# Lecture 4 Inferences About Mean

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# T-Test of Univariate Normal Population Mean

- In this lecture, we concern about the inference of a mean vector.
- Let us start with the one-sample $t$-test for a univariate normal population and consider the following hypothesis:

$$H_0 : \mu = \mu_0, \quad H_1 : \mu \neq \mu_0$$

- Suppose we collect a random sample $\{X_i\}_{i=1}^n$ from the normal population with mean $\mu$. Then the test statistic is

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}, \quad \bar{X} = \frac{1}{n}\sum_{i=1}^n X_i, \; s^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$$

- This test statistic follow $t$-distribution with degree of freedom $(n-1)$ under the null hypotheis, i.e. $t \overset{H_0}{\sim} t(n-1)$.

- Reject $H_0$ when $|t| > t_{1-\alpha/2}(n-1)$ at the confidence level of $\alpha 100\%$.
- This is equivalent to considering the $F$ test statistic

$$F = t^2 = n(\bar{X} - \mu_0) s^{-2} (\bar{X} - \mu_0) \sim F(1, n-1)$$

- Note the region of rejecting $H_0$ is

$$\mu_0 \in (-\infty, \bar{x} - t_{1-\alpha/2}(n-1)s/\sqrt{n}) \cup (\bar{x} + t_{1-\alpha/2}(n-1)s/\sqrt{n}, +\infty)$$

- Or equivalently, the $(1-\alpha)100\%$ confidence interval for $\mu$ is

$$\mu \in [\bar{x} - t_{1-\alpha/2}(n-1)s/\sqrt{n}, \bar{x} + t_{1-\alpha/2}(n-1)s/\sqrt{n}]$$

# Multivariate Generalization of $F$ Test Statistic

- Now we generalize the above $F$ test statistic to multivariate case.
- Recall sample mean $\bar{\mathbf{X}} = \frac{\mathbf{1}_n^T}{n}\mathbf{X}$ and sample covariance $\mathbf{S} = \frac{1}{n-1}\mathbf{X}^T(\mathbf{I}_n - \mathbf{J})\mathbf{X}$.
- When the sample $\mathbf{X}_{n\times p}$ is taken from multivariate normal, i.e. $\mathbf{X}_i \overset{iid}{\sim} N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we have

$$\bar{\mathbf{X}} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n), \quad (n-1)\mathbf{S} \sim W_{n-1}(\boldsymbol{\Sigma}), \quad n(\bar{\mathbf{X}} - \boldsymbol{\mu})^T\mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \dot{\sim} \chi^2_p.$$

- The the quadratic form $T^2 = n(\bar{\mathbf{X}} - \boldsymbol{\mu}_0)^T\mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}_0)$ is called *Hotelling's $T^2$ statistic* which follows a $F$-distribution

$$T^2 = n(\bar{\mathbf{X}} - \boldsymbol{\mu}_0)^T\mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}_0) \overset{H_0}{\sim} \frac{(n-1)p}{n-p}F(p, n-p)$$

- In general, $T^2(p, n-1) = N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})^T[W_{p,n-1}(\boldsymbol{\Sigma})/(n-1)]^{-1}N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

# Hotelling's $T^2$ Test

**Example 5.2 (Testing a multivariate mean vector with $T^2$)** Perspiration from 20 healthy females was analyzed. Three components, $X_1$ = sweat rate, $X_2$ = sodium content, and $X_3$ = potassium content, were measured, and the results, which we call the *sweat data*, are presented in Table 5.1.

Test the hypothesis $H_0: \boldsymbol{\mu}' = [4, 50, 10]$ against $H_1: \boldsymbol{\mu}' \neq [4, 50, 10]$ at level of significance $\alpha = .10$.

