MATHS – IV

Descartes rule of sign

If $f(x) \in C[a,b]$ and $f(a)$ & $f(b)$ have opp signs; then $\exists$ a $c \in (a,b)$ s.t $f(c) = 0$.

* Direct Methods :

(i) $a + bx = 0$

$x = -a/b \quad , \; b \neq 0$

(ii) $ax^2 + bx + c = 0 \qquad a \neq 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

(iii) $ax^3 + bx^2 + cx + d = 0 \qquad x = ??$

(iv) $ax^4 \; - - - \qquad x = ??$ (No direct method)

*) **Multiple Root** - root repeated more than once.

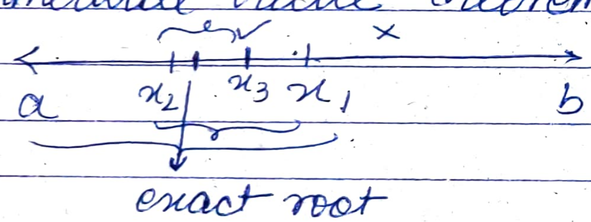
*) **Multiplicity** - no of times the root is repeated.

If $f(x) = 0$ can be written as $f(x) = (x-a)^m g(x) = 0$ where $g(x)$ is bounded and $g(a) \neq 0$ then 'a' is a multiple root of multiplicity 'm'; if $m = 1$ $x = a$ is simple roo

~~If $f(x)$ has simple root b~~
~~(a) $f(x) = (x-b)^2 g(x$~~

If $f(x)$ has simple root b:
$f(x) = (x-b)g(x)$, $g(b) \neq 0$ (If this is not given; $f(x) = (x-b)\, g(x)$ ✓)

⊛ **BISECTION METHOD** - Repeated application of intermediate value theorem.



Let a root $\in (a,b) = I_0$ & $f(a)f(b) < 0$

$x_1 = \dfrac{a+b}{2}$

$I_1 = (a, x_1)$: $f(a)f(x_1) < 0$

$x_2 = \dfrac{a+x_1}{2}$

★ Continue the process ; After k time we either find root & find an interval of $\dfrac{b-a}{2^k}$ which contains the root

Ex – approximation of $\sqrt{3}$ to 2 decimal place (stopping criteria).

$\sqrt{3} \Rightarrow x^2 - 3 = 0$ ✓

Drawback – ① It can be never ending process if stopping criteria isn't given

② Convergence is linear ⇒ slow.

Condition – ① $f(x)$ is cts from $[a, b]$

② $f(a)f(b) < 0$ to find solⁿ of $f(x) = 0$.

Algorithm:

① Set $a_1 = a$, $b_1 = b$

② Set $i = 1$

③ $x_i = \frac{1}{2}(a_i + b_i)$

④ If x_1 is satisfactory approx goto step 10.

If not goto step 5.

⑤ If $f(x_i)f(a_i) > 0$ goto step 6

If $f(x_i)f(a_i) < 0$ goto step 8.

⑥ $a_{i+1} = x_i$

$b_{i+1} = b_i$

⑦ Add 1 to i, go to step 3.

⑧ Set $a_{i+1} = a_i$

$b_{i+1} = x_i$

⑨ Add 1 to go to step 3

⑩ The procedure is complete.

★) A number α is called root or zero of $f(x)=0$ if $f(\alpha)=0$ and number α^* is said to be an approximation root / zero of $f(x)=0$ with an error $\varepsilon>0$ if, $|\alpha-\alpha^*| - \varepsilon$

Recall that: a method is convergent if sequence $x_0, x_1, x_2 \ldots$ ($\langle x_n \rangle$) converges by the true / exact solution

i.e given $\varepsilon>0$, $\exists$ a $n_0 \in n$ s.t

$|x_n-\alpha| < \varepsilon \quad \forall \; n \geq n_0.$

x_1, x_2 iterate

$\boxed{\text{Let } e_n = x_n-\alpha : e_n \text{ is called error in } n^{th} \text{ iterate}}$

★ A method which yields the iterates x_1, x_2, x_n is ~~cal~~ said to have to have ~~root~~ rate of convergence 'p' if 'p' is largest positive real number s.t $\lim_{n\to\infty} \frac{e_{n+1}}{|e_n|^p} = \lambda$ (constant)

p = 1 ; convergence is linear.

p = 2 ; convergence is quadratic.

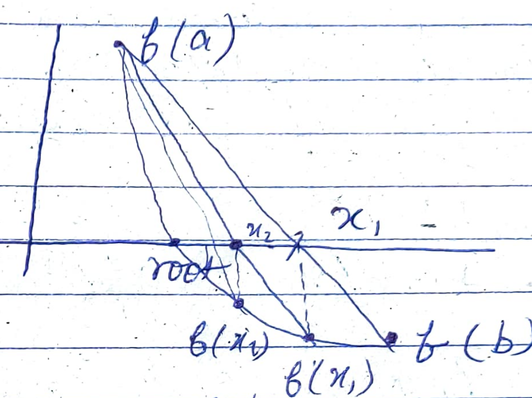
★ $\boxed{\text{asymptotic error constant}}$

$\boxed{\text{★ Convergence of Bisection method is linear}}$

Advantages : easy as you just have to know IVT.

⊛ Regula Falsi Method also called False Position Method :

Equation of chord joining the two points $[a, f(a)]$, $[b, f(b)]$



$$\boxed{y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)}$$

The method consists of replacing the ~~point~~ part of curve between the points $[a, f(a)]$ & $[b, f(b)]$ by means of joining these points and the point of intersection of chord with x-axis as an approximation of root.
Hence, $(x, 0)$ is point of intersection with x-axis and is first approx :

$$\boxed{0 - f(a) = \frac{f(b)-f(a)}{b-a}(x_1-a)}$$

$$\boxed{x_1 = a - \frac{(b-a)}{f(b)-f(a)} f(a)}$$

Now if $f(x_1)$ & $f(a)$ are of opposite signs then root lies in (a, x_1).

and if we replace b by x_1 in ① then we get obtain next approximation ; else replace by x_1.
The process is continued till the root is obtained to the desired accuracy
↳ stopping criteria
disadvantage → $\boxed{\text{never ending}}$

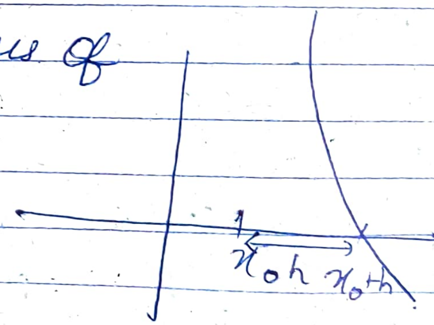
⊛ NEWTON RAPHSON METHOD

Let x_0 be an approximation to the root $f(x)=0$ and $x_1 = x_0 + h$ be exact root $\therefore f(x_1) = 0$

$$f(x_0+h) = 0$$

$$f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_3) + \cdots = 0$$

neglecting 2nd & higher powers of since h is small



$$h = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 + h$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$$x_2 = x_1 + h$$

