Preliminaries

The following is a convenient package that carries the essential part of our assumption that the rational barycentric representation (R, P^*, S^*) of $\mathcal{V} = (\mathcal{I}, \mathcal{O}, \mathcal{F})$ is $\mathbf{Q}$-regular:

Definition 5.0.1. A ***standard situation*** is a vector $D = (n, P, P^*, S^*, S^{Cl})$ consisting of a non-negative integer n, a rational polytope $P \subseteq \mathbf{R}^n$, the set P^* of rational points of P, a $\mathbf{Q}$-regular decomposition $S^* = \{r_a^*\}_{a\in\Delta}$ of P^*, and $S^{Cl} = \{r_a^{*Cl}\}_{a\in\Delta}$.

Assuming that $D = (n, P, P^*, S^*, S^{Cl})$ is a standard situation and $\mathcal{V}$ is an abstract anonymous voting system, we show, in Sections 5.1–5.3 respectively, that:

(1) if S^* represents $\mathcal{V}$, then S^{Cl} represents $\mathcal{V}$.
(2) S^{Cl} is "extended $\mathbf{Q}$-regular" (defined below).
(3) S^{Cl} is $\mathbf{Q}$-neat.

It follows immediately that for consistent and connected systems the extended barycentric representation is a mean neat representation.

Definition 5.0.2. A decomposition S of a set $P \subseteq \mathbf{R}^n$ is ***extended* Q*-regular*** if it is regular and for every rational line L the regions of $S|L$ have rational endpoints.

Basic notions 5.0.3. Our proofs will use the following notions and notations of convexity theory: an ***affine combination*** of finitely many points of $\mathbf{R}^n$ with corresponding ***affine coefficients***; a ***convex combination*** of finitely many points of $\mathbf{R}^n$ with corresponding ***convex coefficients***; an ***affine subspace*** of $\mathbf{R}^n$; the ***affine span*** **Aff(*X*)** and ***convex hull C(X)*** of a set $X \subseteq \mathbf{R}^n$; an ***affinely independent*** set $X \subseteq \mathbf{R}^n$; an ***affine basis*** for an affine subspace of $\mathbf{R}^n$; the ***affine dimension***, **dim *(X)*** of a set $X \subseteq \mathbf{R}^n$, which is the same as dim[Aff(X)]; the ***relative open ball*** $B_\delta^A(p) = \{q \in A \,|\, \|p - q\| < \delta\}$ for a set A $\subseteq \mathbf{R}^n$ and corresponding ***relative*** (or ***induced***) ***topology*** on A; and the ***relative interior r.i.(X)*** of a set $X \subseteq \mathbf{R}^n$. We will also use ***r.i.*** $_A$ ***(X)*** to denote the interior of X as calculated by the relative topology on A; thus, $r.i.(X) = r.i._{\mathrm{Aff}(X)}(X)$.

Lemma 5.0.1 (*Rational Convexity of Regions and Subregions*). *If* $D = (n, P, P^*, S^*, S^{Cl})$ *is a standard situation,* $\{u_1^*, \ldots, u_j^*\} \subseteq S^*$ *with* $j \geq 1$*, and* $U^* = \bigcap_{i=1}^{j} u_i^*$ *then* $C(U^*)^* \subseteq U^*$.

Proof. For $y^* \in C(U^*)^*$, choose $\{x_1^*, \ldots, x_m^*\} \subseteq U^*$ with $y^* \in C(\{x_1^*, \ldots, x_m^*\})$ and m minimal. Then there are unique convex coefficients c_i for which $y^* = \sum_{i=1}^m c_i x_i^*$ and it follows that the c_i are rational numbers. Using $\mathbf{Q}$-regularity it is straightforward to prove by induction on m that $y^* = \sum_{i=1}^m c_i x_i^* \in U^*$. ■

Lemma 5.0.2. *Let* $D = (n, P, P^*, S^*, S^{Cl})$ *be a standard situation, and* u^* *be a region of* S^**. Then* $u^{*\,Cl}$ *is convex.*

Proof. Straightforward. ■

Lemma 5.0.3. *If $X \subseteq \mathbf{R}^n$ with $|X| > 1$, then $r.i.(C(X))$ is dense in $C(X)$.*

