CLASSIFICATION OF COMPLEX SINGULARITIES WITH NON-DEGENERATE NEWTON BOUNDARY

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ABSTRACT. In his groundbreaking work on classification of singularities with regard to right and stable equivalence of germs, Arnold has listed normal forms for all isolated hypersurface singularities over the complex numbers with either modality ≤ 2 or Milnor number ≤ 16. Moreover, he has described an algorithmic classifier, which determines the type of a given such singularity. In the present paper, we extend Arnold's work to a large class of singularities which is unbounded with regard to modality and Milnor number. We develop an algorithmic classifier, which determines a normal form for any corank ≤ 2 singularity which is equivalent to a germ with non-degenerate Newton boundary in the sense of Kouchnirenko. In order to realize the classifier, we prove a normal form theorem: Suppose K is a μ-constant stratum of the jet space which contains a germ with a non-degenerate Newton boundary. We first observe that all germs in K are equivalent to some germ with the same fixed non-degenerate Newton boundary. We then prove that all right-equivalence classes of germs in K can be covered by a single normal form obtained from a regular basis of an appropriately chosen special fiber. All algorithms are implemented in the library `arnold.lib` for the computer algebra system SINGULAR.

1. INTRODUCTION

In (Arnold, 1976; Arnold et al., 1985) Arnold has classified all isolated hypersurface singularities for which the Milnor number ≤ 16 or for which the modality is ≤ 2. He has given normal forms in the sense of polynomial families with moduli parameters such that every stable equivalence class of function germs contains at least one, but only finitely many, elements of these families. We refer to such elements as normal form equations. In Arnold's work and in the rest of the paper each normal form contains representatives of all the stable equivalence classes of a specific μ-constant stratum. For a specific germ f, we refer to a normal form covering the μ-constant stratum containing f, as a normal form of f. Two germs are stably equivalent if they are right-equivalent after the direct addition of a non-degenerate quadratic form (see Section 5, Arnold (1976)). Two function germs $f, g \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$, where $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, are right-equivalent, written $f \sim g$, if there is a $\mathbb{C}$-algebra automorphism ϕ of $\mathbb{C}\{x_1, \ldots, x_n\}$ such that $\phi(f) = g$. Using the Splitting Lemma (Lemma 9.2.10, de Jong and Pfister (2000)), any germ with an isolated singularity at the origin can be written, after choosing a suitable coordinate system, as the sum of two functions on disjoint sets of variables. One function that is called the non-degenerate part, is a non-degenerate quadratic form, and the other part, called the residual part, is in $\mathfrak{m}^3$. The Splitting Lemma is implemented in the computer algebra system SINGULAR (Decker et al., 2019; Greuel and Pfister, 2008) as part of the library `classify.lib` (Krüger, 1997). Given the classification of singularities of function germs, one can easily derive the classification of the corresponding hypersurface singularities (up to contact equivalence,

which in addition to right equivalence allows for multiplication of the defining equation by a constant), see, for example, Marais and Steenpaß (2015).

In (Arnold et al., 1985), Arnold has made his classification explicit by describing an algorithmic classifier, which is based on a case-by-case analysis formulated in a series of 105 theorems. This approach determines the type of the singularity in the sense of its normal form. The algorithm does not determine the values of the moduli parameters, that is, no normal form equation is given. Arnold's classifier is implemented in the SINGULAR library `classify.lib`. In (Böhm, Marais, and Pfister, 2016a) a classifier has been developed for complex singularities of modality ≤ 2 and corank ≤ 2, which computes, for a given input polynomial, a normal form equation in its equivalence class. For an implementation see the SINGULAR library `classify2.lib` Böhm, Marais, and Pfister (2016b).

Classification of complex singularities has a multitude of practical and theoretical applications. In particular, the classification of real singularities as for example developed in (Marais and Steenpaß, 2015, 2016; Böhm, Marais, and Steenpaß, 2020) is based on determining first the complex type of the singularity. Recent applications arise in high-energy physics in the study of singularities of Baikov polynomials (Lee and Pomeransky, 2013), which are a key object in the computation of integration-by-parts identities for Feynman integrals (Böhm et al., 2018).

In the present paper, we develop a classification algorithm for all isolated hypersurface singularities of corank ≤ 2 which are right-equivalent to a germ with a non-degenerate Newton boundary in the sense of Kouchnirenko's definition in Kouchnirenko (1976). This algorithm determines for a given input polynomial f representing a μ-constant stratum a polynomial normal form (that is, a family of polynomials with polynomial, in fact linear, dependence on the moduli parameters) which covers the whole μ-constant stratum of f.

In order to realize the classifier, we prove a normal form theorem: Suppose K is a μ-constant stratum of the jet space which contains a germ with a non-degenerate Newton boundary. We observe that all germs in K are equivalent to some germ with the same fixed non-degenerate Newton boundary. We then prove that all right-equivalence classes of germs in K can be covered by a single normal form obtained from a so-called regular basis of the Milnor algebra of an appropriately chosen special fiber .[1] All our algorithms are implemented in the SINGULAR library `arnold.lib` Böhm, Marais, and Pfister (2020).

This paper is structured as follows: In Section 2, we give the fundamental definitions and provide the prerequisites on singularities and their classification. In Section 3, we prove that a normal form of the μ-constant stratum of a given germ which is equivalent to a germ with non-degenerate Newton boundary can be obtained from the Newton boundary and a regular basis of the Milnor algebra (see Theorem 3.20). In Section 4 we prove a self-contained fact which is required to complete the proof of the normal form theorem. We show that, if two germs have the same characteristic exponents and intersection numbers, and one of them has a non-degenerate Newton boundary, then also the other one is right-equivalent to a germ with non-degenerate Newton boundary and same Newton polygon. Moreover, we prove that if a germ f is equivalent to a germ with a non-degenerate Newton boundary, the Newton polygon of this germ is uniquely determined by the characteristic exponents and intersection numbers of the branches of f, provided we require the normalization condition. In Section 5, we develop an algorithm which transforms an input singularity to a germ with a non-degenerate Newton boundary, in case this is possible (see Algorithm 4). In Section 6, we use the result of Algorithm 4 to compute the modality for the input polynomial, as well as a regular basis of the Milnor algebra (see Algorithms 5 and 6). Finally, in Section 7, relying on Theorem 3.20, we use Algorithm 4 and Algorithm 6 to compute for a given input polynomial f a normal form which covers its μ-constant stratum (see Algorithm 7).

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[1]We remark, that we have also developed an algorithm for determining a normal form equation for f, that is, a representative of its right-equivalence class by specifying the values of the moduli parameters in our normal form family. We also have developed an algorithm to enumerate all normal form families up to a specified Milnor number. These algorithms will be presented in follow-up papers.

2. Definitions and Preliminary Results

In this section we give some basic definitions and results, as well as some notation that will be used throughout the paper. We denote by $\mathbb{C}\{x_1, \ldots, x_n\}$ the ring of power series convergent in open neighborhoods of $(0, \ldots, 0)$. We start with a short account on weighted jets, filtrations, and Newton polygons. See also (Arnold, 1974) and (de Jong and Pfister, 2000).

Notation 2.1. We write $\mathrm{mon}(x_1, \ldots, x_n)$ for the multiplicative monoid of monomials in the variables $x_1, \ldots, x_n$. Given $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ and $m \in \mathrm{mon}(x_1, \ldots, x_n)$, we write $\mathrm{coeff}(f, m)$ for the coefficient of m in f.

Definition 2.2. Let $w = (c_1, \ldots, c_n) \in \mathbb{N}^n$ be a weight on the variables $(x_1, \ldots, x_n)$. The w-weighted degree on $\mathrm{mon}(x_1, \ldots, x_n)$ is given by $w\text{-}\deg(\prod_{i=1}^{n} x_i^{s_i}) := \sum_{i=1}^{n} c_i s_i$. If the weight of all variables is equal to 1, we refer to the weighted degree of a monomial m as the standard degree of m and write $\deg(m)$ for $w\text{-}\deg(m)$. We use the same notation for terms of polynomials.

We call a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ **quasihomogeneous** or **weighted homogeneous** of degree d with respect to the weight w if $w\text{-}\deg(t) = d$ for any term t of f.

Definition 2.3. Let $w = (w_1, \ldots, w_s) \in (\mathbb{N}^n)^s$ be a finite family of weights on the variables $(x_1, \ldots, x_n)$. For any monomial (or term) $m \in \mathbb{C}[x_1, \ldots, x_n]$, we define the **piecewise weight** with respect to w as

$$w\text{-}\deg(m) \quad := \quad \min_{i=1,\ldots,s} w_i\text{-}\deg(m).$$

A polynomial f is called **piecewise homogeneous** of degree d with respect to w if $w\text{-}\deg(t) = d$ for any term t of f.

Definition 2.4. Let w be a (piecewise) weight on $\mathrm{mon}(x_1, \ldots, x_n)$.

(1) Let $f = \sum_{i=0}^{\infty} f_i$ be the decomposition of $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ into weighted homogeneous summands f_i of w-degree i. If $f_i = 0$ for all $i > d$ and $f_d \neq 0$, then we write $w\text{-}\deg(f) = d$. The **(piecewise) weighted j-jet** of f with respect to w is

$$w\text{-}\mathrm{jet}(f, j) := \sum_{i=0}^{j} f_i.$$

The sum of terms of f of lowest w-degree is the **principal part** of f with respect to w. The **order** $w\text{-}\mathrm{ord}(f)$ of f with respect to w is the degree of the principal part of f.

(2) We say that a power series in $\mathbb{C}\{x_1, \ldots, x_n\}$ has **filtration** $d \in \mathbb{N}$ with respect to w if all its monomials are of w-weighted degree d or higher. We write E_d^w for the sub-vector space of power series in $\mathbb{C}\{x_1, \ldots, x_n\}$ of filtration d with respect to w. The sub-vector spaces E_d^w for $d \in \mathbb{N}$ define a filtration on $\mathbb{C}\{x_1, \ldots, x_n\}$.

If the weight of each variable is 1, we write E_d instead of E_d^w and $\mathrm{jet}(f, j)$ instead of $w\text{-}\mathrm{jet}(f, j)$.

Definition 2.5. A power series $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$ is **k-determined** if

$$f \sim \mathrm{jet}(f, k) + g \qquad \text{for all } g \in E_{k+1},$$

where $\sim$ denotes right-equivalence. We define the **determinacy** $\mathrm{dt}(f)$ of f as the minimum number k such that f is k-determined.

Definition 2.6. Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$. The **Jacobian ideal** $\mathrm{Jac}(f) \subset \mathbb{C}\{x_1, \ldots, x_n\}$ of f is generated by the partial derivatives of f. The **local algebra** Q_f of f is the residue class ring of the Jacobian ideal of f. The **Milnor number** of f is the $\mathbb{C}$-vector space dimension of Q_f.

It is a well-known fact, that all function germs are finitely determined:

Theorem 2.7. *(de Jong and Pfister, 2000, Theorem 9.1.4) Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x_1, \ldots, x_n\}$. If $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \cdot \mathrm{Jac}(f)$ then f is k-**determined**. In particular, isolated singularities are finitely determined, and if $k \geq \mu(f) + 1$ then f is k-determined.*

Remark 2.8. Theorem 2.7 implies that every right-equivalence class of an isolated singularity has a polynomial representative. Note that since a result analogous to Theorem 2.7 is true also for formal power series, every right-equivalence class of formal power series with an isolated singularity has a polynomial (hence convergent) representative.

