A COMBINATORIAL APPROACH FOR COMPUTING INTEGRAL BASES

J. BOEHM, W. DECKER, S. LAPLAGNE, G. PFISTER

Abstract. In [2], we gave an algorithm for computing integral bases in function fields of plane curves which has quasi-optimal runtime [4]. Our approach is local-to-global in the sense that we first compute a local contribution to the normalization at each singularity of the curve, and then put these local contributions together. To find a local contribution at a given singularity in the first step, we rely on Puiseux expansions and Hensel lifting to find a contribution to the normalization at each branch of the singularity, and then merge these contributions using Chinese remaindering. In this paper, we present a combinatorial approach for merging which often performs better in practise, notably in cases where the curve has a large number of branches at one of its singularities.

1. Introduction

In this paper, we develop an algorithm to compute the integral closure $\bar{A}$ in terms of an integral basis for the coordinate ring A of an algebraic curve. We focus on the case where the curve is defined over a field K of characteristic zero. Let $f \in K[X, Y]$ be an irreducible polynomial in two variables, let $C \subset \mathbb{A}^2(K)$ be the affine plane curve defined by f, and let

$$A = K[C] = K[X, Y]/\langle f(X, Y)\rangle$$

be the *coordinate ring* of C. We write x and y for the residue classes of X and Y modulo f, respectively. Throughout the paper, we suppose that f is monic in Y (due to Noether normalization, this can always be achieved by a linear change of coordinates). Then the *function field* of C is

$$K(C) = \mathrm{Q}(A) = K(x)[y] = K(X)[Y]/\langle f(X, Y)\rangle,$$

where x is a separating transcendence basis of $K(C)$ over K, and y is integral over $K[x]$, with integrality equation $f(x, y) = 0$. Indeed, we have the isomorphism $\mathrm{Q}(K[x][y]) \to K(x)[y]$ defined by mapping $1/h(x, y) \mapsto b(x, y)/x^c$, where $X^c = af + bh \in K[X][Y]$ is a representation which arises from a Bézout identity in $K(X)[Y]$ by clearing denominators.

Since A is integral over $K[x]$, $\overline{A}$ is equal to the integral closure of $K[x]$ in $K(C)$. Hence, $\overline{A}$ can be represented both as an A-module or as a module over $K[x] \cong K[X]$. As noted in [2, Remark 2], $\overline{A}$ is a free $K[x]$-module with the rank given by:

$$n := \deg_Y(f) = [K(C) : K(x)].$$

Definition 1.1. *An* integral basis *for $\overline{A}$ over $K[x]$ is a set $b_0, \dots, b_{n-1}$ of free generators for $\overline{A}$ over $K[x]$:*

$$\overline{A} = K[x]b_0 \oplus \cdots \oplus K[x]b_{n-1}.$$

Remark 1.2. (See [2, Remark 4].) In the above setting, there exist polynomials $p_i \in K[X][Y]$ monic of degree i in Y and polynomials $d_i \in K[X]$ such that

$$\left\{1, \frac{p_1(x,y)}{d_1(x)}, \ldots, \frac{p_{n-1}(x,y)}{d_{n-1}(x)}\right\}$$

is an integral basis for $\overline{A}$ over $K[x]$.

In [2], we gave an algorithm to compute integral bases for $\overline{A}$ which is local-to-global in the following sense: First, for each prime ideal P of the singular locus $\mathrm{Sing}(A)$, find a local contribution to the integral basis at P. Second, add up these local contributions. In this paper, we are concerned with improving the first step. For simplicity of our presentation, we assume that $\mathrm{Sing}(A)$ consists of just one prime ideal P, and that this prime ideal defines a K-rational singularity of C which is located at the origin[1].

In this setting, consider the decomposition

$$f = f_0 \widetilde{f} = f_0 f_1 \cdots f_r \tag{1}$$

given by the Weierstrass preparation theorem (see [1], [3]). Here, $f_0 \in K[[X]][Y]$ is a unit in $K[[X,Y]]$, and $f_1, \ldots, f_r \in K[[X]][Y]$ are irreducible Weierstrass polynomials to which we refer as the *branches* of f (over K, centered at P). If $g = f$, $g = \widetilde{f}$, or g is one of the branches f_i, and

$$B_g = K[[x]][y] = K[[X]][Y]/\langle g \rangle,$$

then it makes sense to speak of integral bases for $\overline{B_g}$ over $K[[x]]$ (see [2]).

Integral bases for the $\overline{B_{f_i}}$ can be computed using Puiseux series and Hensel lifting as discussed in [2]. Merging the computed bases gives us an integral basis for $\overline{B_{\widetilde{f}}}$. From this, we can read off an integral basis for $\overline{B_f}$ as described in [2, Proposition 41]. This basis is the desired local contribution to the integral basis for $\overline{A}$ at P.

In our previous paper, the merging of the integral bases for the $\overline{B_{\widetilde{f}_i}}$ is done using Chinese remaindering. This can be inefficient if there is a large number of branches of f at P. Indeed, to find the cofactors required for the Chinese remainder step, we may need to develop the involved Puiseux expansions up to a high order. Moreover, summing up the result may lead to generators which are not in triangular shape, and computing the triangular shape can be computationally expensive.

In this paper we present a modification of the algorithm in [2] which uses a combinatorial procedure to obtain an integral basis handling all the different conjugacy classes at once, and directly return the generators in triangular shape. On our way to explaining this, we first recall some formulae for valuations.

1.1. **Valuations.** Let $L\{\{X\}\}$ be the field of Puiseux series. A nonzero element $f \in L\{\{X\}\}$ can be uniquely written as

$$f = \sum_{k=k_0}^{\infty} c_k X^{k/n},$$

[1]If there are more singularities or conjugated singularities, the techniques presented in [2] allow us to reduce the problem to the case dealt with here.

with $c_k \in L$ for all $k \geq k_0$, $c_{k_0} \neq 0$, and $n \in \mathbb{N}$. The *valuation* of f is $\upsilon_X(f) = \frac{k_0}{n}$. The corresponding *valuation ring*

$$L\{\{X\}\}_{\upsilon \geq 0} = \bigcup_{k=1}^{\infty} L[[X^{1/k}]]$$

consists of all Puiseux series with non-negative exponents only. Henceforth, it will be denoted by $\mathcal{P}_X$.

Definition 1.3 (Valuation of a polynomial at a Puiseux expansion)**.** *If $q \in L\{\{X\}\}[Y]$ is any polynomial in Y with coefficients in $L\{\{X\}\}$, the* valuation *of q at $\gamma \in L\{\{X\}\}$ is defined to be*

$$\upsilon_\gamma(q) = \upsilon_X(q(\gamma)).$$

By the properties of valuations, we obtain

$$\upsilon_\gamma(pq) = \upsilon_\gamma(p) + \upsilon_\gamma(q).$$

Definition 1.4 (Valuation of a polynomial at another polynomial)**.** *Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be the set of Puiseux expansions of a polynomial $g \in L\{\{X\}\}[Y]$. The* valuation *of a polynomial $q \in L\{\{X\}\}[Y]$ at g is defined to be*

$$\upsilon_g(q) = \min_{1 \leq i \leq m} \upsilon_{\gamma_i}(q),$$

which we also denote by $\upsilon_\Gamma(q)$.

