Numerical Methods HW 3

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I am assuming a cubic domain; j_1, j_2, & j_3 represent the three orthogonal directions; k, l & m are the wave numbers; i is the imaginary constant. A value of C less than zero isn't possible because $\Delta x, \Delta t$, and α are all greater than 0.

Problem 5

FTCS 2D:

$T^n_{j_1,j_2} = G^n e^{ikj_1\Delta x} e^{ilj_2\Delta x}$:

$$\frac{G^{n+1}e^{ikj_1\Delta x}e^{ilj_2\Delta x} - G^n e^{ikj_1\Delta x}e^{ilj_2\Delta x}}{\Delta t} =$$
$$G^n \frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x}e^{ilj_2\Delta x} + e^{ik(j_1-1)\Delta x}e^{ilj_2\Delta x} + e^{ik(j_1)\Delta x}e^{il(j_2+1)\Delta x}$$
$$+e^{ik(j_1)\Delta x}e^{il(j_2-1)\Delta x} - 4e^{ikj_1\Delta x}e^{ilj_2\Delta x}]$$

An $e^{ikj_1\Delta x}e^{ilj_2\Delta x}$ term cancels from both sides:

$$\frac{G^{n+1} - G^n}{\Delta t} = G^n \frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} - 4]$$

Solve for $\frac{G^{n+1}}{G^n}$ and substituting $\cos(\theta)$ for exponentials:

$$\frac{G^{n+1}}{G^n} = 1 + C[2cos(k\Delta x) + 2cos(l\Delta x) - 4]$$

Worst case for k and l is when both cosine terms are -1:

$$\frac{G^{n+1}}{G^n} = 1 + C[-2 - 2 - 4] = 1 - 8C$$

For stability: -1 $\leq$ 1 - 8C $\leq$ 1 or -2 $\leq$ -8C $\leq$ 0; Therefore, 0 $\leq$ C $\leq$ $\frac{1}{4}$

FTCS 3D:

$\mathrm{T}^n_{j_1,j_2} = \mathrm{G}^n \mathrm{e}^{ikj_1\Delta x}\mathrm{e}^{ilj_2\Delta x}\mathrm{e}^{imj_3\Delta x}$:

$$\frac{G^{n+1}e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x} - G^n e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}}{\Delta t} =$$

$$G^n\frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}+e^{ik(j_1-1)\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}+e^{ik(j_1)\Delta x}e^{il(j_2+1)\Delta x}e^{imj_3\Delta x}$$
$$+e^{ik(j_1)\Delta x}e^{il(j_2-1)\Delta x}e^{imj_3\Delta x}+e^{ik(j_1)\Delta x}e^{ilj_2\Delta x}e^{im(j_3+1)\Delta x}+e^{ik(j_1)\Delta x}e^{ilj_2\Delta x}e^{im(j_3-1)\Delta x}$$
$$-6e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}]$$

An $\mathrm{e}^{ikj_1\Delta x}\mathrm{e}^{ilj_2\Delta x}\mathrm{e}^{imj_3\Delta x}$ term cancels from both sides:

$$\frac{G^{n+1} - G^n}{\Delta t} = G^n\frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} + e^{im\Delta x} + e^{-im\Delta x} - 6]$$

Solve for $\frac{G^{n+1}}{G^n}$ and substituting cos(θ) for exponentials:

$$\frac{G^{n+1}}{G^n} = 1 + C[2cos(k\Delta x) + 2cos(l\Delta x) + 2cos(m\Delta x) - 6]$$

Worst case for k, l and m is when all cosine terms are -1:

$$\frac{G^{n+1}}{G^n} = 1 + C[-2 - 2 - 2 - 6] = 1 - 12C$$

For stability: -1 $\leq$ 1 - 12C $\leq$ 1 or -2 $\leq$ -12C $\leq$ 0; Thus, 0 $\leq$ C $\leq$ $\frac{1}{6}$

Conclusion:
FTCS is only stable when C is smaller than $\frac{1}{2n}$ where n is the number of dimensions.

Problem 6

BECS 1D:

$T^n_{j_1,j_2} = G^n e^{ikj_1\Delta x}$:

$$\frac{G^{n+1}e^{ikj_1\Delta x} - G^n e^{ikj_1\Delta x}}{\Delta t} =$$
$$G^{n+1}\frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x} + e^{ik(j_1-1)\Delta x} - 2e^{ikj_1\Delta x}]$$

An $e^{ikj_1\Delta x}$ term cancels from both sides:

$$\frac{G^{n+1} - G^n}{\Delta t} = G^{n+1}\frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} - 2]$$

Solve for $\frac{G^{n+1}}{G^n}$ and substituting $\cos(\theta)$ for exponentials:

$$\frac{G^{n+1}}{G^n} = [1 - C(2cos(k\Delta x) - 2)]^{-1}$$

Worst case for k is when cosine term is -1:

$$\frac{G^{n+1}}{G^n} = [1 - C(-2-2)]^{-1} = [1+4C]^{-1}$$

For stability: -1 $\leq$ [1+4C]$^{-1}$ $\leq$ 1 or -[1+4C] $\leq$1$\leq$ [1+4C]; Thus, 0 $\leq$ C