There is a bijection between the right-equivalence classes in the convergent and the formal sense: By the Artin approximation theorem (Artin (1969)), two convergent power series which are formally equivalent are also equivalent via a convergent transformation.

Definition 2.9. *Let $w \in \mathbb{N}^n$ be a single weight. A power series $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ is called **semi-quasihomogeneous** with respect to w if its principal part with respect to w is **non-degenerate**, that is, has finite Milnor number.[2] The principal part of f is then called the **quasihomogeneous part** of f.*

There are similar concepts of jets and filtrations for coordinate transformations:

Definition 2.10. *Let ϕ be a $\mathbb{C}$-algebra automorphism of $\mathbb{C}\{x_1, \ldots, x_n\}$ and let w be a single weight on $\mathrm{mon}(x_1, \ldots, x_n)$.*

(1) For $j > 0$ we define $w\text{-}\mathrm{jet}(\phi, j) := \phi_j^w$ as the automorphism given by

$$\phi_j^w(x_i) := w\text{-}\mathrm{jet}(\phi(x_i), w\text{-}\deg(x_i) + j) \quad \text{for all } i = 1, \ldots, n.$$

If the weight of each variable is equal to 1, that is, $w = (1, \ldots, 1)$, we write ϕ_j for ϕ_j^w.

(2) ϕ has filtration d if, for all $\lambda \in \mathbb{N}$,

$$(\phi - \mathrm{id})E_\lambda^w \subset E_{\lambda+d}^w.$$

Remark 2.11. Note that $\phi_0(x_i) = \mathrm{jet}(\phi(x_i), 1)$ for all $i = 1, \ldots, n$. Furthermore note that ϕ_0^w has filtration ≤ 0, and that, for $j > 0$, ϕ_j^w has filtration j if $\phi_{j-1}^w = \mathrm{id}$.

Definition 2.12. *Let $f = \sum_{i_1, \ldots, i_n} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{C}\{x_1, \ldots, x_n\}$. We call*

$$\mathrm{mon}(f) \quad := \quad \{x_1^{i_1} \cdots x_n^{i_n} \mid a_{i_1, \ldots, i_n} \neq 0\}$$

and

$$\sup(f) \quad := \quad \{i_1 \cdots i_n \mid a_{i_1, \ldots, i_n} \neq 0\}$$

*the **monomials** of f and the **support** of f, respectively. Let*

$$\Gamma_+(f) \quad := \bigcup_{x_1^{i_1} \cdots x_n^{i_n} \in \mathrm{supp}(f)} ((i_1, \ldots, i_n) + \mathbb{R}_+^n)$$

*and let $\Gamma(f)$ be the boundary in $\mathbb{R}_+^n$ of the convex hull of $\Gamma_+(f)$. We call $\Gamma(f)$ the **Newton boundary** of f. Then:*

*(1) Compact segments of $\Gamma(f)$ are called **facets** of $\Gamma(f)$[3]. If Δ is a facet, then the set of monomials of f lying on Δ is denoted by $\mathrm{supp}(f, \Delta)$ and the sum of the terms lying on Δ by $\mathrm{jet}(f, \Delta)$. Moreover, we write $\mathrm{supp}(\Delta)$ for the set of monomials corresponding to the lattice points of Δ. We use the same notation for a set of facets, considering the monomials lying on the union of the facets.*

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[2]We say that f is **(semi-)quasihomogeneous** if there exists a weight w such that f is (semi-)quasihomogeneous with respect to w.

[3]This term is used since they are codimension 1 faces of $\Gamma_+(f)$

(2) Any facet Δ induces a weight $w(\Delta)$ on $\mathrm{mon}(x_1, \ldots, x_n)$ in the following way: If Δ has the normal vector $-(w_{x_1}, \ldots, w_{x_n})$, in lowest terms, and $w_{x_1}, \ldots, w_{x_n} > 0$, we set

$$w(\Delta)\text{-}\deg(x_1) = w_{x_1}, \ldots, w(\Delta)\text{-}\deg(x_n) = w_{x_n}.$$

(3) If $w_1, \ldots, w_s$ are the weights associated to the facets of $\Gamma(f)$ ordered by increasing slope, there are unique minimal integers $\lambda_1, \ldots, \lambda_s \geq 1$ such that the piecewise weight with respect to $w(f) := (\lambda_1 w_1, \ldots, \lambda_s w_s)$ is constant on $\Gamma(f)$. We denote this constant by $d(f)$.

(4) Let Δ_1 and Δ_2 be facets with weights w_1 and w_2, respectively, and let w be the piecewise weight defined by w_1 and w_2. Let d be the w-degree of the monomials on Δ_1 and Δ_2. Then $\mathrm{span}(\Delta_1, \Delta_2)$ is the Newton polygon associated to the sum of all monomials of (w_1, w_2)-degree d.

(5) Suppose $\Gamma(f)$ has at least one facet. A monomial m lies strictly underneath, on or above $\Gamma(f)$, if the $w(f)$-degree of m is less than, equal to or greater than $d(f)$, respectively.

Definition 2.13. *Suppose f has finite Milnor number. A homogeneous basis $\{e_1, \ldots, e_\mu\}$ of the local algebra of f is* **regular** *with respect to the filtration given the piecewise weight w, if, for each $D \in \mathbb{N}$, the elements of the basis of degree D with respect to w are independent modulo the sum of $\mathrm{Jac}(f)$ and the space $E^w_{>D}$ of functions of filtrations bigger than D.*

For convenience we use the following notation: A **regular basis for** *f is a basis of Q_f which is regular with respect to the filtration given by $w(f)$.*

Remark 2.14. *Suppose f has finite Milnor number.*

(1) *For any $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ there exists a finite basis of Q_f with monomial representatives (take the monomials not appearing in the lead ideal of a Gröbner basis of $\mathrm{Jac}(f)$). We refer to this basis as a* **monomial basis** *of Q_f.*

(2) *A monomial basis of Q_f is regular for f if and only if the images of the degree D elements in the basis form a basis for $E^w_D/(E^w_D \cap \mathrm{Jac}(f) + E^w_{>D})$ for each D, where $w = w(f)$.*

Remark 2.15. *In the case that f is semi-quasihomogeneous, any basis of Q_f is a regular basis for f.*

More generally, the following result is proven as Proposition 9.4 in Arnold (1974).

Proposition 2.16. *For each $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ with finite Milnor number, there exists a finite regular basis for f, in fact, one consisting entirely out of monomials.*

The following remark describes how to obtain a regular basis:

Remark 2.17. *Under $E^w_D/(E^w_D \cap \mathrm{Jac}(f)) \to E^w_D/((E^w_D \cap \mathrm{Jac}(f)) + E^w_{>D})$, every element of a monomial basis of the target has a unique preimage with a monomial representative of degree D. Let R_D be a distinct set of monomial representatives of degree D of these preimages. Then $\bigcup_D R_D$ is a regular basis for f. We may restrict the union to the D with $E^w_D \not\subset \mathrm{Jac}(f)$.*

Definition 2.18. *A germ $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ is called* **convenient** *if the Newton polygon meets all coordinate axes.*

Definition 2.19. *We say that a convenient germ $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ has* **non-degenerate Newton boundary** *if for every facet Δ of $\Gamma(f)$ the saturation[4] of $\mathrm{jet}(f, \Delta)$ has finite Milnor number.*

Theorem 2.20. *(Kouchnirenko (1976)) Let $f \in \mathbb{C}\{x, y\}$ be a germ and suppose that its Newton polygon meets the x- and y-axis at $(a, 0)$ and $(0, b)$, respectively. Denoting by S the area of the polygon formed by the x- and y-axis of the Newton boundary, we have $\mu(f) \geq 2 \cdot S - a - b + 1$. If f has a non-degenerate Newton boundary, then $\mu(f) = 2 \cdot S - a - b + 1$. We define the* **Newton number** *of f as $N(f) = 2 \cdot S - a - b + 1$.*

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[4]The saturation of a polynomial $g \in \mathbb{C}[x, y]$ is the polynomial $\mathrm{sat}(g) := h \in \mathbb{C}[x, y]$ with $g = h \cdot x^n y^m$ with n, m maximal. Note that $\mathrm{sat}(g)$ is a generator of the ideal $\langle g \rangle : \langle x, y \rangle^\infty := \{h \in \mathbb{C}[x, y] \mid h \cdot \langle x, y \rangle^n \subset \langle g \rangle \text{ for some } n \geq 1\}$ in $\mathbb{C}[x, y]$.

Definition 2.21. *The μ-**constant stratum** of a germ $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ with Milnor number $\mu = \mu(f)$ is the connected component of the $(\mu + 1)$-jet space (that is, the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ of degree $\leq \mu + 1$) with fixed Milnor number μ which contains $\mathrm{jet}(f, \mu + 1)$.*

Our work will be based on Arnold's notion of a normal form (see Arnold et al. (1985)):

Definition 2.22. *Let $K \subset \mathbb{C}\{x_1, \ldots, x_n\}$ be a union of equivalence classes with respect to right-equivalence. A **normal form** for K is given by a smooth[5] map*

$$\Phi : \mathcal{B} \longrightarrow \mathbb{C}[x_1, \ldots, x_n] \subset \mathbb{C}\{x_1, \ldots, x_n\}$$

of a finite-dimensional $\mathbb{C}$-linear space $\mathcal{B}$ into the space of polynomials for which the following three conditions hold:

(1) $\Phi(\mathcal{B})$ intersects all equivalence classes of K,
(2) the inverse image in $\mathcal{B}$ of each equivalence class is finite,
(3) $\Phi^{-1}(\Phi(\mathcal{B}) \setminus K)$ is contained in a proper hypersurface in $\mathcal{B}$.

*The elements of the image of Φ are called **normal form equations**.*
*We say that a normal form is a **polynomial normal form** if the map Φ is polynomial.*

Example 2.23. Referring to Arnold's list of normal forms, $f = x^4 + y^4$ is of complex type X_9. The μ-constant stratum of f is covered by the complex normal form $\Phi : \mathbb{C} \to \mathbb{C}[x, y]$, $\Phi(a) = x^4 + ax^2y^2 + y^4$. For example, the germ $g = x^4 + \epsilon x^3 y + y^4$, for given ϵ, is in the μ-constant stratum of f. Hence, there exists a $\mathbb{C}$-algebra automorphism ϕ_1 such that $\phi_1(g) = x^4 + ax^2y^2 + y^4$, for some $a \in \mathbb{C}$. In this case there also exists a $\mathbb{C}$-algebra automorphism ϕ_2 such that $\phi_2(g) = x^4 - ax^2y^2 + y^4$.

Those $a \in \mathbb{C}$ for which $x^4 + ax^2y^2 + y^4$ is not in the μ-constant stratum of f are contained in a proper hypersurface. In the case of X_9, this hypersurface is given by the equation $a^2 - 4 = 0$. Precisely for these values of a, the Milnor number of $x^4 + ax^2y^2 + y^4$ is infinite, that is, the singularity is not isolated.