From the definitions, we obtain the following formulae.

Lemma 1.5. *Let $\gamma \in L\{\{X\}\}$ and let $q \in L\{\{X\}\}[Y]$ be a monic polynomial of degree $d \geq 1$ in Y. If $q = (Y - \eta_1(X)) \cdots (Y - \eta_d(X))$ is the factorization of q in $L\{\{X\}\}[Y]$, then*

$$\upsilon_\gamma(q) = \sum_{j=1}^{d} \upsilon_X(\gamma - \eta_j).$$

For a polynomial $g \in L\{\{X\}\}[Y]$ with Puiseux expansions $\{\gamma_1, \ldots, \gamma_m\}$,

$$\upsilon_g(q) = \min_{1 \leq i \leq m} \sum_{j=1}^{d} \upsilon_X(\gamma_i - \eta_j).$$

1.2. **Polynomials with maximal valuation.** We recall two results from [2] that are central for our combinatorial approach.

The first lemma says that if we look for a polynomial $p \in \mathcal{P}_X[Y]$ with maximal valuation at g, then we can always take a polynomial p whose Puiseux expansions are a subset of the expansions of g.

Lemma 1.6 ([2, Lemma 21])**.** *Let $g \in K[[X]][Y]$ be a square-free monic polynomial of degree $m \geq 1$ in Y, with Puiseux expansions $\gamma_1, \ldots, \gamma_m$. Fix an integer d with $1 \leq d \leq m-1$. If $\mathcal{A} \subset \{1, \ldots, m\}$ is a subset of cardinality d, set*

$$\operatorname{Int}(\mathcal{A}) = \min_{i \notin \mathcal{A}} \left(\sum_{j \in \mathcal{A}} \upsilon_X(\gamma_i - \gamma_j) \right).$$

Choose a subset $\widetilde{\mathcal{A}} \subset \{1,\dots,m\}$ *of cardinality* d *such that* $\mathrm{Int}(\widetilde{\mathcal{A}})$ *is maximal among all* $\mathrm{Int}(\mathcal{A})$ *as above, and set* $\widetilde{p}_d = \prod_{j\in\widetilde{\mathcal{A}}}(Y-\gamma_j) \in \mathcal{P}_X[Y]$. *Then* $\upsilon_g(\widetilde{p}_d) = \mathrm{Int}(\widetilde{\mathcal{A}})$, *and this number is the maximal valuation* $\upsilon_g(q)$, *for* $q \in L\{\{X\}\}[Y]$ *monic of degree* d *in* Y.

For $d = m-1$, we call $E(g) := \lfloor \upsilon_g(\widetilde{p}_d)\rfloor$ the maximal integrality exponent with respect to g. It is the maximum exponent of the denominators in an integral basis of g.

For our combinatorial approach, it will be easier to work in the ring $\mathcal{P}_X[Y]$ and once we determine which is an optimal subset of expansions for each degree, we construct a polynomial in $K[X][Y]$ using the following lemma, for which we recall also the proof since it gives a constructive way to go from $\mathcal{P}_X[Y]$ to $K[X][Y]$.

Lemma 1.7. *Suppose* $g \in K[[X]][Y]$ *is a square-free monic polynomial of degree* $m \geq 1$ *in* Y. *Let* $1 \leq d \leq m-1$ *be an integer, and denote by* R *any of the rings* $K[X]$, $K[X]_{\langle X\rangle}$, $K[[X]]$, $K((X))$, $\mathcal{P}_X$, *or* $L\{\{X\}\}$. *For a polynomial* $q \in R[Y]$ *which is monic of degree* d *in* Y, *the maximal valuation* $\upsilon_g(q)$ *is independent of the choice of* R *from the above list.*

Proof. For any R as in the statement, there are natural inclusions $K[X] \subset R \subset L\{\{X\}\}$. The valuation $\upsilon_g(q)$ is thus defined for any $q \in R[Y]$ and it is sufficient to prove that there is a polynomial $p_d \in K[X][Y]$ which maximizes the valuation $\upsilon_g(p)$ among all p in $L\{\{X\}\}[Y]$ of degree d. To see this, recall from Lemma 1.6 that there exist $\widetilde{p}_d = \prod_{j\in\widetilde{\mathcal{A}}}(Y-\gamma_j) \in \mathcal{P}_X[Y]$ which maximizes the valuation over $L\{\{X\}\}$ in degree d. There exists an integer k such that $\widetilde{p}_d \in L[[X^{1/k}]][Y]$. Truncating every γ_j to degree $\upsilon_g(\widetilde{p}_d)$, we obtain a polynomial $\overline{p}_d = \prod_{j\in\widetilde{\mathcal{A}}}(Y-\overline{\gamma}_j) \in L[X^{1/k}][Y]$ with the same valuation as $\widetilde{p}_d$. Applying the trace map for $L(X^{1/k})$ over $L(X)$ to the Y-monic polynomial $\overline{p}_d$, and dividing by the (integer) leading coefficient, we obtain a monic polynomial $p'_d \in L[X][Y]$ of degree d in Y with $\upsilon_g(p'_d) \geq \upsilon_g(\widetilde{p}_d)$[2]. Considering p'_d as a polynomial in X and Y with coefficients in L and adjoining these coefficients to K, we obtain an extension $K \subset K'$ of finite fields such that p'_d is in $K'[X][Y]$. We then apply the trace map of the extension $K \subset K'$ to p'_d and divide the result by its leading coefficient. This gives a monic polynomial $p_d \in K[X][Y]$ of degree d in Y satisfying $\upsilon_g(p_d) \geq \upsilon_g(\widetilde{p}_d)$. Note that by Lemma 1.6 and the choice of $\widetilde{p}_d$, the inequality is in fact an equality since $\widetilde{p}_d$ maximizes the valuation over $L\{\{X\}\}$. □

2. One Puiseux block

Let $\Gamma \subset \mathcal{P}_X$ be the set of all Puiseux expansions of f. The Puiseux blocks of f are a partition of the Puiseux expansions of f such that in each set the first non–rational term of every expansion is identical or conjugated. We assume first that f has only one Puiseux block.

To compute an integral basis of f, we compute for each $0 \leq d < \deg(f)$ a monic polynomial $p \in K[X][Y]$ of degree d with maximal valuation at f among all monic polynomials of degree d.

[2] Note that the trace map sends $X^{1/k}$ to zero.