Computer calculations provide

$$\bar{\mathbf{x}} = \begin{bmatrix} 4.640 \\ 45.400 \\ 9.965 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} 2.879 & 10.010 & -1.810 \\ 10.010 & 199.788 & -5.640 \\ -1.810 & -5.640 & 3.628 \end{bmatrix}$$

and

$$\mathbf{S}^{-1} = \begin{bmatrix} .586 & -.022 & .258 \\ -.022 & .006 & -.002 \\ .258 & -.002 & .402 \end{bmatrix}$$

We evaluate

$$T^2 =$$

$$20\,[4.640 - 4, \quad 45.400 - 50, \quad 9.965 - 10] \begin{bmatrix} .586 & -.022 & .258 \\ -.022 & .006 & -.002 \\ .258 & -.002 & .402 \end{bmatrix} \begin{bmatrix} 4.640 - 4 \\ 45.400 - 50 \\ 9.965 - 10 \end{bmatrix}$$

$$= 20\,[.640, \quad -4.600, \quad -.035] \begin{bmatrix} .467 \\ -.042 \\ .160 \end{bmatrix} = 9.74$$

# Hotelling's $T^2$ Statistic

- Hotelling's $T^2$ statistics is invariant to affine transformation. That is, if $\mathbf{Y}_{p\times 1} = \mathbf{C}_{p\times p}\mathbf{X}_{p\times 1} + \mathbf{d}_{p\times 1}$ with $\mathbf{C}$ nondegenerate, then

$$T^2_{\mathbf{Y}} = T^2_{\mathbf{X}}$$

# Likelihood Ratio Test

- The above Hotelling's $T^2$ test statistic can also be derived from likelihood ratio test (LRT),
- Recall the MLE $\hat{\boldsymbol{\mu}} = \bar{\mathbf{X}}, \hat{\boldsymbol{\Sigma}} = \frac{n-1}{n}\mathbf{S}$ of an MVN $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is the maximum of

$$\max_{\boldsymbol{\mu},\boldsymbol{\Sigma}} L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{np/2}|\hat{\boldsymbol{\Sigma}}|^{-n/2}e^{-np/2}$$

where $L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{np/2}|\boldsymbol{\Sigma}|^{-n/2}\exp\{-\frac{1}{2}\mathrm{tr}[(\mathbf{X}-\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})^T]\}$.

- By similar argument of MLE for $\hat{\boldsymbol{\Sigma}}$, we have

$$\max_{\boldsymbol{\Sigma}} L(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}) = (2\pi)^{np/2}|\hat{\boldsymbol{\Sigma}}_0|^{-n/2}e^{-np/2}$$

where $\hat{\boldsymbol{\Sigma}}_0 = \frac{1}{n}\sum_{i=1}^n(\mathbf{x}_i - \boldsymbol{\mu}_0)(\mathbf{x}_i - \boldsymbol{\mu}_0)^T = \frac{1}{n}(\mathbf{X}-\boldsymbol{\mu}_0)^T(\mathbf{X}-\boldsymbol{\mu}_0)$.

# Hotelling's $T^2$ Test as LRT

- Now we consider the following statistic for LRT $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ against $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$:

$$\Lambda = \frac{\max_{\boldsymbol{\Sigma}} L(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})}{\max_{\boldsymbol{\mu}, \boldsymbol{\Sigma}} L(\boldsymbol{\mu}, \boldsymbol{\Sigma})} = \left( \frac{|\hat{\boldsymbol{\Sigma}}|}{|\hat{\boldsymbol{\Sigma}}_0|} \right)^{n/2}$$

- The statistic $\Lambda^{2/n}$ is called *Wilk's lambda*.
- Based on the MVN assumption, i.e. $\mathbf{X}_i \overset{iid}{\sim} N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we have

$$\Lambda^{2/n} = \left[ 1 + \frac{T^2}{n-1} \right]^{-1}$$

- Hint: consider the determinant of $\mathbf{A} = \begin{bmatrix} (n-1)\mathbf{S} & \sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu}_0) \\ \sqrt{n}(\bar{\mathbf{X}} - \boldsymbol{\mu}_0)^T & -1 \end{bmatrix}$.

