1·2. Gradient of a Scalar Field.

Let $\phi(x, y, z)$ be any continuously differentiable scalar function. The gradient of scalar function ϕ is mathematically defined as

$$\text{grad } \phi = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z} \quad ...(1)$$

$$\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \nabla \phi$$

where $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is called 'del' or *nabla*.

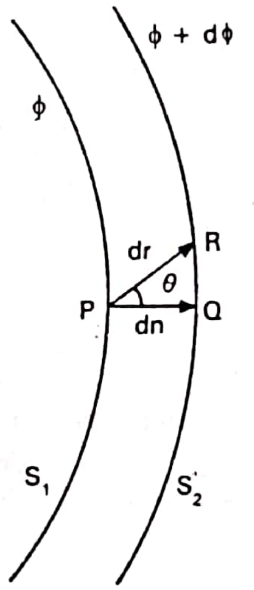

Fig. 1.9

Physical Interpretation. In the scalar field let there be two level surfaces S_1 and S_2 very close to each other characterised by scalar functions ϕ and $\phi + d\phi$ respectively. Consider two points P and R on the level surfaces* S_1 and S_2 respectively. Let $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ be the position vectors of P and R respectively relative to any arbitary origin, then $\overrightarrow{PR} = d\mathbf{r}$.

If co-ordinates of P and R are (x, y, z) $(x + dx,\ y + dy,\ z + dz)$ respectively, then

$$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz \quad ...(2)$$

As the values of scalar function at $P(x, y, z)$ and $R(x + dx,\ y + dy,\ z + dz)$ are ϕ and $\phi + d\phi$, we may write

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \quad ...(3)$$

$$= \left(\mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}\right)\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$

$$= (\nabla \phi)\cdot d\mathbf{r} \quad ...(4)$$

using (1) and (2)

In particular if we consider that the point R (*i.e.* $d\mathbf{r}$) lies on the level surface S_1, then $d\phi = 0$.

so that

$$(\nabla \phi)\cdot d\mathbf{r} = 0 \quad ...(5)$$

thereby showing that *the vector* $\nabla \phi$ *is normal to the level surface* S_1.

In dn represents the distance along the normal from point P to the surface S_2, then

*A level surface is one which has same value of scalar function at each point.

$$dn = PQ = dr\cos\theta = \hat{n}\cdot d\mathbf{r} \qquad ...(6)$$

where $\hat{n}$ is a unit vector normal to the surface S_1 at P. If we propagate from P to Q, the value of scalar function ϕ increases by an amount $d\phi$; therefore we may write

$$d\phi = \frac{\partial\phi}{\partial n}\,dn = \frac{\partial\phi}{\partial n}\,\hat{n}\cdot d\mathbf{r} \text{ using (6)} \qquad ...(7)$$

Comparing (4) and (7), we get

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial n}\,\hat{n} \qquad ...(8)$$

thus *the gradient of a scalar function ϕ at any point is a vector whose magnitude is equal to the rate of change of scalar function ϕ along the normal to the level surface and whose direction is normal to the level surface at the point.*

As $\frac{\partial\phi}{\partial n}\hat{n}$ gives the greatest rate of increase of ϕ with respect to space variables, therefore grad ϕ may be defined as follows :

The gradient of a scalar function ϕ is a vector whose magnitude is equal to maximum rate of change of scalar function ϕ with respect to space variables and whose direction is along that change.

If u and v are scalar differentiable functions, then it may be easily seen that

$$\text{grad}\,(u+v) = \text{grad}\,u + \text{grad}\,v \qquad ...(9)$$

$$\text{grad}\,(uv) = u\,\text{grad}\,v + v\,\text{grad}\,u. \qquad ...(10)$$

1·3. Line, Surface and Volume Integrals.

(i) Line Integral. The integral of a point function along a curve is called the *line integral.*

Let $\mathbf{r} = \mathbf{r}(t)$ be the equation of a curve. If ϕ and $\mathbf{A}$ are the scalar and vector fields respectively and $d\mathbf{r}$ is the vector increment of length, then we may encounter the integrals

$$\int_C \phi\, d\mathbf{r} \qquad ...(1)$$

$$\int_C \mathbf{A}\cdot d\mathbf{r} \qquad ...(2)$$

$$\int_C \mathbf{A}\times d\mathbf{r} \qquad ...(3)$$

each of which being as *line integral* along the curve C that may be open or closed. The results of integration are respectively a vector, a scalar and a vector.

To compute any of the integrals, the method of attack will be to reduce the vector integrals, into scalar integrals with which one is assumed to be familiar.

As ϕ is a scalar function and $d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$, integral (1) immediately reduces to

$$\int_C \phi\, d\mathbf{r} = \mathbf{i}\int_C \phi(x,y,z)\,dx + \mathbf{j}\int_C \phi(x,y,z)\,dy + \mathbf{k}\int_C \phi(x,y,z)\,dz \qquad ...(4)$$

the three integrals on R.H.S. of (4) are ordinary scalar integrals and to avoid complications we shall assume that they are Riemann integrals. The integral with respect to x can not be evaluated unless y and z are known in terms of x and similarly for the integrals with respect to y and z. This simply means that the path of integration C must be specified. Unles the integrand has special properties that lead the integral to depend only on the value of end points, the value will depend upon the particular choice of C.

The line integrals (2) and (3) may be interpreted in the similar fashion and like integral (1) they are dependent, in general, on the choice of the path.

Line integral (2) in most commonly used in vector analysis, it is called the *line interal of the tangential component of vector* **A** *along the curve* **C.** If **A** (vector functions of position) represents the force **F** acting on the particle, then the line integral *i.e.* $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the work done by the force.

An important special case arises in scalar calculus when the function to be integrated is an exact differential where the value of integral is independent of the path. In vector analysis let

$$\mathbf{A} = \text{grad}\ \phi = \nabla \phi$$

Then

$$\int_P^Q \mathbf{A} \cdot d\mathbf{r} = \int_P^Q \nabla \phi \cdot d\mathbf{r}$$

$$= \int_P^Q \left\{ \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right\}$$

$$= \int_P^Q d\phi = \phi_Q - \phi_P$$

where P and Q are initial are final points of the curve with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2).

If the integration is taken around a closed curve, then $Q = P$, so that

$$\int_P^P \mathbf{A} \cdot d\mathbf{r} = \int_P^P \nabla \phi \cdot d\mathbf{r} = \oint \nabla \phi \cdot d\mathbf{r} = \phi_P - \phi_P = 0$$

where the symbol $\oint$ denotes the integral along the boundary of closed curve.