Our strategy is to compute a factorization of p. If η is a Puiseux expansion of p and $\{\eta_1, \dots, \eta_s\}$ is the conjugacy class of η for the extension $K[X][Y] \hookrightarrow \mathcal{P}_X[Y]$, then $q = \prod_{i=1}^{s}(Y - \eta_i)$ is a factor of p. By Lemma 1.6, we can assume that any expansion η of p is a truncation of an expansion γ of f. Moreover, we can assume that there exists $\gamma \in \Gamma$ such that $\eta = \overline{\gamma}^{<t}$ for t an extended characteristic exponent of γ or $\eta = \overline{\gamma}^{\leq N}$, for N the integrality exponent of f.

Following [2, Algorithm 6], let $\Delta = \{\delta_1, \dots, \delta_m\}$ be the set of Puiseux expansions in a conjugacy class of f. In Algorithm 1 we provide a procedure to compute all possible factors of p coming from this class of expansions.

Algorithm 1 `PolynomialFactors`

Input: $\Delta = \{\delta_1, \dots, \delta_m\}$ the set of Puiseux expansions at the origin in a conjugacy class of f, developed up to the integrality exponent $N := E(f)$ of f.

Output: A set $Q \subset K[X][Y]$ of all the possible factors of an integral basis element coming from the conjugacy class Δ.

1: Let $\{t_1, \dots, t_s\}$ be the extended characteristic exponents of the expansions.
2: **for all** $t \in \{t_1, \dots, t_s\}$ **do**
3: Let $\rho_1, \dots, \rho_{\overline{m}}$ be the pairwise different elements in $\left\{\overline{\delta}_1^{<t}, \dots, \overline{\delta}_m^{<t}\right\}$.
4: Set
$$q_t := \prod_{i=1}^{\overline{m}}(Y - \rho_i(X)).$$
5: For N the integrality exponent of f, set $\overline{f_\Delta}^{\leq N} := \prod_{i=1}^{m}(Y - \overline{\delta_i}^{\leq N}(X))$.
6: **return** $Q = \{q_{t_1}, \dots, q_{t_s}, \overline{f_\Delta}^{\leq N}\}$

Example 2.1. Let $f = (Y^4 + 2X^3Y^2 + 2X^5Y + X^6 + 1/4X^7) + Y^5 \in \mathbb{Q}[X, Y]$. The Puiseux expansions of g are

$$\begin{aligned}
\gamma_1 &= IX^{3/2} + (-1/2I - 1/2)X^{7/4} + \dots, \\
\gamma_2 &= IX^{3/2} + (1/2I + 1/2)X^{7/4} + \dots, \\
\gamma_3 &= -IX^{3/2} + (1/2I - 1/2)X^{7/4} + \dots, \\
\gamma_4 &= -IX^{3/2} + (-1/2I + 1/2)X^{7/4} + \dots, \\
\gamma_5 &= -1 + \dots
\end{aligned}$$

where I is a root of $Z^2 + 1$.

There is only one class of expansions at the origin, $\Delta = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. The characteristic exponents are $3/2$ and $7/4$. The integrality exponent is 4. Applying Algorithm 1, we obtain the following polynomials:

$$\begin{aligned}
q_{3/2} &= Y, \\
q_{7/2} &= (Y - IX^{3/2})(Y + IX^{3/2})) = Y^2 + X^3, \\
\overline{f_\Delta}^{\leq 4} &= Y^4 + 2X^3Y^2 + 2X^5Y + X^6 + 1/4X^7.
\end{aligned}$$

Computing these polynomials for all the conjugacy classes of f we get all possible factors of p. The next step is to determine the multiplicity of these factors in the factorization of p so that the resulting polynomial has the desired degree and maximal valuation. We do this by exhaustive search among all possible combinations. The key argument for our algorithm in this case is that the valuation of $\gamma \in \Gamma$ at a polynomial $q \in K[X][Y]$ is always the same for all expansions γ in the same conjugacy class. We obtain Algorithm 2.

Algorithm 2 `IntegralElementOneBlock`

Input: $\Delta_1, \ldots, \Delta_s$ the conjugacy classes of Puiseux expansions at the origin of a monic polynomial $f \in K[X][Y]$, developed up to the maximum integrality exponent of f; a non-negative integer d, $0 \leq d \leq n = \deg_Y(\tilde{f})$.

Output: a polynomial $p \in K[X][Y]$ of degree d of maximal valuation at the set of expansions $\Delta = \Delta_1 \cup \cdots \cup \Delta_s$ among all monic polynomials of degree d; $o \in \mathbb{Q}_{\geq 0}$, the valuation of p at f.

1: For each $1 \leq i \leq s$, let $P_i = \texttt{PolynomialFactors}(\Delta_i)$, the polynomials factors corresponding to Δ_i.

2: Consider the set $\{p_1, \ldots, p_m\} = \cup_{i=1}^s P_i \subset K[X][Y]$ of all the polynomials obtained from all the conjugacy classes, and let $d_1, \ldots, d_m$ be the corresponding degrees.

3: Define $C = \{(c_1, \ldots, c_m) \in \mathbb{Z}^m_{\geq 0} : c_1 d_1 + \ldots c_m d_m = d\}$, the set of all possible m-tuples.

4: For each $\boldsymbol{c} \in C$, compute the valuation of $p_{\boldsymbol{c}} = p_1^{c_1} \cdots p_m^{c_m}$ at $\tilde{f}$ by the second formula in Lemma 1.5.

5: **return** $(p, v_{\tilde{f}}(p))$, for p the polynomial with maximal valuation at $\tilde{f}$ among all the polynomials computed.

We have seen in Lemma 1.7 that the maximal valuation over monic polynomials in $K[X][Y]$ of a given degree d is the same as the maximal valuation over polynomials in $\mathcal{P}_X[Y]$ of degree d. Hence Algorithm 2 provides an effective way to compute this valuation, which we call $o(g, d)$ or $o(\Gamma, d)$ for Γ the set of Puiseux expansions of g.

Example 2.2. Let $f = (Y^2+X^3)(Y^4+2X^3Y^2+2X^5Y+X^6+1/4X^7)+Y^7 \in \mathbb{Q}[X,Y]$. The Puiseux expansions of f are

$$\begin{aligned}
\gamma_1 &= IX^{3/2} + (-1/2I - 1/2)X^{7/4} + \ldots, \\
\gamma_2 &= IX^{3/2} + (1/2I + 1/2)X^{7/4} + \ldots, \\
\gamma_3 &= -IX^{3/2} + (1/2I - 1/2)X^{7/4} + \ldots, \\
\gamma_4 &= -IX^{3/2} + (-1/2I + 1/2)X^{7/4} + \ldots, \\
\gamma_5 &= IX^{3/2} + 1/4IX^{5/2} + \ldots, \\
\gamma_6 &= -IX^{3/2} - 1/4IX^{5/2} + \ldots, \\
\gamma_7 &= -1 + \ldots.
\end{aligned}$$