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

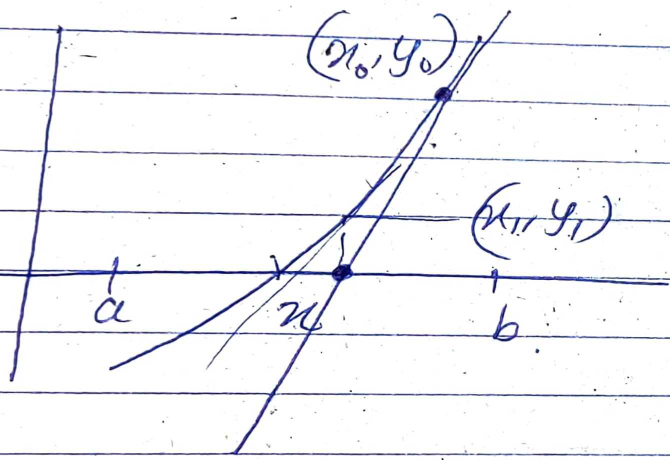
$$\Rightarrow \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$n = 0, 1, 2$

★ It is useful in cases of large values of $f'(x)$ when graph of $f(x)$ which crossing x-axis is nearly vertical.

Advantage = ① simplicity

Drawbacks = ① divide by zero
② Differentiability
③ Convergence geometrically it is etc.



also called:
Method of Tangent

Let x_0 be initial approxn then equation of tangent to $y = f(x)$ is:

$$y - f(x_0) = f'(x_0)\,(x - x_0)$$

$$x_1 = x - \frac{f(x_0)}{f'(x_0)}$$ where $(x, 0)$ is pt of intersection with x-axis.

*) Find cube root of N using NR method

$x = \sqrt[3]{N}$

$x^3 = N \Rightarrow \boxed{x^3 - N = 0}$ $f(x)$

$f'(x) = 3x^2$

$f'(N) = 3N^2$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n - \frac{x_N^3 - N}{3x_n^2}$

$$\boxed{x_{n+1} = \frac{2x_n^3 + n}{3x_n^3}}$$

To find first initial approximation Use Bisection method for x_0 *

$x^3 - 2x + 5 = 0$ (to 3 decimal places)

↳ $(-2, -3)$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

⊛ SECANT METHOD - improvement of RF method

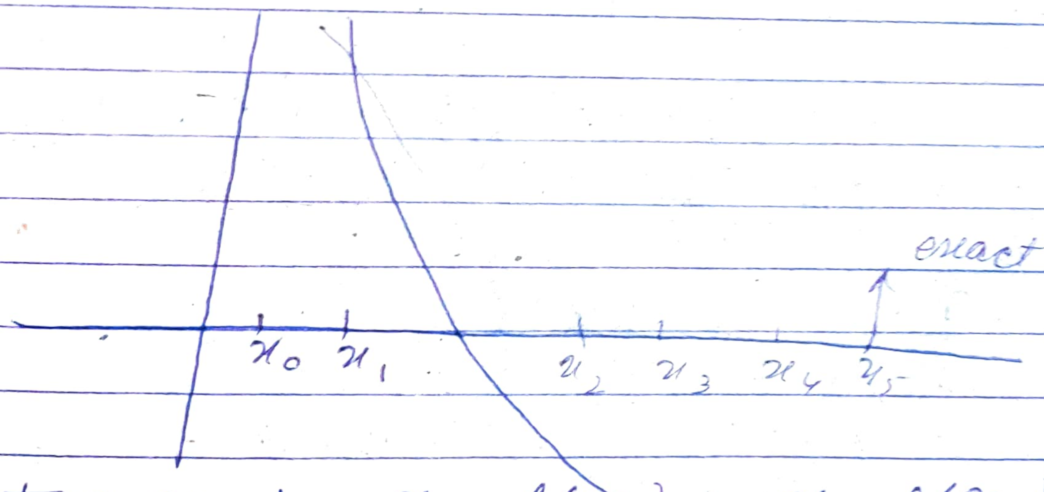
$f(a) f(b) < 0$

In secant method we need two initial approx.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

$$\boxed{x_{n+1} = x_n - \frac{(x_n - x_{n-1}) f(x_n)}{f(x_n) - f(x_{n-1})}}$$



join two point $x_0, f(x_0)$ $x_1, f(x_1)$ and extend the line get the third approx where line meets x-axis

⊛ SIGNIFICANT DIGITS - digits used to express a number

Ex 3.1416 — 5 SD

0.66667 — 5 SD

4.0687 — 5 SD

0.00023 — 2 SD.

n SD - you can list total n digits beginning with left most non zero digit

① **Chopping | Truncating** - simply cut tail of req no.

1.412313478212...

└ 1.4123 — by chopping

② **Rounding off** - discard all digits to right of nth digit if (n+1)th digit

(a) <5 leave nth undisturbed

b) >5 add 1 to nth.

c) =5 (i) Add 1 to nth if n is odd

(ii) undisturbed if n is even

Q. 3.1455123 — SD

↳ 3.14 — 3SD

↳ 3.146 — 4SD.

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⊛ ERRORS :

① **Inherent error** - errors which are already present on statement of problem
ex - wrong data collection.

② **Rounding** - Arise during process of rounding off.

③ **Truncation** - Arise due to :
(i) truncation
(ii) Replacement of an infinite series by finite one

$$x_T = e^x = 1 + x + \frac{x^2}{2!} \ldots$$

$$x_A = 1 + x + \frac{x^2}{2!}$$

$$\boxed{E_T = x_T - x_A}$$ Truncation

⊛.

Error - $E = x_T - x_A$

Absolute error - $E_a = |x_T - x_A|$

Relative error - $E_r = \frac{|x_T - x_A|}{|x_T|}$

Percentage error $= 100 \times \varepsilon_r$

$$= \frac{100\ \varepsilon_a}{|x_T|}$$

$$= 100 \times \frac{|x_T - x_A|}{|x_T|}$$

Initial approx - the value for which iterative sequence starts (x_0).

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*) $Ax = b$ where;

$A \in \mathbb{R}^{n \times n}$, $x \in R^n$ $b \in R^n$.

set of real square matrix of order n; A is real $n \times n$ matrix
b & x are real n-vector.

Direct Methods
Gauss Elimination
Gauss Jordan
Factorization Method

Iterative method
Gauss - Jacobi
Gauss - Seidel
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GAUSS ELIMINATION - $Ax = b$

Elementary op^n : make it $Ux = c$ — upper triangular matrix $\begin{bmatrix} & \diagdown \\ 0 & \end{bmatrix}$
Apply back substitution method to get x.