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# Confidence Regions

- We extend the concept of univariate *confidence interval* to a multivariate *confidence region*.
- Let $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ be a vector of unknown population parameter. A *confidence region (CR)* of $\boldsymbol{\theta}$ based on sample $\mathbf{X}$ at $100(1-\alpha)\%$ confidence level, denoted as $R(\mathbf{X})$, is defined as

$$\Pr[\boldsymbol{\theta} \in R(\mathbf{X})] = 1 - \alpha$$

- Recall Hotelling's $T^2 = n(\bar{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu}) \sim \frac{(n-1)p}{n-p} F(p, n-p)$.
- Therefore CR for $\boldsymbol{\mu}$ is computed based on

$$\Pr\left[T^2 \leq \frac{(n-1)p}{n-p} F_{1-\alpha}(p, n-p)\right] = 1 - \alpha$$

# Confidence Region of Population Mean Vector



- The CR of MVN mean vector $\boldsymbol{\mu}$ is determined by

$$(\bar{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1} (\bar{\mathbf{X}} - \boldsymbol{\mu}) = c^2 = \frac{(n-1)p}{n(n-p)} F_{1-\alpha}(p, n-p)$$

- This is an ellipsoid centered at $\bar{\mathbf{X}}$ and having axes $\pm\sqrt{\lambda_i} c \mathbf{v}_i$ with eigen-paris $\{\lambda_i, \mathbf{v}_i\}$ of $\mathbf{S}$.

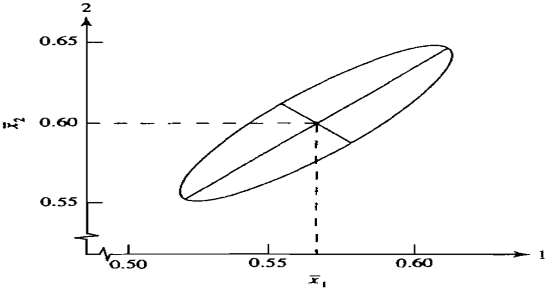


**Figure 5.1** A 95% confidence ellipse for $\mu$ based on microwave-radiation data.

# Simultaneous Confidence Intervals

- CR is a joint statement of all the plausible values of population parameter, e.g. mean $\boldsymbol{\mu}$.
- Often we are concerned about separate confidence statements holding simultaneously, i.e. *simultaneous confidence intervals (SCI)*.
- We consider a linear combination of random vector $X \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: $Z = \mathbf{a}^T X \sim N(\mathbf{a}^T\boldsymbol{\mu}, \mathbf{a}^T\boldsymbol{\Sigma}\mathbf{a})$.
- Given a random sample $\mathbf{X}_{n\times p}$, we have corresponding sample $\mathbf{Z} = \mathbf{X}\mathbf{a}$, and hence $\bar{\mathbf{Z}} = \bar{\mathbf{X}}\mathbf{a}$ and $s_Z^2 = \mathbf{a}^T\mathbf{S}\mathbf{a}$.
- Therefore, the $100(1-\alpha)\%$ CI for $\mu_Z = \mathbf{a}^T\boldsymbol{\mu}$ can be obtained based on $|t| = \left|\frac{\bar{\mathbf{Z}}-\mu_Z}{s_Z/\sqrt{n}}\right| \leq t_{1-\alpha/2}(n-1)$:

$$\bar{\mathbf{X}}\mathbf{a} - t_{1-\alpha/2}(n-1)\sqrt{\mathbf{a}^T\mathbf{S}\mathbf{a}/n} \leq \mu_Z \leq \bar{\mathbf{X}}\mathbf{a} + t_{1-\alpha/2}(n-1)\sqrt{\mathbf{a}^T\mathbf{S}\mathbf{a}/n} \quad (*)$$