The integrality exponent of f is 8. There are two classes of Puiseux expansions: $\Delta_1 = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $\Delta_2 = \{\gamma_5, \gamma_6\}$, and both classes are in the same Puiseux block. Applying Algorithm 1 to Δ_1 we obtain the factors $\{Y, Y^2+X^3, \overline{f_{\Delta_1}}^{\leq 8}\}$. Applying Algorithm 1 to Δ_2 we obtain the factors $\{Y, \overline{f_{\Delta_2}}^{\leq 8}\}$. Now we apply Algorithm 2 for every $0 \leq d \leq 6$. We obtain the following elements:

$$
\begin{aligned}
&p_0 = 1, \upsilon_{\tilde{f}}(p_0) = 0 \\
&p_1 = Y, \upsilon_{\tilde{f}}(p_1) = 3/2 \\
&p_2 = Y^2 + X^3, \upsilon_{\tilde{f}}(p_2) = 3/2 + 7/4 = 13/4 \\
&p_3 = Y(Y^2 + X^3), \upsilon_{\tilde{f}}(p_3) = 3/2 + 3/2 + 7/4 = 19/4 \\
&p_4 = \overline{f_{\Delta_1}}^{\leq 8}, \upsilon_{\tilde{f}}(p_4) = 13/2 \\
&p_5 = Y \cdot \overline{f_{\Delta_1}}^{\leq 8}, \upsilon_{\tilde{f}}(p_5) = 13/2 + 3/2 = 8
\end{aligned}
$$

3. Direct approach

We consider now the case of a polynomial $f \in K[X][Y]$ whose Puiseux expansions at the origin are grouped into several Puiseux blocks. Let Γ be the set of all Puiseux expansions of f and let $\Pi_1, \ldots, \Pi_s$ be the Puiseux blocks of f. For each Puiseux block Π_i, let f_i be the corresponding factor of f in $K[[X]][Y]$ (that is, $f_i = \prod_{\gamma \in \Pi_i}(Y - \gamma)$). Let m_i be the cardinal of Π_i (and hence also the Y-degree of f_i).

We address first the (theoretical) problem of finding for each $0 \leq d < n$ a polynomial $p_d \in \mathcal{P}_X[Y]$ of maximal valuation at $\tilde{f}$ among all polynomials of degree d. We know that we can take the expansions of p_d as a subset of the expansions of $\tilde{f}$, hence we can factorize

$$p_d = p_{(1)} \cdot \cdots \cdot p_{(s)},$$

where $p_{(i)} \in \mathcal{P}_X[Y]$, $1 \leq i \leq s$, is a polynomial whose Puiseux expansions are a subset of the expansions of Π_i.

Note that although $\upsilon_\gamma(pq) = \upsilon_\gamma(p) + \upsilon_\gamma(q)$ for a single Puiseux expansion γ, it is not true in general that $\upsilon_g(pq) = \upsilon_g(p) + \upsilon_g(q)$ for a polynomial g, so even if we fix the degrees $c_1, \ldots, c_s$ of the polynomials $p_{(1)}, \ldots, p_{(s)}$, we cannot directly split the problem into smaller problems, one for each branch. In [2] we used the Chinese remainder theorem to merge the integral bases for the branches. In this section we will compute the polynomials p_d, $0 \leq d < n$ by exhaustive search over all possible tuples of degrees $(c_1, \ldots, c_s)$. In the next section we will show how to optimize the strategy using a combinatorial approach.

We recall the valuation formula from Lemma 1.5. For $q \in \mathcal{P}_X[Y]$ of degree d with Puiseux expansions $\{\eta_1, \ldots, \eta_d\}$,

$$\upsilon_\gamma(q) = \sum_{j=1}^{d} \upsilon_X(\gamma - \eta_j).$$

By the definition of Puiseux blocks, we deduce that if $\gamma \in \Gamma$ is not in Π_j then $\upsilon_\gamma(p_{(j)})$ only depends on the degree c_j of $p_{(j)}$ and not on the specific expansions of $p_{(j)}$. Since $\upsilon_X(\gamma - \eta)$ is the same for any $\gamma \in \Pi_i$ and $\eta \in \Pi_j$, $i \neq j$, we note υ_{ij} this value. We obtain the following formulae.

Lemma 3.1. *Let $\Pi_1, \dots, \Pi_s$ be the Puiseux blocks of a polynomial $f \in K[X][Y]$. Let $p_{(1)}, \dots, p_{(s)} \in \mathcal{P}_X[Y]$ be monic polynomials of degree $c_1, \dots, c_s$ respectively such that for all $1 \leq i \leq s$, the Puiseux expansions of $p_{(i)}$ are a subset of the expansions in Π_i. Then, if $\gamma \in \Pi_i$ and $\eta \in \Pi_j$,*

$$\upsilon_\gamma(p_{(j)}) = c_j \upsilon_{ij} \quad \text{and} \quad \upsilon_\eta(p_{(i)}) = c_i \upsilon_{ij}.$$

For $p = p_{(1)} \cdots p_{(s)}$ and any $\gamma \in \Pi_i$,

$$\boxed{\upsilon_\gamma(p) = \left(\sum_{j \neq i} c_j \upsilon_{ij}\right) + \upsilon_\gamma(p_{(i)})}$$

For $p = p_{(1)} \cdots p_{(s)}$ as in the lemma, we call $\boldsymbol{c} = (c_1, \dots, c_s)$ the multiplicity of p with respect to the sets $\Pi_1, \dots, \Pi_s$. As observed before, only $\upsilon_\gamma(p_{(i)})$ depends on the actual Puiseux expansions of p and not on the number of them in each block.

For any $0 \leq k < m_i := \#\Pi_i$, we note $\tilde{\tilde{p}}_{(i,k)}$ the polynomial in $\mathcal{P}_X[Y]$ of degree c_i in Y of maximal valuation at f_i, whose Puiseux expansions are a subset of the expansions of f_i. By the observation above, if we fix the degrees $c_1, \dots, c_s$ of the polynomials

$$p_{(1)}, \dots, p_{(s)},$$

then the best choice for $p = p_{(1)} \cdots p_{(s)}$ is to take $p_{(i)} := \tilde{\tilde{p}}_{(i,c_i)}$.

For $\boldsymbol{c} = (c_1, \dots, c_s)$ $(0 \leq c_i \leq m_i)$, we define

$$\tilde{\tilde{p}}_{\boldsymbol{c}} = \tilde{\tilde{p}}_{(1,c_1)} \cdots \tilde{\tilde{p}}_{(s,c_s)},$$

a polynomial with maximal valuation at f among all polynomials with multiplicity $(c_1, \dots, c_s)$.

Hence for determining the polynomial $p \in \mathcal{P}_X[Y]$ of degree d of maximal valuation at f among all monic polynomials of degree d it is enough to consider all tuples $\boldsymbol{c} = (c_1, \dots, c_s)$ such that $c_1 + \cdots + c_s = d$, compute for each of these tuples the valuation at $\tilde{f}$ of the polynomial $\tilde{\tilde{p}}_{\boldsymbol{c}} = \prod_{i=1}^s \tilde{\tilde{p}}_{(i,c_i)}$ and take the one with maximal valuation.

