UNIT III

Integral Calculus

3.1 INTRODUCTION

In this section we study the double and triple integrals (multiple integrals) are defined in the same way as the definite integral of a function of a single variable along with the applications. We also discuss two special functions, 'Beta function' and 'Gamma function' defined in the form of definite integrals.

3.2 MULTIPLE INTEGRALS

In this topic we discuss a repeated process of integration of a function of two and three variables referred to as

double integrals: $\iint f(x, y)\, dx\, dy$ and

triple integrals: $\iiint f(x, y, z)\, dx\, dy\, dz.$

3.3 DOUBLE INTEGRALS

The double integral of a function $f(x, y)$ over a region D in R^2 is denoted by $\iint_D f(x, y)\, dxdy$

Let $f(x, y)$ be a continuous function in R^2 defined on a closed rectangle

$$R = \{(x, y)/a \leq x \leq b \text{ and } c \leq y \leq d\}$$

For any fixed $x \in [a, b]$ consider the integral $\int_c^d f(x, y)\, dy$.

The value of this integral depends on x and we get a new function of x. This can be integrated depends on x and, we get $\int_a^b \left[\int_c^d f(x, y)\, dy\right] dx$. This is called an **"iterated integral".**

Similarly, we can define another

$$\int_c^d \left[\int_a^b f(x, y)\, dx\right] dy$$

For continuous function $f(x, y)$, we have

$$\iint_R f(x, y)\, dx\, dy = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy$$

If $f(x, y)$ is continuous on a bounded region S and S is given by

$S = \{(x, y)/a \le x \le b$ and $\phi_1(x) \le y \le \phi_2(x)\}$, where ϕ_1 and ϕ_2 are two continuous functions on $[a, b]$ then

$$\iint_S f(x, y)\, dx\, dy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(y)} f(x, y)\, dy\right] dx$$

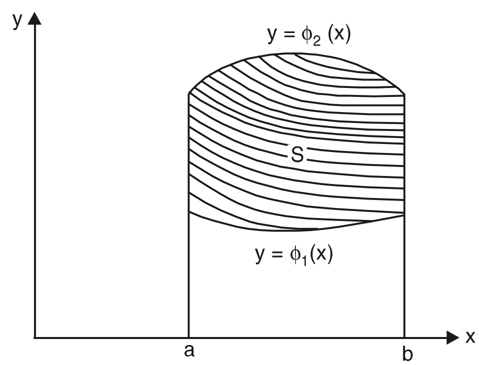

Fig. 3.1

The iterated integral in the R.H.S. is also written in the form

$$\int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy$$

Similarly, if $S = \{(x, y)/c \le y \le d$
and $\phi_1(y) \le x \le \phi_2(y)\}$

then
$$\iint_S f(x, y)\, dx\, dy = \int_c^d \left[\int_{\phi_1(y)}^{\phi_2(y)} f(x, y)\, dx\right] dy$$

If S cannot be written in neither of the above two forms we divide S into finite number of sub-regions such that each of the subregions can be represented in one of the above forms and we get the double integral over S by adding the integrals over these subregions.

WORKED OUT EXAMPLES

1. *Evaluate:* $I = \int_0^1 \int_0^2 xy^2\, dy\, dx.$

Solution
$$I = \int_0^1 \left[\int_0^2 xy^2\, dy\right] dx$$

$$= \int_0^1 \left[\frac{xy^3}{3}\right]_0^2 dx \qquad \text{(Integrating w.r.t. } y \text{ keeping } x \text{ constant)}$$

$$= \frac{1}{3}\int_0^1 8x\, dx$$

$$= \frac{1}{3}\left[\frac{8x^2}{2}\right]_0^1 = \frac{4}{3}.$$

2. *Evaluate:* $\int_0^1 \int_1^2 xy\, dy\, dx.$

Solution. Let I be the given integral

Then, $$I = \int_0^1 x \left\{ \int_1^2 y\, dy \right\} dx$$

$$= \int_0^1 x \cdot \left[\frac{y^2}{2} \right]_1^2 dx. = \frac{3}{2} \int_0^1 x\, dx = \frac{3}{4}.$$

3. *Evaluate the following:*

(*i*) $\int_1^2 \int_2^3 e^{x+y}\, dy\, dx$ (*ii*) $\int_0^1 \int_{x^2}^{x} dy\, dx$

(*iii*) $\int_1^2 \int_y^{3y} (x+y)\, dx\, dy$ (*iv*) $\int_0^{\pi} \int_0^{\cos\theta} r \sin\theta\, dr\, d\theta$

Solution. Let I be the given integral. Then

(*i*) $$I = \int_1^2 e^x \left(\int_2^3 e^y\, dy \right) dx$$

$$= \int_1^2 e^x \left[e^y \right]_2^3 dx$$

$$= \int_1^2 e^x \left(e^3 - e^2 \right) dx$$

$$= \left(e^3 - e^2 \right) \int_1^2 e^x\, dx$$

$$= \left(e^3 - e^2 \right) \left[e^x \right]_1^2$$

$$I = \left(e^3 - e^2 \right) \left(e^2 - e^1 \right).$$

(*ii*) $$I = \int_0^1 \left\{ \int_{x^2}^{x} dy \right\} dx$$

$$= \int_0^1 [y]_{x^2}^{x}\, dx = \int_0^1 \left(x - x^2 \right) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

(*iii*) $$I = \int_1^2 \left\{ \int_y^{3y} \left(x + y \right) dx \right\} dy$$

$$= \int_1^2 \left[\frac{x^2}{2} + xy \right]_{x=y}^{x=3y} dy$$

$$= \int_1^2 6y^2\, dy = \left[2y^3 \right]_1^2 = 14$$

(*iv*) $$I = \int_0^\pi \sin\theta \left\{ \int_0^{\cos\theta} r\,dr \right\} d\theta$$

$$= \int_0^\pi \sin\theta \left[\frac{r^2}{2} \right]_0^{\cos\theta} d\theta$$

$$= \frac{1}{2} \int_0^\pi \sin\theta \cdot \cos^2\theta \, d\theta$$

where $\cos\theta = t$

$-\sin\theta \, d\theta = dt$

$\therefore \quad \sin\theta \, d\theta = -\, dt$

$$= \frac{1}{2} \int_0^\pi t^2 \cdot (-\, dt)$$

$$= \frac{-1}{2} \left[\frac{t^3}{3} \right]_0^\pi$$

$$= \frac{-1}{6} [\cos\theta]_0^\pi$$

$$= \frac{-1}{6} (-1 - 1) = \frac{1}{3} \cdot$$

4. *Evaluate:* $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1 + x^2 + y^2} \cdot$

Solution $$I = \int_0^1 \left\{ \int_0^{\sqrt{1+x^2}} \frac{dy}{1 + x^2 + y^2} \right\} dx$$

where $a^2 = 1 + x^2$ or $a = \sqrt{1 + x^2}$

$$= \int_0^1 \left\{ \int_0^a \frac{dy}{a^2 + y^2} \right\} dx$$

$$= \int_0^1 \left[\frac{1}{a} \tan^{-1} \frac{y}{a} \right]_0^a dx$$

$$= \int_0^1 \frac{1}{a} \cdot \frac{\pi}{4} dx = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{x^2 + 1}}$$

$$= \frac{\pi}{4} \left[\log \left\{ x + \sqrt{x^2 + 1} \right\} \right]_0^1$$

$$= \frac{\pi}{4} \log \left(\sqrt{2} + 1 \right)$$

Note : $\displaystyle\frac{\pi}{4} \left[\log \left\{ x + \sqrt{x^2 + 1} \right\} \right]_0^1 = \frac{\pi}{4} \left[\sin h^{-1} x \right]_0^1 = \frac{\pi}{4} \sin h^{-1}(1)$

5. *Evaluate:* $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\left(x^2+y^2+z^2\right)dz\,dy\,dx.$

Solution $$I = \int_{x=-c}^{c}\int_{y=-b}^{b}\int_{z=-a}^{a}\left(x^2+y^2+z^2\right)dz\,dy\,dx$$

Integrating w.r.t. z, x and y – constant.

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2z+y^2z+\frac{z^3}{3}\right]_{z=-a}^{a}dy\,dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2(a+a)+y^2(a+a)+\left(\frac{a^3}{3}+\frac{a^3}{3}\right)\right]dy\,dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left(2ax^2+2ay^2+\frac{2a^3}{3}\right)dy\,dx$$

Integrating w.r.t. y, x – constant.

$$= \int_{x=-c}^{c}\left[2ax^2y+\frac{2ay^3}{3}+\frac{2a^3}{3}\,y\right]_{y=-b}^{b}dx$$

$$= \int_{x=-c}^{c}\left[2ax^2(b+b)+\frac{2a}{3}\left(b^3+b^3\right)+\frac{2a^3}{3}(b+b)\right]dx$$

$$= \int_{x=-c}^{c}\left[4ax^2b+\frac{4ab^3}{3}+\frac{4a^3b}{3}\right]dx$$

$$= \left[4ab\left(\frac{x^3}{3}\right)+\frac{4ab^3}{3}(x)+\frac{4a^3b}{3}(x)\right]_{-c}^{c}$$

$$= 4ab\left(\frac{2c^3}{3}\right)+\frac{4ab^3}{3}\cdot(2c)+\frac{4a^3b}{3}(2c)$$

$$= \frac{8abc^3}{3}+\frac{8ab^3c}{3}+\frac{8a^3bc}{3}$$

$$I = \frac{8abc}{3}\left(a^2+b^2+c^2\right)\cdot$$

6. *Evaluate:* $\int_{0}^{a}\int_{0}^{x}\int_{0}^{x+y} e^{x+y+z}\,dz\,dy\,dx.$

Solution $$I = \int_{x=0}^{a}\int_{y=0}^{x}\int_{z=0}^{x+y} e^{x+y}\cdot e^{z}\,dz\,dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \cdot \left[e^{z}\right]_{0}^{x+y} dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left(e^{x+y} - 1\right) dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} \left(e^{2x} \cdot e^{2y} - e^{x} \cdot e^{y}\right) dy\,dx$$

$$= \int_{x=0}^{a} \left\{ e^{2x} \left[\frac{e^{2y}}{2}\right]_{0}^{x} - e^{x} \left[e^{y}\right]_{0}^{x} \right\} dx$$

$$= \int_{x=0}^{a} \left\{ \frac{e^{2x}}{2}\left(e^{2x} - 1\right) - e^{x}\left(e^{x} - 1\right) \right\} dx$$

$$= \int_{x=0}^{a} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x} \right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x} \right]_{0}^{a}$$

$$= \left(\frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} \right) - \left(\frac{1}{8} - \frac{3}{4} + 1 \right)$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{3}{8}$$

$$I = \frac{1}{8}\left(e^{4a} - 6e^{2a} + 8e^{a} - 3\right)\cdot$$

7. *Evaluate:* $\int_{0}^{\log 2} \int_{0}^{x} \int_{0}^{x + \log y} e^{x+y+z}\, dz\, dy\, dx.$

Solution

$$I = \int_{x=0}^{\log 2} \int_{y=0}^{x} \int_{z=0}^{x + \log y} e^{x+y}.\, e^{z}\, dz\, dy\, dx$$

$$= \int_{x=0}^{\log 2} \int_{y=0}^{x} e^{x+y} \left[e^{z}\right]_{0}^{x+\log y} dy\,dx$$

$$= \int_{x=0}^{\log 2} \int_{y=0}^{x} e^{x+y} \left[e^{x+\log y} - 1\right] dy\,dx$$

$$= \int_{x=0}^{\log 2} \int_{y=0}^{x} e^{x+y} \left[e^{x} \cdot e^{\log y} - 1\right] dy\,dx$$

But $e^{\log y} = y$

$\therefore$

$$I = \int_{x=0}^{\log 2} \int_{y=0}^{x} \left(e^{2x} \cdot y \cdot e^{y} - e^{x}\, e^{y}\right) dy\,dx$$

$$= \int_{x=0}^{\log 2} \left[e^{2x} \left(y\, e^{y} - e^{y} \right) - e^{x}\, e^{y} \right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{\log 2} \left[e^{2x} \left\{ \left(x\, e^{x} - e^{x} \right) - (0-1) \right\} - e^{x} \left(e^{x} - 1 \right) \right] dx$$

$$= \int_{x=0}^{\log 2} \left(x\, e^{3x} - e^{3x} + e^{2x} - e^{2x} + e^{x} \right) dx$$

$$= \int_{x=0}^{\log 2} \left(x\, e^{3x} - e^{3x} + e^{x} \right) dx$$

$$= \left[x \,.\, \frac{e^{3x}}{3} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^{x} \right]_{0}^{\log 2}$$

$$= \left[\frac{x\, e^{3x}}{3} - \frac{4\, e^{3x}}{9} + e^{x} \right]_{0}^{\log 2}$$

$$= \left[\frac{\log 2 \,.\, e^{3\log 2}}{3} - 0 \right] - \frac{4}{9} \left(e^{3\log 2} - 1 \right) + \left(e^{\log 2} - 1 \right)$$

$$= \frac{8\log 2}{3} - \frac{4}{9}(8-1) + (2-1)$$

$$= \frac{8\log 2}{3} - \frac{28}{9} + 1$$

Thus, $$I = \frac{8\log 2}{3} - \frac{19}{9}.$$

8. *Evaluate:* $\int_{0}^{a} \int_{0}^{\sqrt{a^2 - y^2}} \sqrt{a^2 - x^2 - y^2}\; dx\, dy.$

Solution $$I = \int_{0}^{a} \left\{ \int_{0}^{\sqrt{a^2 - y^2}} \sqrt{\left(a^2 - y^2 \right) - x^2}\; dx \right\} dy$$

$$= \int_{0}^{a} \left\{ \int_{0}^{b} \sqrt{b^2 - x^2} \right\} dy$$

where $$b^2 = a^2 - y^2$$

$$= \int_{0}^{a} \left[\frac{x}{2} \sqrt{b^2 - x^2} + \frac{b^2}{2} \sin^{-1} \frac{x}{b} \right]_{0}^{b} dy$$

$$= \int_{0}^{a} \frac{b^2}{2} \,.\, \frac{\pi}{2}\, dy = \frac{\pi}{4} \int_{0}^{a} \left(a^2 - y^2 \right) dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_{0}^{a} = \frac{\pi a^3}{6}$$

EXERCISE 3.1

I. Evaluate the following double integrals:

1. $\int_0^3 \int_1^2 xy\,(x+y)\,dy\,dx$ **[Ans.** 24**]**

2. $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y)\,dy\,dx$ **[Ans.** 2**]**

3. $\int_0^1 \int_0^{x^2} ce^y\,dy\,dx$ $\left[\textbf{Ans. } \frac{1}{2}e-1\right]$

4. $\int_0^1 \int_0^{1-x} xy\,dy\,dx$ $\left[\textbf{Ans. } \frac{1}{24}\right]$

5. $\int_0^1 \int_0^{1-x} (x+y)^2\,dy\,dx$ $\left[\textbf{Ans. } \frac{1}{4}\right]$

6. $\int_0^1 \int_0^{y^2} e^{x/y}\,dx\,dy$ $\left[\textbf{Ans. } \frac{1}{2}\right]$

7. $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2y\,dy\,dx$ $\left[\textbf{Ans. } \frac{a^5}{15}\right]$

8. $\int_0^\infty \int_y^\infty x\,e^{-x^2/y}\,dx\,dy$ $\left[\textbf{Ans. } \frac{1}{2}\right]$

9. $\int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dx\,dy$ $\left[\textbf{Ans. } \frac{16}{3}a^2\right]$

10. $\int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} \frac{a\,r}{\sqrt{a^2-r^2}}\,dr\,d\theta$ $\left[\textbf{Ans. } a^2\left(\frac{\pi}{2}-1\right)\right]$

II. Evaluate the triple integrals:

1. $\int_0^2 \int_1^3 \int_1^2 xy^2z\,dz\,dy\,dx$ **[Ans.** 26**]**

2. $\int_{-3}^3 \int_0^1 \int_1^2 (x+y+z)\,dx\,dy\,dz$ **[Ans.** 12**]**

3. $\int_0^1 \int_0^1 \int_0^y xyz\,dx\,dy\,dz$ $\left[\textbf{Ans. } \frac{1}{16}\right]$

4. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\,dz\,dx\,dy$ $\left[\textbf{Ans. } \frac{4}{35}\right]$

5. $\int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^2 xyz\,dz\,dy\,dx$ $\left[\textbf{Ans. } \frac{3}{8}\right]$

6. $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\,dx\,dz$ **[Ans.** 8π**]**

7. $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)\,dy\,dx\,dz$ **[Ans.** 0**]**

8. $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{(a^2-r^2)/a} r\,dr\,d\theta$ $\left[\textbf{Ans. } \frac{5\pi a^3}{64}\right]$

9. $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz\,dy\,dx}{(1+x+y+z)^3}$ $\left[\textbf{Ans. } \frac{1}{2}\left(\log 2-\frac{5}{8}\right)\right]$

10. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx$ $\left[\textbf{Ans. } \frac{a^6}{48}\right]$

3.3.1 Evaluation of a Double Integral by Changing the Order of Integration

In the evaluation of the double integrals sometimes we may have to change the order of integration so that evaluation is more convenient. If the limits of integration are variables then change in the order of integration changes the limits of integration. In such cases a rough idea of the region of integration is necessary.

3.3.2 Evaluation of a Double Integral by Change of Variables

Sometimes the double integral can be evaluated easily by changing the variables.

Suppose x and y are functions of two variables u and v.

i.e., $x = x\,(u, v)$ and $y = y\,(u, v)$ and the Jacobian

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

Then the region A changes into the region R under the transformations

$$x = x\,(u, v) \text{ and } y = y\,(u, v)$$

Then $$\iint_A f(x, y)\,dx\,dy = \iint_R f(u, v)\,J\,du\,dv$$

If $$x = r\cos\theta,\ y = r\sin\theta$$

$$J = \frac{\partial(x, u)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\therefore \quad \iint_A f(x, y)\,dx\,dy = \iint_R F(r, \theta)\,r\,dr\,d\theta. \quad \text{...(1)}$$

3.3.3 Applications to Area and Volume

1. $\iint_R dx\,dy$ = Area of the region R in the Cartesian form.

2. $\iint_R r \cdot dr\,d\theta$ = Area of the region R in the polar form.

3. $\iiint_V dx\,dy\,dz$ = Volume of a solid.

4. Volume of a solid (in polars) obtained by the revolution of a curve enclosing an area A about the initial line is given by

$$V = \iint_A 2\pi r^2 \sin\theta \cdot dr\,d\theta.$$

5. If $z = f(x, y)$ be the equation of a surface S then the surface area is given by

$$\iint_A \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\,dy$$

Where A is the region representing the projection of S on the xy-plane.

WORKED OUT EXAMPLES

Type 1. Evaluation over a given region

1. *Evaluate* $\iint_R xy\,dx\,dy$ *where R is the triangular region bounded by the axes of coordinates and the line* $\frac{x}{a} + \frac{y}{b} = 1.$

Solution. R is the region bounded by $x = 0$, $y = 0$ being the coordinates axes and $\frac{x}{a} + \frac{y}{b} = 1$ being the straight line through $(0, a)$ and $\left(0, b\left(1 - \frac{x}{a}\right)\right)$

when x is held fixed and y varies from 0 to $b\left(1 - \frac{x}{a}\right)$

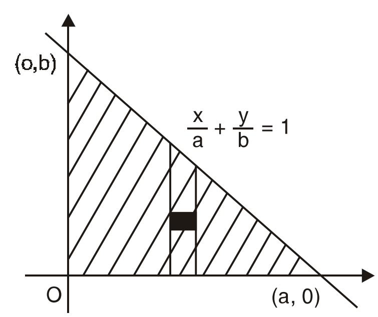

Fig. 3.2

$$\therefore \quad \frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow \quad \frac{y}{b} = 1 - \frac{x}{a}$$

$$\Rightarrow \quad y = b\left(1 - \frac{x}{a}\right)$$

$$\therefore \quad \iint_R xy\,dx\,dy = \int_{x=0}^{a} \left\{ \int_{y=0}^{b\left(1-\frac{x}{a}\right)} xy\,dy \right\} dx$$

$$= \int_0^a x \cdot \left[\frac{y^2}{2}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= \int_0^a \left\{ x \cdot \frac{b^2}{2}\left(1 - \frac{x}{a}\right)^2 \right\} dx$$

$$= \frac{b^2}{2} \int_0^a \left(x - 2\frac{x^2}{a} + \frac{x^3}{a^2} \right) dx$$

$$= \frac{b^2}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3a} + \frac{x^4}{4a^2} \right]_0^a$$

$$= \frac{b^2}{2}\left[\frac{a^2}{2} - \frac{2}{3}a^2 + \frac{1}{4}a^2\right]$$

$$= \frac{a^2b^2}{24}$$

2. *Evaluate* $\iint xy\, dx\, dy$ *over the area in the first quadrant bounded by the circle* $x^2 + y^2 = a^2$.

Solution $\iint xy\, dx\, dy = \int_{x=0}^{a}\left[\int_{y=0}^{\sqrt{a^2-x^2}} xy\, dy\right]dx$ $\left\{\begin{aligned} \because x^2 + y^2 &= a^2 \\ \Rightarrow y^2 &= a^2 - x^2 \\ y &= \sqrt{a^2 - x^2}\end{aligned}\right.$

$$= \int_0^a x\cdot\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a x\left(\frac{a^2-x^2}{2}\right)dx$$

$$= \frac{1}{2}\int_0^a \left(a^2x - x^3\right)dx$$

$$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8}.$$

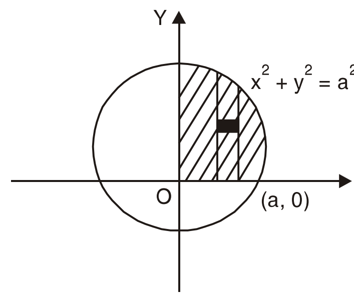

Fig. 3.3

3. *Evaluate* $\iint_R x\, dx\, dy$ *where R is the region bounded by* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and lying in the first quadrant.*

Solution. From the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

x changes from 0 to a and y changes from 0 to $\frac{b}{a}\sqrt{a^2 - x^2}$

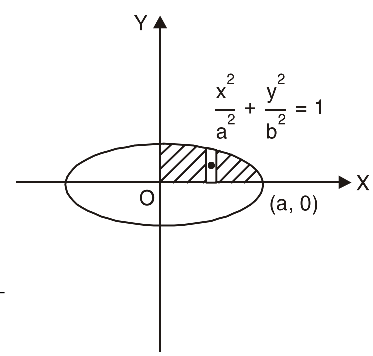

Fig. 3.4

$$\iint_R x\, dx\, dy = \int_{x=0}^{a}\left\{\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} x\, dy\right\}dx$$

$$= \int_0^a x\,[y]_0^{\frac{b}{a}\sqrt{a^2-x^2}}\, dx$$

$$= \int_0^a \left(x\,\frac{b}{a}\sqrt{a^2-x^2} \right) dx$$

Putting $x = a \sin \theta$, $dx = a \cos \theta\, d\theta$

$$\sqrt{a^2 - x^2} = a \cos \theta$$

$\therefore$ θ varies from 0 to $\pi/2$

$$= \frac{b}{a}\int_0^{\pi/2} a \sin\theta \cdot a \cos\theta \cdot a\cos\theta\, d\theta$$

$$= a^2 b \int_0^{\pi/2} \sin\theta \cos^2\theta\, d\theta$$

$$= a^2 b \times \frac{1}{3} = \frac{a^2 b}{3}\cdot$$

4. *Evaluate* $\iint_R xy\,dx\,dy$ *where R is the region in the first quadrant included between* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and* $\frac{x}{a} + \frac{y}{b} = 1$.

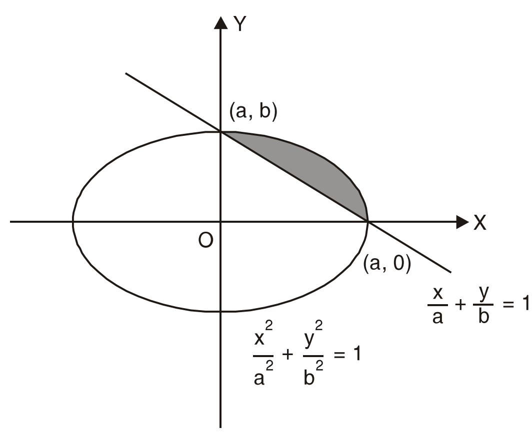

Fig. 3.5

Solution $$\frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow \quad y = b\left(1 - \frac{x}{a}\right)$$

$$= \frac{b}{a}(a - x)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad (\because\ y \geq 0)$$

$$\therefore \quad \iint_R xy\,dx\,dy = \int_0^a \left\{ \int_{y=\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dy \right\} dx$$

$$= \int_0^a x \left[\frac{y^2}{2} \right]_{\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_0^a x \left[\frac{b^2}{a^2}\left(a^2 - x^2\right) - \frac{b^2}{a^2}\left(a - x\right)^2 \right] dx$$

$$= \frac{b^2}{2a^2} \int_0^a \left(2ax^2 - 2x^3\right) dx$$

$$= \frac{b^2}{2a^2} \left[2a\,\frac{x^3}{3} - \frac{x^4}{2} \right]_0^a$$

$$= \frac{b^2}{2a^2} \left[\frac{2a^4}{3} - \frac{a^4}{2} \right] = \frac{a^2\,b^2}{12}.$$

5. *Evaluate* $\iint_R xy^2\,dx\,dy$ *where R is the Triangular region bounded by y = 0, x = y and x + y = 2.*

Solution. Given

$$y = 0,\ x = y,\ x + y = 2$$

where $y = 0,\ y + y = 2$

$\Rightarrow \quad 2y = 2$

$\Rightarrow \quad y = 1$

where $x = y,\ x = 2 - y$

$\therefore$ y varies from 0 to 1

x varies from y to $2 - y$

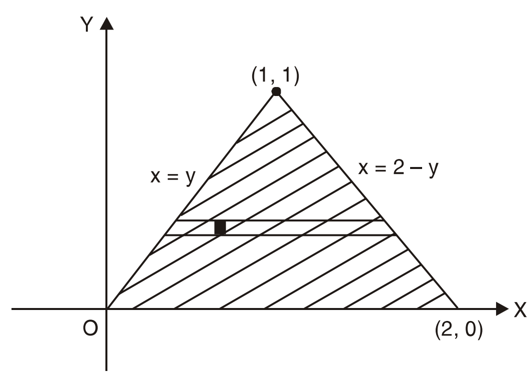

Fig. 3.6

$$\iint_R xy^2\,dx\,dy = \int_{y=0}^{1} \int_{x=y}^{2-y} xy^2\,dx\,dy$$

$$= \int_{y=0}^{1} y^2 \left[\frac{x^2}{2} \right]_{x=y}^{2-y} dy$$

$$= \frac{1}{2} \int_0^1 y^2 \left\{ (2-y)^2 - y^2 \right\} dy$$

$$= \frac{1}{2} \int_0^1 y^2 \left(4 - 4y\right) dy$$

$$= \frac{1}{2} \left[\frac{4}{3} y^3 - y^4 \right]_0^1 = \frac{1}{6}.$$

6. *Evaluate* $\iint_R xy(x+y)\,dx\,dy$ *over the region between* $y = x^2$ *and* $y = x$.

Solution. The bounded curves are $y = x^2$ and $y = x$. The common points are given by solving the two equations.

So, we have

$$x^2 = x = x(x-1) = 0$$

$$\Rightarrow \qquad x = 0 \text{ or } 1$$

when $x = 0$, we have $y = 0$ and

when $x = 1$, $y = 1$ (from $y = x$)

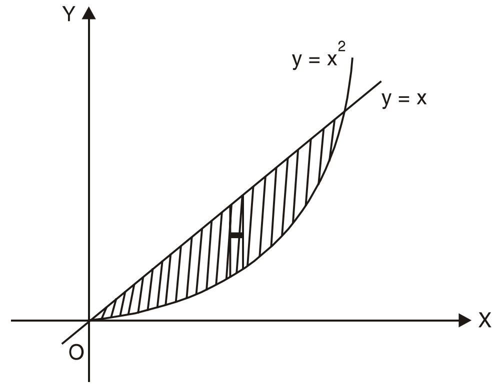

Fig. 3.7

$$\therefore \quad \iint_R xy(x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_0^1 x\left[\frac{xy^2}{2}+\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 x\left\{x\left(\frac{x^2}{2}-\frac{x^4}{2}\right)+\left(\frac{x^3}{3}-\frac{x^6}{3}\right)\right\}dx$$

$$= \int_0^1\left(\frac{5}{6}x^4-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$= \left[\frac{5}{6}\,\frac{x^5}{5}-\frac{x^7}{14}-\frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{6}-\frac{1}{14}-\frac{1}{24} = \frac{3}{56}.$$

7. *Evaluate* $\iint_R xy\,dx\,dy$ *where R is the region bounded by the x-axis, ordinate at* $x = 2a$ *and* $x^2 = 4ay$.