Proof. Given $x_1 \in C(X)$, expand $\{x_1\}$ to an affine basis $X' = \{x_1, x_2, \ldots x_{k+1}\} \subseteq C(X)$ for $\mathrm{Aff}(X) = \mathrm{Aff}(C(X))$. Each convex combination $y = \sum_{i=1}^{k+1} c_i x_i$ using strictly positive convex coefficients $c_i > 0$ satisfies $y \in \mathrm{r.i.}(C(X))$, and it is clear that we can construct such points y arbitrarily close to x_1. ∎

Lemma 5.0.4. *If $X \subseteq \mathbf{R}^n$ and $\mathrm{Aff}(X)$ is a rational affine subspace of $\mathbf{R}^n$ then $(r.i.(C(X)))^*$ is dense in $C(X)$.*

Proof. Rational points are dense in $\mathrm{Aff}(X)$, so via Lemma 5.0.3 we can find a rational point $w^* \in \mathrm{r.i.}(C(X))$ as close as we desire to any $x \in C(X)$. ∎

Lemma 5.0.5. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation, with $u^*, t^* \in S^*$. Assume that $P \subseteq \mathrm{Aff}(u^*)$ and $u^* - t^* \neq \varnothing$. Then $\dim(u^* \cap t^*) < \mathrm{Aff}(u^*)$.*

Proof. If not, let $X^* = \{x_1, \ldots, x_{j+1}\} \subseteq u^* \cap t^*$ be an affine basis for $\mathrm{Aff}(u^*)$. As $X^* \subseteq u^* \cap t^*$, $C(X^*)^* \subseteq u^* \cap t^*$. Choose rational points $b^* \in r.i._{\mathrm{Aff}(u^*)}(C(X^*))$ and $d^* \in u^* - t^*$. Clearly, $[b^*, d^*]^* \cap t^* \cap u^* = [b^*, c^*]$, with $c^* \in (b^*, d^*)$. Thus $b^*, d^* \in u^*$, $c^* \in t^*$, and $d^* \notin t^*$, contradicting $\mathbf{Q}$-regularity. ∎

5.1. Representation

The definition of the rational barycentric representation guarantees that it *represents* $\mathcal{V}$: for every $p \in (\mathbf{Z}^+)^{\mathcal{I}}$ and $a \in O$, $\bar{p} \in {r^*}_a \Leftrightarrow a \in \mathcal{F}(p)$. Here we show that the same holds when $(r_a^*)^{Cl}$ replaces r_a^*.

Proposition 5.1.1 (*"No New Rational Points"*). *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation. Then $S^* = S^{Cl}|P^*$.*

Proof. Clearly, each region r^* of S^* is contained in the corresponding region $r^{*Cl} \in S^{Cl}$, whence $r^* \subseteq (r^{*Cl} \cap P^*)$. The reverse containment establishes the proposition, and follows immediately from the following two claims.

Claim 1. Let y^* be a rational point in $\mathrm{r.i.}(r^{*Cl})$. Then $y^* \in r^*$.

Proof of claim 1. Choose $\delta > 0$ with $B_\delta^{\mathrm{Aff}(r^*)}(y^*) \subseteq r^{*Cl}$. Choose points $a_1, \ldots, a_k$ in $B_\delta^{\mathrm{Aff}(r^*)}(y^*)$ so that

- $\{a_1, \ldots, a_k\}$ is an affine basis for A, and
- $y^* \in \mathrm{r.i.}(C(\{a_1, \ldots, a_k\}))$.

As each a_j lies in r^{*Cl}, we can replace each a_j with a point $a_j^* \in r^*$ that is as close to a_j as we like. By choosing them close enough, it follows that

- $\{a_1^*, \ldots, a_k^*\}$ is an affine basis for A, and
- $y^* \in \mathrm{r.i.}(C(\{a_1^*, \ldots, a_k^*\}))$.

Now as $y^* \in C(r^*)^*$, $y \in r^*$ by Lemma 5.0.1.

Claim 2. Let y^* be a rational point in r^{*Cl}. Then $y^* \in r^*$.

Proof of claim 2. Construct an affine basis $\{y^*, a_2^*, \ldots, a_k^*\}$ for A with $\{a_2^*, \ldots, a_k^*\} \subseteq r^*$. For any point z of A, let C_z denote the convex hull of $\{z, a_2^*, \ldots, a_k^*\}$.

Subclaim 2.1. Any member z^* of r^* sufficiently close to y^* satisfies

- $\{z^*, a_2^*, \ldots, a_k^*\}$ is an affine basis of A, and
- $\mathrm{r.i.}(C_{z^*}) \subseteq \mathrm{r.i.}(r^{*Cl})$.

