Theory of Electromagnetic Waves

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The aim of this chapter is to give a brief introduction to the theory of electromagnetic waves.

Chapter objectives

1. **To understand the basic vector calculus of electrodynamics:**

   Derivative: gradient, divergence, curl; Integration: line, surface and volume integration; Basic theorems of electrodynamics: Gauss divergence and Stoke's theorem.

2. **To understand the basic laws of electricity and magnetism:**

   Laws of electricity: Coulomb's law and Gauss's law in electricity.
   Laws of magnetism: Biot - Savart's law, Gauss's law in magnetism and Ampere's circuital law.
   Laws of electromagnetism: Faraday's law of electromagnetic induction.
   Equation of continuity, modified Ampere's circuital law and displacement current.

3. **To derive Maxwell's equations:**

   Derivation, interpretation and physical significance

4. **To derive Electromagnetic wave equation:**

   Derivation and solution of wave equation, propagation in different media

5. **To discuss Poynting vector and theorem for understanding em energy propagation:**

   Definition, derivation of the theorem and interpretation

INTRODUCTION

The phenomenon of electricity and magnetism were studied independently until the observation of Oersted, Ampere, Gauss, Biot, Savart and others about the magnetic effects of electric current. The combined phenomena associated with electricity and magnetism was termed as electromagnetism. This was developed further by the discovery of electromagnetic induction by Michel Faraday and independently by Joseph Henry. The various observations in electromagnetism were synthesized and put into mathematical form by James Maxwell. He also introduced some new ideas of his own and put electromagnetism on a sound theoretical basis. He combined all the known facts about electromagnetism into a set of four equations, now known as ***Maxwell's electromagnetic equations***. These equations correlate experimental observations in vast and widely differing areas, and also predict new results which have been successfully verified. In this chapter, an overview of the various laws of electromagnetism which led to the Maxwell's equations has been presented from fundamental ideas. The laws of electromagnetism, in general, and Maxwell's equations, in particular, can be formulated in a compact and elegant manner by using the language of vector calculus. This chapter begins with a brief discussion of vector calculus.

1. The basic vector calculus of electrodynamics

The electric field $\vec{E}$, magnetic induction $\vec{B}$, magnetic intensity $\vec{H}$, electric displacement $\vec{D}$, electric current density $\vec{j}$, magnetic vector potential $\vec{A}$ and other vector quantities occurring in electromagnetism are, in general, functions of position and time. In contrast with constant vectors, the magnitude or direction (or both) of the above vector quantities may change with position and time. Hence these are called **vector fields**. Similarly, scalar quantities such as the electrostatic potential, electric charge density, electromagnetic energy density etc. which are, in general, functions of position and time are called **scalar fields.**

The various laws of electromagnetism describe how the different vector and scalar fields change with position and time. The laws are expressed mathematically in terms of derivatives of the fields with respect to space coordinates and time. The derivatives with respect to space coordinates are expressed in terms of gradient, divergence and curl operators.

1.1 Derivative

(i) Time derivative

If $\vec{A}$ is expressed in terms of its cartesian components $\vec{A}$ (t)= $i\ A_x$(t) + $j\ A_y$(t)+ $k\ A_z$(t), then the time derivative is defined by

$$\frac{d\vec{A}}{dt} = i\ \frac{dA_x}{dt} + j\ \frac{dA_y}{dt} + k\ \frac{dA_z}{td} \tag{1.1}$$

The rules of calculus are obeyed by time derivative.

$$\frac{d}{dt}\left(c\,\vec{A}\right) = c\,\frac{d\vec{A}}{dt}, \text{ where c is a constant}$$

$$\frac{d}{dt}\left(f\,\vec{A}\right) = f\,\frac{d\vec{A}}{dt} + \frac{df}{dt}\,\vec{A}, \text{ where f is a scalar field}$$

For two vector fields $\vec{A}$ and $\vec{B}$

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$$

Direction of time derivative

The change of a vector field with time may occur due to change in its magnitude or direction or both.

If $\vec{A}$ (t) $=\hat{A}$ A , where A is the magnitude of the vector and $\hat{A}$ is the unit vector along $\vec{A}$,

$$\frac{d\vec{A}}{dt} = \frac{d\hat{A}}{dt}\,A + \hat{A}\,\frac{dA}{dt} \qquad (1.2)$$

If both the magnitude and direction of the vector change with time $\frac{d\vec{A}}{dt}$ makes an arbitrary angle with $\vec{A}$ depending on the relative values of the two terms on the r.h.s of Eq(1.2)

If only the magnitude changes with time but direction is constant, $\frac{d\hat{A}}{dt}$=0 and Eq (1.2) becomes

$$\frac{d\vec{A}}{dt} = \hat{A}\,\frac{dA}{dt}$$

So the time derivative in along the vector $\vec{A}$.

If only the direction changes with time but magnitude is constant, $\frac{dA}{dt}$ = 0 and eq. (1.2) becomes, $\frac{d\vec{A}}{dt} = \frac{d\hat{A}}{dt}$ A

Since $\hat{A}$ is a unit vector, $\hat{A}.\hat{A}$=1.Hence

$$\frac{d}{dt}(\hat{A}.\,\hat{A}) = \frac{d\hat{A}}{dt}\,.\,\hat{A} + \hat{A}\,.\,\frac{d\hat{A}}{dt} = 2\hat{A}\,.\,\frac{d\hat{A}}{dt} = 0 \qquad (1.3)$$

But both $\hat{A}$ and $\frac{d\hat{A}}{dt}$ are non-zero. So, from Eq (1.3) it is concluded that $\frac{d\hat{A}}{dt}$ is perpendicular to $\hat{A}$. In this case, the time derivative is perpendicular to $\hat{A}$. Thus the direction of the time derivative of a vector field may be along the vector, Perpendicular to the vector, or may make an arbitrary angle with the vector depending on the nature of its time dependence.

Example – 1 A vector field is given by $\vec{A}$ (t) = $\hat{\imath}\, a \sin \omega\, t\; + \hat{\jmath}$ b $e^{i\,\omega\, t}$, where a , b and ω are constants .

Find the time derivative of the vector field.

Solution: $\frac{d\vec{A}}{dt} = \hat{\imath}\,\frac{d}{dt}$(a sin ω t) + $\hat{\jmath}\,\frac{d}{dt}$(b $e^{i\omega t}$) = $\hat{\imath}$ a ω cos ω t + $\hat{\jmath}$ b ω $e^{i\omega t}$

(ii) Gradient of a scalar field

The change of a scalar field with position is described in terms of gradient operator. If V is a scalar field, its gradient is defined as Grad $V = \vec{\nabla}\, V$

$$= (\hat{\imath}\,\frac{d}{dx} + \hat{\jmath}\,\frac{d}{dy} + \hat{k}\,\frac{d}{dz})\, V$$

$$= \hat{\imath}\,\frac{dV}{dx} + \hat{\jmath}\,\frac{dV}{dx} + \hat{k}\,\frac{dV}{dz} \tag{1.4}$$

The operator $\vec{\nabla} \equiv \hat{\imath}\,\frac{d}{dx} + \hat{\jmath}\,\frac{d}{dy} + \hat{k}\,\frac{d}{dz}$ is called the *del* operator. It is a vector deferential operator. Although V is scalar, $\vec{\nabla}\, V$ is a vector. The change of a scalar field with position can be expressed in terms of its gradient as given below.

Let the value of a scalar field be V at position $\vec{r}$ and $V + dV$ at position $\vec{r}$ +d$\vec{r}$. The scalar field changes by dV as the position changes from $\vec{r}$ to $\vec{r}$ + d$\vec{r}$. Since V depends on the coordinates x, y and z, we can write

$$dV = \frac{dv}{dx}\,dx + \frac{dv}{dy}\,dy + \frac{dv}{dz}\,dz \tag{1.5}$$

Since d$\vec{r}$ = $\hat{\imath}\ dx + \hat{\jmath} dy + \hat{k} dz$, we have

$$\vec{\nabla}\,\text{V}.\ \text{d}\vec{r} = (\ \hat{\imath}\,\frac{dV}{dx} + \hat{\jmath}\,\frac{dV}{dy} + \hat{k}\,\frac{dV}{dz}\ ) . (\ \hat{\imath}\text{dx} + \hat{\jmath}\text{dy} + \hat{k}\text{dz}\ )$$

$$= \frac{dV}{dx}\,dx + \frac{dV}{dy}\,dy + \frac{dV}{dz}\,dz \tag{1.6}$$

Comparing eq. (1.5) with eq. (1.6), we can write $dV = \vec{\nabla}\, V$ .d$\vec{r}$ (1.7)

This expresses the change dV of the scalar field V in terms of its gradient.

Direction of $\vec{\nabla}\, V$:

From eq. (7), it can be noticed that the magnitude of change dV depends on the relative directions of $\vec{\nabla}V$ and d$\vec{r}$. If one starts from a point $\vec{r}$ where the scalar field has value V, the change d$\vec{r}$ in position can take place along any direction. We can write Eq. (1.7) as

$$\text{dV} = \left|\vec{\nabla}\, V\right| |\text{d}\vec{r}\,|\ \cos\theta,$$

where, θ is the angle between the direction of $\vec{\nabla}V$ and d$\vec{r}$. The change dV is maximum when cos $\theta = 1$ or, $\theta = 0$, i.e, when d$\vec{r}$ is along $\vec{\nabla}V$. So, $\vec{\nabla}V$ *has the direction along which V has the maximum rate of change*. On the other hand $dV = 0$, if $\theta = \pi/2$, i.e, d$\vec{r}$ is normal to $\vec{\nabla}V$. So the value of V remains constant, as one move in a direction normal to $\vec{\nabla}V$. In other words, $\vec{\nabla}V$ is normal to the surface of constant V. For example, the electric field $\vec{E}$ and electric potential V are related by $\vec{E} = -\vec{\nabla}V$. The electric field $\vec{E}$ is normal to the equipotential surface (the surface of constant potential) and is directed along which V changes most rapidly.

If V and W are two scalar fields, the following relations are satisfied.

(1) $\vec{\nabla}(V+W) = \vec{\nabla}V + \vec{\nabla}W$
(2) $\vec{\nabla}(V.W) = \overline{(\vec{\nabla}V)}W + V\overline{(\vec{\nabla}W)}$

Example-2 Find $\vec{\nabla}f$ where $f = a\,x^2 + b\,y^2 + cz$. Given a, b and c are constants.

$$\vec{\nabla}f = \hat{\imath}\frac{df}{dx} + \hat{\jmath}\frac{df}{dy} + \hat{k}\frac{df}{dz}$$

$$= \hat{\imath}\,2ax + \hat{\jmath}\,2by + \hat{k}\,c.$$

Example 3 The temperature of an inhomogeneous solid is T(r) = A $x\,y$ + B z^2, where A and B are constants. Find the temperature gradient in the solid.

Sol: ∇T = i A y + j Ax + k 2B z.

Example-4 The position vector of a body is $\vec{r} = \hat{\imath}\,x + \hat{\jmath}\,y + \hat{k}\,z$. Find grad r.

Sol: $r = (x^2+y^2+z^2)^{1/2}$

$$\frac{dr}{dx} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2\,x\,\hat{\imath} \qquad \frac{dr}{dy} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2\,y\hat{\jmath}$$

$$\frac{dr}{dz} = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}\,2\,z\,\hat{k}$$

Grad r = $(x^2+y^2+z^2)^{-\frac{1}{2}}(\vec{r})$. Grad r = unit vector of $\vec{r}$.