Method of eval $[A|b] \sim [U|c]$

ex
$$ax + by = r \longrightarrow (\text{I}) \quad \text{— pivotal eq}^n$$
$$cx + dy = s \longrightarrow (\text{II}) \quad \longrightarrow \text{pivot eq}^n$$

$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{b}{\begin{bmatrix} r \\ s \end{bmatrix}}$$

step 1
Eliminate x from second eq
$$eq^n(2) \leftarrow eq(2) - \frac{c}{a}\, eq(1)$$

$$ax + by = r$$
$$\left(d - \frac{c}{a} b\right) y = \left(s - \frac{c}{a}\right) r$$

Step 2 : $y = \dfrac{s - c/a\, r}{d - bc/a}$

put the value of y to get x

else:

$$\begin{array}{cc} A & x \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} \end{array} = \begin{bmatrix} r \\ s \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{cc|c} a & b & r \\ c & d & s \end{array}\right]$$

$$R_2 \leftarrow R_2 - \frac{c}{a} R_1$$

$$\left[\begin{array}{cc|c} a & b & r \\ 0 & d - cb/a & s \end{array}\right]$$

$$= [U|c]$$

$$ax + by = r$$

$$d - \frac{cb}{a}\, y = s_1$$

$$\boxed{y = \frac{s_1}{d - cb/a}}$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_n x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + x_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \vdots \quad - \quad a_{2n}x_{2n} = b_n$$

$$\begin{bmatrix} a_{21} & - & a_n \\ \vdots & & \\ a_{n1} & - & x_{2n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\Rightarrow$ ~~[A|b]~~ $\sim$ [U|C]

Gauss Elimination : upper triangular system & unknowns are found by back substitution.

I) eliminate x from all eq^n except one the first eq^n is pivotal eq^n & a_1 = first pivot.

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &\ \ d_3 \end{aligned} \Rightarrow \begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ b_2' y + c_2' z &= d_2 \\ b_3' y + c_3' z &= d_3 \end{aligned}$$

I) eliminate y from third equation

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ b_2' y + c_2' z &= d_2' \\ c_3'' z &= d_3'' z \end{aligned}$$

(back substitution)

elementary row op^n.

else :

$$\overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \overset{x}{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} r \\ s \end{bmatrix}$$

$$[A|b] = \left[\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{matrix} r \\ s \end{matrix}\right]$$

$$R_2 \leftarrow R_2 - \frac{c}{a} R_1$$

$$\left[\begin{matrix} a & b \\ 0 & d - cb/a \end{matrix} \middle| \begin{matrix} r \\ s \end{matrix}\right]$$

$$= [U|c]$$

$$ax + by = r$$

$$d - \frac{cb}{a} y = s_1$$

$$\boxed{y = \frac{s_1}{d - cb/a}}$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_n x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + x_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots \quad a_{2n}x_{2n} = b_n$$

$$\begin{bmatrix} a_{21} & \cdots & a_n \\ \vdots & & \\ a_{n1} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

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$\Rightarrow [A|b] \sim [U|C]$

Gauss Elimination : upper triangular system & unknowns are found by back substitution.

I) eliminate x from all eq^n except one the first eq^n is pivotal eq^n & a_1 = first pivot.

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z & \; d_3 \end{aligned} \Rightarrow \begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ b_2' y + c_2' z &= d_2 \\ b_3' y + c_3' z &= d_3 \end{aligned}$$

II) eliminate y from third equation

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ b_2' y + c_2' z &= d_2' \\ c_3'' z &= d_3'' z \end{aligned}$$

(back substitution)

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elementary row op^n.

labour of back substitution is saved at cost of additional calculations.

Gauss Jordan Method - modification of GEM. elimination of unknown is performed in eqn below but in above also reducing system to diagonal matrix (each involving one unknown)

Factorization Method - every matrix A can be expressed as product of lower triangular matrix provided all principal minors of A are non-singular i.e $A = [a_{ij}]$

$\boxed{\text{Doolittle's Method}}$ / Crout.

It fails if any $l_{ii} = 0$

- superior to Gauss elimination.
- finding inverse
- software for computers.

Iterative Method

Gauss

★ Jacobi -

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

If a_1, b_2, c_3 are large compared to other coeff.

then: $x = k_1 - l_1 y - m_1 z$

$y = k_2 - l_2 x - m_2 z$

$z = k_3 - l_3 x - m_3 y$

initial approx x_0, y_0, z_0 (each = 0)

$x_1 = k_1, \quad y_1 = k_2, \quad z_1 = k_3$

$x_2 = k_1 - l_1 y_1 - m_1 z_1$

$y_2 = k_2 - l_2 x_1 - m_2 z_1$

$z_2 = k_3 - l_3 x_1 - m_3 y_1$

repeated till same / stopping condition

faster convergence
converges if abs value of largest eq^n is > sum of ~~val~~ abs val of remaining coeff.

★ **Gauss - Seidal Iteration method** modification of method of Jacobi

first: $x_0 \Rightarrow y_0 = 0 \quad z_0 = 0 \quad = x_0$

$y_0 \Rightarrow x = x_0, \quad z_0 = 0 \quad = y_0$

$z_0 \Rightarrow x = x_0, \quad y = y_0$

Successive Over Relaxation

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Residuals

$$R_x = d_1 - a_1 x - b_1 y - c_1 z$$
$$R_y = d_2 - a_2 x - b_2 y - c_2 z$$
$$R_z = d_3 - a_3 x - b_3 y - c_3 z$$

Assume $x = y = z = 0$

Op^n table

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

① At each step numerically largest residual is reduced to almost zero.

② R_x to be reduced by p : x should be increased by P/a,

③ When all the residuals have been reduced to almost zero the increment in x, y, z are added separately to give desired solution

$\Sigma \delta x = 6.13$, $\Sigma \delta y = 4.31$, $\Sigma \delta z = 3.23$.