# Simultaneous Confidence Intervals

- Note, the CI for the component mean, e.g. $\mu_j$ can be obtained by setting $\mathbf{a} = \mathbf{e}_j = [\underbrace{0, \cdots, 0}_{j-1}, 1, \underbrace{0, \cdots, 0}_{p-j}]^T$.
- However, the confidence associated with all of the statements taken together is not $1 - \alpha$.
- It would be desirable to associate a 'collective' confidence coefficient of $1 - \alpha$ with the CIs generated by any $\mathbf{a}$. For this purpose, we consider

$$\max_{\mathbf{a}} t^2 = \max_{\mathbf{a}} \frac{n(\mathbf{a}^T(\bar{\mathbf{X}} - \boldsymbol{\mu}))^2}{\mathbf{a}^T\mathbf{S}\mathbf{a}} = n(\bar{\mathbf{X}} - \boldsymbol{\mu})^T\mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}) = T^2$$

- Therefore, SCI can be obtained based on the previous Hotelling's test statistics $T^2 \sim \frac{(n-1)p}{n-p} F(p, n-p)$.

# Simultaneous $T^2$ Confidence Intervals

- For $j = 1, \cdots, p$, successfully choose $\mathbf{a} = \mathbf{e}_j$ to obtain CI for $\mu_j$ simultaneously:

$$\bar{\mathbf{X}}_1 - \sqrt{\frac{(n-1)p}{n(n-p)} F_{1-\alpha}(p, n-p)} s_1 \le \mu_1 \le \bar{\mathbf{X}}_1 + \sqrt{\frac{(n-1)p}{n(n-p)} F_{1-\alpha}(p, n-p)} s_1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\bar{\mathbf{X}}_j - \sqrt{\frac{(n-1)p}{n(n-p)} F_{1-\alpha}(p, n-p)} s_j \le \mu_j \le \bar{\mathbf{X}}_j + \sqrt{\frac{(n-1)p}{n(n-p)} F_{1-\alpha}(p, n-p)} s_j$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\bar{\mathbf{X}}_p - \sqrt{\frac{(n-1)p}{n(n-p)} F_{1-\alpha}(p, n-p)} s_p \le \mu_p \le \bar{\mathbf{X}}_p + \sqrt{\frac{(n-1)p}{n(n-p)} F_{1-\alpha}(p, n-p)} s_p$$

# Simultaneous CIs As Projections of CR

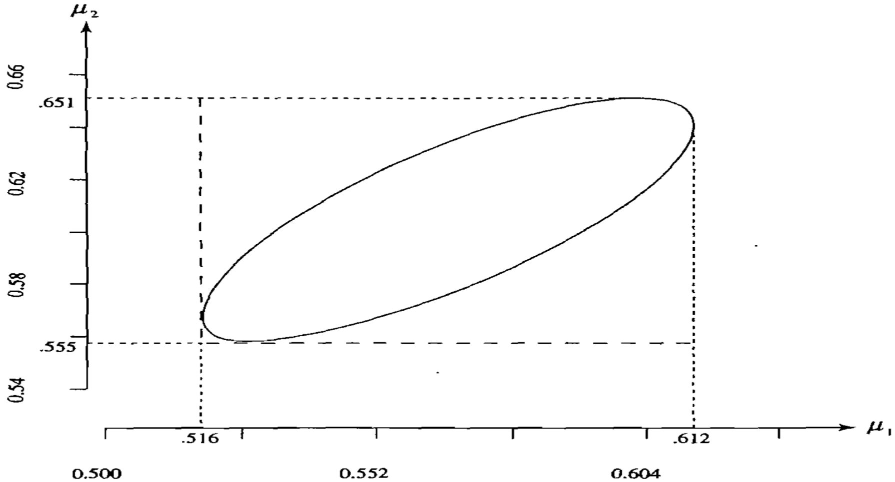


**Figure 5.2** Simultaneous $T^2$-intervals for the component means as shadows of the confidence ellipse on the axes—microwave radiation data.