BECS 2D:

$T^n_{j_1,j_2} = G^n e^{ikj_1\Delta x} e^{ilj_2\Delta x}$:

$$\frac{G^{n+1}e^{ikj_1\Delta x}e^{ilj_2\Delta x} - G^n e^{ikj_1\Delta x}e^{ilj_2\Delta x}}{\Delta t} =$$
$$G^{n+1}\frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x}e^{ilj_2\Delta x} + e^{ik(j_1-1)\Delta x}e^{ilj_2\Delta x} + e^{ik(j_1)\Delta x}e^{il(j_2+1)\Delta x}$$
$$+e^{ik(j_1)\Delta x}e^{il(j_2-1)\Delta x} - 4e^{ikj_1\Delta x}e^{ilj_2\Delta x}]$$

An $e^{ikj_1\Delta x}e^{ilj_2\Delta x}$ term cancels from both sides:

$$\frac{G^{n+1} - G^n}{\Delta t} = G^{n+1}\frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} - 4]$$

Solve for $\frac{G^{n+1}}{G^n}$ and substituting cos(θ) for exponentials:

$$\frac{G^{n+1}}{G^n} = [1 - C(2cos(k\Delta x) + 2cos(l\Delta x) - 4)]^{-1}$$

Worst case for k and l is when both cosine terms are -1:

$$\frac{G^{n+1}}{G^n} = [1 - C(-2-2-4)]^{-1} = [1+8C]^{-1}$$

For stability: -1 $\leq$ [1+8C]$^{-1}$ $\leq$ 1 or -[1+8C] $\leq$1$\leq$ [1+8C]; Thus, 0 $\leq$ C

BECS 3D:

$T^n_{j_1,j_2}$ = $G^n e^{ikj_1\Delta x} e^{ilj_2\Delta x} e^{imj_3\Delta x}$:

$$\frac{G^{n+1}e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x} - G^n e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}}{\Delta t} =$$

$$G^{n+1}\frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}+e^{ik(j_1-1)\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}+e^{ik(j_1)\Delta x}e^{il(j_2+1)\Delta x}e^{imj_3\Delta x}$$
$$+e^{ik(j_1)\Delta x}e^{il(j_2-1)\Delta x}e^{imj_3\Delta x}+e^{ik(j_1)\Delta x}e^{ilj_2\Delta x}e^{im(j_3+1)\Delta x}+e^{ik(j_1)\Delta x}e^{ilj_2\Delta x}e^{im(j_3-1)\Delta x}$$
$$-6e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}]$$

An $e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}$ term cancels from both sides:

$$\frac{G^{n+1}-G^n}{\Delta t} = G^{n+1}\frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} + e^{im\Delta x} + e^{-im\Delta x} - 6]$$

Solve for $\frac{G^{n+1}}{G^n}$ and substituting cos(θ) for exponentials:

$$\frac{G^{n+1}}{G^n} = [1 - C(2cos(k\Delta x) + 2cos(l\Delta x) + 2cos(m\Delta x) - 6)]^{-1}$$

Worst case for k, l and m is when both cosine terms are -1:

$$\frac{G^{n+1}}{G^n} = [1 - C(-2-2-2-6)]^{-1} = [1+12C]^{-1}$$

For stability: -1 $\leq$[1+12C]$^{-1}$$\leq$ 1 or -[1+12C] $\leq$1$\leq$ [1+12C]; Thus, 0 $\leq$ C

Conclusion:
BECS is always stable.

Problem 7

Crank-Nicolson 1D:

$T^n_{j_1,j_2} = G^n e^{ikj_1\Delta x}$:

$$\frac{G^{n+1}e^{ikj_1\Delta x} - G^n e^{ikj_1\Delta x}}{\Delta t} =$$
$$\frac{1}{2}[G^n \frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x} + e^{ik(j_1-1)\Delta x} - 2e^{ikj_1\Delta x}]$$
$$+G^{n+1}\frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x} + e^{ik(j_1-1)\Delta x} - 2e^{ikj_1\Delta x}]]$$

An $e^{ikj_1\Delta x}$ term cancels from both sides:

$$\frac{G^{n+1} - G^n}{\Delta t} = \frac{1}{2}[G^n \frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} - 2] + G^{n+1}\frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} - 2]]$$

Solve for $\frac{G^{n+1}}{G^n}$ and substituting $\cos(\theta)$ for exponentials:

$$\frac{G^{n+1}}{G^n} = \frac{2 + C[2cos(k\Delta x) - 2]}{2 - C[2cos(k\Delta x) - 2]}$$

Worst case for k is when cosine term is -1:

$$\frac{G^{n+1}}{G^n} = \frac{2 + C(-2-2)}{2 - C(-2-2)} = \frac{2-4C}{2+4C}$$

For stability: $-1 \leq \frac{2-4C}{2+4C} \leq 1$ or $-[2+4C] \leq [2-4C] \leq [2+4C]$; Thus, $0 \leq C$