The polynomials $\tilde{\tilde{p}}_{(i,c_i)}$ cannot be effectively computed because they involve infinite series. We note $\tilde{p}_{(i,c_i)}$ the polynomial in $K[X][Y]$ of degree c_i in Y of maximal valuation at f_i, which can be computed using Algorithm 2. The formula

$$\upsilon_\gamma(\tilde{p}_{(j,c_j)}) = c_j \upsilon_{ij}$$

still holds for $\gamma \in \Pi_i$, $i \neq j$, because the truncations in the expansions in $\tilde{p}_{(j,c_j)}$ only occur at degrees equal or higher than the first extended characteristic exponent.

We conclude that

$$\tilde{p}_{\boldsymbol{c}} = \tilde{p}_{(1,c_1)} \cdots \tilde{p}_{(s,c_s)},$$

is a polynomial in $K[X][Y]$ with maximal valuation at $\tilde{f}$ among all polynomials with multiplicity $(c_1, \dots, c_s)$. Using these polynomials, we obtain Algorithm 3 to compute effectively the elements of an integral basis of $\tilde{f}$.

Algorithm 3 `ExhaustiveSearch`

Input: $\Pi_1, \dots, \Pi_s$ the Puiseux blocks of expansions at the origin of a polynomial $f \in K[X,Y]$ monic in Y; $0 \le d < n = \deg_Y(\tilde{f})$.

Output: $p \in \mathcal{P}_X[Y]$ of Y–degree d of maximal valuation at $\tilde{f}$; $o \in \mathbb{Q}_{\ge 0}$, the valuation of p at $\tilde{f}$.

$m_i = \#\Pi_i$ for $i = 1, \dots, s$, the number of expansions in each Puiseux block

$C_d = \{(c_1, \dots, c_s) \in \mathbb{Z}^s_{\ge 0} : 0 \le c_i \le m_i,\ c_1 + \cdots + c_s = d\}$

for all $\boldsymbol{c} = (c_1, \dots, c_s) \in C_d$ **do**

  for $i = 1, \dots, s$ **do**

    $\tilde{p}_{(i,c_i)} =$ `IntegralElementOneBlock`(Π_i, c_i).

  $p_{\boldsymbol{c}} = \tilde{p}_{(1,c_1)} \cdots \tilde{p}_{(s,c_s)}$

  $v_{\tilde{f}}(p_{\boldsymbol{c}}) = \min_{1\le i\le s} \left\{ \left(\sum_{j\ne i} c_j v_{ij} \right) + v_{\Pi_i}(\tilde{p}_{(i,c_i)}) \right\}$.

$p^* = p_{\boldsymbol{c}}$ for $\boldsymbol{c} \in C_d$ such that $v_{\tilde{f}}(p_{\boldsymbol{c}})$ is maximal

return $(p^*, v_{\tilde{f}}(p^*))$.

Example 3.2. Let $f = (Y^3 - X^2)(Y^2 + X^3)(Y^4 + 2X^3Y^2 + 2X^5Y + X^6 + 1/4X^7) + Y^{10} \in \mathbb{Q}[X,Y]$. The Puiseux expansions of f are

$$\begin{aligned}
\gamma_1 &= IX^{3/2} + (-1/2I - 1/2)X^{7/4} + \dots,\\
\gamma_2 &= IX^{3/2} + (1/2I + 1/2)X^{7/4} + \dots,\\
\gamma_3 &= -IX^{3/2} + (1/2I - 1/2)X^{7/4} + \dots,\\
\gamma_4 &= -IX^{3/2} + (-1/2I + 1/2)X^{7/4} + \dots,\\
\gamma_5 &= IX^{3/2} + 1/4IX^{5/2} + \dots,\\
\gamma_6 &= -IX^{3/2} - 1/4IX^{5/2} + \dots,\\
\gamma_7 &= \alpha_1 X^{2/3} - 1/3\alpha_1 X^{4/3} + \dots,\\
\gamma_8 &= \alpha_2 X^{2/3} - 1/3\alpha_2 X^{4/3} + \dots,\\
\gamma_9 &= \alpha_3 X^{2/3} - 1/3\alpha_3 X^{4/3} + \dots,\\
\gamma_{10} &= -1 + \dots,
\end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3$ are the roots of $Z^3 - 1$.

The integrality exponent of f is 10. There are 3 classes of expansions $\Delta_1 = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, $\Delta_2 = \{\gamma_5, \gamma_6\}$ and $\Delta_3 = \{\gamma_7, \gamma_8, \gamma_9\}$, and 2 blocks $\Pi_1 = \Delta_3$ and $\Pi_2 = \Delta_1 \cup \Delta_2$. By similar computations as in Example 2.2, applying Algorithm 2 to Π_2 we obtain the same elements and valuations as in that example. For Π_1, applying Algorithm 2, we obtain the elements $1, Y, Y^2, \overline{f_{\Delta_3}}^{\le 10}$ whose valuations at Π_1 are $0, 2/3, 4/3, 10$ respectively.

Now we apply Algorithm 3 to combine the two blocks. For example, for $d = 5$ we test all combinations of degrees (c_1, c_2) with $c_1 + c_2 = 5$ and $c_1 \le 3$. We obtain that the element with highest valuation at the origin is achieved for $\boldsymbol{c} = (3, 2)$. The corresponding polynomial is $p_{\boldsymbol{c}} = \overline{f_{\Delta_3}}^{\le 10} \cdot (Y^2 + X^3)$ and the valuation at f_0 is $\frac{13}{4} + 3\frac{2}{3} = \frac{21}{4}$.

4. Combinatorial approach

Let $f \in K[X][Y]$ and let $\Pi_1, \ldots, \Pi_s$ be the Puiseux blocks of expansions f at the origin, with cardinalities $m_1, \ldots, m_s$, as before. To apply Algorithm 3 for computing the element of the integral basis of f of degree d we must run over all tuples in $C_d = \{(c_1, \ldots, c_s) \in \mathbb{Z}^s_{\geq 0} : 0 \leq c_i \leq m_i,\, c_1 + \cdots + c_s = d\}$. This can be very slow when f has a large number of Puiseux classes, since the number of tuples to test grows exponentially with the number of Puiseux blocks. We explain in this section how to find the optimal $(c_1, \ldots, c_s) \in C_d$ in an efficient way. Instead of considering all tuples of s elements, we will always consider ordered pairs and proceed iteratively.