~~Note~~

~~★) If A is a positive semidefinite then $A = LU$ is unique~~

~~★ Compute the no of op on Crouts / Doolittle~~

Newton Raphson

$1/N = x_{n+1} = x_n (2 - N x_n)$

$\sqrt{N} = x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$

$1/\sqrt{N} = x_{n+1} = \frac{1}{2}\left(x_n + 1/N x_n\right)$

$\sqrt[k]{N} = x_{n+1} = \frac{1}{k}\left((k-1) x_n + \frac{N}{x_n^{k-1}}\right)$

Mathematics - 4

I)

- **FINITE DIFFERENCES** - deals with changes that take place on value of f^n due to finite changes in the independent variable

The change may be of equal interval / unequal interval

- DIFFERENCES
  - forward diff
  - backward diff
  - Central diff

I)

- **Forward Difference** : Let $x = x_0$,

$$x_1 = x_0 + h$$
$$x_2 = x_0 + 2h = x_1 + h$$
$$\vdots$$
$$x_n = x_0 + nh$$

$$\& \quad y = y_0$$
$$y_1 = f(x_1) = f(x_0 + h)$$
$$\vdots$$
$$y_n = f(x_0 + nh) = f(x_n)$$

The differences $y_1 - y_0$, $y_2 - y_1$, ---- $y_n - y_{n-1}$ denoted by Δy_0, Δy_1, Δy_2 ---- Δy_{n-1} are called FIRST FORWARD DIFFERENCES where Δ - forward diff^n operator

$$\left.\begin{array}{ll} \Delta y_r = y_{r+1} - y_r & \text{- first} \\ \Delta^2 y_r = \Delta y_{r+1} - \Delta y_r & \text{- second} \\ \Delta^3 y_3 = \Delta^2 y_{r+1} - \Delta^2 y_r & \text{- third} \end{array}\right\} \text{forward diff}^n$$

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$
$$\Delta^2 y_r = y_{r+2} - y_{r+1} - y_{r+1} + y_r$$
$$\Delta^2 y_r = y_{r+2} - 2y_{r+1} + y_r$$

In a difference table;

x = argument | y_0 = first entry, leading term

y = function | $\Delta y_0, \Delta^2 y_0$ = leading differences.

(*) DIAGONAL DIFF^n TABLE

value of x	value of y	I diff	II diff	III diff	IV diff
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

$$\Delta^4 y_0 - 4y_3 + 6y_2 - 4y_1 + y_0$$
$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

Given

x	2	4	6	8	10
y	5	10	17	29	15

$\Delta f(6) = 12$ ✓ $\Delta^2 f(4) = 5$ ✓ $\Delta^3 f(2) = 3$.

$$\Delta^3 f(2) = \Delta^2 f(4) - \Delta^2 f(2)$$
$$= \Delta f(6) - \Delta f(4) - (\Delta f(4) - \Delta f(2))$$
$$= 12 - 7 - 7 + 5$$
$$= 3$$

$$\frac{\Delta f(9) - \Delta f(4)}{12 - 7}$$

Properties

~~Given~~ ★ 1) Δ is linear

$$\Delta[\alpha f(x) + \beta g(x)] = \alpha[\Delta f(x)] + \beta[\Delta g(x)]$$

2) Δ satisfies index law ;

$$\Delta^m \Delta^n f = \Delta^{m+n} f \qquad m, n \in \mathbb{N}$$

$$\Delta(f \cdot \phi) = f(x+h)\Delta\phi(x) + \phi(x)\Delta f(x)$$

II) Backward diff (∇)

first $\nabla r = y_r - y_{r-1}$

second $\nabla^2 r = \nabla y_r - \nabla y_{r-1}$

~~third~~ kth $\nabla^k r = \nabla^{k-1}{}_r - \nabla^{k-1}{}_{r-1}$

starts with ∇y_1

$$\boxed{\nabla^n y_n = y_n - n y_{n-1} + \frac{n(n-1)}{2} y_{n-2} + \cdots (-1)^n y_0}$$

Backward / Horizontal diff table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
x_1	y_1	∇y_1	$\nabla^2 y_2$		
x_2	y_2	∇y_2	$\nabla^2 y_3$	$\nabla^3 y_3$	
x_3	y_3	∇y_3			
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\Delta^3 y_4$	$\nabla^4 y_4$

1	8
3	12
5	21
7	36
9	62

$\nabla^2 f(7) = = 6$?

$\nabla^3 f(9) = 5$

$\nabla f(3) = 4$

Relation between ∇ & Δ

$\Delta y_0 = y_1 - y_0 = \nabla y_1$

$\Delta^2 y_0 = y_2 - 2y_1 + y_0 = \nabla^2 y_2$

In general:

$$\boxed{\Delta^n y_i = \nabla^n y_{i+n}}$$

$$\boxed{\nabla^n y_i = \Delta^n y_{i-n}}$$

n = order of diff^n

i = no of tabloid value

⊛ Central difference

$\delta y(x) = y(x+h/2) - y(x-h/2)$ $\quad h=1$

$\delta y_{1/2} = y_1 - y_0$

$\delta y_{3/2} = y_2 - y_1$

$\delta^2 y_1 = \delta_{3/2} - \delta_{1/2}$

$\delta^2 y_2 = \delta_{5/2} - \delta_{4/2}$

$$\boxed{y_n - y_{n-1} = \delta y_{n-(1/2)} = \delta y_{\frac{2n-1}{2}}}$$

⊛ Central diffⁿ table :

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
x_1	y_1	$\delta y_{1/2}$			
x_2	y_2	$\delta y_{3/2}$	$\delta^2 y_1$	$\delta^3{}_{3/2}$	$\delta^4{}_{2}$
x_3	y_3	$\delta y_{5/2}$	$\delta^2 y_2$	$\delta^3{}_{5/2}$	
x_4	y_4	$\delta y_{7/2}$	$\delta^3 y_3$		

$5/2 - 1/2 = 4/2 = 2$

Ex $\Delta e^x = e^{x+1} - e^x$
$= (e-1)\, e^x$

Find missing y_x

y_x	0	0	1	1	
Δy_x	0	1	2	3	?

$y_1 - y_0 = \Delta y_0 = 0 \Rightarrow y_1 = 0$

$y_2 - y_1 = \Delta y_1 = 1 \Rightarrow y_2 = 1$

$y_3 - y_2 = \Delta y_2 = \quad \Rightarrow y_3 = 1$

0

begin — Forward
end — Backward
centre — Stirling / Bessel

II)

* INTERPOLATION

If $y = f(x)$, $x_0 \leq x \leq x_n$ is given, then we can compute $y = f(x)$ for certain values of x say $x_0, x_1, x_2, x_3, ----, x_n$.

Our problem is the converse part of the above stated fact ie suppose $y = f(x)$ is given for certain values of x say $x_1 ---- x_n$ then compute $y = f(x)$

ie suppose $(x_0, y_0), (x_1, y_1), (x_2, y_2) ---- (x_n, y_n)$ are given then objective is to find $y = f(x)$

One can find another f^n. $\phi(x)$ such that $\phi(x) = f(x)$ at $x = x_1, x_2, ---- x_n$.

The process for finding such ϕ is called Interpolation. If $\phi(x)$ is a polynomial then it is called as an interpolating polynomial

* Weirshrass theorem - If $f(x) \in [a, b]$ then for any $\varepsilon > 0$ $\exists$ a polynomial $P(x)$ s.t

$$|f(x) - P(x)| < \varepsilon \quad \forall \; x \in (a, b)$$

Questions - ① How will we ~~define~~ derive the closeness of approximation & how it will be measured.
② How to decide an approx as the best approx or not ?