Definition 2.24. *(Arnold et al. (1985)) The **modality** of a germ $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ is the least number such that a sufficiently small neighborhood of $\mathrm{jet}(f, k)$, with k an upper bound on the determinacy of f, can be covered by a finite number of m-parameter families of orbits under the right-equivalence action on the k-jet space.*

A general question is how the number of parameters in a normal form relates to the modality. In the cases where Arnold has given a normal form, the two numbers are known to agree (this follows e.g. from the results in Arnold et al. (1985)). In the subsequent section, for each corank 2 singularity with non-degenerate Newton boundary, we give a normal form covering its μ-constant stratum. We will see, in particular, that for corank 2 singularities with non-degenerate Newton boundary, the number of parameters in the normal form equals the modality.

3. Normal forms for Isolated Singularities with a non-degenerate Newton Boundary

Arnold has constructed normal forms for the μ-constant strata of (in particular) all singularities of corank 2 and modality ≤ 2. He has also associated a type to each μ-constant stratum of these singularities (then also referred to as the type of a germ in the stratum). Restricting to the non-degenerate cases of corank 2 and modality ≤ 2, the type of a μ-constant stratum can be labeled by the constant Newton polygon of the respective normal form covering the stratum (noting that all of Arnold's normal forms have different Newton polygons).

Taking up this pattern in the more general setting of any corank 2 germ with non-degenerate Newton boundary, our goal in this section is to show that the μ-constant stratum of each such germ can be covered by a single normal form, and to explicitly specify such a normal form. Without loss of generality we can restrict to convenient germs. Our argument is based on the concept of an unfolding, and makes use of Greuel et al. (2007) and Boubakri et al. (2011).

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[5]That is, infnitely often differentiable.

Remark 3.1. In a fixed normal form, all normal form equations for different values of the parameters corresponding to points in the μ-constant stratum under consideration have the same Newton polygon. Except for the Newton polygon, our normal form only depends on a choice of a regular basis on and above the Newton polygon for a specifically chosen special fiber. If two μ-constant strata have normal forms with the same Newton polygon, they contain the same special fiber, hence must be equal. While a μ-constant stratum can have normal forms with different Newton polygons, we will see that all possible Newton polygons are related via right-equivalence, and we can impose a normalization condition to fix one of them. So the type of a μ-constant stratum can be labeled by the (up to normalization unique) Newton polygon of the normal form.

Definition 3.2. A convergent power series $F \in \mathbb{C}\{x,t\} = \mathbb{C}\{x_1,\ldots,x_n,t_1,\ldots,t_k\}$ is called an **unfolding** of $f \in \mathbb{C}\{x_1,\ldots,x_n\}$ if $F(x,0) = f(x)$. After choosing a representative $U \times T \to \mathbb{C}$ for F (again denoted by F) with open neighbourhoods of the origin $U \subset \mathbb{C}^n$ and $T \subset \mathbb{C}^k$ we use the notation

$$F_t(x) = F(x,t),$$

for the corresponding family F_t of holomorphic functions (or convergent power series) parametrized by $t \in T$. The neighborhood T is called the base of F.

Definition 3.3. Two unfoldings $F_1 : U \times T \to \mathbb{C}$ and $F_2 : U \times T \to \mathbb{C}$ are equivalent if one is taken into the other under the action of right-equivalence, that is, under the action of the Lie group G of $\mathbb{C}$-algebra automorphisms from $\mathbb{C}\{x_1,\ldots,x_n\}$ to itself, that smoothly depends on $t \in T$. In other words, there exist a smooth map-germ $g : (T,0) \to (G,e)$, where e is the identity element of G, such that

$$F_1(t) = g(t)F_2(t).$$

Definition 3.4. Let $F_1 : U \times T \to \mathbb{C}$ and $F_2 : U \times T' \to \mathbb{C}$ be unfoldings of $f \in \mathbb{C}\{x_1,\ldots,x_n\}$. Then F_2 is induced by F_1, if there exists a smooth map $\theta : (T',0) \to (T,0)$ such that the pullback of F_1 is F_2, that is $F_2 = (\mathrm{id}_U,\theta)^*(F_1)$.

Definition 3.5. An unfolding F of $f \in \mathbb{C}\{x_1,\ldots,x_n\}$ is called **versal**[6] if every unfolding of f is equivalent to one induced by F. A versal unfolding for which the base has the smallest possible dimension is called **miniversal**. An unfolding preserving the Milnor number is called an **equisingular** unfolding.

Definition 3.6. (Gabriélov, 1974, §3, Definition 2) Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x,y\}$ be a germ of modality m and let $\mathrm{F}_t(x,y)$ be a miniversal unfolding. The **proper modality** of f is the dimension at the origin of the set of values t for which $\mathrm{F}_t(x,y)$ has a singular point with Milnor number $\mu(f)$.

In fact, we have:

Theorem 3.7. (Gabriélov, 1974, §3, Theorem 6) The modality and the proper modality of a germ $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x,y\}$ coincide.

Definition 3.8. (Arnold, 1974) For a germ $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x,y\}$ with a non-degenerate Newton boundary, the **inner modality** is the number of all monomials in a regular basis for Q_f lying on or above $\Gamma(f)$.

We now connect Definitions 3.6 and 3.8. In Greuel et al. (2007), Corollary 2.71 gives a local description of the μ-constant stratum of a germ with a non-degenerate Newton boundary with respect to contact equivalence. The following result, which gives an equivalent version in terms of right equivalence, is an easy consequence thereof.

Theorem 3.9. Let $f \in \mathbb{C}\{x,y\}$ be a germ with a non-degenerate Newton boundary at the origin, then a miniversal, equisingular unfolding is given by

$$F(x,y,t) = f + \sum_{i=1}^{m} t_i g_i,$$

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[6]Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x,y\}$ and $e_1,\ldots,e_{\mu(f)+1}$ be a basis of Q_f and $\lambda = (\lambda_1,\ldots,\lambda_{\mu(f)+1})$ then $\mathrm{F}_\lambda(x,y) = f + \sum \lambda_i e_i$ is a versal unfolding.

where m is the modality of f, and $g_1, \ldots, g_m$ represent a regular basis for Q_f on and above $\Gamma(f)$.

It follows from the above result that:

Theorem 3.10. *For a germ $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x,y\}$ with a non-degenerate Newton boundary the proper modality and inner modality coincide.*

The following general observation on the behaviour of bases of the local algebra in families is made in (Arnold, 1974).

Lemma 3.11. *Assume that a family of smooth function germs f depending smoothly on a finite number of parameters has 0 as an isolated singularity with the same Milnor number μ for all values of the parameters of the family. Then every basis of the local algebra Q_f of the function corresponding to the value 0 of the parameter remains a basis for nearby values of the parameter.*

Proof. Let $A = \mathbb{C}\{t\}$, $t = (t_1, \ldots, t_n)$, let $F \in R = A\{x,y\}$ define the family, write $f_t(x,y) = F(x,y,t)$, and let $M(t) := R/\operatorname{Jac}(f_t)$. By assumption, the Milnor number $\mu(t) := \dim_\mathbb{C}(M(t))$ of the germ defined by f_t is finite. Let $m_1, \ldots, m_k \in R$ be monomials inducing a basis of $M(0)$. By Theorem 2.7, the $M(t)$ are locally finitely presented, hence form a coherent sheaf $\mathcal{M}$ on $\operatorname{Spec}(A)$. By the Lemma of Nakayama, the monomials $m_1, \ldots, m_k$ forming a basis of the fiber $M(0)$ extend to sections of $\mathcal{M}$ forming a generating system of the stalk at 0. These sections are defined in a Euclidean open neighborhood of 0. Since $\mathcal{M}$ is coherent, the sections also induce a generating system of every fiber $M(s)$ for $s \in U$. Since μ is assumed to be constant, $m_1, \ldots, m_k$ form a basis of $M(s)$ for $s \in U$.[7]

$\square$

The proof of the lemma in particular yields the well-known fact, that the Milnor number is upper semi-continuous in families, that is, $\mu(s) \leq \mu(0)$. In the case of germs with non-degnerate Newton boundary, we can refine the argument showing that a regular basis stays regular:

Proposition 3.12. *Let f_0 be a germ with a non-degenerate Newton boundary $\Gamma(f_0)$ and let f be a germ with the same Newton polygon as f_0 and non-degenerate Newton boundary. Then for f sufficiently close[8] to f_0, the monomials in $\operatorname{mon}(x,y)$ representing a regular basis for f_0 with respect to the filtration defined by $\Gamma(f_0) = \Gamma(f)$ also represent a regular basis for f with respect to the same filtration.*

Proof. Let $\mu = \mu(f_0)$, $w = w(f_0)$, $A = \mathbb{C}[t]$, $t = (t_1, \ldots, t_n)$, and let $F \in A\{x,y\}$ be an unfolding of f_0 covering the subspace of germs with Newton polygon $\Gamma(f_0)$ of the $(\mu+1)$-jet space, and write $f_t(x,y) = F(x,y,t)$. For the A-module E_D^w generated by the monomials m of $w\text{-}\deg(m) \geq D$ in the variables x,y, consider the vector spaces

$$M_D(t) := E_D^w/(E_D^w \cap \operatorname{Jac}(f_t) + E_{>D}^w).$$

and let $m_1, \ldots, m_k \in E_D^w$ be monomials inducing a basis of $M_D(0)$. The $M_D(t)$ form a coherent sheaf $\mathcal{M}_D$ on $\operatorname{Spec}(A)$. Arguing as in the proof of Lemma 3.11, the sections also induce a generating system of every fiber $M_D(s)$ for $s \in U$. In particular, we have $\dim_\mathbb{C} M_D(s) \leq \dim_\mathbb{C} M_D(0)$. On the other hand,

$$\bigoplus_D M_D(s) = \mathbb{C}\{x,y\}/\operatorname{Jac}(f_s) \text{ and } \dim \bigoplus_D M_D(s) = \mu = \dim \bigoplus_D M_D(0),$$

which implies that $\dim_\mathbb{C} M_D(0) = \dim_\mathbb{C} M_D(s)$ for all D. Therefore a regular basis for f_0 is a regular basis for f_s.

$\square$

---

[7]We can formulate the proof in a more elementary way: Since $\mathcal{M}$ is locally finitely presented, that is, the cokernel of a matrix (which varies smoothly in terms of t), the monomials $m_1, \ldots, m_k$ (which do not vary in terms of t) form at $t = 0$ a basis of a transversal space of the image of the presentation matrix, hence, they stay locally a generating system of a transversal space.

[8]Sufficiently close refers to the Euclidean distance in the $(\mu + 1)$-jet space.

As a next step in finding a normal form for the μ-constant stratum of a germ with a non-degenerate Newton boundary, we will observe that the topological type of all germs in the μ-constant stratum is the same. We will conclude that all germs can be transformed to germs with the same Newton polygon, and that for each of the these germs the Newton boundary is non-degenerate.