We leave the proof of this subclaim to the reader.

Now let $w^* = \sum_{i=2}^{k} \left(\frac{1}{k-1}\right) a_i^*$ so that w^* is a convex combination of $a_2^*, \ldots, a_k^*$ using strictly positive convex coefficients, and consider the line L through w^* and y^*.

Subclaim 2.2. The open interval (w^*, y^*) of L lies entirely inside $\mathrm{r.i.}(r^{*Cl})$.

Proof of subclaim. As $y^* \in r^{*Cl}$, we can find points z^* of r^* as close as desired to y^*. For each such z^*, there is a point v_z^* such that $\mathrm{r.i.}(C_{z^*}) \cap (w^*, y^*) = (w^*, v_z^*)$, and so by subclaim 2.1, $(w^*, v_z^*) \subseteq \mathrm{r.i.}(r^{*Cl})$. By choosing z^* sufficiently close to y^* we can force v_z^* to be as close as we wish to y^*, so $(w^*, y^*) \subseteq \mathrm{r.i.}(r^{*Cl})$.

Thus, by claim 1 $(w^*, y^*)^* \subseteq r^*$. By $\mathbf{Q}$-regularity, $L \cap r^*$ is a $\mathbf{Q}$-closed interval $[a^*, b^*]^* \supseteq (w^*, y^*)^*$. So $y^* \in [a^*, b^*]^* \subseteq r^*$, and $y^* \in r^*$ as desired. ∎

Lemma 5.1.2. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation, u^* be a region of S^*, and A be any rational affine subspace such that $A = \mathrm{Aff}(u^{*Cl} \cap A)$. Then $u^* \cap$ r.i. $(u^{*Cl} \cap A)$ is dense in $u^{*Cl} \cap A$.*

Proof. By Lemma 5.0.2, $u^{*\,Cl} \cap A = C(u^{*Cl} \cap A)$. Also, $\mathrm{Aff}[(u^*)^{Cl} \cap A] = A$ so $\mathrm{Aff}[(u^*)^{Cl} \cap A]$ is a rational affine subspace. Hence by Lemma 5.0.4, $[\mathrm{r.i.}(u^{*Cl} \cap A)]^*$ is dense in $u^{*Cl} \cap A$. But by Proposition 5.1.1, $[\mathrm{r.i.}(u^{*Cl} \cap A)]^* \subseteq u^*$. ∎

Corollary 5.1.3. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation, and u^* be a region of S^*. Then $u^* \cap$ r.i. (u^{*Cl}) is dense in u^{*Cl}.*

Proof. Take A to be $\mathrm{Aff}(u^{*Cl}) = \mathrm{Aff}(u^*)$ in Lemma 5.1.2. ■

Proposition 5.1.4. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation, u^* be a region of S^*, and A be a rational affine subspace of $\mathbf{R}^n$. Then $(u^* \cap A)^{Cl} = (u^*)^{Cl} \cap A$.*

Proof. Suppose that (P^*, S^*, u^*, A) represents a failure of this lemma. Then both $(P^* \cap \mathrm{Aff}(u^*), S^*|\mathrm{Aff}(u^*), u^*, A)$, and $(P^*, S^*, u^*, A \cap \mathrm{Aff}(u^*))$ also represent failures, so any failure triggers a ***smooth*** failure: one satisfying $\mathrm{Aff}(u^*) = \mathrm{Aff}(P^*)$ and $A \subseteq \mathrm{Aff}(u^*)$. Clearly, any smooth failure satisfies $\dim(A) < \dim(u^*)$.

If (P^*, S^*, u^*, A) is a smooth failure satisfying $\dim(A) < \dim(u^*) - 1$, let B be any rational affine subspace with $A \subseteq B \subseteq \mathrm{Aff}(u^*)$ and $\dim(A) < \dim(B) < \dim(u^*)$. Then either (P^*, S^*, u^*, B) or $(P^* \cap B, S^*|B, u^* \cap B, A)$ must be a (smooth) failure, else (P^*, S^*, u^*, A) is not itself a failure. So any smooth failure triggers a smooth and ***sequential*** failure: one satisfying $\dim(u^*) = 1 + \dim(A)$.