Example-5 If $\phi = 3x^2y - y^3z^2$; find grad ϕ at the point (1,-2,-1).

Sol: grad $\phi = \vec{\nabla}\phi$

$$= (\hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz})(3x^2y - y^3z^2)$$

$$= \hat{\imath}\frac{d}{dx}(3x^2y - y^3z^2) + \hat{\jmath}\frac{d}{dy}(3x^2y - y^3z^2) + \hat{k}\frac{d}{dz}(3x^2y - y^3z^2)$$

$$== \hat{\imath}6xy + \hat{\jmath}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

grad ϕ at (1,-2,-1) = $\hat{\imath}(6)(1)(-2) + \hat{\jmath}[(3(1)-3(4)(1)] + \hat{k}(-2)(-8)(-1)$

$$= -12\,\hat{\imath} - 9\,\hat{\jmath} - 16\,\hat{k}$$

Example-6 Find the directional derivatives of $x^2y^2z^2$ at the point (1, 1, -1) in the direction of the tangent to the curve $x = e^t$, y= sin 2t + 1, z = 1- cos t, at t = 0.

Sol: Let $\phi = x^2y^2z^2$

Directional Derivative of $\phi = \vec{\nabla}\phi$

$$= \left(\hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)(x^2y^2z^2)$$

$$\vec{\nabla}\phi = 2xy^2z^2\,\hat{\imath} + 2yx^2z^2\,\hat{\jmath} + 2zx^2y^2\,\hat{k}$$

Directional Derivative of ϕ at (1, 1, -1)

$$= 2\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$$

$$\vec{r} = \hat{\imath}\,x + \hat{\jmath}\,y + \hat{k}\,z = e^t\,\hat{\imath} + (\sin 2t + 1)\,\hat{\jmath} + (1 - \cos t)\,\hat{k}$$

Tangent vector $\vec{T} = \frac{d\vec{r}}{dt} = e^t\,\hat{\imath} + 2\cos 2t\,\hat{\jmath} + \sin t\,\hat{k}$

Tangent (at t = 0) = $e^o\,\hat{\imath} + 2(\cos 0)\,\hat{\jmath} + (\sin 0)\,\hat{k} = \hat{\imath} + 2\,\hat{\jmath}$

Required directional derivative along tangent = $(2\hat{\imath} + 2\hat{\jmath} - 2\hat{k}).\frac{\hat{\imath} + 2\,\hat{\jmath}}{\sqrt{1+4}}$

$$= \frac{2+4+0}{\sqrt{5}} = \frac{6}{\sqrt{5}}$$

Example-7 Find the unit normal to the surface $xy^3z^2 = 4$ at (-1,-1, 2).

Sol: Let $\phi(x,y,z) = x\,y^3z^2 - 4$

We know that $\boldsymbol{\nabla}\phi$ is the vector normal to the surface $\phi(x,y,z) = c$.

Normal vector = $\vec{\nabla}\phi = \left(\hat{\imath}\frac{d\phi}{dx} + \hat{\jmath}\frac{d\phi}{dy} + \hat{k}\frac{d\phi}{dz}\right)$

$$= \hat{\imath}\frac{d}{dx}(x\,y^3z^2) + \hat{\jmath}\frac{d}{dy}(x\,y^3z^2) + \hat{k}\frac{d}{dz}(x\,y^3z^2)$$

Normal vector = $y^3z^2\,i + 3xy^2z^2\,\hat{\jmath} + 2xy^3z\,\hat{k}$

Normal vector at (-1,-1, 2) = $-4\hat{\imath} - 12\hat{\jmath} + 4\hat{k}$

Unit vector normal to the surface at (-1,-1, 2)

$$= \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{-4\hat{\imath} - 12\hat{\jmath} + 4\hat{k}}{\sqrt{16+144+16}} = -\frac{1}{\sqrt{11}}(\hat{\imath} + 3\hat{\jmath} - \hat{k})$$

Example -8 Find the unit vector normal to the surface $x^2+y^2 = z$ at a point (1,2,5).

Sol: Let $\phi = x^2+y^2 - z$

$$\text{Gradient } \phi = \vec{\nabla}\phi = \left(\hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)(x^2+y^2 - z)$$

$$= 2x\hat{\imath} + 2y\hat{\jmath} - \hat{k}$$

$$(\text{Gradient } \phi)_{1,2,5} = 2\hat{\imath} + 4\hat{\jmath} -- \hat{k}$$

$$\text{Unit normal vector} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2\hat{\imath}+4\hat{\jmath}-\hat{k}}{\sqrt{4+16+1}} = \frac{2}{\sqrt{21}}\hat{\imath} + \frac{4}{\sqrt{21}}\hat{\jmath} - \frac{\hat{k}}{\sqrt{21}}$$

Example -9 Find the rate of change of $\phi = xyz$ in the direction normal to the surface

$x^2y + y^2x + yz^2 = 3$ at the point (1, 1, 1).

Sol: Rate of change of $\phi = \vec{\nabla}\phi$

$$= \left(\hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)(xyz)$$

$$= \hat{\imath}\, yz + \hat{\jmath}\, xz + \hat{k}\, xy$$

Rate of change of ϕ at (1, 1, 1) $= (\hat{\imath} + \hat{\jmath} + \hat{k})$

Normal to the surface $\psi = x^2y + y^2x + yz^2 - 3$ is given as

$$\vec{\nabla}\psi = \left(\hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)(x^2y + y^2x + yz^2 - 3)$$

$$= \hat{\imath}(2xy + y^2) + \hat{\jmath}(x^2+2xy+z^2) + \hat{k}\, 2yz$$

$$\overline{(\vec{\nabla}\psi)}_{(1,1,1)} = 3\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$$

$$\text{Unit nomal} = \frac{3\hat{\imath} + \widehat{4\jmath} + 2\hat{k}}{\sqrt{9+16+4}}$$

$$\text{Required rate of change of } \phi = (\hat{\imath} + \hat{\jmath} + \hat{k}).\frac{(3\hat{\imath} + \widehat{4\jmath} + \widehat{2k})}{\sqrt{9+16+4}} = \frac{3+4+2}{\sqrt{29}} = \frac{9}{\sqrt{29}}$$

Example-10 The position vector of a body is $\vec{r} = \hat{\imath}\, x + \hat{\jmath}\, y + \hat{k}\, z$. Show that

(1) grad $r = \frac{\vec{r}}{r}$ (2) grad $\frac{1}{r} = \frac{\vec{r}}{r^3}$

Sol: (1) $\vec{r} = \hat{\imath}\, x + \hat{\jmath}\, y + \hat{k}\, z$

$$r = \sqrt{x^2 + y^2 + z^2} \text{ so } r^2 = x^2+y^2+z^2$$

$$2r\frac{dr}{dx} = 2x \quad \text{so} \quad \frac{dr}{dx} = \frac{x}{r}$$

$$\text{Similarly } \frac{dr}{dy} = \frac{y}{r} \text{ and } \frac{dr}{dz} = \frac{z}{r}$$

$$\text{grad } r = \vec{\nabla} r = \left(\hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right) r$$

$$= \left(\hat{\imath}\frac{dr}{dx} + \hat{\jmath}\frac{dr}{dy} + \hat{k}\frac{dr}{dz}\right) = \frac{\hat{\imath}\, x + \hat{\jmath}\, y + \hat{k}\, z}{r} = \frac{\vec{r}}{r}$$

(2) $$\text{grad } \frac{1}{r} = \left(\hat{\imath}\frac{d}{dx} + \hat{\jmath}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right)\left(\frac{1}{r}\right)$$

$$= \hat{\imath}\left(-\frac{1}{r^2}\frac{x}{r}\right) + \hat{\jmath}\left(-\frac{1}{r^2}\frac{y}{r}\right) + k\left(-\frac{1}{r^2}\frac{z}{r}\right)$$

$$= \frac{\hat{\imath}\, x + \hat{\jmath}\, y + \hat{k}\, z}{r^3} = \frac{\vec{r}}{r^3}$$

(iii) Divergence of a vector field

The divergence of a vector field $\vec{A}$ is given by

$$div\vec{A} = \vec{\nabla}.\vec{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z\right)$$

$$= \left(\hat{\imath}\frac{\partial A_x}{\partial x} + \hat{\jmath}\frac{\partial A_y}{\partial y} + \hat{k}\frac{\partial A_z}{\partial z}\right) \quad (1.8)$$

Note-1:

$\vec{\nabla}.\left(\vec{A} + \vec{B}\right) = \vec{\nabla}.\vec{A} + \vec{\nabla}.\vec{B}$, Where $\vec{A}$ and $\vec{B}$ are vector fields.

$\vec{\nabla}.\left(V\vec{A}\right) = \left(\vec{\nabla}V\right).\vec{A} + V\vec{\nabla}.\vec{A}$, Where V is a scalar field and A is a vector fields.

Note-2:

$\vec{\nabla}.\vec{A}$ is a scalar quantity. It may be positive, negative or zero. If the divergence of a vector field vanishes everywhere, it is called a *solenoidal field.* If divergence of a vector field comes as +ve, it is a source field and if it comes as –ve, it is a sink field.

Note-3:

$\vec{\nabla}.\vec{A}$ is a measure of the how much the vector A spreads out or diverges from the point.

Example-10

Evaluate $\vec{\nabla}.\vec{F}$, where $\vec{F} = \vec{i}\, ax^2 + \vec{j}\cos(by) - \vec{k}\, 2z$. Given that a and b are constants.

Solution: $\vec{\nabla}.\vec{F} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$

$$= \left(\frac{\partial(ax^2)}{\partial x} + \frac{\partial(\cos(by))}{\partial y} + \frac{\partial(-2z)}{\partial z}\right)$$

$$= 2ax - b\sin(by) - 2$$

Example-11

Evaluate $\vec{\nabla}.\vec{F}$ where $\vec{A} = \vec{r}\, V$ Here V is a scalar field and $\vec{r}$ is the position vector.

Sol: $\vec{\nabla}.\vec{A} = (\vec{\nabla}.\vec{r})V + \vec{r}.\vec{\nabla}V$

Since $\vec{r} = \vec{i}\, x + \vec{j} y + \vec{k} z$, $\quad \vec{\nabla}.\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

Also, $\vec{\nabla} V = \vec{r}\frac{\partial V}{\partial r}$; So $\vec{\nabla}.\vec{A} = 3V + \vec{r}.\hat{r}\frac{\partial V}{\partial r} = r\hat{r}.\hat{r}\frac{\partial V}{\partial r} = 3V + r\frac{\partial V}{\partial r}$, (Since $\hat{r}.\hat{r} = 1$)

(iv) Curl of a vector field

The curl of a vector field $\vec{A}$ is given by: Curl$\vec{A} = \vec{\nabla} \times \vec{A}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad (1.9)$$

This is the cross product of the $\vec{\nabla}$ operator with the vector field and hence is a vector. If ***s*** is a scalar field and $\vec{A}$ and $\vec{B}$ are vector fields. Then

(i) $\vec{\nabla} \times (\vec{A} + \vec{B}) = (\vec{\nabla} \times \vec{A}) + (\vec{\nabla} \times \vec{B})$

(ii) $\vec{\nabla} \times (s\vec{A}) = (\vec{\nabla}s) \times \vec{A} + s(\vec{\nabla} \times \vec{A})$

If the curl of a vector filed vanishes everywhere, it is called an *irrotational field.*

Example-12

Evaluate, $\vec{\nabla} \times \vec{r}$, where $\vec{r}$ is a position vector.