Thus if $\mathbf{A} = \text{grad}\ \phi$, then the line integral $\int_P^Q \mathbf{A} \cdot d\mathbf{r}$ depends only on initial and final values of ϕ and is independent of the path and also $\oint \mathbf{A} \cdot d\mathbf{r} = 0$.

Conversely if $\oint \mathbf{A} \cdot d\mathbf{r} = 0$, then there must exist a scalar point function ϕ such that $\mathbf{A} = \text{grad}\ \phi$.

If the line integral $\int_C \mathbf{A} \cdot d\mathbf{r}$ is independent of path, then the vector field $\mathbf{A}(x, y, z)$ is said to be *conservative* or lamellar or noncurl vector field.

From above discussion it follows that :

A vector field **A** *is said to be conservative if and only if there exists a scalar point function* ϕ *such that* $\mathbf{A} = \text{grad}\ \phi$.

Conversely if a vector field **A** is derivable from a scalar point function ϕ according to relation $\mathbf{A} = \text{grad}\ \phi$, then vector field **A** is a conservative field.

2. Surface Integral. Let S be any surface divided into infinitesimal elements each of which may be considered as a vector $d\mathbf{S}$. Then if ϕ and **A** are scalar and vector fields repectively, the surface integrals may be encountered as

$$\iint_S \phi\ d\mathbf{S} \qquad \text{...(5)}$$

$$\iint_S \mathbf{A} \cdot d\mathbf{S} \qquad \text{...(6)}$$

$$\iint_S \mathbf{A} \times d\mathbf{S} \qquad \text{...(7)}$$

The results of integration are respectively a vector, a scalar and a vector.

Often the area element $d\mathbf{S}$ may be written as $d\mathbf{S} = \hat{n}\, dS$ where $\hat{n}$ is a unit (normal) vector to indicate the positive direction. There are two conventions for chosing the positive direction. Firstly if the surface is a closed surface, we agree to take the outward normal as positive. Secondly if the surface is an open surface, the positive normal depends on the direction in which the perimeter of the open surface is traversed. If the right hand fingers are placed in the direction of travel around the perimeter, the positive normal is indicated by the thumb of the right hand.

The surface integral $\iint_S \mathbf{A} \cdot d\mathbf{S}$ is called the flow or flux of the vector function $\mathbf{A}$ through the given surface S. For example if $\mathbf{A} = \rho \mathbf{v}$ where ρ is the density and $\mathbf{v}$ the velocity of fluid, then the surface integral $\iint_S \rho \mathbf{v} \cdot d\mathbf{S}$ denotes the amount of fluid flowing through the given surface in unit time.

Volume Integrals. Let $dV = dx\,dy\,dz$ denote the element of volume. As volume element dV is a scalar, there are only two possible volume integrals

$$\iiint_V \phi \, dV \qquad ...(8)$$

$$\iiint_V \mathbf{A} \, dV \qquad ...(9)$$

the results of integration are respectively a scalar and a vector.

It is often convenient to convert multiple integrals into others with fewer integrals sign. This is done by two important theorems viz Gauss' divergence theorem and Stoke's theorem of vector analysis.

1·4. Divergence of a Vector Function

Let there exist a vector field $\mathbf{A}$ in a certain region of space. The divergence of the vector field $\mathbf{A}$ at certain point $P(x, y, z)$ is defined as the outward flux of the vector field $\mathbf{A}$ per unit volume enclosed, through an infinitesimal closed surface surrounding the point P.

If τ is the infinitesimal volume enclosed by an infinitely small closed surface S surrounding the point $P(x, y, z)$, then the divergence of the vector field $\mathbf{A}$ at point P is defined as

$$\text{div}\,\mathbf{A} = \lim_{\tau \to 0} \frac{\iint_S \mathbf{A} \cdot d\mathbf{S}}{\tau} \qquad ..(1)$$

The divergence of a vector function is a scalar quantity. It should be noted that divergence itself is simply an operator and has no physical meaning in itself. After operating on suitable physical vector functions, it represents various significant physical scalar quantities. To understand the physical meaning of the divergence of a vector field, we consider the following two examples :

1. The divergence of fluid velocity $\mathbf{v}$ at a point represents the quantity of fluid flowing out per second per unit volume enclosed by infinitesimal closed surface surrounding that point. If div $\mathbf{v}$ at a point is negative, it means that fluid is constantly flowing towards that point and thus there exists a sink for the fluid. If div $\mathbf{v}$ at a point is positive, it indicates the existence of the sources of fluid at that point.

2. The divergence of current density $\mathbf{J}$ at a point is numerically equal to the quantity of electric charge flowing out per second per unit volume through an infinitesimal closed surface surrounding that point. Taking the law of conservation of charge into considration, div $\mathbf{J}$ at any point is equal to the rate of decrease of charge density in the neighbourhood of that point, thus

$$\text{div}\,\mathbf{J} = -\frac{\partial \rho}{\partial t}$$

or

$$\text{div}\,\mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \qquad ...(2)$$

This equation is known as equation of continuity and represents the law of *conservation of charge.*

If the divergence of any vector function in a region is zero ((*i.e.* div $\mathbf{A} = 0$), this means that the flux of the vector function entering any element of this region is equal to that leaving it. Any vector $\mathbf{A}$, satisfying the condition div $\mathbf{A} = 0$, is called a solenoidal vector.

The divergence in cartesian co-ordinates : Consider an infinitesimal rectangular box (fig. 1.10), with sides Δx, Δy and Δz and one corner at the point $P(x, y, z)$ in the region of any vector function $\mathbf{A}$ with rectangular faces perpendicular to co-ordinate axes. The rectangular box, in all, has six faces, two of them are perpendicular to X-axis having area $\Delta y\ \Delta z$ each, other two are perpendicular to Y-axis having area $\Delta x\ \Delta z$ each and the last two are perpendicular to Z-axis having area $\Delta x\ \Delta y$ each. If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors and A_x, A_y, A_z are components of vector function $\mathbf{A}$ at the middle point of the box along X, Y and Z axes respectively, the vector function $\mathbf{A}$ at this point is expressed in cartesian co-ordinates as

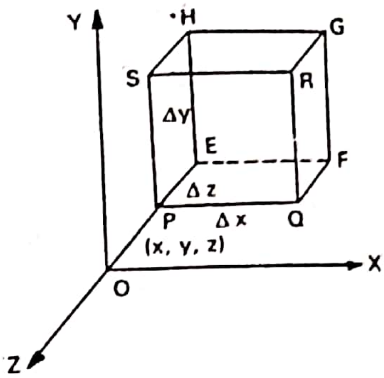

Fig. 1.10

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z.$$

To arrive at a cartesian expression for the divergence of $\mathbf{A}$ *i.e.* (div $\mathbf{A}$), consider two opposite faces $PEHS$ and $QFGR$ of this box, each of area $\Delta y\ \Delta z$ perpendicular to X-axis.