Solution

When $\qquad x = 2a$ and $x^2 = 4ay$

$\therefore \qquad 4a^2 = 4ay$

$\Rightarrow \qquad y = a$

$\therefore$ The point of intersection of

$$x = 2a \text{ and } x^2 = 4ay \text{ is } (2a,\ a)$$

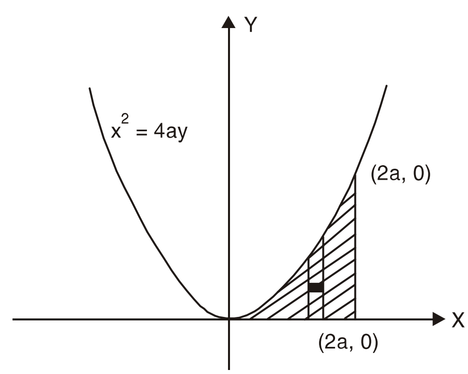

Fig. 3.8

Now $\qquad \iint_R xy\,dx\,dy = \int_{x=0}^{2a}\int_{y=0}^{\frac{x^2}{4a}} xy\,dy\,dx$

$$= \int_0^{2a} x \left[\frac{y^2}{2} \right]_0^{\frac{x^2}{4a}} dx$$

$$= \int_0^{2a} \frac{x^5}{32a^2} dx = \left[\frac{x^6}{32a^2 \times 6} \right]_0^{2a} = \frac{a^4}{3}$$

Type 2. Evaluation of a double integral by changing the order of integration

1. *Change the order of integration and hence evaluate* $\int_0^a \int_0^{2\sqrt{ax}} x^2 \, dy \, dx$.

Solution $\qquad y = 2\sqrt{ax}$

$\Rightarrow \qquad y^2 = 4ax$

when $x = a$ on $y^2 = 4ax$, $y^2 = 4a^2$

$\Rightarrow \qquad y = \pm 2a$

So, on $y = 2\sqrt{ax}$, $y = 2a$ when $x = a$

The integral is over the shaded region.

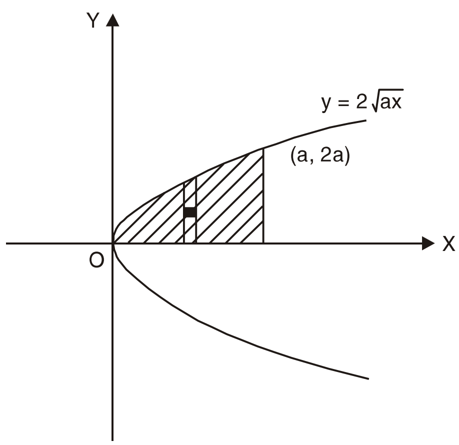

Fig. 3.9

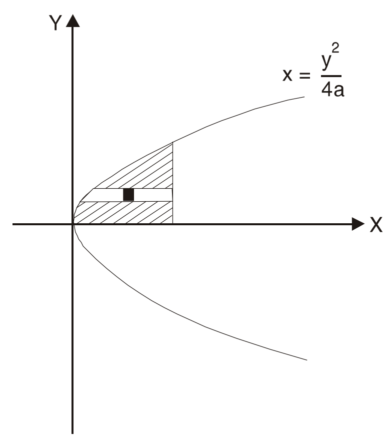

Fig. 3.10

$$\int_0^a \int_0^{2\sqrt{ax}} x^2 \, dy \, dx = \int_{y=0}^{2a} \int_{x=\frac{y^2}{4a}}^{a} x^2 \, dx \, dy \qquad \text{(By changing the order)}$$

$$= \int_0^{2a} \left[\frac{x^3}{3} \right]_{\frac{y^2}{4a}}^{a} dy$$

$$= \int_0^{2a} \left(\frac{a^3}{3} - \frac{y^6}{192\, a^3} \right) dy$$

$$= \left[\frac{a^3}{3}\, y - \frac{y^7}{192 a^3 \times 7} \right]_0^{2a}$$

$$= \frac{2a^4}{3} - \frac{2^7\, a^4}{192 \times 7}$$

$$= a^4 \left(\frac{2}{3} - \frac{2}{21} \right) = \frac{4}{7}\, a^4.$$

2. *Change the order of integration and hence evaluate* $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$.

Solution $\quad y = \sqrt{2 - x^2}$

$\Rightarrow \quad y^2 = 2 - x^2$

$\Rightarrow \quad x^2 + y^2 = 2$

This circle and $y = x$ meet if $x^2 + x^2 = 2$

$\therefore \quad 2x^2 = 2 \Rightarrow x = 1$

So, (1, 1) is the meeting point.

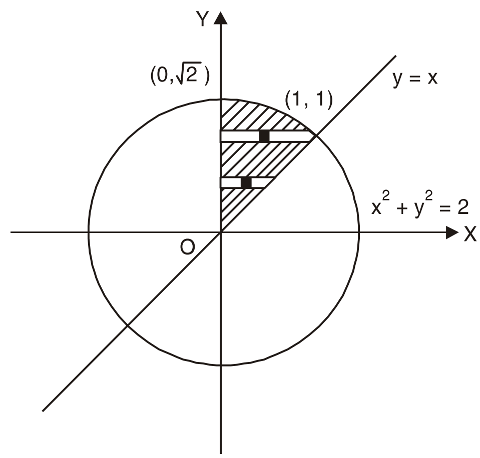

Fig. 3.11

Now $\quad I = \displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$

$$= \int_{y=0}^{\sqrt{2}} \int_{x=0}^{\phi(y)} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

where $\phi(y) = \begin{cases} y & \text{for } 0 \le y \le 1 \\ \sqrt{2-y^2} & \text{for } 1 \le y \le \sqrt{2} \end{cases}$

(Note that $x = \phi(y)$ is the R.H.S. boundary of the shaded region)

So, the required integral is

$$I = \int_{y=0}^{1} \int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \int_{y=1}^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

$$= \int_0^1 \left[x^2 + y^2 \right]_0^y dy + \int_1^{\sqrt{2}} \left[\sqrt{x^2+y^2} \right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 \left(\sqrt{2}\, y - y \right) dy + \int_1^{\sqrt{2}} \left(\sqrt{2} - y \right) dy$$

$$= \left[(\sqrt{2}-1)\frac{y^2}{2}\right]_0^1 + \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= \frac{\sqrt{2}-1}{2} + \sqrt{2}(\sqrt{2}-1) - \left(\frac{2}{2}-\frac{1}{2}\right)$$

$$= 1 - \frac{1}{\sqrt{2}}.$$

3. *Change the order of integration and hence evaluate* $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx.$

Solution. The region of integration is the portion of the first quadrant between $y = x$ and the y-axis. So, by changing the order of integration.

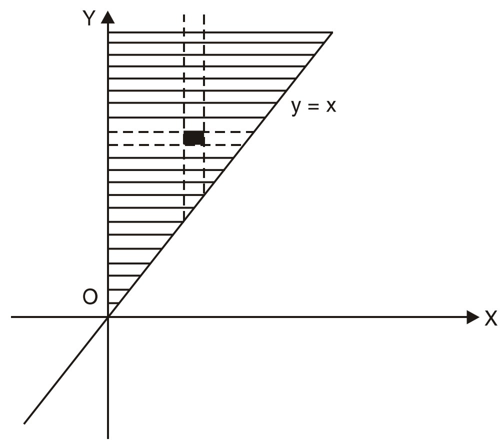

Fig. 3.12

$$\int_{x=0}^\infty \int_{y=x}^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_{y=0}^\infty \int_{x=0}^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}[x]_0^y\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left[-e^{-y}\right]_0^\infty = 1.$$

4. *Change the order of integration and hence evaluate* $\int_{y=0}^3 \int_{x=1}^{4-y} (x+y)\, dx\, dy.$

Solution $\quad x = 4 - y \quad \Rightarrow \quad x + y = 4$

Limits for x are from 1 to $4 - y$

when $\quad x = 1$ on $x + y = 4$

we have $\quad 1 + y = 4 \;\Rightarrow\; y = 3$

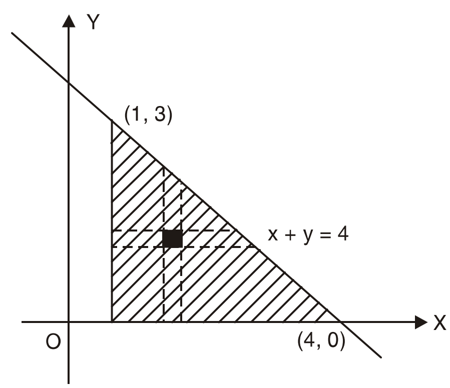

Fig. 3.13

So, $\displaystyle\int_0^3 \int_1^{4-y} (x+y)\, dx\, dy = \int_{x=1}^4 \int_0^{4-x} (x+y)\, dy\, dx$

by changing the order of integration.

$$= \int_1^4 \left[xy + \frac{y^2}{2}\right]_0^{4-x} dx$$

$$= \int_1^4 \left[x(4-x) + \frac{(4-x)^2}{2}\right] dx$$

$$= \int_1^4 \left(8 - \frac{1}{2}x^2\right) dx$$

$$= \left[8x - \frac{x^3}{6}\right]_1^4 = \frac{27}{2}.$$

5. *Change the order of integration and hence evaluate* $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$.

Solution $x = \sqrt{4-y}$

$\Rightarrow$ $x^2 = 4 - y$

$y = 4 - x^2$, a parabola.

Here, the limits 1 and $\sqrt{4-y}$ are for x, 0 and 3 are for y.

When $x = 1$, on $y = 4 - x^2$, $y = 3$

Now, $$\int_{y=0}^{3} \int_{x=1}^{\sqrt{4-y}} (x+y)\, dx\, dy = \int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\, dy\, dx$$

(By changing the order of integration)

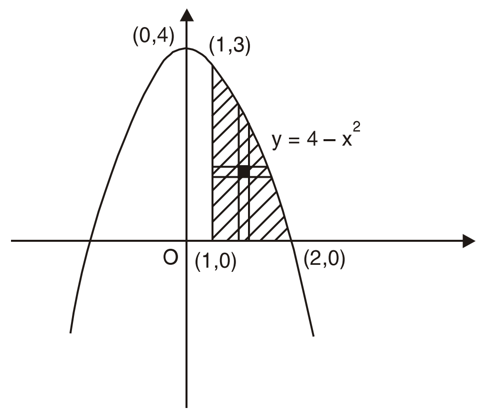

Fig. 3.14

$$= \int_1^2 \left[xy + y^2/2\right]_0^{4-x^2} dx$$

$$= \int_1^2 \left(4x - x^3 + 8 - 4x^2 + \frac{x^4}{2}\right) dx$$

$$= \left[2x^2 - \frac{x^4}{4} + 8x - \frac{4}{3}x^3 + \frac{x^5}{10}\right]_1^2$$

$$= 6 - \frac{15}{4} + 8 - \frac{28}{3} + \frac{31}{10} = \frac{241}{60}.$$

Type 3. Evaluation by changing into polars

1. *Evaluate* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$ *by changing to polar coordinates.*

Solution. In polars we have $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$

Since x, y varies from 0 to ∞

r also varies from 0 to ∞

In the first quadrant 'θ' varies from 0 to $\pi/2$

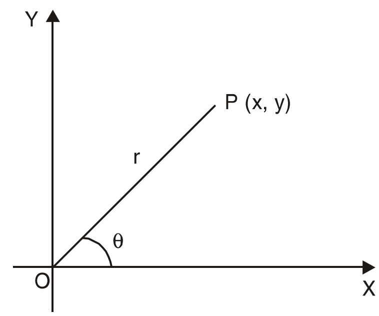

Fig. 3.15

Thus $$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\theta$$

Put $$r^2 = t \qquad \therefore\ r\, dr = \frac{dt}{2}$$

t also varies from 0 to ∞

$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} \frac{dt}{2}\, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \left[-e^{-t}\right]_0^{\infty} d\theta$$

$$= \frac{-1}{2} \int_0^{\pi/2} (0-1)\, d\theta$$

$$= +\frac{1}{2} \int_0^{\pi/2} 1 . d\theta$$

$$= \frac{+1}{2} [\theta]_0^{\pi/2} = \frac{+1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

2. *Evaluate* $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$ *by changing into polars.*

Solution $$I = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$$

$x = \sqrt{a^2-y^2}$ or $x^2 + y^2 = a^2$ is a circle with centre origin and radius a. Since, y varies from 0 to a the region of integration is the first quadrant of the circle.

In polars, we have $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \qquad x^2 + y^2 = r^2$

i.e., $\qquad r^2 = a^2$

$\Rightarrow \qquad r = a$

Also $x = 0$, $y = 0$ will give $r = 0$ and hence we can say that r varies from 0 to a. In the first quadrant θ varies from 0 to $\pi/2$, we know that $dx\, dy = r\, dr\, d\theta$

$$\therefore \qquad I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r \sin\theta \, r \, r \, dr \, d\theta$$

$$= \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^3 \sin\theta \, dr \, d\theta$$

$$= \int_{r=0}^{a} r^3 \left(-\cos\theta\right)_0^{\pi/2} dr$$

$$= \int_{0}^{a} -r^3 \left(0-1\right) dr = \left[\frac{r^4}{4}\right]_0^a = \frac{a^4}{4}$$

$$I = \frac{a^4}{4}.$$

Type 4. Applications of double and triple integrals

1. *Find the area of the circle $x^2 + y^2 = a^2$ by using double integral.*

Solution

Since, the circle is symmetric about the coordinates axes, area of the circle is 4 times the area *OAB* as shown in Figure.

For the region *OAB*, y varies from 0 to $\sqrt{a^2 - x^2}$ and x varies from 0 to a.

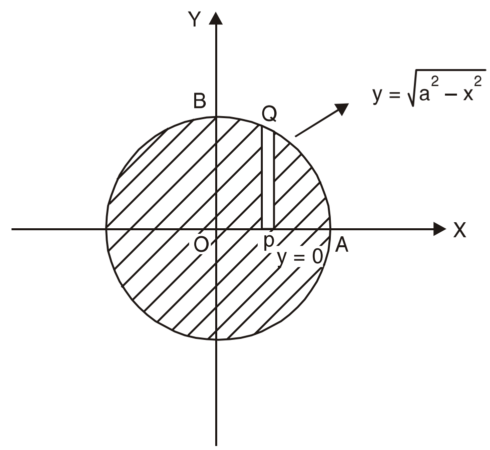

Fig. 3.16

$$\therefore \text{ Area of the circle } = 4 \int_{0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} dy \, dx$$

$$= 4 \int_{0}^{a} \left[y\right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= 4 \int_{0}^{a} \sqrt{a^2 - x^2} \, dx$$

$$= 4 \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = \pi a^2 \text{ sq. units}$$

2. *Find by double integration the area enclosed by the curve $r = a\,(1 + \cos\theta)$ between $\theta = 0$ and $\theta = \pi$.*

Solution $\qquad$ Area $= \displaystyle\iint r \, dr \, d\theta$

where r varies from 0 to $a\,(1 + \cos\theta)$ and θ varies from 0 to π

$$\int \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

i.e.,
$$A = \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$
$$= \int_{\theta=0}^{\pi} \left[\frac{r^2}{2}\right]_{r=0}^{a(1+\cos\theta)} d\theta$$
$$= \frac{1}{2}\int_{0}^{\pi} a^2 (1+\cos\theta)^2\, d\theta$$
$$= \frac{a^2}{2}\int_{0}^{\pi} \left\{2\cos^2\left(\frac{\theta}{2}\right)\right\}^2 d\theta$$
$$= 2a^2 \int_{0}^{\pi} \cos^4\left(\frac{\theta}{2}\right) d\theta$$

Put $\theta/2 = \phi,\ d\theta = 2d\phi$

and ϕ varies from 0 to $\pi/2$

$$\therefore \quad A = 2a^2 \int_{0}^{\pi/2} \cos^4\phi \cdot 2d\phi$$
$$= 4a^2 \int_{0}^{\pi/2} \cos^4\phi \cdot d\phi$$
$$= 4a^2 \cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad \text{(by the reduction formula)}$$

Area, $A = 3\pi a^2/4$ sq. units.

3. *Find the value of* $\iiint z\,dx\,dy\,dz$ *where V is the hemisphere* $x^2 + y^2 + z^2 = a^2,\ z \geq 0.$

Solution

Let
$$I = \iiint_V z\,dx\,dy\,dz$$
$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} z\,dz\,dy\,dx$$
$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$$= \frac{1}{2}\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(a^2-x^2-y^2\right)dy\,dx$$

$$= \frac{1}{2}\int_{x=-a}^{a}\left[\left(a^2-x^2\right)y-\frac{y^3}{3}\right]_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dx$$

$$= \frac{1}{2}\cdot\frac{4}{3}\int_{-a}^{a}\left(a^2-x^2\right)^{3/2}dx$$

$$= \frac{2}{3}\cdot 2\int_{0}^{a}\left(a^2-x^2\right)^{3/2}dx$$

Put $x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

θ varies from 0 to $\pi/2$

$$= \frac{4}{3}\int_{\theta=0}^{\pi/2}\left(a^2\cos^2\theta\right)^{3/2}a\cos\theta\,d\theta$$

$$= \frac{4a^4}{3}\int_{0}^{\pi/2}\cos^4\theta\,d\theta$$

$$= \frac{4a^4}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \qquad \text{(By applying reduction formula)}$$

$$= \frac{\pi a^4}{4}$$

Thus, $I = \dfrac{\pi a^4}{4}$.

4. *Using multiple integrals find the volume of the ellipsoid* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 1$.

Solution

The volume (V) is 8 times in the first octant (V_1)

i.e., $V = 8V_1 = 8\iiint dz\,dy\,dx$

z varies from 0 to $c\sqrt{1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}}$

y varies from 0 to $(b/a)\ \sqrt{a^2 - x^2}$

x varies from 0 to a

$$V = 8V_1 = 8 \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{(b/a)\sqrt{a^2-x^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\, dy\, dx$$

$$= 8c \int_{x=0}^{a} \int_{y=0}^{(b/a)\sqrt{a^2-x^2}} \frac{1}{b}\sqrt{b^2\left\{1-\left(\frac{x^2}{a^2}\right)\right\}-y^2}\, dy\, dx$$

We shall use $\int \sqrt{\alpha^2 - y^2}\, dy = \dfrac{y\sqrt{\alpha^2-y^2}}{2} + \dfrac{\alpha^2}{2}\sin^{-1}\left(\dfrac{y}{\alpha}\right)$

where $\alpha^2 = b^2\ \{1 - x^2/a^2\} = b^2\ (a^2 - x^2)/a^2$

$$\therefore \qquad V = \frac{8c}{b} \int_{x=0}^{a} \int_{y=0}^{\alpha} \sqrt{\alpha^2 - y^2}\, dy\, dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \left[\frac{y\sqrt{\alpha^2-y^2}}{2} + \frac{\alpha^2}{2}\sin^{-1}\left(\frac{y}{\alpha}\right)\right]_0^{\alpha} dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} 0 + \frac{\alpha^2}{2}\left[\sin^{-1}(1) - \sin^{-1}(0)\right] dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \frac{\pi}{2} \cdot \frac{1}{2} \frac{b^2}{a^2}\left(a^2 - x^2\right) dx$$

$$= \frac{2bc\pi}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{2bc\pi}{a^2} \cdot \frac{2a^3}{3} = \frac{4\pi\, abc}{3}$$

Thus the required volume $(V) = \dfrac{4\pi abc}{3}$ cubic units.

EXERCISE 3.2

1. Evaluate $\iint_R xy^2\, dx\, dy$ over the region bounded by $y = x^2$, $y = 0$ and $x = 1$. $\left[\textbf{Ans. } \frac{1}{24}\right]$

2. Evaluate $\iint_R xy\,(x + y)\, dx\, dy$ taken over the region bounded by the parabolas $y^2 = x$ and $y = x^2$. $\left[\textbf{Ans. } \frac{3}{28}\right]$

3. Evaluate $\iint_R x^2 y\, dx\, dy$ over the region bounded by the curves $y = x^2$ and $y = x$. $\left[\textbf{Ans. } \frac{1}{35}\right]$

4. Evaluate $\iint_R xy\, dx\, dy$ where R is the region in the first quadrant bounded by the line $x + y = 1$. $\left[\textbf{Ans. } \frac{1}{6}\right]$

Evaluate the following by changing the order of integration (5 to 9)

5. $\displaystyle\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} \left(x^2 + y^2\right) dy\, dx.$ $\left[\textbf{Ans. } \frac{a^3}{28} + \frac{a}{20}\right]$

6. $\displaystyle\int_0^a \int_0^{2\sqrt{ax}} x^2\, dx\, dy.$ $\left[\textbf{Ans. } \frac{4a^4}{7}\right]$

7. $\displaystyle\int_{-a}^a \int_0^{\sqrt{a^2 - x^2}} (a - x)\, dy\, dx.$ $\left[\textbf{Ans. } \frac{\pi a^3}{2}\right]$

8. $\displaystyle\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dy\, dx}{\sqrt{y^4 - a^2x^2}}.$ $\left[\textbf{Ans. } \frac{\pi a^2}{6}\right]$

9. $\displaystyle\int_0^a \int_{\frac{x^2}{a}}^{2a - x} xy\, dy\, dx.$ $\left[\textbf{Ans. } \frac{3a^4}{8}\right]$

10. Evaluate $\displaystyle\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} x^2\, dy\, dx$ by transforming into polar coordinates. $\left[\textbf{Ans. } \frac{5\pi a^4}{8}\right]$

11. Find the area of the cardioid $r = a\,(1 + \cos\theta)$ by double integration. $\left[\textbf{Ans. } \frac{3\pi\, a^2}{2}\right]$

12. Find the volume of the region bounded by the cylinder $x^2 + y^2 = 16$ and the planes $z = 0$ and $z = 3$. $[\textbf{Ans. } 48\pi]$

3.4 BETA AND GAMMA FUNCTIONS

In this topic we define two special functions of improper integrals known as Beta function and Gamma function. These functions play important role in applied mathematics.

3.4.1 Definitions

1. The Beta function denoted by B (m, n) or β (m, n) is defined by

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, (m, n > 0) \qquad ...(1)$$

2. The Gamma function denoted by $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1}\cdot e^{-x}\,dx \qquad ...(2)$$

3.4.2 Properties of Beta and Gamma Functions

1. $\beta(m, n) = \beta(n, m)$

2. $$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \qquad ...(3)$$

3. $$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \qquad ...(4)$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta\, d\theta$$

4. $$\beta\left[\frac{p+1}{2}, \frac{q+1}{2}\right] = 2\int_0^{\pi/2} \sin^{p}\theta \cos^{q}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{q}\theta \cos^{p}\theta\, d\theta \qquad ...(5)$$

5. $\Gamma(n + 1) = n\ \Gamma(n)$ ...(6)

6. $\Gamma(n + 1) = n!$, if n is a $+ve$ real number.

Proof 1. We have

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{m-1}\left[1-(1-x)\right]^{n-1}\,dx$$

$$\text{Since } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^1 (1-x)^{m-1}(1-1+x)^{n-1}\,dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= \beta(n, m)$$

Thus, $\beta(m, n) = \beta(n, m)$

Hence (1) is proved.

(2) By definition of Beta function,

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Substituting $x = \dfrac{1}{1+t}$ then $dx = \dfrac{-1}{(1+t)^2}\,dt$ when $x = 0$, $t = \infty$ and when $x = 1$, $t = 0$.

Therefore,

$$\beta(m, n) = \int_\infty^0 \left[\frac{1}{1+t}\right]^{m-1}\left[1-\frac{1}{1+t}\right]^{n-1}\left\{\frac{-1}{(1+t)^2}\,dt\right\}$$

$$= \int_\infty^0 \left(\frac{1}{1+t}\right)^{m-1}\left(\frac{t}{1+t}\right)^{n-1}\left\{\frac{-1}{(1+t)^2}\,dt\right\}$$

$$= \int_0^\infty \frac{t^{n-1}}{(1+t)^{m-1+n-1+2}}\,dt$$

$$\beta(m, n) = \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}}\,dt = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\text{Similarly, } \beta(n, m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Since, $\beta(m, n) = \beta(n, m)$, we get

$$\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+1}}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

(3) By definition of Beta functions

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Substitute $x = \sin^2\theta$ then $dx = 2\sin\theta\cos\theta\, d\theta$

Also when $x = 0,\ \theta = 0$

when $x = 1,\ \theta = \dfrac{\pi}{2}$

$$\therefore \quad \beta(m, n) = \int_0^{\pi/2} \left(\sin^2\theta\right)^{m-1} \cdot \left(1-\sin^2\theta\right)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \left(\cos^2\theta\right)^{n-1} \cdot \sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

Since, $\beta(m, n) = \beta(n, m)$, we have

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta\cos^{2m-1}\theta\, d\theta$$

(**4**) Substituting $2m - 1 = p$ and $2n - 1 = q$

So that $m = \dfrac{p+1}{2},\quad n = \dfrac{q+1}{2}$ in the above result, we have

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^q\theta\cos^p\theta\, d\theta$$

(1) Substituting $q = 0$ in the above result, we get

$$\beta\left[\frac{p+1}{2}, \frac{1}{2}\right] = 2\int_0^{\pi/2} \sin^p\theta\, d\theta = 2\int_0^{\pi/2} \cos^p\theta\, d\theta \cdot$$

(2) Substituting $p = 0$ and $q = 0$ in the above result

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} d\theta = \pi$$

(5) Replacing n by $(n + 1)$ in the definition of gamma function.

$$\Gamma(n) = \int_0^\infty x^{n-1} \cdot e^{-x}\, dx$$

where $n = (n + 1)$

$$\Gamma(n + 1) = \int_0^\infty x^n \cdot e^{-x}\, dx$$

On integrating by parts, we get

$$\Gamma(n + 1) = \left[x^n \cdot \left(-e^{-x}\right)\right]_0^\infty - \int_0^\infty \left(-e^{-x}\right) \cdot n\, x^{n-1}\, dx$$

$$= 0 + n \int_0^\infty e^{-x}\, x^{n-1}\, dx = n\,\Gamma(n).$$

$$\left[\text{since } \lim_{x \to \infty} \frac{x^n}{e^x} = 0, \text{ if } n > 0\right]$$

Thus, $\boxed{\Gamma(n + 1) = n\,\Gamma(n),}$ for $n > 0$

This is called the recurrence formula, for the gamma function.

(6) If n is a positive integer then by repeated application of the above formula, we get

$$\begin{aligned}
\Gamma(n + 1) &= n\,\Gamma(n) \\
&= n\,\Gamma(n - 1 + 1) \\
&= n\,(n - 1)\,\Gamma(n - 1) \text{ (using above result)} \\
&= n\,(n - 1)\,(n - 2)\,\Gamma(n - 2) \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots \\
&= n\,(n - 1)\,(n - 2)\ldots\ldots 1\,\Gamma(1) \\
&= n!\;\Gamma(1)
\end{aligned}$$

But $$\Gamma(1) = \int_0^\infty x^0\, e^{-x}\, dx$$

$$= -\left[e^{-x}\right]_0^\infty = -(0 - 1) = 1$$

Hence $\Gamma(n + 1) = n!$, if n is a positive integer.

For example

$$\Gamma(2) = 1! = 1,\ \Gamma(3) = 2! = 2,\ \Gamma(4) = 3! = 6$$

If n is a positive fraction then using the recurrence formula $\Gamma(n + 1) = n\,\Gamma(n)$ can be evaluated as follows.

(1) $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

(2) $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

(3) $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$

$$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{15}{8}\Gamma\left(\frac{1}{2}\right).$$

3.4.3 Relationship between Beta and Gamma functions

The Beta and Gamma functions are related by

$$\beta\,(m,\,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad ...(7)$$

Proof. We have $\Gamma\,(n) = \int_0^\infty x^{n-1}\cdot e^{-x}\,dx$

Substituting $x = t^2$, $dx = 2t\,dt$, we get

$$\Gamma\,(n) = \int_0^\infty \left(t^2\right)^{n-1} e^{-t^2}\cdot 2t\,dt$$

$$= 2\int_0^\infty t^{2n-1}\cdot e^{-t^2}\,dt$$

$$\Gamma\,(n) = 2\int_0^\infty x^{2n-1}\,e^{-x^2}\,dx \qquad ...(i)$$

Replacing n by m, and 'x' by 'y', we have

$$\Gamma\,(m) = 2\int_0^\infty y^{2m-1}\,e^{-y^2}\,dy \qquad ...(ii)$$

Hence

$$\Gamma\,(m)\cdot\Gamma\,(n) = \left\{2\int_0^\infty x^{2n-1}\,e^{-x^2}\,dx\right\}\left\{2\int_0^\infty y^{2m-1}\,e^{-y^2}\,dy\right\}$$

$$= 4\int_0^\infty\int_0^\infty x^{2n-1}e^{-x^2}\,y^{2m-1}\cdot e^{-y^2}\,dx\,dy$$

$$= 4\int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)} x^{2n-1}\, y^{2m-1}\, dx\, dy \qquad ...(iii)$$

We shall transform the double integral into polar coordinates.