Now assume that $G = (P^*, S^*, u^*, A)$ is a smooth and sequential failure that is ***minimal***: $k = \dim(A)$ is as small as possible among such failures. We will obtain a contradiction by constructing a failure with a lower k. Note that $k > 0$ by Proposition 5.1.1. Clearly $(u^*)^{Cl} \cap A \supseteq (u^* \cap A)^{Cl}$, so choose any point $y \in (u^*)^{Cl} \cap A - (u^* \cap A)^{Cl}$.

Claim. $\dim[(u^*)^{Cl} \cap A] < k$.

Proof of claim. If not, then $A = \mathrm{Aff}[(u^*)^{Cl} \cap A]$, so by Lemma 5.1.2, $u^* \cap \mathrm{r.i.}(u^{*Cl} \cap A)$ is dense in $u^{*Cl} \cap A$. Thus $y \in (u^* \cap A)^{Cl}$, contradicting our assumption.

As $y \in (u^*)^{Cl} \cap A$, we know by Corollary 5.1.3 that there are points x^* in $u^* \cap \mathrm{r.i.}(u^{*Cl})$ arbitrarily close to y, but by assumption this is false if we additionally require $x^* \in A$. Let x^* be any element of $u^* \cap \mathrm{r.i.}(u^{*Cl}) - A$.

Our failure was sequential, so $\mathrm{Aff}(u^*) = \mathrm{Aff}(A \cup \{x^*\})$. Thus, there are points of $\mathrm{Aff}(A \cup \{x^*\}) \cap u^* \cap \mathrm{r.i.}(u^{*Cl})$ arbitrarily close to y. For each natural number $n > 0$, choose a (rational) point $w_n^* \in \mathrm{Aff}(A \cup \{x^*\}) \cap u^* \cap \mathrm{r.i.}(u^{*Cl})$ with $\|w_n^* - y\| < 1/n$, and choose a rational point v_n^* on A with $\|v_n^* - y\| < 1/n$. For all sufficiently large n, $v_n^* \notin u^*$ and we consider the rational line segment $[v_n^*, w_n^*]^*$. As $w_n^* \in u^*$, $u^* \cap [v_n^*, w_n^*]^* = [p_n^*, w_n^*]^*$ for some point p_n^* lying strictly between v_n^* and w_n^*. Note that $\|p_n^* - y\| < 1/n$. As P^* contains $[v_n^*, p_n^*]^*$, by $\mathbf{Q}$-regularity there exists some region t_n^* of S^* such that $[v_n^*, w_n^*]^* \cap t_n^* = [q_n^*, p_n^*]^*$ where $q_n^* \in [v_n^*, p_n^*)^*$. Thus $w_n^* \notin t_n^*$.

Via the pigeonhole principle choose a region t^* of S^* with $t^* = t_n^*$ for each integer $n \in K$, where K is infinite. Let $V = \mathrm{Aff}(\{p_n^* | n \in K\})$, so that $\mathcal{V}$ contains points of $t^* \cap u^*$ arbitrarily close to y.

By Lemma 5.0.5, $\dim(V) < k + 1$. The rational affine subspace $W = A \cap V$ satisfies $\dim(W) < k$, and $y \in (u^* \cap V)^{Cl} \cap W - (u^* \cap V \cap W)^{Cl}$, so $G' = (P^* \cap V, S^*|V, u^* \cap V, W)$ is a smooth failure with $\dim(W) < k$. The process by which G' triggers a smooth sequential failure G'' does not increase the dimension of the fourth coordinate of the failure, so the dimension of the fourth coordinate of G'' is $<k$, contradicting G's minimality. ■

5.2. Extended $\mathbf{Q}$-regularity

Proposition 5.2.1. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation. Then S^{Cl} is regular.*

Corollary 5.2.2. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation. Then S^{Cl} is extended $\mathbf{Q}$-regular.*

Proof of Corollary 5.2.2. Via Proposition 5.1.1, regularity implies extended $\mathbf{Q}$-regularity. ■