The flux emerging or outward flux through the face $QFGR$,

$$= \int_{QFGR} \overline{\mathbf{A}}_2 \cdot d\mathbf{S} = (\mathbf{i}\overline{A}_{2x} + \mathbf{j}\overline{A}_{2y} + \mathbf{k}\overline{A}_{2z}) \cdot (\Delta y\ \Delta z\ \mathbf{i})$$

$$= \overline{\mathbf{A}}_{2x} \cdot \Delta y\ \Delta z\,.$$

where $\overline{\mathbf{A}}_2$ is the average of the vector function over the surface $QFGR$, $\overline{A}_{2x}$ is the average of X-component of vector function over the face $QFGR$.

Similarly the outward flux through the face $PEHS$

$$= \int_{PEHS} \overline{\mathbf{A}}_1 \cdot d\mathbf{S} = (\mathbf{i}\overline{A}_{1x} + \mathbf{j}\overline{A}_{1y} + \mathbf{k}\overline{A}_{1z}) \cdot (-\Delta y\ \Delta z\ \mathbf{i})$$

$$= -\overline{A}_{1x} \cdot \Delta y\ \Delta z\,.$$

where $\overline{A}_{1x}$ is the average of X-component of vector function over the face $PEHS$.

Thus the net outward flux of vector function $\mathbf{A}$ through the two faces perpendicular to X-axis

$$= (\overline{A}_{2x} - \overline{A}_{1x})\ \Delta y\ \Delta z\,. \qquad \text{...(1)}$$

Although A_x does not remain uniform over the two faces considered ; but if the box is small, the difference between averages of A_x over the two faces (*i.e.* $\overline{A}_{2x} - \overline{A}_{1x}$) is approximately equal to the difference between the values of A_x at P and Q. Thus

$$\overline{A}_{2x} - \overline{A}_{1x} = A_x(x + \Delta x, y, z) - A_x(x, y, z)$$

$$= \frac{\partial A_x}{\partial x}\,\Delta x$$

where $\dfrac{\partial A_x}{\partial x}$ is the rate of variation of A_x with distance along X-axis

Thus the net outward flux of vector function $\mathbf{A}$ through the two faces perpendicular to X-axis from (1) is,

$$= \frac{\partial A_x}{\partial x}\,\Delta x\ \Delta y\ \Delta z$$

Similarly the net outward flux of **A** through the two faces *PQFE* and *SRGH* perpendicular to *Y*-axis is,

$$= \frac{\partial A_y}{\partial y} \Delta x \ \Delta y \ \Delta z$$

and net outward flux of **A** thorugh two faces *PQRS* and *FEGH*, perpendicular to *Z*-axis is,

$$= \frac{\partial A_z}{\partial z} \Delta x \ \Delta y \ \Delta z$$

Therefore the net outward flux of vector field **A** through the whole of the box of infinitesimal volume $\Delta x \ \Delta y \ \Delta z$ is given by,

$$\delta\phi = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \Delta x \ \Delta y \ \Delta z$$

Now div **A** at any point, which is the flux enclosed per unit infinitesimal volume surrounding that point, is given by

$$\text{div } \mathbf{A} = \lim_{\delta V (= \delta x \delta y \delta z) \to 0} \frac{\delta\phi}{\delta V} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{...(3)}$$

This is the required expression for div **A** in cartesian co-ordintes, div **A** may also be written as

$$\text{div } \mathbf{A} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (\mathbf{i} A_x + \mathbf{j} A_y + \mathbf{k} A_z)$$

$$= \nabla \cdot \mathbf{A}$$

where (∇) del is a vector operator defined as

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}.$$

Thus for divergence (div) we may write '*del-dot*' (∇ •).

1·5. Curl of a Vector Function.

To understand the meaning of the curl of a vector function, consider a small plane ara (say rectangular) in the region of *non-lamellar vector field.* (For non lamellar field **A**, div **A** $\neq$ 0). Then the area is perpendicular to the vector field as in position 1, the vector field is normal to each side of the rectangle and hence the line integral along all sides is zero. In position 2, area is parallel to the field, the line integrals along *BC* and *DA* are zero, while the line integrals along *AB* and *CD* have finite values. Thus the line integral around the boundary of the area has finite value, since the value of the vector field along the upper and lower edges is assumed to be different. Thus the value of the line integral around a closed path depends upon the orientation of the small vector area considered in the region of the vector field. *There is a certain orientation of the area, for which the line integral of the vector field is maximum.*

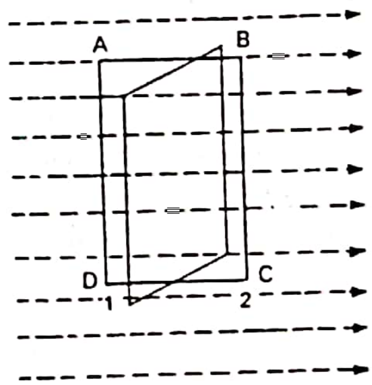

Fig. 1.11

The *curl of a vector field* at any point is defined as a vector quantity whose magntiude is equal to the maximum line integral per unit area along the boundary of an infinitesimal test area at that point and whose direction is perpendicular to the plane of the test area.

If **A** is any vector field at any point *P* and δ**S** an infinitesimal test area at point *P*, then curl **A** at point *P* is defined as

$$|\text{curl } \mathbf{A}| = \lim_{\delta S \to 0} \frac{\left[\oint \mathbf{A} \cdot d\mathbf{r}\right]_{maximum}}{\delta S}$$

or
$$\text{curl } \mathbf{A} = \lim_{\delta S \to 0} \frac{\oint \mathbf{A} \cdot d\mathbf{r}}{\delta S} \cdot \hat{n}. \quad ...(i)$$

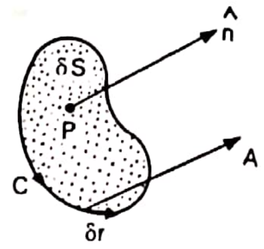

Fig. 1.12

where $\hat{\mathbf{n}}$ is the unit vector along the normal to the plane of test area (determined by the right hand screw rule).

The curl of a vector field is sometimes called *circulation* or *rotation* (or simply rot). If the fluid velocity vector **v** has a 'curl' somewhere, this means that the velocity fluid, over and above the general motion in a certain direction, has something like $\begin{smallmatrix}\leftarrow\uparrow\\ \downarrow\rightarrow\end{smallmatrix}$ or $\begin{smallmatrix}\rightarrow\downarrow\\ \uparrow\leftarrow\end{smallmatrix}$ superimposed on it in that region, because the existence of 'curl' of the vector field at a point of space indicates the circulation or vorticity there.

