Lecture 3 Areal Data Models

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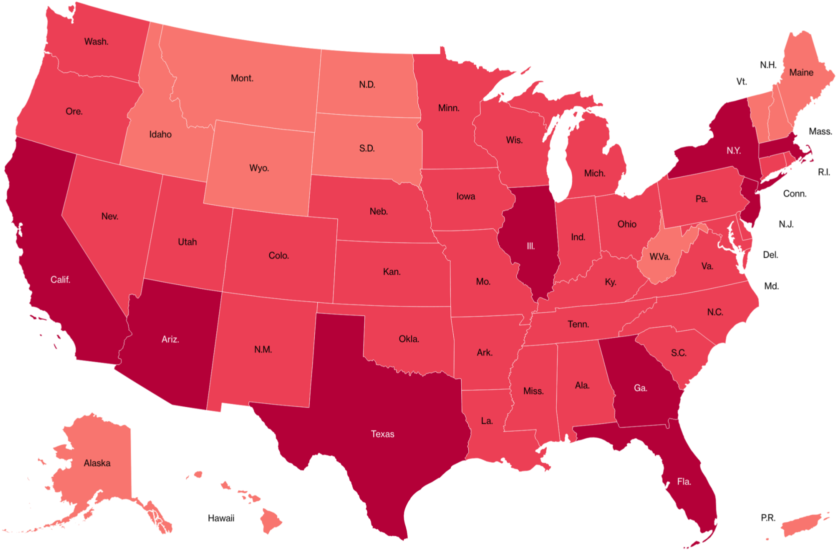
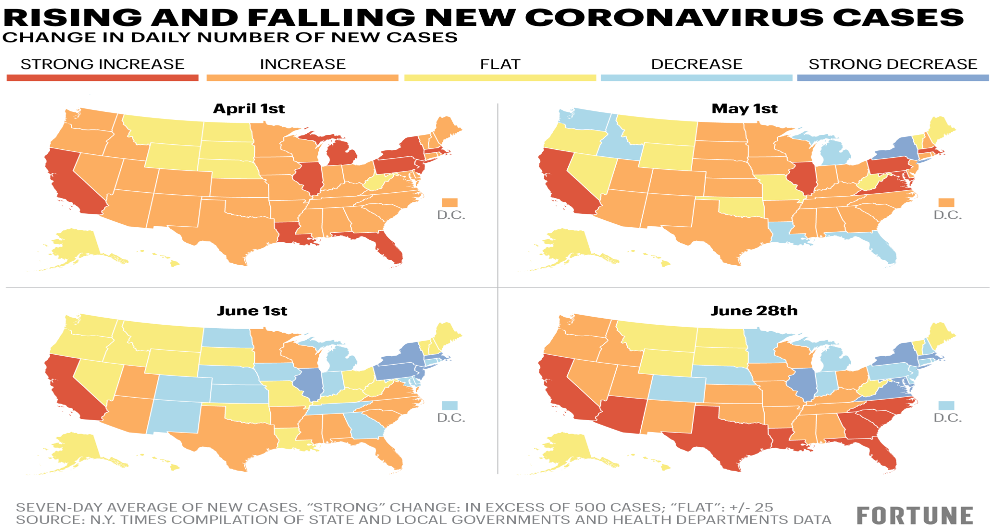
Spatial Problems



Spatiotemporal Problems



Areal data models

In the context of areal units the general inferential issues are the following:

1. Is there spatial pattern? If so, how strong is it?
2. Do we want to smooth the data? If so, how much?
3. For a new areal unit or set of units, how can we infer about what data values we expect to be associated with these units? This is the so-called *modifiable areal unit problem (MAUP)*.

We will explore both descriptive and model-based approaches in this lecture.