Substitute $x = r\cos\theta$, $y = r\sin\theta$ then we have $dx\,dy = r\,dr d\theta$

As x and y varies from 0 to ∞, the region of integration entire first quadrant. Hence, θ varies from 0 to $\dfrac{\pi}{2}$ and r varies from 0 to ∞ and also $x^2 + y^2 = r^2$

Hence (iii) becomes,

$$\Gamma(m)\,\Gamma(n) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\left(r\cos\theta\right)^{2n-1}\left(r\sin\theta\right)^{2m-1}\cdot r\,d\theta\, dr$$

$$= 4\int_{r=0}^{\infty} r^{2(m+n)-1}\, e^{-r^2}\, dr \times \int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta\cos^{2n-1}\theta\, d\theta \qquad ...(iv)$$

Substituting $r^2 = t$, in the first integral. We get,

$$\int_{r=0}^{\infty} r^{2(m+n)-1}\, e^{-r^2}\, dr = \frac{1}{2}\int_0^{\infty} t^{m+n-1}\, e^{-t}\, dt$$

$$= \frac{1}{2}\,\Gamma\left(m+n\right)$$

and from (iv), $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{1}{2}\,\beta\left(m,n\right)$

Therefore (iv) reduces to $\Gamma\left(m\right)\Gamma\left(n\right) = \Gamma\left(m+n\right)\beta\left(m,n\right)$

Thus, $$\beta\,(m,\,n) = \frac{\Gamma\left(m\right)\Gamma\left(n\right)}{\Gamma\left(m+n\right)}\cdot$$ **Hence proved.**

Corollary. To show that $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

Putting $m = n = \dfrac{1}{2}$ in this result, we get

$$\beta\left[\frac{1}{2},\frac{1}{2}\right] = \frac{\Gamma\left[\frac{1}{2}\right]\cdot\Gamma\left[\frac{1}{2}\right]}{\Gamma\left[1\right]}$$

But $\Gamma\,(1) = 1$

$$\therefore \quad \beta\left[\frac{1}{2},\frac{1}{2}\right] = \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 \qquad ...(8)$$

Now consider $\beta\,(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$

Now we have from (8), L.H.S.

$$\beta\left[\frac{1}{2}, \frac{1}{2}\right] = 2\int_0^{\frac{\pi}{2}} \sin^0\theta\cos^0\theta\, d\theta = 2\,[\theta]_0^{\frac{\pi}{2}} = \pi$$

$$\pi = \Gamma\left\{\frac{1}{2}\right\}^2 \quad \therefore \quad \Gamma\left[\frac{1}{2}\right] = \sqrt{\pi}\,.$$

WORKED OUT EXAMPLES

1. *Evaluate the following:*

(*i*) $\dfrac{\Gamma(7)}{\Gamma(5)}$ (*ii*) $\dfrac{\Gamma(5/2)}{\Gamma(3/2)}$ (*iii*) $\dfrac{\Gamma(8/3)}{\Gamma(2/3)}$

Solution

(*i*) $$\frac{\Gamma(7)}{\Gamma(5)} = \frac{6!}{4!} = 30$$

(*ii*) $$\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{3}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{3}{2}$$

(*iii*) $$\frac{\Gamma\left(\frac{8}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\Gamma\left(\frac{5}{3}+1\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\frac{5}{3}\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}$$

$$= \frac{\frac{5}{3}\Gamma\left(\frac{2}{3}+1\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\frac{5}{3}\times\frac{2}{3}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}$$

$$= \frac{10}{9}.$$

2. *Evaluate:*

(*i*) $\displaystyle\int_0^{\infty} x^4 e^{-x}\, dx$ (*ii*) $\displaystyle\int_0^{\infty} x^6 e^{-3x}\, dx$ (*iii*) $\displaystyle\int_0^{\infty} x^2 e^{-2x^2}\, dx$.

Solution

(*i*)
$$\int_0^\infty x^4\, e^{-x}\, dx = \int_0^\infty x^{5-1}\, e^{-x}\, dx = \Gamma(5) = 4!$$
$$= 24$$

(*ii*)
$$\int_0^\infty x^6\, e^{-3x}\, dx$$

Substituting
$$3x = t \Rightarrow x = \frac{t}{3} \text{ then } dx = \frac{dt}{3}$$

$$\int_0^\infty x^6\, e^{-3x}\, dx = \int_0^\infty \left(\frac{t}{3}\right)^6 \cdot e^{-t} \cdot \frac{dt}{3}$$
$$= \frac{1}{3^7}\int_0^\infty t^6\, e^{-t}\, dt = \frac{1}{3^7}\int_0^\infty t^{7-1}\, e^{-t}\, dt$$
$$= \frac{1}{3^7}\Gamma(7)$$
$$= \frac{1}{3^7} 6! = \frac{80}{243}$$

(*iii*) Substitute
$$2x^2 = t \Rightarrow x^2 = \frac{t}{2} \Rightarrow x = \sqrt{\frac{t}{2}}$$

Then
$$2x\, dx = \frac{dt}{2}$$
$$dx = \frac{dt}{4x} = \frac{\sqrt{2}\, dt}{4\sqrt{t}}$$

$$\int_0^\infty x^2\, e^{-2x^2}\, dx = \int_0^\infty \frac{t}{2} \cdot e^{-t} \cdot \frac{\sqrt{2}}{4\sqrt{t}}\, dt$$
$$= \frac{\sqrt{2}}{8}\int_0^\infty t^{\frac{1}{2}}\, e^{-t}\, dt$$
$$= \frac{\sqrt{2}}{8}\int_0^\infty t^{\frac{3}{2}-1}\, e^{-t}\, dt \qquad \left[\because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$
$$= \frac{\sqrt{2}}{8}\Gamma\left(\frac{3}{2}\right)$$
$$= \frac{\sqrt{2}}{8}\Gamma\left(\frac{1}{2}+1\right) = \frac{\sqrt{2}}{8} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{2}}{8} \cdot \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{2\pi}}{16}$$

3. *Evaluate the following:*

(*i*) $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$ (*ii*) $\int_0^\infty x^4\, e^{-x^2}\, dx$ (*iii*) $\int_0^\infty x^{\frac{1}{4}}\, e^{-\sqrt{x}}\, dx$

(*iv*) $\int_0^\infty x^{\frac{-3}{2}}\left(1-e^{-x}\right) dx$ (*v*) $\int_0^\infty 3^{-4x^2}\, dx.$

Solution

(*i*) Substitute $x^3 = t$ so that $3x^2\, dx = dt$

where $x = t^{\frac{1}{3}},\ dx = \dfrac{dt}{3t^{\frac{2}{3}}}$

when $x = 0,\ t = 0$ and when $x = \infty,\ t = \infty$

Hence,
$$\int_0^\infty \sqrt{x}\, e^{-x^3} dx = \int_0^\infty t^{\frac{1}{6}}\, e^{-t}\, \frac{dt}{3t^{\frac{2}{3}}}$$
$$= \frac{1}{3}\int_0^\infty t^{\frac{-1}{2}}\, e^{-t}\, dt$$
$$= \frac{1}{3}\int_0^\infty t^{\frac{1}{2}-1}\, e^{-t}\, dt$$
$$= \frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{1}{3}\sqrt{\pi}\,.$$

(*ii*) Substitute, $x^2 = t \;\Rightarrow\; x = t^{\frac{1}{2}} = \sqrt{t}$

So that $2x\, dx = dt \;\Rightarrow\; dx = \dfrac{dt}{2\sqrt{t}}$

Hence,
$$\int_0^\infty x^4\, e^{-x^2}\, dx = \int_0^\infty t^2\, e^{-t}\, \frac{dt}{2\sqrt{t}}$$
$$= \frac{1}{2}\int_0^\infty t^{\frac{3}{2}}\, e^{-t}\, dt$$
$$= \frac{1}{2}\int_0^\infty t^{\frac{5}{2}-1}\, e^{-t}\, dt$$
$$= \frac{1}{2}\Gamma\left(\frac{5}{2}\right)$$
$$= \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{8}\sqrt{\pi}\,.$$

(*iii*) Substitute, $\sqrt{x} = t \Rightarrow x = t^2,\ dx = 2t\ dt$

Hence, $$\int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}}\, dx = \int_0^\infty t^{1/2} e^{-t}\, 2t\, dt$$

$$= 2\int_0^\infty t^{\frac{3}{2}} e^{-t}\, dt$$

$$= 2\int_0^\infty t^{\frac{5}{2}-1} e^{-t}\, dt$$

$$= 2\,\Gamma\left(\frac{5}{2}\right)$$

$$= 2 \cdot \frac{3}{2} \cdot \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}\,.$$

(*iv*) On integrating by parts, we get

$$\int_0^\infty x^{\frac{-3}{2}} \left(1 - e^{-x}\right) dx = \left[\left(1 - e^{-x}\right)\left(-2x^{\frac{-1}{2}}\right)\right]_0^\infty - \int_0^\infty \left(-2x^{\frac{-1}{2}}\right) e^{-x}\, dx$$

$$= 0 + 2\int_0^\infty x^{\frac{-1}{2}} e^{-x}\, dx$$

$$= 2\int_0^\infty x^{\frac{1}{2}-1} e^{-x}\, dx$$

$$= 2\,\Gamma\left(\frac{1}{2}\right)$$

$$= 2\sqrt{\pi}\,.$$

(*v*) Since $a = e^{\log a},\ a > 0$, we have

$$3^{-4x^2} = \left[e^{\log 3}\right]^{-4x^2} = e^{-(4\log 3)x^2}$$

$$\int_0^\infty e^{-4x^2}\, dx = \int_0^\infty e^{-(4\log 3)x^2}\, dx$$

Setting $(4\ \log\ 3) = x^2 = t$ we get,

$$x^2 = \frac{t}{4\log 3} \Rightarrow x = \frac{\sqrt{t}}{2\sqrt{\log 3}}$$

$$(4\ \log\ 3)\ 2x\ dx = dt$$

$$(4\ \log\ 3)\ 2 \cdot \frac{\sqrt{t}}{2\sqrt{\log 3}}\, dx = dt$$

$$\sqrt{t}\cdot 4\sqrt{\log 3}\,dx = dt \qquad \Rightarrow dx = \frac{1}{\sqrt{t}\cdot 4\sqrt{\log 3}}\,dt$$

$$\int_0^\infty 3^{-4x^2}\,dx = \int_0^\infty e^{-t}\cdot\frac{1}{\sqrt{t}\cdot 4\sqrt{\log 3}}\,dt$$

$$= \frac{1}{4\sqrt{\log 3}}\int_0^\infty t^{\frac{-1}{2}}\,e^{-t}\,dt$$

$$= \frac{1}{4\sqrt{\log 3}}\int_0^\infty t^{\frac{1}{2}-1}\,e^{-t}\,dt$$

$$= \frac{1}{4\sqrt{\log 3}}\,\Gamma\left(\frac{1}{2}\right)$$

$$\int_0^\infty 3^{-4x^2}\,dx = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$$

4. *Evaluate:*

(*i*) $\int_0^1 (\log\ x)^4\,dx$ (*ii*) $\int_0^1 (x\log x)^3\,dx$

(*iii*) $\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}}$.

Solution

Substitute $\log x = -t$ so that $x = e^{+t}$

Also $\frac{1}{x}\,dx = -dt$ or $dx = -x\,dt = -e^{-t}\,dt$

when $x = 0$, $t = -\log 0 = \infty$ and

when $x = 1$, $t = -\log 1 = 0$ (note that $\log 0 = -\infty$)

(*i*) Hence $\int_0^1 (\log x)^4\,dx = \int_\infty^0 (-t)^4\cdot -e^{-t}dt$

$$= \int_0^\infty t^4\,e^{-t}\,dt$$

$$= \int_0^\infty t^{5-1}\,e^{-t}\,dt$$

$$= \Gamma(5) = 4\,! = 24.$$

(*ii*) $\int_0^1 (x\log x)^3\,dx = \int_\infty^0 \left[e^{-1}\,(-t)\right]^3\left(-e^{-t}\,dt\right)$

$$= -\int_0^\infty t^3\,e^{-4t}\,dt$$

Put $\quad 4t = u \quad \Rightarrow \quad 4dt = du$

$$dt = \frac{1}{4}du$$

$$\therefore \quad \int_0^1 (x \log x)^3\, dx = -\int \left(\frac{u}{4}\right)^3 \cdot e^{-u} \cdot \frac{1}{4}du$$

$$= \frac{-1}{(4)^4}\int_0^\infty u^3 \cdot e^{-u}\, du$$

$$= \frac{-1}{256}\int_0^\infty u^{4-1}\, e^{-u}\, du$$

$$= \frac{-1}{256}\Gamma(4) = -\frac{3!}{256} = \frac{-3}{128}.$$

(*iii*)
$$\int_0^1 \frac{1}{\sqrt{\log\left(\frac{1}{x}\right)}}\, dx = \int_\infty^0 \frac{-e^{-t}\, dt}{\sqrt{t}}$$

$$= \int_0^\infty t^{\frac{-1}{2}}\, e^{-t}\, dt$$

$$= \int_0^\infty t^{\frac{1}{2}-1}\, e^{-t}\, dt$$

$$= \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

5. *Prove that* $\int_0^\infty a^{-bx^2}\, dx = \dfrac{\sqrt{\pi}}{2\sqrt{b \log a}}$ *where a and b are positive constants.*

Solution

Now, $\quad \int_0^\infty a^{-bx^2}\, dx = \int_0^\infty \left\{e^{\log a}\right\}^{-bx^2} \quad$ since $a = e^{\log a}$

$$= \int_0^\infty e^{-(b \log a)x^2}\, dx$$

Substitute $\quad (b \log a)\, x^2 = t,\ dx = \dfrac{dt}{(b \log a)\cdot 2x}$

So that, $\quad x = \dfrac{\sqrt{t}}{\sqrt{b \log a}}$

$$\therefore \quad dx = \frac{dt}{2\sqrt{t}\,\sqrt{b \log a}}$$

$$\int_0^\infty e^{-bx^2} dx = \int_0^\infty e^{-t} \cdot \frac{dt}{2\sqrt{t}\,\sqrt{b \log a}}$$

$$= \frac{1}{2\sqrt{b\log a}} \int_0^\infty t^{\frac{-1}{2}} e^{-t}\, dt$$

$$= \frac{1}{2\sqrt{b\log a}} \int_0^\infty t^{\frac{1}{2}-1} e^{-t}\, dt$$

$$= \frac{1}{2\sqrt{b\log a}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2\sqrt{b\log a}} .$$

6. *Prove that* $\int_0^\infty x^m e^{-ax^n}\, dx = \frac{1}{na^{\frac{(m+1)}{n}}} \Gamma\left(\frac{m+1}{n}\right)$, *where m and n are positive constants.*

Solution

Substitute $\qquad ax^n = t$ so that $x = \left(\frac{t}{a}\right)^{\frac{1}{n}}$

Then $\qquad dx = \frac{1}{na^{\frac{1}{n}}} \cdot t^{\frac{1}{n}-1}\, dt$

Therefore,

$$\int_0^\infty x^m e^{-ax^n}\, dx = \int_0^\infty \left[\left(\frac{t}{a}\right)^{\frac{1}{n}}\right]^m e^{-t} \cdot \frac{t^{\frac{1}{n}-1}}{na^{\frac{1}{n}}}\, dt$$

$$= \frac{1}{na^{(m+1)/n}} \int_0^\infty t^{\frac{(m+1)}{n-1}} e^{-t}\, dt$$

$$= \frac{1}{na^{(m+1)/n}} \Gamma\left[\frac{m+1}{n}\right].$$

7. *Prove that* $\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n \cdot n!}{(m+1)^{n+1}}$, *where n is a positive integer and m > –1.*

Solution

Substitute $\qquad \log x = -t$ or $x = e^{-t}$

Then $\qquad dx = -e^{-t}\, dt$

when $x = 0$, $t = \infty$ and when $x = 1$, $t = 0$.

Therefore,

$$\int_0^1 x^m (\log x)^n\, dx = \int_\infty^0 \left(e^{-t}\right)^m (-t)^n \cdot \left(-e^{-t}\right) dt$$

$$= (-1)^n \int_0^\infty t^n \, e^{-(m+1)t} \, dt$$

$$= (-1)^n \int_0^\infty \left\{\frac{u}{m+1}\right\}^n e^{-u} \cdot \frac{du}{m+1}$$

since setting $(m + 1)\, t = u$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty u^n \, e^{-u} \, du$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^\infty u^{(n+1-1)} e^{-u} du$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \Gamma(n+1)$$

$$= \frac{(-1)\, n!}{(m+1)^{n+1}} \text{ where } \Gamma\,(n + 1) = n!.$$

8. *Prove that*

$$(i) \int_0^\infty x^{n-1} \, e^{-ax} \cos bx \, dx = \frac{\Gamma(n)}{\left(a^2+b^2\right)^{\frac{n}{2}}} \cos\left\{n \tan^{-1} \frac{b}{a}\right\}$$

$$(ii) \int_0^\infty x^{n-1} \, e^{-ax} \sin bx \, dx = \frac{\Gamma(n)}{\left(a^2+b^2\right)^{\frac{n}{2}}} \sin\left\{n \tan^{-1} \frac{b}{a}\right\}.$$

Solution

Consider $I = \int_0^\infty x^{n-1} \, e^{-ax} \, e^{ibx} \, dx$

$$= \int_0^\infty x^{n-1} \, e^{-(a-ib)x} \, dx$$

Substitute $(a - ib)\, x = t$, so that $dx = \dfrac{dt}{a-ib}$

$$x = \frac{t}{a-ib}$$

Hence, $I = \int_0^\infty \left\{\dfrac{t}{a-ib}\right\}^{n-1} e^{-t} \cdot \dfrac{dt}{a-ib}$

$$= \frac{1}{(a-ib)^n} \int_0^\infty t^{n-1} \, e^{-t} \, dt$$

$$= \frac{1}{(a-ib)^n} \Gamma(n)$$

$$I = \frac{(a+ib)^n \cdot \Gamma(n)}{(a^2+b^2)^n} \qquad ...(1)$$

Since $\quad a + ib = r(\cos\theta + i\sin\theta)$

where $\quad r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Hence (1) reduces to,

$$I = \frac{\Gamma(n)\left[r(\cos\theta + i\sin\theta)\right]^n}{(a^2+b^2)^n}$$

Apply De Moivre's theorem

$$= \frac{\Gamma(n)\, r^n (\cos n\theta + i\sin n\theta)}{(a^2+b^2)^n}$$

$$= \frac{\Gamma(n) \cdot (a^2+b^2)^{n/2} (\cos n\theta + i\sin n\theta)}{(a^2+b^2)^n}$$

$$= \frac{\Gamma(n)}{(a^2+b^2)^{n/2}} (\cos n\theta + i\sin n\theta)$$

On equating the real and imaginary parts, we get

(*i*) $$\int_0^\infty x^{n-1} e^{-ax} \cos bx\, dx = \frac{\Gamma(n)}{(a^2+b^2)^{n/2}} \cos n\theta$$

(*ii*) $$\int_0^\infty x^{n-1} e^{-ax} \sin bx\, dx = \frac{\Gamma(n)}{(a^2+b^2)^{n/2}} \sin n\theta$$

where $\theta = \tan^{-1}\frac{b}{a}$.

9. *Evaluate*

(i) β *(3, 5)* *(ii)* β *(3/2, 2)* *(iii)* β *(1/3, 2/3).*

Solution

Using the relation $\beta(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

(*i*) $$\beta(3, 5) = \frac{\Gamma(3)\,\Gamma(5)}{\Gamma(3+5)} = \frac{2!\,4!}{7!} = \frac{1}{105}$$

(*ii*)
$$\beta\left[\frac{3}{2}, 2\right] = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(2)}{\Gamma\left(\frac{3}{2}+2\right)} = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(2)}{\Gamma\left(\frac{7}{2}\right)}$$
$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right).1!}{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{4}{15}$$

(*iii*)
$$\beta\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{\Gamma\left(\frac{1}{3}\right).\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)}$$
$$= \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

where $\Gamma(n)\ \Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$
$$= \frac{\pi}{\sin\frac{\pi}{3}}$$
$$= \frac{2\pi}{\sqrt{3}}.$$

10. *Evaluate each of the following integrals*

(*i*) $\int_0^1 x^4\left(1-x^3\right)dx$ (*ii*) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$

(*iii*) $\int_0^a y^4\sqrt{a^2-y^2}\,dy$ (*iv*) $\int_0^1 \sqrt{\frac{1-x}{x}}\,dx$

(*v*) $\int_0^2 \left(4-x^2\right)^{3/2}dx$

Solution

(*i*)
$$\int_0^1 x^4(1-x)^3\,dx = \beta(5,4) = \frac{\Gamma(5)\,\Gamma(4)}{\Gamma(5+4)} = \frac{4!\,3!}{8!} = \frac{1}{280}$$

(*ii*) Substitute $x = 2t$

Then $dx = 2dt$

$$\therefore \quad \int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx = \int_0^1 \frac{4t^2}{\sqrt{2-2t}}\,2\,dt$$

$$= 4\sqrt{2} \int_0^1 t^2 (1-t)^{-1/2}\, dt$$

$$= 4\sqrt{2}\, \beta\left[3, \frac{1}{2}\right]$$

$$= 4\sqrt{2}\, \frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left[3+\frac{1}{2}\right]}$$

$$= 4\sqrt{2}\, \frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{64\sqrt{2}}{15}$$

(*iii*) Substitute $y^2 = a^2 t$

or $y = a\sqrt{t}$

$$dy = \frac{a}{2\sqrt{t}}\, dt$$

Given integral becomes,

$$\int_0^1 \left(a\sqrt{t}\right)^4 \sqrt{a^2 - a^2 t}\, \frac{a}{2}\, \frac{dt}{\sqrt{t}} = \frac{a^6}{2} \int_0^1 t^{3/2} (1-t)^{1/2}\, dt$$

$$= \frac{a^6}{2}\, \beta\left[\frac{5}{2}, \frac{3}{2}\right]$$

$$= \frac{a^6}{2}\, \frac{\Gamma\left(\frac{5}{2}\right).\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)} = \frac{\pi a^6}{32}.$$

(*iv*)
$$\int_0^1 \sqrt{\frac{1-x}{x}}\, dx = \int_0^1 x^{-1/2} (1-x)^{1/2}\, dx$$

$$= \beta\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)} = \frac{\pi}{2}.$$

(*v*) Substitute $x^2 = 4t$

or $x = 2\sqrt{t}$

then $$dx = \frac{dt}{\sqrt{t}}$$

Given integral reduces to,

$$\int_0^1 (4-4t)^{3/2} \frac{dt}{\sqrt{t}} = 8\int_0^1 t^{-1/2}(1-t)^{3/2}\, dt$$

$$= 8\,\beta\left[\frac{1}{2}, \frac{5}{2}\right]$$

$$= 8\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)}$$

$$= \frac{8\sqrt{\pi}\,.\,\frac{3}{2}\,.\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{2}$$

$$= 3\,\pi.$$

11. *Evaluate each of the following integrals:*

(i) $\int_0^{\pi/2} sin^6\,\theta\, d\theta$ *(ii)* $\int_0^{\pi} cos^4\,\theta\, d\theta$

(iii) $\int_0^{\pi/2} sin^4\,\theta\, cos^5\,\theta\, d\theta$ *(iv)* $\int_0^{\pi/2} sin^{1/2}\,\theta\, cos^{3/2}\,\theta\, d\theta$

(v) $\int_0^{\pi/2} \sqrt{tan\,\theta}\; d\theta.$

Solution

From the relation

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta$$

$$\therefore \qquad \int_0^{\pi/2} \sin^q\theta\cos^q\theta\, d\theta = \frac{1}{2}\beta\left[\frac{p+1}{2}, \frac{q+1}{2}\right]$$

(*i*) Taking $p = 6$

$q = 0$

we get $$\int_0^{\pi/2} \sin^6\theta\, d\theta = \frac{1}{2}\beta\left[\frac{6+1}{2}, \frac{0+1}{2}\right]$$

$$= \frac{1}{2}\beta\left[\frac{7}{2}, \frac{1}{2}\right]$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{7}{2}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma(4)} = \frac{5\pi}{32}.$$

(*ii*) $$\int_0^{\pi}\cos^4\theta\,d\theta = 2\int_0^{\pi/2}\cos^4\theta\,d\theta = 2\,.\,\frac{1}{2}\,\beta\left[\frac{0+1}{2},\frac{4+1}{2}\right]$$

$$= \beta\left[\frac{1}{2},\frac{5}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)} = \frac{3\pi}{8}.$$

(*iii*) Here $p = 4$

$q = 3$ from the above relation

$$\int_0^{\pi/2}\sin^4\theta\cos^5\theta\,d\theta = \frac{1}{2}\,\beta\left[\frac{4+1}{2},\frac{5+1}{2}\right]$$

$$= \frac{1}{2}\,\beta\left[\frac{5}{2},3\right]$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{\Gamma\left(\frac{11}{2}\right)} = \frac{8}{315}$$

(*iv*) Here $p = \frac{1}{2},\ q = \frac{3}{2}$

$$\int_0^{\pi/2}\sin^{1/2}\theta\cos^{3/2}\theta\,d\theta = \frac{1}{2}\,\beta\left[\frac{\frac{1}{2}+1}{2},\ \frac{\frac{3}{2}+1}{2}\right]$$

$$= \frac{1}{2}\,\beta\left[\frac{3}{4},\ \frac{5}{4}\right]$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{1}$$

$$= \frac{1}{8}\,\Gamma\left(1-\frac{1}{4}\right)\ \Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{8}\,\frac{\pi}{\sin\frac{\pi}{4}} \qquad \left(\text{where } \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right)$$

$$= \frac{\sqrt{2}\,\pi}{8}.$$

(*v*) $$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$= \frac{1}{2}\beta\left[\frac{\frac{1}{2}+1}{2}, \frac{\frac{-1}{2}+1}{2}\right] = \frac{1}{2}\beta\left[\frac{3}{4}, \frac{1}{4}\right]$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{4}\right)\,\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{2}\Gamma\left(1-\frac{1}{4}\right)\,\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}.$$

12. *Evaluate: (i)* $\int_0^{\infty} \frac{x\,dx}{1+x^6}$ *(ii)* $\int_0^{\infty} \frac{y^2\,dy}{1+y^4}$.