Since the line integral of a conservative field **A** around any closed path is zero *i.e.*

$$\oint \mathbf{A} \cdot d\mathbf{r} = 0,$$

therefore *the conservative vector fields have a zero curl at all points of space.* That is why such fields are known as *non-curl fields or lamellar vector fields.*

An example of conservative or lamellar vector field is the electrostatic field **E**, therefore curl $\mathbf{E} = 0$.

The curl in cartesian co-ordinates : Let us assume an infinitesimal rectangular area *EFGH* with sides Δx and Δy parallel to *X–Y* plane in the region of a vector function **A** (fig. 1.13). In order to find the curl **A** at point *P* within the rectangle, we have to calculate the line integral of the vector function **A** around the boundary of the rectangle.

We assign the positive direction according to right hand rule, when fingertips of right hand point in the direction of positive sense of $\delta\mathbf{r}$, the thumb points in the direction of positive normal to the surface (*i.e.*, the direction of the curl **A**). According to this rule, the line interal of **A** when taken along the circulation *EFGH* will give us *z*-component of curl **A**. Let the co-ordinates of *E* be (x, y, z).

If A_x, A_y and A_z are cartesian components of **A** at *P*, then

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z.$$

The line integral of the vector field along the path *EF*

$$= \int_{EF} \bar{\mathbf{A}} \cdot d\mathbf{r} = (\mathbf{i}\bar{A}_{1x} + \mathbf{j}\bar{A}_{1y} + \mathbf{k}\bar{A}_{1z}) \cdot (\mathbf{i}\,\Delta x)$$

$$= \bar{A}_{1x}\,\Delta x$$

where $\bar{A}_{1x}$ is the average value of *X*-component of the vector function over the path *EF*.

The line integral of the vector field along the path *GH*

$$= \int_{GH} \bar{\mathbf{A}}_2 \,.\, d\mathbf{r}$$

$$= (\mathbf{i}\bar{A}_{2x} + \mathbf{j}\bar{A}_{2y} + \mathbf{k}\bar{A}_{3z}) \,.\, (-\mathbf{i}\,\Delta x)$$

$$= -\bar{A}_{2x} \,.\, \Delta x$$

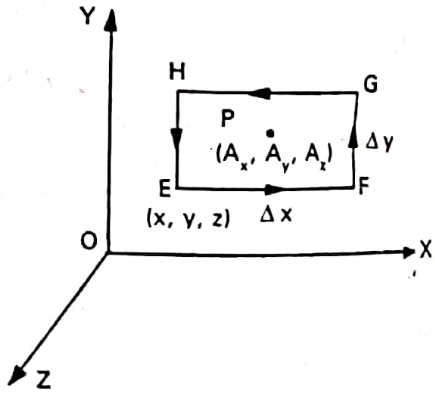

Fig. 1.13

where $\bar{A}_{2x}$ is the average of *X*-component of vector function over the path *GH*.

Hence the contribution to the line integral $\oint_{EFGH} \mathbf{A} \cdot d\mathbf{r}$ from these two parts (*EF* and *GH*) parallel to *X*-axis is,

$$-(\bar{A}_{2x} - \bar{A}_{1x})\,\Delta x. \quad ...(1)$$

As the rectangle is infinitely small, the difference between the averages of A_x (*i.e.* $\bar{A}_{2x} - \bar{A}_{1x}$) along these two paths may be approximated to the difference between the values of A_x at *E* and *H*. Thus

$$\overline{A}_{2x} - \overline{A}_{1x} = A_{Hx} - A_{Ex} = A_x\ (x, y+\Delta y, z) - A_x\ (x, y, z)$$

$$= \frac{\partial A_x}{\partial y}\ \Delta y. \quad ...(2)$$

Hence, from eqn. (1) the contribution to the line integral $\oint_{EFGH} \mathbf{A} \cdot d\mathbf{r}$ from the path EF and GH

$$= -\frac{\partial A_x}{\partial y}\ \Delta y\, \Delta x \quad ...(3)$$

Similarly the line integral of vector field along path $FG = +\overline{A}_{3y}.\ \Delta y$, and the line integral of the vector field along path $HE = -\overline{A}_{4y}\ \Delta y$.

Hence contribution to the line integral $\oint_{EFGH} \mathbf{A} \cdot d\mathbf{r}$ from the paths FG and HE is

$$= (\overline{A}_{3y} - \overline{A}_{4y})\ \Delta y = \{A_y\ (x+\Delta x, y, z) - A_y\ (x, y, z)\}\ \Delta y$$

$$= \frac{\partial A_y}{\partial x}\ \Delta x\ \Delta y \quad ...(4)$$

Therefore the line integral around the whole rectangle $EFGH$ from eqns. (3) and (4), is given by

$$T = \oint_{EFGH} \mathbf{A} \cdot d\mathbf{r} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \Delta x\ \Delta y. \quad ...(5)$$

Now curl $\mathbf{A}$ at any point is defined as the line integral per unit infinitesimal area at that point along its boundary and is directed along positive normal to the area. Here the area of the rectangle is $\Delta x\ \Delta y.$ whose positive normal is along z-direction. Hence from eqn. (5) we get z-component of curl $\mathbf{A}$ as

$$(\text{curl } \mathbf{A})_z = \underset{\delta S\ (\Delta x\ \Delta y) \to 0}{\text{limit}} \frac{\delta T}{\delta S} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \quad ...(6)$$

If we rotate the rectangle in YZ plane with its positive normal along X-axis, we get X-component of line integral in the similar manner as,

$$(\text{curl } \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \quad ...(7)$$

Similarly if the rectangle lies in Z–X plane with its positive normal along Y-axis, we get Y-component of curl $\mathbf{A}$ as,

$$(\text{curl } \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \quad ...(8)$$

Summing up the results given in eqns. (6), (7) and (8), we get

$$\text{curl } \mathbf{A} = \mathbf{i}\ (\text{curl } \mathbf{A})_x + \mathbf{j}\ (\text{curl } \mathbf{A})_y + \mathbf{k}\ (\text{curl } \mathbf{A})_z$$

$$= \mathbf{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \quad ...(9)$$

$$= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times (\mathbf{i} A_x + \mathbf{j} A_y + \mathbf{k} A_z)$$

$$= \nabla \times \mathbf{A}. \quad ...(10)$$

Thus for curl, we may write *'del-cross'* ($\nabla \times A$).

Eqn. (9) or (10) represents the curl $\mathbf{A}$ in terms of cartesian co-ordinates which may also be written in the determinant form as,

$$\text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \quad ...(11)$$

Thus for curl we may write del-cross ($\nabla \times$).

1·7. Gauss Divergence Theorem.