Definition 3.13. *Let $f, g \in \mathbb{C}\{x, y\}$ be germs with singular points at 0. Then f is* **topologically equivalent** *to g when there are neighbourhoods U and V of 0 in $\mathbb{C}^2$ and a homeomorphism $\phi : U \to V$ with $\phi(0) = 0$ such that for the zero sets X of f and Y of g we have*

$$\phi(X \cap U) = Y \cap V.$$

The **topological type** *of f is the equivalence class of f.*

Theorem 3.14. *(Brieskorn, Knörrer (1986), Theorem 15) Let $f \in \mathbb{C}\{x, y\}$ and $g \in \mathbb{C}\{x, y\}$ be germs with singularities at 0. Then f and g are topologically equivalent if and only if there exists a bijection between the irreducible branches of f and g (that is, the factors of f and g in $\mathbb{C}\{x, y\}$) such that*
- *the corresponding characteristic exponents are the same, and*
- *the intersection numbers between corresponding branches coincide.*

Theorem 3.15. *(Lê, Ramanujan (1976), Theorem 2.1) Let $F(x, t)$ be a family of μ-constant polynomials in $x = (x_1, \ldots, x_n)$, $n \neq 3$, with isolated singularity at 0 and coefficients which are smooth complex valued functions of $t \in I = [0, 1]$. Then the topological type of the singularities is constant in the family.*

Remark 3.16. Since the μ-constant stratum of a plane curve singularity is smooth, see Wahl (1974) and (Greuel et al., 2007, Theorem II.2.61, Corollary II.2.67) for an alternative proof, the μ-constant stratum can locally be parametrized by polynomial families depending smoothly on a finite number of parameters. Hence, by Theorem 3.15, the topological type is constant on the μ-constant stratum. By Theorem 3.14, this implies that the characteristic exponents and intersection numbers of the Puiseux expansions are constant on the μ-constant stratum.

Remark 3.17. If two germs f_1 and f_2 have the same characteristic exponents and intersection numbers, and f_1 has a non-degenerate Newton boundary, then f_2 is right-equivalent to a germ with non-degenerate Newton boundary and same Newton polygon as f_1. Moreover, any germ with maximal Newton number in the right-equivalence class of f_2 has a non-degenerate Newton boundary. Note that the converse is also true by Theorem 2.20.

For clarity of the presentation, the proof of this fact will be given in the next section in Proposition 4.17 and Corollary 4.18. We conclude from the remark that, if the μ-constant stratum contains a germ with non-degenerate Newton boundary, then every germ in the μ-constant stratum is equivalent to a germ with non-degenerate Newton boundary. In particular, Lemma 3.9 gives the existence of local polynomial families covering the whole μ-constant stratum. Combining our observations we obtain:

Theorem 3.18. *Let $f \in \mathbb{C}\{x, y\}$ be a convenient germ with non-degenerate Newton boundary Γ. Then all the germs in the μ-constant stratum of f are equivalent to a germ with the Newton polygon Γ and non-degenerate Newton boundary.*

We use the following result of Boubakri et al. (2011) to prove the main result of the section.

Proposition 3.19. *(Boubakri et al. (2011), Corollary 4.6) Let $f \in \mathbb{C}\{x, y\}$ be a convenient germ with a non-degenerate Newton boundary. Let f_0 be the principal part of f and let $\{e_1, \ldots, e_n\}$ be the set of all monomials in a regular basis for f_0 lying above $\Gamma(f_0)$. Then there are α_i such that*

$$f \sim f_0 + \sum_{i=1}^{n} \alpha_i e_i.$$

We now give the main result of this section:

Theorem 3.20. *Let f be a convenient germ with non-degenerate Newton boundary. Let f_0 be the sum of the terms of f lying on the vertex points of $\Gamma(f)$, and let $\{e_1, \ldots, e_n\}$ be the set of all monomials in a regular basis for f_0 lying on or above $\Gamma(f)$. Then the family*

$$f_0 + \sum_{i=1}^n \alpha_i e_i,$$

is a normal form of the μ-constant stratum containing f. Restricting the parameters $\alpha_1, \ldots, \alpha_n$ to values such that every germ $f_0 + \sum_{i=1}^n \alpha_i e_i$ has a non-degenerate Newton boundary and the same Newton polygon as that of f, we obtain all germs in the μ-constant stratum of f.

Proof. We first show that every germ g in the μ-constant stratum of f is equivalent to a germ in the family $f_0 + \sum_{i=1}^n \alpha_i e_i$. Note that by Theorem 3.18, every germ in the μ-constant stratum of f is equivalent to a germ with the same Newton polygon as f and non-degenerate Newton boundary.

Let B be a regular basis of f_0. By Proposition 3.12, B is also a regular basis for every germ with the same Newton boundary in a sufficiently small neighborhood M' of f_0 in the μ-constant stratum. By Theorem 3.9, there exists a Euclidean open neighborhood M'' of f_0 contained in the μ-constant stratum of f_0 such that all the germs in M'' are equivalent to at least one of the germs in the family $f_0 + \sum_{i=1}^n \alpha_i e_i$. Let $M := M' \cap M''$.

Since g is equivalent to a germ with the same Newton polygon as f and with non-degenerate Newton boundary, we may assume without loss of generality that $\Gamma(g) = \Gamma(f)$. We denote the sum of the terms of g on $\Gamma(g)$ by g_0.

As a linear combination of all monomials on the Newton boundary of f_0, we obtain a germ $f' \in M$ with the same monomials as g_0 and with regular basis B. Hence, by varying coefficients of the terms on the Newton polygon, we can construct a path τ in the germ space from g_0 to f' with constant Newton polygon. Since being degenerate is a Zariski closed condition, there are only finitely many points on the path such that the Newton boundary is degenerate. Moreover, if we denote by $\mathrm{reg}(h)$ the set of all regular bases of the germ h, then for all germs h in a Zariski open subset $U \subset \tau$, the set $\mathrm{reg}(h)$ is the same. Since, by Proposition 3.12, $\mathrm{reg}(h)$ is constant on a Euclidean open neighborhood of g_0, we have $g_0 \in U$. Hence, the set $A := \{h \in \tau \mid h \text{ Newton non-degenerate}, \mathrm{reg}(h) = \mathrm{reg}(g_0)\} \subset \tau$ is Zariski open. By construction, $\tau \cap M$ is contained in the subspace of M of germs with the same Newton polygon as f_0 (and with regular basis B). This implies that the intersection $A \cap \tau \cap M$ is non-empty, so $B \in \mathrm{reg}(g_0)$.

Hence, writing $e_1, \ldots, e_n$ for the elements of B above the Newton polygon and $e_{n+1}, \ldots, e_r$ for the elements of B on the Newton polygon[9], Proposition 3.19 implies that $g \sim g_0 + \sum_{i=1}^n \alpha_i' e_i$ with some α_i'. We will now show that $g_0 + \sum_{i=1}^n \alpha_i' e_i \sim f_0 + \sum_{i=1}^r \alpha_i e_i$ with some α_i, which proves our claim. In fact, we will show that, after modifying g without loss of generality by a right-equivalence keeping the Newton polygon of g, there are α_i'' with $g_0 = f_0 + \sum_{i=n+1}^r \alpha_i'' e_i$.

By the second part of Remark 3.17 (which refers to Proposition 4.17 in the next section), any germ with maximal Newton number in the right-equivalence class of a germ with non-degenerate Newton boundary also has a non-degenerate Newton boundary. So if g' is right-equivalent to g with $\Gamma(g') = \Gamma(g)$, then $N(g') = N(g) = \mu(g) = \mu(g')$ (using Theorem 2.20), so g' has maximal Newton number, and thus a non-degenerate Newton boundary. Hence, a right-equivalence keeping the Newton polygon also keeps non-degeneracy of the Newton boundary.

Write $w := w(f_0) = w(g_0)$ and $d := d(f_0) = d(g_0)$. The only elements of $\mathrm{Jac}(f_0)$ which have terms of w-degree d are of the form $c_1 x \frac{\partial f_0}{\partial x}$, $c_2 y \frac{\partial f_0}{\partial y}$, $c_3 y^{n_1} \frac{\partial f_0}{\partial x}$ and $c_4 x^{n_2} \frac{\partial f_0}{\partial y}$ or are linear combinations thereof. We can make a connection between terms occuring in a basis of $E_d^w / (\mathrm{Jac}(f_0) + E_{>d}^w)$ and terms occuring in g by the following argument:

(1) If there is an n_1 such that $\mathrm{supp}(c_3 y^{n_1} \frac{\partial f_0}{\partial x})$ contains a monomial $x^\alpha y^\beta$ of w-degree d (note that there can be at most one such monomial), the corresponding term in g can be modified by a right equivalence of the form $x \mapsto x + c y^{n_1}, y \mapsto y$, possibly generating terms of higher w-degree. We distinguish two cases:

---

[9]Note that there are also elements in B under the Newton polygon, which are not relevant here.

(a) If $\alpha \neq 0$, then $x^\alpha y^\beta$ is not a vertex monomial (a monomial corresponding to vertices of the Newton polygon): Removing a vertex monomial by the above transformation would increase the area under the Newton polygon and thus, according to Theorem 2.20, change the Milnor number.

(b) If $\alpha = 0$, then via the above transformation we may assume that the coefficient of y^β in g is normalized to 1.

(2) If there is a monomial of $c_4 x^{n_2} \frac{\partial f_0}{\partial y}$ of w-degree d, we can agrue in the same way as in (1), using a right equivalence of the form $y \mapsto y + cx^{n_2}$, $x \mapsto x$.

(3) Depending on whether $x\frac{\partial f_0}{\partial x}$ and $y\frac{\partial f_0}{\partial y}$ are $\mathbb{C}$-linearly independent in E_d^w or not, the elements $x\frac{\partial f_0}{\partial x}$ and $y\frac{\partial f_0}{\partial y}$ of $\mathrm{Jac}(f_0)$ can remove one or two vertex monomials in a basis of $E_d^w / E_{>d}^w$. After modifying g by a suitable right equivalence of the form $x \mapsto ax$, $y \mapsto by$ (which again does not change the Newton polygon of g), we may assume that the (one or two) vertex monomials have coefficients normalized to 1.

We can, hence, conclude that every monomial of w-degree d is either in B or does not appear in g or is a vertex monomial of g with coefficient 1. Hence, as claimed, there are α_i'', with $g_0 = f_0 + \sum_{i=n+1}^r \alpha_i'' e_i$.

We now show that the exceptional locus in the family is given by a hypersurface in the base space, and after removing this hypersurface, we still obtain up to right-equivalence all germs in the μ-constant stratum: First note, that if we restrict $\alpha_1, \ldots, \alpha_n$ to values such that every germ $f_0 + \sum_{i=1}^n \alpha_i e_i$ has a non-degenerate Newton boundary and Newton polygon $\Gamma(f)$, by Theorem 2.20, we obtain only germs in the μ-constant stratum of f. Moreover, this restriction amounts to considering parameter values in the complement of a hypersurface, which is the union of the hypersurfaces where the vertex terms vanish and the discriminant conditions for the individual faces. On the other hand, none of the germs corresponding to parameter values in the hypersurface is in the μ-constant stratum of f: Vanishing of a vertex monomial, by Theorem 2.20, leads to a larger Milnor number. Assume the Newton boundary of a germ $f_1 = f_0 + \sum_{i=1}^n \alpha_i e_i$ becomes degenerate (with no vertex monomial is vanishing), and the germ is right-equivalent to a germ f_1' with non-degenerate Newton boundary and Milnor number $\mu(f_1') = \mu(f)$. Then, by Theorem 3.18, we may assume that $\Gamma(f_1') = \Gamma(f_0)$. Hence, we obtain $\mu(f_1) = \mu(f_1') = N(f_1') = N(f_0) = N(f_1)$. In particular f_1 has maximal Newton number in its right-equivalence class. Then, as above, the second part of Remark 3.17 implies that f_1 is non-degenerate, a contradiction.