While the process of finding values of y for some value for some x outside given range is called EXTRAPOLATION

⊛ Newton's Forward Interpolation

Let $(x_0, y_0), (x_1, y_1) \ldots (x_n, y_n)$ be calculated points where x_is are equally spaced

Claim - To construct a function / polynomial $\phi(x)$ of degree not higher than 'n' which coincide ~~the~~ with values of y_i at those points x_i

$$\phi(x_i) = y_i \qquad x_i = 0, 1, 2, \ldots n$$

Since $\phi(x)$ is of degree n we assume

$$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots$$
$$\ldots a_n(x-x_0)(x-x_1) \ldots (x-x_{n-1})$$

★ To find a_i $\quad i = 0, 1, \ldots n$
from the initial conditions;

$y_i = \phi(x_i) \quad i = 0, 1, 2, \ldots n$

$y_0 = \phi(x_0)$

$$\boxed{y_0 = a_0}$$

$y_1 = \phi(x_1) = a_0 + a_1(x_1 - x_0) = a_0 + a_1 h$
$(a_0 = y_0)$

$y_1 = y_0 + a_1 h$

$$a_1 = \frac{y_1 - y_0}{h}$$

$$\boxed{a_1 = \frac{\Delta y_0}{h}}$$

Similarly; $\boxed{a_2 = \frac{\Delta^2 y_0}{2!\,h^2}}$ $\left(\frac{y_2 - 2y_1 + y_0}{2h^2}\right)$

$$\boxed{a_n = \frac{\Delta^n y_0}{n!\,h^n}}$$

$$\therefore \phi(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta y_0^2}{2!\,h^2}(x - x_0)(x - x_1)$$
$$+ \ldots + \frac{\Delta^n y_0}{n!\,h^n}(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

Newton's forward interpolation

Any $x = x_0 + ph$ where $p \in R$
$x - x_0 = ph$ (equally spaced)

$x - x_1 = x - x_0 + x_0 - x_1$
$x - x_1 = ph + (-h)$
$\boxed{x - x_1 = h(p-1)}$

$$\boxed{x - x_{n-1} = [p-(n-1)]h = (p-n+1)h}$$

★★★

$$\boxed{\phi(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \frac{p(p-1)\cdots[p-(n-1)]}{n!}\Delta^n y_0}$$

~~we use when find~~

Find $y(1.2)$ of

x	0	1	2	3	4
y	1	1.5	2.2	3.1	4.3

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		.5			
1	1.5	.7	.2		
2	2.2			0	
		.9	.2		.1
3	3.1			.1	
		1.2	.3		
4	4.3				

$p = \frac{1.2 - 1}{1} = 0.2$

$$\left(\begin{aligned} 1.2 &= x_0 + ph \\ &= 1 + p(1) \\ p &= 1.2 - 1/1 \end{aligned} \right) \quad h = 1$$

$y(1.2) \simeq \phi(1.2) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0$

$= 1.5 + 0.2 \times .7 + \underline{\qquad}$

$= 1.624$

$\simeq 1.6$

☆ Book

Newton's interpolation formula is used to find $f(x)$ for $x = x_0 + ph$ (both forward & backward)

$$\boxed{y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 ---}$$

☆ This formula is used for interpolating the values of y near the beginning of a set of Tabulated value & extrapolating values of y a little backward of Y (left of y_0)

~~Newton's~~

$$\boxed{y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + ---}$$

This formula is used for interpolating values of y near the end of set of tabulated values & extrapolating a little ahead of y_n (right of y_n)

$$\boxed{x = x_n + ph}$$

Newton's Backward Interpolation formula

$$\boxed{y_n = a_0}$$

$$\boxed{a_1 = \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_n}{h}}$$

!

$$\boxed{a_n = \frac{\nabla^n y_n}{n!\, h^n}}$$

$$\boxed{\phi(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!\, h^2}(x - x_n)(x - x_{n-1}) - - - - - - - \frac{\nabla^n y_n}{n!\, h^n}(x - x_n) - - - (x - x_1)}$$

Modified version -

$$x = x_n + ph$$

$$x - x_n = ph$$

$$x - x_{n-1} = x - (x_n - h)$$

$$= x - x_n + h$$

$$\boxed{x - x_{n-1} = (p+1)h}$$

$$\boxed{\phi(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n - - - - \frac{p(p+1)(p+2) - - - (p+n-1)}{n!}\nabla^n y_n}$$

☆ Lagrange's Interpolation

Interpolation for unequally spaced values of x:
there are two:-
(i) Lagrange's interpolation formula
(ii) Newton's general interpolation formula with divided differences

---

Lagrange
$y = f(x)$ takes values $y_0, y_1, y \ldots y_n$ corresponding to $x = x_0, x_1, \ldots x_n$ then:

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1 + \ldots \frac{(x-x_0)\ldots(x-x_{n-1})}{(x_n-x_0)\ldots(x_n-x_{n-1})} y_n.$$

Lagrange's interpolation formula for n points is polynomial of degree $(n-1)$ which is known as Lagrangian polynomial and is very simple to implement to computer.

This fomula can also be used to split given f^n into partial fraction

On dividing both sides by $(x-x_0)(x-x_1)\ldots(x-x_n)$

$$\frac{f(x)}{(x-x_0)(x_0-x_1)\ldots(x_0-x_n)} = \frac{y_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} \cdot \frac{1}{x_0-x_0}$$

$$+ \ldots \frac{y_n}{(x_n-x_0)(x_n-x_1)\ldots(n-x_{n-1})} \frac{1}{x_0-x_n}$$

* Eq^n of line

$$y-y_1 = \frac{y_1-y_0}{x_1-x_0}(x-x_1) \Rightarrow y_1$$

$$y(x_1-x_0) = (y_1-y_0)(x-x_1) + y_1(x_1-x_0)$$

$$y = \frac{x-x_1}{x_0-x_1} y_0 + \frac{x-x_0}{x_1-x_0} y_1$$

$$= \sum_{i=0}^{1} l_i(x) y_i$$

$$\sum_{i=0}^{1} l_1(x_j) = l_0(x) + l_1(x) = 1$$

$$l_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & i \neq j \end{cases}$$

We can extend this for $(n+1)$ pt^n to get Lagrange's Interpolat formula.

★) <u>Divided difference</u> -

$$\underbrace{[x_0 , x_1]}_{\text{denotion}} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

$$\boxed{[x_0, x_1, x_2 \cdots x_n] = \frac{[x_1, x_2 \cdots x_n] - [x_0, x_1, \cdots x_{n-1}]}{x_n - x_0}}$$

★) $[x_0, x_0] = y'(x_0)$ if $y(x)$ is diff.