For this approach we group the Puiseux classes in sets by the initial term. All the Puiseux classes with the same (or conjugated) initial term are grouped in the same set. Let $\Gamma_1, \ldots, \Gamma_s$ be the resulting sets of Puiseux expansions of f, and let $f_1, \ldots, f_s$ be the corresponding polynomials (this classification is similar to Puiseux segments defined in [2, Section 7.2] except that if two classes have the same initial exponent but the corresponding coefficients are different, they are grouped in different sets). We assume that the sets $\Gamma_1, \ldots, \Gamma_s$ are ordered by the initial exponent increasingly (the order between sets with the same initial exponent is not important).

4.1. Theoretical approach.

As before, for each $0 \leq d < n$, we look first for a polynomial $p_d \in \mathcal{P}_X[Y]$ with maximal valuation at $\tilde{f}$. We now that any such polynomial can be factorized as

$$p_d = p_{(1)} \cdot \cdots \cdot p_{(s)},$$

where $p_{(i)} \in \mathcal{P}_X[Y]$, $1 \leq i \leq s$, is a polynomial whose Puiseux expansions are a subset of the expansions of Γ_i.

The key property for the combinatorial approach is that if $1 \leq i < j \leq s$, then for any $\gamma \in \Gamma_i$ and $\eta \in \Gamma_j$,

$$\upsilon_X(\gamma - \eta) = \upsilon_X(\gamma),$$

because the initial term of γ has smaller or equal degree than the initial term of η (and if they have the same degree, they have different coefficients). We define $\upsilon_i = \upsilon_X(\gamma)$, for any $\gamma \in \Gamma_i$.

We obtain the following formulae (compare with Lemma 3.1).

Lemma 4.1. *Let $\Gamma_1, \ldots, \Gamma_s$ be the sets of Puiseux expansions of a polynomial $f \in K[X][Y]$, and assume that for $i < j$ the valuation of the expansions in Γ_i is smaller or equal than the valuation of the expansions in Γ_j. Let $p_{(1)}, \ldots, p_{(s)} \in \mathcal{P}_X[Y]$ be polynomials of degree $c_1, \ldots, c_s$ respectively such that for all $1 \leq i \leq s$, the Puiseux expansions of $p_{(i)}$ are a subset of Γ_i. Then, for $i < j$, if $\gamma \in \Gamma_i$ and $\eta \in \Gamma_j$,*

$$\upsilon_\gamma(p_{(j)}) = c_j \upsilon_X(\gamma) = c_j \upsilon_i \quad \text{and} \quad \upsilon_\eta(p_{(i)}) = c_i \upsilon_X(\gamma) = c_i \upsilon_i.$$

For $p = p_{(1)} \cdots p_{(s)}$ and any $\gamma \in \Gamma_i$,

$$\boxed{\upsilon_\gamma(p) = \left(\sum_{j<i} c_j \upsilon_j\right) + \upsilon_\gamma(p_{(i)}) + \left(\sum_{j>i} c_j \upsilon_i\right).}$$

As in Lemma 3.1, only $\upsilon_\gamma(p_{(i)})$ depends on the actual Puiseux expansions of p and not only on the number of expansions in each set. Hence, for fixed multiplicities $\boldsymbol{c} = (c_1, \ldots, c_s)$, a polynomial with maximal valuation at f is

$$\tilde{\tilde{p}}_{\boldsymbol{c}} := \prod_{i=1}^{s} \tilde{\tilde{p}}_{(i,c_i)},$$

where $\tilde{\tilde{p}}_{(i,k)}$ is the polynomial in $\mathcal{P}_X[Y]$ of degree k in Y of maximal valuation at Γ_i, whose expansions are a subset of the expansions of f_i.

For our combinatorial approach, we define $\Theta_i := \Gamma_i \cup \cdots \cup \Gamma_s$, $1 \leq i \leq s$. For any subset N_1 of c_1 expansions of Γ_i and any subset N_2 of c_2 expansions of Θ_{i+1}, if we define

$$q_1 = \prod_{\gamma \in N_1} (Y - \gamma), \quad q_2 = \prod_{\eta \in N_2} (Y - \eta), \quad \text{and } q = q_1 q_2,$$

we have

$$\upsilon_\gamma(q_2) = c_2 \upsilon_i \text{ and } \upsilon_\eta(q_1) = c_1 \upsilon_i,$$

for any $\gamma \in \Gamma_i$ and $\eta \in \Theta_{i+1}$, by the formulae we obtained before.

Since $\upsilon_{\Gamma_i}(q_1)$ is the minimum of $\upsilon_{\gamma_i}(q_1)$ for $\gamma_i \in \Gamma_i$, we obtain that

$$\min_{\gamma \in \Gamma_i} \upsilon_\gamma(q) = \upsilon_{f_{\Gamma_i}}(q_1) + c_2 \upsilon_i.$$

Similarly,

$$\min_{\eta \in \Theta_{i+1}} \upsilon_\eta(q) = c_1 \upsilon_i + \upsilon_{\Theta_{i+1}}(q_2).$$

We obtain the following formula.

Lemma 4.2. *For $q = q_1 q_2$ as above,*

$$\boxed{\upsilon_{\Theta_i}(q) = \min\{\upsilon_{\Gamma_i}(q_1) + c_2 \upsilon_i, c_1 \upsilon_i + \upsilon_{\Theta_{i+1}}(q_2)\}.}$$

Remark 4.3. In this formula, only $\upsilon_{\Gamma_i}(q_1)$ and $\upsilon_{\Theta_{i+1}}(q_2)$ depend on the actual expansions and not only on the number of expansions. Hence, if we fix the degrees c_1, c_2 of q_1, q_2 respectively, we can split the problem of computing the polynomial q with maximal valuation at f_{Θ_i} into the two smaller problems of computing the polynomial q_1 with maximal valuation at f_{Γ_i} and the polynomial q_2 with maximal valuation at Θ_{i+1}.

4.2. **Effective algorithm.** We will use Remark 4.3 to determine, for $0 \leq c \leq m_i + \cdots + m_s$, the polynomial $p_{\Theta_i}(c)$ in $K[X][Y]$ of Y-degree c with maximal valuation at $f_i \cdots f_s$, by decreasing induction on i, starting with $i = s$.

As with the formulae in the previous sections, Lemma 4.2 is still valid if we replace the polynomials $q_1, q_2 \in \mathcal{P}_X[Y]$ with polynomials $\bar{q}_1, \bar{q}_2 \in K[X][Y]$ whose Puiseux expansions are truncations of the expansions in q_1, q_2 at degrees equal or higher than the first extended characeteristic exponents.

For each $1 \leq i \leq s$ and $1 \leq c_i \leq m_i$, we define $p_{\Gamma_i}(c_i) := \tilde{p}_{(i,c_i)} \in K[X][Y]$ (as defined in Section 3). We can compute $\tilde{p}_{(i,c_i)}$ as before by exhaustive search using Algorithm 3 or, if the Puiseux set contains several Puiseux blocks, we can apply recursively the combinatorial approach we develop now, as we will see below.