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# One-at-a-Time Confidence Intervals





- For $j = 1, \cdots, p$, successfully choose $\mathbf{a} = \mathbf{e}_j$ in (*) to obtain CI for $\mu_j$ one at a time:

$$
\begin{aligned}
\bar{\mathbf{X}}_1 - t_{1-\alpha/2}(n-1)s_1/\sqrt{n} \le \mu_1 \le \bar{\mathbf{X}}_1 + t_{1-\alpha/2}(n-1)s_1/\sqrt{n} \\
\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\bar{\mathbf{X}}_j - t_{1-\alpha/2}(n-1)s_j/\sqrt{n} \le \mu_j \le \bar{\mathbf{X}}_j + t_{1-\alpha/2}(n-1)s_j/\sqrt{n} \\
\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
\bar{\mathbf{X}}_p - t_{1-\alpha/2}(n-1)s_p/\sqrt{n} \le \mu_p \le \bar{\mathbf{X}}_p + t_{1-\alpha/2}(n-1)s_p/\sqrt{n}
\end{aligned}
$$

- What is the issue?

# One-at-a-Time Confidence Intervals



- For $j = 1, \cdots, p$, successfully choose $\mathbf{a} = \mathbf{e}_j$ in (*) to obtain CI for $\mu_j$ one at a time:

$$\begin{aligned}
\bar{\mathbf{X}}_1 - t_{1-\alpha/2}(n-1)s_1/\sqrt{n} &\le \mu_1 \le \bar{\mathbf{X}}_1 + t_{1-\alpha/2}(n-1)s_1/\sqrt{n} \\
&\vdots \\
\bar{\mathbf{X}}_j - t_{1-\alpha/2}(n-1)s_j/\sqrt{n} &\le \mu_j \le \bar{\mathbf{X}}_j + t_{1-\alpha/2}(n-1)s_j/\sqrt{n} \\
&\vdots \\
\bar{\mathbf{X}}_p - t_{1-\alpha/2}(n-1)s_p/\sqrt{n} &\le \mu_p \le \bar{\mathbf{X}}_p + t_{1-\alpha/2}(n-1)s_p/\sqrt{n}
\end{aligned}$$

- What is the issue?
- The probability of them holding simultaneously

$$\Pr[\bar{\mathbf{X}}_j - t_{1-\alpha/2}(n-1)s_j/\sqrt{n} \le \mu_j \le \bar{\mathbf{X}}_j + t_{1-\alpha/2}(n-1)s_j/\sqrt{n},\ 1 \le j \le p] = (1-\alpha)^p$$

# The Bonferroni Method

- Often, we are concerned about a limited number, $m$, of linear combinations of means, i.e. $\mathbf{a}_1^T\boldsymbol{\mu}, \cdots, \mathbf{a}_m^T\boldsymbol{\mu}$.
- Let $C_i$ be the $100(1-\alpha_i)\%$ CI for $\mathbf{a}_i^T\boldsymbol{\mu}$, i.e. $\Pr[\mathbf{a}_i^T\boldsymbol{\mu} \in C_i] = 1-\alpha_i$. Then we have

$$\begin{aligned}\Pr[\mathbf{a}_i^T\boldsymbol{\mu} \in C_i,\ j=1,\cdots,p] &= 1-\Pr[\exists i_0,\ s.t.\ \mathbf{a}_{i_0}^T\boldsymbol{\mu} \notin C_{i_0}] \\ &\geq 1-\sum_{i=1}\Pr[\mathbf{a}_i^T\boldsymbol{\mu} \notin C_i] = 1-\sum_{i=1}^{m}\alpha_i\end{aligned}$$

- Specifically, setting $\alpha_i = \frac{\alpha}{m}$ for $i=1,\cdots,m$ with $m=p$ we get the SCI for means with confidence level (at least) $1-\alpha$:

$$\bar{\mathbf{X}}_j - t_{1-\alpha/(2p)}(n-1)s_j/\sqrt{n} \leq \mu_j \leq \bar{\mathbf{X}}_j + t_{1-\alpha/(2p)}(n-1)s_j/\sqrt{n}, \quad j=1,\cdots,p$$