$\underbrace{[x_0, \cdots x_0]}_{r+1 \text{ times}} = y^r(x_0)$ if $y(x)$ is diff

★) $\boxed{[x_0, x_1] = [x_1, x_0]}$ symmetrical

(independent of order)

$(x_0\ x_1, x_2) = [x_2\ x_1\ x_0] =$ —— ——

- If tabulated f^n is nth degree ; $\Delta^n y_0$ is constant
  nth divided diff will also be constant.

* In case of equally spaced trees

$x_n - x_{n-1} = x_i - x_{i-i} = h$

$[x_1, x_2] = \frac{y_2 - y_1}{h}$

$\boxed{[x_0, x_1] = \frac{y_1 - y_0}{h}} = \boxed{\frac{\Delta y_0}{h}}$

$[x_0, x_1, x_2] = \frac{[x_1, x_0] - [x_0, x_1]}{x_2 - x_0}$

$= \frac{y_2 - y_1 - y_1 + y_0}{2h^2}$

$= \frac{y_2 - 2y_1 + y_0}{2!\,h^2} = \frac{\Delta^2 y_0}{2!\,h^2}$

$[x_0, x_1, \ldots x_n] = \frac{\Delta^n y_0}{h^2 n!}$

* Newton's general Interpolation Formula

$[x_1, x_0] = \frac{y - y_0}{x - x_0}$

$$\Rightarrow \quad y = y_0 + (x - x_0)[x, x_0] \longrightarrow ①$$

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0\, x_1]}{x - x_1}$$

$$\Rightarrow [x, x_0] = [x_0\, x_1] + (x - x_1)[x, x_0, x_1]$$

putting values in ①

$$y = y_0 + (x - x_0)[x_0\, x_1] + (x - x_0)(x - x_1)[x\, x_0\, x_1$$

$$\vdots$$

$$\boxed{y = y_0 + (x - x_0)[x_0\, x_1] + (x - x_0)(x - x_1)[x_0\, x_1, x_2] + \ldots (x - x_0) \ldots (x - x_n)[x, x_0, x_1 \ldots x_n]}$$

$$= \phi_n(x) + E(x)$$ — error / remainder

★ Lagrange formula has drawback - if another interpolation value were inserted then interpolation coeff are req to be recalculated which is solved by Newton's method.

★ $x \leftrightarrow y$ in Lagrange → inverse interpolation.

(*) CURVE FITTING : (best fit)

- problems of fitting curve are subject to errors
- To derive an approx. f^n that broadly fits the general trend of data without necessarily passing through individual pts
- The curve is drawn in such a way that the error b/w data points & curve is least. this process is called Method of least square

(2)

Graphically, plot the data points in a graph paper then join the line passing the data pts (few)
⇒ the line maynot be unique

If $y_i = f(x_i) + \epsilon_i$ (y have approx values) (experimental cases).

But if we use poly interpolation, error would further reproduce. In such cases, ~~the~~ we use Curve fitting technique

Book

If there are pairs of observed values the it is possible to given data to an eq^n of n arbitrary constant & solve n simultaneous eq^n; but if $< n$, we can use four methods - GRAPHICAL, METHOD OF LEAST SQUARE, METHOD OF GROUP AVERAGE, METHOD OF MOMENTS.

- Graphical methods fails to give values of unknowns uniquely & accurately. (others do)
- Method of least squares) is probably the best method to fit unique curve. (easy implementation on computer, widely applicable

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☆) Linear Curve Fitting -

$$y = a + bx$$

process of finding a line which passes through the tabulated pts.

Suppose $(x_0, y_0), (x_1, y_1) \ldots (x_m, y_m)$ be $m+1$ data points,
let $y = a+bx$ be the st line that fits given data

$$e_1 = |y - (a+bx_1)|$$
$$e_2 = |y - (a+bx_2)|$$
$$e_m = |y - (a+bx_m)|$$

Curve fitting tells to :- $\sum_{k=1}^{m} e_k = \min(\phi(a,b))$

can be solved using LPP. But method of calculus has not approached because $\phi'(a,b)$ may not be diff always.

I) Straight line

$y = a + bx$

① $\Sigma y = na + b\Sigma x$

n → no of tabular val.

② $\Sigma xy = a\Sigma x + b\Sigma x^2$

NORMA EQ^n

(Σxy, Σy, Σx, Σx^2 are calculated from table)

- Solve these as simultaneous eq^n of a & b
- Substitute a & b in $y = a + bx$ ← req ans

II) Parabola $\quad y = a + bx + cx^2$

Normal eq^n ① $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

solve & substitute

Thus, $y = a + bx + cx^2 + \ldots\ldots kx^{m-1}$ can be calculated by normal eq^n.

* Normal eq^n ~~mult~~ is found for every ~~coeff~~ constant ~~coeff~~ (a, b, c etc) by multiplying its coeff & adding.

$a + bx + cx^2 = y$

for c : coeff $= x^2$ → $x^2y = ax^2 + bx^3 + cx^4$

summing

$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

---

* Fitting of other curves

① $y = ax^b$

$\log_{10} y = \log_{10} a + b\log_{10} x$

$Y = A + bX$

$$\left.\begin{aligned} \Sigma Y &= nA + b\Sigma X \\ \Sigma XY &= A\Sigma X + b\Sigma X^2 \end{aligned}\right\} NE,$$

A, b to be found.

$A = \log_{10} a$

find a

(II) $y = ae^{bx}$

$\log y = \log a + bx \log e$

$Y = a + Bx$

$$\left.\begin{aligned} \Sigma Y &= nA + B\Sigma x \\ \Sigma xY &= A\Sigma X + B\Sigma x^2 \end{aligned}\right\} NE$$

(III) $xy^n = b$ $(pv^r = k)$

$\log_{10} x + a \log_{10} y = \log_{10} b$

$$\boxed{\log_{10} y = \frac{1}{a}\log_{10} b - \frac{1}{a}\log_{10} x}$$

$$Y = A + \underset{(-1/a)}{B}X$$

reduced to st line

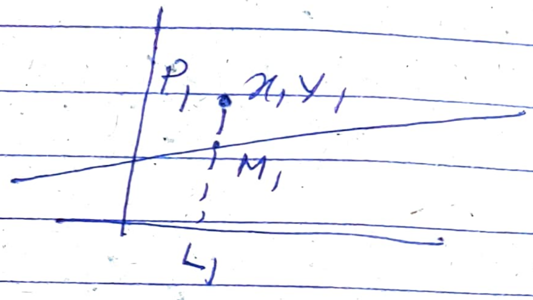
⊛ Method of Group Averages

$y = a + bx$

Let (x_1, y_1) —— (x_n, y_n) are set of n observations that fit st line quite closely.

· At $x = x_1$

$Y = Y_1 = L_1 P_1$



expected $= L_1 M_1$

error = observed − expected

$= Y_1 - (a + b x_1) = \underline{\underline{M_1 P_1}}$

,

|

$M_n P_2$

some are +ve / some −ve.