Finally, we show that our family $f_\alpha = f_0 + \sum_{i=1}^n \alpha_i e_i$ meets every right-equivalence class in only finitely many points. Fix a parameter value β. Then for any parameter value γ with $f_\gamma \sim f_\beta$, by (Greuel et al., 2018, Part II, Proposition 2.48(i)) it follows that in a sufficiently small neighborhood of γ in the parameter space, the set of all γ' with $f_{\gamma'} \sim f_\beta$ is closed. Since the parameter space is paracompact, it admits a locally finite cover of such neighborhoods. Hence, the set of all γ' in the parameter space with $f_{\gamma'} \sim f_\beta$ is closed. So, if we assume that there are infinitely many such γ', there exists a non-constant one parameter subfamily $f_t = f_0 + \sum_{i=1}^n \alpha_i(t) e_i$ with $f_t \sim f_\beta$. Then, by (de Jong and Pfister, 2000, Theorem 9.1.5), it follows that there is an $e_i \in \mathrm{Jac}(f)$, a contradiction.

$\square$

Remark 3.21. Theorem 3.20 also is true in the formal sense.

Proof. Any formal power series f with finite Milnor number is (via a formal transformation) right-equivalent to a polynomial (see Remark 2.8). The first part of the proof of Theorem 3.20 applied to this polynomial shows that f is right-equivalent to a germ in the family $\mathrm{NF}(f)$. In order to see that this family intersects every right-equivalence class of formal power series only in finitely many points, note that, by the second part of the proof of Theorem 3.20, it intersects the corresponding right-equivalence class of convergent power series in only finitely many points

and that all germs in $\mathrm{NF}(f)$ are polynomials, and hence are convergent. The claim then follows by the Artin approximation theorem. $\qquad\square$

We finish the current section by discussing in which sense the normal form associated by Theorem 3.20 to a germ which is equivalent to a germ with non-degenerate Newton boundary is unique, and how it can be used to label μ-constant strata. We first observe:

Remark 3.22. Any μ-constant stratum containing a germ with a non-degenerate Newton boundary has a normal form as constructed in Theorem 3.20 with fixed Newton polygon. Any such normal form depends only on the vertices of the Newton polygon Γ and the choice of a regular basis. For fixed Newton polygon Γ, different choices of regular bases lead to normal forms of the same μ-constant stratum. However, for the same μ-constant stratum there can be normal forms with different non-degenerate Newton boundaries. Hence, like Arnold, we associate a **type** to each μ-constant stratum and then make a specific choice of a normal form for each type T. This choice amounts to fixing a Newton polygon and a regular basis of monomials on and above the Newton polygon. For distinguishing between different types, it is sufficient to know the Newton polygon of the normal form.

To achieve uniqueness of the Newton polygon associated to a fixed type (in order to label types by Newton polygons), the Newton polygon may be chosen according to the following normalization condition, which can be satisfied via a right-equivalence for any germ with non-degenerate Newton boundary. Moreover, the normalization condition is satisfied for all germs in the normal form produced by Theorem 3.20

Remark 3.23. We will refer to a weighted homogeneous non-unit in $\mathbb{C}[x,y]$ which is linear in one variable as **linear homogeneous**, otherwise we refer to it as non-linear homogeneous.

Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x,y\}$ be a convenient germ with non-degenerate Newton boundary, and let Δ be a facet of $\Gamma(f)$, and write $w = w(\Delta)$. If in the factorization of $\mathrm{jet}(f,\Delta) = w\text{-}\mathrm{jet}(f,d(f))$ in $\mathbb{C}[x,y]$ would occur any linear homogeneous factor except x or y with exponent larger than one, or if it has a non-linear factor of multiplicity larger than one, then $w\text{-}\mathrm{jet}(f,d(f))$ would have degenerate saturation. Hence, we can factorize $w\text{-}\mathrm{jet}(f,d(f))$ in $\mathbb{C}[x,y]$ as

$$w\text{-}\mathrm{jet}(f,d(f)) = x^a \cdot y^b \cdot g_1 \cdots g_n \cdot \widetilde{g},$$

where a,b are integers, $g_1,\ldots,g_n$ are linear homogeneous polynomials not associated to x or y , and $\widetilde{g}$ is a product of non-associated irreducible non-linear homogeneous polynomials. If $w(x) = w(y)$, we can change f via a right-equivalence such that for the resulting germ $a \neq 0$ and $b \neq 0$. If $w(x) > w(y)$, we can change f such that for the resulting germ $a = 0$ only if $n = 0$. If $w(x) < w(y)$, we can change f such that for the resulting germ $b = 0$ only if $n = 0$.

Proof. Note that if $w(x) > w(y)$ and $a = 0$ (i.e., Δ meets the y-axis), then for $n \neq 0$ there is a right-equivalence with $g_1 \mapsto x$, $y \mapsto y$. The case $w(x) < w(y)$, $b = 0$ can be treated in a similar way. The same applies if $w(x) = w(y)$ for the cases $a = 0$, $b \neq 0$ and $a \neq 0$, $b = 0$. If $a = b = 0$, then by $f \in \mathfrak{m}^2$ we have $n \geq 2$, hence there is a right-equivalence with $g_1 \mapsto x$, $g_2 \mapsto y$. $\qquad\square$

Definition 3.24. *Let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x,y\}$ be a convenient germ with non-degenerate Newton boundary, and let Δ be a facet of $\Gamma(f)$, and write $w = w(\Delta)$. Then $\mathrm{jet}(f,\Delta)$ factorizes in $\mathbb{C}[x,y]$ as*

$$\mathrm{jet}(f,\Delta) = x^a \cdot y^b \cdot g_1 \cdots g_n \cdot \widetilde{g},$$

*where a,b are integers, $g_1,\ldots,g_n$ are linear homogeneous polynomials not associated to x or y, and $\widetilde{g}$ is a product of non-associated irreducible non-linear homogeneous polynomials. We say that f is **normalized** with respect to the facet Δ, if*

$$
\begin{aligned}
w(x) = w(y) && \implies && a,b \neq 0 \\
w(x) > w(y) \quad and \quad a = 0 && \implies && n = 0 \\
w(x) < w(y) \quad and \quad b = 0 && \implies && n = 0
\end{aligned}
$$

This observation can be used to choose a natural normalization for germs with non-degenerate Newton boundary:

Lemma 3.25. *There exists a right-equivlance such that f is normalized with respect to all facets of the Newton polygon.*

Proof. For any facet not meeting the coordinate axes, the normalization condition is empty since both a and b are non-zero. The claim follows then by the following iteration: If there is only one facet meeting a coordinate axis (hence by convenience both axes), we apply Remark 3.23. If there are two facets Δ_1 and Δ_2 meeting the x- and y-axis, respectively, then by Remark 3.23 there are right-equivalences ϕ_1 and ϕ_2 which normalize the jets of Δ_1 and Δ_2 individually. Applying the composition of ϕ_1 and ϕ_2 in any order will normalize the jets of both facets: Without loss of generality, we may assume that $\phi_1(x) = x$, $\phi_1(y) = y + c_1 \cdot x^{n_2}$ and $\phi_2(x) = x + c_2 \cdot y^{n_2}$, $\phi_2(y) = y$ with $c_1, c_2 \in \mathbb{C}$. Hence, by convexity of the Newton polygon, ϕ_1 does not change the jet of any facet except Δ_1, analogously ϕ_2 does not change the jet of any facet except Δ_2. Truncating at the determinacy and repeating the process either normalizes (after a single iteration) or deletes (after possibly several iterations) facets meeting a coordinate axis. The process, hence, determines in finitely many steps a germ which is normalized with respect to all faces (but may not be convenient). $\qquad\square$

Since the definitions and results in the standard literature are typically formulated for convenient germs, it makes sense to relate our observations to the setting of convenient germs. To do so, we define:

Definition 3.26. *A facet of the Newton polygon of a germ is called a **smooth face** if the saturation of its jet is smooth.*

Remark 3.27. A germ f which is normalized with respect to every facet of the Newton polygon may not be convenient. However, adding the terms x^d or y^d with $d = \mu(f) + 2$, if necessary, yields a convenient germ which is right-equivalent to f. In this way, we obtain a convenient germ, which is normalized with respect to all facets of the Newton polygon except for smooth facets meeting a coordinate axis. If a germ with non-degenerate Newton boundary satisfies this property, we say that it satisfies the **normalization condition**.[10]

Remark 3.28. For any germ which is equivalent to a germ with non-degenerate Newton boundary, the germs with non-degenerate Newton boundary in its right-equivalence class which satisfy the normalization condition have a unique Newton polygon (up to permuation of x and y).

This follows directly from the following remark, since characteristic exponents and intersection numbers are invariant under right-equivalence.

Remark 3.29. If a germ is right-equivalent to a germ with non-degenerate and normalized Newton polygon Γ, then the characteristic exponents and intersection numbers of the branches of f uniquely determine Γ (up to permutation of the variables).

For clarity of the presentation, the proof will be given in the next section in Corollary 4.12. More generally, combining Remark 3.29 with Remark 3.17, and Lemma 3.25, we obtain:

Theorem 3.30. *In a μ-constant stratum which contains a germ with a non-degenerate Newton boundary, every right-equivalence class contains a normalized germ, and all germs in the μ-constant stratum satisfying the normalization condition have the same Newton polygon (up to permutation of the variables).*

Throughout the paper we will choose normal forms according to the normalization condition, which means that any germ in the image of the normal form which is in the μ-constant stratum satisfies the normalization condition. We denote the chosen normal form corresponding to the

---

[10]Note that a germ satisfying the normalization condition may or may not have smooth facets meeting a coordinate axis, that is, there may exist non-smooth facets meeting a coordinate axis.

type T by $\mathrm{NF}(T)$. For $b \in \mathrm{par}(\mathrm{NF}(T)) := \Phi^{-1}(K)$ with K as in Definition 2.22, we write $\mathrm{NF}(T)(b) := \Phi(b)$ for the corresponding normal form equation.