Solution: curl $\vec{r} = \vec{\nabla} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{j}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial z}{\partial y}\right)$$

=0; (Since the Cartesian coordinate x, y and z are independent of one another, each of the derivatives vanishes)

So, $\vec{\nabla} \times \vec{r} = 0$

Example-13

Evaluate $\vec{\nabla} \times \vec{A}$ for a vector filed given by $\vec{A} = \vec{i}\,2xy + \vec{j}\,xy^2 - \vec{k}e^{az}x^2$, where a is a constant.

Sol: $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \hat{i}\left(\frac{\partial(-e^{az}x^2)}{\partial y} - \frac{\partial(xy^2)}{\partial z}\right) + \hat{j}\left(\frac{\partial(2xy)}{\partial z} - \frac{\partial(-e^{az}x^2)}{\partial x}\right) + \hat{k}\left(\frac{\partial(xy^2)}{\partial x} - \frac{\partial(2xy)}{\partial y}\right)$$

$$= \hat{i}(0-0) + \hat{j}(0 + 2xe^{az}) + \hat{k}(y^2 - 2x) \quad = \hat{j}(2xe^{az}) + \hat{k}(y^2 - 2x)$$

Example-14 Find the divergence and curl of $\vec{v} = (xyz)\,\hat{i} + (3x^2y)\,\hat{j} + (xz^2 - y^2z)\,\hat{k}$ at (2,-1,1)

Sol: We have $\vec{v} = (xyz)\,\hat{i} + (3x^2y)\,\hat{j} + (xz^2 - y^2z)\,\hat{k}$

Div $\vec{v} = \vec{\nabla}.\vec{v} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z) = yz + 3x^2 + 2xz - y$

Div $\vec{v}$ at (2,-1,1) = -1+12+4-1 = 14 Ans.

Curl $\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (xyz) & (3x^2y) & (xz^2 - y^2z) \end{vmatrix} = -2yz\,\hat{i} - (z^2 - xy)\,\hat{j} + (6xy - xz)\,\hat{k}$

Curl $\vec{v}$ at (2,-1,1) $= 2\,\hat{i} - 3\,\hat{j} - 14\,\hat{k}$ Ans.

Example-15 Prove that $(y^2-z^2+3yz-2x)\,\vec{i} + (3xz+2xy)\,\hat{j} + (3xy-2xz+2z)\,\hat{k}$ is both solenoidal and irrotational.

Sol: Let $\vec{F} = (y^2-z^2+3yz-2x)\,\vec{i} + (3xz+2xy)\,\hat{j} + (3xy-2xz+2z)\,\hat{k}$

For solenoidal, we have to prove $\mathrm{Div}\vec{F} = 0$.

Now $\vec{\nabla}.\vec{F} = (\hat{\imath}\,\frac{d}{dx} + \hat{\jmath}\,\frac{d}{dy} + \hat{k}\,\frac{d}{dz}).[(y^2-z^2+3yz-2x)\,\vec{i} + (3xz+2xy)\,\hat{j} + (3xy-2xz+2z)\,\hat{k}\,]$

$= -2+2x-2x+2 \quad = 0$

Thus, $\vec{F}$ is solenoidal. For irrotational, we have to prove Curl $\vec{F} = 0$

Now Curl $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 - z^2 + 3yz - 2x) & (3xz + 2xy) & (3xy - 2xz + 2z)\,) \end{vmatrix}$

$= (3x-3x)\,\vec{i} - (-2z+3y-3y+2z)\,\vec{j} + (3z+2y-2y-3z)\,\vec{k}$

$= 0$

Thus, $\vec{F}$ is irrotational.

(v) Successive operation of the $\vec{\nabla}$ operator:

Since $\vec{\nabla}$ is a vector differential operator it can operate successively on scalar or vector fields.

(a) The Laplacian

$$\vec{\nabla}.\vec{\nabla} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

Or, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \hat{\jmath}\frac{\partial^2}{\partial y^2} + \hat{k}\frac{\partial^2}{\partial z^2}$

This is called the *Laplacian operator*. This scalar operator can operate either on a scalar field or a vector filed. If V and $\vec{A}$ are scalar and vector fields respectively,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{1.10}$$

And $\nabla^2\vec{A} = \frac{\partial^2\vec{A}}{\partial x^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \frac{\partial^2\vec{A}}{\partial z^2}$

(b) Curl gradient of a scalar

The gradient of a scalr field V is

$$\vec{\nabla}V = \left(\hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

$$\vec{\nabla} \times \vec{\nabla}V = = \hat{\imath}\left(\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial y}\right)\right) + \hat{\jmath}\left(\frac{\partial}{\partial z}\left(\frac{\partial V}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z}\right)\right) + \hat{k}\left(\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right)\right)$$

$$= \hat{\imath}\left(\frac{\partial^2 V}{\partial y\,\partial z} - \frac{\partial^2 V}{\partial z\,\partial y}\right) + \hat{\jmath}\left(\frac{\partial^2 V}{\partial z\,\partial x} - \frac{\partial^2 V}{\partial x\,\partial z}\right) + \hat{k}\left(\frac{\partial^2 V}{\partial x\,\partial y} - \frac{\partial^2 V}{\partial y\,\partial x}\right)$$

If the second derivative is continuous, the order of differentiation is immaterial, i.e. $\frac{\partial^2 V}{\partial y\,\partial z} = \frac{\partial^2 V}{\partial z\,\partial y}$ etc. So, each term in the bracket vanishes.

$$\therefore \vec{\nabla} \times \vec{\nabla}V = 0$$

The curl of gradient of a scsalr fiel is zero. This result has the following important significance.

If $\vec{\nabla} \times \vec{A} = 0$, then vector field $\vec{A}$ can always be expressed aas the gradient of a scalar field , i.e. $\vec{A} = \vec{\nabla}V$. Conversely, if a vector field is the gradient of a scalar, its curl vanishes. For example, the electric field $\vec{E}$ is the –ve gradient of a potential V.

$$\vec{E} = -\vec{\nabla}V$$

So, $\vec{\nabla} \times \vec{E} = 0$

(c) Divergence of curl of vector field

The curl of a vector field $\vec{A}$ is

$$\vec{\nabla} \times \vec{A} = \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

So, $\vec{\nabla}.\vec{\nabla} \times \vec{A} = = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$

$$= \frac{\partial^2 A_z}{\partial x\,\partial y} - \frac{\partial^2 A_y}{\partial x\,\partial z} + \frac{\partial^2 A_x}{\partial y\,\partial z} - \frac{\partial^2 A_z}{\partial y\,\partial x} + \frac{\partial^2 y}{\partial z\,\partial x} - \frac{\partial^2 A_x}{\partial z\,\partial y} = 0$$

(Since the order of differentiation in immaterial)

$$\vec{\nabla}.\vec{\nabla} \times \vec{A} = 0$$

So, if a vector field is the curl of another vector , its divergence is zero. Conversely, if the divergence of a vector field is zero, it can always be expressed as the curl of a vector.

(d) Curl of curl of vector field

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2\vec{A}$$

(e) Divergence of cross product of two vectors

$$\vec{\nabla}.\left(\vec{A} \times \vec{B}\right) = \vec{B}.\left(\vec{\nabla} \times \vec{A}\right) - \vec{A}.(\vec{\nabla} \times \vec{B})$$

Example-16

If a scalar field f satisfies the relation, $\nabla^2 f = 0$ show that $\vec{\nabla} f$ is both solenoidal and irrotational.

Sol: A vector field $\vec{A}$ is solenoidal if $\vec{\nabla}.\vec{A} = 0$ and it is irrotational if $\vec{\nabla} \times \vec{A} = 0$.

Since $\vec{\nabla} \times \vec{\nabla} f = 0$ for any f, $\vec{\nabla}$f is irrotational.

Further $\nabla^2 f = 0 \equiv \vec{\nabla}.\vec{\nabla} f = 0$.

So $\vec{\nabla}$f is solenoidal.

1.2 INTEGRATION

(i) Line integration

The line integral of a vector field $\vec{A}$, between two points *a* and *b*, along a given path is

$$= \int_a^b \overrightarrow{A.\vec{dl}} \qquad (1.11)$$

Where $\vec{dl}$ is a vector length element of the path between *a* and *b*. The line integral of a vector field is a scalar quantity. The line integral depends, in general, on the initial point *a*, the final point *b* and the path of integration. If the integral is independent of the path of integration and depends only on the initial and final points, the corresponding vector field is called a conservative field. The line integral of a conservative field $\vec{A}$ along a closed path vanishes,

$$\oint \vec{A}.\vec{dl} = 0$$

The symbol $\oint$ signifies that the integration is along a closed path. An example of the line integral of a vector field is the work done by a variable force on a particle to move it from one point P to another point Q along a given path.

$$W = \int_p^q \vec{F}.\vec{dl}$$

If the work done by the force is independent of the path, the force is a conservative force.

Example 17. Calculate the line integral of the function $v = y^2 \hat{i} + 2x(y+1)\hat{j}$ from the point $O = (1,1,0)$ to the point $b = (2,2,0)$, along the path -1: O to A then to B and path-2: O to B directly in fig 1. What is $\oint v.dI$ for the loop that goes along path-1 and along path-2?

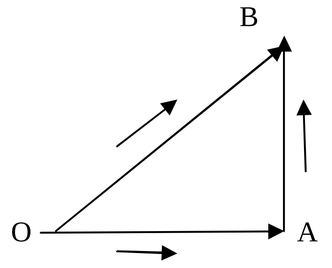


Fig.1

Solution: Here initial point is O and final point is B. The final point is reached in two ways. Let us see if line integration in moving from O to B in two ways gives same line integration or not. As

$$dI = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}.$$

Path (1): O to B consists of two parts.

(i) Along the horizontal path (O to A): y = 1, z =0 and $dy = dz = 0$ so,

$$\int v.dI = \int_1^2 [y^2\hat{i} + 2x(y+1)\,\hat{j}\,].dx\hat{i} = 1 \qquad \text{(As } y = 1)$$

(ii) On the vertical path (A to B): $dx = dz = 0$ so,

$$\int v.dI = \int_1^2 [y^2\,\hat{i} + 2x(y+1)\hat{j})].dy\hat{j} = \int_1^2 2x(y+1)\,dy = 10 \qquad \text{(As x=2)}$$

By path (1) then, $\int v.dI = 1 + 10 = 11$

Path (2): From O to B, $x = y, dx = dy$ and $dz = 0$ so, $dI = dx\,\hat{i} + dx\,\hat{j}$

$$\int_o^B v.dI = \int_1^2 [x^2 + 2x(x+1)]dx = (x^3 + x^2)\Big|_1^2 = 10$$

Thus, line integration along two different paths comes out different. That is line integration depends on the path of integration. For the loop that goes from O to A, then A to B and then to O, now becomes,

$$\oint v.dI = 11 - 10 = 1$$

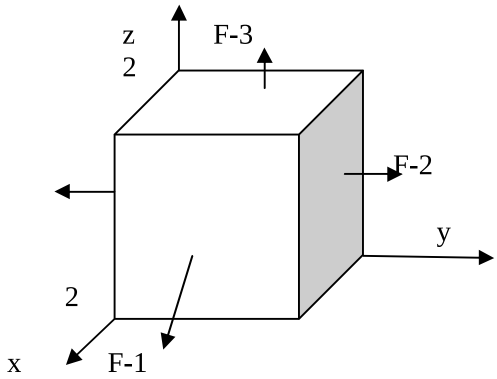
It can be now declared that the given vector is not a conservative vector field.