Gauss divergence theorem enables us to transform volume integral of the divergence of the vector function into surface integral of the vector field and vice-versa. The theorem states.

The flux of a vector field A over any closed surface S is equal to the volume integral of the divergence of the vector field over the volume V enclosed by the surface S i.e.

$$\iint_S \mathbf{A} \cdot d\mathbf{S} = \iiint_V \text{div } \mathbf{A} \, dV.$$

Let us consider a finite volume V of any shape enclosed by the surface S (fig. 1.14a) in the region of any vector function $\mathbf{A}$.

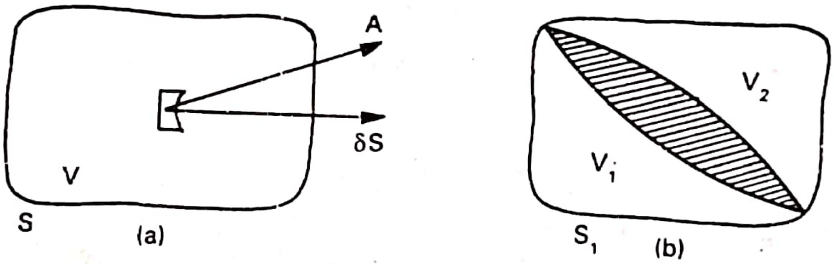

Fig. 1.14

The flux diverging from the surface S of volume V is,

$$\phi = \int_S \mathbf{A} \cdot d\mathbf{S}.$$

Let us now divide the volume V into two parts of volumes V_1 and V_2 enclosed by surfaces S_1 and S_2 respectively (fig. 1.14 b). The flux emerging out of surface $S_1 = \int_{S_1} \mathbf{A} \cdot d\mathbf{S}_1$

and the flux emerging out of surface $S_2 = \int_{S_2} \mathbf{A} \cdot d\mathbf{S}_2$

As the shaded surface in fig. (1.14b) is common to both the volumes, the flux emerging out of it for both the volumes being equal and in opposite directions cancel each other when considered together. The rest of the surfaces S_1 and S_2 is identical to the original surface S. Thus obviously

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$$\phi = \int_S \mathbf{A} \cdot d\mathbf{S} = \int_{S_1} \mathbf{A} \cdot d\mathbf{S}_1 + \int_{S_2} \mathbf{A} \cdot d\mathbf{S}_2 \qquad ...(1)$$

Simiarly if we divide the volume V into a large number of parts $V_1, V_2 \ldots V_i \ldots V_N$ enclosed by surfaces $S_1, S_2 \ldots S_i \ldots S_N$ respectively, we must have

$$\phi = \int_S \mathbf{A} \cdot d\mathbf{S} = \int_{S_1} \mathbf{A} \cdot d\mathbf{S}_1 + \int_{S_2} \mathbf{A} \cdot d\mathbf{S}_2 + \ldots$$

$$+ \int_{S_i} \mathbf{A} \cdot d\mathbf{S}_i + \ldots + \int_{S_N} \mathbf{A} \cdot d\mathbf{S}_N$$

$$= \sum_{i=1}^{N} \int_{S_i} \mathbf{A} \cdot d\mathbf{S}_i$$

We may also write $\longrightarrow (\mathrm{div}\,\vec{A})_i = \dfrac{\iint_{S_i} \vec{A} \cdot d\vec{a}}{V_i}$

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \sum_{i=1}^{N} V_i \frac{\int_{S_i} \mathbf{A} \cdot d\mathbf{S}_i}{V_i}$$

If N is sufficiently large, then volume V_i becomes infinitely small, *i.e.*, if N tends to infinity, V_i tends to zero and in the limit we may write

$$\lim_{V_i \to 0} \frac{\int_{S_i} \mathbf{A} \cdot d\mathbf{S}_i}{V_i} = \mathrm{div}\,\mathbf{A} \qquad ...(3)$$

$\Rightarrow \lim_{V_i \to 0} \int_{S_i} \mathbf{A} \cdot d\mathbf{S}_i = \mathrm{div}\,\vec{A} \cdot V_i$

and convert the summation into integration writing dV for infinitely small volume V_i; so that eqn. (3) in the limit, may be written as

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \int_V \mathrm{div}\,\mathbf{A} \cdot dV \qquad ...(4)$$

This equation is called *Gauss's divergence theorem and is used to convert the volume integral of the divergance of the vector field into surface integral of the vector field and vice-versa.*

1.8. Stoke's Theorem.

Stoke's theorem enables us to transform the surface integral of the curl of the vector field $\mathbf{A}$ into the line integral of that vector field $\mathbf{A}$ over the boundary C of that surface and vice-versa. The theorem states.

The flux of the curl of a vector function* $\mathbf{A}$ *over any surface* S *of any shape is equal to the line integral of the vector field* $\mathbf{A}$ *over the boundary* C *of that surface i.e.

$$\iint_S \mathrm{curl}\,\mathbf{A} \cdot d\mathbf{S} = \int_C \mathbf{A} \cdot d\mathbf{r}.$$

To derive Stoke's theorem, let us consider a surface S with C its boundary. Let us calculate the line integral of the vector function $\mathbf{A}$ around the boundary C of the surface S.

The line integral of $\mathbf{A}$ around the boundary C of surface S

$$= \oint_C \mathbf{A} \cdot d\mathbf{r}.$$

Now let us divide the surface S into two parts of surfaces S_1 and S_2, having boundaries C_1 and C_2 respectively (fig. 1.15 b).

The line integral of $\mathbf{A}$ around the boundary of S_1

$$= \oint_{C_1} \mathbf{A} \cdot d\mathbf{r}_1$$

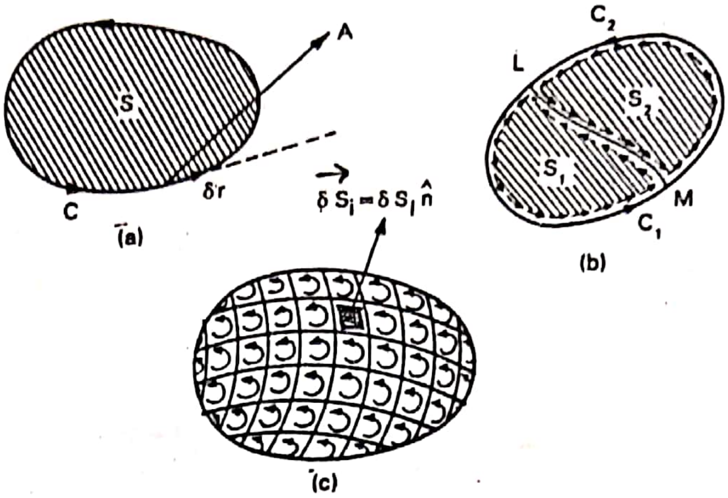

Fig. 1.15

and the line integral of $\mathbf{A}$ around the boundary of S_2

$$= \oint_{C_2} \mathbf{A} \cdot d\mathbf{r}_2 .$$

As the thick line LM is common boundary to both the surfaces and is traversed in opposite directions in the two integrals (fig. 1.15 b), the integral of $\mathbf{A}$ along this common boundary, being equal and opposite, cancel each other when considered together. The rest of the boundaries C_1 and C_2 is identical to the original boundary C. Thus obviously