Proof of Proposition 5.2.1. A ***failure*** of Proposition 5.2.1 is a tuple $H = (n, P, P^*, S^*, S^{Cl}, r^*, u^*, x, y, z, u^*, L)$ in which $D = (n, P, P^*, S^*, S^{Cl})$ is a standard situation; $r^*, u^* \in S^*$; $x, z \in r^{*Cl}$; L is the line through x and z; $y \in (x, z)$; and $y \in u^{*Cl}$ while x and z are not both members of u^{*Cl}. Given Definition 5.0.2, $\mathbf{Q}$-regularity follows from showing that no such failures exist. A failure H is ***level*** if $P^* \subseteq \mathrm{Aff}(r^{*Cl}) = \mathrm{Aff}(r^*)$ and the ***dimension*** of a level failure is the common value of $\dim(P^*) = dim(\mathrm{Aff}(r^{*Cl}))$. If $H = (n, P, P^*, S^*, S^{Cl}, r^*, u^*, x, y, z, u^*, L)$ is a failure then $H' = (n, P \cap \mathrm{Aff}(r^*), P^* \cap \mathrm{Aff}(r^*), S^*|\mathrm{Aff}(r^*), S^{Cl}|\mathrm{Aff}(r^*), r^*, u^* \cap \mathrm{Aff}(r^*), x, y, z, L)$ is a level failure, so it suffices to rule out level failures.

By way of contradiction, let H be level failure that is ***minimal***: its dimension k is the smallest possible among level failures. We will first rule out $k = 1$, and then show that if $k > 1$ then H triggers a level failure H'' with $\dim(H'') < \dim(H)$.

Assume that $k = 1$. Then P^* is a closed $\mathbf{Q}$-segment $[a^*, b^*]^*$ of a rational line M. From $\mathbf{Q}$-regularity, the regions r^* of S^* are $\mathbf{Q}$-closed intervals $[c^*, d^*]^*$ and any two distinct regions that intersect have, as their sole point of intersection, a (rational) endpoint of each. As the rational points are dense in M, $P = [a^*, b^*]$, and each $r^{*Cl} \in S^{Cl}$ is a closed segment $[c^*, d^*]$ of M with rational endpoints. Thus every two distinct regions of S^{Cl} that intersect have, as their sole point of intersection, a (rational) endpoint of each. Finally, $L = M$ as they share the points $x \neq z$, so H is not a failure.

Assume that $k > 1$.

Case 1. Assume that for each $\varepsilon > 0$, there exists a $z^\circ \in \mathrm{r.i.}(r^{*Cl}) \cap (y, z)$ with $\|z^o - z\| < \frac{\varepsilon}{2}$. Then for each such ε, choose δ in $\left(0, \frac{\varepsilon}{2}\right)$ with $B_\delta^{\mathrm{Aff}(r^*)}(z^\circ) \subseteq r^{*Cl}$. Choose $\eta > 0$ with

$$\eta < \delta \left(\frac{\frac{1}{2}\|x - y\|}{\|z - y\| + \frac{1}{2}\|x - y\|} \right).$$

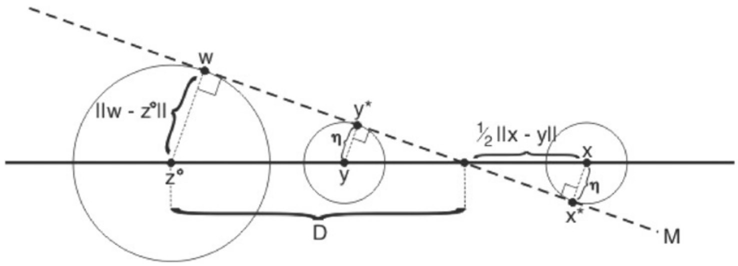

Fig. 1. Let $D = \|z^\circ - y\| + \frac{1}{2}\|x - y\|$. Then, in the limiting worst case shown, $\eta = \|x - x^\circ\| = \|y - y^\circ\|$, and $\frac{\|w-z^\circ\|}{D} = \frac{\eta}{\frac{1}{2}\|x-y\|} < \frac{\delta}{D}$, so $\|w - z^\circ\| < \delta$.

Choose $y^* \in u^*$ with $\|y^* - y\| < \eta$ and $x^* \in r^*$ with $\|x^* - x\| < \eta$, and let M be the (rational) line through y^* and x^*. By proportional triangles (see Fig. 1), the closest point w of M to z° is in $B_\delta^{\mathrm{Aff}(r^*)}(z^\circ)$. Thus M contains a rational point $z^* \in r^* \cap B_\delta^{\mathrm{Aff}(r^*)}(z^\circ)$.

By **Q**-rationality, as $x^*, z^* \in r^*$ and $y^* \in u^* \cap (x^*, z^*)$, both x^* and z^* lie in u^*. As $\|z - z^o\| < \frac{\varepsilon}{2}$ and $\|z^o - z^*\| < \frac{\varepsilon}{2}$, $\|z - z^*\| < \varepsilon$. As both x and z are arbitrarily close to points in u^*, both x and z are in u^{*Cl}, contradicting our choice of H as a failure.