Definition 2.12 extends then to types:

Definition 3.31. *Let T be a singularity type of a germ with a non-degenerate Newton boundary. We call*

$$\mathrm{supp}(T) \quad := \quad \mathrm{supp}(\mathrm{NF}(T)(b))$$

*where $b \in \mathrm{par}(\mathrm{NF}(T))$ is generic, the **support** of T. In the same way, we define the **Newton boundary** $\Gamma(T)$ of the type, and the notations $\Gamma_+(T)$, $d(T)$, and $\mathrm{supp}(T, \Delta)$. If for a given type T, $w\text{-}\mathrm{jet}(\mathrm{NF}(T)(b), j)$ is independent of $b \in \mathrm{par}(\mathrm{NF}(T))$, we denote it by $w\text{-}\mathrm{jet}(T, j)$.*

4. The correspondence between Characteristic Exponents, Intersection Numbers and normalized non-degenerate Newton boundaries

In this section, we will prove the results stated in Remarks 3.17 and 3.29. So we will show that, if two germs have the same characteristic exponents and intersection numbers, and one of them has a non-degenerate Newton boundary, then also the other one is right-equivalent to a germ with non-degenerate Newton boundary and same Newton polygon. Moreover, we will prove that if a germ f is equivalent to a germ with a non-degenerate Newton boundary, this Newton boundary is uniquely determined by the characteristic exponents and intersection numbers of the branches of f, provided we require the normalization condition as specified in Definition 3.24. The results will be given in Corollaries 4.12 and 4.18.

We first recall some facts on Puiseux expansions, which play a key role in the proofs.

Theorem 4.1. *(de Jong and Pfister, 2000, Theorem 5.1.1) Let $0 \neq f \in \mathbb{C}\{x, y\}$ be irreducible. Then there exist $x(t), y(t) \in \mathbb{C}\{t\}$ such that*

(i) $f(x(t), y(t)) = 0$,
(ii) $\dim_{\mathbb{C}}(\mathbb{C}\{t\}/\mathbb{C}\{x(t), y(t)\}) < \infty$.

*The tuple $(x(t), y(t))$ is called a **parametrization** of f.*

Parametrizations can be computed using the Newton-Puiseux algorithm. A germ $f \in \mathbb{C}\{x, y\}$ is called y-general, if f is not divisible by x, and f is called y-general of order b, if $f(0, y)$ has a zero of order b. The field of Puiseux series over $\mathbb{C}$ is $\mathbb{C}\{\{x\}\} = \bigcup_{k=1}^{\infty} \mathbb{C}((x^{\frac{1}{k}}))$. The valuation ring $\mathcal{P}(x) = \mathbb{C}\{\{x\}\}_{v \geq 0} = \bigcup_{k=1}^{\infty} \mathbb{C}\{x^{\frac{1}{k}}\}$ with respect to the canonical valuation ν consists of all Puiseux series with non-negative exponents only. By the Newton-Puiseux theorem (see, for example, (Greuel et al., 2007, Theorem I.3.3)), any y-general germ $f \in \mathbb{C}\{x\}[y]$, can be factorized as $f = c \cdot \prod_{i=1}^{d}(y - \phi_i(x))$ with $c \in \mathbb{C}$ and the Puiseux expansions $\phi_i \in \mathcal{P}(x)$. By passing to an irreducible factor of f and translating the singularity to the origin, it is enough to consider irreducible Weierstrass polynomials. In this case, the structure of the Puiseux expansions has a straight-forward description.

Theorem 4.2. *(de Jong and Pfister, 2000, Theorem 5.1.7) Let $f \in \mathbb{C}\{x\}[y]$ be an irreducible Weierstraß polynomial of y-degree n. Let ϵ be a primitive n-th root of unity. Then there exists a power series $\eta \in t \cdot \mathbb{C}\{t\}$ such that f factorizes in $\mathbb{C}\{x^{\frac{1}{n}}\}[y]$ as*

$$f = \prod_{i=1}^{n}(y - \eta(\epsilon^i x^{\frac{1}{n}})).$$

In particular, all Puiseux expansions of f are contained in $\mathbb{C}\{x^{\frac{1}{n}}\}$.

A Puiseux expansion of a y-general germ can be computed using the Newton-Puiseux algorithm, which we recall for further reference as Algorithm 1. Note that, by branching into all possible choices of facets Δ in the algorithm, one can obtain all Puiseux expansions of f.

Remark 4.3. (1) Starting out with the field $K_{(0)} = \mathbb{C}((x))$, this field is enlarged in the i-th iteration of the algorithm to $K_{(i+1)} = K_{(i)}(x^{\frac{1}{p_0 \cdots p_i}})$.

Algorithm 1 Newton-Puiseux algorithm

Input: A y-general germ $f \in \langle x, y \rangle \subset \mathbb{C}\{x\}[y]$ and a bound $l \geq 0$.
Output: A Puiseux expansion ψ of f up to order l

1: $i := 0$, $f^0 := f$, $x_0 := x$, $y_0 := y$, $K_{(0)} = \mathbb{C}((x_0))$
2: **while** true **do**
3:     Let Δ be a facet of $\Gamma(f^{(i)})$ with slope $-\frac{p_i}{q_i}$ in lowest terms
4:     $w := (p_i, q_i)$
5:     Let a_i be a root of the univariate polynomial $w\text{-}\mathrm{jet}(f^{(i)}, d_w)(1, y_{i+1}) \in \mathbb{C}[y_{i+1}]$
6:     With new variables x_{i+1}, y_{i+1} and s_{i+1}, set

$$f^{(i+1)} \quad := \quad \frac{1}{x_{i+1}^{d_w}} f^{(i)}(x_{i+1}^{p_i}, x_{i+1}^{q_i}(a_i + y_{i+1})) \in \mathbb{C}\{x_{i+1}\}[y_{i+1}]$$

$$s_i \quad := \quad x_i^{\frac{q_i}{p_i}}(a_i + s^{(i+1)}) \in K_{(i+1)}[s^{i+1}],$$

    where $K_{(i+1)} = K_{(i)}[x_{i+1}]/(x_{i+1}^{p_i} - x_i)$
7:     **if** $f^{(i+1)} \in \mathbb{C}[y_{i+1}]$ or $\frac{q_0}{p_0} + \frac{q_1}{p_0 p_1} + \ldots + \frac{q_i}{p_0 \cdot \ldots \cdot p_i} \geq l$ **then**
8:         Replace successively $s^{(j+1)}$ in $s^{(j)}$ for $j = 0, \ldots, i$ and obtain

$$s^{(0)} \quad := \quad x^{\frac{q_0}{p_0}}\left(a_0 + x_1^{\frac{q_1}{p_1}}\left(a_1 + \ldots + x_i^{\frac{q_i}{p_i}}(a_i + s^{(i+1)})\right)\right)$$

$$= \quad x^{\frac{q_0}{p_0}}\left(a_0 + x^{\frac{q_1}{p_0 p_1}}\left(a_1 + \ldots + x^{\frac{q_i}{p_0 \cdot \ldots \cdot p_i}}(a_i + s^{(i+1)})\right)\right)$$

        and set $s^{(i+1)} = 0$ in the expression.
9:         **return** $s^{(0)}$
10:    **else**
11:        $i := i + 1$

---

(2) Suppose f is y-general and n is minimal such that all Puiseux expansions of f are in $\mathbb{C}(x^{1/n})$. Setting $x(t) = t^n$ and $y(t) = \psi(t^n)$ with ψ as returned by Algorithm 1, we obtain a parametrization of f as in Theorem 4.1 developed up to order l.

(3) If f is irreducible and y-general of order b, then $n = b$ (Greuel et al., 2007, Proposition I.3.4).

Definition 4.4. *(de Jong and Pfister, 2000, Definition 5.2.14) For $\gamma \in \mathcal{P}(x)$, let $n \in \mathbb{N}$ be minimal with $\gamma \in \mathbb{C}\{x^{\frac{1}{n}}\}$, and write $\gamma = \sum_{j \geq m} b_j x^{\frac{j}{n}}$, with coefficients $b_j \in \mathbb{C}$, $b_m \neq 0$. We define*

$$k_0 \quad := \quad \min\{n, m\},$$

$$k_1 \quad := \quad \begin{cases} \min\{j \mid b_j \neq 0, \ \gcd\{j, k_0\} < k_0\} & \text{if } n \leq m, \\ \min\{j \mid b_j \neq 0, \ \gcd\{j, k_0\} < k_0\} + n - m & \text{if } n > m \text{ and } m \mid n, \\ n & \text{if } n > m \text{ and } m \nmid n, \end{cases}$$

and for $\nu \geq 2$,

$$k_\nu \quad := \quad \begin{cases} \min\{j \mid b_j \neq 0, \ \gcd\{j, k_0, \ldots, k_{\nu-1}\} < \gcd\{k_0, \ldots, k_{\nu-1}\}\} & \text{if } n \leq m, \\ \min\{j \mid b_j \neq 0, \ \gcd\{j, k_0\} < k_0\} + n - m & \text{if } n > m \text{ and } m \mid n, \\ n & \text{if } n > m \text{ and } m \nmid n, \end{cases}$$

*The number k_ν is called the ν-th **characteristic exponent** of γ. Note that, by construction, there is a minimal g such that $\{k_\nu \mid \nu \in \mathbb{N}_0\} = \{k_0, \ldots, k_g\}$. Also note that in all cases $k_0 < k_1 < \ldots < k_g$ and $\gcd(k_0, \ldots, k_g) = 1$.*

*If there is a maximal $m_0 \geq 0$ with $\sum_{j=m}^{m_0} b_j x^{\frac{j}{n}} \in \mathbb{C}\{x\}$, then we define the **rational part** of γ as the polynomial $\gamma^{rat} = \sum_{j=m}^{m_0} b_j x^{\frac{j}{n}}$, otherwise we set $\gamma^{rat} = \gamma$.*

Remark 4.5. (1) The above definition can also be formulated in terms of a parametrization satisfying conditions (i) and (ii) in Theorem 4.1 (see de Jong and Pfister (2000)). After applying the $\mathbb{C}$-algebra automorphism $x \mapsto y$, $y \mapsto x$, if necessary, we may assume that $n \leq m$. In this case, n is the first characteristic exponent of γ. Considering higher characteristic exponents, $j \in \mathbb{N}$ is a characteristic exponent if and only if for some $r \in \mathbb{N}$ we have $x^{\frac{m}{n}} + x^{\frac{m+1}{n}} + \ldots + x^{\frac{i-1}{n}} \in \mathbb{C}[x^{\frac{1}{r}}]$ and $x^{\frac{m}{n}} + x^{\frac{m+1}{n}} + \ldots + x^{\frac{j}{n}} \notin \mathbb{C}[x^{\frac{1}{r}}]$. That is, there is an i in Algorithm 1 such that $\frac{p_i}{q_i} = \frac{j}{n}$ and $K_{(i+1)}$ is a proper extension of $K_{(i)}$.

(2) By Theorem 4.2, all Puisuex expansions of an irreducible Weierstrass polynomial have the same characteristic exponents.

(3) Suppose f is an irreducible Weierstraßpolynomial, and $\phi_1, \ldots, \phi_d$ are its Puiseux expansions in terms of y, and G is the Galois group of the splitting extention $\mathbb{C}(x) \subset \mathbb{C}(x)[\phi_1, \ldots, \phi_d]$. Then the characteristic exponents of f determine the length of the orbit under G of any jet of a Puiseux expansion.

