ON COMPUTING THE MILNOR NUMBER AND DELTA INVARIANT OF AN ISOLATED SINGULARITY

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Abstract.

Algorithm 1 Transform the standard homogeneous d-jet of a polynomial, where d is the maximal filtration of the polynomial, to a polynomial with a non-degenerate saturation.

Input: A polynomial $f \in \mathfrak{m}^3, f \in \mathbb{Q}[x,y]$ with maximal filtration d.
Output: A polynomial $h \in \mathbb{K}[x,y]$ defined over an extension field $\mathbb{Q} \subset \mathbb{K} \subset \overline{\mathbb{K}}$ such that h is right-equivalent to f and the facet formed by $\operatorname{sat}(\operatorname{jet}(f,d))$ is of minimum lenght. .

1: $g := \operatorname{jet}(f,d)$.
2: Factorize $g = c \cdot g_1^{l_1} \cdot \ldots \cdot g_n^{l_n}$, where $l_1 \geq l_2 \geq \cdots \geq l_n$, $c \in \mathbb{Q}$, $g_1 \in \mathbb{K}[x,y]$ with $\mathbb{K} = \mathbb{Q}$ in case $n = 1$, and $g_1, \ldots, g_n \in \overline{\mathbb{K}}[x,y]$ linear homogeneous and coprime, $g_1, g_2 \in \mathbb{K}[x,y]$ with $[\mathbb{K} : \mathbb{Q}]$ minimal (among all admissible choices of g_1, g_2), in case $n \geq 2$.
3: **if** $n = 1$ **then**
4: Apply a linear automorphism ϕ to f sending $g_1 \mapsto x$.
5: **else**
6: Apply $g_1 \mapsto x$, $g_2 \mapsto y$ to f.
7: **return** ϕ, f

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Algorithm 2 Transform a non-standard weighted homogeneous d-jet of a polynomial, where d is the maximum weighted filtration of the polynomial, to a polynomial with a non-degenerate saturation.

Input: A polynomial $f \in \mathfrak{m}^3$, $f \in \mathbb{K}[x, y]$, where $\mathbb{K}$ is an extension field of $\mathbb{Q}$, and a weight $w = (w_1, w_2)$ with $w_1 \neq w_2$ the weight of one of the facets of the Newton boundary of f.

Output: A polynomial $h \in \mathbb{L}[x, y]$ defined over an extension field $\mathbb{K} \subset \mathbb{L} \subset \overline{\mathbb{K}}$, such that h is right-equivalent to f, where $\mathrm{jet}(h, w)$ is a polynomial (possibly term) with a non-degenerate saturation, if such an h exists, and false otherwise.

if $w_1 < w_2$ **then**
  $g := \mathrm{sat}(\mathrm{jet}(f, w), x)$.
if $w_1 > w_2$ **then**
  $g := \mathrm{sat}(\mathrm{jet}(f, w), y)$.
Factorize $g = c g_1^{l_1} \cdots g_n^{l_n} \widetilde{g}$, where $c \in \mathbb{K}$, $l_1 \geq l_2 \geq \cdots \geq l_n$ and $g_1 \ldots, g_n \in \overline{\mathbb{K}}[x, y]$ weighted linear homogeneous and coprime with n maximal, $\widetilde{g} \in \mathbb{K}[x, y]$ a product of non-associated irreducible polynomials in $\overline{\mathbb{K}}[x, y]$, and $g_1 \in \mathbb{L}[x, y]$ with $[\mathbb{L} : \mathbb{K}]$ minimal.
if $l_2 > 1$ **then**
  return (`false`,f)
if $\widetilde{g}$ is degenerate **then**
  return (`false`,f)
if $w_1 < w_2$ **and** $n \geq 1$ **then**
  Apply right equivalence to f which sends $g_1 \mapsto y, x \mapsto x$.
if $w_1 > w_2$ **and** $n \geq 1$ **then**
  Apply right equivalence to f which sends $g_1 \mapsto x, y \mapsto y$.
return (ϕ, f)

Algorithm 3 Determine a cut-off bound for a germ in order to determine the highest Newton Number in its equivalence class.