As P^* is a rational polytope, we can list rational hyperplanes $h_1, \ldots, h_m$ of $\mathbf{R}^n$ such that P is the intersection of $\mathrm{Aff}(r^*)$ with m closed half-spaces corresponding to the h_i, and such that m is minimal. Each h_i

- contains $\mathrm{Aff}(r^*)$, or
- is disjoint from $\mathrm{Aff}(r^*)$, or
- intersects $\mathrm{Aff}(r^*)$ in a rational affine subspace g_i of dimension $k - 1$ or less,

but each h_i on our list is of the last type, or it could safely have been struck from the list, so we obtain a new list $T = g_1, \ldots, g_m$ of these subspaces.

Choosing an $\varepsilon > 0$ for which no point z° (with the Case 1 property) exists, we obtain a non-empty interval (z, z') of L contained in $r^{*Cl} - \mathrm{r.i.}(r^{*Cl})$. For each w in this interval and each positive integer n, choose rational points $w_n^* \in B_{\frac{1}{n}}^{\mathrm{Aff}(r^*)}(w) - r^*$ and (using Corollary 5.1.3) $v_n^* \in B_{\frac{1}{n}}^{\mathrm{Aff}(r^*)}(w) \cap r^* \cap \mathrm{r.i.}(r^{*Cl})$; note that $[w_n^*, v_n^*] \subseteq B_{\frac{1}{n}}^{\mathrm{Aff}(r^*)}(w)$. If $w_n^* \notin P^*$, choose t_n^* to be a subspace $g_i \in T$ that separates v_n^* from w_n^* in $\mathrm{Aff}(r^*)$, with $\{p_n^*\} = [v_n^*, w_n^*] \cap g_i$. If $w_n^* \in P^*$, $[v_n^*, w_n^*] \cap r^* = [v_n^*, p_n^*]^*$ for some $p_n^* \in (v_n^*, w_n^*) \cap r^*$. Choose $t_n^* \in S^*$ with $p_n^* \in r^* \cap t_n^*$ and $v_n^* \notin t_n^*$. As there are finitely many possible choices for t_n^*, choose some value $t(w)$ that is equal to t_n^* for infinitely many n. If $t(w)$ is one of the rational subspaces g_i, then $w \in g_i$, and if $t(w) = t^* \in S^*$, then $w \in \mathrm{Aff}(r^* \cap t^*)$. As $t(w)$ takes on only finitely many possible values, there exist $w_1 \neq w_2$ in (z, z') with $t(w_1) = t(w_2) = t_0$.

If $t_0 = g_i$ then $L \subseteq g_i$, $dim(g_i) < dim(\mathrm{Aff}(r^*))$, and g_i serves as the desired affine subspace A of $\mathrm{Aff}(r^*)$. If $t_0 = t^*$ then $L \subseteq \mathrm{Aff}(r^* \cap t^*)$. By Lemma 5.0.5, $\dim(r^* \cap t^*) < dim(r^*)$, with any of the points v_n^* witnessing $r^* - t^* \neq \varnothing$. So $\mathrm{Aff}(r^* \cap t^*)$ serves as the desired affine subspace A of $\mathrm{Aff}(r^*)$. ■

5.3. Neat separation via rational hyperplanes

From Corollary 5.2.2, we know that in any standard situation, S^{Cl} is extended **Q**-regular.
From S^{Cl}'s regularity the first three claims of Lemma 2.2 of [1] then immediately yield the following:

Corollary 5.3.1. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation. Then the relative interiors of the regions of S^{Cl} are pairwise disjoint, every pair of distinct regions of S^{Cl} can be properly separated by a hyperplane, and each region of S^{Cl} is a polytope.*

The fourth claim of Lemma 2.2 of [1] then shows that S^{Cl} is neat. Here we modify that proof to show, from extended **Q**-regularity, that S^{Cl} is **Q**-neat.

Definition 5.3.1. A set $F \subseteq r$ is a ***face*** of the polytope r if for every $x, z \in r$ and $y \in (x, z) \cap F$, $x \in F$ and $z \in F$.

Definition 5.3.2. A ***facet*** of a dimension-j polytope r is a face of r with dimension $j - 1$.

Definition 5.3.3. The ***algebraic difference*** $r -_{\mathrm{ALG}} u$ of two subsets of $\mathbf{R}^n$ is the set $\{x - y | x \in r \text{ and } y \in u\}$ of all algebraic differences of their respective elements.