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{A} \cdot d\mathbf{r}_1 + \oint_{C_2} \mathbf{A} \cdot d\mathbf{r}_2 . \qquad \text{...(1)}$$

Simiarly if we divide the surface S into a large number of parts $S_1, S_2 \ldots, S_i \ldots S_N$ having boundaries $C_1, C_2, \ldots, C_i \ldots, C_N$ respectively (fig. 1.15 c), then

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{A} \cdot d\mathbf{r}_1 + \oint_{C_2} \mathbf{A} \cdot d\mathbf{r}_2 + \ldots$$

$$\ldots + \oint_{C_i} \mathbf{A} \cdot d\mathbf{r}_i + \ldots + \oint_{C_N} \mathbf{A} \cdot d\mathbf{r}_N$$

$$= \sum_{i=1}^{N} \oint_{C_i} \mathbf{A} \cdot d\mathbf{r}_i . \qquad \text{...(2)}$$

This may also be written as

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \sum_{i=1}^{N} S_i \frac{\int_{C_i} \mathbf{A} \cdot d\mathbf{r}_i}{S_i} . \qquad \text{...(3)}$$

If N is sufficiently large, then suface area S_i becomes infinitely small, *i.e.*, if N tends to infinity, S_i tends to zero and in the limit we may write definition of curl $\mathbf{A}$ as

$$\lim_{S_i \to 0} \frac{\int_{C_i} \mathbf{A} \cdot d\mathbf{r}_i}{S_i} = \text{curl}\, \mathbf{A} \cdot \hat{n}$$

and convert the summation into integration writing dS for infinitely small area S_i, so that eqn. (3), in the limit, may be written as

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_C \text{curl } \mathbf{A} \cdot \hat{n} d\mathbf{S}$$

or

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_C \text{curl } \mathbf{A} \cdot d\mathbf{S} \qquad ...(4)$$

This equation is called **Stoke's theorem.** It is necessary for vector feild $\mathbf{A}$ to be well behavedc continuous function. Stoke's theorem ralates the line integral of a vector function to the surface integral of the curl of the vector, whereas Gauss's theorem relates the surface integral of the vector to the volume integral of the divergence of the vector.

8

Maxwell's Equations and Electromagnetic Waves

8.1. Introduction

In the preceding chapters we have dealt with *steady state problems* in electrostatics and magnetostatics treating electric and magnetic phenomenon independent of each other. The only link between them was the fact that electric currents which produced magnetic fields are basically electric in nature, being charges in motion. Now if we wish to consider more general problems in which field quantities may depend upon time, the almost independent nature of electric and magnetic phenomenon disappears. Time-varying magnetic fields give rise to electric field and vice-versa. We then must speak of *electromagnetic fields* rather than electric and magnetic fields. The behaviour of time dependent electromagnetic fields is described by a set of equations known as *Maxwell's equations.* These equations are mathematical abstractions of experimental results.

In this chapter we shall steady the formulation of *Maxwell's equations* along with their general properties and the basic conservation law of charge and energy.

8.2. Equation of Continuity

Conservation of charge : According to principle of conservation of charge *the net amount of charge in an isolated system remains constant.* For generality let us assume that the charge density is a function of time. Then the principle of conservation of charge may be stated as follows :

If the net charge crossing a surface bounding a closed volume is not zero, then the charge density within the volume must change with time in such a manner that the time rate of increase of charge within the volume equals the net rate of flow charge into the volume. This statement of conservation of charge in a medium may be expressed by the equation of continuity which may be derived as follows :

Let S be the surface enclosing a volume V and let $d\mathbf{S}$ be a small element of this surface. The direction of $d\mathbf{S}$ is taken to be that of the outward normal. If $\mathbf{J}$ is the current density (*i.e.* current per unit area placed normal of direction of current flow) at a point on surface element $d\mathbf{S}$, then $\mathbf{J} \cdot d\mathbf{S}$ represents the charge per unit time leaving volume V across $d\mathbf{S}$. Therefore the *time rate at which charge leaves the volume V* bounded by entire surface S is given by

$$\int_S \int \mathbf{J} \cdot d\mathbf{S}.$$

If q is charge contained is V, then according to charge conservation law, the above integral must be equal to $-dq/dt$, where dq/dt represents the time rate of flow of charge into V, thus

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$$\iint \mathbf{J} \cdot d\mathbf{S} = -\frac{dq}{dt}. \quad \text{...(1)}$$

But $$q = \iint_V \int \rho \, dv,$$

where ρ is the charge density and dv is an element of volume.

Therefore equation (1) takes the form

$$\int_S \int \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt} \iint_V \int \rho \, dv.$$

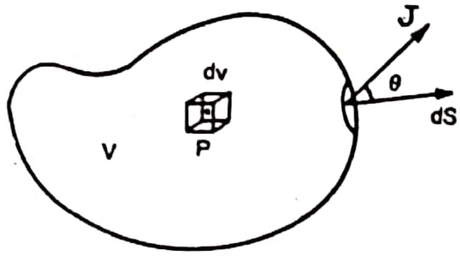

Fig. 8.1

Since the order of differentiation and integration is interchangeable, therefore

$$\int_S \int \mathbf{J} \cdot d\mathbf{S} = -\iiint \frac{\partial \rho}{\partial t} \, dv, \quad \text{...(2)}$$

But from Gauss divergence

$$\int_S \int \mathbf{J} \cdot d\mathbf{S} = \iint_V \int \text{div } \mathbf{J} \, dv \quad \text{...(3)}$$

Comparing (2) and (3), we get

$$\int_V \iint \text{div } \mathbf{J} \, dv = -\iiint \frac{\partial \rho}{\partial t} \, dv.$$

or $$\int_V \iint \left(\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dv = 0.$$

Since volume is arbitrary, therefore integrand must be zero

i.e. $$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0.$$

This is the required *equation of continuity and expresses the conservation of charge.*