(ii) Surface Integration

For a vector function $\vec{F}$ which encloses a surface S, the surface integration is defined as $\iint \vec{F}.\vec{ds}$. The surface integral of a vector field is a scalar quantity. An example of surface integral of a vector field is the flux ϕ_E of electric field over a given area S.

$$\phi_E = \iint \vec{E}.\vec{ds}$$

Example 18 Calculate the surface integral of $E = 2xz\,\hat{i} + (x+2)\,\hat{j} + y(z^2-3)\,\hat{k}$ over five sides (excluding the bottom) of cubical box shown below.



Solution:

1. Face F-1: $\int E.da = \int [2xz\,\hat{i} + (x+2)\,\hat{j} + y(z^2-3)\,\hat{k}].dy\,dz\,\hat{\imath}$

   so, $\int E.da = 4\int_0^2 dy \int_0^2 zdz = 16$ as, $x = 2$

2. Back of face F-1: $\int E.da = \int [2xz\,\hat{i} + (x+2)\,\hat{j} + y(z^2-3)\,\hat{k}].(dy\,dz - \hat{\imath})$

   so, $\int E.da = 0$ as, $x = 0$

3. Face, F-2: $y = 2 \quad da = dzdz\,\hat{y} \quad v.da = (x+2)dxdz$

   So, $\int E.da = \int_0^2 (x+2)dx \int_0^2 dz = 12$

4. Back of face F-2: $y = 0 \quad da = -dzdz\,\hat{y} \quad v.da = -(x+2)dxdz$

   So, $\int E.da = -\int_0^2 (x+2)dx \int_0^2 dz = -12$

5. Face-3: $z = 2 \quad da = dxdy\,\hat{z} \quad v.da = y(z^2-3)dxdy = ydxdy$ so

   $$\int E.da = -\int_0^2 dx \int_0^2 ydy = 4$$

The total flux is $\int_s E.da = 16 + 0 + 12 - 12 + 4 = 20$ unit

(iii) Volume Integration

For any vector function $\vec{F}$ in which encloses a volume V, the Volume integration = $\iiint \vec{F}\, dV$. The volume integration of vector is a vector.

Example-19 If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$,then find its volume integration in a region bounded by surface (x=0 , x=2 ; y=0 , y=4 ; $z=x^2$, z=2).

Solution: $\iiint \vec{F}\, dV = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\, dx\, dy\, dz$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^{2} (2z\hat{i} - x\hat{j} + y\hat{k})\, dz$$

$$= \int_0^2 dx \int_0^4 dy\ [\, z2i - xzj + yz\hat{k}\,]_{x^2}^{2}$$

$$= \int_0^2 dx \int_0^4 dy\ [4\hat{i} - x^4\hat{i} - 2x\hat{j} + x^3\hat{j} + 2y\hat{k} - yx^2\hat{k}]$$

$$= \int_0^2 dx\ [4y\hat{i} - x^4y\hat{i} - 2xy\hat{j} + x^3y\hat{j} + y^2k - \frac{x^2y^2}{2}k]_0^4$$

$$= \int_0^2 [16\hat{i} - 4x^4\hat{i} - 8x\hat{j} + 4x^3\hat{j} + 16k - 8x^2k]\, dx$$

$$= \frac{32}{15}(3\hat{i} + 5\hat{k})$$ Ans.

1.3 BASIC THEOREMS OF ELECTRODYNAMICS

(i) Gauss divergence theorem:

This is a very useful vector integral theorem. It relates volume integral with surface integral. According to Gauss divergence theorem, the volume integral of divergence of a vector $\vec{A}$ over a given volume V is equal to the surface integral of the vector over a closed area enclosing the volume.

$$\iiint \vec{\nabla}.\vec{A}\ dv = \iint \vec{A}\,.d\vec{s} \qquad (1.12)$$

The applicability of the theorem can be understood by doing some numericals.

Example-20 Show that $\oint \vec{r}.\overrightarrow{ds}$ = 3V, where $\vec{r}$ is position vector equal to $x\hat{\imath} + y\hat{\jmath} + z\,\hat{k}$ and S is a closed area enclosing volume V.

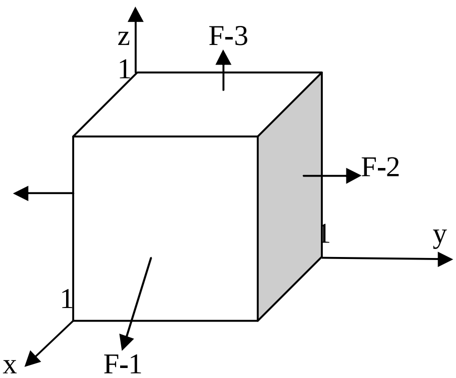
Sol: Gauss divergence theorem is $\iiint \vec{\nabla}.\vec{r}\;\; dv = \iint \vec{r}\,.\,d\vec{s}$.

Now $\vec{\nabla}.\vec{r} = \frac{dx}{dx} + \frac{dy}{dy} + \frac{dz}{dz}$ = 1+1+1=3

So, left side of above equation becomes, $\iiint \vec{\nabla}.\vec{r}\;\; dv = 3\iiint dv$ = 3V.

Thus, $\iint \vec{r}\,.\,d\vec{s}$ = 3V

Example 21 Check the divergence theorem using the function $v = y^2\hat{i} + (2xy + z^2)\,\hat{j} + 2yz\,\hat{k}$. and the unit cube situated at the origin.



Sol: We have $\nabla.v = 2(x + y)$

So, $\iiint \vec{\nabla}.v\; dv = \iiint 2(x + y)\; dx\; dy\; dz$

$$= 2\int_0^1\int_0^1\int_0^1 (x + y)dxdydz$$

$$\int_0^1 (x + y)dx = \frac{1}{2} + y \quad \int_0^1 (\frac{1}{2} + y)dy = 1 \quad \int_0^1 1dz = 1$$

$\iiint \vec{\nabla}.v\; dv = 2$

To evaluate the surface integral we must consider separately the six side of the cube:

(i) Face-1: $\iint v.da = \iint_{0,0}^{1,1} [\, y^2\hat{\imath} + (2xy + z^2)\hat{\jmath} + 2yz\,\hat{k}].dy\; dz\;\hat{\imath} = \frac{1}{3}$

(ii) Opposite to face-1: : $\iint v.da = \iint_{0,0}^{1,1} [\, y^2\hat{\imath} + (2xy + z^2)\hat{\jmath} + 2yz\,\hat{k}].dy\; dz(-\,\hat{\imath}) = -\frac{1}{3}$

(iii) Face-2: $\int v.da = \int_0^1 \int_0^1 (2x + z^2) dxdz = \frac{4}{3}$

(iv) Opposite to face-2: $\int v.da = -\int_0^1 \int_0^1 z^2 dxdz = -\frac{1}{3}$

(v) Face-3: $\int v.da = \int_0^1 \int_0^1 2y dxdy = 1$

(vi) Opposite to face-3: $\int v.da = -\int_0^1 \int_0^1 0 dxdy = 0$

So the total flux is $\oint_S v.da = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2$

Hence Gauss divergence theorem is verified.

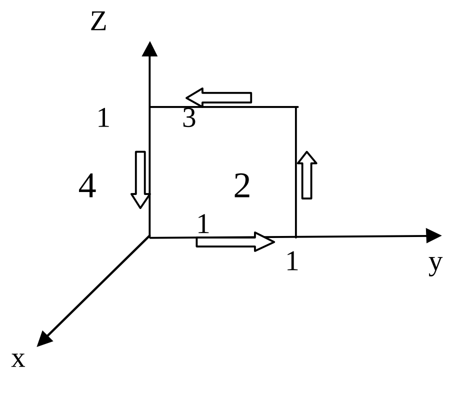
(ii) Stokes theorem

The Stoke's theorem relates the surface integral of a vector field with line integral. According to this theorem, the surface integral of the curl of a vector field $\vec{A}$ over a given area S is equal to the line integral of the vector along the boundary C of the area.

$$\iint_s \vec{\nabla} \times \vec{A} \cdot \overrightarrow{ds} = \oint_c \vec{A} \cdot \overrightarrow{dl} \tag{1.13}$$

Here the closed curve C is the boundary of the surface S, which may have any shape. For a closed surface, C=0. Hence the surface integral of the curl of a vector, over a closed surface, vanishes.

Example 22 Check the Stoke's theorem using the function $\vec{A} = (2xz + 3y^2)\hat{\jmath} + (4yz^2)\hat{k}$ for the square surface shown in the below.



Sol Here $\vec{\nabla} \times \vec{A} = (4z^2 - 2x)\hat{\imath} + (2z)\hat{k}$ and $\overrightarrow{ds} = dy\,dz\,\hat{\imath}$

So, $\iint_s \vec{\nabla} \times \vec{A} \cdot \overrightarrow{ds} = \iint_{0,0}^{1,1} ((4z^2 - 2x)\hat{\imath} + (2z)\hat{k}) . (dy\,dz\,\hat{\imath})$

$= \iint_{0,0}^{1,1} (4z^2 - 2x) dy\,dz$

$= \iint_{0,0}^{1,1} (4\,z^2)dy\,dz$ since $x=0$ for this surface.
$= 4/3$

The right hand side is line integral, it consists of four segments.

(i) Along arrow 1:

$$\int \vec{A}\cdot\overrightarrow{dl} = \int (\,2\,x\,z + 3\,y^2)\hat{j} + (\,4\,y\,z^2)\hat{k}\;\cdot dy\hat{j}$$
$$= \int_0^1 3y^2\;dy$$
$$= 1$$

(ii) Along arrow 2:

$$\int \vec{A}\cdot\overrightarrow{dl} = \int (\,2\,x\,z + 3\,y^2)\hat{j} + (\,4\,y\,z^2)\hat{k}\;\cdot dz\hat{k}$$
$$= \int_0^1 4z^2\;dz$$
$$= 4/3$$

(iii) Along arrow3:

$$\int \vec{A}\cdot\overrightarrow{dl} = \int (\,2\,x\,z + 3\,y^2)\hat{j} + (\,4\,y\,z^2)\hat{k}\;\cdot dy\,\hat{k}$$
$$= \int_1^0 3y^2\;dy$$
$$= -1$$

(iv) Along arrow 4:

$$\int \vec{A}\cdot\overrightarrow{dl} = \int (\,2\,x\,z + 3\,y^2)\hat{j} + (\,4\,y\,z^2)\hat{k}\;\cdot dz(-\,\hat{k})$$
$$= \int_1^0 0\;dz \qquad \text{as y=0}$$
$$= 0$$

So, $\int \vec{A}\cdot\overrightarrow{dl} = 4/3$

Hence, Stoke's theorem is checked.

2. The basic laws of electricity and magnetism

(i) Laws of electricity

(a) Coulomb's law

The electrostatic force of attraction or repulsion on a point charge q_1 due to another charge q_2, separated by a distance r in vacuum is given by Coulomb's law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1 q_2}{r^2}\,\hat{r} \qquad (2.1)$$

where $\hat{r}$ is a unit vector from q1 to q2 and ε_0 = electric permittivity of vacuum = $8.85\times 10^{-12}\ C^2\ N^{-1}\ m^{-2}$. The constant $\frac{1}{4\pi\varepsilon_0}$ = $9.0 \times 10^9\ N\ m^2 / C^2$. In a medium, Coulomb's law is modified to the form,

$$\vec{F} = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2}\hat{r}, \qquad (2.2)$$

where ε is the electric permittivity of the medium. $\varepsilon = \varepsilon_r\varepsilon_0$. Here $\varepsilon_r = \varepsilon/\varepsilon_0$ is called the relative permittivity or dielectric constant of the medium. It is a positive number greater than 1.