Lemma 5.3.2. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation, $r^* \in S$, and F be a facet of r^{*Cl}. Then $\mathrm{Aff}(F)$ is rational.*

Proof. As $F = [\text{r.i.}(F)]^{Cl}$, it suffices to show that there are points of F^* arbitrarily close to any point $y \in \text{r.i.}(F)$. Let $\varepsilon > 0$ be arbitrary, and without loss of generality assume that ε is small enough so that no point in $B_\varepsilon(y)$ lies on any face of r other than F. Now choose rational points z^* and w^* in $B_\varepsilon^{\text{Aff}(r^*)}(y)$ with $z^* \in \text{r.i.}(r^{*Cl})$ and $w^* \notin r^{*Cl}$. Then the entire line segment $[z, w]$ lies in $B_\varepsilon^{\text{Aff}(r^*)}(y)$ with one endpoint in r^{*Cl}'s interior. It follows from extended **Q**-regularity that $r^{*Cl} \cap [z^*, w^*] = [z^*, v^*]$ with v^* rational. Now v^* is the desired rational point in $F \cap B_\varepsilon(y)$. ■

Lemma 5.3.3. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation and $r^* \in S^*$. Then the vertices of the polytope r^{*Cl} are rational.*

Proof. Each vertex v is an intersection of the finitely many rational affine subspaces $\text{Aff}(F)$ for the facets F containing v. So v is the unique solution of a system of linear equations, each equation having integer coefficients and constants. ■

Lemma 5.3.4. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation and $r^*, u^* \in S^*$. Then $r^{*Cl} -_{\text{ALG}} u^{*Cl}$ is a polytope with rational vertices.*

Proof. Each extreme point of $r^{*Cl} -_{\text{ALG}} u^{*Cl}$ is the difference $x_r - x_u$ of extreme points of r^{*Cl} and u^{*Cl} respectively. But the extreme points of r^{*Cl} (or of u^{*Cl}) are its finitely many rational vertices. Thus $r^{*Cl} -_{\text{ALG}} u^{*Cl}$ has finitely many extreme points, each of which is rational. By the Krein–Milman theorem, as $r^{*Cl} -_{\text{ALG}} u^{*Cl}$ is bounded it is the convex hull of its extreme points, each of which is a rational differences $x_r^* - x_u^*$. ■

Lemma 5.3.5. *Let h and A be rational affine subspaces with $h \subseteq A$ and $\dim(A) = \dim(h) + 1$. Then there is a rational hyperplane h' of $\mathbf{R}^n$ with $A \cap h' = h$.*

Proof. Choose any rational point v^* on $A - h$. Let w^* be the orthogonal projection of v^* onto h. Then w^* is rational with $v^* - w^*$ normal to h, and h consists of all points x of A with $(x - w^*) \perp (v^* - w^*)$. So we can take h' to be the (rational) hyperplane through w^* normal to $v^* - w^*$. ■

Lemma 5.3.6. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation with $r^*, u^* \in S^*$ and $r^* \neq u^*$. Then r^{*Cl} and u^{*Cl} can be properly separated by a rational hyperplane h.*

Proof. *Case* 1. Assume that $0 \notin r^{*Cl} -_{\text{ALG}} u^{*Cl}$. Then as r^{*Cl} and u^{*Cl} are disjoint closed polytopes, choose any hyperplane g separating them strictly. Obtain the desired h by perturbing the constants in the equation of g so that they have rational values so close to the original values that h still separates r and u strictly.