Solution

(*i*) Let $x^6 = t$ or $x = t^{1/6}$

$$dx = \frac{1}{6} t^{-5/6}\, dt$$

The given integral becomes,

$$\int_0^{\infty} \frac{t^{\frac{1}{6}}\left(\frac{1}{6}\right) t^{\frac{-5}{6}}\, dt}{1+t} = \frac{1}{6}\int_0^{\infty} \frac{t^{\frac{-2}{3}}}{1+t}\, dt$$

$$= \frac{1}{6}\int_0^{\infty} \frac{t^{\frac{1}{3}-1}}{(1+t)^{2/3+1/3}}\, dt$$

$$= \frac{1}{6}\beta\left[\frac{1}{3}, \frac{2}{3}\right]$$

$$\left[\text{Using the relation, } \beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx\right]$$

$$= \frac{1}{6}\,\frac{\Gamma\left(\frac{1}{3}\right)\,\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)}$$

$$= \frac{1}{6}\,\Gamma\left(\frac{1}{3}\right).\Gamma\left(1-\frac{1}{3}\right)$$

$$= \frac{1}{6}\cdot\frac{\pi}{\sin\left(\frac{\pi}{3}\right)} = \frac{\pi}{3\sqrt{3}}.$$

(*ii*) Substituting $y^4 = t$, or $y = t^{1/4}$

then $dy = \frac{1}{4}\, t^{\frac{-3}{4}}\, dt$

so that,

$$\int_0^\infty \frac{y^2\,dy}{1+y^4} = \int_0^\infty \frac{\left(t^{\frac{1}{4}}\right)^2 \left(\frac{1}{4}\right) t^{\frac{-3}{4}}\,dt}{1+t}$$

$$= \frac{1}{4}\int_0^\infty \frac{t^{\frac{-1}{4}}}{1+t}\,dt$$

$$= \frac{1}{4}\int_0^\infty \frac{t^{\frac{3}{4}-1}}{(1+t)^{\frac{3}{4}+\frac{1}{4}}}\,dt$$

$$= \frac{1}{4}\,\beta\left[\frac{3}{4},\ \frac{1}{4}\right]$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}\,\frac{\pi}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{2\sqrt{2}}.$$

13. *Prove that* $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi.$

Solution

L.H.S. $\int_0^{\pi/2} \sin^{\frac{-1}{2}}\theta\, d\theta \times \int_0^{\pi/2} \sin^{\frac{1}{2}}\theta\, d\theta$

$$= \frac{1}{2}\beta\left[\frac{-\frac{1}{2}+1}{2}, \frac{0+1}{2}\right] \times \frac{1}{2}\beta\left[\frac{\frac{1}{2}+1}{2}, \frac{0+1}{2}\right]$$

$$= \frac{1}{4}\beta\left[\frac{1}{4}, \frac{1}{2}\right] \times \beta\left[\frac{3}{4}, \frac{1}{2}\right]$$

$$= \frac{1}{4}\left[\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]\left[\frac{\Gamma\left(\frac{3}{4}\right)\ \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\right]$$

$$= \frac{1}{4}\ \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 \Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \pi. \qquad \left(\text{since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

14. *Prove that* $\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}}\, dx \times \int_0^{\infty} x^2\, e^{-x^4}\, dx = \frac{\pi}{4\sqrt{2}}.$

Solution

Substituting $x^2 = t$ or $x = \sqrt{t}$ in the first integral,

we get $dx = \frac{dt}{2\sqrt{t}}$

$$I_1 = \int_0^{\infty} \frac{e^{-t}}{\left(t^{1/2}\right)^{\frac{1}{2}}} \cdot \frac{1}{2\sqrt{t}}\, .dt$$

$$= \frac{1}{2}\int_0^{\infty} t^{\frac{-3}{4}}\, e^{-t}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} t^{\frac{1}{4}-1}\, e^{-t}\, dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)$$

Taking $x^4 = u$ or $x = u^{1/4}$ in the second integral, we obtain, $dx = \frac{1}{4} u^{\frac{-3}{4}}\, du$

$$I_2 = \int_0^\infty x^2 e^{-x^4}\, dx$$

$$= \int_0^\infty \left(x^{1/4}\right)^2 . e^{-u} . \frac{1}{4} u^{\frac{-3}{4}}\, du$$

$$= \frac{1}{4}\int_0^\infty u^{\frac{-1}{4}} e^{-u}\, du$$

$$= \frac{1}{4}\int_0^\infty u^{\frac{3}{4}-1} e^{-u}\, du$$

$$= \frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$\therefore$ The given integral $= I_1 \times I_2$

$$= \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

$$= \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{8} . \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{4\sqrt{2}}.$$

15. *Prove that* $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\, dx \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}.$

Solution. In the first integral setting $x^2 = \sin\theta$ or $x = \sqrt{\sin\theta}$ we get $dx = \dfrac{\cos\theta}{2\sqrt{\sin\theta}}\, d\theta$ when $x = 1$, $\theta = \dfrac{\pi}{2}$, when $x = 0$, $\theta = 0$.

Therefore,

$$I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\, dx = \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} . \frac{\cos\theta}{2\sqrt{\sin\theta}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} . \frac{\cos\theta}{\sqrt{\sin\theta}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\beta\left[\frac{\frac{1}{2}+1}{2},\ \frac{0+1}{2}\right]$$

$$= \frac{1}{4}\beta\left(\frac{3}{4},\ \frac{1}{2}\right)$$

In the second integral substitute $x^2 = \tan\theta$ or $x = \sqrt{\tan\theta}$ then $dx = \dfrac{\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}}$.

when $x = 0,\ \theta = 0$
when $x = 1,\ \theta = \pi/4$

Hence, $$I_2 = \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}}\cdot\frac{\sec^2\theta}{2\sqrt{\tan\theta}}\,.\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{\sec\theta}{\sqrt{\tan\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\sin\theta\cdot\cos\theta}}$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\left(\frac{1}{2}\right)\sin 2\theta}}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\sin 2\theta}}$$

where $t = 2\theta,\ dt = 2\cdot d\theta,\ \Rightarrow d\theta = \dfrac{1}{2}\ dt$

$\theta = 0,\ t = 0$, when $\theta = \dfrac{\pi}{4},\ t = \dfrac{\pi}{2}$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{1}{2}\cdot\frac{dt}{\sqrt{\sin t}}$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{1}{\sqrt{\sin t}}\,dt$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2}(\sin t)^{-1/2}\,dt$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} \beta\left[\frac{\frac{-1}{2}+1}{2}, \frac{0+1}{2}\right]$$

$$= \frac{1}{4\sqrt{2}} \beta\left[\frac{1}{4}, \frac{1}{2}\right]$$

$\therefore$ The given integral is

$$I_1 \times I_2 = \frac{1}{16\sqrt{2}} \beta\left[\frac{3}{4}, \frac{1}{2}\right] \cdot \beta\left[\frac{1}{4}, \frac{1}{2}\right]$$

$$= \frac{1}{16\sqrt{2}} \frac{\Gamma\left(\frac{3}{4}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \cdot \frac{\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\pi}{4\sqrt{2}}.$$

16. *Show that* $\int_{-1}^{1} (1+x)^{p-1} (1-x)^{q-1}\, dx = 2^{p+q-1} \beta(p, q).$

Solution

Substitute: $1 + x = 2t,\ dx = 2dt$

when $x = -1,\ t = 0$, when $x = 1,\ t = 1$

Given Integral,

$$= \int_0^1 (1+x)^{p-1} (1-x)^{q-1}\, dx \qquad \text{(where } x = 2t - 1)$$

$$= \int_0^1 (2t)^{p-1} \left[1 - (2t-1)\right]^{q-1} 2\, dt$$

$$= 2^{p+q-1} \int_0^1 t^{p-1} (1-t)^{q-1}\, dt$$

$$= 2^{p+q-1} \beta(p, q).$$

17. *Show that* $\int_a^b (x-a)^{m-1} (b-x)^{n-1}\, dx = (b-a)^{m+n-1} \beta(m, n).$

Solution

Substitute $x - a = (b - a)\, t$

so that $dx = (b - a)\, dt$

when $x = a,\ t = 0$ and

when $x = b,\ t = 1$

$$\therefore \int_a^b (x-a)^{m-1} (b-x)^{n-1}\, dx = \int_0^1 \left[(b-a)\, t\right]^{m-1} \left[b - a - (b-a)\, t\right]^{n-1} (b-a)\, dt$$

$$= (b-a)^{m+n-1} \int_0^1 t^{m-1}(1-t)^{n-1}\, dt$$

$$= (b-a)^{m+n-1}\, \beta\,(m, n). \quad \textbf{Hence proved.}$$

18. *Prove that* $\displaystyle\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$.

Solution

From the relation,

$$\beta\,(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \qquad \text{...(1)}$$

In the second integral on R.H.S. of (1)

Substitute $x = 1/t$ so that $dx = -dt/t^2$

when $x = 1,\ t = 1$, and

when $x = \infty,\ t = 0$

Hence,
$$\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(\frac{-dt}{t^2}\right)$$

$$= -\int_1^0 \frac{1}{t^{m-1}}\cdot\frac{t^{m+n}}{(1+t)^{m+n}}\cdot\frac{dt}{t^2}$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Therefore from (1), we get

$$\beta\,(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx \quad \textbf{Hence proved.}$$

19. *Prove that* $\displaystyle\int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{\left(a\cos^2\theta + b\sin^2\theta\right)^{m+n}}\,d\theta = \frac{1}{2a^m b^n}\,\beta(m, n).$

Solution. Let I be the given integral,

then
$$I = \int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{\left(\cos^2\theta\right)^{m+n}\left(a + b\tan^2\theta\right)}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^{2n-1}\theta \sec^2\theta\, d\theta}{\left(a + b\tan^2\theta\right)^{m+n}}$$

Substituting $\tan\theta = t$, we get $\sec^2\theta\, d\theta = dt$

when $\theta = 0$, $t = 0$ and

when $\theta = \pi/2$, $t = \infty$

Then
$$I = \int_0^{\infty} \frac{t^{2n+1}}{\left(a + bt^2\right)^{m+n}}\,dt$$

Now substitute
$$bt^2 = ay \text{ or } t = \frac{\sqrt{a}}{\sqrt{b}}\sqrt{y}$$

so that
$$dt = \frac{\sqrt{a}}{\sqrt{b}}\,\frac{dy}{2\sqrt{y}}$$

Limits remain the same.

Hence,
$$I = \int_0^{\infty} \frac{\left(\sqrt{a}\sqrt{y}/\sqrt{b}\right)^{2n-1}}{\left(a + by\right)^{m+n}} \cdot \frac{\sqrt{a}\,dy}{\sqrt{b}\,2\sqrt{y}}$$

$$= \frac{1}{2a^m b^m} \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \frac{1}{2a^m b^m}\,\beta(m, n)$$

where $\beta\,(m,\ n) = \displaystyle\int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$ **Hence proved.**

EXERCISE 3.3

1. Evaluate

(1) $\dfrac{\Gamma(7)}{2\,\Gamma(4)\,\Gamma(3)}$ $\left[\textbf{Ans. } 30\right]$

(2) $\dfrac{\Gamma(3)\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$ $\left[\textbf{Ans. } \dfrac{16}{105}\right]$

(3) $\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)$ $\left[\textbf{Ans. } \dfrac{3\pi\sqrt{\pi}}{8}\right]$

(4) $\dfrac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{4}{3}\right)}$ $\left[\textbf{Ans. } \dfrac{4}{3}\right]$

(5) $\dfrac{\Gamma(3)\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}$. $\left[\textbf{Ans. } \dfrac{16}{315}\right]$

2. Evaluate

(1) $\Gamma\left(\dfrac{-5}{2}\right)$ $\left[\textbf{Ans. } \dfrac{-8\sqrt{\pi}}{15}\right]$

(2) $\Gamma\left(\dfrac{-7}{2}\right)$ $\left[\textbf{Ans. } \dfrac{16\sqrt{\pi}}{105}\right]$

(3) $\Gamma\left(\dfrac{-9}{2}\right)$ $\left[\textbf{Ans. } \dfrac{-32\sqrt{\pi}}{945}\right]$

(4) $\Gamma\left(\dfrac{-1}{3}\right)$. $\left[\textbf{Ans. } -3\,\Gamma\left(\dfrac{2}{3}\right)\right]$

3. Evaluate

(1) $\displaystyle\int_0^\infty x^5 e^{-x}\,dx$ $\left[\textbf{Ans. } 120\right]$

(2) $\displaystyle\int_0^\infty \sqrt{x}\, e^{-x}\,dx$ $\left[\textbf{Ans. } \dfrac{\sqrt{\pi}}{2}\right]$

(3) $\displaystyle\int_0^\infty x^{3/2} e^{-x}\,dx$ $\left[\textbf{Ans. } \dfrac{3\sqrt{\pi}}{4}\right]$

(4) $\displaystyle\int_0^\infty x^3 e^{-2x}\,dx$ $\left[\textbf{Ans. } \dfrac{3}{8}\right]$

(5) $\displaystyle\int_0^\infty x^6 e^{-2x}\,dx$ $\left[\textbf{Ans. } \dfrac{45}{8}\right]$

(6) $\displaystyle\int_0^\infty x^5 e^{-x^2}\,dx$ $\left[\textbf{Ans. } \dfrac{105\sqrt{\pi}}{8}\right]$

(7) $\displaystyle\int_0^\infty e^{-x^3}\,dx$ $\left[\textbf{Ans. } \dfrac{1}{3}\Gamma\left(\dfrac{1}{3}\right)\right]$

(8) $\displaystyle\int_0^1 (\log x)^3\,dx$ $\left[\textbf{Ans. } -6\right]$

(9) $\displaystyle\int_0^1 (x\log x)^4\,dx$ $\left[\textbf{Ans. } \dfrac{94}{625}\right]$

(10) $\displaystyle\int_0^1 \left(\log\frac{1}{x}\right)^{3/2} dx$ $\left[\textbf{Ans. } -2\sqrt{\pi}\right]$

(11) $\displaystyle\int_0^\infty 2^{-3x^2}\,dx$ $\left[\textbf{Ans. } \dfrac{\sqrt{\pi}}{2\sqrt{3\log 2}}\right]$

(12) $\displaystyle\int_0^1 x^2\left(\log\frac{1}{x}\right)^3 dx$. $\left[\textbf{Ans. } \dfrac{2}{27}\right]$

4. Show that $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\, dt = \sqrt{\frac{\pi}{s}}$, $s > 0$.

5. Prove that $\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$, $n > 0$.

6. Prove that $\Gamma(n) = a^n \int_0^\infty e^{-ax} x^{n-1}\, dx$.

7. Prove that $\Gamma(n) = 2a^n \int_0^\infty x^{2n-1} e^{-ax^2}\, dx$.

8. Prove that

(1) $\int_0^\infty x^{n-1} \cos ax\, dx = \frac{1}{a^n}\Gamma(n)\cos\left(\frac{n\pi}{2}\right)$

(2) $\int_0^\infty x^{n-1} \sin ax\, dx = \frac{1}{a^n}\Gamma(n)\sin\left(\frac{n\pi}{2}\right)$.

[**Hint.** choose $a = 0$, $b = a$, in solved example 8.]

9. Evaluate:

(1) $\beta(4, 3)$ $\left[\textbf{Ans. } \frac{1}{60}\right]$

(2) $\beta\left(\frac{3}{2}, \frac{5}{2}\right)$ $\left[\textbf{Ans. } \frac{\pi}{6}\right]$

(3) $\beta\left(\frac{7}{2}, \frac{1}{2}\right)$ $\left[\textbf{Ans. } \frac{5\pi}{16}\right]$

(4) $\beta\left(\frac{1}{4}, \frac{1}{2}\right)$ $\left[\textbf{Ans. } \left\{\Gamma\left(\frac{1}{4}\right)\right\}^2 \Big/ \sqrt{2}\,\pi\right]$

(5) $\beta\left(\frac{5}{6}, \frac{1}{6}\right)$. $[\textbf{Ans. } 2\pi]$

10. Evaluate the following integrals:

(1) $\int_0^1 x^{3/2}(1-x)^{1/2}\, dx$ $\left[\textbf{Ans. } \frac{\pi}{16}\right]$

(2) $\int_0^1 \sqrt{\frac{x}{1-x}}\, dx$ $\left[\textbf{Ans. } \frac{\pi}{2}\right]$

(3) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx$ $\left[\textbf{Ans. } \frac{64\sqrt{2}}{15}\right]$

(4) $\int_0^4 u^{3/2}(4-u)^{5/2}\, du$ $[\textbf{Ans. } 12\pi]$

(5) $\int_0^3 \frac{dx}{\sqrt{3x - x^2}}$ $[\textbf{Ans. } \pi]$

(6) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ $\left[\textbf{Ans. } \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}\right]$

(7) $\int_0^1 x^3\left(1-\sqrt{x}\right)dx$ $\left[\textbf{Ans. } \dfrac{1}{21}\right]$ (8) $\int_0^a x^4\sqrt{a^2-x^2}\,dx$ $\left[\textbf{Ans. } \dfrac{\pi a^6}{32}\right]$

(9) $\int_0^2 x\left(8-x^3\right)^{1/3}dx$ $\left[\textbf{Ans. } \dfrac{16\pi}{9\sqrt{3}}\right]$ (10) $\int_0^1 \sqrt{1-x^4}\,dx$ $\left[\textbf{Ans. } \dfrac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]$

11. Evaluate each of the following integrals:

(1) $\int_0^{\pi/2}\sin^5\theta\,d\theta$ $\left[\textbf{Ans. } \dfrac{8}{15}\right]$ (2) $\int_0^{\pi/2}\cos^7\theta\,d\theta$ $\left[\textbf{Ans. } \dfrac{16}{35}\right]$

(3) $\int_0^{\pi/2}\sin^3\theta\cos^4\theta\,d\theta$ $\left[\textbf{Ans. } \dfrac{2}{105}\right]$ (4) $\int_0^{\pi/2}\sin^{1/2}\theta\cos^{7/2}\theta\,d\theta$ $\left[\textbf{Ans. } \dfrac{5\sqrt{2}\,\pi}{64}\right]$

(5) $\int_0^{\pi/2}\sin^{1/3}\theta\cos^{-1/3}\theta\,d\theta$ $\left[\textbf{Ans. } \dfrac{\pi}{\sqrt{3}}\right]$ (6) $\int_0^{\pi/2}\dfrac{d\theta}{\sqrt{\cos\theta}}$ $\left[\textbf{Ans. } \dfrac{1}{2}\dfrac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]$

(7) $\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta$ $\left[\textbf{Ans. } \dfrac{\pi}{\sqrt{2}}\right]$ (8) $\int_0^{\pi/2}\dfrac{\sqrt[3]{\sin^8\theta}}{\sqrt{\cos\theta}}\,d\theta$ $\left[\textbf{Ans. } \dfrac{60}{13}\dfrac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\sqrt{\pi}}\right]$

(9) $\int_0^{\pi/2}\tan^q\theta\,d\theta,\ 0<p<1.$ $\left[\textbf{Ans. } \dfrac{\pi}{2}\sec\left(\dfrac{p\pi}{2}\right)\right]$

12. Evaluate each of the following integrals:

(1) $\int_0^\infty \dfrac{dx}{1+x^4}$ $\left[\textbf{Ans. } \dfrac{\pi\sqrt{2}}{4}\right]$ (2) $\int_0^\infty \dfrac{dx}{\sqrt{x}\,(1+x)}$ $[\textbf{Ans. } \pi]$

(3) $\int_0^\infty \dfrac{x^2\,dx}{\left(1+x^4\right)^3}.$ $\left[\textbf{Ans. } \dfrac{5\sqrt{2}\,\pi}{128}\right]$

13. Evaluate:

(1) $\int_1^3 \dfrac{dx}{\sqrt{(x-1)(3-x)}}$ $[\textbf{Ans. } \pi]$

(2) $\int_0^7 \sqrt[4]{(7-x)(x-3)}\,dx$ $\left[\textbf{Ans. } \dfrac{2\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{3\sqrt{\pi}}\right]$

14. Show that

(1) $\int_0^\infty x\, e^{-x^2}\, dx \times \int_0^\infty x^2\, e^{-x^4}\, dx = \dfrac{\pi}{16\sqrt{2}}$

(2) $\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx \times \int_0^\infty \dfrac{e^{-x^2}}{\sqrt{x}}\, dx = \pi$

(3) $\int_0^\infty x^2\, e^{-x^4}\, dx \times \int_0^\infty e^{-x^4}\, dx = \dfrac{\pi}{8\sqrt{2}}$

(4) $\int_0^\infty \sin^p \theta\, d\theta \times \int_0^\infty \sin^{p+1} \theta\, d\theta = \dfrac{\pi}{2\left(P+1\right)}$.

15. Prove that $\int_0^\infty \dfrac{x^{n-1}}{\left(x+a\right)^{m+n}}\, dx = \dfrac{1}{a^n}\, \beta\left(m, n\right)$.

ADDITIONAL PROBLEMS *(From Previous Years VTU Exams.)*

1. *Evaluate* $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y\, dx\, dy.$

Solution. We have

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} x^3 y\, dx\, dy$$

i.e.,

$$I = \int_{y=0}^{1} y \left[\frac{x^4}{4}\right]_{x=0}^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{4} \int_{y=0}^{1} y \left(1-y^2\right)^2 dy$$

$$= \frac{1}{4} \int_{y=0}^{1} y \left(1-2y^2+y^4\right) dy$$

$$= \frac{1}{4} \int_{y=0}^{1} \left(y-2y^3+y^5\right) dy$$

$$= \frac{1}{4} \left[\frac{y^2}{2} - \frac{2y^4}{4} + \frac{y^6}{6}\right]_{y=0}^{1}$$

$$= \frac{1}{4}\left[\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right]$$

$$I = \frac{1}{24}.$$

2. *Change the order of integration and hence evaluate* $\int_0^1 \int_{\sqrt{y}}^1 dx\, dy$.

Solution

Let $\quad I = \int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} dx\, dy$

On changing the order of integration,

$$x = \sqrt{y} \quad \Rightarrow \quad x^2 = y$$

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} dy\, dx$$

$$= \int_{x=0}^{1} [y]_0^{x^2}\, dx = \int_{x=0}^{1} x^2\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$I = \frac{1}{3}.$$

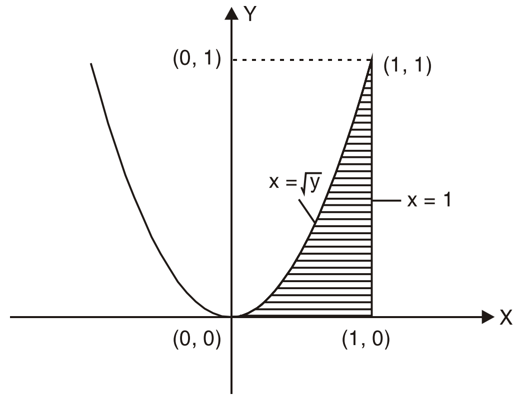

Fig. 3.1

3. *Change the order of integration and evaluate* $\int_0^3 \int_0^{\sqrt{4-y}} (x+y)\, dx\, dy$.

Solution. Refer page no. 122. Example 5.

4. *Change the order of integration and hence evaluate* $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\, dy\, dx$.

Solution. We have

$$I = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} xy\, dy\, dx$$

We have $\quad \dfrac{x^2}{4a} = 2\sqrt{ax} \quad$ or $\quad x^4 = 64a^3x$

i.e., $\;x(x^3 - 64x^3) = 0 \qquad \Rightarrow \quad x = 0$ and $x = 4a$

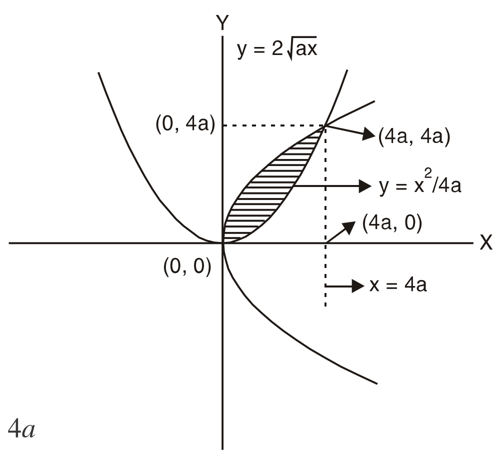

Fig. 3.2

From $y = \dfrac{x^2}{4a}$, we get $y = 0$ and $y = 4a$.

Thus the points of intersection of the parabola $y = \dfrac{x^2}{4a}$ and $y = 2\sqrt{ax}$ are (0, 0) and ($4a$, $4a$) on changing the order of integration we have y varying from 0 to $4a$ and x varying from $\dfrac{y^2}{4a}$ to $2\sqrt{ay}$.

Thus

$$I = \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} xy \, dx \, dy$$

$$= \int_{y=0}^{4a} y \cdot \left[\frac{x^2}{2}\right]_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_{y=0}^{4a} y\left[4ay - \frac{y^4}{16a^2}\right] dy$$

$$= \frac{1}{2}\int_{y=0}^{4a} \left(4ay^2 - \frac{y^5}{16a^2}\right) dy$$

$$= \frac{1}{2}\left[\frac{4ay^3}{3} - \frac{1}{16a^2}\cdot\frac{y^6}{6}\right]_{y=0}^{4a}$$

$$= \frac{1}{2}\left[4a\left(\frac{64a^3}{3}\right) - \frac{1}{96a^2}\left(4096\, a^6\right)\right]$$

$$= \frac{1}{2}\left[\frac{256a^3}{3} - \frac{128a^4}{3}\right]$$

$$= \frac{64a^4}{3}$$

$$I = \frac{64a^4}{3}.$$

5. *Find the value of* $\iint xy\left(x+y\right) dx\, dy$ *taken over the region enclosed by the curves* $y = x$ *and* $y = x^2$.

Solution. Refer page no. 118. Example 6.

6. *Change the order of integration and hence evaluate* $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$.

Solution. Refer page no. 120. Example 2.

7. *Change the order of integration and hence evaluate* $\int_{y=0}^{3}\int_{x=1}^{4-y}(x+y)\,dx\,dy$.

Solution. Refer page no. 121. Example 4.

8. *With usual notation show that* $\beta(m, n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$.

Solution. Refer page no. 133.

9. *Show that* $\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}\,dx = 2^{p+q-1}\,\beta\,(p,\,q)$.

Solution. Refer page no. 153. Example 16.

10. *Using Beta and Gamma functions evaluate* $\int_{0}^{\pi/2}\sqrt{\sin\theta}\,d\theta \times \int_{0}^{\pi/2}\dfrac{d\theta}{\sqrt{\sin\theta}}$.

Solution. Refer page no. 150. Example 13.

OBJECTIVE QUESTIONS

1. The area bounded by the curves $y^2 = x - 1$ and $y = x - 3$ is

(*a*) 3 (*b*) $\dfrac{7}{2}$

(*c*) $\dfrac{9}{2}$ (*d*) $\dfrac{7}{3}$. **[Ans. *c*]**

2. The volume of the tetrahedron bounded by the coordinate planes and the plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c} = 1$ is

(*a*) $\dfrac{abc}{2}$ (*b*) $\dfrac{abc}{3}$

(*c*) $\dfrac{abc}{6}$ (*d*) $\dfrac{24}{abc}$. **[Ans. *c*]**

3. For $\int_0^\infty\int_x^\infty f(x, y)\,dx\,dy$, the change or order is

(*a*) $\int_x^\infty\int_0^\infty f(x, y)\,dx\,dy$ (*b*) $\int_0^\infty\int_y^\infty f(x, y)\,dx\,dy$

(*c*) $\int_0^\infty\int_0^y f(x, y)\,dx\,dy$ (*d*) $\int_0^\infty\int_0^x f(x, y)\,dx\,dy$. **[Ans. *c*]**

4. The value of the integral $\int_{-2}^{2} \frac{dx}{x^2}$ is

(*a*) 0 (*b*) 0.25

(*c*) 1 (*d*) ∞. **[Ans.** *d***]**

5. $\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$ is equal to

(*a*) $\frac{3}{4}$ (*b*) $\frac{3}{8}$

(*c*) $\frac{3}{5}$ (*d*) $\frac{3}{7}$. **[Ans.** *b***]**

6. $\int_0^2 \int_0^x (x+y)\, dx\, dy =$ ....

(*a*) 4 (*b*) 3

(*c*) 5 (*d*) None of these. **[Ans.** *a***]**

7. $\int_0^1 \int_0^{1-x} dx\, dy$ represents.....

[Ans. Area of the triangle having vertices (0, 0), (0, 1), (1, 0)**]**

8. $\int_0^\infty e^{-x^2} dx$ = ...

(*a*) $\frac{\sqrt{\pi}}{2}$ (*b*) $\frac{\pi}{2}$

(*c*) $\frac{\pi}{4}$ (*d*) None of these. **[Ans.** *a***]**

9. $\beta\left(\frac{1}{2}, \frac{1}{2}\right)$ = ...

(*a*) 3.1416 (*b*) 5.678

(*c*) 2 (*d*) None of these. **[Ans.** *a***]**

10. $\Gamma(3.5)$ = ...

(*a*) $\frac{15}{8}$ (*b*) $\frac{15}{8}\sqrt{\pi}$

(*c*) $\frac{10}{7}$ (*d*) None of these. **[Ans.** *b***]**

11. The surface area of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 8z - 2 = 0$ is

(a) $59\ \pi$ (b) $60\ \pi$

(c) $92\ \pi$ (d) None of these. **[Ans.** c**]**

12. $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dz\, dy\, dx$ = ...

(a) 26 (b) 28

(c) 30 (d) 50. **[Ans.** a**]**

13. $\iint_R dx\, dy$ is

(a) Area of the region R in the Cartesian form

(b) Area of the region R in the Polar form

(c) Volume of a solid

(d) None of these. **[Ans.** a**]**

14. $\Gamma\left(\frac{1}{2}\right)$ is

(a) $\sqrt{\pi}$ (b) $\frac{\sqrt{\pi}}{2}$

(c) 2 (d) None of these. **[Ans.** a**]**

15. $\Gamma\left(\frac{-7}{2}\right)$ is

(a) $\frac{16}{15}$ (b) $\frac{16}{315}$

(c) $\frac{16}{18}$ (d) None of these. **[Ans.** b**]**

16. $\Gamma\ (n + 1)$ is

(a) n (b) $n + 1$

(c) $(n + 1)\ !$ (d) $n\ !$. **[Ans.** d**]**

17. $\beta\left(\frac{7}{2}, \frac{-1}{2}\right)$ is

(a) $\frac{-15\pi}{8}$ (b) $\frac{15}{8}$

(c) $\frac{\pi}{8}$ (d) $\frac{15\pi}{8}$. **[Ans.** a**]**

18. $\int_0^{\infty} x^{\frac{3}{2}} e^{-x}\, dx$ is

(*a*) $\frac{3}{4}$ (*b*) $\frac{3\sqrt{\pi}}{4}$

(*c*) $\frac{\sqrt{\pi}}{4}$ (*d*) None of these. **[Ans.** *b***]**

19. $\int_0^{\pi/2} \sin^6 \theta\, d\,\theta$ is

(*a*) $\frac{5\pi}{32}$ (*b*) $\frac{5}{32}$

(*c*) $\frac{\pi}{32}$ (*d*) None of these. **[Ans.** *a***]**

20. $\int_0^{\pi/2} \sin^4 \theta \cos^3 \theta\, d\,\theta$ is

(*a*) $\frac{5\pi}{32}$ (*b*) $\frac{16}{35}$

(*c*) $\frac{2}{35}$ (*d*) $\frac{6}{35}$. **[Ans.** *c***]**

❑❑❑

UNIT IV

Vector Integration and Orthogonal Curvilinear Coordinates

4.1 INTRODUCTION

In the chapter we shall define line integrals, surface integrals and volume integrals which play very important role in Physical and Engineering problems. We shall show that a line integral is a natural generalization of a definite integral and surface integral is a generalization of a double integral.