The current is called stationary if there is no accumulation of charge at any point *i.e.* for stationary current $\partial \rho / \partial t = 0$ at all points. Therefore the criterion for stationary flow is

$$\text{div } \mathbf{J} = \nabla \cdot \mathbf{J} = 0. \quad \text{...(5)}$$

8.3. Maxwell's Postulate ; Displacement Current.

From Amperes circuital law, we have

$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$

$$\oint \mathbf{H} \cdot d\mathbf{l} = I \quad \text{...(1)}$$

$\vec{H} = \frac{\vec{B}}{\mu_0}$

(refer equation (3) of section 5.8)

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}. \quad \text{..(3)}$$

$$\therefore \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}. \quad \text{...(4)}$$

But from Stoke's theorem

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \text{curl } \mathbf{H} \cdot d\mathbf{S}$$

Comparing (3) and (4), we get

$$\int_S \text{curl } \mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

i.e. $$\int_S (\text{curl } \mathbf{H} - \mathbf{J}) \cdot d\mathbf{S} = 0. \quad \text{...(5)}$$

As the surface is arbitrary, therefore integrand must vanish *i.e.*

$$\text{curl } \mathbf{H} - \mathbf{J} = 0$$

or

$$\text{curl } \mathbf{H} = \mathbf{J}. \quad ...(6)$$

Let us examine the validity of this equation for time-varying fields. Since div of curl of any vector quantity is always zero. therefore div curl $\mathbf{H} = 0$. Then equation (6) implies

$$\text{div } \mathbf{J} = 0. \quad ...(7)$$

Now continuity equation is

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \quad ...(8a)$$

i.e.

$$\text{div } \mathbf{J} = -\frac{\partial \rho}{\partial t}. \quad ...(8b)$$

According to this equation div $\mathbf{J} = 0$ only if $\partial\rho/\partial t = 0$ *i.e.*, charge density is static. Thus we conclude that Ampere's equation (1) is valid only for steady state conditions and is insufficient for the cases of time-varying fields. Hence to include time-varying fields Ampere's law must be modified. Maxwell investigated mathematically how one could alter Ampere's equaion (1) so as the make it consistent with the equation of continuity. Maxwell assumed that the definition for current densiy $\mathbf{J}$ is incomplete and hence some thing say, $\mathbf{J}_d$ must be added to it. Then total current density which must be solenoidal, becomes $C = \mathbf{J} + \mathbf{J}_d$.

Using this postulate, equation (6) becomes

$$\text{curl } \mathbf{H} = \mathbf{C} = \mathbf{J} + \mathbf{J}_d \quad ..(9)$$

In order to identify $\mathbf{J}_d$ let us take the divergence of equation (9) :

$$\text{div curl } \mathbf{H} = \text{div } (\mathbf{J} + \mathbf{J}_d)$$

But div curl $\mathbf{H} = 0$ since div of curl of any vector is always zero ; therefore we get

$$\text{div } (\mathbf{J} + \mathbf{J}_d) = 0$$

or

$$\text{div } \mathbf{J} + \text{div } \mathbf{J}_d = 0$$

or

$$\text{div } \mathbf{J}_d = -\text{div } \mathbf{J}. \quad ...(10)$$

But div $\mathbf{J} = -\partial\rho/\partial t$ from equation of continuity, hence equaion (10) becomes

$$\text{div } \mathbf{J}_d = \frac{\partial \rho}{\partial t} \quad ...(11)$$

But Gauss theorem in differential form gives

$$\text{div } \mathbf{D} = \rho \quad ...(12)$$

Using this equation (11) may be written as

$$\text{div } \mathbf{J}_d = \frac{\partial}{\partial t} (\text{div } \mathbf{D})$$

$$= \text{div} \left(\frac{\partial \mathbf{D}}{\partial t}\right)$$

This gives

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} \quad ...(13)$$

Therefore the modified from the Ampere's law is

$$\boxed{\text{curl } \mathbf{H} = \mathbf{J} + \mathbf{J}_d = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \quad ...(14)$$

The term which Maxwell added to Ampere's law to include time varying fields is known as *displacement current* because it arises when electric displacement vector $\mathbf{D}$ changes with time. By addition this term

Maxwell assumed that this term (displacement current) is as effective as the *conduction current* **J** for producing magnetic field.

Characteristics of displacement current

(i) Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current since it is not linked with the motion of charges. For example displacement current has a finite value even in a perfect vaccum where there are no charges at all.

(ii) The magnitude of displacement current is equal to rate of change of electric displacement vector *i.e.* $\mathbf{J}_d = \partial \mathbf{D}/\partial t$.

(iii) Displacement current serves the purpose to make the total current continuous across the discontinuity in a conduction current. As an example, a battery charging a capacitor produces a closed current loop in terms of total current $\mathbf{J}_{total} = \mathbf{J} + \mathbf{J}_d$.

(iv) Displacement current in a good conductor is negligible as compared to the conduction current at any frequency less than optical frequencies ($\approx 10^{15}$ Hertz).

With the postulate of displacement current Maxwell was able to derive his theory of electromagnetic waves. We may consider the experimental observation of such waves, with the properties predicted, as the experimental basis for *Maxwell's* postulate. Furthermore we shall show that this postulate has a reasonable physical interpretation.

8.4. Physical Interpretation of Maxwell's Postulate

Maxwell's original explanation of displacement current was puzzling, therefore for convenience we shall consider another point of view.

The modified form of Ampere's law may be expressed as

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}. \quad ...(1)$$

Just as $\oint \mathbf{E} \cdot d\mathbf{l}$ represents electromotive force in electrostatics, the magnetomotive force (m.m.f.) around the path *C* is

$$\text{m.m.f.} = \oint_C \mathbf{H} \cdot d\mathbf{l} \qquad \text{(refer section 5.29)}. \quad ...(2)$$

Substituting this into (1), we get

$$\text{m.m.f.} = \oint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}. \quad ...(3)$$

Equations (1) and (2) indicate that there are two ways of producing a magnetic intensity, one with a ordinary conduction current density **J**, as observed by Oersted and postulated by Ampere, and the other by means of time varying electric displacement, as postulated by Maxwell. Since $\mathbf{D} \propto \mathbf{E}$ for air or vacuum, we may say that *a changing electric field gives rise to a magnetic field.* This is the converse of Faraday's discovery that a changing magnetic field gives rise to an electric field.

If we take the case where $\mathbf{J} = 0$ everywhere, then equation (1) becomes

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \oint_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}, \quad (\mathbf{J} = 0). \quad ..(4)$$

This equaion indicates that time varying electric displacement produces a magnetic field : thus varifying Maxwell's postulate.

Direct observation of a magenetic field produced by a changing electric field is difficult. We can not look for an *induced magnetic current* because there are neither free poles nor conductors for magnetic currents. Also we cannot maintain a constant value of $\partial \mathbf{D}/\partial t$ long enough to measure the resulting magnetic field, as we would that due to a steady current. Thus Maxwell's postulate is not as susceptible to direct experimental verification as is Faraday's law. That is why it was the last fundamental law of classical electromagnetism to discover.