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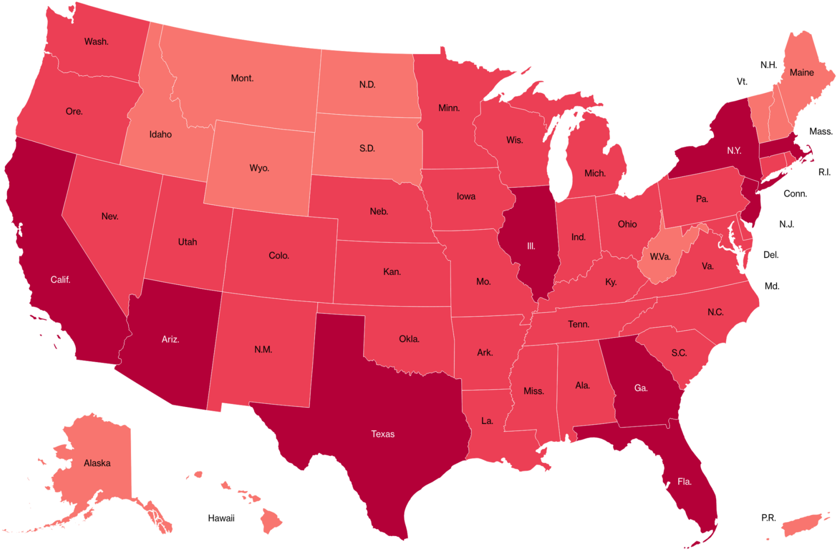
Exploratory data analysis (EDA)

- The primary concept *proximity matrix* W for areal units $1, 2, \cdots, n$ is defined by setting entries w_{ij} spatially connect units i and j ($w_{ii} = 0$).
- Binary choice: $w_{ij} = 1$ if i and j share common boundary; otherwise 0.
- 'Distance': e.g. decreasing function of intercentroidal distance between the units, binary values based on truncated distance or K nearest neighborhood.
- W can be standardized as $\widetilde{W}$ with $\tilde{w}_{ij} = w_{ij}/w_{i+}$ where $w_{i+} = \sum_j w_{ij}$. $\widetilde{W}$ is row stochastic, i.e. $\widetilde{W}\mathbf{1} = \mathbf{1}$.
- Divide distances into bins $(0, d_1], (d_1, d_2], \cdots$ and define k-th order neighbors of unit i as all units with distances in $(d_{k-1}, d_k]$. We can define k-th order proximity matrix $W^{(k)}$ based on k-th order neighbors.

Exploratory data analysis (EDA)



- There are two standard statistics to measure the spatial association (Ripley, 1981).
- Moran's I:

$$I = \frac{n \sum_i \sum_j w_{ij}(Y_i - \bar{Y})(Y_j - \bar{Y})}{\left(\sum_{i \neq j} w_{ij}\right) \sum_i (Y_i - \bar{Y})^2} \tag{1}$$

- Under the null model where Y_i are i.i.d., $I \dot{\sim} N(-1/(n-1), \mathrm{Var}(I))$ with

$$\mathrm{Var}(I) = \frac{n^2(n-1)S_1 - n(n-1)S_2 - 2S_0^2}{(n+1)(n-1)^2 S_0^2}$$

where $S_0 = \sum_{i \neq j} w_{ij}$, $S_1 = \frac{1}{2}\sum_{i \neq j}(w_{ij} + w_{ji})^2$, $S_2 = \sum_k (\sum_j w_{kj} + \sum_i w_{ik})^2$.

- Geary's C:

$$C = \frac{n \sum_i \sum_j w_{ij}(Y_i - Y_j)^2}{\left(\sum_{i \neq j} w_{ij}\right) \sum_i (Y_i - \bar{Y})^2} \tag{2}$$

- $C \dot{\sim} N(1, \mathrm{Var}(C))$ under the null model.

Exploratory data analysis (EDA)

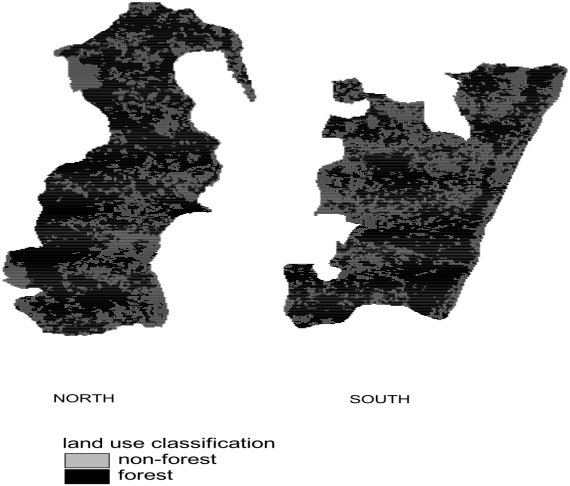


Figure 3.2 *Rasterized north and south regions (1 km × 1 km) with binary land use classification overlaid.*