Line integrals can be transformed into double integrals or into surface integrals and conversely. Triple integrals can be transformed into surface integrals. The corresponding integral theorems of Gauss, Green and Stokes are discussed.

The concept of Gradient, Divergence, Curl and Laplacian already discussed in the known Cartesian system. These will be discussed in a general prospective in the topic orthogonal curvilinear coordinates.

4.2 VECTOR INTEGRATION

4.2.1 Vector Line Integral

If $\vec{F}$ is a force acting on a particle at a point P whose positive vector is r on a curve C then $\int_C \vec{F} \cdot \vec{dr}$ represents physically the total work done in moving the particle along C.

Thus, total work done is $\int_C \vec{F} \cdot \vec{dr} = 0$

WORKED OUT EXAMPLES

1. *If* $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$. *Evaluate* $\int_C \vec{F} \cdot \vec{dr}$ *along the curve* $y = x^3$ *in the x-y plane from (1, 1) to (2, 8).*

Solution. We have $\vec{F} = (5xy - 6x^2)\,\vec{i} + (2y - 4x)\,\vec{j}$ and $\vec{r} = xi + yj$ will give

$$\vec{dr} = dxi + dyj$$

$$\therefore \quad \vec{F} \cdot \vec{dr} = (5xy - 6x^2)\, dx + (2y - 4x)\, dy$$

Since $y = x^3$ we have $dy = 3x^2\, dx$ and varies from 1 to 2

$$\int_C \vec{F} \cdot \vec{dr} = (5x \cdot x^3 - 6x^2)\, dx + (2 \cdot x^3 - 4x) \cdot 3x^2\, dx$$

$$= \int_1^2 \left(5x^4 - 6x^2 + 6x^5 - 12x^3\right) dx$$

$$= \left[x^5 - 2x^3 + x^6 - 3x^4\right]_1^2 = 35$$

2. *If* $\vec{F} = (3x^2 + 6y)\, \vec{i} - 14yz\, \vec{j} + 20\, xz^2\, \vec{k}$. *Evaluate* $\int_C \vec{F} \cdot \vec{dr}$ *from (0, 0, 0) to (1, 1, 1) along the path* $x = t,\ y = t^2,\ z = t^3$.

Solution $\quad \vec{F} = (3x^2 + 6y)\, \vec{i} - 14yz\vec{j} + 20xz^2\, \vec{k}.$

$$\vec{dr} = dxi + dyj + dzk$$

$$\therefore \quad \vec{F} \cdot \vec{dr} = (3x^2 + 6y)\, dx - 14yz\, dy + 20xz^2 dz$$

Since $\quad x = t,\ y = t^2,\ z = t^3$

we obtain $\quad dx = dt,\ dy = 2t\, dt,\ dz = 3t^2\, dt$

$$\therefore \quad \vec{F} \cdot \vec{dr} = (3t^2 + 6t^2)\, dt - (14t^5)\, 2tdt + (20t^7)\, 3t^2\, dt$$

i.e., $\quad \vec{F} \cdot \vec{dr} = (9t^2 - 28t^6 + 60t^9)\, dt\ ;\ 0 \le t \le 1$

$\therefore$ t varies from 0 to 1

$$\int_C \vec{F} \cdot \vec{dr} = \int_{t=0}^{1} \left(9t^2 - 28t^6 + 60t^9\right) dt$$

$$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1$$

$$= 3 - 4 + 6$$

$$= 5.$$

3. *Evaluate:* $\int_C \vec{F} \cdot \vec{dr}$ *where* $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ *and C is given by* $\vec{r} = t\,\vec{i} + t^2\,\vec{j} + t^3\vec{k}; 0 \le t \le 1.$

Solution $\quad \vec{F} = yz\,\vec{i} + zx\vec{j} + xy\vec{k}$

$$\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$$

$$\therefore \quad \vec{dr} = dt\vec{i} + 2tdt\,\vec{j} + 3t^2 dt\,\vec{k}$$

$$\therefore \quad \vec{F} \cdot \vec{dr} = yz\, dt + zx \times 2tdt + xy3t^2.dt$$

where $\quad \vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\therefore \quad x = t, \quad y = t^2, \quad z = t^3$$

$$\therefore \quad \vec{F}\cdot\vec{dr} = t^5\,dt + 2t^5\,dt + 3t^5\,dt$$

$$= (t^5 + 2t^5 + 3t^5)\,dt$$

$$\therefore \quad \vec{F}\cdot\vec{dr} = 6t^5\,dt$$

$\therefore$ t varies from 0 to 1

$$\int_C \vec{F}\cdot\vec{dr} = \int_0^1 6t^5\,dt$$

$$= \left[6\frac{t^6}{6}\right]_0^1$$

$$= \left[t^6\right]_0^1 = 1 - 0$$

$$= 1.$$

4. *Evaluate* $\int_C \vec{F}\cdot\vec{dr}$ *where* $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ *and C is given by* $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le \pi$.

Solution $\quad \vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

Here,

$$x = \cos t \quad \Rightarrow \quad dx = -\sin t\,dt$$

$$y = \sin t \quad \Rightarrow \quad dy = \cos t\,dt$$

$$z = t \quad \Rightarrow \quad dz = dt$$

$$\vec{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k},\ 0 \le t \le \pi$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_{t=0}^{\pi}\left(x^2dx + y^2dy + z^2dz\right)$$

$$= \int_0^{\pi}\left\{-\cos^2 t\,\sin t\,dt + \sin^2 t\cos t\,dt + t^2dt\right\}$$

$$= \left[\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3} + \frac{t^3}{3}\right]_0^{\pi}$$

$$= -\frac{1}{3} - \frac{1}{3} + 0 + \frac{\pi^2}{3} = \frac{\pi^2 - 2}{3}$$

5. *If* $\vec{F} = x^2\vec{i} + xy\vec{j}$. *Evaluate* $\int_C \vec{F}\cdot\vec{dr}$ *from (0, 0) to (1, 1) along (i) the line* $y = x$ *(ii) the parabola* $y = \sqrt{x}$.

Solution $\quad \vec{F} = x^2\vec{i} + xy\vec{j}$

$$\vec{dr} = dx\vec{i} + dy\vec{j}$$

$$\vec{F}\cdot\vec{dr} = x^2dx + xy\,dy$$

(i) Along $y = x$; we have $0 \le x \le 1$ and $dy = dx$.

$$\therefore \quad \int_C \vec{F} \cdot \vec{dr} = \int_{x=0}^{1} \left(x^2 + x^2\right) dx$$

$$= \int_{x=0}^{1} 2x^2\, dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}$$

(ii) Along $y = \sqrt{x}$, $y^2 = x \Rightarrow 2y\, dy = dx$, $0 \le y \le 1$

$$\int_C \vec{F} \cdot \vec{dr} = \int_{y=0}^{1} \left(2y^5 + y^3\right) dy$$

$$= \left[\frac{y^6}{3} + \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

6. *Use the line integral, compute work done by a force* $\vec{F} = (2y+3)\vec{i} + xz\,\vec{j} + (yz - x)\vec{k}$ *when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve* $x = 2t^2$, $y = t$, $z = t^3$.

Solution Work done $= \int_C \vec{F} \cdot \vec{dr}$

where, $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$

Here,

$$x = 2t^2 \quad \Rightarrow \quad dx = 4t\ dt$$

$$y = t \quad \Rightarrow \quad dy = dt$$

$$z = t^3 \quad \Rightarrow \quad dz = 3t^2\ dt$$

t varies from 0 to 1 $(\because\ y = t)$

$$\vec{dr} = dx\,\vec{i} + dy\,\vec{j} + dz\vec{k}$$

$$= 4tdt\,\vec{i} + dt\,\vec{j} + 3t^2 dt\,\vec{k}$$

$$\int_C \vec{F} \cdot \vec{dr} = \int_{t=0}^{1} \left\{(2t+3)4t\ dt + 2t^5 dt + \left(t^4 - 2t^2\right) 3t^2 dt\ \right\}$$

$$= \int_0^1 \left(12t + 8t^2 - 6t^4 + 2t^5 + 3t^6\right) dt$$

$$= \left[6t^2 + \frac{8}{3}t^3 - \frac{6}{5}t^5 + \frac{1}{3}t^6 + \frac{3}{7}t^7\right]_0^1$$

$$= \frac{288}{35}.$$

7. *Find the work done in moving a particle once around an ellipse C in the xy-plane, if the ellipse has centre at the origin with semi-major axis 4 and semi-minor axis 3 and if the force field is given by*

$$\vec{F} = \left(3x-4y+2z\right)\vec{i}+\left(4x+2y-3z^2\right)\vec{j}+\left(2xz-4y^2+z^3\right)\vec{k}.$$

Solution. Here path of integration C is the ellipse whose equation is $\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$ and its parametric equations are $x = 4\cos t$, $y = 3\sin t$. Also t varies from 0 to 2π since C is a curve in the xy-plane, we have $z = 0$

$$\therefore \quad \vec{F} = \left(3x-4y\right)\vec{i}+\left(4x+2y\right)\vec{j}$$

and

$$\vec{dr} = dxi + dyj$$

$$\vec{F}\cdot\vec{dr} = \left[\left(3x-4y\right)i+\left(4x+2y\right)j\right]\cdot\left[dxi+dyj\right]$$

$$\vec{F}\cdot\vec{dr} = (3x-4y)\,dx + (4x+2y)\,dy$$

$$x = 4\cos t \quad \Rightarrow \quad dx = -4\sin t\,dt$$

$$y = 3\sin t \quad \Rightarrow \quad dy = 3\cos t\,dt$$

$\therefore$ t varies from 0 to 2π

$$\int_C \vec{F}\cdot\vec{dr} = \int_0^{2\pi}\left\{\left(12\cos t-12\sin t\right)\left(-4\sin t\right)dt+\left(16\cos t+6\sin t\right)\cdot 3\cos t\,dt\right\}$$

$$= \int_0^{2\pi}\left(48-30\sin t\cos t\right)dt \qquad \left(\because \sin t\cos t = \frac{\sin 2t}{2}\right)$$

$$= \int_0^{2\pi}\left(48-15\sin 2t\right)dt$$

$$= \left[48t+\frac{15}{2}\cos 2t\right]_0^{2\pi}$$

$$= 96\pi.$$

8. *If* $\vec{F} = xy\vec{i}-z\vec{j}+x^2\vec{k}$. *Evaluate* $\int_C \vec{F}\times\vec{dr}$ *where C is the curve* $x = t^2$, $y = 2t$, $z = t^3$ *from* $t = 0$ *to* 1.

Solution

$$\vec{F} = xy\vec{i}-z\vec{j}+x^2\vec{k} \text{ and}$$

$$\vec{dr} = dx\vec{i}+dy\vec{j}+dz\vec{k}$$

Hence

$$\vec{F}\times\vec{dr} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ xy & -z & x^2 \\ dx & dy & dz \end{vmatrix}$$

$$\vec{F}\cdot\vec{dr} = -(zdz + x^2dy)\ \vec{i} - (xy\ dz - x^2dx)\ \vec{j} + (xydy + zdx)\ \vec{k}$$

where
$$x = t^2 \Rightarrow dx = 2tdt$$
$$y = 2t \Rightarrow dy = 2dt$$
$$z = t^3 \Rightarrow dz = 3t^2dt$$

t varies from 0 to 1

$$= \left\{-\left(3t^5 + 2t^4\right)\vec{i} - 4t^5\vec{j} + \left(4t^3 + 2t^4\right)\vec{k}\right\}dt$$

$$\therefore \quad \int_C \vec{F}\cdot\vec{dr} = -\vec{i}\int_0^1 \left(3t^5 + 2t^4\right)dt - 4\vec{j}\int_0^1 t^5 dt + \vec{k}\int_0^1\left(4t^3 + 2t^4\right)dt$$

$$= -\vec{i}\left[\frac{3t^6}{6} + \frac{2t^5}{5}\right]_0^1 - 4\vec{j}\left[\frac{t^6}{6}\right]_0^1 + \vec{k}\left[\frac{4t^4}{4} + \frac{2t^5}{5}\right]_0^1$$

$$= \frac{-9}{10}\vec{i} - \frac{2}{3}\vec{j} + \frac{7}{5}\vec{k}.$$

9. *If* $\phi = 2xyz^2$, $\vec{F} = xy\vec{i} - z\vec{j} + x^2\vec{k}$ and C is the curve: $x = t^2$, $y = 2t$, $z = t^3$ *from* $t = 0$ *to* $t = 1$ *evaluate the following line integrals. (i)* $\int_C \phi\cdot\vec{dr}$ *(ii)* $\int_C \vec{F}\times\vec{dr}$.

Solution
$$\vec{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$
$$x = t^2 \Rightarrow dx = 2tdt$$
$$y = 2t \Rightarrow dy = 2dt$$
$$z = t^3 \Rightarrow dz = 3t^2dt$$

$$\therefore \quad \vec{dr} = \left(2t\vec{i} + 2\vec{j} + 3t^2\vec{k}\right)dt$$

$$\phi = 2xyz^2$$
$$\phi = 2\cdot t^2\cdot 2t\cdot t^6 = 4t^9$$

$$\therefore \quad \phi\cdot d\vec{r} = (8t^{10}\ i + 8t^9\ j + 12t^{11}\ k)\ dt$$

$$(i) \quad \int_C \phi\cdot d\vec{r} = \int_{t=0}^{1}\left(8t^{10}i + 8t^9 j + 12t^{11}k\right)dt$$

$$= \left[\frac{8t^{11}}{11}\right]_0^1 i + \left[\frac{8t^{10}}{10}\right]_0^1 j + \left[\frac{12t^{12}}{12}\right]_0^1 k$$

Thus,
$$\int_C \phi\cdot d\vec{r} = \frac{8}{11}\vec{i} + \frac{4}{5}\vec{j} + \vec{k}$$

$$(ii) \quad \vec{F} = 2t^3\vec{i} - t^3\vec{j} + t^4\vec{k}$$

$$\vec{F}\times\vec{dr} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix}$$

$$= -(3t^5 + 2t^4)\,\vec{i} - (6t^5 - 2t^5)\,\vec{j} + (4t^3 + 2t^4)\,\vec{k}$$

$$= -(3t^5 + 2t^4)\,\vec{i} - 4t^5\,\vec{j} + (4t^3 + 2t^4)\,\vec{k}$$

$$\int_C \vec{F} \times \vec{dr} = \int_0^1 \left\{\left(3t^5 + 2t^4\right)\vec{i} - 4t^5\vec{j} + \left(4t^3 + 2t^4\right)\vec{k}\right\} dt$$

$$= -\left[\frac{t^6}{2} + \frac{2t^5}{5}\right]_0^1 \vec{i} - 4\left[\frac{t^6}{6}\right]_0^1 \vec{j} + \left[t^4 + \frac{2t^5}{5}\right]_0^1 \vec{k}$$

$$\int_C \vec{F} \times \vec{dr} = -\frac{9}{10}\vec{i} - \frac{2}{3}\vec{j} + \frac{7}{5}\vec{k}.$$

EXERCISE 4.1

1. If $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$. Evaluate $\int_C \vec{F} \cdot \vec{dr}$ along the curve C given by $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$. **[Ans.** 303**]**

2. Evaluate $\int_C \vec{F} \cdot \vec{dr}$ where $\vec{F} = (2x + y)\vec{i} + (3y - x)\vec{j} + yz\,\vec{k}$ and C is the curve $x = 2t^2$, $y = t$, $z = t^3$ from $t = 0$ to $t = 2$. $\left[\textbf{Ans. } \frac{227}{42}\right]$

3. Evaluate $\int_C \vec{F} \cdot \vec{dr}$ where $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and C is the portion of the curve $r = a\cos t\vec{i} + b\sin t\vec{j} + ct\vec{k}$ from $t = 0$ to $t = \frac{\pi}{2}$. **[Ans.** 0**]**

4. Evaluate $\int_C \vec{F} \cdot \vec{dr}$ where $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and C is the arc of the curve $r = ti + t^2 j + t^3 k$ from $t = 0$ to $t = 1$. **[Ans.** 1**]**

5. Find the total work done in moving a particle once round a circle C in the xy-plane if the curve has centre at the origin and radius 3 and the force field is given by

$$\vec{F} = (2x - y + z)\,\vec{i} + (x + y - z^2)\,\vec{j} + (3x - 2y + 4z)\,\vec{k}.$$ **[Ans.** 8π**]**

6. If $\vec{F} = 2y\vec{i} - z\vec{j} + x\vec{k}$ and C is the circle $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \frac{\pi}{2}$ evaluate $\int_C \vec{F} \times d\vec{k}$. $\left[\textbf{Ans. } \left(2 - \frac{\pi}{4}\right)i + \left(\pi - \frac{1}{2}\right)j\right]$

4.3 INTEGRAL THEOREM

4.3.1 Green's Theorem in a Plane

This theorem gives the relation between the plane, surface and the line integrals.

Statement. If R is a closed region in the xy-plane bounded by a simple closed curve C and $M(x, y)$ and $N(x, y)$ are continuous functions having the partial derivatives in R then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

4.3.2 Surface integral and Volume integral

Surface Integral

An integral evaluated over a surface is called a surface integral. Consider a surface S and a point P on it. Let $\vec{A}$ be a vector function of x, y, z defined and continuous over S.

In $\hat{n}$ is the unit outward normal to the surface S and P then the integral of the normal component of $\vec{A}$ at P (*i.e.*, $\vec{A} \cdot \hat{n}$) over the surface S is called the surface integral written as

$$\iint_S A \cdot \hat{n}\, ds$$

where ds is the small element area. To evaluate integral we have to find the double integral over the orthogonal projection of the surface on one of the coordinate planes.

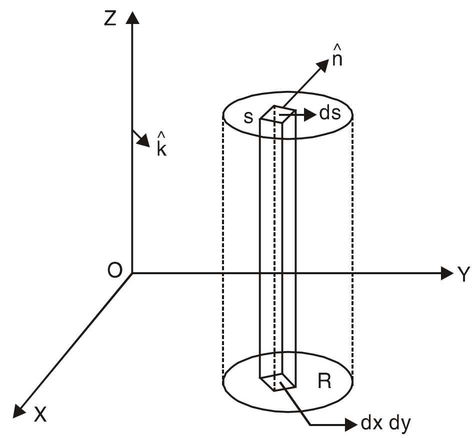

Fig. 4.1

Suppose R is the orthogonal projection of S on the XOY plane and $\hat{n}$ is the unit outward normal to S then it should be noted that $\hat{n} \cdot \hat{k}\, ds$ ($\hat{k}$ being the unit vector along z-axis) is the projection of the vectorial area element $\hat{n}\, ds$ on the XOY plane and this projection is equal to $dx\,dy$ which being the area element in the XOY plane. That is to say that $\hat{n} \cdot \hat{k}\, ds = dx\,dy$. Similarly, we can argue to state that $\hat{n} \cdot \hat{j}\, ds = dz\,dx$ and $\hat{n} \cdot \hat{i}\, ds = dy\,dz$. All these three results hold good if we write $\hat{n}\, ds = dy\,dz\,i + dz\,dx\,j + dx\,dy\,k$.

Sometimes we also write

$$d\vec{s} = \hat{n}\,ds = \sum dy\,dz\,i$$

Volume Integral

If V is the volume bounded by a surface and if $F(x, y, z)$ is a single valued function defined over V then the volume integral of $F(x, y, z)$ over V is given by $\iiint_V F\,dv$. If the volume is divided into sub-elements having sides dx, dy, dz then the volume integral is given by the triple integral $\iiint F(x, y, z)\,dx\,dy\,dz$ which can be evaluated by choosing appropriate limits for x, y, z.

4.3.3 Stoke's Theorem

Statement. If S is a surface bounded by a simple closed curve C and if $\vec{F}$ is any continuously differentiable vector function then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \hat{n}\, ds = \iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{n}\, ds$$

4.3.4 Gauss Divergence Theorem

Statement. If V is the volume bounded by a surface S and $\vec{F}$ is a continuously differentiable vector function then

$$\iiint_V \text{div } \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, dS$$

where $\hat{n}$ is the positive unit vector outward drawn normal to S.

WORKED OUT EXAMPLES

1. *Verify Green's theorem in the plane for* $\int_C \left[\left(3x^2 - 8y^2\right) dx + \left(4y - 6xy\right) dy\right]$ *where C is the boundary for the region enclosed by the parabola* $y^2 = x$ *and* $x^2 = y$.

Solution. We shall find the points of intersection of the parabolas

$$y^2 = x \text{ and } x^2 = y$$

i.e., $$y = \sqrt{x} \text{ and } y = x^2$$

Equating both, we get

$$\sqrt{x} = x^2 \Rightarrow x = x^4$$

or $$x - x^4 = 0$$

$$x\,(1 - x^3) = 0$$

$\therefore$ $$x = 0,\ 1$$

and hence $y = 0, 1$ the points of intersection are (0, 0) and (1, 1).

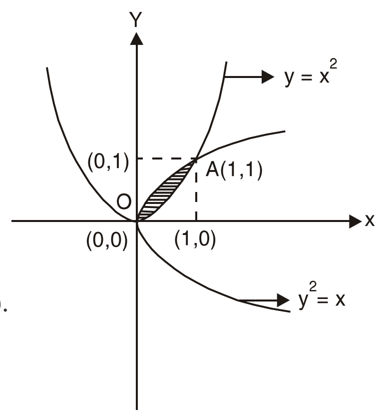

Fig. 4.2

Let $$M = 3x^2 - 8y^2,\ N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y \quad \frac{\partial N}{\partial x} = -6y$$

By Green's theorem,

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$\text{L.H.S} = \int_C Mdx + Ndy$$

$$= \int_{OA} \left(M\, dx + N\, dy\right) + \int_{AO} \left(M\, dx + N\, dy\right) = I_1 + I_2$$

Along OA: $y = x^2 \quad dy = 2x dx,$

x varies from 0 to 1

$$I_1 = \int_{x=0}^{1} \left(3x^2 - 8x^4\right) dx + \left(4x^2 - 6x^3\right) 2x\, dx$$

$$= \int_{x=0}^{1} \left(3x^2 + 8x^3 - 20x^4\right) dx$$

$$= \left[x^3 + 2x^4 - 4x^5\right]_0^1 = -1$$

Along AO: $y^2 = x \Rightarrow dx = 2y\, dy,$

y varies from 1 to 0

$$I_2 = \int_{y=1}^{0} \left(3y^4 - 8y^2\right) 2y\, dy + \left(4y - 6y^3\right) dy$$

$$= \int_{1}^{0} \left(4y - 22y^3 + 6y^5\right) dy$$

$$= \left[2y^2 - \frac{11}{2} y^4 + y^6\right]_1^0 = \frac{5}{2}$$

Hence, $\text{L.H.S.} = I_1 + I_2 = -1 + \frac{5}{2} = \frac{3}{2}$

Also $\text{R.H.S.} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (-6y + 16y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y\, dy\, dx$$

$$= \int_{x=0}^{1} \left[\frac{10y^2}{2}\right]_{y=x^2}^{\sqrt{x}} dx$$

$$= 5\int_{x=0}^{1} \left(x - x^4\right) dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_{x=0}^{1}$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{2}$$

$\therefore$ $\text{L.H.S.} = \text{R.H.S.} = \frac{3}{2}.$ **Hence verified.**

2. *Verify Green's theorem in the plane for* $\int_C \left\{\left(xy + y^2\right) dx + x^2 dy\right\}$ *where C is the closed curve of the region bounded by* $y = x$ *and* $y = x^2$.

Solution. We shall find the points of intersection of $y = x$ and $y = x^2$.

Equating the R.H.S.

$\therefore \quad x = x^2 \Rightarrow \quad x - x^2 = 0$

$x(1 - x) = 0$

$x = 0, 1$

$\therefore \quad y = 0, 1$ and hence (0, 0), (1, 1) are the points of intersection.

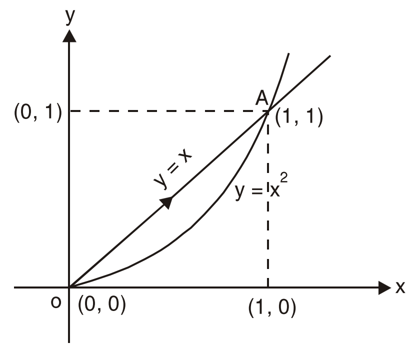

Fig. 4.3

We have Green's theorem in a plane,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

The line integral,

$$\int_C \left(xy + y^2\right) dx + x^2 dy = \int_{OA} \left\{\left(xy + y^2\right) dx + x^2 dy\right\} + \int_{AO} \left\{\left(xy + y^2\right) dx + x^2 dy\right\}$$

$$= I_1 + I_2$$

Along OA, we have $y = x^2$, $\therefore$ $dy = 2x\,dx$ and x varies from 0 to 1.

$$I_1 = \int_{x=0}^{1} \left(x \cdot x^2 + x^4\right) dx + x^2 \cdot 2x\,dx$$

$$= \int_{x=0}^{1} \left(3x^3 + x^4\right) dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along AO, we have $y = x$ $\therefore$ $dy = dx$

x varies from 1 to 0

$$I_2 = \int_{1}^{0} \left(x \cdot x + x^2\right) dx + x^2\,dx$$

$$= \int_{1}^{0} 3x^2\,dx = \left[x^3\right]_1^0 = -1$$

Hence, $\quad$ L.H.S. $= I_1 + I_2 = \dfrac{19}{20} - 1 = \dfrac{-1}{20}$

Also $\quad$ R.H.S. $= \displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

where $\quad N = x^2 \qquad M = xy + y^2$

$$\frac{\partial N}{\partial x} = 2x \qquad \frac{\partial M}{\partial y} = x + 2y$$

R is the region bounded by $y = x^2$ and $y = x$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{x} (2x - x - 2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[xy - y^2\right]_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[\left(x^2 - x^2\right) - \left(x^3 - x^4\right)\right] dx$$

$$= \int_{x=0}^{1} \left(x^4 - x^3\right) dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

$\therefore \quad$ L.H.S. $=$ R.H.S. $= \dfrac{-1}{20}$. **Hence verified.**

3. *Apply Green's theorem in the plane to evaluate* $\int_C \left[\left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy\right]$ *where C is the curve enclosed by the x-axis and the semicircle* $x^2 + y^2 = 1$.

Solution. The region of integration is bounded by AB and the semicircle as shown in the figure.

By Green's theorem,

$$\int_C \left[M dx + N dy\right] = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \qquad ...(1)$$

Given $\int_C \left[\left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy\right]$

where $\quad M = 2x^2 - y^2, \qquad N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = 2x$$

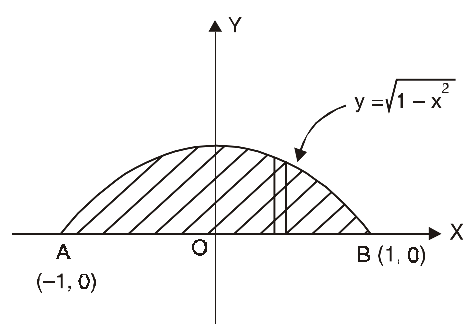

Fig. 4.4

From the equation (1),

$$\int_C \left[\left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy\right] = \iint_R \left(2x + 2y\right) dx\, dy$$

In the region, x varies from – 1 to 1 and y varies from 0 to $\sqrt{1-x^2}$

$$= 2 \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left(x + y\right) dy\, dx$$

$$= 2 \int_{x=-1}^{1} \left[xy + \frac{y^2}{2}\right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= 2 \int_{x=-1}^{1} \left[x\sqrt{1-x^2} + \frac{1}{2}\left(1-x^2\right)\right] dx$$

Since, $x\sqrt{1-x^2}$ is odd and $(1 - x^2)$ is even function

$$= 0 + 2\int_0^1 \left(1 - x^2\right) dx$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{4}{3}.$$

4. *Evaluate* $\int_C \left(xy - x^2\right) dx + x^2 y\, dy$ *where C is the closed curve formed by y = 0, x = 1 and y = x (i) directly as a line integral (ii) by employing Green's theorem.*

Solution

(i) Let $M = xy - x^2$, $N = x^2y$

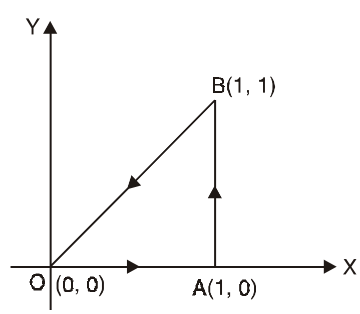

Fig. 4.5

$$\int_C Mdx + Ndy = \int_{OA} \left(Mdx + Ndy\right) + \int_{AB} \left(Mdx + Ndy\right) + \int_{BO} \left(Mdx + Ndy\right)$$

(a) Along OA: $y = 0 \Rightarrow dy = 0$ and x varies from 0 to 1.

