STP598sta: Spatiotemporal Analysis

Homework 3

Name: Your name; NetID: Your ID

Due 11:59pm Sunday November 10, 2024

Question 1

Compute the coherence (generalized correlation), $\frac{\mathrm{cov}(Y_\ell(\mathbf{s}),Y_{\ell'}(\mathbf{s}+\mathbf{h}))}{\sqrt{\mathrm{cov}(Y_\ell(\mathbf{s}),Y_\ell(\mathbf{s}+\mathbf{h}))\mathrm{cov}(Y_{\ell'}(\mathbf{s}),Y_{\ell'}(\mathbf{s}+\mathbf{h}))}}$:

(a) for the cross-covariance $\Sigma_{\mathbf{Y}(\mathbf{s}),\mathbf{Y}(\mathbf{s}')} = C(\mathbf{s}-\mathbf{s}') = \sum_{j=1}^{p} \rho_j(\mathbf{s}-\mathbf{s}')T_j$.

(b) for the cross-covariance $C(\mathbf{s}-\mathbf{s}') = \sum_{u=1}^{r} \rho_u(\mathbf{s}-\mathbf{s}')T^{(u)}$.

Question 2

Let $Y(\mathbf{s}) = (Y_1(\mathbf{s}), Y_2(\mathbf{s}))^T$ be a bivariate process with a stationary cross-covariance matrix function

$$C(\mathbf{s}-\mathbf{s}') = \begin{pmatrix} c_{11}(\mathbf{s}-\mathbf{s}') & c_{12}(\mathbf{s}-\mathbf{s}') \\ c_{12}(\mathbf{s}'-\mathbf{s}) & c_{22}(\mathbf{s}-\mathbf{s}') \end{pmatrix}$$

and a set of covariates $\mathbf{x}(\mathbf{s})$. Let $\mathbf{y} = (\mathbf{y}_1^T, \mathbf{y}_2^T)^T$ be the $2n \times 1$ data vector, with $\mathbf{y}_1^T = (y_1(\mathbf{s}_1), \cdots, y_1(\mathbf{s}_n))^T$ and $\mathbf{y}_2^T = (y_2(\mathbf{s}_1), \cdots, y_2(\mathbf{s}_n))^T$.

(a) Show that the cokriging predictor has the form

$$\mathrm{E}[Y_1(\mathbf{s}_0)|\mathbf{y}] = \mathbf{x}^T(\mathbf{s}_0)\boldsymbol{\beta} + \boldsymbol{\gamma}^T\Sigma^{-1}(\mathbf{y} - X\boldsymbol{\beta})$$

, with appropriate definitions of $\boldsymbol{\gamma}$ and Σ.

(b) Show further that if $\mathbf{s}_k$ is a site where $y_l(\mathbf{s}_k)$ is observed, then for $l = 1, 2$, $\mathrm{E}[Y_l(\mathbf{s}_k)|\mathbf{y}] = y_l(\mathbf{s}_k)$ if and only if $\tau_l^2 = 0$.

Question 3

For a moving average process of the form

$$x_t = w_{t-1} + 2w_t + w_{t+1}$$

where w_t are independent with zero means and variance σ_w^2, determine the autocovariance and autocorrelation functions as a function of lag $h = s - t$ and plot the ACF as a function of h.

Question 4

In this problem, we explore the difference between a random walk and a trend stationary process.

(a) Generate four series that are random walk with drift, $x_t = \delta t + \sum_{i=1}^{t} w_j$, of length $n = 100$ with $\delta = 0.01$ and $\sigma_w = 1$. Call the data x_t for $t = 1, \cdots, 100$. Fit the regression $x_t = \beta t + w_t$ using least squares. Plot the data, the true mean function (i.e. $\mu_t = 0.01t$) and the fitted line, $\hat{x}_t = \hat{\beta} t$, on the same graph. Hint: The following `R` code may be useful.

```r
par(mfrow=c(2,2), mar=c(2.5,2.5,0,0)+.5, mgp=c(1.6,.6,0)) # set up
for (i in 1:4){
  x = ts(cumsum(rnorm(100,.01,1)))         # data
  regx = lm(x~0+time(x), na.action=NULL)  # regression
  plot(x, ylab='Random Walk w Drift')     # plots
  abline(a=0, b=.01, col=2, lty=2)        # true mean (red - dashed)
  abline(regx, col=4)                     # fitted line (blue -  solid)
}
```

(b) Generate four series of length $n = 100$ that are linear trend plus noise, say $y_t = 0.01t + w_t$, where t and w_t are as in part (a). Fit the regression $y_t = \beta t + w_t$ using least squares. Plot the data, the true mean function (i.e. $\mu_t = 0.01t$) and the fitted line, $\hat{y}_t = \hat{\beta}t$, on the same graph.

(c) Comment (what did you learn from this assignment).