If Maxwell's postulate is converse of Faraday's law, then the question arises why is there not converse of Ampere's law ? If we put this quesiton in slight different manner we may say, why does equation (3) contains two terms, while the corresponding equation for e.m.f.

$$e = -\frac{d\phi}{dt} = -\iint \mathbf{B} \cdot d\mathbf{S} \qquad ...(5)$$

has but one term ?

The answer is that the term missing in equation (5) involves *a current density of magnetic current* or a flow of magnetic poles of one sign and since isolated poles of one sign and magnetic currents due to them have no physical significance, therefore the term analogous to **J** in equation (3) and the converse of Ampere's law do not exist. Therefore we must realise the fact that the fundamental role of electric charges leads to certain lack of symmetry in our equations.

8.5. Maxwell's Equations and Their Empirical Basis.

There are four fundamental equations of electromagnetism known as *Maxwell's equations* which may be written in *differential form* as

1. $\nabla \cdot \mathbf{D} = \rho$ (Differential form of Gauss law in electrostatics)
2. $\nabla \cdot \mathbf{B} = 0$ (Differential form of Gauss law in magnetostatics)
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ Differential form of Faraday's law of electromagnetic induction
4. $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ Maxwell's modification of Ampere's law.

In above equations the notation have the following meanings :

D = electric displacement vector in coulomb/m^2.

ρ = charge density of coul/m^3.

B = magnetic induction in weber/m^2.

E = electric field intensity in volt/m or n/coul.

H = magnetic field intensity in amp/m-turn.

Each of Maxwell's equations represents a generalisation of certain experimental observations : Equations (1) represents the differential form of Gauss's law in electrostatics which in turn derives from Coulomb's law. Equation (2) represents Gauss's law in magnetostatics which is usually said to represent the fact that isolated magnetic poles do not exist in our physical world. Equation (3) represents differential form of Faraday's law of electromagnetic induction and finally equation (4) represents Maxwell's modification of Ampere's law to include time varying fields.

It is clear that the Maxwell's equations represent mathematical expression of certain experimental results. As already pointed out these equations can not be verified directly, however their application to any situation can be verified. As a result of extensive experimental work, Maxwell's equations are now known to apply to almost all macroscopic situations and they are usually used, must like conservation of momentum, as guiding principles.

8.6. Derivation of Maxwell's Equation

1. Derivation of first Equation Div $\mathbf{D} = \nabla \cdot \mathbf{D} = \rho$

Let us consider a surface S bounding a volume V in a dielectric medium. In a dielecric medium total charge consists of free charge plus polarisation charge. If ρ and ρ_p are the charge densiies of free charge and polarisation charge at a point in a small volume element dV, then *Gauss' law* can be expressed as

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V (\rho + \rho_p)\, dV$$

But polarisation charge density $\rho_p = -\operatorname{div} \mathbf{P}$, therefore above equaions takes the form

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V (\rho - \operatorname{div} \mathbf{P})\, dV$$

i.e.

$$\int_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_V \rho\, dV - \int_V \operatorname{div} \mathbf{P}\, dV$$

Using Gauss divergence theorem to change surface integral into volume integral, we get

$$\int_V \operatorname{div} (\varepsilon_0 \mathbf{E})\, dV = \int_V \rho\, dV - \int_V \operatorname{div} P\, dV$$

i.e.

$$\int_V \operatorname{div} (\varepsilon_0 \mathbf{E} + \mathbf{P})\, dV = \int_V \rho\, dV \qquad ...(1)$$

But $\varepsilon_0 \mathbf{E} + \mathbf{P} = \mathbf{D}$ = electric displacement vector.

Therefore equation (1) becomes

$$\int_V \operatorname{div} \mathbf{D}\, dV = \int_V \rho\, dV$$

$$\int_S \operatorname{div} (\mathbf{D} - \rho)\, dV = 0$$

Since this equation is true for all volumes, therefore the integrand in this equation must vanish *i.e.*

$$\operatorname{div} \mathbf{D} - \rho = 0$$

or

$$\operatorname{div} \mathbf{D} = \rho \quad i.e. \quad \nabla \cdot \mathbf{D} = \rho$$

2. Derivation of Second Equation div $\mathbf{B} = \nabla \cdot \mathbf{B} = 0$

Since isolated magnetic poles and magnetic currents due to them have no physical significance : therefore magnetic lines of force in general are either closed curves or go off to infinity. Consequently the number of magnetic lines of force entering any arbitrary closed surface is exactly the same as leaving it. It means that the flux of magnetic induction **B** across any closed surface is always zero, *i.e.*

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Using Gauss divergence theorem to change surface integral into volume integral, we get

$$\int_V \operatorname{div} \mathbf{B}\, dV = 0$$

As the surface bounding the volume is arbitrary, therefore this equation holds only if the integrand vanishes *i.e.*

$$\operatorname{div} \mathbf{B} = 0 \text{ or } \nabla \cdot \mathbf{B} = 0.$$

Note. For an alternative derivation of div $\mathbf{B} = 0$ refer section 5.10 of chapter 5.

3. Derivation of third equation

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

[According to Faraday's law of electromagnetic induction it is known that e.m.f. induced in a closed loop is defined as negative rate of change of magnetic flux *i.e.*

$$e = -\frac{d\phi}{dt}$$

But magnetic flux $\phi = \int_S \mathbf{B} \cdot d\mathbf{S}$ where S is any surface having loop as boundary

$$e = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$= -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \quad \text{....(2)}$$

(Since surface is fixed in space, hence only **B** changes with time).

But e.m.f. 'e' can also be computed by calculating the work done in carrying a unit charge round the closed loop C. Thus if **E** is the electric field intensity at a small element $d\mathbf{l}$ of loop, we have

$$e = \int_C \mathbf{E} \cdot d\mathbf{l} \quad \text{...(3)}$$

Comparing equations (2) and (3), we get

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \quad \text{...(4)}$$

Using Stoke's theorem to change line integral into surface integral, we get [$\because \int_C E \cdot dl = \int_S \text{curl} E \cdot dS$]

$$\int_S \text{curl}\, \mathbf{E} \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

or

$$\int_S \left(\text{curl}\, \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S} = 0. \quad \text{...(5)}$$

Since surface is arbitrary, therefore equation (5) holds only if the intergrand vanishes *i.e.*

$$\text{curl}\, \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

or

$$\text{curl}\, \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad i.e. \quad \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}.$$]

4. Derivation of fourth equation

$$\text{curl}\, \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

For its derivation refer section 8.3.