(b) Along AB: $x = 1 \Rightarrow dx = 0$ and y varies from 0 to 1.

(c) Along BO: $y = x \Rightarrow dy = dx$ and x varies from 1 to 0.

$$\therefore \quad \int_C (M\,dx + N\,dy) = \int_{x=0}^{1} -x^2\,dx + \int_{y=0}^{1} y\,dy + \int_{x=1}^{0} x^3\,dx$$

$$= -\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 + \left[\frac{x^4}{4}\right]_1^0$$

$$= -\frac{1}{3} + \frac{1}{2} - \frac{1}{4} = \frac{-1}{12}$$

Thus $$\int_C (xy - x^2)\,dx + x^2 y dy = \frac{-1}{12}$$

(ii) We have Green's theorem,

$$\int_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\text{R.H.S.} = \iint_R (2xy - x)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} (2xy - x)\,dy\,dx \quad \text{(from the figure)}$$

$$= \int_{x=0}^{1} \left[xy^2 - xy\right]_{y=0}^{x}$$

$$= \int_{x=0}^{1} \left[x^3 - x^2\right] dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{4} - \frac{1}{3} = \frac{-1}{12}$$

$$\therefore \quad \text{R.H.S.} = \frac{-1}{12}.$$

5. *Verify Stoke's theorem for the vector* $\vec{F} = (x^2 + y^2)\, i - 2xyj$ *taken round the rectangle bounded by* $x = 0,\ x = a,\ y = 0,\ y = b$.

Solution

By Stoke's theorem : $$\int_C \vec{F} \cdot \vec{dr} = \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\,dS$$

$$\vec{F} = (x^2 + y^2)\, i - 2xyj$$

$$\vec{dr} = dxi + dyj$$

$$\vec{F} \cdot \vec{dr} = (x^2 + y^2)\, dx - 2xy\, dy$$

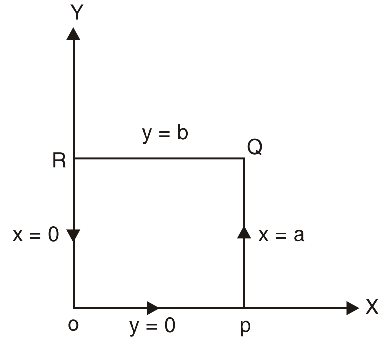

Fig. 4.6

(1) Along OP: $y = 0$, $dy = 0$, x varies from 0 to a

$$\int_{OP} \vec{F} \cdot \vec{dr} = \int_0^a x^2\, dx = \frac{a^3}{3}$$

(2) Along PQ: $x = a$, $dx = 0$; y varies from 0 to b

$$\int_{PQ} \vec{F} \cdot \vec{dr} = \int_0^b 2ay\, dy = ab^2$$

(3) Along QR: $y = b$, $dy = 0$; x varies from a to 0

$$\int_{QR} \vec{F} \cdot \vec{dr} = \int_a^0 \left(x^2 - b^2\right) dx = \left[\frac{x^3}{3} - b^2 x\right]_a^0 = ab^2 - \frac{a^3}{3}$$

(4) Along RO: $x = 0$, $dx = 0$; x varies from b to 0

$$\int_{RO} \vec{F} \cdot \vec{dr} = \int (0 - 0)\, dy = 0$$

$$\text{L.H.S.} = \int_C \vec{F} \cdot \vec{dr} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} + 0$$

$$= 2ab^2$$

Now, $$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\,\vec{k}$$

For the surface, $S \cdot \vec{n} = \vec{k}$

$\therefore$ $\text{curl } \vec{F} \cdot \hat{n} = 4y$

$$\text{R.H.S.} = \iint_S curl\, \vec{F} \cdot \hat{n}\, dS = \int_0^a \int_0^b 4y\, dy\, dx$$

$$= \int_0^a 4\left[\frac{y^2}{2}\right]_0^b dx$$

$$= 2b^2 \int_0^a dx$$

$$= 2ab^2$$

L.H.S. = R.H.S.

Hence, the Stoke's theorem is verified.

6. *Verify Stoke's theorem for the vector field* $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2 z\vec{k}$ *over the upper half surface of* $x^2 + y^2 + z^2 = 1$ *bounded by its projection on the xy-plane.*

Solution $$\int_C \vec{F} \cdot \vec{dr} = \iint_S curl\, \vec{F} \cdot \hat{n}\, dS \qquad \text{(Stoke's theorem)}$$

C is the circle: $x^2 + y^2 = 1,\ z = 0$ (xy-plane)

i.e., $x = \cos t,\ y = \sin t,\ z = 0$

$$r = x\vec{i} + y\vec{j} \text{ where } 0 \le \theta \le 2\pi$$

$$\vec{dr} = dx\vec{i} + dy\vec{j}$$

where, $$\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2 z\vec{k}$$

$$\therefore \quad \vec{F} \cdot \vec{dr} = (2x - y)\, dx \qquad (\because\ z = 0)$$

$$\text{L.H.S.} = \int_C \vec{F} \cdot \vec{dr} = \int_C (2x - y)\, dx$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\, dt$$

$$= \int_0^{2\pi} (\sin^2 t - 2\cos t \sin t)\, dt$$

$$= \int_0^{2\pi} (\sin^2 t - \sin 2t)\, dt$$

$$= \int_0^{2\pi} \left\{\frac{1}{2}(1 - \cos 2t) - \sin 2t\right\} dt$$

$$= \left[\frac{t}{2} - \frac{\sin 2t}{4} + \frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= \left(\frac{1}{2} - \frac{1}{2}\right) + (\pi - 0) = \pi$$

Hence, $$\vec{F} \cdot \vec{dr} = \pi \qquad \text{...(1)}$$

Also, $$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \vec{i}\,(-2yz + 2yz) - \vec{j}\,(0) + \vec{k}\,(0 + 1)$$

$$= \vec{k}$$

$$\therefore \quad \overrightarrow{dS} = \hat{n} dS$$

$$= dydz\ i + dzdx\ j + dxdy\ k$$

Hence, $\quad$ R.H.S. $= \iint_S curl\ \vec{F} \cdot \hat{n} dS = \iint dx\, dy$

$$= \pi \qquad \text{...(2)}$$

$\because \iint dx\, dy$ represents the area of the circle $x^2 + y^2 = 1$ which is π.

Thus, from (1) and (2) we conclude that the theorem is verified.

7. *If* $\vec{F} = 3y\vec{i} - xz\vec{j} + yz^2\vec{k}$ *and S is the surface of the paraboloid* $2z = x^2 + y^2$ *bounded by* $z = 2$, *show by using Stoke's theorem that* $\iint_S curl\ \vec{F} \cdot \hat{n}\, ds = 20\,\pi$.

Solution. If $z = 0$ then the given surface becomes $x^2 + y^2 = 4$.

Hence, C is the circle $x^2 + y^2 = 4$ in the plane $z = 2$

i.e., $\quad x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$

Hence by Stoke's theorem, we have

$$\int_C \vec{F} \cdot \overrightarrow{dr} = \iint_S curl\ \vec{F} \cdot \hat{n}\, dS$$

L.H.S. put $\quad \vec{F} = 3y\vec{i} - xz\vec{j} + yz^2\vec{k},\ \overrightarrow{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\therefore \quad \int_C \vec{F} \cdot \overrightarrow{dr} = \int_C \left(3y\, dx - xz\, dy + yz^2\, dz\right)$$

where $\quad z = 2,\ dz = 0$

$$\therefore \quad \int_C \vec{F} \cdot \overrightarrow{dr} = \int_C \left(3y\, dx - 2x\, dy\right)$$

$$x = 2\cos t \quad \Rightarrow \quad dx = -2\sin t\, dt$$

$$y = 2\sin t \quad \Rightarrow \quad dy = 2\cos t\, dt$$

$$\therefore \quad \int_C \vec{F} \cdot \overrightarrow{dr} = \int_{2\pi}^{0} 6\sin t\left(-2\sin t\right) dt - 4\cos t\left(2\cos t\right) dt$$

Since, the surface S lies below the curve C

$$= -\int_{2\pi}^{0} \left(12\sin^2 t + 8\cos^2 t\right) dt$$

$$= \int_{0}^{2\pi} \left(12\sin^2 t + 8\cos^2 t\right) dt$$

$$= 48\int_0^{\frac{\pi}{2}} \sin^2 t\,dt + 32\int_0^{\frac{\pi}{2}} \cos^2 t\,dt$$

$$= 48\cdot\frac{\pi}{4} + 32\cdot\frac{\pi}{4} = 20\,\pi$$

$$\therefore \qquad = \iint_S curl\,\vec{F}\cdot\hat{n}\,dS = 20\,\pi$$

Hence proved.

8. *Using divergence theorem, evaluate* $\iint_S \vec{F}\cdot\hat{n}dS$ *where* $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ *and S is the surface of the cube bounded by* $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

Solution. We have divergence theorem:

$$\iiint_V div\,\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,dS$$

Now

$$\text{div}\ \vec{F} = \nabla\cdot\vec{F}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(4xzi - y^2 j + yzk\right)$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}\left(-y^2\right) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$= 4z - y$$

Hence, by divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \text{div}\,\vec{F}\cdot dV$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z - y)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\left[2z^2 - yz\right]_{z=0}^{1} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}(2 - y)\,dy\,dx$$

$$= \int_{x=0}^{1}\left[2y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_{x=0}^{1} \left[2 - \frac{1}{2}\right] dx$$

$$= \int_{x=0}^{1} \frac{3}{2}\, dx$$

$$= \frac{3}{2}[x]_0^1 = \frac{3}{2}.$$

9. *Using divergence theorem, evaluate* $\iint_S \vec{F}\cdot\hat{n}\, dS$ *where* $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ *and S is the surface of the solid cut off by the plane* $x + y + z = a$ *from the first octant.*

Solution. Now

$$div\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}\left(x^2\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)$$

$$= 2x + 2y + 2z$$

$$= 2\,(x + y + z)$$

Hence, by divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \text{div}\,\vec{F}\cdot dV$$

$$= \iiint_V 2\left(x + y + z\right) dV$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}\left(x + y + z\right) dz\,dy\,dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\left[\left(x + y\right)z + \frac{1}{2}z^2\right]_{z=0}^{a-x-y} dy\,dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\frac{1}{2}\left[a^2 - \left(x + y\right)^2\right] dy\,dx$$

$$= \int_{x=0}^{a}\left[a^2 y - \frac{\left(x + y\right)^3}{3}\right]_{y=0}^{a-x} dx$$

$$= \frac{1}{3}\int_{x=0}^{a}\left(2a^3 - 3a^2 x + x^3\right) dx$$

$$= \frac{1}{3}\left[2a^3 x - 3a^2\frac{x^2}{2} + \frac{x^4}{4}\right]_0^a$$

$$\iint_S \vec{F}\,\hat{n}\cdot dS = \frac{1}{4}a^4.$$

10. *Using divergence theorem, evaluate* $\iint_S \vec{F} \cdot \hat{n}\, dS$ *where* $\vec{F} = x^3 i + y^3 j + z^3 k$ *and S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$.

Solution.

$$\begin{aligned} \text{div}\, \vec{F} &= \frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(y^3\right) + \frac{\partial}{\partial z}\left(z^3\right) \\ &= 3x^2 + 3y^2 + 3z^2 \\ &= 3\,(x^2 + y^2 + z^2) \end{aligned}$$

$\therefore$ by divergence theorem, we get

$$\begin{aligned} \iint_S \vec{F} \cdot \hat{n}\, d\,S &= \iiint_V \text{div}\, \vec{F}\, dV \\ &= \iiint_V 3\left(x^2 + y^2 + z^2\right) dx\, dy\, dz \qquad \text{...(1)} \end{aligned}$$

Since, V is the volume of the sphere we transform the above triple integral into spherical polar coordinates (r, θ, ϕ).

For the spherical polar coordinates (r, θ, ϕ), we have

$$x^2 + y^2 + z^2 = r^2 \text{ and } dx\, dy\, dz = dV$$

$$\therefore \qquad dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Also, $0 \le r \le a$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$

Therefore (1) reduces to,

$$\begin{aligned} \iint_S \vec{F} \cdot \hat{n}\, dS &= 3 \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(r^2\right) r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= 3 \int_{r=0}^{a} r^4 dr \times \int_{\theta=0}^{\pi} \sin\theta\, d\theta \times \int_{\phi=0}^{2\pi} d\phi \\ &= 3 \times \left[\frac{r^5}{5}\right]_{r=0}^{a} \times \left[-\cos\theta\right]_{\theta=0}^{\pi} \times \left[\phi\right]_{\phi=0}^{2\pi} \\ &= \frac{3a^5}{5} \times \left(-\cos\pi + 1\right) \times 2\pi \\ &= \frac{12}{5}\pi a^5. \end{aligned}$$

11. *Evaluate* $\iint_S \left(yzi + zxj + xyk\right) \cdot \hat{n}\, dS$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant.*

Solution. The given surface is $x^2 + y^2 + z^2 = a^2$, we know that $\nabla\phi$ is a vector normal to the surface $\phi\,(x, y, z) = c$.

Taking $\qquad \phi\,(x, y, z) = x^2 + y^2 + z^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$= 2xi + 2yj + 2zk$$

$\therefore$ unit vector normal $\quad \hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{2(xi + yj + zk)}{\sqrt{2^2(x^2 + y^2 + z^2)}}$$

$$= \frac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{xi + yj + zk}{a} \qquad (\because\ x^2 + y^2 + z^2 = a^2)$$

Also, if $\quad \vec{F} = yzi + zxj + xyk$

$$\vec{F}\cdot\hat{n} = \frac{1}{a}(xyz + yzx + zxy)$$

$$= \frac{3\,xyz}{a} \qquad ...(1)$$

Projection the given surface on the xy-plane, we get $dx\,dy = \hat{n}\cdot\hat{k}\,dS$

$$\therefore \qquad dS = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{\frac{z}{a}} = \frac{a\,dx\,dy}{z} \qquad ...(2)$$

From (1) and (2)

$$\iint_S \vec{F}\cdot\hat{n}\cdot dS = \iint_R \frac{3\,xyz}{a}\cdot\frac{a\,dx\,dy}{z}$$

$$= \iint_R 3\,xy\,dx\,dy$$

The region R of integration is the quadrant of the circle $x^2 + y^2 = a^2$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = 3\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2 - x^2}} xy\,dy\,dx$$

$$= 3\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{y=0}^{\sqrt{a^2 - x^2}} dx$$

$$= \frac{3}{2}\int_0^a x\left(a^2 - x^2\right)dx$$

$$= \frac{3}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{3}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{3a^4}{8}$$

Thus $\iint_S \left(yzi + zxj + xyk\right)\cdot \hat{n}dS = \frac{3a^4}{8}.$

12. *Evaluate* $\iint_S \left(axi + byj + czk\right)\cdot \hat{n}\, d\, S$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 1$.

Solution. Let $\vec{F} = axi + byj + czk$

we have $\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V div\,\vec{F}\cdot dV$

$$div\cdot\vec{F} = \nabla\cdot\vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(axi + byj + czk\right)$$

$$= \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)$$

$$= (a + b + c)$$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V \left(a+b+c\right)dV$$

$$= (a + b + c)\ V \qquad ...(1)$$

where V is the volume of the sphere with unit radius and $V = \frac{4}{3}\pi r^3$ for a sphere of radius r.

Here, since we have $r = 1$, $V = \frac{4}{3}\pi$

Thus, $\iint_S \vec{F}\cdot\hat{n}\, dS = \frac{4\pi}{3}\left(a+b+c\right).$

EXERCISE 4.2

1. If $\vec{F} = axi + byj + czk$ and S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ by using divergence theorem. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, dS$. $\left[\textbf{Ans. } \frac{4}{3}\pi\right]$

2. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, dS$ where $\vec{F} = 4xy\vec{i} + yz\vec{j} - xz\vec{k}$ and S is the surface of the cube bounded by the planes $x = 0$, $x = 2$, $y = 0$, $y = 2$, $z = 0$, $z = 2$. $[\textbf{Ans. } 32]$

3. If $\vec{F} = y^2z^2i + z^2x^2j + x^2y^2k$, evaluate $\iint_S \vec{F} \cdot \hat{n}\, dS$ where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane and bounded by this plane. $\left[\textbf{Ans. } \frac{\pi}{12}\right]$

4. Use Gauss divergence theorem to evaluate $\iint_S \vec{F} \cdot \hat{n}\, dS$ where $\vec{F} = (x^2 - z^2)\, i + 2xyj + (y^2 + z^2)\, k$ where S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. $[\textbf{Ans. } 3]$

5. Verify Green's theorem in plane for $\int_C \left(3x^2 - 8y^2\right) dx + \left(4y - 6xy\right) dy$, where C is the boundary of the triangle formed by the lines $x = 0$, $y = 0$ and $x + y = 1$. $\left[\textbf{Ans. } \frac{5}{3}\right]$

6. Verify Green's Theorem in the plane for $\int_C \left(x^2 + y^2\right) dx - 2xy\, dy$, where C is the rectangle bounded by the lines $x = 0$, $y = 0$, $x = a$, $y = b$. $\left[\textbf{Ans. } -2ab^2\right]$

7. Using Green's theorem, evaluate $\int_C \left(\cos x \sin y - xy\right) dx + \sin x \cos y\, dy$, where C is the circle $x^2 + y^2 = 1$. $[\textbf{Ans. } 0]$

8. Evaluate by Stoke's theorem $\int_C \left(yzdx + xzdy + xydz\right)$, where C is the curve $x^2 + y^2 = 1$, $z = y^2$. $[\textbf{Ans. } 0]$

9. Verify Stoke's theorem for the function $\vec{F} = zi + xj + yk$, where C is the unit circle in the xy-plane bounding the hemisphere $Z = \sqrt{1 - x^2 - y^2}$. $[\textbf{Ans. } \pi]$

10. If $\vec{F} = yi + z^3xj - y^3zk$ and C is the circle $x^2 + y^2 = 4$ in the plane $z = \frac{3}{2}$. Evaluate $\int_C \vec{F} \cdot \vec{dr}$ by Stoke's theorem. $\left[\textbf{Ans.} \frac{19\pi}{2}\right]$

11. Evaluate $\int_C \vec{F} \cdot dr$ by Stoke's theorem $\vec{F} = y^2 i + x^2 j - (x + 2)\, k$ and C is the boundary of the triangle with vertices (0, 0, 0) (1, 0, 0) and (1, 1, 0). $\left[\textbf{Ans.} \frac{1}{3}\right]$

4.4 ORTHOGONAL CURVILINEAR COORDINATES

4.4.1 Definition

Let the coordinates (x, y, z) of any point be expressed as functions of (u_1, u_2, u_3), so that $x = x\,(u_1, u_2, u_3)$, $y = y\,(u_1, u_2, u_3)$, $z = z\,(u_1, u_2, u_3)$ then u, v, w can be expressed in terms of x, y, z, as $u_1 = u\,(x, y, z)$, $u_2 = v\,(x, y, z)$ and $u_3 = w\,(x, y, z)$. And also if $\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \neq 0$ then the system of coordinates (u_1, u_2, u_3) will be an alternative specification of the Cartesian system (x, y, z) and (u_1, u_2, u_3) are called the curvilinear coordinates of the point.

If one of the coordinates is kept constant say $u_1 = c$, then

and the locus of (x, y, z) is a surface which is called a coordinate surface. Thus, we have three families of coordinate system corresponding to $u_1 = c$, $u_2 = c$, $u_3 = c$.

Suppose $u_1 = c$, $u_2 = c$ and $u_3 \neq c$ in that case locus obtained is called a coordinate curve and also there are such families.

The tangent to the coordinate curves at the point p and the three coordinate axes of the curvilinear systems.

The direction of these axes vary from point to point and hence the unit associated with them are not constant.

When the coordinate surfaces are mutually perpendicular to each other, the three curves define an orthogonal system and $(u, v, w) = (u_1, u_2, u_3)$ are called orthogonal curvilinear coordinates.

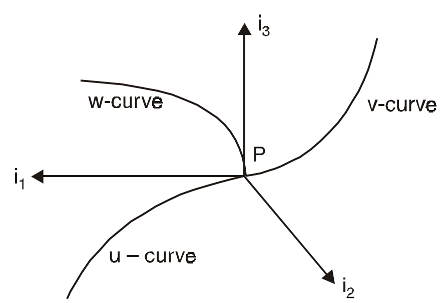

Fig. 4.7

4.4.2 Unit Tangent and Unit Normal Vectors

The position vector of point p (x, y, z) is $\vec{r} = xi_1 + yi_2 + zi_3$ where i_1, i_2, i_3 are unit vectors along the tangent to the three coordinate curves.

$$\therefore \quad i_1 \cdot i_2 = i_3 \cdot i_2 = i_3 \cdot i_1 = 0$$

$$\text{and} \quad i_1 \times i_2 = i_3,\ i_2 \times i_3 = i_1,\ i_3 \times i_1 = i_2$$

$$\therefore \quad \vec{r}\,(u_1, u_2, u_3) = x\,(u_1, u_2, u_3)\, i_1 + y\,(u_1, u_2, u_3)\, i_2 + z\,(u_1, u_2, u_3)\, i_3$$

$$\therefore \qquad \overrightarrow{dr} = \frac{\partial \overrightarrow{r}}{\partial u_1} du_1 + \frac{\partial \overrightarrow{r}}{\partial u_2} du_2 + \frac{\partial \overrightarrow{r}}{\partial u_3} du_3$$

Then $\vec{r}\ (u_1, u_2, u_3)$ is a vector point, function of variables u, v, w.

The unit tangent vector i_1 along the tangent to u-curve at P is

$$i_1 = \frac{\dfrac{\partial \overrightarrow{r}}{\partial u_1}}{\left|\dfrac{\partial \overrightarrow{r}}{\partial u_1}\right|}$$

If $\left|\dfrac{\partial \overrightarrow{r}}{\partial u_1}\right| = h_1$ which is called scalar factor, then $\dfrac{\partial r}{\partial u_1} = h_1 i_1$.

Similarly, unit tangent vectors along v-curve and w-curves are

$$i_2 = \frac{\dfrac{\partial \overrightarrow{r}}{\partial u_2}}{\left|\dfrac{\partial \overrightarrow{r}}{\partial u_2}\right|} = \frac{\dfrac{\partial \overrightarrow{r}}{\partial u_2}}{h_2}$$

$$\therefore \qquad \frac{\partial \overrightarrow{r}}{\partial u_2} = h_2\, i_2$$

$$i_3 = \frac{\dfrac{\partial \overrightarrow{r}}{\partial u_3}}{\left|\dfrac{\partial \overrightarrow{r}}{\partial u_3}\right|} = \frac{\dfrac{\partial \overrightarrow{r}}{\partial u_3}}{h_3}$$

$$\therefore \qquad \frac{\partial \overrightarrow{r}}{\partial u_3} = h_3\, i_3$$

$$\therefore \qquad \overrightarrow{dr} = \frac{\partial \overrightarrow{r}}{\partial u_1} du_1 + \frac{\partial \overrightarrow{r}}{\partial u_2} du_2 + \frac{\partial \overrightarrow{r}}{\partial u_3} du_3$$

$$= h_1 du_1 i_1 + h_2 du_2 i_2 + h_3 du_3 i_3$$

Then length of the arc dS is given by

$$dS^2 = \overrightarrow{dr} \cdot \overrightarrow{dr} = h_1^2 du_1^2 + h_2^2\ du_2^2 + h_3^2\ du_3^2$$

Example 1. *Find the square of the element of arc length in cylindrical coordinates and determine the corresponding Lommel constants.*

Solution. The position vector, $\vec{r}$ is cylindrical coordinates is

$$\vec{r} = r\cos\theta\vec{i} + r\sin\theta\vec{j} + z\vec{k}$$

Then $$d\vec{r} = \frac{\partial \vec{r}}{\partial r} dr + \frac{\partial \vec{r}}{\partial \theta} d\theta + \frac{\partial \vec{r}}{\partial z} dz$$

$$= \left(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}\right) dr + \left(-r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j}\right) d\theta + \vec{k}\, dz$$

$$= \left(\cos\theta\, dr - r\sin\theta\, d\theta\right)\vec{i} + \left(\sin\theta\, d\theta + r\cos\theta\, d\theta\right)\vec{j} + \vec{k}\, dz$$

From the relation,

$$dS^2 = \overrightarrow{dr} \cdot \overrightarrow{dr} = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

we have,

$$dS^2 = \overrightarrow{dr} \cdot \overrightarrow{dr} = (\cos\theta\ dr - r\ \sin\theta\ d\theta)^2 + (\sin\theta\ dr + r\ \cos\theta\ d\theta)^2 + dz^2$$

$$= (dr)^2 + r^2(d\theta)^2 + dz^2$$

The Lommel's constant, also called scale factors are $h_1 = 1$, $h_2 = r$, $h_3 = 1$.

Example 2. *Find the volume element dv in cylindrical coordinates.*

Solution. The volume element in orthogonal curvilinear coordinates u_1, u_2, u_3 is given by

$$u_1 = r, \qquad u_2 = \theta, \qquad u_3 = z,$$

In cylindrical coordinates $h_1 = 1, \quad h_2 = r, \quad h_3 = 1.$

Then $dv = h_1 h_2 h_3 du_1 du_2 du_3$

$$u_1 = r \quad \Rightarrow \quad du_1 = dr,$$

$$u_2 = \theta \quad \Rightarrow \quad du_2 = d\theta,$$

$$u_3 = z_1 \quad \Rightarrow \quad du_3 = dz,$$

$\therefore$ $dv = (1)\cdot(r) \cdot (1)\cdot\ dr\ d\theta\ dz$

Then, $dv = r\ dr\ d\theta\ dz.$

4.4.3 The Differential Operators

In this section, we shall express the gradient, divergence and curl in terms of orthogonal curvilinear coordinates u_1, u_2, u_3. Then, the Laplacian can be expressed as the divergence of a gradient by the chain rule, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u_1} \frac{\partial u_1}{\partial x} + \frac{\partial f}{\partial u_2} \cdot \frac{\partial u_2}{\partial x} + \frac{\partial f}{\partial u_3} \cdot \frac{\partial u_3}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u_1} \cdot \frac{\partial u_1}{\partial y} + \frac{\partial f}{\partial u_2} \cdot \frac{\partial u_2}{\partial y} + \frac{\partial f}{\partial u_3} \cdot \frac{\partial u_3}{\partial y}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u_1} \cdot \frac{\partial u_1}{\partial z} + \frac{\partial f}{\partial u_2} \cdot \frac{\partial u_2}{\partial z} + \frac{\partial f}{\partial u_3} \cdot \frac{\partial u_3}{\partial z}$$

In rectangular Cartesian coordinate system

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

where f is a scalar function. In engineering problems this f is usually a potential link

Velocity potential or electric potential or gravitational potential.