Crank-Nicolson 2D:

$T^n_{j_1,j_2} = G^n e^{ikj_1\Delta x} e^{ilj_2\Delta x}$:

$$\frac{G^{n+1}e^{ikj_1\Delta x}e^{ilj_2\Delta x} - G^n e^{ikj_1\Delta x}e^{ilj_2\Delta x}}{\Delta t} =$$
$$\frac{1}{2}[G^n\frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x}e^{ilj_2\Delta x} + e^{ik(j_1-1)\Delta x}e^{ilj_2\Delta x}$$
$$+e^{ik(j_1)\Delta x}e^{il(j_2+1)\Delta x} + e^{ik(j_1)\Delta x}e^{il(j_2-1)\Delta x} - 4e^{ikj_1\Delta x}e^{ilj_2\Delta x}]$$
$$+G^{n+1}\frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x}e^{ilj_2\Delta x} + e^{ik(j_1-1)\Delta x}e^{ilj_2\Delta x}$$
$$+e^{ik(j_1)\Delta x}e^{il(j_2+1)\Delta x} + e^{ik(j_1)\Delta x}e^{il(j_2-1)\Delta x} - 4e^{ikj_1\Delta x}e^{ilj_2\Delta x}]]$$

An $e^{ikj_1\Delta x}e^{ilj_2\Delta x}$ term cancels from both sides:

$$\frac{G^{n+1} - G^n}{\Delta t} = \frac{1}{2}[G^n\frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} - 4]$$
$$+G^{n+1}\frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} - 4]]$$

Solve for $\frac{G^{n+1}}{G^n}$ and substituting $\cos(\theta)$ for exponentials:

$$\frac{G^{n+1}}{G^n} = \frac{2 + C[2cos(k\Delta x) + 2cos(l\Delta x) - 4]}{2 - C[2cos(k\Delta x) + 2cos(l\Delta x) - 4]}$$

Worst case for k and l is when both cosine terms are -1:

$$\frac{G^{n+1}}{G^n} = \frac{2 + C(-2 - 2 - 4)}{2 - C(-2 - 2 - 4)} = \frac{2 - 8C}{2 + 8C}$$

For stability: $-1 \leq \frac{2-8C}{2+8C} \leq 1$ or $-[2+8C] \leq [2-8C] \leq [2+8C]$; Thus, $0 \leq C$

Crank-Nicolson 3D:

$T^n_{j_1,j_2} = G^n e^{ikj_1\Delta x} e^{ilj_2\Delta x} e^{imj_3\Delta x}$:

$$\frac{G^{n+1}e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x} - G^n e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}}{\Delta t} =$$
$$\frac{1}{2}[G^n \frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x} + e^{ik(j_1-1)\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}$$
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$$+e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{im(j_3+1)\Delta x} + e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{im(j_3-1)\Delta x} - 6e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}]$$
$$+G^{n+1}\frac{\alpha}{\Delta x}[e^{ik(j_1+1)\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x} + e^{ik(j_1-1)\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}$$
$$+e^{ik(j_1)\Delta x}e^{il(j_2+1)\Delta x}e^{imj_3\Delta x} + e^{ik(j_1)\Delta x}e^{il(j_2-1)\Delta x}e^{imj_3\Delta x}$$
$$+e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{im(j_3+1)\Delta x} + e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{im(j_3-1)\Delta x} - 6e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}]]$$

An $e^{ikj_1\Delta x}e^{ilj_2\Delta x}e^{imj_3\Delta x}$ term cancels from both sides:

$$\frac{G^{n+1}-G^n}{\Delta t} = \frac{1}{2}[G^n\frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} + e^{im\Delta x} + e^{-im\Delta x} - 6]$$
$$+G^{n+1}\frac{\alpha}{\Delta x}[e^{ik\Delta x} + e^{-ik\Delta x} + e^{il\Delta x} + e^{-il\Delta x} + e^{im\Delta x} + e^{-im\Delta x} - 6]]$$

Solve for $\frac{G^{n+1}}{G^n}$ and substituting $\cos(\theta)$ for exponentials:

$$\frac{G^{n+1}}{G^n} = \frac{2 + C[2cos(k\Delta x) + 2cos(l\Delta x) + 2cos(m\Delta x) - 6]}{2 - C[2cos(k\Delta x) + 2cos(l\Delta x) + 2cos(m\Delta x) - 6]}$$

Worst case for k and l is when both cosine terms are -1:

$$\frac{G^{n+1}}{G^n} = \frac{2 + C(-2-2-2-6)}{2 - C(-2-2-2-6)} = \frac{2-12C}{2+12C}$$

For stability: -1 $\leq \frac{2-12C}{2+12C} \leq$ 1 or -[2+12C]$\leq$[2-12C]$\leq$[2+12C]; Thus, 0 $\leq$ C

Conclusion:
Crank-Nicolson is always stable.