Exploratory data analysis (EDA)

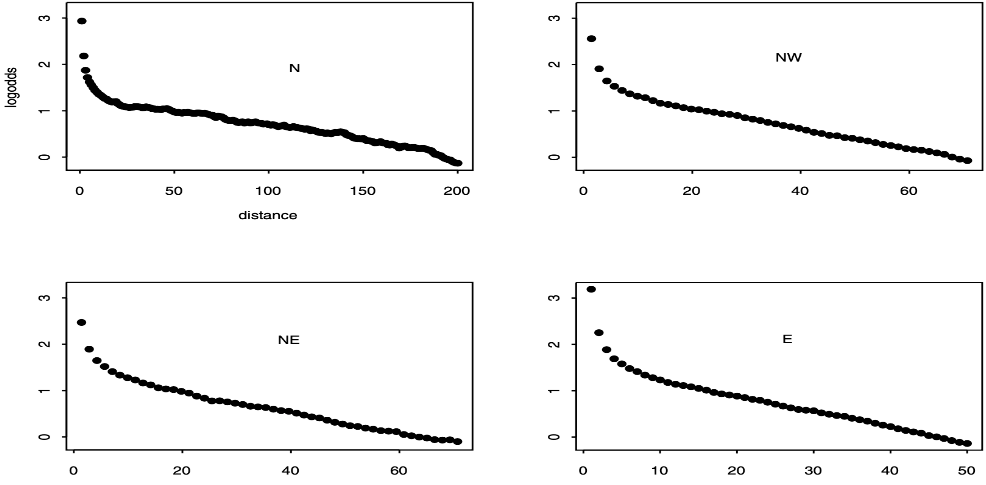


Figure 3.3 *Land use log-odds ratio versus distance in four directions.*

Exploratory data analysis (EDA)

- One could also investigate a choropleth map by smoothing Y_i's.
- The proximity matrix W provides a smoother: $\widehat{Y}_i = \sum_j w_{ij} Y_j / w_{i+}$.
- However, $\widehat{Y}_i$ ignores Y_i. We might revise it to be

$$\widehat{Y}_i^* = (1-\alpha) Y_i + \alpha \widehat{Y}_i \tag{3}$$

  where $\alpha \in (0,1)$.
- One can refer to general *filters*.

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Full conditional distribution

- Given $p(y_1, \cdots, y_n)$, the so-called *full conditional* distributions, $p(y_i|y_j, j \neq i)$, $i = 1, \cdots, n$, are uniquely determined.
- Brook's lemma (1964) proves the converse and constructively retrieve the unique joint distribution from these full conditionals.
- *Compatible* conditionals, *proper* conditionals (improper joint).
- Brook's lemma:

$$
\begin{aligned}
p(y_1, \cdots, y_n) = & \frac{p(y_1|y_2, \cdots, y_n)}{p(y_{10}|y_2, \cdots, y_n)} \cdot \frac{p(y_2|y_{10}, y_3, \cdots, y_n)}{p(y_{20}|y_{10}, y_3, \cdots, y_n)} \\
& \cdots \frac{p(y_n|y_{10}, , \cdots, y_{n-1,0})}{p(y_{n0}|y_{10}, , \cdots, y_{n-1,0})} \cdot p(y_{10}, \cdots, y_{n0})
\end{aligned} \tag{4}
$$