Using chain rule this becomes,

$$\nabla f = \frac{\partial f}{\partial u_1}\left[\frac{\partial u_1}{\partial x} i + \frac{\partial u_1}{\partial y} j + \frac{\partial u_1}{\partial z} k\right]$$

$$+ \frac{\partial f}{\partial u_2}\left[\frac{\partial u_2}{\partial x} i + \frac{\partial u_2}{\partial y} j + \frac{\partial u_2}{\partial z} k\right]$$

$$\frac{\partial f}{\partial u_3}\left[\frac{\partial u_3}{\partial x} i + \frac{\partial u_3}{\partial y} j + \frac{\partial u_3}{\partial z} k\right]$$

$$= \frac{\partial f}{\partial u_1}\nabla u_1 + \frac{\partial f}{\partial u_2}\nabla u_2 + \frac{\partial f}{\partial u_3}\nabla u_3$$

But $$\nabla u_1 = \frac{1}{h}\hat{e}_1,\ \nabla u_2 = \frac{1}{h_2}\hat{e}_2,\ \nabla u_3 = \frac{1}{h_3}\hat{e}_3.$$

Then the gradient of f, in orthogonal curvilinear coordinates, is

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1} e_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2} e_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3} e_3.$$

WORKED OUT EXAMPLES

1. *Show that* $\left[\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}\right] = h_1\ h_2\ h_3 = \frac{1}{[\nabla u_1, \nabla u_2, \nabla u_3]}.$

Solution. By definition of unit vectors,

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial r}{\partial u_1},\ \hat{e}_2 = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2},\ \hat{e}_3 = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial u_3} \quad ...(1)$$

we also know that

$$\nabla u_1 = \frac{1}{h_1}\hat{e}_1,\ \nabla u_2 = \frac{1}{h_2}\hat{e}_2,\ \nabla u_3 = \frac{1}{h_3}\hat{e}_3 \quad ...(2)$$

Then using (1), we have

$$\left[\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}\right] = [h_1\hat{e}_1, h_2\hat{e}_2, h_3\hat{e}_3]$$

$$= [h_1h_2h_3][\hat{e}_1\hat{e}_2\hat{e}_3]$$

$$= h_1h_2h_3 \quad ...(3)$$

Similarly from (2), we have

$$[\nabla u_1, \nabla u_2, \nabla u_3] = \left[\frac{1}{h_1}\hat{e}_1, \frac{1}{h_2}\hat{e}_2, \frac{1}{h_3}\hat{e}_3\right]$$

$$= \frac{1}{h_1 h_2 h_3}[\hat{e}_1, \hat{e}_2, \hat{e}_3] = \frac{1}{h_1 h_2 h_3} \qquad ...(4)$$

From (3) and (4), we obtain

$$\left[\frac{\partial r}{\partial u_1}, \frac{\partial r}{\partial u_2}, \frac{\partial r}{\partial u_3}\right] = h_1 h_2 h_3 = \frac{1}{[\nabla u_1, \nabla u_2, \nabla u_3]}$$

as the required solution.

2. *Find the* ∇r^m.

Solution. The position vector of a point (x, y, z) from $\frac{m}{2}$, the origin is

$$\vec{r} = xi + yj + zk$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

Then

$$\nabla r^m = mr^m \cdot \left(\frac{\vec{r}}{r^2}\right) = mr^{m-2}$$

$$\vec{r} = mr^{m-2}(xi + yj + zk).$$

3. *Find the gradient of* $f = x^2y + zy^2 + xz^2$ *in curvilinear coordinates.*

Solution. In curvilinear coordinates (u_1, u_2, u_3) the given function f takes the form

$$f = u_1^2 u_2 + u_3 u_2^2 + u_1 u_3^2$$

$$\frac{\partial f}{\partial u_1} = 2u_1 u_2 + u_3^2, \quad \frac{\partial f}{\partial u_2} = 2u_3 u_2 + u_1^2$$

$$\frac{\partial f}{\partial u_3} = u_2^2 + 2u_1 u_3$$

The gradient formula is

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3$$

Sub. $\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}$ and $\frac{\partial f}{\partial u_3}$, we get

$$\nabla f = \frac{1}{h_1}\left(2u_1u_2 + u_3^2\right)\hat{e}_1 + \frac{1}{h_2}\left(u_1^2 + 2u_3u_2\right)\hat{e}_2 + \frac{1}{h_3}\left(u_2^2 + 2u_1u_3\right)\hat{e}_3$$

which is the required gradient.

EXERCISE 4.3

1. Find the gradient of the following functions in cylindrical polar coordinates.

(*i*) $xy + yz + zx$,

(*ii*) $x(y + z) - y(z - x) + z(x + y)$,

(*iii*) $\exp(x^2 + y^2 + z^2)$.

$$\left[\begin{aligned} \textbf{Ans.}\ (i)\ & \left(r\sin 2\theta + z\sin\theta + z\cos\theta\right)\hat{e}_r + \left(-r\sin^2\theta + z\cos\theta - z\sin\theta\right)\hat{e}_\theta \\ & + r\left(\sin 2\theta + \cos\theta\right)\hat{z}. \\ (ii)\ & 2\left(r\sin 2\theta + z\cos\theta\right)\hat{e}_r + \left[(1+r)\cos^2\theta - z\left(\cos\theta + \sin\theta\right)\right]\hat{e}_\theta \\ & + r\left(\cos\theta + \sin\theta\right)\hat{e}_z \end{aligned}\right]$$

2. Find ∇f in spherical polar coordinates

when (*i*) $f = xy + yz + zx$

(*ii*) $f = x(y + z) + y(z - x) + z(x + y)$

$$\left[\begin{aligned} \textbf{Ans.}\ (i)\ & r\sin\theta\left(\sin\theta\sin 2\theta + 2\cos\theta\sin\theta + 2r\cos\theta\cos\theta\right)\hat{e}_r \\ & + r\left(\sin\theta\cos\theta\sin 2\theta + \cos 2\theta\sin\theta + \cos 2\theta\cos\theta\right)\hat{e}_r \\ & + r\left(\sin\theta\cos 2\theta + r\cos\theta\cos\theta - \cos\theta\sin\theta\right)\hat{e}_y. \\ (ii)\ & 2r\left[\left(\sin 2\theta\hat{e}_r + \cos 2\theta\hat{e}_\theta\right)\left(\cos\theta + \sin\theta\right) + \cos\theta\left(\cos\theta - \sin\theta\right)\hat{e}_\theta\right] \end{aligned}\right]$$

4.4.4. Divergence of a Vector

We now derive the expression for the divergence of a vector. In this orthogonal curvilinear coordinate system (u_1, u_2, u_3) the unit vector $\vec{F}$ can be expressed as

$$\vec{F} = F_1\hat{e}_1 + F_2\hat{e}_2 + F_3\hat{e}_3 \qquad ...(1)$$

By the vector equation can be written as

$$\begin{aligned} \vec{F} = {} & h_2h_3\,\mathrm{F}_1\,(\nabla u_2 \times \nabla u_3) + h_3h_1\,F_2(\nabla u_3 \times \nabla \mathrm{u}_1) + \\ & h_1h_2F_3\,(\nabla u_1 \times \nabla u_2) \end{aligned} \qquad ...(2)$$

Then the divergence of $\vec{F}$ is

$$\begin{aligned} \nabla\cdot\vec{F} &= \nabla\cdot[h_2h_3F_1(\nabla u_2 \times \nabla u_3)] + \nabla\cdot[h_3h_1F_2\,(\nabla u_3 \times \nabla u_1)] \\ &\qquad + \nabla u_3\,[h_1h_2F_3\,(\nabla u_1 \times \nabla u_2)] \\ &= \nabla(h_2h_3F_1)\cdot(\nabla u_2 \times \nabla u_3) + \nabla(h_1h_2F_2)\,.\,(\nabla u_3 \times \nabla u_1) \\ &\qquad + \nabla(h_3h_1F_2)\,.\,(\nabla u_1 \times \nabla u_2) \\ &= (h_2h_3F_1)\ \nabla\cdot(\nabla u_2 \times \nabla u_3) + h_3h_1F_2\,\nabla\,.\,(\nabla u_3 \times \nabla u_1) \\ &\qquad + h_1h_2F_3\ \nabla\,.\,(\nabla u_1 \times \nabla u_2) \end{aligned}$$

$$\nabla\cdot(\nabla u \times \nabla V) = \nabla V\cdot(\nabla \times \nabla u) - \nabla u\cdot(\nabla \times \nabla V) = 0$$

Using (1) and $\vec{A}\cdot\left(\vec{A}\times\vec{B}\right)$ = 0 for any pair of vectors, we have

$$\begin{aligned}\nabla\cdot\vec{F} &= \frac{\partial}{\partial u_1}\left(h_2h_3F_1\right)\nabla u_1\cdot\left(\nabla u_2\times\nabla u_3\right)\\ &+\frac{\partial}{\partial u_2}\left(h_3h_1F_2\right)\nabla u_2\cdot\left(\nabla u_3\times\nabla u_1\right)\\ &+\frac{\partial}{\partial u_3}\left(h_1h_2F_2\right)\nabla u_3\cdot\left(\nabla u_1\times\nabla u_2\right)\cdot\end{aligned}$$

But
$$\begin{aligned}\hat{e}_1\cdot\left(\hat{e}_2\times\hat{e}_3\right) &= \hat{e}_2\cdot\left(\hat{e}_3\times\hat{e}_1\right)=\hat{e}_3\cdot\left(\hat{e}_1\times\hat{e}_2\right)\\ &= h_1h_2h_3\ \nabla u_1\ (\nabla u_2\times\nabla u_3) = h_1h_2h_3\ \nabla u_2\ (\nabla u_3\times\nabla u_1)\\ &= h_1h_2h_3\ \nabla u_3\cdot(\nabla u_1\times\nabla u_2) = 1\end{aligned}$$

$\therefore$
$$\nabla.\vec{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(h_2h_3F_1\right)+\frac{\partial}{\partial u_2}\left(h_3h_1F_2\right)+\frac{\partial}{\partial u_3}\left(h_1h_2F_3\right)\right]$$

WORKED OUT EXAMPLES

1. *If f and g are continuously differentiable show that* $\nabla f\times\nabla g$ *is a solenoidal.*

Solution. A vector $\vec{F}$ is solenoidal if $\nabla\cdot\vec{F}$ = 0. To show that $\nabla f\times\nabla g$ is solenoidal. First we show that $\nabla f\times\nabla g$ can be expressed as a curl of a vector. We can show this using the identity.

$$\begin{aligned}\nabla\times(f\nabla g) &= \nabla f\times\nabla g + f\nabla + \nabla g\\ &= \nabla f\times\nabla g \qquad (\because \text{ curl glad is zero})\end{aligned}$$

Operating divergence on this and using the identity

$$\nabla\times\nabla(f\nabla g) = 0 \qquad (\because \text{ div curl of any vector is zero})$$

This gives $\quad\nabla.(\nabla f\times\nabla g) = 0$

Hence, $\nabla f\times\nabla g$ is solenoidal and hence proved.

2. *Find* $\nabla.(\nabla r^m)$

Solution. Let $\vec{r} = xi + yj + zk$ be the positive vector so that $r = \left(x^2+y^2+z^2\right)^{\frac{1}{2}}$

This gives
$$\nabla r^m = mr^{m-2}\vec{r}$$

$\therefore$
$$\nabla.\nabla r^m = \frac{\partial}{\partial x}\left(mr^{m-2}x\right)+\frac{\partial}{\partial y}\left(mr^{m-2}y\right)+\frac{\partial}{\partial z}\left(mr^{m-2}z\right) \qquad ...(1)$$

Differentiating by part, we get

$$\frac{\partial}{\partial x}\left(mr^{m-2}x\right) = m(m-2)\,r^{m-2}\cdot\frac{\partial r}{\partial x}x+mr^{m-2}$$

But
$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2+y^2+z^2\right)^{\frac{-1}{2}}2x$$

$$= x\left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}}$$

$$= \frac{x}{r}$$

Similarly, $$\frac{\partial}{\partial x}\left(mr^{m-2} \cdot x\right) = m(m-2)r^{m-4} x^2 + mr^{m-2}$$

$$\frac{\partial}{\partial y}\left(mr^{m-2} y\right) = m(m-2)\, r^{m-4}\, y^2 + mr^{m-2}$$

$$\frac{\partial}{\partial z}\left(mr^{m-2} z\right) = m(m-2)\, r^{m-4}\, z^2 + mr^{m-2}$$

Equation (1) becomes, then

$$\nabla\cdot(\nabla r^m) = m(m-2)\, r^{m-4}\, r^2 + 3mr^{m-2}$$

$$= m(m+1)\, r^{m-2}.$$

EXERCISE 4.4

1. Show that

(*i*) $\nabla \cdot \vec{r} = 3$ (*ii*) $\nabla \cdot \left(\vec{r} \times \vec{a}\right) = 0$

(*iii*) $\nabla \cdot \left(\vec{a} \times \vec{r}\right) = 0$ (*iv*) $\nabla \cdot \left(\vec{x} \times \vec{y}\right) = \vec{y} \cdot \nabla \vec{x} - \vec{x} \cdot \nabla \times \vec{y}$

2. Prove that

(*i*) $\nabla \cdot \nabla f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$ (*ii*) $\nabla \cdot \nabla \times \vec{F} = 0$

4.4.5 Curl of a Vector

Let $$\vec{F} = F_1\hat{e}_1 + F_2\hat{e}_2 + F_3\hat{e}_3$$

$$= h_1F_1\nabla u_1 + h_2F_2\nabla u_2 + h_3F_3\nabla u_3$$

Then, $$\nabla \times \vec{F} = \nabla \times (h_1F_1\nabla u_1) + \nabla \times (h_2F_2\nabla u_2) + \nabla \times (h_3F_3\, \nabla u_3)$$

$$= \nabla\, (h_1F_1) \times \nabla u_1 + h_1F_1\nabla \times \nabla u_1 \times \nabla(h_2F_2) \times \nabla u_2 + h_2F_2\nabla + \nabla u_2 + \nabla(h_3F_3) \times \nabla u_3 + h_3F_3\, \nabla \times \nabla u_3$$

But curl of a gradient is zero

$$\nabla \times \vec{F} = \nabla(h_1F_1) \times \nabla u_1 + \nabla(h_2F_2) \times \nabla u_2 + \nabla(h_3F_3) \times \nabla u_3 \qquad ...(1)$$

But $$\nabla(h_1F_1) \times \nabla u_1 = \left[\frac{\partial}{\partial u_1}\left(h_1F_1\right) \cdot \frac{\hat{e}_1}{h} + \frac{\partial}{\partial u_2}\left(h_2F_2\right)\frac{\hat{e}_2}{h_2} + \frac{\partial}{\partial u_3}\left(h_3F_3\right)\frac{\hat{e}_3}{h_3}\right] \cdot \frac{\hat{e}_1}{h_1}$$

Using the fact that $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ are the orthogonal unit vectors, we have

$$\nabla(h_1F_1) \times \nabla u_1 = \frac{1}{h_1h_3}\frac{\partial}{\partial u_3}(h_1F_1)\,\hat{e}_2 - \frac{1}{h_1h_2}\frac{\partial}{\partial u_2}(h_1F_1)\,\hat{e}_2$$

Similarly, we can show that

$$\nabla(h_2F_2) \times \nabla u_2 = \frac{1}{h_1h_2}\frac{\partial}{\partial u_1}(h_2F_2)\cdot\hat{e}_3 - \frac{1}{h_2h_3}\frac{\partial}{\partial u_1}(h_2F_2)\cdot\hat{e}_1$$

and

$$\nabla(h_3F_3) \times \nabla u_3 = \frac{1}{h_2h_3}\frac{\partial}{\partial u_2}(h_3F_3)\cdot\hat{e}_3 - \frac{1}{h_1h_3}\frac{\partial}{\partial u_2}(h_3F_3)\cdot\hat{e}_2$$

Then the equation (1) becomes

$$\nabla\times\vec{F} = \frac{1}{h_1h_3}\left[\frac{\partial}{\partial u_2}(h_3F_3) - \frac{\partial}{\partial u_3}(h_2F_2)\right]\hat{e}_1 + + \frac{1}{h_1h_3}\left[\frac{\partial}{\partial u_3}(h_1F_1) - \frac{\partial}{\partial u_1}(h_1F_1)\right]$$

$$+ \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2F_2) - \frac{\partial}{\partial u_2}(h_1F_1)\right]\hat{e}_3$$

$$= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$

WORKED OUT EXAMPLES

1. *Show that* $\nabla\times\left(f\,\vec{F}\right) = \nabla Q\times\vec{F} + f\,\nabla\times\vec{f}$.

Solution. By definition

$$\nabla\times\left(f\,\vec{F}\right) = \sum\hat{i}\times\frac{\partial}{\partial x}\left(f\,\vec{F}\right)$$

$$= \sum\hat{i}\times\left[\frac{\partial}{\partial x}\left(f\,\vec{F}\right) + f\,\frac{\partial\vec{F}}{\partial x}\right]$$

$$= \sum\hat{i}\times\frac{\partial f\,\vec{F}}{\partial x} + \sum\hat{i}\times f\,\frac{\partial\vec{F}}{\partial x}$$

$$= \sum\hat{i}\,\frac{\partial f}{\partial x}\times\vec{F} + \left[\sum\hat{i}\times\frac{\partial\vec{F}}{\partial x}\right]f$$

$$= \nabla f\times\vec{F} + f\,\nabla\times\vec{F}.$$

2. *Show that* $\nabla \times \left(\vec{A} \times \vec{B}\right) = \vec{A}\,\nabla \cdot \vec{B} - \vec{B}\,\nabla \cdot \vec{A} + \vec{B} \cdot \nabla\,\vec{A} - \vec{A} \cdot \nabla\,\vec{B}$.

Solution. By definition

$$\begin{aligned}
\nabla \times \left(\vec{A} \times \vec{B}\right) &= \sum \hat{i} \times A \times \frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x} \times \vec{B} \\
&= \sum \left[\hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right] \vec{A} - \sum \left(\hat{i} \cdot \vec{A}\right) \cdot \frac{\partial \vec{B}}{\partial x} + \sum \left(\hat{i} \cdot \vec{B}\right) \frac{\partial \vec{A}}{\partial x} \\
&\quad - \left(\sum \hat{i}\, \frac{\partial \vec{A}}{\partial x}\right) \cdot \vec{B} - \sum \hat{i} \cdot \left(\frac{\partial B}{\partial x}\right) \\
&\quad - \sum \hat{i} \cdot \left(\frac{\partial \vec{A}}{\partial x}\right) \vec{B} + \sum \left(\hat{i} \cdot \vec{B}\right) \frac{\partial \vec{A}}{\partial x} - \sum \left(\hat{i} . \vec{A}\right) \frac{\partial \vec{B}}{\partial x} \\
&= \vec{A}\,\nabla \cdot \vec{B} - \vec{B} - \vec{B}\,\nabla \cdot \vec{A} + \vec{B} \cdot \nabla\,\vec{A} - A \cdot \nabla\,\vec{B}
\end{aligned}$$

3. *Show that* $\nabla \times \nabla\, r^m = 0$.

Solution. We know that

$$\nabla r^m = m\, r^{m-2} \left(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\right). \text{ Then}$$

$$\nabla \times \nabla r^m = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ mr^{m-2}x & mr^{m-2}y & mr^{m-2}z \end{vmatrix}$$

The coefficient of $\hat{i}$ in this determinant

$$\begin{aligned}
&= \frac{\partial}{\partial y}\left(mr^{m-2}z\right) - \frac{\partial}{\partial z}\left(mr^{m-2}y\right) \\
&= m(m-2)\, r^{m-3}\, \frac{yz}{r} - m(m-2)\, r^{m-3}\, \frac{zy}{r} = 0
\end{aligned}$$

Hence, $\nabla \times \nabla r^m = 0$.

EXERCISE 4.5

1. If $\vec{F} = \left(x + y + 1\right)\hat{i} + \hat{j} - \left(x + y\right)\hat{k}$ then show that $\vec{F} \cdot \nabla \times \vec{F} = 0$.

2. If $\vec{F} = \nabla\,(x^3 + y^3 + z^3 - 3xyz)$, then find $\nabla \times F$.

3. Show that $\nabla \times \vec{r} = 0$ and $\nabla \times \left(\vec{r} \times \vec{a}\right) = -2\vec{a}$ when $\vec{r}$ is the position vector and $\vec{a}$ is constant vector.

4. Show that $\nabla \times \left(\vec{r} \times \vec{a}\right) \times \vec{b} = 2\vec{b} \times \vec{a}$.

4.4.6. Expression for Laplacian $\nabla^2 \psi$

We now consider the expression for the Laplacian $\nabla^2 \psi$. We know that

$$\nabla^2 \psi = \nabla . (\nabla \psi)$$

This, using the expression for gradient and divergence, becomes,

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2\, h_3}{h_1}\frac{\partial \psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial u_3}\right)\right] \qquad ...(1)$$

4.4.7. Particular Coordinate System

In many practical applications we need differential operator in a particular coordinate system. Some of them will be discussed in this section.

(1) Cartesian Coordinates

The Cartesian coordinate system form a particular case of the orthogonal curvilinear coordinates $(u_1\ u_2\ u_3)$ in which $u_1 = x$, $u_2 = y$, $u_3 = z$ such that

$$\frac{\partial \vec{r}}{\partial u_1} = \frac{\partial}{\partial x}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \hat{i}$$

Similarly,

$$\frac{\partial \vec{r}}{\partial u_2} = \frac{\partial r}{\partial y} = \hat{j} \text{ and}$$

$$\frac{\partial \vec{r}}{\partial u_3} = \frac{\partial r}{\partial z} = \hat{k}$$

Then, for Cartesian coordinates system

$$h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right| = \left|\hat{i}\right| = 1,$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right| = \left|\hat{j}\right| = 1,$$

and

$$h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right| = 1$$

So that, the unit tangent (or base) vectors, becomes

$$\hat{e}_1 = \hat{i}$$

$$\hat{e}_2 = \hat{j}$$

and

$$\hat{e}_3 = \hat{k}$$

Note that $\hat{e}_1 . \hat{e}_2 = \hat{i} . \hat{j} = 0$ and so on

and $\hat{e}_1 \times \hat{e}_2 = \hat{i} \times \hat{j} = \hat{k} = \hat{e}_3$ and so on.

Hence the Cartesian coordinate system is a right hand orthogonal coordinate system.

Then the arc length in this system is

$$dr^2 = dx^2 + dy^2 + dz^2$$

and the volume element is

$$dr = dx\ dy\ dz$$

The area element in the yz-plane perpendicular to the x-axis is $dy\ dz$, in the zx perpendicular to the y-axis is $dz\ dx$ and in the xy-plane perpendicular to z-axis is $dx\ dy$.

(2) Cylindrical Polar Coordinates

Let (r, θ, z) be the cylindrical coordinates of the point p. The three surfaces through $p : r = u_1$, = constant = c_1; $\theta = u_2$ = constant = c_2 and $z = u_3$ = constant = c_3 are respectively, the cylinder through p coaxial with oz, half plane through oz making an angle θ with the coordinate plane xoz; and the planes perpendicular to oz and distance z from it.

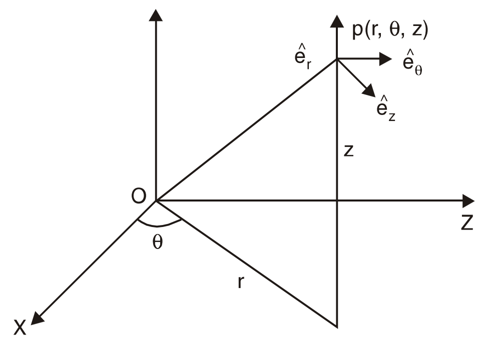

Fig. 4.8

The coordinates (r, θ, z) are related to the Cartesian coordinates (x, y, z) through the relation $x = r \cos\theta$, $y = r \sin\theta$, $z = z$.

The coordinate surface for any set of constants C_1, C_2 and C_3 are orthogonal. Therefore, the cylindrical system is orthogonal.

Here,
$$u_1 = r$$
$$u_2 = \theta$$
$$u_3 = z \text{ and}$$

position vector $\vec{r}$ in this is

$$\vec{r} = r\cos\theta\ \hat{i} + r\sin\theta\ \hat{j} + z\ \hat{k}$$

Hence,
$$\frac{\partial \vec{r}}{\partial u_1} = \frac{\partial \vec{r}}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta\ \hat{i} + r\sin\theta\ \hat{j} + z\ \hat{k})$$
$$= \cos\theta\ \hat{i} + \sin\theta\ \hat{j}$$
$$\frac{\partial \vec{r}}{\partial u_2} = \frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\ \hat{i} + r\cos\theta\ \hat{j}$$
$$\frac{\partial \vec{r}}{\partial u_3} = \frac{\partial \vec{r}}{\partial z} = \hat{k}$$

Then the Lommel constants in this case

$$h_1 = \left|\frac{\partial r}{\partial u_1}\right| = \left|\frac{\partial \vec{r}}{\partial r}\right| = \left(\cos^2\theta + \sin^2\theta\right)^{1/2} = 1$$

$$h_2 = \left|\frac{\partial r}{\partial u_2}\right| = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = (r^2\sin^2\theta + r^2\cos^2\theta)^{1/2} = r$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right| = \left|\frac{\partial \vec{r}}{\partial z}\right| = 1$$

In this case the unit vector $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_z$ in the increasing direction of (r, θ, z) are respectively take the form:

$$\hat{e}_r = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1} = \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e}_\theta = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2} = \frac{1}{2}\frac{\partial \vec{r}}{\partial \theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{e}_z = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial u_3} = \frac{\partial \vec{r}}{\partial z} = \hat{k}$$

From these, it follows that

$$\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_\theta \cdot \hat{e}_z = \hat{e}_z \cdot \hat{e}_r = 0$$

$$\hat{e}_r \times \hat{e}_\theta = (\cos\theta\,\hat{i} + \sin\theta\, j) \times (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$= (\cos^2\theta + \sin^2\theta)\,\hat{k} = \hat{k} = \hat{e}_z$$

$$\hat{e}_r \times \hat{e}_z = (\sin\theta\,\hat{j} + \cos\theta\,\hat{i}) = \hat{e}_r$$

$$\hat{e}_z \times \hat{e}_r = (\cos\theta\,\hat{j} - \sin\theta\,\hat{i}) = \hat{e}_\theta$$

The element of arc dl in cylindrical coordinates will be

$$dl^2 = dr^2 + r^2\, d\theta^2 + dz^2.$$

The conditions for $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_z$ show that the cylindrical polar coordinates system is a right handed orthogonal coordinate system.

The differential operators in this coordinate system take the form

$$\nabla Q = \frac{\partial Q}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial Q}{\partial \theta}\hat{e}_\theta + \frac{\partial Q}{\partial z}\hat{e}_z$$

$$\nabla \cdot \vec{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \vec{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_r}{\partial \theta}\right)\hat{e}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right)\hat{e}_\theta$$

Area elements on the coordinate surfaces and the volume elements in this coordinate systems are

$$dS_r = rd\theta\, dz,\ dS_\theta = dz\, dr$$

$$dS_z = rdr\, d\theta,\ dr = rdr\, d\theta\, dz.$$

(*iii*) Spherical Polar Coordinates

If (r, θ, Q) are the spherical polar coordinates, then in this system

$$x = r\sin\theta\cos\theta$$

$$y = r\sin\theta\sin\theta$$

$$z = r\cos\theta$$

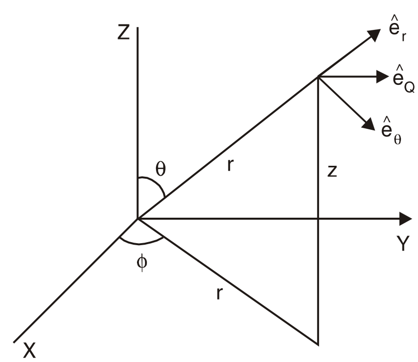

Fig. 4.9

with $0 \le r$, $0 \le \theta \le \pi$ and $0 \le Q \le 2\pi$.

Here,
$$u_1 = r$$
$$u_2 = \theta$$
$$u_3 = Q \text{ and}$$

$$\vec{r} = r \sin\theta \cos\phi\, \hat{i} + r \sin\theta \cos\phi\, \hat{j} + r\cos\theta\, \hat{k}$$

Then
$$\frac{\partial r}{\partial u_1} = \frac{\partial \vec{r}}{\partial r} = \sin\theta \cos\phi\, \hat{i} + \sin\theta \sin\phi\, \hat{j} + \cos\theta\, \hat{k}$$
$$\frac{\partial r}{\partial u_2} = \frac{\partial \vec{r}}{\partial \theta} = r \sin\theta \cos\phi\, \hat{i} + r\cos\theta \sin\phi\, \hat{j} - \sin\theta\, \hat{k}$$
$$\frac{\partial r}{\partial u_3} = \frac{\partial \vec{r}}{\partial \phi} = r \sin\theta \sin\phi\, \hat{i} + r \sin\theta \cos\phi\, \hat{j}$$

In this case the formed constants are

$$h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right| = \left|\frac{\partial \vec{r}}{\partial r}\right| = [\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta]^{1/2} = 1$$
$$h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right| = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = r\,[\cos^2\theta\,(\cos^2\phi + \sin^2\phi) + \sin^2\phi]^{1/2} = r$$
$$h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right| = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = r\sin\theta\,(\cos^2\phi + \sin^2\phi)^{1/2} = r\sin\theta$$

Then the unit vectors are

$$\hat{e}_r = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1} = \frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\,\hat{k}$$
$$\hat{e}_\theta = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2} = \frac{1}{r}\frac{\partial \vec{r}}{\partial \theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$
$$\hat{e}_\phi = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial u_3} = \frac{1}{r\sin\theta}\frac{\partial r}{\partial \phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

From these, we have
$$\hat{e}_r \cdot \hat{e}_\theta = \sin\theta\cos\theta\,(\cos^2\phi + \sin^2\phi) - \cos\theta\sin\theta = 0$$
$$\hat{e}_\theta \cdot \hat{e}_\phi = \cos\theta\,(-\sin\phi\cos\phi + \sin\phi\cos\phi) = 0$$
$$\hat{e}_\phi \cdot \hat{e}_r = \sin\theta\,(-\sin\phi\cos\phi + \cos\phi\sin\phi) = 0.$$

Also
$$\hat{e}_r \times \hat{e}_\theta = \cos\theta\,(\sin^2\theta + \cos^2\theta)\,j - \sin\phi\,(\sin^2\theta + \cos^2\theta)\,i$$
$$= \hat{e}_\phi$$
$$\hat{e}_\theta \times \hat{e}_\phi = \cos\theta\,(\cos^2\phi + \sin^2\phi)\,\hat{k} + \sin\theta\sin\phi\,\hat{j} + \sin\theta\cos\phi\,\hat{i} = \hat{e}_r$$
$$\hat{e}_\phi \times \hat{e}_r = -\sin\theta\,(\sin^2\phi + \cos^2\phi)\,\hat{k} + \sin\phi\cos\theta\,\hat{j} + \cos\theta\cos\phi\,\hat{i} = \hat{e}_\theta.$$

These conditions show that the spherical polar coordinates system is a right handed orthogonal system.

