

Lecture 4

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Hotelling's T² as

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Large Sample Inference of Mean Vector

Lecture 4 Inferences About Mean

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T-Test of Univariate Normal Population Mean

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Large Sample Inference abou a Population Mean Vector

Large Sample Inference of Mean Vector • In this lecture, we concern about the inference of a mean vector.

• Let us start with the one-sample *t*-test for a univariate normal population and consider the following hypothesis:

$$H_0: \mu = \mu_0, \quad H_1: \mu \neq \mu_0$$

• Suppose we collect a random sample $\{X_i\}_{i=1}^n$ from the normal population with mean μ . Then the test statistic is

$$t = \frac{\bar{X} - \mu_0}{s / \sqrt{n}}, \quad \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i, \ s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

• This test statistic follow t-distribution with degree of freedom (n-1) under the null hypotheis, i.e. $t \stackrel{H_0}{\sim} t(n-1)$.



F-Test of Univariate Normal Population Mean

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• Reject H_0 when $|t| > t_{1-\alpha/2}(n-1)$ at the confidence level of $\alpha 100\%$.

• This is equivalent to considering the F test statistic

$$F = t^2 = n(\bar{X} - \mu_0)s^{-2}(\bar{X} - \mu_0) \sim F(1, n - 1)$$

• Note the region of rejecting H_0 is

$$\mu_0\in (-\infty, \bar{x}-t_{1-\alpha/2}(n-1)s/\sqrt{n})\cup (\bar{x}+t_{1-\alpha/2}(n-1)s/\sqrt{n}, +\infty)$$

• Or equivalently, the $(1-\alpha)100\%$ confidence interval for μ is

$$\mu \in [\bar{x} - t_{1-\alpha/2}(n-1)s/\sqrt{n}, \bar{x} + t_{1-\alpha/2}(n-1)s/\sqrt{n}]$$



Multivariate Generalization of F Test Statistic

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• Now we generalize the above F test statistic to multivariate case.

• Recall sample mean $\bar{\mathbf{X}} = \frac{\mathbf{1}_n^T}{n} \mathbf{X}$ and sample covariance $\mathbf{S} = \frac{1}{n-1} \mathbf{X}^T (\mathbf{I}_n - \mathbf{J}) \mathbf{X}$.

• When the sample $\mathbf{X}_{n \times p}$ is taken from multivariate normal, i.e. $\mathbf{X}_i \stackrel{iid}{\sim} N_p(\mu, \mathbf{\Sigma})$, we have

$$\bar{\mathbf{X}} \sim N_{p}(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n), \quad (n-1)\mathbf{S} \sim W_{n-1}(\boldsymbol{\Sigma}), \quad n(\bar{\mathbf{X}}-\boldsymbol{\mu})^{\mathsf{T}}\mathbf{S}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\mu}) \stackrel{\cdot}{\sim} \chi_{p}^{2}.$$

• The the quadratic form $T^2 = n(\bar{\mathbf{X}} - \mu_0)^T \mathbf{S}^{-1}(\bar{\mathbf{X}} - \mu_0)$ is called *Hotelling's* T^2 statistic which follows a F-distribution

$$T^2 = n(\bar{\mathbf{X}} - \boldsymbol{\mu}_0)^T \mathbf{S}^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu}_0) \stackrel{H_0}{\sim} \frac{(n-1)p}{n-p} F(p, n-p)$$

• In general, $T^2(p,n-1) = N_p(\mu,\mathbf{\Sigma})^T[W_{p,n-1}(\mathbf{\Sigma})/(n-1)]^{-1}N_p(\mu,\mathbf{\Sigma}).$



Hotelling's T^2 Test

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Large Sample Inference of Mean Vector **Example 5.2 (Testing a multivariate mean vector with T^2)** Perspiration from 20 healthy females was analyzed. Three components, X_1 = sweat rate, X_2 = sodium content, and X_3 = potassium content, were measured, and the results, which we call the *sweat data*, are presented in Table 5.1.

Test the hypothesis H_0 : $\mu' = [4, 50, 10]$ against H_1 : $\mu' \neq [4, 50, 10]$ at level of significance $\alpha = .10$.