Input: $g \in \mathbb{Q}[x, y]$.
Output: A number c.

if $\Gamma(g)$ cut the y-axis **then**
  Let $(0, c_y)$ be the vertex of $\Gamma(g)$ on the y-axis.
else
  if $\Gamma(f)$ has a vertex $(1, c_1)$ on the $x = 1$-axis **then**
    Let m be the slope of the facet with vertex (x, c_1).
    $c_y = \lceil m \rceil + c_1$.
  else
    $c_y = -1$.
if $\Gamma(g)$ cut the x-axis **then**
  Let $(0, c_x)$ be the vertex of $\Gamma(g)$ on the x-axis.
else
  if $\Gamma(f)$ has a vertex $(c_2, 1)$ on the $y = 1$-axis **then**
    Let m be the slope of the facet with vertex $(c_2, 1)$.
    $c_x = \lceil m \rceil + c_2$.
  else
    $c_x = -1$
if $c_x = -1$ or $c_y = -1$ **then**
  $c = -1$
else
  $c = \max(c_x, c_y)$
return (c)

Algorithm 4 Determining the number of branches of an isolated singularity with the maximum Newton Number in its equivalence class

Input: $g \in \mathbb{Q}[x, y]$, g has maximum Newton Number in its equivalence class.
Output: A number nb.

Let $\Delta_1, \ldots, \Delta_n$ be the facets of $\Gamma(g)$, with respective weights $w_1, \ldots, w_n$.
Let $nb = 0$.
for $i = 1, \ldots, n$ **do**
  Let $h = \mathrm{sat}(\mathrm{jet}(g, \Delta_i))$.
  Let n_x and n_y be the x- and y- intercepts of h.
  $nb = nb + \gcd(n_x, n_y)$
return (nb)

Algorithm 5 Determining the highest Newton Number in the equivalence class of a germ with an Isolated Singularity.

Input: A polynomial germ $f \in \mathbb{Q}[x, y]$, $f \in \mathfrak{m}^3$, of corank 2 with finite Milnor number.
Output: A polynomial g which is equivalent to f with the highest Newton Number in its equivalence class, as well as the Newton Number of g.