Definition 4.6. *Let $f, g \in \mathbb{C}\{x, y\}$ be irreducible germs with no common component. Then the* **intersection number** *of f and g is*

$$i(f, g) := \dim(\mathbb{C}\{x, y\}/\langle f, g \rangle).$$

Lemma 4.7. *(de Jong and Pfister, 2000, Lemma 5.1.5) Let $f, g \in \mathfrak{m} \subset \mathbb{C}\{x, y\}$ with g irreducible. If $t \mapsto (x(t), y(t))$ is a parametrization of g, then*

$$
\begin{aligned}
i(f, g) &= \operatorname{ord}_t f((x(t), y(t))) \\
&= \sup\{m \in \mathbb{N} \mid t^m \text{ divides } f((x(t)), (y(t)))\}.
\end{aligned}
$$

Theorem 4.8. *(de Jong and Pfister, 2000, Theorem 5.1.17) Let $f \in \mathbb{C}\{x, y\}$ be convenient, let $\Delta_1, \ldots, \Delta_r$ be the facets of $\Gamma(f)$, and d_i be the slope of Δ_i. Then $f = f_1 \cdot \ldots \cdot f_r$, where f_i is convenient and $\Gamma(f_i)$ has only one facet of slope d_i for all $i = 1, \ldots, r$.*

Lemma 4.9. *Let $f \in \mathfrak{m} \subset \mathbb{C}\{x, y\}$ be a convenient germ with a non-degenerate Newton boundary. Let $\Delta_1, \ldots, \Delta_r$ be the facets of $\Gamma(f)$, let d_i be the slope of Δ_i, and let $f = f_1 \cdot \ldots \cdot f_r$ be the factorization of f as in Theorem 4.8 such that $\Gamma(f_i)$ is convenient with only one facet of slope d_i, which we denote by $\overline{\Delta}_i$. Then the following holds true:*

(1) If f_i does not have a weighted linear factor, then

$$d_i = -\frac{k_0^i}{k_1^i} \quad \text{or} \quad d_i = -\frac{k_1^i}{k_0^i},$$

where k_0^i and k_1^i are the first two characteristic exponents of a Puiseux expansion of f_i.

(2) With notation and assumptions as in (1) and assuming in addition that $d_j \neq -1$ for some $j \neq i$, the slope d_i is uniquely determined by the characteristic exponents k_0^i, k_1^i, the intersection number of any branch of f_i with any branch of f_j, and the first characteristic exponent k_0^j of a Puiseux expansion of that branch.

(3) Suppose f_i has a weighted linear factor, and denote this factor by f_i'. If for some j we have $d_j \geq d_i \geq -1$ or $-1 \geq d_i \geq d_j$, and f_j has a factor f_j' with $f_i' \neq f_j'$, then d_i is determined by the intersection number of f_i' and f_j', and the first characteristic exponent k_0^j of a Puiseux expansion of f_j'.

(4) If $d_i \leq -1$, then the sum of the first characteristic exponents associated to the irreducible branches of f_i is the x-degree of the saturation of f_i. If $d_i \geq -1$, then the sum yields the y-degree of the saturation of f_i.

Proof. (1) By Theorem 4.8 we have $w(f_i)\text{-}\mathrm{jet}(f_i, d(f_i)) = \mathrm{sat}(\mathrm{jet}(f, \Delta_i))$. Since f has a non-degenerate Newton boundary, $w(f_i)\text{-}\mathrm{jet}(f_i, d(f_i))$ is non-degenerate. This implies that $w(f_i)\text{-}\mathrm{jet}(f_i, d(f_i))$ factorizes into $w(f_i)$-homogeneous irreducible polynomials, each factor with multiplicity one. Let γ be a Puiseux expansion of any of the factors. Via a right equivalence exchanging the variables, we may assume that γ satisfies $n \leq m$ in Definition 4.4. By Remark 4.3(3), we obtain that f_i is a y-general polynomial of y-degree $n = k_0^i$. Since f_i

is not smooth, we have that $n > 1$. Hence, a field extension over $\mathbb{C}$ of degree > 1 is needed in line 6 of the first iteration of Algorithm 1, which implies by Remark 4.5(1) that the slope of $\overline{\Delta}_i$ is $d_i = -\frac{k_0^i}{k_1^i}$.

(2) Suppose f_i' and f_j' are branches of f_i and f_j, respectively. Modulo switching the variables, we may assume that $w(y) > w(x)$ for $w = w(\Delta_j)$. Denoting the y-degree of $\mathrm{jet}(f_j, \overline{\Delta}_j)$ by n, and using that $\mathrm{jet}(f_j, \overline{\Delta}_j)$ is y-general of order n, Remark 4.3(3) implies that $n = k_0^j$.

If f_i' is standard homogeneous, then $d_i = -k_0^i/k_1^i = -k_1^i/k_0^i$, and there is nothing to prove. Depending on $\tilde{w} = w(\Delta_i)$ we have the following two cases: If $\tilde{w}(x) < \tilde{w}(y)$ then $d_i = -k_0^i/k_1^i$, and a parametrization of f_i' is given by

$$x(t) := t^{k_1^i} + \text{ higher order terms}, \qquad y(t) := t^{k_0^i}. \tag{1}$$

while for $\tilde{w}(x) > \tilde{w}(y)$, we have $d_i = -k_1^i/k_0^i$, and a parametrization of f_i' is

$$x(t) := t^{k_0^i}, \qquad y(t) := t^{k_1^i} + \text{ higher order terms}, \tag{2}$$

If (1) is a parametrization of f_i', then $i(f_i', f_j') = n \cdot k_0^i = k_0^j \cdot k_0^i$. If (2) is a parametrization of f_i', then $i(f_i', f_j') > n \cdot k_0^i = k_0^j \cdot k_0^i$, which distinguishes between the two cases.

(3) Without loss of generality, $\mathrm{jet}(f_i', \Delta_i)$ is of the form $ay + bx^n$. Then $-1 \geq d_i \geq d_j$, and f_i' has a parametrization of the form $x(t) = t$, $y(t) = c \cdot t^n +$ higher order terms in t, where c is a non-zero constant. Since the y-intercept of f_j' is k_0^j, we obtain that $i(f_i', f_j') = n \cdot k_0^j$. Here, we use that $d_j \leq d_i$ and f has a non-degenerate Newton boundary, which implies that when plugging the parametrization of f_i' into that of f_j', the lowest non-vanishing order is $n \cdot k_0^j$. We hence obtain that $d_i = -k_0^j/i(f_i', f_j')$.

(4) Any irreducible factor g of f_i is x- and y-general. Hence, by Remark 4.3(3), the first characteristic exponent of the Puiseux expansions of g is either the x-degree or the y-degree of f, depending on whether the slope of the facet is ≤ -1 or ≥ -1, respectively.

$\square$

Remark 4.10. Suppose that, in Lemma 4.9, the factor f_i does not satisfy any of the assumptions in (1)–(4). Then f_i is a smooth branch, and the face Δ_i meets a coordinate axis, without loss of generality the y-axis. There are right equivalences which do not change the jets of the faces Δ_j, $j \neq i$ such that the resulting germ has a face Δ_i with arbitrarily small slope. Moreover, we can achieve for Δ_i any y-intercept such that $d_i \geq d_j$ for all j.

Proof. We have to consider the case that in Lemma 4.9(3) there does not exist a branch f_j' with the required properties. Then $\mathrm{jet}(f, \Delta_i)$ is of the form $a \cdot x \cdot y^r + b \cdot y^{n+r}$ with constants $a, b \neq 0$. Applying the right equivalence $x \mapsto x - \frac{b}{a}y^n$, $y \mapsto y$, does not change $\mathrm{jet}(f, \Delta_j)$ for $j \neq i$, but strictly increases the y-intercept of $\mathrm{jet}(f, \Delta_i)$. By repeating the process, we can achieve that the y-intercept is larger than the determinacy of f, which means that the pure y-power term and, hence, the face Δ_i can be removed by a right-equivalence. Then a face Δ_i with any y-intercept k such that $d_i \geq d_j$ can be created by applying a transformation of the form $x \mapsto x + y^{k-r}$, $y \mapsto y$ to the germ.[11]

$\square$

We collect the observations of the lemma:

Proposition 4.11. *Let $f \in \mathfrak{m} \subset \mathbb{C}\{x, y\}$ be a germ with non-degenerate Newton boundary, and let $f = g_1 \cdots g_s$ be a factorization of f into irreducible branches, sorted by increasing slope. Then the intersection numbers and characteristic exponents of the factors g_i determine the Newton polygon of each g_i, except that of g_1 or g_s in case the facet containing the respective factor is smooth.*[12]

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[11]Note that $k > r$, since the original face Δ_i satisfied $d_i \geq d_j$. We may, however, have $k < r + n$ if this is allowed by the convexity condition $d_i \geq d_j$.

[12]Equivalently, the factor is the only one of its slope and is weighted linear.

Proof. Follows directly from Lemma 4.9.   □

Corollary 4.12. *Suppose that $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ is equivalent to a germ with non-degenerate and normalized Newton polygon Γ. Then the characteristic exponents and intersection numbers of the branches of f uniquely determine Γ (up to permutation of the variables).*

Proof. Follows from Proposition 4.11 since imposing the normalization condition fixes the slope of the Newton polygon of g_1 and g_s in case the facet containing the respective factor is smooth.   □

We now turn to the non-degeneracy statement. We first note:

Proposition 4.13. *Let $f \in \mathbb{C}\{x, y\}$ be a germ with non-degenerate Newton boundary. Then no two Puiseux series of f agree up to first non-zero order.*

Proof. If two Puiseux expansions agree up to first non-zero order, then the jet of the corresponding facet has a multiple factor and, hence, is degenerate.   □

Corollary 4.14. *Let $f \in \mathbb{C}\{x, y\}$ be an irreducible Weierstrass polynomial, let $\phi_1, \ldots, \phi_d$ be the Puiseux expansions of f in terms of y, let G be the Galois group of the splitting extension $\mathbb{C}(x) \subset \mathbb{C}(x)[\phi_1, \ldots, \phi_d]$, and denote by r the lowest order of a non-vanishing term of the expansions. Then f has non-degenerate Newton boundary if and only if $\mathrm{jet}(\phi_i, r)$, $i = 1, \ldots, d$ form an orbit of length d under the action of G.*

Proof. Immediate from Proposition 4.13.   □

Corollary 4.15. *Let $f \in \mathbb{C}\{x, y\}$ be a germ with non-degenerate Newton boundary. Then every Puiseux series of f with respect to y is either in $\mathbb{C}\{x\}$ or does not have a rational part.*

Proof. If a branch has y-degree ≥ 2, then it has at least two conjugate Puiseux series, which have the same rational part. By Proposition 4.13, the rational part has to vanish.   □

Lemma 4.16. *Assume that $g \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ has finite Milnor number and maximal Newton number in its right-equivalence class.*

(1) $\mathrm{jet}(g, \Delta)$ is divisible by y^2 in case $w(y) \geq w(x)$, and
(2) $\mathrm{jet}(g, \Delta)$ is divisible by x^2 in case $w(x) \geq w(y)$.

Proof.   (1) Suppose $\mathrm{jet}(g, \Delta)$ is not divisible by y^2. One then can apply a right-equivalence to the germ to make the jet divisible by y^2 by mapping a linear homogeneous factor of the jet with exponent ≥ 2 to y. If the resulting Newton polygon is not convenient, one can add the monomial $y^{\mu(g)+2}$ to make the germ convenient. We thus obtain a germ $\tilde{g}$ which is right-equivalent to g and has $N(\tilde{g}) > N(g)$, contradicting our assumption on g.
   (2) Replace y by x in (1).