Computer calculations provide

$$\bar{\mathbf{x}} = \begin{bmatrix} 4.640 \\ 45.400 \\ 9.965 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 2.879 & 10.010 & -1.810 \\ 10.010 & 199.788 & -5.640 \\ -1.810 & -5.640 & 3.628 \end{bmatrix}$$

and

$$\mathbf{S}^{-1} = \begin{bmatrix} .586 & -.022 & .258 \\ -.022 & .006 & -.002 \\ .258 & -.002 & .402 \end{bmatrix}$$

We evaluate

$$T^2 =$$

$$20[4.640 - 4, 45.400 - 50, 9.965 - 10] \begin{bmatrix} .586 & -.022 & .258 \\ -.022 & .006 & -.002 \\ .258 & -.002 & .402 \end{bmatrix} \begin{bmatrix} 4.640 - 4 \\ 45.400 - 50 \\ 9.965 - 10 \end{bmatrix}$$

$$= 20[.640, -4.600, -.035] \begin{bmatrix} .467 \\ -.042 \\ .160 \end{bmatrix} = 9.74$$

Hotelling's T^2 Statistic

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Large Sample Inference of Mean Vector • Hotelling's T^2 statistics is invariant to affine transformation. That is, if $\mathbf{Y}_{p\times 1} = \mathbf{C}_{p\times p}\mathbf{X}_{p\times 1} + \mathbf{d}_{p\times 1}$ with \mathbf{C} nondegenerate, then

$$T_{\mathbf{Y}}^2 = T_{\mathbf{X}}^2$$

Likelihood Ratio Test

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Large Sample Inference

• The above Hotelling's T^2 test statistic can also be derived from likelihood ratio test (LRT).

• Recall the MLE $\hat{\mu} = \bar{\mathbf{X}}, \hat{\mathbf{\Sigma}} = \frac{n-1}{n} \mathbf{S}$ of an MVN $N_n(\mu, \mathbf{\Sigma})$ is the maximum of

$$\max_{\boldsymbol{\mu}, \boldsymbol{\Sigma}} L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{np/2} |\hat{\boldsymbol{\Sigma}}|^{-n/2} e^{-np/2}$$

where
$$L(\mu, \mathbf{\Sigma}) = (2\pi)^{np/2} |\mathbf{\Sigma}|^{-n/2} \exp\{-\frac{1}{2} tr[(\mathbf{X} - \mu)\mathbf{\Sigma}^{-1}(\mathbf{X} - \mu)^T]\}.$$

• By similar argument of MLE for $\hat{\Sigma}$, we have

$$\max_{\mathbf{\Sigma}} L(\mu_0, \mathbf{\Sigma}) = (2\pi)^{np/2} |\hat{\mathbf{\Sigma}}_0|^{-n/2} e^{-np/2}$$

where
$$\hat{\mathbf{\Sigma}}_0 = \frac{1}{n} \sum_{i=1}^n (\mathbf{x}_i - \boldsymbol{\mu}_0) (\mathbf{x}_i - \boldsymbol{\mu}_0)^T = \frac{1}{n} (\mathbf{X} - \boldsymbol{\mu}_0)^T (\mathbf{X} - \boldsymbol{\mu}_0)$$
.



Hotelling's T^2 Test as LRT

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Large Sample Inference of Mean Vector • Now we consider the following statistic for LRT $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$:

$$\Lambda = \frac{\max_{\boldsymbol{\Sigma}} L(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})}{\max_{\boldsymbol{\mu}, \boldsymbol{\Sigma}} L(\boldsymbol{\mu}, \boldsymbol{\Sigma})} = \left(\frac{|\hat{\boldsymbol{\Sigma}}|}{|\hat{\boldsymbol{\Sigma}}_0|}\right)^{n/2}$$

- The statistic $\Lambda^{2/n}$ is called Wilk's lambda.
- Based on the MVN assumption, i.e. $\mathbf{X}_i \stackrel{iid}{\sim} N_p(\mu, \mathbf{\Sigma})$, we have

$$\Lambda^{2/n} = \left[1 + \frac{T^2}{n-1}\right]^{-1}$$

• Hint: consider the determinant of $\mathbf{A} = \begin{bmatrix} (n-1)\mathbf{S} & \sqrt{n}(\mathbf{X} - \boldsymbol{\mu}_0) \\ \sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu}_0)^T & -1 \end{bmatrix}$.



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Confidence Regions

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Hotelling's T2 of Multivariate Normal

Confidence Regions

Large Sample Inference

 We extend the concept of univariate confidence interval to a multivariate confidence region.