- Denote ∂_i as the set of neighbors of unit i. An areal process Y_i is referred as a *Markov random field (MRF)* (Besag 1974, Kaiser and Cressie 2000) if

$$
p(y_i|y_j, j \neq i) = p(y_i|y_j, j \in \partial_i) \tag{5}
$$

Gibbs distribution

- A *clique* is a set of cells (indices) such that each element is a neighbor of every other element.
- A *potential* of order k is a function of k exchangeable arguments.
- $p(y_1, \cdots, y_n)$ is a *Gibbs distribution* if it is a function of the Y_i only through potentials on cliques:

$$p(y_1, \cdots, y_n) \propto \exp\left\{ \gamma \sum_k \sum_{\boldsymbol{\alpha} \in \mathcal{M}_k} \phi^{(k)}(y_{\alpha_1}, y_{\alpha_2}, \cdots, y_{\alpha_k}) \right\} \tag{6}$$

where $\phi^{(k)}$ is a potential of order k, $\mathcal{M}_k$ is the collection of all subsets of size k from $\{1, 2, \cdots, n\}$, $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_k)$ indexes this set, and $\gamma > 0$ is a scale parameter.

Gibbs distribution

- The *Hammersley-Clifford Theorem* (Clifford 1990) demonstrates that if we have an MRF, then its joint distribution is a Gibbs distribution.
- Geman and Geman (1984) provides essentially the converse of the Hammersley-Clifford theorem: A Gibbs distribution determines an MRF.
- Sampling a MRF is reduced to sampling its associated Gibbs distribution, hence coining the term 'Gibbs sampler'.
- With cliques of order 1, we consider for continuous data on $\mathbb{R}^1$

$$p(y_1, \cdots, y_n) \propto \exp\left\{-\frac{1}{2\tau^2}\sum_{i,j}(y_i - y_j)^2 I(i \sim j)\right\} \tag{7}$$

- It is a Gibbs distribution on potentials of order 1 and 2 and that

$$p(y_i | y_j, j \neq i) = N\left(\sum_{j\in\partial_i} y_j/m_i, \tau^2/m_i\right) \tag{8}$$

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Conditionally autoregressive (CAR) models

- We begin with the Gaussian (*autonormal*) case. Suppose

$$Y_i|y_j, j \neq i \sim N\left(\sum_j b_{ij} y_j, \tau_i^2\right), \quad i = 1, \cdots, n \tag{9}$$

- By Brook's Lemma, we have

$$p(y_1, \cdots, y_n) \propto \exp\left\{-\frac{1}{2}\mathbf{y}' D^{-1}(I - B)\mathbf{y}\right\} \tag{10}$$

where $B = (b_{ij})$ and $D = \mathrm{diag}\{\tau_i^2\}$.

- $\mathbf{Y} \sim N(\mathbf{0}, \Sigma_{\mathbf{y}} = (I - B)^{-1} D)$?

$$\frac{b_{ij}}{\tau_i^2} = \frac{b_{ji}}{\tau_j^2} \quad \textit{for all } i, j \tag{11}$$

Conditionally autoregressive (CAR) models

- Setting $b_{ij} = w_{ij}/w_{i+}$ and $\tau_i^2 = \tau^2/w_{i+}$, we have

$$p(y_i|y_j, j \neq i) = N\left(\sum_j w_{ij}y_j/w_{i+}, \tau^2/w_{i+}\right) \tag{12}$$

- Therefore we have the joint distribution (intrinsically autoregressive, IAR)

$$p(y_1, \cdots, y_n) \propto \exp\left\{-\frac{1}{2\tau^2}\mathbf{y}'(D_w - W)\mathbf{y}\right\} = \exp\left\{-\frac{1}{2\tau^2}\sum_{i\neq j} w_{ij}(y_i - y_j)^2\right\} \tag{13}$$

where $D_w = \mathrm{diag}\{w_{i+}\}$.

- $(D_w - W)\mathbf{1} = 0$. $\Sigma_{\mathbf{y}} =?$
- Redefine $\Sigma_{\mathbf{y}}^{-1} = D_w - \rho W > 0$ for chosen $\rho \in (1/\lambda_{(1)}, 1/\lambda_{(n)})$.