Case 2. Assume that $0 \in r^{*Cl} -_{\text{ALG}} u^{*Cl}$. Via Corollary 5.3.1, choose a hyperplane h' that properly separates r^{*Cl} and u^{*Cl}. Choose m and c with $m \bullet x \geq c$ for all $x \in r^{*Cl}$, $m \bullet y \leq c$ for all $y \in u^{*Cl}$, and $m \bullet z \neq c$ for some $z \in r^{*Cl} \cup u^{*Cl}$. Let f be the hyperplane through 0 with normal m. Then $r^{*Cl} -_{\text{ALG}} u^{*Cl}$ is not contained in f, but is contained in one of the two closed half-spaces formed by f. Thus 0 lies on the "relative boundary" of $r^{*Cl} -_{\text{ALG}} u^{*Cl}$ (the topological boundary in the relative topology of $\text{Aff}(r^{*Cl} -_{\text{ALG}} u^{*Cl})$), whence 0 lies on some facet F of $r^{*Cl} -_{\text{ALG}} u^{*Cl}$. By Lemma 5.3.2, $\text{Aff}(F)$ is a rational affine subspace. Using Lemma 5.3.5, extend F to a rational hyperplane g through 0, with $g \cap \text{Aff}(r^{*Cl} -_{\text{ALG}} u^{*Cl}) = \text{Aff}(F)$. The rational normal α of g satisfies $\alpha \bullet x \geq \alpha \bullet y$ for all $x \in r^{*Cl}$ and $y \in u^{*Cl}$. At some (rational) vertex v_0^* of r^{*Cl}, $x \mapsto \alpha \bullet x$ achieves its minimum value on r^{*Cl}. Let $\beta = \alpha \bullet v_0$. Then β is rational, $\alpha \bullet x \geq \beta$ for all $x \in r^{*Cl}$, and $\alpha \bullet y \leq \beta$ for all $y \in u^{*Cl}$. Equation $\alpha \bullet x = \beta$ provides our rational properly separating hyperplane h. ■

Theorem 5.3.7. *Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation. Then S^{Cl} is **Q**-neat.*

Proof. We will use ***dim(D)*** to denote $\dim(P)$. The argument is by induction on $j = \mathit{dim}(D)$, with the dimension $n \geq j$ of the ambient space held fixed. If $j = 1$, then $\text{Aff}(P)$ is a rational line L, $S^{Cl} = S^{Cl}|L$ is neat by Proposition 1.12 (from [1]) and Definition 1.13 (from [1]) of regularity. The neatly separating hyperplanes $h_{a,L}$ in the 1.12 proof of are rational, as follows. Each endpoint a^* of an interval of S^{Cl} is rational. As L is rational we may take $b^* - a^*$ as the normal to $h_{a,L}$, where $b^* \in L^*$ with $b^* \neq a^*$. The resulting equation $(b^* - a^*) \bullet x = (b^* - a^*) \bullet a^*$ for $h_{a,L}$ has a rational constant, and all coefficients rational. So S^{Cl} is **Q**-neat.

Assume that S^{Cl} is **Q**-neat for every standard situation of dimension j or less. Let $D = (n, P, P^*, S^*, S^{Cl})$ be a standard situation with $\dim(P) = j + 1$, and $r = r^{*Cl}$ and $u = u^{*Cl}$ be distinct regions of S^{Cl}. We need to neatly separate r and u with a rational hyperplane. Via Lemma 5.3.6, choose a rational, properly separating hyperplane h' with equation $\alpha \bullet x = \beta$ and assume that $\alpha \bullet x \geq \beta$ on r.

If h' neatly separates r and u we are done. If $r \cap h' = \varnothing$ then any sufficiently small rational increase in β yields a parallel rational hyperplane h'' that strictly separates r and u, while if $u \cap h = \varnothing$ a small rational decrease in β achieves strict separation.

So, assume that $\varnothing \neq r \cap h' \neq u \cap h' \neq \varnothing$ (see Figure 3 of [1]). The restriction $S^{Cl}|h'$ is an extended **Q**-regular closed decomposition of the rational polytope $P \cap h'$, and $\dim(P \cap h') < \dim(P)$. By induction, choose a rational hyperplane k' of $\mathbf{R}^n$ that neatly separates the distinct regions $r \cap h'$ and $u \cap h'$ of $S^{Cl}|h'$, and let $k = k' \cap h'$. As in [1], k is a codimension 2 affine subspace of $\mathbf{R}^n$. Clearly, k is rational. Define v and w as in [1], and let h_ε be the hyperplane obtained by rotating h' about k, taking v toward w through an angle of ε. As in [1], for any sufficiently small value of $\varepsilon > 0$, h_ε neatly separates r and u.

It remains only to show that there are arbitrarily small values of ε for which h_ε is rational. Choose an affine basis $X^* = \{x_1^*, \ldots, x_{n-1}^*\}$ of k. We will show that there are arbitrarily small values of ε for which h_ε contains a rational point $x_n^* \notin k$, for then $X^* \cup \{x_n^*\}$ must be an affine basis for h_ε, whence h_ε is rational. But this is clear—no matter how small is $\varepsilon > 0$, as the angle increases from 0 to ε, the rotating h' sweeps out a region of $\mathbf{R}^n$ whose non-empty interior is disjoint from k, and contains rational points of $\mathbf{R}^n - k$. ■

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