• Let $\theta \in \Theta$ be a vector of unknown population parameter. A confidence region (CR) of θ based on sample **X** at $100(1-\alpha)\%$ confidence level, denoted as $R(\mathbf{X})$, is defined as

$$\Pr[\boldsymbol{\theta} \in R(\mathbf{X})] = 1 - \alpha$$

- Recall Hotelling's $T^2 = n(\bar{\mathbf{X}} \mu)^T \mathbf{S}^{-1}(\bar{\mathbf{X}} \mu) \sim \frac{(n-1)p}{p-p} F(p, n-p)$.
- Therefore CR for μ is computed based on

$$\Pr\left[T^2 \le \frac{(n-1)p}{n-p}F_{1-\alpha}(p,n-p)\right] = 1 - \alpha$$



Confidence Region of Population Mean Vector

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Large Sample Inference

ullet The CR of MVN mean vector $oldsymbol{\mu}$ is determined by

$$(\bar{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu}) = c^2 = \frac{(n-1)p}{n(n-p)} F_{1-\alpha}(p, n-p)$$

• This is an ellipsoid centered at $\bar{\mathbf{X}}$ and having axes $\pm \sqrt{\lambda_i} c \mathbf{v}_i$ with eigen-paris $\{\lambda_i, \mathbf{v}_i\}$ of \mathbf{S} .

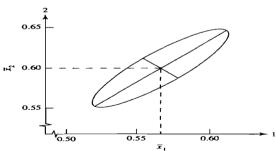


Figure 5.1 A 95% confidence ellipse for μ based on microwaveradiation data.

Simultaneous Confidence Intervals

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ullet CR is a joint statement of all the plausible values of population parameter, e.g. mean μ .

- Often we are concerned about separate confidence statements holding simultaneously, i.e. *simultaneous confidence intervals (SCI)*.
- We consider a linear combination of random vector $X \sim N_p(\mu, \Sigma)$: $Z = \mathbf{a}^T X \sim N(\mathbf{a}^T \mu, \mathbf{a}^T \Sigma \mathbf{a})$.
- Given a random sample $\mathbf{X}_{n \times p}$, we have corresponding sample $\mathbf{Z} = \mathbf{X}\mathbf{a}$, and hence $\mathbf{\bar{Z}} = \mathbf{\bar{X}}\mathbf{a}$ and $s_{\mathcal{Z}}^2 = \mathbf{a}^T \mathbf{S}\mathbf{a}$.
- Therefore, the $100(1-\alpha)\%$ CI for $\mu_Z = \mathbf{a}^T \boldsymbol{\mu}$ can be obtained based on $|t| = \left|\frac{\bar{\mathbf{z}} \mu_Z}{s_Z/\sqrt{n}}\right| \le t_{1-\alpha/2}(n-1)$:

$$\mathbf{ar{X}}\mathbf{a} - t_{1-lpha/2}(n-1)\sqrt{\mathbf{a}^T\mathbf{S}\mathbf{a}/n} \le \mu_Z \le \mathbf{ar{X}}\mathbf{a} + t_{1-lpha/2}(n-1)\sqrt{\mathbf{a}^T\mathbf{S}\mathbf{a}/n} \quad (*)$$

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Large Sample Inference

• Note, the CI for the component mean, e.g. μ_j can be obtained by setting $\mathbf{a} = \mathbf{e}_j = [\underbrace{0, \cdots, 0}_{i-1}, 1, \underbrace{0, \cdots, 0}_{p-i}]^T$.

- However, the confidence associated with all of the statements taken together is not $1-\alpha$.
- It would be desirable to associate a 'collective' confidence coefficient of $1-\alpha$ with the CIs generated by any **a**. For this purpose, we consider

$$\max_{\mathbf{a}} t^2 = \max_{\mathbf{a}} \frac{n(\mathbf{a}^T (\bar{\mathbf{X}} - \boldsymbol{\mu}))^2}{\mathbf{a}^T \mathbf{S} \mathbf{a}} = n(\bar{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu}) = T^2$$

• Therefore, SCI can be obtained based on the previous Hotelling's test statistics $T^2 \sim \frac{(n-1)p}{n-p} F(p, n-p)$.