As the first step, we set $p_{\Theta_s}(c) = p_{\Gamma_s}(c)$ for $0 \leq c \leq m_s$. Proceeding inductively, once we have determined $p_{\Theta i+1}(c)$ for all $0 \leq c \leq m_{i+1}+\cdots+m_s$, we want to compute $p_{\Theta_i}(c)$ for all $0 \leq c \leq m_i + \cdots + m_s$.

Using Lemma 4.2 and Remark 4.3 we can compute inductively

$$o(\Theta_i, c) = \max_{\substack{c_1+c_2=c \\ c_1 \leq m_i}} \upsilon_{\Theta_i}(p_{\Gamma_i}(c_1) p_{\Theta_{i+1}}(c_2))$$

and define $p_{\Theta_i}(c)$ as the polynomial for which the maximum is obtained. We obtain Algorithm 4.

Algorithm 4 `IntegralBasisIterative`

Input: $\Gamma_1, \ldots, \Gamma_s$ the sets of Puiseux expansions at the origin of a polynomial $f \in K[X, Y]$ monic in Y of degree n, ordered increasingly by the order of the expansions, developed up to the integrality exponent N of f; $m_i, 1 \leq i \leq s$, the cardinal of Γ_i.

Output: $\{(p_0, o_0), \ldots, (p_n, o_n)\}$ such that $p_d \in K[X][Y]$ has Y–degree d and maximal valuation at f among all polynomials of Y-degree d and $o_d \in \mathbb{Q}_{\geq 0}$, $o_d = \upsilon_f(p_d)$.

if $s = 1$ **then**
  return $\{(\tilde{p}_{(1,c)}, \upsilon_f(\tilde{p}_{(1,c)}) = \texttt{ExhaustiveSearch}(\Gamma_1, c))\}_{c=0,\ldots,n}$
else
  $\Theta_s = \Gamma_s$
  $\{(p_{(\Theta_s,c)}, o(\Theta_s, c))\}_{c=0,\ldots,m_s} = \texttt{IntegralBasisIterative}(\Theta_s)$
  for $i = s-1, \ldots, 1$ **do**
    $\Theta_i = \Gamma_i \cup \Theta_{i+1}$, $f_{\Theta_i} = f_{\Gamma_i} f_{\Theta_{i+1}}$
    $\{(p_{(\Gamma_i,c)}, o(\Gamma_i, c))\}_{c=0,\ldots,m_i} = \texttt{IntegralBasisIterative}(\Gamma_i)$
    for $0 \leq d \leq m_i + \cdots + m_s$ **do**
      $C_d = \{(c_1, c_2) \in \mathbb{Z}^2_{\geq 0} \mid c_1 + c_2 = d, 0 \leq c_1 \leq m_i, 0 \leq c_2 \leq m_{i+1} + \cdots + m_s\}$
      $o(\Theta_i, d) = \max_{(c_1,c_2) \in C_d} \upsilon_{\Theta_i}(p_{\Gamma_i}(c_1) p_{\Theta_{i+1}}(c_2))$
      $p_{(\Theta_i,d)}$ = the polynomial for which the maximum is obtained
  return $\{(p_{(\Theta_1,d)}, o(\Theta_1, d))\}_{0 \leq d \leq n}$.

We note that with this approach the number of cases to test grows linearly with the number of conjugacy classes, which is much more efficient than the previous approach with exponential growth.

We apply the algorithm to an example.

Example 4.4. Let $f = (Y^3 - X^2)(Y^4 + 2X^3Y^2 + 2X^5Y + X^6 + 1/4X^7)(Y^2 - X^5) + Y^{10} \in \mathbb{Q}[X, Y]$. The Puiseux expansions of f are

$$\begin{aligned}
\gamma_1 &= \alpha_1 X^{2/3} - 1/3\alpha_1 X^{4/3} + \dots,\\
\gamma_2 &= \alpha_2 X^{2/3} - 1/3\alpha_2 X^{4/3} + \dots,\\
\gamma_3 &= \alpha_3 X^{2/3} - 1/3\alpha_3 X^{4/3} + \dots,\\
\gamma_4 &= IX^{3/2} + (-1/2I - 1/2)X^{7/4} + \dots,\\
\gamma_5 &= IX^{3/2} + (1/2I + 1/2)X^{7/4} + \dots,\\
\gamma_6 &= -IX^{3/2} + (1/2I - 1/2)X^{7/4} + \dots,\\
\gamma_7 &= -IX^{3/2} + (-1/2I + 1/2)X^{7/4} + \dots,\\
\gamma_8 &= X^{5/2} + 1/2X^{29/2} + \dots,\\
\gamma_9 &= -X^{5/2} - 1/2X^{29/2} + \dots,\\
\gamma_{10} &= -1 + \dots,
\end{aligned}$$

where α_i, $1 \le i \le 3$, are the roots of $Z^3 - 1$.

There are 3 classes of Puiseux expansions at the origin, and each class is a different set: $\Gamma_1 = \{\gamma_1, \gamma_2, \gamma_3\}$, $\Gamma_2 = \{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$ and $\Gamma_3 = \{\gamma_8, \gamma_9\}$. We have $m = 9$, $m_1 = 3$, $m_2 = 4$ and $m_3 = 2$, and the integrality exponent is 10.

Hence $\Theta_3 = \Gamma_3$ and the elements of the local integral basis corresponding to Θ_3 are

$$\{(1, 0), (Y, 5/2), (\overline{f_{\Gamma_3}}^{\le 10}, 10)\}.$$

For $i = 2$, we set $\Theta_2 = \Gamma_2 \cup \Gamma_3$. The elements of the local integral basis corresponding to Γ_2 are

$$\{(1, 0), (Y, 1), (Y^2 + X^3, 13/4), (Y \cdot (Y^2 + X^3), 19/4), (\overline{f_{\Gamma_2}}^{\le 10}, 10)\}.$$

Now we compute the elements with maximal valuation at Θ_2, testing for each $0 \le d \le 6$ all ordered pairs (c_1, c_2) such that $d = c_1 + c_2$. We obtain the following polynomials:

$$\begin{aligned}
&p_0 = 1, \upsilon_{\Theta_2}(p_0) = 0,\\
&p_1 = Y, \upsilon_{\Theta_2}(p_1) = 3/2,\\
&p_2 = Y^2, \upsilon_{\Theta_2}(p_2) = 3,\\
&p_3 = Y \cdot (Y^2 - X^3), \upsilon_{\Theta_2}(p_3) = 7/4 + 3/2 + 3/2 = 19/4,\\
&p_4 = (Y^2 - X^3)\overline{f_{\Gamma_3}}^{\le 10}, \upsilon_{\Theta_2}(p_4) = 7/4 + 3/2 + 3/2 + 3/2 = 25/4,\\
&p_5 = Y \cdot \overline{f_{\Gamma_2}}^{\le 10}, \upsilon_{\Theta_2}(p_5) = 3/2 + 3/2 + 3/2 + 3/2 + 5/2 = 17/2,\\
&p_6 = \overline{f_{\Gamma_2}}^{\le 10}\overline{f_{\Gamma_3}}^{\le 10}, \upsilon_{\Theta_2}(p_6) = 10.
\end{aligned}$$