(b) Electric field

The *electric field* $\vec{E}$ at a point is defined as the limiting value of the net electrostatic force per unit charge, on a test charge Δq as the charge tends to zero

$$\vec{E} = \lim_{\Delta q \to 0}\frac{\vec{F}}{\Delta q} \qquad (2.3)$$

The magnitude of the test charge is infinitely small; otherwise the original electric field would be changed due to the field of the test charge. The electric field is represented geometrically by electric lines of force which are continous curve such that the tangent to the line of force at a point gives the direction of the electric field at that point. The SI unit of electric field is volt / meter(V/m) or Newton/Coulomb(N/C).

(c) The electric displacement $\vec{D}$

The *electric displacement* $\vec{D}$ at a point is related to the electric field by $\vec{D} = \varepsilon\vec{E}$ where ε is the electric permittivity of the medium. In vacuum, $\vec{D} = \varepsilon_0\vec{E}$. The SI unit of electric displacement D is C/m^2.

(d) Electric flux ϕ_E

The concept of electric flux was developed by Gauss. The flux of any vector field $\vec{A}$ over a given surface S is the surface integral of the field over the area

$$\phi_E = \iint_s \vec{A}\cdot\overrightarrow{ds}.$$

The electric flux ϕ_E over a surface S is the surface integral of the electric field over the surface,

$$\phi_E = \iint_s \vec{E}\cdot\overrightarrow{ds}. \qquad (2.4)$$

Example 23. Evaluate the flux of a vector field $\vec{E} = \hat{\imath}\,xy + \hat{\jmath}\,2x^2 + \hat{k}\,xz$, over a plane area on the XZ plane bounded by x = 0, x = a; z = 0 and z = b.

Solution: Since the surface is on the XZ plane, the normal to the area is along the Y-axis.

So, $d\vec{S} = \hat{n}\, dS = \hat{j}$ dxdz. Flux = $\phi = \iint_s \vec{E} \cdot \overrightarrow{ds}$.

$$= \int_{x=0}^{a} \int_{z=0}^{b} (\hat{\imath}\, xy + \hat{\jmath}\, 2x^2 + \hat{k}\, xz) \cdot \hat{\jmath}\; dx\, dz$$

$$= \int_{x=0}^{a} \int_{z=0}^{b} 2x^2\; dx\, dz$$

$$= 2\int_{x=0}^{a} x^2\, dx \int_{z=0}^{b} dz \quad = \frac{2}{3} a^3 b$$

(e) Gauss's law in Electrostatics

The electric flux over a closed surface is related to the net electric charge enclosed by it. The connection between them was established by Gauss. According to Gauss' law the total electric flux ϕ_E over a closed surface is equal to $1/\varepsilon_0$ times the net charge enclosed by the surface.

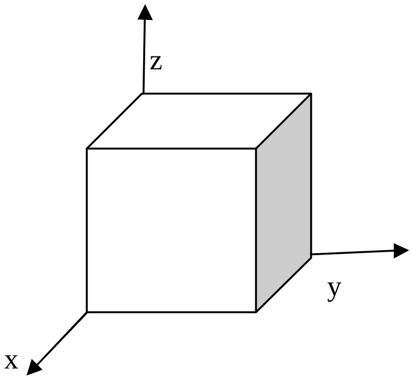
$$\phi_E = \oint_s \vec{E} \cdot d\vec{S} = \frac{q_{net}}{\varepsilon_0} \tag{2.5}$$

The net charge q_{net} is the algebraic sum of the charges enclosed by the surface. The surface S is usually called the *Gaussian surface.*

Note: The following points need to be noted.

1. The charges enclosed by the surface may be point charges or continuously distributed charges. They may be positive or negative charges.
2. The net charge q_{net} may be positive, negative or zero. Accordingly, the net electric flux over the area may be *outward, inward or zero*.
3. The net electric flux over a surface does not depend on the relative position or state of motion of the charges as long as they are within the surface.
4. The electric flux does not depend on the shape or size of the Gaussian surface as long as the charges are enclosed by it. This feature is extremely useful in practical application of Gauss' law. The Gaussian surface can be chosen to have a suitable geometrical shape over which the flux can be evaluated in a simple way.
5. **Limitation of gauss' law**: since the electric flux is a scalar quantity, Gauss' law enables one to find only the magnitude of the electric field. The direction is to be determined from other considerations.
6. Gauss's law in a dielectric medium: In a medium, Gauss' law takes the form,
$$\phi_E = \oint_s \vec{E} \cdot d\vec{S} = \frac{q_{net}}{\varepsilon} \tag{2.6}$$
where ε is the permittivity of the medium.
7. Since $\vec{D} = \varepsilon \vec{E}$ in a medium and $\vec{D} = \varepsilon_0 \vec{E}$ in vacuum, we get from (2.6),
$$\oint_s \vec{D} \cdot d\vec{S} = q_{net}$$

Problem 24 A cubical box of side $1A^0$ contains an electron at center. Find out electric flux from any one face of the box.



Sol: ϕ = net charge / ε_0

$= 1.6 \times 10^{-19}$ C / ε_0

Flux through one face = ϕ / 6

$= 1.6 \times 10^{-19}$ / 6 ε_0 C/ m^2

(iii) The laws of magnetism

The magnetic field (or magnetic induction or magnetic flux density) is defined in terms of the force $\overrightarrow{F_B}$ experienced by a test charge q moving in the field with velocity $\vec{v}$.

$$\overrightarrow{F_B} = q\, \vec{v} \times \vec{B}$$

The SI unit of magnetic field is Tesla (T). 1 T = 1 $\frac{N}{A.m}$ = 1 $\frac{Wb}{m^2}$ = 10^4 gauss

(a) Ampere's circuital law

Ampere's circuital law (formulated by Andre Marie Ampere) relates the distribution of magnetic field along a closed loop with the net electric current enclosed by the closed loop. According to this law, the line integral of magnetic field along a closed loop is equal to μ_0 times the net electric current enclosed by the loop

$$\oint_c \vec{B}.\overrightarrow{dl} = \mu_0 I_{net} \qquad (2.7)$$

Where C is a closed path enclosed the current and I_{net} is the algebraic sum of currents enclosed by the loop C. The closed loop C is called the *Ampere loop*, which can be of any shape as long as it encloses the currents. The magnetic field can be evaluated choosing a convenient shape of the Amperian loop.
Ampere's circuital law can be expressed in term of magnetic intensity as follow

$$\oint_s \vec{H} \cdot \overrightarrow{dl} = I_{net} \qquad (2.8)$$

In a medium, Eq. (2.7) and (2.8) are modified by replacing μ_0 by μ

(b) Faraday's law of Electromagnetic induction

Michael Faraday established experimentally that an e.m.f. is induced in a closed conducting loop if the magnetic flux linked with the surface enclosed by the loop change with time. The magnitude of the e.m.f. depends on the rate at which the flux changes. This is depicted quantitatively in Faraday's law of electromagnetic induction, according to which "the e.m.f. ε induced in a conducting loop is equal to the negative of the rate of change of magnetic flux through the surface enclosed by the loop".

$$\varepsilon = -\frac{\partial \phi_n}{\partial t} \tag{2.9}$$

The induced e.m.f. is the line integral of electric field along the loop

$$\varepsilon = \oint_C \vec{E}.\vec{dl}$$

If S is the surface enclosed by the loop the magnetic flux through S is

$$\phi_m = \oint_S \vec{B}.\vec{dS}$$

So Eq. (4) become

$$\oint_C \vec{E}.\vec{dl} = \frac{\partial}{\partial t}\oint_S \vec{B}.\vec{dS} \tag{2.10}$$

This is faraday's law of electromagnetic induction in term of *E* and *B*.

(c) Equation of continuity

The electric current density $\vec{j}$ (current per unit area) and electric charge densityρ ρ (charge per unit volume) are related by the equation of continuity, which follows from the conservation of charge in a given volume. The electric current through a closed surface S is

$$I = \iint \vec{j}\cdot\overrightarrow{ds} \tag{2.11}$$

The electric current *I* is the rate of decrease of electric charge, $I = -\frac{dq}{dt}$

But, $\quad q = \iiint \rho\, dv$

So, $\quad I = -\frac{d}{dt}\iiint \rho\, dv \quad (2.12)$

From eq. (2.11) and (2.12), we get $\quad \iint \vec{j}\cdot\overrightarrow{ds} = -\frac{d}{dt}\iiint \rho\, dv \quad (2.13)$

Using Gauss divergence theorem, the left side can be converted into $\iint \vec{j}\cdot\overrightarrow{ds} = \iiint (\vec{\nabla}\cdot\vec{j})\, dv$

So, $\quad \iiint (\vec{\nabla}\cdot\vec{j})dv = -\frac{d}{dt}\iiint \rho\, dv$

Or, $\quad \iiint(\vec{\nabla}\cdot\vec{j} + \frac{d\rho}{dt}) = 0$

The vanishing of integrand gives the equation of continuity,

$$\vec{\nabla}\cdot\vec{j} + \frac{d\rho}{dt} = 0 \tag{2.14}$$

The Gauss law in electrostatic and magneto-statics and Ampere's circuital law describe the steady state behavior of the electric and magnetic fields. However, when the fields change with time, it is observed that (i) a time varying magnetic field gives rise to electric field and (ii) a time varying electric field produces a magnetic field. These are described by Faraday's law of electromagnetic induction and Maxwell's idea of *displacement current* respectively.

Problem 25 The current density at a point in ionosphere is $\vec{j}$ = 2x$\hat{i}$ -3y $\hat{j}$ + 5z$\hat{k}$. Find out the charge density at that point of the ionosphere.

Sol: $\vec{\nabla} \cdot \vec{j} + \frac{d\rho}{dt} = 0$; As, $\nabla . j = \nabla .(2x\hat{i} - 3y\hat{j} + 5z\hat{k})$ = 2-3+5 = 4

So, $\frac{d\rho}{dt} = -4$ This gives ρ = - 4 t + C. where C is a const.