This method is based on assumption that sum of residuals is zero, to find a & b we require 2 eq^n, so we divide data in two groups − first containing k obs.

(x_1, y_1) —— —— (x_k, y_k)

& second $n-k$

(x_{k+1}, y_{k+1}) —— (x_n, y_n)

Assuming sum of errors = 0

$y_1 - (a + bx_1) + \{y_2 - (a+bx_2) + \_\_\_ = 0$

$y_{k+1} - (a + bx_{k+1}) + \_\_\_ = 0$

Simplifying,

$$\left\{\begin{aligned} \frac{y_1 + \dots + y_k}{k} &= a + b\,\frac{x_1 + \dots x_k}{k} \\ \frac{y_{k+1} + \dots + y_n}{n-k} &= a + b\,\frac{x_{k+1} \dots + x_n}{n-k} \end{aligned}\right\} \text{solve to get } a, b.$$

Avg values $\left.\begin{aligned} &\frac{1}{k}(y_1 + \dots + y_k) \\ &\frac{1}{k}(x_1 + \dots x_k) \end{aligned}\right\}$ in first group.

Drawback - diff grouping of obs will give diff answer of a & b.

In practice divide data so that each grp contains almost equal no of observation

Fitting parabola

$y = a + bx + cx^2$

$y_1 = a + bx_1 + cx_1^2$ (let)

$y - y_1 = b(x - x_1) + (x^2 - x_1^2)^2$

$$\frac{y - y_1}{x - x_1} = b + c(x + x_i)$$

$x + x_1 = X$, $\frac{y - y_1}{x - x_1} = Y$

$Y = b + cX$

linear fom can be found as prev.

UNIT-3

- If x is equispaced and $\frac{dy}{dx}$ is req at near beginning of table : we use Newton's forward formula
  if required at end : we use Newton's backward formula
  middle of table : Stirling / Bessel's Formula

If x is not equispaced we use Newton's divided difference formula.

(A) Derivative using forward diff formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta y_0^2 + \ldots$$

Differentiating w.r.t p

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots$$

$$p = \frac{x-x_0}{h} \Rightarrow \frac{dp}{dx} = \frac{1}{h}$$

$$\frac{dy}{dp} \times \frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 \ldots\right] \longrightarrow (1)$$

At $x = x_0$, $\underline{p=0}$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$\therefore \boxed{\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots\right]}$$

Diffⁿ ① wrt x;

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 + \dots\right] \times 1/h$$

Now

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0\right]}$$

$(p=0)$

Similarly

$$\boxed{\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \dots\right]}$$

Otherwise, $1+\Delta = E = e^{hD}$

$$hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 \dots$$

$$D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right]$$

$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 + \cdots\right]^2 = \text{same}$$

$Dy_0 = 0$

(B) <u>Derivatives using backward diff^n formula.</u>

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla y_n^2 + \cdots$$

Diff^n both sides wrt p.

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots$$

$$p = \frac{x - x_n}{h} = \frac{dp}{dx} = \frac{1}{h}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\cdot\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right.$$

Diffⁿ w.r.t x.

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right) \cdot \frac{dp}{dx}$$

$\frac{1}{h}$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots\right]$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right]}$$

$$\boxed{\left[\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n\right]}$$

Otherwise, $1-\nabla = E^{-1} = e^{-hD}$

$$-hD = \log(1-\nabla)$$

$$-hD = -\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \cdots\right)$$

$$D = \frac{1}{h}\left(\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \cdots\right)$$

same

(c) Derivative using Central difference formula

Stirling formula =

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\Delta^2 y_{-1} + \frac{3p^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)\right.$$

$$\left. + \frac{2p^3 - p}{12}\Delta^4 y_{-2} + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\,\frac{\Delta^5 y_{-2} + \Delta^5 \ldots}{2}\right.$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots\right]$$

③ Derivative using Newton's divided diffⁿ formula

$$f(x) = f(x_0) + (x-x_0)f(x_0 - x_1) + (x-x_0)(x-x_1)f(x_0\, x_1\, x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\, f(x_0\, x_1\, x_2\, x_3)$$
$$+ (x-x_0)(x-x_1)(x-2)(x-3)\, f(x_0\, x_1, x_2\, x_3\, x_4) + \ldots$$

$$f'(x) = f(x_0\, x_1) + (2x - x_0 - x_1)\, f(x_0\, x_1, x_2)$$
$$+ 3x^2 - 2x(x_0 + x_1 + x_2) + (x_0x_1 + x_1x_2 + x_2x_0)$$
$$\times f(x_0, x_1\, x_2\, x_3)$$

$$+ 4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3) + 2x(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2) - [x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3]$$
$$\times f(x_0\, x_1, x_2\, x_3\, x_4) + \ldots$$

Substitute & find answer

★ Alway make table ★ — form, backw, central, divided diff.

★ Numerical Integration - process of evaluating a definite integral from a set of tabulated values of integrand $f(x)$ is called NI

When applied to f^n of single variable this process is called QUADRATURE

Like diff^n it is done by—
(I) Representing $f(x)$ in interpolation formula
(II) Integrating it b/w limits (given).

I) Newton's Cotes Quadrature Formula

$$I = \int_a^b f(x)\,dx$$

dividing interval (a, b) into n subintervals of width h, $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ + --- $x_n = x_0 + nh = b$

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = h\int_0^n f(x_0 + rh)\,dr$$

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$$\left(\text{putting} \quad \begin{aligned} x &= x_0 + rh \\ dx &= h\,dr \end{aligned}\right)$$

$$I = h \int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \cdots \right] dr$$

(Newton's forward interpolation formula)

<u>On integrating</u>,

$$I = nh\Bigg[\left(y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0\right)$$

$$+ \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} +$$

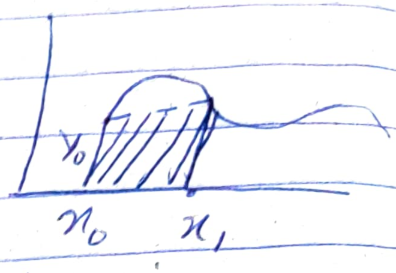
$$+ \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!}$$

$$+ \left(\frac{n^6}{7} - \frac{15}{6}n^5 + 17n^4 - \frac{225}{4}n^3 + \frac{274n^2}{3}\right.$$

$$\left. - 60n\right)\frac{\Delta^6 y_0}{6!} + \cdots \Bigg]$$

Quadrature rules

$n=1$, TRAPEZOIDAL RULE



- straight line
- polynomial of first order so
- that diff n of order higher than first become zero

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}\left(y_0 + y_1\right)$$

$$\int_{x_0}^{x_0+2h} f(x)\,dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}\left(y_1 + y_2\right)$$

$$\vdots$$

$$\int_{x_0(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left(y_{n-1} + y_n\right)$$