# The Bonferroni Method of Multiple Comparisons

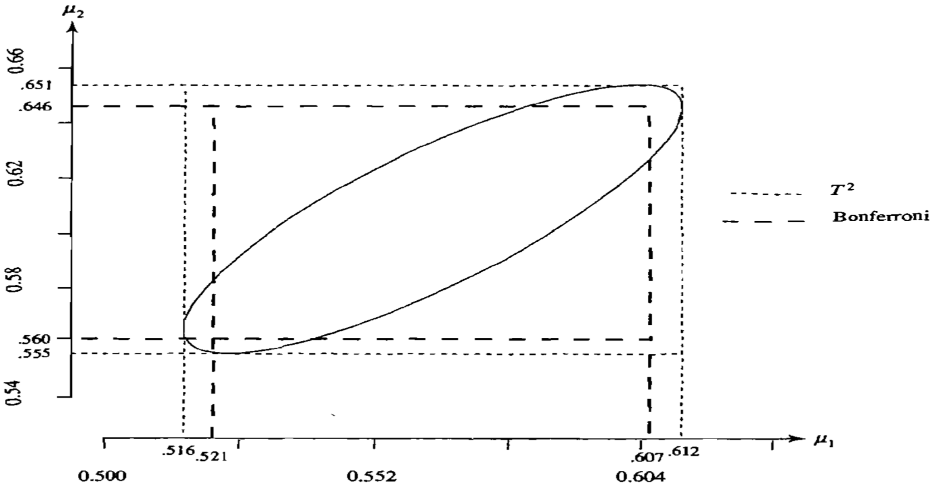


**Figure 5.4** The 95% $T^2$ and 95% Bonferroni simultaneous confidence intervals for the component means—microwave radiation data.

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# Large Sample Inference of Mean Vector

- Recall that for a sample $\mathbf{X}_i \overset{iid}{\sim} (\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we have for large $n$

$$n(\bar{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \dot{\sim} \chi^2_p.$$

- We can consider large sample inference of mean vector $\boldsymbol{\mu}$ regardless of the original distribution.
- Consider the hypothesis test $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ against $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$ and reject $H_0$ at the level of significance of $\alpha$ if

$$n(\bar{\mathbf{X}} - \boldsymbol{\mu}_0)^T \mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}_0) > \chi^2_{1-\alpha}(p)$$

- Alternatively, we can consider the (approximate) $100(1-\alpha)\%$ CR of $\boldsymbol{\mu}$ based on

$$\Pr[n(\bar{\mathbf{X}} - \boldsymbol{\mu})^T \mathbf{S}^{-1}(\bar{\mathbf{X}} - \boldsymbol{\mu}) \leq \chi^2_{1-\alpha}(p)] = 1 - \alpha$$

# Large Sample Simultaneous Confidence Interval



- For $\mathbf{X}_i \overset{iid}{\sim} (\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we can also consider the (approximate) SCI for $\mu_Z = \mathbf{a}^T\boldsymbol{\mu}$ when $n - p$ is large

$$\bar{\mathbf{X}}\mathbf{a} - \sqrt{\chi^2_{1-\alpha}(p)}\sqrt{\mathbf{a}^T\mathbf{S}\mathbf{a}/n} \le \mu_Z \le \bar{\mathbf{X}}\mathbf{a} + \sqrt{\chi^2_{1-\alpha}(p)}\sqrt{\mathbf{a}^T\mathbf{S}\mathbf{a}/n} \quad (*)$$

- We can consider the one-at-a-time confidence interval when $n$ is large

$$\bar{\mathbf{X}}_j - z_{1-\alpha/2}\sqrt{s_{ii}/n} \le \mu_j \le \bar{\mathbf{X}}_j + z_{1-\alpha/2}\sqrt{s_{ii}/n}, \quad j = 1, \cdots, p$$

- The Bonferroni SCI is

$$\bar{\mathbf{X}}_j - z_{1-\alpha/(2p)}\sqrt{s_{ii}/n} \le \mu_j \le \bar{\mathbf{X}}_j + z_{1-\alpha/(2p)}\sqrt{s_{ii}/n}, \quad j = 1, \cdots, p$$