(d) Modified Ampere's law

Maxwell while studying the electromagnetic laws noticed that there was something strange in the Ampere's law. The strange thing is the following. He first expressed Ampere's law in differential form.

$$\oint_c \vec{B}.\vec{dl} = \mu_0 I_{net}$$

The LHS can be converted into a surface integral by the help of stake's theorem. It gives

$$\iint \vec{\nabla} \times \vec{B} \cdot \vec{ds} = \mu_0 I_{net}$$

As, we know $I_{net} = \iint \vec{j} \cdot \vec{ds}$

So, above equation becomes

$$\iint \vec{\nabla} \times \vec{B} \cdot \vec{ds} = \mu_0 \iint \vec{j} \cdot \vec{ds}$$

This gives, $$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \qquad (2.15)$$

This is the differential form of Ampere's law. If Ampere's law is correct, then it should be applied for all types of currents. To examine correctness, Maxwell applied divergence on both sides of the Ampere's law eq. (2.15)

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 (\vec{\nabla} \cdot \vec{j})$$

As divergence of a curl is always zero, the left side of above equation is zero. This gives,

$$\vec{\nabla} \cdot \vec{j} = 0$$

But equation of continuity tells that $\vec{\nabla} \cdot \vec{j} + \frac{d\rho}{dt} = 0$. Using $\vec{\nabla} \cdot \vec{j} = 0$, equation of continuity reduces to $\frac{d\rho}{dt} = 0$. But this certainly cannot in general be zero because we know that the charges move and vary with time. In other words, Ampere's law needs $\frac{d\rho}{dt} = 0$.This tells us that

Ampere's law is incorrect but correct when $\frac{d\rho}{dt} = 0$. This means Ampere law is valid for steady current only. The law is not valid for varying currents. Maxwell proposed that Ampere's law is not correct for moving charges and suggested that for changing charge distributions, Ampere's law should be

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + ?$$

which will be valid for all type of charges and currents. To know this unknown, Maxwell suggested a proposal. The proposal is similar to the electric field due to a changing magnetic field (Faraday's law of induction) there would be a magnetic field due to the changing electric field, i.e,

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{dE}{dt}$$

Using this as the missing term, we get

$$\boldsymbol{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{dE}{dt}} \qquad (2.16)$$

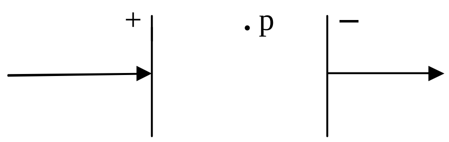
To check validity of his proposal, he take divergence on both sides

$$\begin{aligned}
\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B}\right) &= \vec{\nabla} \cdot (\mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{dE}{dt}) \\
&= \vec{\nabla} \cdot (\mu_0 \vec{J}) + \vec{\nabla} \cdot \left(\mu_0 \varepsilon_0 \frac{dE}{dt}\right) \\
&= \mu_0 (\vec{\nabla} \cdot \vec{J} + \varepsilon_0 \frac{d}{dt} \left(\vec{\nabla} \cdot E\right)) \\
&= \mu_0 (\vec{\nabla} \cdot \vec{J} + \frac{d}{dt} \rho)
\end{aligned}$$

Maxwell first equation $\vec{\nabla} \cdot E$ =0 is used here. As div (curl B) is zero, hence left hand side is zero. The right hand side is also zero by virtue of the equation of continuity condition. Equation (2.16) is known as modified Ampere's law which is valid for all type of currents. Maxwell called the extra term in equation (2.16) as displacement current density. How did Maxwell derive the extra term?

(e) Maxwell's displacement current

The conduction current (in metals) and convection current (in electrolytic solutions and ionized gases) produced observable magnetic fields which can be evaluated by using Bio-Savart law or Ampere's circuital law. However when the electric field in a region (in vacuum or in a medium) changes with time, the time varying electric field also produce a magnetic field. Maxwell associated a current (the displacement current) with the time varying electric field. Consider a parallel plate capacitor being charged by a cell.



Parallel plate capacitor

During the charging process, the electric fields between the capacitor plates changes with time. It observed that a magnetic field exit between the plates, as long as the electric field changes, even if there is no current between the plates. In order to resolve this inconsistency, Maxwell introduced the concept of displacement current associated with time varying electric field between the capacitor plates. If q is the electric charge on the capacitor plates and A is the area of each plate, the electric field in the gap between the plates is

$$E = \frac{q}{\varepsilon_0 A}$$

$$\frac{dE}{dt} = \frac{1}{\varepsilon_0 A}\frac{dq}{dt}$$

$$= \frac{1}{\varepsilon_0 A} I_d$$

Or, $$I_d = \varepsilon_0 A \frac{dE}{dt} \quad (2.17)$$

This is the displacement current between the plates. The displacement current exits as long as the electric field changes with time. When the plates of capacitor are fully charged, the electric field attains a constant value and hence the displacement current vanishes. In general whenever there is a time varying electric field, a displacement current exists.

$$I_d = \varepsilon_0 \frac{d}{dt}\iint \vec{E}\cdot\overrightarrow{ds}$$

Or, $$I_d = \varepsilon_0 \frac{d\emptyset_e}{dt} \quad (2.18)$$

Distinction between displacement current and conduction current

1. The conduction current originates from the actual flow of charge carriers in metals or in other conducting medium. On the other hand, the displacement current is a fictitious current, which can exit in vacuum or in any medium (even in the absence of free charge carriers), if there exists a time varying electric field there.

2. The conduction current obeys ohm’s law and depends on the resistance and potential difference of the conductor. The displacement current, on the other hand, depends on the electric permittivity of the medium and the rate at which the electric field changes with time.

3. The displacement current can show all characters of a current except heating effect as this current is not produced by flow of charge carriers which produces resistance.

Relative magnitude of displacement current and conduction current

Consider an alternating electric field $E = E_0 \sin\omega\, t$ propagating in a medium. The conduction current density due to this is $j = \sigma E = \sigma E_0 \sin\omega\, t$. The displacement current density is $j_d = \frac{d\,(\varepsilon_0 E)}{dt} = \omega\, \varepsilon_0\, E_0 \cos\omega\, t$. So, there is a phase difference of $\pi/2$ between the conduction current and displacement current. Further, the ratio of the peak values is $\frac{j}{j_d} = \frac{\sigma}{\varepsilon_{0\,\omega}}$. Thus, ratio of peak values depends on the frequency with which the electric field alternates. For a copper conductor in vacuum $\frac{\sigma}{\varepsilon_0} = 10^{19}$, so, the ratio is of the order of $\frac{10^{19}}{\omega}$. So, for most of the conductor, the ratio is very large even for reasonably high frequency. However, if the frequency exceeds 10^{19}Hz, the displacement current will be dominant. So, the normal conductors will behave as dielectric at extremely high frequencies.

Example 26 A parallel plate capacitor having circular plates of radius 5.5 cm is being charged. Calculate the displacement current if the rate of change of electric field between the plates is $1.5 \times 10^{10}\ \frac{V}{m.\ s}$.

Sol: The displacement current is $I_d = \varepsilon_0 A\ \frac{dE}{dt} = \varepsilon_0 \pi r^2\ \frac{dE}{dt}$

$$I_d = (8.8 \times 10^{-12}\ C^2N^{-1}m^{-2}) \cdot (3.14) \cdot (5.5 \times 10^{-2}\ m) \cdot (1.5 \times 10^{10}\ V/ms)$$

$$I_d = 0.126\ \mu\text{A}$$

(f) Modification of Ampere's circuital law

In order to incorporate the effect of time varying electric field, the Ampere's circuital law is to be modified by adding by the displacement current I_d with the conduction current I_c

$$\oint \vec{B} \cdot \vec{dl} = \mu_0\ (\ I_c + I_d) \tag{2.19}$$

3. Maxwell's electromagnetic equations

The various laws of electromagnetism were pulled together and were cast into four equation involving time and space derivative of electric and magnetic fields. These equations are known as Maxwell's electromagnetic equations and are given below

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{3.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3.2}$$

$$\vec{\nabla} \cdot \vec{E} = -\frac{dB}{dt} \tag{3.3}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J_c} + \mu_0\ \frac{d}{dt} \varepsilon_0\, \vec{E} \tag{3.4}$$

The above four equations are the famous Maxwell's electromagnetic equations in vacuum medium in presence of charges and currents. They are the differential forms of (i) Gauss' law in electrostatics, (ii) Gauss' law in magnetism (iii) Faraday's law of electromagnetic induction and, (iv) Generalized from of Ampere's circuital law respectively.

(i) Derivation of first Maxwell's equation

According to Gauss's law of Electrostatics $\phi = \frac{1}{\varepsilon_0}$q

But, q = $\iiint \rho\ dV$ and $\phi = \iint E$ dS

So, $\iint E \text{ ds} = \frac{1}{\varepsilon_0}\iiint \rho \, dV$

Applying Gauss's divergence theorem to the left hand side of above equation

$$\iiint divE \, dV = \frac{1}{\varepsilon_0}\iiint \rho \, dV$$

Or, $\iiint (div\, E - \frac{\rho}{\varepsilon_0}) \, dv = 0.$

Above equation gives

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{3.1}$$

This is first Maxwell's equation.

(ii) Derivation of second Maxwell's equation

Gauss's law of magneto statics states that isolated magnetic pole does not exist. That is magnetic poles exist in coupled form i.e, in dipole form (north and South Pole separated by some distance). So, for a north pole, there will be a south pole. Then in a material the net number of magnetic mono pole will be zero (North Pole can be thought as + ve and South Pole can be thought as – ve).

So the magnetic flux linked with a closed surface will be zero, i.e, $\iint B.$ ds = 0

Applying Gauss's divergence theorem to left hand side of above equation, we get $\iiint (\vec{\nabla} \cdot \vec{B}) \, dV = 0.$

This equation is true when integrand itself is zero.

That is $$\vec{\nabla} \cdot \vec{B} = 0 \tag{3.2}$$

This is second Maxwell's equation.

(iii) Derivation of third Maxwell's equation

From Faraday's law of electromagnetic induction e = - $\frac{d\phi}{dt}$

When magnetic flux linked with a surface is changed then emf is induced

Then $$e = -\frac{d\phi}{dt} = -\frac{d}{dt}\iint B \text{ ds} \tag{3.5}$$

If E is electric field produced due to change in magnetic flux then

$$e = \oint E \, dl \tag{3.6}$$

Comparing equations (3.5) and (3.6)

$$\oint E \, dl = -\frac{d}{dt}\iint B \text{ ds}$$

Applying Stokes curl theorem to the left hand side of above equation

$$\iint curl\, E\, ds \;=\; -\frac{d}{dt}\iint B \, \mathrm{ds}$$

Comparing the both sides of above equation we get-

$$\vec{\nabla} \times \vec{E} = -\frac{dB}{dt} \tag{3.3}$$

This is third Maxwell's equation.