Finally we consider $\Theta_1 = \Gamma_1 \cup \Theta_2$ and for each $0 \le d \le 9$ we consider all tuples (c_1, c_2) with $d = c_1 + c_2$ and $c_1 \le 3$. For example, for $d = 4$, we consider the tuples

$$p_{(0,4)} = (Y^2 - X^3)\overline{f_{\Gamma_3}}^{\leq 10}, \upsilon_{\Theta_1}(p_{(0,4)}) = 4 \cdot 2/3 = 8/3$$
$$p_{(1,3)} = Y \cdot Y \cdot (Y^2 - X^3), \upsilon_{\Theta_1}(p_{(1,3)}) = 4 \cdot 2/3 = 8/3$$
$$p_{(2,2)} = Y^2 \cdot Y^2, \upsilon_{\Theta_1}(p_{(1,3)}) = 4 \cdot 2/3 = 8/3$$
$$p_{(3,1)} = \overline{f_{\Gamma_1}}^{\leq 10} \cdot Y, \upsilon_{\Theta_1}(p_{(1,3)}) = 3 \cdot 2/3 + 3/2 = 7/2.$$

The best element for $d = 4$ is then $p_4 = \overline{f_{\Gamma_3}}^{\leq 10} \cdot Y$, with integrality exponent $\lfloor 7/2 \rfloor = 3$.

4.3. **Expansions with common rational part.** We consider now the case of a set Γ_i of expansions containing more than one Puiseux block. In this case, the expansions have the same initial term, which is rational. Hence, we can subtract from all the expansions in the set the common rational part. After removing the common rational part, the expansions will not be all in the same set, and we can apply Algorithm 4 to the resulting sets. To keep this presentation simple, we give an example of this case, but we do not introduce the corresponding modifications in Algorithm 4.

Example 4.5. Consider the polynomial $f = ((Y-X)^2 - X^3)((Y-X)^2 - X^5) + (Y-X)^5$ with Puiseux expansions at the origin

$$\gamma_1 = X + X^{3/2} - 1/2X^3 + \dots \qquad \gamma_3 = X + X^{5/2} + 1/2X^7 + \dots$$
$$\gamma_2 = X - X^{3/2} - 1/2X^3 + \dots \qquad \gamma_4 = X - X^{5/2} + 1/2X^7 + \dots$$

where $\{\gamma_1, \gamma_2\}$ is a conjugacy class and $\{\gamma_3, \gamma_4\}$ is another conjugacy class, and both classes are in the same set.

All the expansion have X as common rational part. After removing this common part, we obtain

$$\eta_1 = X^{3/2} + X^2 + \dots \qquad \eta_3 = X^{5/2} + X^3 + \dots$$
$$\eta_2 = -X^{3/2} + X^2 + \dots \qquad \eta_4 = -X^{5/2} + X^3 + \dots$$

Now, $N_1 = \{\eta_1, \eta_2\}$ is a set of expansion and $N_2 = \{\eta_3, \eta_4\}$ is another set, so we can apply Algorithm 4 for $\Theta = N_1 \cup N_2$.

The integrality exponent is 5. We obtain the following elements:

$$p_0 = 1, \upsilon_{\Theta_1}(p_0) = 0,$$
$$p_1 = Y, \upsilon_{\Theta_1}(p_1) = 3/2,$$
$$p_2 = Y^2, \upsilon_{\Theta_1}(p_2) = 3,$$
$$p_3 = Y\overline{f_{N_1}}^{\leq 5}, \upsilon_{\Theta_1}(p_0) = 3 + 5/2 = 11/2.$$

Replacing Y by $Y - X$ in these polynomials, we obtain the elements of maximal valuation at f.

5. Timings

In this section we measure timings in some examples. We generate examples with several branches.

(1) $f = (y^4 + 3x^3y + x^4)(y^7 + 6x^4y^3 + 2xy + x^7)(y^5 + 7xy - 4x^2)(y^3 + x^2)(y^2 - x^3) + y^{30}$
(2) $f = (y^4 + 3x^3y + x^4)(y^7 + 6x^4y^3 + 2xy + x^7)(y^5 + 7xy - 4x^2)(y^3 + x^2)(y^2 - x^3) + y^{100}$
(3) $f = (y^4 + 3x^3y + x^4)(y^7 + 6x^4y^3 + 2xy + x^7)(y^9 + 7xy2 - 4x^2)(y^3 + x^2)(y^2 - x^3) + y^{30}$
(4) $f = (y^4+x^4)(y^7+2xy+x^2)(y^5+7x^3)(y^3+x^2)(y^3-x^2)(y^2-x^3)+y^{30}$
(5) $f = (y^4+x^4)(y^7+2xy+x^2)(y^5+7x^3)(y^3+x^2)(y^3-x^2)(y^2-x^3)+y^{100}$
(6) $f = (y^4+3x^3y+x^4)(y^7+6x^4y^3+2xy+x^7)(y^5+7x^3)(y^3-4x^2)(y^3+x^2)(y^2-x^3)+y^{30}$
(7) $f = (y^4+3x^3y+x^4)(y^7+6x^4y^3+2xy+x^7)(y^5+7x^3)(y^3-4x^2)(y^3+x^2)(y^2-x^3)+y^{100}$

No.	Branches	Y-degree	Combinatorial	Chinese remainder	Maple
(1)	5	30	1	7	3
(2)	5	100	1	1	45
(3)	5	30	1	7	4
(4)	6	30	1	198	2.4
(5)	6	100	1	2	41.1
(6)	6	30	2	238	3.5
(7)	6	100	2	2	64.7

We observe that the combinatorial approach presented in this work is always fast for these examples. The approach merging the local contributions to the integral basis using the Chinese Remainder theorem is slow when the polynomial has low degree. Surprisingly, it is very fast when the polynomial has large degree. The reason seems to be that in the case of large degree the Groebner basis computations are faster because the large degree monomials are separated from the low degree ones. When compared with Maple, we observe that our approach is always faster than Maple. The computations in Maple are very sensible to the degree of the polynomial.

References

[1] Shreeram S. Abhyankar. *Algebraic geometry for scientists and engineers.* Providence, RI: American Mathematical Society, 1990.

[2] Janko Böhm, Wolfram Decker, Santiago Laplagne, and Gerhard Pfister. Computing integral bases via localization and Hensel lifting. *J. Symb. Comput.*, 109:283–324, 2022.

[3] Theo de Jong and Gerhard Pfister. *Local analytic geometry. Basic theory and applications.* Braunschweig: Vieweg, 2000.

[4] R. Matsumoto. On computing the integral closure. *Comm. in Algebra*, 28:401–405, 2000.