Simultaneous T^2 Confidence Intervals

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Large Sample Inference of Mean Vector • For $j=1,\cdots,p$, successfully choose $\mathbf{a}=\mathbf{e}_j$ to obtain CI for μ_j simultaneously:

$$\bar{\mathbf{X}}_1 - \sqrt{\frac{(n-1)p}{n(n-p)}} F_{1-\alpha}(p,n-p) s_1 \leq \mu_1 \leq \bar{\mathbf{X}}_1 + \sqrt{\frac{(n-1)p}{n(n-p)}} F_{1-\alpha}(p,n-p) s_1$$

$$\mathbf{\bar{X}}_j - \sqrt{\frac{(n-1)p}{n(n-p)}} F_{1-\alpha}(p,n-p) s_j \leq \mu_j \leq \mathbf{\bar{X}}_j + \sqrt{\frac{(n-1)p}{n(n-p)}} F_{1-\alpha}(p,n-p) s_j$$

$$\bar{\mathbf{X}}_p - \sqrt{\frac{(n-1)p}{n(n-p)}} F_{1-\alpha}(p,n-p) s_p \leq \mu_p \leq \bar{\mathbf{X}}_p + \sqrt{\frac{(n-1)p}{n(n-p)}} F_{1-\alpha}(p,n-p) s_p$$



Simultaneous CIs As Projections of CR

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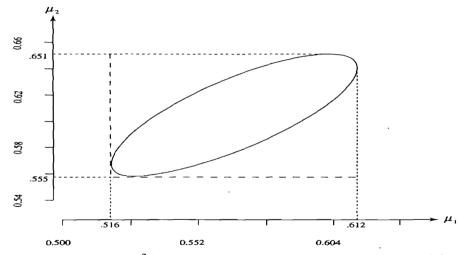


Figure 5.2 Simultaneous T^2 -intervals for the component means as shadows of the confidence ellipse on the axes—microwave radiation data.



One-at-a-Time Confidence Intervals

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Large Sample Inference

• For $j=1,\cdots,p$, successfully choose $\mathbf{a}=\mathbf{e}_j$ in (*) to obtain CI for μ_j one at a time:

$$egin{aligned} ar{\mathbf{X}}_1 - t_{1-lpha/2}(n-1)s_1/\sqrt{n} &\leq \mu_1 \leq ar{\mathbf{X}}_1 + t_{1-lpha/2}(n-1)s_1/\sqrt{n} \ &dots &dots &dots \ ar{\mathbf{X}}_j - t_{1-lpha/2}(n-1)s_j/\sqrt{n} \leq \mu_j \leq ar{\mathbf{X}}_j + t_{1-lpha/2}(n-1)s_j/\sqrt{n} \ &dots &dots &dots \ ar{\mathbf{X}}_p - t_{1-lpha/2}(n-1)s_p/\sqrt{n} \leq \mu_p \leq ar{\mathbf{X}}_p + t_{1-lpha/2}(n-1)s_p/\sqrt{n} \end{aligned}$$

• What is the issue?



One-at-a-Time Confidence Intervals

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• For $j=1,\cdots,p$, successfully choose $\mathbf{a}=\mathbf{e}_j$ in (*) to obtain CI for μ_j one at a time:

$$egin{aligned} ar{\mathbf{X}}_1 - t_{1-lpha/2}(n-1)s_1/\sqrt{n} &\leq \mu_1 \leq ar{\mathbf{X}}_1 + t_{1-lpha/2}(n-1)s_1/\sqrt{n} \ &dots &dots \end{aligned}$$

 $\bar{\mathbf{X}}_{p} - t_{1-\alpha/2}(n-1)s_{p}/\sqrt{n} \le \mu_{p} \le \bar{\mathbf{X}}_{p} + t_{1-\alpha/2}(n-1)s_{p}/\sqrt{n}$

$$egin{aligned} ar{\mathbf{X}}_j - t_{1-lpha/2}(n-1)s_j/\sqrt{n} &\leq \mu_j \leq ar{\mathbf{X}}_j + t_{1-lpha/2}(n-1)s_j/\sqrt{n} \ &dots &dots &dots &dots \end{aligned}$$

• What is the issue?

- -

Pr $[ar{\mathbf{X}}_j - t_{1-lpha/2}(n-1)s_j/\sqrt{n} \leq \mu_j \leq ar{\mathbf{X}}_j + t_{1-lpha/2}(n-1)s_j/\sqrt{n}, \ 1 \leq j \leq p] = (1-lpha)^p$ Mean Vector

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Large Sample Inference of Mean Vector • Often, we are concerned about a limited number, m, of linear combinations of means, i.e. $\mathbf{a}_1^T \mu, \dots, \mathbf{a}_m^T \mu$.