(iv) Derivation of fourth Maxwell's equation

The modified Ampere's circuital law for time varying fields is

$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 (I_c + I_d)$$

$$= \mu_0 (\iint \vec{J_c} \cdot \overrightarrow{ds} + \varepsilon_0 \frac{d}{dt} \iint \vec{E} \cdot \overrightarrow{ds})$$

$$= \mu_0 (\iint \vec{J_c} \cdot \overrightarrow{ds} + \iint \frac{d}{dt} \varepsilon_0 \vec{E} \cdot \overrightarrow{ds})$$

$$= \iint \mu_0 (\vec{J_c} + \frac{d}{dt} \varepsilon_0 \vec{E}) \cdot \overrightarrow{ds}$$

Or, $$\oint \vec{B} \cdot \overrightarrow{dl} = \iint \mu_0 (\vec{J_c} + \frac{d}{dt} \varepsilon_0 \vec{E}) \cdot \overrightarrow{ds}$$

Applying stoke's theorem on left side, we get

$$\iint (\vec{\nabla} \times \vec{B}) \cdot \overrightarrow{ds} = \iint \mu_0 (\vec{J_c} + \frac{d}{dt} \varepsilon_0 \vec{E}) \cdot \overrightarrow{ds}$$

Or, $$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J_c} + \mu_0 \frac{d}{dt} \varepsilon_0 \vec{E} \tag{3.4}$$

This is fourth Maxwell's equation.
Maxwell's equations can be written in term of D and H as

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{3.7}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{3.8}$$

$$\vec{\nabla} \times \vec{D} = -\mu\varepsilon \frac{\overrightarrow{dH}}{dt} \tag{3.9}$$

$$\vec{\nabla} \times \vec{H} = \vec{J_c} + \frac{d}{dt} \vec{D} \tag{3.10}$$

In vacuum, ε and μ are to be replaced by ε_0 and μ_0 respectively. In absence of charge, ρ = 0 and in absence of carrier $\vec{j} = 0$. So in vacuum in absence of charge and current, Maxwell's equations are

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{dB}{dt}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \frac{d}{dt} \varepsilon_0 \vec{E}$$

Maxwell's equations in integral from

The Maxwell's equations Eq. (3.1) to (3.4) are in differential form. They were obtained from the laws of electromagnetism. The Maxwell's equations can be expressed in integral form by taking volume and surface integral of both sides, and using the integral theorems of vector calculus. The integral forms of Maxwell's equations are:

$$\iint_s \vec{E} \cdot \overrightarrow{ds} = \frac{1}{\varepsilon_0} \iiint_v \rho \, dv$$

$$\iint_s \vec{B} \cdot \overrightarrow{ds} = 0$$

$$\oint_c \vec{E} \cdot \overrightarrow{dl} = -\iint_s \frac{d\vec{B}}{dt} \cdot \overrightarrow{ds}$$

$$\oint_c \vec{B} \cdot \overrightarrow{dl} = \iint_s \left(\mu \vec{J} + \mu\varepsilon_0 \frac{d\vec{E}}{dt}\right) \cdot \overrightarrow{ds}$$

Interpretation

i) Eq. (3.2) and (3.3) has the same from in vacuum or in a medium. They are also unaffected by the presence of free charge or currents. They are usually called constraints equation for electric and magnetic fields.

ii) Eq. (3.1) and (3.4) depend on the presence or absence of free charge and currents and also on the medium.

iii) The divergence equation given by Eq. (3.1) and (3.2) are called steady state equation as they do not involve the time dependence of field.

iv) The curl equation given by Eq. (3.3) and (3.4) are time varying equation as they describe the time dependence of the fields.

The significance of Maxwell's equations

1. Maxwell's equations incorporate all the law of electromagnetism, which were developed from experimental observations and were expressed in term of various empirical laws.

2. Maxwell's equations lead to existence of electromagnetic waves which has been confirmed by experimental observations. These equations are consistent with all the observed properties of electromagnetic waves.

3. Maxwell's equations are consistent with the special theory of relativity. (It is worth mentioning that many equations in physics are not consistent with the requirement of the special theory of relativity).

4. Maxwell's equations are used to describe the classical electromagnetic field as well as the quantum theory of interaction of the charge particles with electromagnetic field.

5. Maxwell's equations provided a unified description of the electric and magnetic phenomena which were treated independently.

The development of Maxwell's electromagnetic equations was a great achievement in physics in the nineteenth century. We discuss the consequence of these equations in the next.

Problem 27 The magnetic field at a point is given by, $\vec{B} = ax\,\hat{i} + by\,\hat{j}$ where a and b are in T/m. How are a and b related, if magnetic mono pole does not exist?

Sol: As we know $\nabla.B = 0$

This gives, a x + b y = 0.

This again gives, a = - b y / x.

Problem 28 An electric field in free space has the components E_x =0, E_y=0, $E_z = E_0 \cos(2\pi x / \lambda)\cos\omega t$. Find out the magnetic field $\vec{B}$.

Sol: Maxwell's thisrd equation is curl E = $-\frac{\partial \vec{B}}{\partial t}$.

$$\text{Curl E} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & E_z \end{vmatrix}$$

$= i\, dE_z/dy - dE_z/dx$

$= j(\frac{2\pi}{\lambda})\, E_0 \sin(\frac{2\pi x}{\lambda}) \cos wt.$

Thus, $\partial B/\partial t = -j(\frac{2\pi}{\lambda}) E_0 \sin(\frac{2\pi x}{\lambda}) \cos wt.$

Or, $B = -j(\frac{2\pi}{\lambda\omega}) E_0 \sin(\frac{2\pi x}{\lambda}) \sin wt + C$

$= -j\, \frac{1}{c}\, E_0 \sin(\frac{2\pi x}{\lambda}) \sin wt + C$

4. Electromagnetic waves

Maxwell's equations lead to the wave equation for electric and magnetic fields. The Electromagnetic waves characterized by electric and magnetic fields vectors are usually classified by gamma rays, X-rays, ultra violet rays, infra red rays, microwaves, radio waves etc. depending on their frequencies. These waves carry energy and momentum, and travel in vacuum with the same speed. However, in a medium they travel with different speeds and may undergo depression, absorption etc. They undergo reflection and refraction at boundaries of two media also.

Wave equation for $\vec{E}$ and $\vec{B}$

Maxwell's equations are a set of coupled first order partial differential equations relating space and time derivatives of various components of electric and magnetic fields. They can be decoupled to obtain the wave equation for electric and magnetic fields. Here ε and μ are the electric permittivity and magnetic permeability of the medium. They may be constant or dependent on position and time, $\varepsilon = \varepsilon(r, t)$ and $\mu = \mu(r, t)$. The form of wave equation in a medium depends on the nature of ε and μ and also on the presence or absence of charges and currents.

(i) Wave equation in free space ($\rho = 0, j = 0$)

Maxwell's equations in a medium in the absence of charges and currents are

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad (4.1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (4.2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{dB}{dt} \qquad (4.3)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{d}{dt} \vec{E} \qquad (4.4)$$

Taking curl on both sides of eq. (4.3), we get

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{d}{dt} \vec{\nabla} \times \vec{B}$$

Since the order of space and time derivatives can be interchanged.

Using the vector relation $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E}$ in above equation, we get

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\frac{d}{dt} \vec{\nabla} \times \vec{B}$$

Using eq. (4.1) and (4.4) in above equation, we get

$$\nabla^2\vec{E} = \frac{d}{dt}\left(\mu_0\varepsilon_0 \frac{d}{dt} \quad \vec{E}\right)$$

Or $$\nabla^2\vec{E} = \mu_0\varepsilon_0 \frac{d^2}{dt^2} \quad \vec{E} \tag{4.5}$$

This is the wave equation for $\vec{E}$. In terms of component of E this equation can be written as

$$\nabla^2 E_x - \mu_0\varepsilon_0 \frac{d^2E_x}{dt^2} = 0,$$

$$\nabla^2 E_y - \mu_0\varepsilon_0 \frac{d^2E_y}{dt^2} = 0$$

And $$\nabla^2 E_z - \mu_0\varepsilon_0 \frac{d^2E_z}{dt^2} = 0$$

Similarly the wave equation for $\vec{B}$ can be obtained by taking curl on both side of eq. (4.4)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = -\mu_0\varepsilon_0 \frac{d}{dt} \vec{\nabla} \times \vec{E}$$

Since the order of space and time derivatives can be interchanged. Using the vector relation $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E}$ in above equation, we get

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right) - \nabla^2\vec{B} = -\mu_0\varepsilon_0 \frac{d}{dt} \vec{\nabla} \times \vec{E}$$

Using eq. (4.2) and (4.3) in above equation, we get

$$\nabla^2\vec{B} = \mu_0\varepsilon_0 \frac{d}{dt}\left(\frac{d}{dt} \quad \vec{B}\right)$$

Or $$\nabla^2\vec{B} = \mu_0\varepsilon_0 \frac{d^2}{dt^2} \quad \vec{B} \tag{4.6}$$

This is the wave equation for $\vec{B}$. Eq. (4.6) can be written in terms of components of $\vec{B}$ as

$$\nabla^2 B_x - \mu_0\varepsilon_0 \frac{d^2B_x}{dt^2} = 0,$$

$$\nabla^2 B_y - \mu_0\varepsilon_0 \frac{d^2B_y}{dt^2} = 0$$

And $$\nabla^2 B_z - \mu_0\varepsilon_0 \frac{d^2B_z}{dt^2} = 0$$

The general wave equation satisfied by a wave is

$$\frac{d^2\psi}{dx^2} = \frac{1}{v^2}\frac{d^2\psi}{dt^2} \tag{4.7}$$

Comparing the wave equation for $\vec{E}$ and $\vec{B}$ with eq. (4.7) we get the speed of electromagnetic wave in vacuum, $v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ .Since $\varepsilon_0 = 8.85\times10^{-12} F/m$ and $\mu_0 = 4\pi\times10^{-7} H/m$. The speed of electromagnetic wave in vacuum is $v = 3\times10^{8} m/s$ which is the speed of light, c in vacuum.

(ii) Wave equation in a charge free and non conducting medium

In a charge free and non conducting medium $\rho = 0$ and $j = 0$. If the electric permittivity ε & magnetic permeability μ of the medium are independent of position and time, the wave equation for $\vec{E}$ and $\vec{B}$ in the medium can be obtained in the same way as that for free space. Since $\varepsilon > \varepsilon_0$ and $\mu > \mu_0$, the speed of electromagnetic wave in the medium will be now $v = \frac{1}{\sqrt{\mu\varepsilon}}$ which is less than c. If ε and μ are dependent on position and time, the wave equation becomes more complicated.

(iii) Wave equation in a charge free but conducting medium

In a charge free region $\rho = 0$. The electric current density, in a conducting medium is $\vec{j} = \sigma\vec{E}$, where σ is the conductivity of medium . If ε and μ are independent of position and time, the Maxwell's equations take the form

$$\vec{\nabla}\cdot\vec{E} = 0 \tag{4.8}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{4.9}$$

$$\vec{\nabla}\times\vec{E} + \frac{d\vec{B}}{dt} = 0 \tag{4.10}$$

$$\vec{\nabla}\times\frac{\vec{B}}{\mu} - \varepsilon\frac{d\vec{E}}{dt} = \sigma\vec{E} \tag{4.11}$$

Taking curl on both sides of eq. (4.10) we get $\vec{\nabla}\times\vec{\nabla}\times\vec{E} + \frac{d}{dt}\vec{\nabla}\times\vec{B} = 0$

Or, $$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} + \frac{d}{dt}\vec{\nabla}\times\vec{B} = 0$$

Using eq. (4.8) and (4.11) in the above, we get,

$$\nabla^2\vec{E} - \mu\varepsilon\frac{d^2\vec{E}}{dt^2} = \mu\sigma\frac{d\vec{E}}{dt} \tag{4.12}$$

Similarly taking curl of both sides of eq. (4.10), we get

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} - \varepsilon\mu\left(\vec{\nabla}\times\frac{d\vec{E}}{dt}\right) = \sigma\mu\left(\vec{\nabla}\times\vec{E}\right)$$

Or, $$\vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B} - \varepsilon\mu\frac{d}{dt}\left(\frac{-d\vec{B}}{dt}\right) = -\sigma\mu\frac{d\vec{B}}{dt}$$

Further, $\vec{\nabla}\cdot\vec{B} = 0$ from eq. (4.9). So, we get,

$$\nabla^2\vec{B} - \varepsilon\mu \frac{d^2\vec{B}}{dt^2} = -\mu\sigma\frac{d\vec{B}}{dt} \tag{4.13}$$

Eq. (4.12) and (4.13) are the wave equation for $\vec{E}$ and $\vec{B}$ in a charge free but conducting medium. In the conducting medium with conductivity σ the term $\sigma\mu\frac{d\vec{E}}{dt}$ and $\sigma\mu\frac{d\vec{B}}{dt}$ are dissipative terms. In a non conducting medium $\sigma = 0$. So that (4.12) and (4.13) reduce to equation (4.5) and (4.6) respectively.