The element of arc length dl is

$$dl^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

The differential operators are given by

$$\nabla \psi = \frac{\partial \psi}{\partial r}\hat{e}_r + \frac{1}{r} + \frac{\partial \psi}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{e}_\phi$$

$$\nabla . \vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, F_\phi\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} \cdot \frac{2}{r} \cdot \frac{\partial \psi}{\partial r} + \frac{1}{r^2 \sin\theta} . \sin\theta \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \theta}{\partial \phi^2}$$

$$\nabla \times \vec{F} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta\, F_\phi\right) - \frac{\partial F_\theta}{\partial \phi}\right]\hat{e}_r +$$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}\left(rF_\phi\right)\right]\hat{e}_\theta +$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{\partial F_\theta}{\partial \theta}\right]\hat{e}_z$$

WORKED OUT EXAMPLES

1. *Verify the equations*

(i) $\nabla\theta + \nabla \times (R \log r) = 0$

(ii) $\nabla \log r + \nabla \times R\theta = 0$

for cylindrical coordinates.

Solution. (*i*) In cylindrical coordinates

$$\nabla r = \hat{e}_1$$

$$\nabla\theta = \frac{1}{r}\hat{e}_r$$

Then $\nabla \times (R \log r) = (\nabla \log r) \times R$

$$= \frac{1}{r}\nabla r \times R$$

$$= \frac{1}{r}\hat{e}_1 \times k$$

$$= \frac{-1}{r}\ \hat{e}_2 = -\nabla\theta$$

$\therefore \qquad \nabla \times R \log r + \nabla\theta = 0$

Hence the result.

(*ii*) By definition,

$$\nabla \log r = \frac{1}{r}\nabla r = \left(\frac{1}{r}\right)\hat{e}_1$$

Similarly, $\qquad \nabla \times (R\ \theta) = \nabla\ \theta \times R$

$$= \frac{1}{r}\ \hat{e}_2 \times R$$

$$= \frac{-1}{r}\hat{e}_1 = -\nabla \log r$$

$\therefore \qquad \nabla \log r + \nabla \times k\theta = 0$

Hence proved.

Example 2. *Determine the scale factors for spherical coordinates. Also find the arc and volume elements.*

Solution. For spherical coordinates

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Hence, $\quad dx = dr \sin\theta \cos\phi + d\,\theta\, r \cos\theta \cos\phi - d\,\phi\, r \sin\theta \sin\phi \qquad ...(1)$

$\quad dy = dr \sin\theta \sin\phi + d\,\theta\, r \cos\theta \sin\phi + d\,\phi\, r \sin\theta \cos\phi \qquad ...(2)$

and $\quad dz = dr \cos\theta - d\,\theta\, r \sin\theta \qquad ...(3)$

Squaring and adding (1), (2) and (3), we have

$$dx^2 + dy^2 + dz^2 = ds^2 = dr^2 + r^2\, d\,\theta^2 + r^2 \sin^2\theta\, d\,\phi^2$$

Hence, $\quad h_1^2 = 1, \Rightarrow h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

or $\quad h_r = 1,\ h_\theta = r,\ h_\varphi = r\sin\theta$

The arc element

$$ds = \sqrt{dr^2 + r^2 d\,\theta^2 + r^2 \sin^2\theta\, d\,\phi^2}$$

Similarly the volume element

$$dV = h_1\, h_2\, h_3\, du_1\, du_2\, du_3$$

$$= r^2 \sin\theta\, dr\, d\,\theta\, dz.$$

Example 3. *Prove that the spherical coordinate system is orthogonal.*

Solution. If $\bar{R}$ be the position vector of a point P (x, y, z), then

$$\bar{R} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$$

Substituting the values of x, y, z we have

$$\bar{R} = r \sin\theta \cos\phi\, \bar{i} + r \sin\theta \sin\phi\, \bar{j} + r \cos\theta\, \bar{k}$$

$$\therefore \quad \frac{\partial \bar{R}}{\partial r} = \sin\theta \cos\phi\, \bar{i} + \sin\theta \sin\phi\, \bar{j} + \cos\theta\, \bar{k}$$

$$\therefore \quad \bar{e}_1 = \bar{e}_r$$

$$= \frac{\dfrac{\partial \bar{R}}{\partial r}}{\left|\dfrac{\partial \bar{R}}{\partial r}\right|} = \sin\theta\cos\phi\, \bar{i} + \cos\theta\sin\phi\, \bar{j} + \cos\theta\, \bar{k} \quad \text{...(1)}$$

Similarly, $\bar{e}_2 = \bar{e}_\theta$

$$= \frac{\dfrac{\partial \bar{R}}{\partial \phi}}{\left|\dfrac{\partial \bar{R}}{\partial \theta}\right|} = \cos\theta\cos\phi\, \bar{i} + \cos\theta\sin\phi\, \bar{j} - \sin\theta\, \bar{k} \quad \text{...(2)}$$

$$\bar{e}_3 = \bar{e}_\varphi = \frac{\dfrac{\partial \bar{R}}{\partial \theta}}{\left|\dfrac{\partial \bar{R}}{\partial \phi}\right|} = -\sin\phi\, \bar{i} + \cos\phi\, \bar{j} \quad \text{...(3)}$$

From (1), (2) and (3)

$$\bar{e}_1 \cdot \bar{e}_2 = \bar{e}_2 \cdot \bar{e}_3 = \bar{e}_3 \cdot \bar{e}_1 = 0$$

Hence the spherical system is orthogonal in which case

$$\bar{e}_r = \bar{E}_r,\ \bar{e}_\theta = \bar{E}_\theta \text{ and } \bar{e}_\varphi = \bar{E}_\varphi.$$

Example 4. *Obtain expression for grad* α, *Div* $\bar{A}$ *and curl* $\bar{A}$ *in spherical coordinates.*

Solution. For spherical coordinates

$$u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi, \quad h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin\theta$$

Let $\bar{A} = A_1 \bar{e}_1 + A_2 \bar{e}_2 + A_3 \bar{e}_3$

(*i*) Expression for grad α,

$$\text{grad } \alpha = \frac{\bar{e}_1}{h_1}\frac{\partial \alpha}{\partial u_1} + \frac{\bar{e}_2}{h_2}\frac{\partial \alpha}{\partial u_2} + \frac{\bar{e}_3}{h_3}\frac{\partial \alpha}{\partial u_3}$$

Substituting for $h_1\ h_2\ h_3$ and u_1, u_2, u_3 we have

$$\text{grad } \alpha = \bar{e}_r \frac{\partial \alpha}{\partial r} + \frac{\bar{e}_\theta}{r}\frac{\partial \alpha}{\partial \theta} + \frac{\bar{e}_\phi}{r \sin\theta}\frac{\partial \alpha}{\partial \phi}$$

(*ii*) Expression for Div $\bar{A}$

$$\text{Div } \bar{A} = \Delta \cdot \bar{A}$$

$$= \frac{1}{h_1\, h_2\, h_3}\left[\frac{\partial}{\partial u_1}(A_1\, h_2\, h_3) + \frac{\partial}{\partial u_2}(A_2\, h_3\, h_1) + \frac{\partial}{\partial u_3}(A_3\, h_1\, h_2)\right]$$

$$= \frac{1}{r_2 \sin\theta}\left[\frac{\partial}{\partial r}(A_1\, r^2 \sin\theta) + \frac{\partial}{\partial \theta}(A_2\, r \sin\theta) + \frac{\partial}{\partial \phi}(A_3\, r)\right]$$

$$= \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}(A_1\, r^2) + r\frac{\partial}{\partial \theta}(A_2 \sin\theta) + r\frac{\partial A_3}{\partial \phi}\right]$$

(*iii*) Expression for curl $\bar{A}$

$$\text{curl } \bar{A} = \frac{1}{h_1\, h_2\, h_3}\begin{vmatrix} h_1\, \bar{e}_1 & h_2\, \bar{e}_2 & h_3\, \bar{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ A_1\, h_1 & A_2\, h_2 & A_3\, h_3 \end{vmatrix} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \bar{e}_r & r\, \bar{e}_\theta & r\sin\theta\, \bar{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_1 & A_2\, r & A_3\, r\sin\theta \end{vmatrix}$$

Example 5. *Show that for spherical polar coordinates (r, θ, φ) curl (cos θ grad φ) =* $grad\left(\frac{1}{r}\right)$.

Solution. We know curl $(\phi\, \bar{A}) = \phi$ curl $\bar{A}$ + grad $\phi \times \bar{A}$. Hence curl {(cos θ) (grad φ)} = cos θ curl grad φ + grad (cos θ) × grad φ.

But curl grad $\phi = 0$

$\therefore$ L.H.S. = grad (cos θ) × grad φ

We know $\quad \text{grad } \alpha = \frac{\bar{e}_1}{h_1}\frac{\partial \alpha}{\partial u_1} + \frac{\bar{e}_2}{h_2}\frac{\partial \alpha}{\partial u_2} + \frac{\bar{e}_3}{h_3}\frac{\partial \alpha}{\partial u_3}$

For spherical coordinates

$$u_1 = r,\ u_2 = \theta\ u_3 = \phi,\quad h_1 = 1,\ h_2 = r,\ h_3 = \sin\theta$$

$$\therefore \quad \text{grad } (\cos\theta) = \bar{e}_1 \frac{\partial}{\partial r}(\cos\theta) + \frac{\bar{e}_2}{r}\frac{\partial}{\partial \theta}(\cos\theta) + \frac{\bar{e}_3}{r\sin\theta}\frac{\partial}{\partial \phi}(\cos\theta)$$

$$= -\frac{\bar{e}_2}{r}\sin\theta \qquad \text{...(1)}$$

Since $\quad \frac{\partial}{\partial r}(\cos\theta) = \frac{\partial}{\partial \phi}(\cos\theta) = 0$

Similarly, $\quad \text{grad}\ \phi = \bar{e}_1 \dfrac{\partial}{\partial r}(\phi) + \dfrac{\bar{e}_2}{r}\dfrac{\partial}{\partial \theta}(\phi) + \dfrac{\bar{e}_3}{r\sin\theta}\left(\dfrac{\partial}{\partial \phi}\phi\right)$

$$= \frac{\bar{e}_3}{r\sin\theta} \qquad ...(2)$$

Hence L.H.S. on using (1) and (2) gives

$$-\frac{\bar{e}_2}{r}\sin\theta \times \frac{\bar{e}_3}{r\sin\theta} = -\frac{\bar{e}_1}{r^2}$$

Since $\quad \bar{e}_2 \times \bar{e}_3 = \bar{e}_1$

Similarly $\quad$ R.H.S. $= \text{grad}\left(\dfrac{1}{r}\right)$

$$\bar{e}_1 \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \bar{e}_2 \frac{\partial}{\partial \theta}\frac{1}{r} + \frac{\bar{e}_3}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r}\right) = \frac{-\bar{e}_1}{r^2}$$

$\therefore \quad$ L.H.S. $=$ R.H.S.

Example 6. *Find* $J\left(\dfrac{x, y, z}{u_1, u_2, u_3}\right)$ *in spherical coordinates.*

Solution. For spherical coordinates $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$ and $x = r \sin\theta \cos\phi$, $y = r \sin\phi \sin\phi$,

$$z = r \cos\theta$$

Hence,

$$J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = J\left(\frac{x, y, z}{r, \theta, \phi}\right)$$

$$= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} & \dfrac{\partial z}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\cos\phi & \cos\phi \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$\begin{aligned} &= \sin\theta \cos\phi\, (r^2 \sin^2\theta \cos\phi) - \sin\theta \sin\phi\, (-\, r^2 \sin^2\theta \sin\phi) \\ &\qquad + \cos\theta\, (r^2 \sin\theta \cos\theta \cos^2\phi + r^2 \cos\theta \sin\theta \sin^2\phi) \\ &= r^2 \sin^3\theta \cos^2\phi + r^2 \sin^3\theta \sin^2\phi + \cos^2\theta \sin\theta\, r^2 \\ &= r^2 \sin^3\theta + r^2 \sin\theta \cos^2\theta \\ &= r^2 \sin\theta. \end{aligned}$$

EXERCISE 4.6

1. Represent $\bar{A} = z\,\bar{i} - 2x\,\bar{j} + y\,\bar{k}$ in cylindrical coordinates. Hence obtain its components in that system. $\left[\textbf{Ans. } A_r = z\cos\theta - r\sin 2\theta,\ A_\theta = -z\sin\theta - 2r\cos^2\theta,\ A_z = r\sin\theta\right]$

2. Prove that for cylindrical system $\dfrac{d}{dt}\bar{e}_\theta = \dfrac{d\theta}{dt}\bar{e}_r$.

3. Obtain expression for velocity $\bar{v}$ and acceleration $\bar{a}$ in cylindrical coordinates.

   [**Hint :** $\bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$. Substitute for x, y, z and $\bar{i}$, $\bar{j}$, $\bar{k}$.

   $$\bar{V} = \frac{dr}{dt} \text{ and } a = \frac{d\bar{\nu}}{dt}$$

   $$\bar{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\bar{e}_r + \left(2\dot{r}\,\dot{\theta} + r\,\theta''\right)\bar{e}_\theta + z''\,\bar{e}_z \qquad \left[\textbf{Ans. } \bar{V} = \dot{r}\,\bar{e}_r + r\,\dot{\theta}\,\bar{e}_\theta + \dot{z}\,\bar{e}_z\right]$$

   Where dots denote differentiation with respect to time t.

4. Obtain an expression for $\Delta^2\,\chi$ in (*i*) cylindrical (*ii*) spherical systems.

   $$\left[\textbf{Ans. } (i)\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial x}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2} + \frac{\partial^2\chi}{\partial z^2}\right.$$

   $$\left.(ii)\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\chi}{\partial r}\right) + \frac{1}{r^2}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\chi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\chi}{\partial\phi^2}\right]$$

5. For spherical coordinates prove that

   (*i*) $\bar{e}_r' = \dot{\theta}\,\bar{e}_\theta + \sin\theta\,\dot{\phi}\,\bar{e}_\phi$

   (*ii*) $\bar{e}_\theta' = -\theta'\,\bar{e}_r + \cos\theta\,\dot{\phi}\,\bar{e}_\phi$

   (*iii*) $\bar{e}\,\phi' = -\sin\theta\,\dot{\phi}\,\bar{e}_r - \cos\theta\,\dot{\phi}\,\bar{e}_\theta$

ADDITIONAL PROBLEMS

1. *Verify Green's theorem for* $\int_c \left(xy + y^2\right)dx + x^2dy$ *where C is the closed curve of the region bounded by the line* $y = x$ *and the parabola* $y = x^2$.

   Solution. Refer page no. 176, Example 2.

2. *Using the divergence theorem evaluate* $\iint_S \vec{F}\cdot\hat{n}\,ds$ *where* $\vec{F} = x^3\,i + y^3\,j + z^3\,k$ *and S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$.

   Solution. Refer page no. 185, Example 10.

3. *Prove that Cylindrical coordinate system is orthogonal.*

   Solution. Refer page no. 200.

4. *Evaluate* $\int_C xy\,dx + xy^2\,dy$ *by Stoke's theorem where C is the square in the x – y plane with vertices (1, 0), (– 1, 0), (0, 1), (0, – 1).*

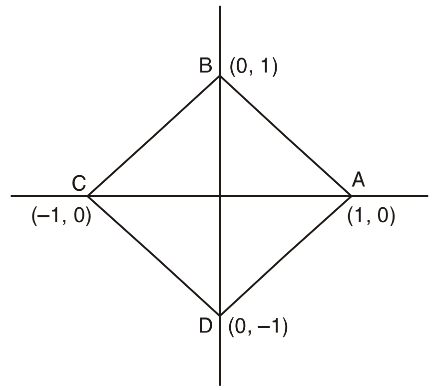

Fig. 4.10

Solution. We have Stoke's theorem

$$\int_C \vec{F}\cdot\vec{dr} = \iint_S \text{curl}\,\vec{F}\cdot\vec{ds}$$

From the given integral it is evident that

$$\vec{F} = xy\,\hat{i} + xy^2\,\hat{j}$$

since,
$$\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Hence,
$$\int_C xy\,dx + xy^2 dy = \int_C \vec{F}\cdot\vec{dr}$$

Which is to be evaluated by applying Stoke's theorem.

Now,
$$\text{curl}\,\vec{F} = \Delta\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

i.e.,
$$\text{Curl}\,\vec{F} = \left(y^2 - x\right)\hat{k}\text{, on expanding the determinant}$$

Further
$$\vec{ds} = dy\,dz\,\hat{i} + dz\,dx\,\hat{j} + dx\,dy\,\hat{k}$$

$$\therefore \quad \iint_S \text{curl}\,\vec{F}\cdot\vec{ds} = \iint \left(y^2 - x\right)dx\,dy$$

It can be clearly seen from the figure that $-1 \le x \le 1$ and $-1 \le y \le 1$

Now,
$$\iint_S \text{curl}\,\vec{F}\cdot\vec{ds} = \int_{x=-1}^{1}\int_{y=-1}^{1}\left(y^2 - x\right)dy\,dx$$

$$= \int_{x=-1}^{1}\left[\frac{y^3}{3} - xy\right]_{y=-1}^{1} dx$$

$$= \int_{x=-1}^{1}\left[\left(\frac{1}{3}+\frac{1}{3}\right) - x\left(1+1\right)\right]dx = \int_{x=-1}^{1}\left(\frac{2}{3} - 2x\right)dx$$

$$= \left[\frac{2}{3}x - x^2\right]_{x=-1}^{1}$$

$$= \frac{2}{3}\left(1+1\right) - \left(1-1\right) = \frac{4}{3}$$

Thus,
$$\int_C xy\,dx + x^2 dy = \frac{4}{3}$$

OBJECTIVE QUESTIONS

1. If $\vec{F}(t)$ has a constant magnitude then

(*a*) $\frac{d}{dt}\vec{F}(t)=0$ (*b*) $\vec{F}(t)\cdot\frac{d\vec{F}(t)}{dt}=0$

(*c*) $\vec{F}(t)\times\frac{d\vec{F}(t)}{dt}=0$ (*d*) $\vec{F}(t)-\frac{d\vec{F}(t)}{dt}=0$ **[Ans.** *b***]**

2. A unit vector normal to the surface $xy^3z^2 = 4$ at the point (– 1, – 1, 2) is

(*a*) $-\frac{1}{\sqrt{11}}\left(\vec{i}+3\vec{j}-\vec{k}\right)$ (*b*) $\frac{1}{\sqrt{11}}\left(\vec{i}+3\vec{j}-\vec{k}\right)$

(*c*) $-\frac{1}{\sqrt{11}}\left(\vec{i}+3\vec{j}+\vec{k}\right)$ (*d*) $\frac{1}{\sqrt{11}}\left(\vec{i}-3\vec{j}+\vec{k}\right)$ **[Ans.** *a***]**

3. The greatest rate of increase of $u = x^2 + yz^2$ at the point (1, – 1, 3) is

(*a*) $\sqrt{79}$ (*b*) $2\sqrt{79}$

(*c*) $\sqrt{89}$ (*d*) $4\sqrt{7}$ **[Ans.** *c***]**

4. The vector grad ϕ at the point (1, 1, 2) where ϕ is the level surface $xy^2 z^2 = 4$ is along

(*a*) normal to the surface at (1, 1, 2) (*b*) tangent to the surface at (1, 1, 2)

(*c*) Z-axis (*d*) $\vec{i}+\vec{j}+2\vec{k}$ **[Ans.** *a***]**

5. Directional derivative is maximum along

(*a*) tangent to the surface (*b*) normal to the surface

(*c*) any unit vector (*d*) coordinate ones **[Ans.** *b***]**

6. If for a vector function $\vec{F}, \text{div}\,\vec{F}=0$ then $\vec{F}$ is called

(*a*) irrotational (*b*) conservative

(*c*) solenoidal (*d*) rotational **[Ans.** *c***]**

7. For a vector function $\vec{F}$, there exists a scalar potential only when

(*a*) $\text{div}\,\vec{F}=0$ (*b*) $\text{grad}\left(\text{div}\,\vec{F}\right)=0$

(*c*) $\text{curl}\,\vec{F}=0$ (*d*) $\vec{F}\,\text{curl}\,\vec{F}=0$ **[Ans.** *c***]**

8. If $\vec{a}$ is a constant vector and $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$, then $\Delta\times\left(\vec{a}\times\vec{r}\right)$ is equal to

(*a*) 0 (*b*) $\vec{a}$

(*c*) $2\vec{a}$ (*d*) $-2\vec{a}$ **[Ans.** *c***]**

9. Which of the following is true

(a) $\text{curl}\left(\vec{A}\cdot\vec{B}\right) = \text{curl}\,\vec{A} + \text{curl}\,\vec{B}$ (b) $\text{div curl}\,\vec{A} = \nabla\,\vec{A}$

(c) $\text{div}\left(\vec{A}\cdot\vec{B}\right) = \text{div}\,\vec{A}\,.\,\text{div}\,\vec{B}$ (d) $\text{div curl}\,\vec{A} = 0$ **[Ans.** d**]**

10. Using the following integral, work done by a force $\vec{F}$ can be calculated:

(a) Line integral (b) Surface integral

(c) Volume integral (d) None of these **[Ans.** a**]**

11. If $\vec{F}$ is the velocity of a fluid particle then $\int_C \vec{F}\cdot\vec{dr}$ represents

(a) work done (b) circulation

(c) flux (d) conservative field **[Ans.** b**]**

12. The well-known equations of poisson and Laplace hold good for every

(a) rotational field (b) solenoidal field

(c) irrotational field (d) compressible field **[Ans.** c**]**

13. If the vector functions $\vec{F}$ and $\vec{G}$ are irrotational, then $\vec{F}\times\vec{G}$ is

(a) irrotational (b) solenoidal

(c) both irrotational and solenoidal (d) none of these **[Ans.** b**]**

14. The gradient of a differentiable scalas field is

(a) irrotational (b) solenoidal

(c) both irrotational and solenoidal (d) none of these **[Ans.** a**]**

15. Gauss Divergence theorem is a relation between

(a) a line integral and a surface integral

(b) a surface integral and a volume integral

(c) a line integral and a volume integral

(d) two volume integrals **[Ans.** b**]**

16. Green's theorem in the plane is applicable to

(a) xy-plane (b) yz-plane

(c) zx-plane (d) all of these **[Ans.** d**]**

17. If all the surfaces are closed in a region containing volume V then the following theorem is applicable

(a) Stoke's theorem (b) Green's theorem

(c) Gauss divergence theorem (d) only (a) and (b) [**Ans.** c]

18. The value of $\int_C \left[(2x-y)\,dx+(x+3y)\,dy\right]$ where $C : x^2 + 4y^2 = 4$ is

(*a*) 0 (*b*) 4

(*c*) 2π (*d*) 4π **[Ans. *d*]**

19. The magnitude of the vector drawn in a direction perpendicular to the surface $x^2 + 2y^2 + z^2 = 7$ at the point (1, – 1, 2) is

(*a*) $\frac{2}{3}$ (*b*) $\frac{3}{2}$

(*c*) 3 (*d*) 6 **[Ans. *d*]**

20. A unit normal to the surface $z = 2xy$ at the point (2, 1, 4) is

(*a*) $2i + 4j - k$ (*b*) $2j + 4j + k$

(*c*) $\frac{1}{\sqrt{21}}(2i+4j-k)$ (*d*) $\frac{1}{\sqrt{21}}(4i+2j-k)$ **[Ans. *c*]**

21. The maximum value of the directional derivative $\phi = x^2 - 2y^2 + 4z^2$ at the point (1, 1, – 1) is

(*a*) $\sqrt{\frac{7}{3}}$ (*b*) 84

(*c*) $6\sqrt{\frac{7}{3}}$ (*d*) $3\sqrt{\frac{3}{7}}$ **[Ans. *c*]**

22. The unit normal to the surface $x^2 + y^2 + z^2 = 1$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is

(*a*) $\frac{1}{\sqrt{2}}\left(\vec{i} - \vec{j}\right)$ (*b*) $\frac{1}{\sqrt{2}}\left(\overrightarrow{-i} + \vec{j}\right)$

(*c*) $\frac{1}{\sqrt{2}}\left(\vec{i} + \vec{j}\right)$ (*d*) None of these **[Ans. *c*]**

23. The unit vector tangent to the curve $x = t$, $y = t^2$, $z = t^3$ at the point (– 1, 1, – 1) is

(*a*) $\frac{1}{\sqrt{14}}(i+2j+3k)$ (*b*) $\frac{1}{\sqrt{14}}(i-2j+3k)$

(*c*) $\frac{1}{\sqrt{3}}(i+j+k)$ (*d*) $\frac{1}{\sqrt{3}}(i-j+k)$ **[Ans. *a*]**

24. The value of curl (grad f), where $f = 2x^2 - 3y^2 + 4z^2$ is

(*a*) $4x - 6y + 8z$ (*b*) $4xi - 6yj + 8zk$

(*c*) 0 (*d*) 3 **[Ans. *c*]**

25. The curl of the gradient of a scalar function $\cup$ is

(*a*) 1 (*b*) $\Delta^2 \cup$

(*c*) $\nabla \cup$ (*d*) 0 **[Ans. *d*]**

26. A force field $\vec{F}$ is said to be conservative if

(*a*) $\text{curl } \vec{F} = 0$ (*b*) $\text{grad } \vec{F} = 0$

(*c*) $\text{div } \vec{F} = 0$ (*d*) $\text{curl}\left(\text{grad } \vec{F}\right) = 0$ **[Ans.** *d***]**

27. The value of the line integral $\int \text{grad}\,(x + y - z)\, dr$ from (0, 1, – 1) to (1, 2, 0) is

(*a*) – 1 (*b*) 3

(*c*) 0 (*d*) No obtainable **[Ans.** *b***]**

28. A necessary and sufficient condition that line integral $\int_C A \cdot dr = 0$ for every closed curve C is that

(*a*) div A = 0 (*b*) curl A = 0

(*c*) div $A \neq 0$ (*d*) curl $A \neq 0$ **[Ans.** *b***]**

29. The value of the line integral $\int_C \left(y^2\, dx + x^2 dy\right)$ where C is the boundary of the square $-1 \leq x \leq 1, -1 \leq y \leq 1$ is

(*a*) 0 (*b*) 2 (x + y)

(*c*) 4 (*d*) $\frac{4}{3}$ **[Ans.** *a***]**

30. The value of the surface integral $\iint_S \left(yz\, dy\, dz + zx\, dzdx + xy\, dx\, dy\right)$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ is

(*a*) $\frac{4\pi}{3}$ (*b*) 0

(*c*) 4π (*d*) 12π **[Ans.** *b***]**

31. Let S be a closed orientable surface enclosing a unit volume. Then the magnitude of the surface integral $\int_S \vec{r} \cdot \hat{n}\, ds$, where $r = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ and $\hat{n}$ is the unit normal to the surface S, equals.

(*a*) 1 (*b*) 2

(*c*) 3 (*d*) 4 **[Ans.** *c***]**

32. If $\overrightarrow{f} = ax\,\vec{i} + by\,\vec{j} + cz\,\vec{k}$, a, b, c constants then $\iint f \cdot dS$ where S is the surface of a unit sphere, is

(*a*) 0 (*b*) $\frac{4}{3}\pi\left(a + b + c\right)$

(*c*) $\frac{4}{3}\pi\left(a + b + c\right)^2$ (*d*) None of these **[Ans.** *b***]**

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