Conditionally autoregressive (CAR) models

- Rewriting the autonormal model

$$\mathbf{Y} = B\mathbf{Y} + \boldsymbol{\epsilon} \tag{14}$$

- If $p(\mathbf{y})$ is proper, then:
  - $\mathbf{Y} \sim N(\mathbf{0}, (I-B)^{-1}D)$, $\boldsymbol{\epsilon} \sim N(\mathbf{0}, D(I-B)^T)$, and $\mathrm{Cov}(\boldsymbol{\epsilon}, \mathbf{Y}) = D$.
  - $1/(\Sigma_{\mathbf{y}}^{-1})_{ii} = \mathrm{Var}(Y_i|Y_j, j \neq i) = \tau_i^2$.
  - $(\Sigma_{\mathbf{y}}^{-1})_{ij} = b_{ij} = 0$ implies $Y_i \perp Y_j | Y_k, k \neq i, j$. We have control on conditional independence (by setting $w_{ij} = 0$)!
- One can introduce regression component to CAR.
- Considering a vector of dependent areal units leads to MCAR model.
- CAR model can be applied to point-level data.

Conditionally autoregressive (CAR) models

- We could also consider non-Gaussian case.

$$p(y_i|y_j, j \neq i) = \exp\left(\{\psi(\theta_i y_i - \chi(\theta_i))\}\right) \tag{15}$$

where $\theta_i = \sum_{j\neq i} b_{ij} y_j$.

- Autologistic model:

$$\log \frac{P(Y_i = 1)}{P(Y_i = 0)} = \mathbf{x}_i^T \gamma + \psi \sum w_{ij} y_j \tag{16}$$

which implies

$$p(y_1, \cdots, y_n) \propto \exp\left(\gamma^T (\sum_i y_i \mathbf{x}_i) + \psi \sum_{i,j} w_{ij} y_i y_j\right) \tag{17}$$

- Potts model

$$P(Y_i = l|Y_j, j \neq i) \propto \exp\left(\psi \sum_{i,j} w_{ij} I(Y_j = l)\right) \tag{18}$$

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Simultaneous autoregressive (SAR) models

- Now we start from $\boldsymbol{\epsilon} \sim N(0, \tilde{D})$ with $\tilde{D} = \mathrm{diag}\{\sigma_i^2\}$. Then

$$\mathbf{Y} = B\mathbf{Y} + \boldsymbol{\epsilon} \sim N(\mathbf{0}, (I - B)^{-1}\tilde{D}(I - B)^{-T}) \tag{19}$$

  where $\mathrm{Cov}(\boldsymbol{\epsilon}, \mathbf{Y}) = \tilde{D}(I - B)^{-1}$.
- $(I - B)$ must be full rank:
  1. $B = \rho W$, W the contiguity matrix $w_{ij} = I(i \sim j)$. $Y_i = \rho \sum_j Y_j I(j \in \partial_i) + \epsilon_i$ with *spatial autoregressive parameter* $\rho \in (1/\lambda_{(1)}, 1/\lambda_{(n)})$.
  2. $B = \alpha \widetilde{W}$, with *spatial autocorrelation parameter* $\alpha \in (-1, 1)$.
- SAR model is introduced in a regression context and is applied to the *residuals* $\mathbf{U} = \mathbf{Y} - X\boldsymbol{\beta}$:

$$\mathbf{U} = B\mathbf{U} + \boldsymbol{\epsilon} \tag{20}$$

Simultaneous autoregressive (SAR) models

- The overall model is then written as follows with B interpolating between an OLS ($B = 0$) regression and a purely spatial model:

$$\mathbf{Y} = B\mathbf{Y} + (I - B)X\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{21}$$

- Assuming $\tilde{D} = \sigma^2 I$, the log-likelihood can be efficiently calculated thus amenable to MLE

$$\frac{1}{2}\log|\sigma^{-1}(I - B)| - \frac{1}{2\sigma^2}(\mathbf{Y} - X\beta)^T(I - B)(I - B)^T(\mathbf{Y} - X\beta) \tag{22}$$

- Extendable to Bayesian setting. No convenient form for full conditional distributions as in CAR.

CAR vs SAR

- Both are spatial models for areal data.
- They are equivalent iff

$$(I - B)^{-1} D = (I - B)^{-1} \tilde{D} (I - B)^{-T} \tag{23}$$

- Cressie (1993) shows that any SAR model can represented as a CAR model; but not vice versa.
- The first-order neighbor correlations increase at a slower rate as a function of ρ in the CAR model than in SAR model.
- Gibbs sampler is usually used for CAR but likelihood based inference is used for SAR.