Relative magnitude of electric and magnetic field

The solution of eq. (4.5) and (4.6) can be written in the form $E_x = E_0 \exp(kx - \omega t)$, and $B_x = B_0 \exp(kx - \omega t)$. Where, E_0 is amplitude of electric field and k is wave vector of the electromagnetic wave. ω is angular frequency of the electromagnetic wave. When these solutions are used in eq. (4.5) and (4.6), we can get

$$c = \frac{E_0}{B_0} \tag{4.14}$$

Where, c is speed of electromagnetic wave in vacuum = $\frac{1}{\sqrt{\varepsilon_0\mu_0}}$. Since $B_0 = \mu_0 H_0$, we have

$$\frac{E_0}{H_0} = \mu_0 c$$

$$= \frac{\mu_0}{\sqrt{\varepsilon_0\mu_0}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = Z_0 \tag{4.15}$$

The quantity Z_0 has the dimension of electric resistance or impedance. It is called impedance of vacuum.

Phase relation between electric & magnetic field

The electric field and magnetic field, in electromagnetic wave, are in phase. When the electric field attains the maximum value, the magnetic field also attains the maximum value. Similarity, they become zero at the same time. Since the magnitude and direction of the electric and magnetic fields in electromagnetic wave are related eq. (4.14) and eq. (4.15), only one of them can be used to describe the electromagnetic wave. Usually, the electric field is chosen for this purpose most of the instruments used to detect electromagnetic wave deal with the electric rather than the magnetic component of the wave. For example, the plane of vibration of the electromagnetic wave (such as light) is taken as the plane as containing the electric field.

5. Electromagnetic energy and Poynting theorem

The Electromagnetic waves carry energy and momentum when they propagate. The conservation of energy in electromagnetic wave phenomena is described by Poynting theorem.

Electromagnetic energy density

The electric energy per unit volume is

$$u_E = \frac{1}{2}\vec{E}\,\vec{D} = \frac{1}{2}\,\varepsilon D^2 \tag{5.1}$$

In vacuum ε is replaced by ε_0. The magnetic energy per unit volume is

$$u_B = \frac{1}{2}\vec{B}\,\vec{H} = \frac{1}{2}\,\mu H^2 \tag{5.2}$$

In vacuum,μ is replaced by μ_0. Electromagnetic energy density is given by

$$u_{EM} = \frac{1}{2}\left(\vec{E}.\vec{B} + \vec{B}\vec{H}\right) = \frac{1}{2}\left(\varepsilon E^2 + \mu H^2\right) \tag{5.3}$$

The total electromagnetic energy in region is obtained by taking the volume integral of the electromagnetic energy density over the volume under consideration.

Poynting vector

The rate of energy transport per unit area in electromagnetic wave is described by a vector called Poynting vector. The Poynting vector $\vec{S}$ defined in terms of electric and magnetic field by

$$\vec{S} = \vec{E} \times \vec{H}$$

$$= \frac{\vec{E} \times \vec{B}}{\mu} \tag{5.4}$$

The Poynting vector measures the flow of energy of electromagnetic wave per sec per square meter normal to direction of wave propagation. Its S.I. unit is watt per square meter. The direction of pointing vector is perpendicular to both electric and magnetic field. It is directed along the direction of propagation of electromagnetic wave.

Poynting theorem

The pointing vector is $\vec{S} = \vec{E} \times \vec{H}$

Taking divergence on both sides, $\vec{\nabla} \cdot \vec{S} = \vec{\nabla} \cdot \overrightarrow{(E \times H)}$

$$= -\vec{H} \cdot (\vec{\nabla} \times \overrightarrow{E}) - E.(\vec{\nabla} \times \vec{H})$$

$$= -\vec{H}\cdot\frac{dB}{dt} - \vec{E}\cdot(j + \frac{dD}{dt})$$

$$= -\left(\vec{E}\cdot\vec{j} + \vec{E}\cdot\frac{dD}{dt} + \vec{H}\cdot\frac{dB}{dt}\right)$$

$$= -(E\cdot j + \frac{1}{2}\ \varepsilon_0 2E\frac{dE}{dt} + \frac{1}{2}\ \mu_0 2H\frac{dH}{dt})$$

$$=-\ (E\cdot j + \frac{1}{2}\ \varepsilon_0\frac{d(E^2)}{dt} + \frac{1}{2}\ \mu_0\frac{d(H^2)}{dt})$$

$$= -\ (\ E\cdot j + \frac{d}{dt}\left(\frac{1}{2}\ \varepsilon_0 E^2 + \frac{1}{2}\ \mu_0 H^2\right)\ )$$

Using the definition of electromagnetic energy density and pointing vector, we can write this equation as

$$\vec{\nabla}\cdot\vec{S} = -\frac{du_{em}}{dt} - \vec{E}\cdot\vec{j} \qquad (5.5)$$

Taking volume integral on both the sides over a given volume,

$$\iiint \vec{\nabla}\cdot\vec{S}\,dv\ =\ -\ \iiint\frac{du_{em}}{dt}dv - \iiint \vec{E}\cdot\vec{j}\,dv$$

Using Gauss divergence theorem, the LHS can be converted to a surface integral over a closed surface S enclosing the volume V. So,

$$\oint_S\ \vec{\nabla}\cdot\vec{S}\,dv\ =\ -\iiint\frac{du_{em}}{dt}dv - \iiint \vec{E}\cdot\vec{j}\,dv \qquad (5.6)$$

This is the **Poynting theorem**.

Interpretation

The LHS is the rate of flow of total e. m. energy through the closed area enclosing the given volume. The first term on the RHS is the rate of change of e.m. energy density. The second term on the RHS is the work done by the e. m. field on the source of current, lost due to heating effect. Thus, poynting theorem is the statement of conservation of energy in electromagnetic field. The pointing vector plays role of flux of e. m. field. In the absence of any source of current $j = 0$. So eq. (5.5) becomes

$$\vec{\nabla}\cdot\vec{S} + \frac{du_{em}}{dt} = 0 \qquad (5.7)$$

This is called the equation of continuity of electromagnetic wave.

Pointing vector and intensity of electromagnetic wave wave

In electromagnetic wave the electric field $\vec{E}$ and magnetic field $\vec{B}$ are time varying fields. So, the Poynting vector is also a time varying quantity. Since the electric and magnetic field are mutually perpendicular ($\theta = 90, \sin 90 = 1$), the magnitude of Poynting vector, from eq. (5.4) is

$$S = E\,H = \frac{EB}{\mu} \qquad (5.8)$$

The eq. (5.8) is in term of instantaneous value of field component and the Pointing vector. Since the electric and magnetic vector is in phase, the ratio of their maximum value is also equal to ratio of their instantaneous value from eq. (4.14). Thus, poyting vector is

$$S = \frac{E^2}{\mu\, c} \qquad (5.9)$$

If $E = E_0 \sin \omega t$, the time average value of Pointing vector is

$$\langle S \rangle = \frac{\langle E_0^2\, sin^2\, \omega\, t \rangle}{\mu\, c} = \frac{E_0^2}{2\, \mu\, c} = \frac{c\, \varepsilon\, E_0^2}{2} = c\, \varepsilon\, E_{rms}^2 \qquad (5.10)$$

Since $\langle sin^2\, \omega\, t \rangle = \frac{1}{2}$ and $E_{rms} = \frac{E_0}{\sqrt{2}}$.The average value of Poynting vector is the intensity of the electromagnetic wave.

$$I = \langle S \rangle = c\, \varepsilon\, E_{rms}^2 \qquad (5.11)$$

It may be note from eq. (5.10) and (5.11) that the intensity of wave i.e., the average value of Poynting vector is the product of energy density and speed of wave.

Example 29 The maximum value of electric field in an e.m. wave, in vacuum, is 800 V/m. find The maximum value of magnetic intensity and the average value of Poynting vector.

Sol: $H_0 = \frac{E_0}{\mu\, c} = \frac{800\, V/m}{(4\, \pi \times 10^{-7}\, H/m)(\, 3 \times 10^8\, m/s)} = 2.12\, A/\text{m}$

$\langle S \rangle = \frac{E_0 H_0}{2} = \frac{(800\, V/m)(2.12\ \, A/m)}{2} = 848\ \text{W/m}^2$

Example 30 A point source emits light with power 250 watt. Find the average value of Poynting vector and the rms value of electric and magnetic fields at the distance of 2 m from the source.

Sol At distance r =2 m, the energy emitted by point source is distributed over the surface of sphere of radius *r* =2 m. the energy following through the unit area per unit time is the intensity and is also equal to the average value of poynting vector.

$$I = \langle S \rangle = \frac{P}{4\pi\, r^2} = \frac{250\, W}{4 \times 3.14 \times\, (2m)^2} = 4.98\ \text{W/m}^2$$

As, $\langle S \rangle = \frac{E_{rms}^2}{\mu\, c}$

This gives $E_{rms} = \sqrt{\mu\, c\, \langle S \rangle}. = \sqrt{4\, \pi\ \times 10^{-7}\ \times 3 \times 10^8 \times 4.98\, W/m^2} = 43.32$ V/m

And, $B_{rms} = \frac{E_{rms}}{c} = \frac{43.32}{3\, \times\, 10^8} = 1.44 \times 10^{-7}\text{T}$

Example 31 Calculate the value of pointing vector on the surface of sun if the power radiated by sun is 4×10^{26} watt while its radius is 7×10^{8} meter.

Sol: Poynting's vector is the power passing through unit surface area. The surface area of the sun is $4\pi r^2 = 4 \times 3.14 \times 49 \times 10^{16} m^2 = 616 \times 10^{16} m^2$

So, the Poynting's vector is = $\frac{4\times10^{26}\, watt}{616\times10^{16} m^2} = 6.5 \times 10^{7}\, \frac{w}{m^2}$

Example 32 The em wave is propagating in free space with electric vector $E(z,t) = 50\cos(\omega t - k\,z)\hat{\imath}$. How much average energy is crossing a circular area of radius 2 meter on XY plane in unit time.

Sol: The above given wave is propagating along +Z direction. The average energy passing perpendicularly through unit area in unit time(i.e. average value of poynting's vector) is given by $\langle S \rangle = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}} \times E_0^2 = \frac{1}{2} \times 2.654 \times 10^{-3} \times 2500 \frac{watt}{m^2} = 3.318 \frac{watt}{m^2}$

Area $\pi r^2 = 3.14 \times 4 = 12.56\, m^2$

Therefore, the amount of energy passing perpendicularly through $12.56\, m^2$ will be $3.318 \frac{watt}{m^2} \times 12.56\, m^2 = 41.67\, watt$

Example 33 The em wave is propagating in free space with electric vector $E(z,t) = 150\cos(\omega t - k\,z)\hat{\imath}$. How much average energy is passing through a rectangular hole of length 3 cm and width 1.5 cm on XY plane in one minute time.

Sol: The above given wave is propagating along +Z direction. The average energy passing perpendicularly through unit area in unit time (i.e. average value of Pynting 's vector) is given by $\langle S \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} \times E_0^2 = \frac{1}{240\times3.14} \times 150^2\, watt/m^2 = 29.84\, watt/m^2$

Area $r^2 = 3 \times 10^{-2} \times 1.5 \times 10^{-2} = 4.5 \times 10^{-4}\, m^2$

Therefore, the amount of energy passing perpendicularly through this area in 60 seconds time will be $1.34 \times 10^{-2} \times 60 = 0.81$ joules.