---

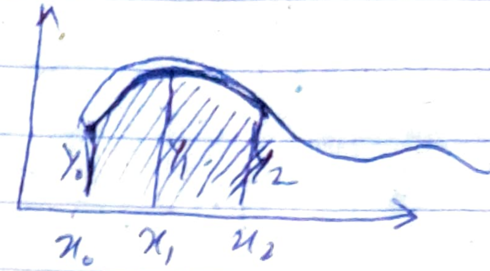
adding these n integrals

Trapezium rule – $$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots y_{n-1})\right]}$$

(The area of each strip (trapezium) is found separately then area under curve & ordinates at x_0 & x_n is approx equal to area of trapeziums)

$n=2$) **Simpson's one-third rule** (Simpson's rule)

Taking curve through (x_0, y_0) (x_1, y_1) (x_2, y_2)



- parabola
- polynomial of second order
- diffⁿ of order >2 vanish.

$$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Δy_0^2

$$\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

⋮

$$\int_{x_0(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \quad ; \quad n \text{ beign even}$$

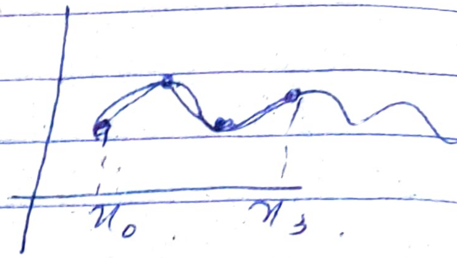
$n =$ even ; Adding

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\Big((y_0+y_n) + \big[4(y_1+y_3+\cdots y_{n-1}) + 2(y_2+y_4+\cdots y_{n-2})\big]\Big)}$$

While applying interval must be divided into even no of equal subintervals, area of two strips at a time

$n=3$ Simpson's three eighth rule

- (x_i, y_i) $\quad i = 0, 1, 2, 3$
- polynomial of third order
- ≥ 3 vanish



$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\Big[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots y_{n-1}) + 2(y_3+y_6+\cdots y_{n-3})\Big]$$

($n =$ multiple of 3)

no of subintervals should be taken multiple of 3

$n=4$ Boole's Rule

· forwith order >4 vanish

$$\int_{x_0}^{x_0+4h} f(x)\,dx = 4h\left(y_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right)$$

$$= \frac{2h}{45}\left(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\right)$$

$$\boxed{\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}\left(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_{12} + \cdots\right)}$$

no of ~~Boo~~ subintervals = multiple of 4.

$n=6$ Weddle's rule

$$\int_{x_0}^{x_0+6h} f(x)\,dx = y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0$$

$$+ \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 y_0 + \frac{1}{6}\cdot\frac{41}{140}\Delta^6 y_0)$$

$$\frac{41}{140}\Delta^6 y_0 \Rightarrow \frac{3}{10}\Delta^6 y_0$$ for ease error is negligible

$$\Rightarrow \int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}\left(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right)$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\left(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} + \cdots\right)$$

Multiple of 6 $\Rightarrow$ subintervals.

★ Most accurate – ① Weddle's rule
② 1/3 Simpson rule.

Application of Simpson rule –

- If various ~~co~~ordinates represent equispaced crosssectional areas then simpson rules gives volume of solid

- useful to civil engineers for calculating amount of Earth that must be moved to fill a depression / make a dam

- If ordinates ~~repre~~ denote vel at equal interval of time, it gives distance travelled

~ END ~

UNIT-4

*) Numerical method for : $\frac{dy}{dx} = f(x, y)$

given $y(x_0) = y_0$.

→ Picard, Taylor => power series in x ; y can be found by direct substitution.

→ Others = set of values of x, y (step-by-step methods).

→ Euler & Runge-Kutta – computing y over limited range of x – values.

→ Milne & Adams Bashforth = y over wider range of x's. and require starting x which is calculated by Picard, Taylor, Runge Kutta.

- Initial value problems – initial condition at initial pt.
- Boundary value problem :- second & > order; condition given at two or more points

I) Picard Method

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx.$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx.$$

$y = y_0$;

$$y_1 = y_0 + \int_{x_0}^{x} f(x_0,y_0)\,dx.$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)\,dx$$

$\vdots$

- only for limited class of equations in which successive integration can be performed easily; simultaneous eq^n / eq^n of higher order.

II) Taylor series method

$$\frac{dy}{dx} = f(x, y)$$

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$y'' = f_x + f_y f'$$

$$y(x) = y_0 + \frac{(x-x_0)^1}{1!} y'_0 + \frac{(x-x_0)^2}{2!} (y'')_0 + \ldots$$

Value of $(y'_0)_0$, $(y'')_0$ can be found by putting $x = x_0$, $y = 0$.

* higher order derivative are a problem

III) Euler → first order

- Use approximate curve of sol^n by tangent in each interval by a sequence of short line

$$\boxed{y_n = y_{n-1} + hf\left(x_0 + \overline{n-1}\,h,\ y_{n-1}\right)}$$

Modified → second second

slope of the curve as the mean of the slopes of tangents at P & P_1

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1)\right]$$

slope of y_1 is not know, y_1 is taken as $y_0 + h(x_0, y_0)$ from Euler's & inserted & y_1 modified value is found; predictor ; corrector

Again $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})\right]$

⋮

till accurate then we proceed for y_2, y_3 ---

$y_2 = y_1 + hf(x_0 + h, y_1)$.

$$y_2 = y_1 + \frac{h}{2}\left[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)\right]$$

IV) Runge ~~Kutta~~. third order

$k_1 = hf(x_0, y_0)$
$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$
$k' = hf(x_0 + h, y_0 + k_1)$
$k_3 = hf(x_0 + h, y_0 + k')$

$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$

weighted mean

Runge - Kutta - fourth order higher order - α h^r

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

$k_1 = hf(x_0, y_0)$
$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$
$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2)$
$k_4 = hf(x_0 + h, y_0 + k_3)$

$$\boxed{k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

↳ operations are identical for linear & non-linear DE.

V) Milne's Method

predictor - corrector method

1) 4 first calculate from Picard / Taylor.

2) Newton's forward interpolation

$$Y_4 = Y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \quad \text{(predictor)}$$

$$f_4 = f(x_0 + 4h, Y_4)$$

Simpson's rule; $Y_4 = Y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

correction

correction.

Date
Page

(ii) Adams - Bashforth Method

$y_{-1} = y(x_0 - h)$
$y_{-2} = y(x_0 - 2h)$
$y_{-3} = y(x_0 - 3h)$

Taylor / Euler / Runge Kutta.

Backward interpolation.

$$y_1^{(p)} = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}).$$

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2}).$$

→ Runge Kutta → most useful.

---

o/c order Milne & Adam Bashforth = 4th order

→ $O(h^5)$ – truncation error

→ 4 values req