$d :=$ maximal filtration of f w.r.t. the standard grading.
Let NN be the Newton Number of f.
$g = f$.
if $\mu(\mathrm{sat}(\mathrm{jet}(g, d))) = \infty$ **then**
    Let g and ϕ be the output of Algorithm 1 applied to g.
    $S_T = \{\phi\}$.
Let c be the bound determined by Algorithm 3 for g.
while $c \leq NN + 1$ **do**
    $NN = 2 * NN$.
    $g = \mathrm{jet}(f, NN + 1)$.
    $g = \phi(g)$.
    Let c be the bound determined by Algorithm 3 for g.
Let δ be the facet formed by $\mathrm{jet}(g, d)$.
$S_0 :=$ the set of monomials of $\mathrm{jet}(g, d)$ that lie on the vertices of $\Gamma(g)$.
$S_1 := \emptyset$
$S_2 := \{\delta\}$
while true do
    Let $\Delta_1, \Delta_2, \ldots, \Delta_v$ be the facets of $\Gamma(f)$ ordered by increasing slope.
    if $\mu(\mathrm{sat}(\mathrm{jet}(g, \Delta_i))) < \infty$ or $\Delta_i \in S_2$, $\forall i = 1, \ldots, v$ **then**
        Determine the Newton Number NN of g.
        Determine the number of Branches, numberOfBranches, of g.
        return (g,$\Gamma(g)$, NN, non-degenerate, numberOfBranches)
    else
        Let m be an element of S_0.
        Let $\delta_1, \ldots, \delta_r$ ($r \leq 2$) be the facets of $\Gamma(g)$, ordered by increasing slope, adjacent to m.
        for i **from** 1 **to** r **do**
            $h := \mathrm{jet}(g, \delta_i)$, $w :=$ the weight defined by h.
            while $\mu(\mathrm{sat}(h)) = \infty$ and $\delta_i \notin S_2$ **do**
                $(g, \phi) :=$ output of Algorithm 2 applied to g and w.
                if $\phi = \texttt{false}$ **then**
                    $S_3 = S_3 \cup \{\delta_i\}$
                    non-degenerate $= 0$.
                else
                    $S_T = S_T \cup \{\phi\}$.
                Let c be the bound determined by Algorithm 3 for g.
                if $c > NN + 1$ **then**
                    Let NN be the Newton Number of g; $g = \mathrm{jet}(f, NN + 1)$.
                    for $i = 1, \ldots, \mathrm{size}(S_T)$ **do**
                        $g = \mathrm{jet}(S_T[i](g), NN + 1)$;
                    Let c be the bound determined by Algorithm 3 for g.
                while $c > NN + 1$ **do**
                    $NN = 2 * NN$; $g = \mathrm{jet}(f, NN + 1)$.
                    for $i = 1, \ldots, \mathrm{size}(S_T)$ **do**
                        $g = \mathrm{jet}(S_T[i](g), NN + 1)$;
                    Let c be the bound determined by Algorithm 3 for g.
                Let $\delta_1, .., \delta_r$ be the facets of $\Gamma(g)$, ordered by increasing slope, adjacent to m.
                $h := \mathrm{jet}(g, \delta_i)$, $w :=$ the weight defined by h.
        Let $\eta_1, \ldots, \eta_r$ be the facets of $\Gamma(g)$, ordered by increasing slope, containing m.
        $S_1 := S_1 \cup \{m\}$
        $g_1 := \mathrm{jet}(g, \eta_1 \cap \ldots \cap \eta_r))$.
        $S_0 :=$ monomials in $(S_0 \cup \mathrm{supp}(g_1)) \setminus S_1$ lying on the vertices of $\Gamma(g)$.

Algorithm 6 Determining the Milnor Number and Delta Invariant of an Isolated Singularity.

Input: A polynomial $f \in \mathfrak{m}^3$, $f \in \mathbb{Q}[x, y]$

Output: The Milnor Number and Delta Invariant of f, if f is an isolated singularity. False, otherwise

return (ϕ, f)
if f has a double factor **then**
    return False
Let (g,$\Gamma(g)$, NN, non-degenerate, nb) be the output of Algorithm 5 applied to f.
$g_\Gamma = \mathrm{jet}(g, \Gamma(g))$.
if non-degenerate == 1 **then**
    nb is the number of facets of $\Gamma(g)$.
    $\delta = \frac{NN+nb-1}{2}$
    return (δ,NN,nb)
else
    while true **do**
        $i = NN + 1$.
        $g = \mathrm{jet}(f, i)$.
        while g has a double factor **do**
            $i = i + 1$.
            $g = \mathrm{jet}(f, i)$.
        Let p be a random prime number > 100.
        Set the ring to $\mathbb{Q}/p[x, y]$.
        Let HC be the degree of the highest corner of the lead ideal of $\mathrm{Jac}(g)$.
        while $HC > i$ **do**
            $i = i + 1$
            Let $g = \mathrm{jet}(f, i)$
            Let HC be the degree of the highest corner of the lead ideal of $\mathrm{Jac}(g)$.
        Let $\mu_p(g)$ be the milnornumber of g.
        Set the ring back to $\mathbb{Q}[x, y]$.
        Let $\mu(f)$ be the milnornumber of $\mathrm{jet}(f, i)$.
        if $\mu(f) = \mu_p(g)$ **then**
            Let nb be the number of branches of g_Γ determined by Algorithm 4.
            $\delta = \frac{\mu(f)+nb-1}{2}$.
            return δ,$\mu(f)$,nb