• Let C_i be the $100(1-\alpha_i)\%$ CI for $\mathbf{a}_i^T \boldsymbol{\mu}$, i.e. $\Pr[\mathbf{a}_i^T \boldsymbol{\mu} \in C_i] = 1-\alpha_i$. Then we have

$$\begin{aligned} \Pr[\mathbf{a}_i^T \boldsymbol{\mu} \in C_i, \ j = 1, \cdots, \rho] &= 1 - \Pr[\exists i_0, \ s.t. \ \mathbf{a}_{i_0}^T \boldsymbol{\mu} \not\in C_{i_0}] \\ &\geq 1 - \sum_{i=1}^m \Pr[\mathbf{a}_i^T \boldsymbol{\mu} \not\in C_i] = 1 - \sum_{i=1}^m \alpha_i \end{aligned}$$

• Specifically, setting $\alpha_i = \frac{\alpha}{m}$ for $i = 1, \dots, m$ with m = p we get the SCI for means with confidence level (at least) $1 - \alpha$:

$$\bar{\boldsymbol{\mathsf{X}}}_j - t_{1-\alpha/(2\rho)}(n-1)s_j/\sqrt{n} \leq \mu_j \leq \bar{\boldsymbol{\mathsf{X}}}_j + t_{1-\alpha/(2\rho)}(n-1)s_j/\sqrt{n}, \quad j=1,\cdots,\rho$$



The Bonferroni Method of Multiple Comparisons

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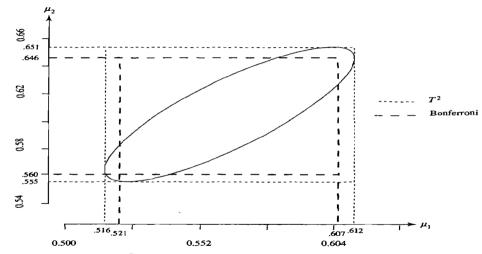


Figure 5.4 The 95% T^2 and 95% Bonferroni simultaneous confidence intervals for the component means—microwave radiation data.



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Large Sample Inference of Mean Vector

• Recall that for a sample $\mathbf{X}_i \stackrel{iid}{\sim} (\mu, \mathbf{\Sigma})$, we have for large n

$$n(\bar{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \stackrel{\cdot}{\sim} \chi_p^2.$$

- We can consider large sample inference of mean vector μ regardless of the original distribution.
- Consider the hypothesis test $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$ and reject H_0 at the level of significance of α if

$$n(\bar{\mathbf{X}} - \boldsymbol{\mu}_0)^T \mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}_0) > \chi^2_{1-\alpha}(p)$$

• Alternatively, we can consider the (approximate) 100(1-lpha)% CR of $oldsymbol{\mu}$ based on

$$\Pr[n(\bar{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \le \chi^2_{1-\alpha}(\boldsymbol{p})] = 1 - \alpha$$



Large Sample Simultaneous Confidence Interval

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Large Sample Inference of Mean Vector

• For $\mathbf{X}_i \stackrel{iid}{\sim} (\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we can also consider the (approximate) SCI for $\mu_Z = \mathbf{a}^T \boldsymbol{\mu}$ when n-p is large

$$\mathbf{\bar{X}}\mathbf{a} - \sqrt{\chi_{1-\alpha}^2(p)}\sqrt{\mathbf{a}^T\mathbf{S}\mathbf{a}/n} \leq \mu_Z \leq \mathbf{\bar{X}}\mathbf{a} + \sqrt{\chi_{1-\alpha}^2(p)}\sqrt{\mathbf{a}^T\mathbf{S}\mathbf{a}/n} \quad (*)$$

• We can consider the one-at-a-time confidence interval when *n* is large

$$oxed{f X}_j - z_{1-lpha/2}^2 \sqrt{s_{ii}/n} \le \mu_j \le ar{f X}_j + z_{1-lpha/2}^2 \sqrt{s_{ii}/n}, \quad j=1,\cdots,p$$

The Bonferroni SCI is

$$\bar{\mathbf{X}}_j - z_{1-\alpha/(2p)}^2 \sqrt{s_{ii}/n} \leq \mu_j \leq \bar{\mathbf{X}}_j + z_{1-\alpha/(2p)}^2 \sqrt{s_{ii}/n}, \quad j = 1, \cdots, p$$