□

The key result is then the following proposition. We provide a proof, which only relies on an analysis of the Puiseux expansions of the germs. For the convenience of the reader we give all details of the proof.

Proposition 4.17. *Suppose that for $f, g \in \mathfrak{m}^2 \subset \mathbb{C}\{x, y\}$ there is bijection between the irreducible branches of f and g such that*

- *the corresponding characteristic exponents are the same, and*
- *the intersection numbers between corresponding branches coincide,*

and suppose that f is equivalent to a germ with non-degenerate Newton boundary, and g has finite Milnor number.

Then any germ in the right-equivalence class of g with maximal Newton number has a non-degenerate Newton boundary. In particular, g is equivalent to a germ with non-degenerate Newton boundary.

Proof. Write $\tilde{f}$ for the germ with non-degenerate Newton boundary which is equivalent to f. Assume that g is not equivalent to a germ with non-degenerate Newton boundary. Choose in the right-equivalence class of g a convenient germ $\tilde{g}$ such that $N(\tilde{g})$ is maximal.

- If there is a facet Δ of $\Gamma(\tilde{g})$ with degenerate jet such that the jet has an irreducible factor q which is not linear homogeneous (and thus all factors, except powers of x and y, are not linear homogeneous), then the jet factorizes as

$$\mathrm{jet}(\tilde{g}, \Delta) = q^a \cdot h,$$

  with $a \geq 2$, and $q \nmid h$.

  Note that, if γ is a Puiseux expansion of a branch of $\tilde{g}$ contributing to q^a, and ϕ is a Puiseux expansion of the corresponding branch of $\tilde{f}$, then $\nu(\gamma) = \nu(\phi)$: Since q is non-linear, the orbit length of $\mathrm{jet}(\gamma, \nu(\gamma))$ is at least two, hence, by Remark 4.5(3), also the orbit length of $\mathrm{jet}(\phi, \nu(\gamma))$ is at least two, so $\nu(\phi) \leq \nu(\gamma)$. On the other hand, since the branch of $\tilde{g}$, and hence of $\tilde{f}$, is non-smooth, Corollary 4.15 implies that ϕ does not have a rational part. So $\mathrm{jet}(\phi, \nu(\phi))$, and hence $\mathrm{jet}(\gamma, \nu(\phi))$, has orbit length at least two, which implies that $\nu(\gamma) \leq \nu(\phi)$, and thus proves our claim.

  Note also, that there are exactly a pairwise different branches of $\tilde{g}$ contributing to q^a: Suppose one branch of y-degree a' contributes a higher order factor q^{a_1}. By Corollary 4.14, the r-jets of the Puiseux expansions of this branch of $\tilde{f}$ form an orbit of length a'. By Remark 4.5(3), the corresponding branch of $\tilde{g}$ has the same orbit lengths of the jets of the Puiseux expansions. Since this orbit length is equal to $\deg_y(q)$, we have $\deg_y(q) = a' = a_1 \cdot \deg_y(q)$, a contradiction.

  Let $\tilde{g}_1$ and $\tilde{g}_2$ be two different branches of $\tilde{g}$ contributing to q^a, and let $\tilde{f}_1$ and $\tilde{f}_2$ be the corresponding branches of $\tilde{f}$. Denoting by γ_2 and ϕ_2 Puiseux expansions of $\tilde{g}_2$ and $\tilde{f}_2$, respectively, we conclude that

$$i(\tilde{g}_1, \tilde{g}_2) > \deg_y(\tilde{g}_1) \cdot \deg_y(\tilde{g}_2) \cdot \nu(\gamma_2) = \deg_y(\tilde{f}_1) \cdot \deg_y(\tilde{f}_2) \cdot \nu(\phi_2) = i(\tilde{f}_1, \tilde{f}_2),$$

  a contradiction.[13]

- Now suppose that there is a facet Δ of $\Gamma(\tilde{g})$ with degenerate jet (after saturation) which factorizes completely into linear homogeneous factors. Write $w = w(\Delta)$. Without loss of generality, we may assume that $w(y) \geq w(x)$. By Lemma 4.16, $\mathrm{jet}(\tilde{g}, \Delta)$ is then divisible by y^2, so it factorizes as

$$\mathrm{jet}(\tilde{g}, \Delta) = l^a \cdot y^b \cdot h,$$

  with $l \neq x, y$ linear homogeneous, $a, b \geq 2$, and $l, y \nmid h$. Consider the factorization of $\tilde{g}$ according to Theorem 4.8. Using that $\mathrm{jet}(\tilde{g}, \Delta)$ is degenerate after saturation, it follows that $\tilde{g}$ has at least one branch $\tilde{g}_1$ which divides the Δ-factor of $\tilde{g}$ and contributes to l^a, and another branch $\tilde{g}_2$ which divides a factor corresponding to a facet of the Newton polygon with larger slope than Δ and contributes to y^b. Let $\tilde{f}_1$ and $\tilde{f}_2$ be the corresponding branches of $\tilde{f}$, according to the given bijection.

  Note that the Puiseux expansions of $\tilde{g}_2$ have a higher vanishing order than that of $\tilde{g}_1$. This implies that $i(\tilde{g}_1, \tilde{g}_2) = \deg_y(\tilde{g}_1) \cdot \deg_y(\tilde{g}_2) \cdot \nu(\gamma_1)$ where γ_1 is a Puiseux expansion of $\tilde{g}_1$. Note also that the Puiseux expansions of $\tilde{g}_1$ have a non-vanishing rational part. Since $\tilde{g}_i$ and $\tilde{f}_i$ have the same characteristic exponents, $\tilde{g}_i$ is smooth if and only if $\tilde{f}_i$ is smooth ($i = 1, 2$). By Remark 4.3(3) and the correspondence of characteristic exponents, it follows that any branch dividing the Δ-factor of $\tilde{g}$ has the same y-degree as the corresponding branch of $\tilde{f}$.

  Without loss of generality, we may assume that $\tilde{g}_1$ is chosen as follows: If, among the branches contributing to the factor l^a, there is a non-smooth branch, we choose $\tilde{g}_1$ as non-smooth. If all the contributing branches are smooth, we choose $\tilde{g}_1$ such that the Puiseux

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[13]The inequality $i(\tilde{g}_1, \tilde{g}_2) > \deg_y(\tilde{g}_1) \cdot \deg_y(\tilde{g}_2) \cdot \nu(\gamma_2)$ follows from the fact that the expansions of $\tilde{g}_1$ and $\tilde{g}_2$ have the same vanishing order and same lowest order term. Note that the role of $\tilde{g}_1$ and $\tilde{g}_2$ can be exchanged in the formula. The same is true for the formula $i(\tilde{f}_1, \tilde{f}_2) = \deg_y(\tilde{f}_1) \cdot \deg_y(\tilde{f}_2) \cdot \nu(\phi_2)$, where equality holds since the expansions of $\tilde{f}_1$ and $\tilde{f}_2$ have same order, but different lowest order terms.

expansions of $\tilde{f}_1$ have maximal vanishing order (among those branches of $\tilde{f}$ which correspond to branches of $\tilde{g}$ contributing to l^a). We may also assume that $\tilde{g}_2$ is chosen, among all branches contributing to the factor y^b, such that it is non-smooth if a non-smooth branch exists, and otherwise such that its Puiseux expansions have minimal vanishing order.

If ϕ_1 is a Puiseux expansion of $\tilde{f}_1$, then

$$\nu(\phi_1) > \nu(\gamma_1),$$

as we observe by considering the following two cases: If $\tilde{g}_1$ is not smooth, then Corollary 4.15 implies that the rational part of the Puiseux expansions of $\tilde{f}_1$ vanishes. Since $\tilde{g}_1$ and $\tilde{f}_1$ have the same characteristic exponents and the rational part of $\tilde{g}_1$ is non-zero, we conclude that $\nu(\phi_1) > \nu(\gamma_1)$. If $\tilde{g}_1$ is a smooth branch, then there is at least one other smooth branch $\tilde{g}_1'$ contributing to l^a. Denote the corresponding branch of $\tilde{f}$ by $\tilde{f}_1'$. Let γ_1' and ϕ_1' be Puiseux expansions of $\tilde{g}_1'$ and $\tilde{f}_1'$, respectively. So $\nu(\gamma_1) = \nu(\gamma_1')$, and, by choice of $\tilde{g}_1$, $\nu(\phi_1) \geq \nu(\phi_1')$. If we would have $\nu(\phi_1) \leq \nu(\gamma_1)$, then

$$i(\tilde{g}_1, \tilde{g}_1') > \nu(\gamma_1) \geq \nu(\phi_1) \geq \nu(\phi_1') = i(\tilde{f}_1, \tilde{f}_1'),$$

using Proposition 4.13 in the last equality. This contradicts our assumption. Hence, again, we can conclude that $\nu(\phi_1) > \nu(\gamma_1)$.

Let now γ_2 and ϕ_2 be Puiseux expansions of $\tilde{g}_2$ and $\tilde{f}_2$, respectively. We observe that

$$\nu(\phi_2) \geq \nu(\gamma_2)$$

by considering the following two cases: If $\tilde{g}_2$ is not smooth, then (just as above) Corollary 4.15 implies that the rational part of the Puiseux expansions of $\tilde{f}_2$ vanishes. Since $\tilde{g}_2$ and $\tilde{f}_2$ have the same characteristic exponents, we conclude that $\nu(\phi_2) \geq \nu(\gamma_2)$. If $\tilde{g}_2$ is a smooth branch, then there is at least one other smooth branch $\tilde{g}_2'$ contributing to y^b. Denote the corresponding branch of $\tilde{f}$ by $\tilde{f}_2'$. Let γ_2' and ϕ_2' be Puiseux expansions of $\tilde{g}_2'$ and $\tilde{f}_2'$, respectively. If $\nu(\phi_2) < \nu(\gamma_2)$, then

$$i(\tilde{g}_2, \tilde{g}_2') \geq \nu(\gamma_2) > \nu(\phi_2) \geq i(\tilde{f}_2, \tilde{f}_2'),$$

using Proposition 4.13 for the last inequality. We thus obtain a contradiction.

If $\nu(\phi_2) > \nu(\phi_1)$, then $i(\tilde{f}_1, \tilde{f}_2) = \deg_y(\tilde{f}_1) \cdot \deg_y(\tilde{f}_2) \cdot \nu(\phi_1)$. The same is true in case $\nu(\phi_2) = \nu(\phi_1)$, since, by Proposition 4.13, for a germ with non-degenerate Newton boundary no two Puiseux expansions agree in lowest order. So we get

$$i(\tilde{f}_1, \tilde{f}_2) = \deg_y(\tilde{f}_1) \cdot \deg_y(\tilde{f}_2) \cdot \nu(\phi_1) > \deg_y(\tilde{g}_1) \cdot \deg_y(\tilde{g}_2) \cdot \nu(\gamma_1) = i(\tilde{g}_1, \tilde{g}_2),$$

which contradicts our assumption, hence, we conclude that $\tilde{g}$ is non-degenerate. On the other hand, if $\nu(\phi_1) > \nu(\phi_2)$, then

$$i(\tilde{f}_1, \tilde{f}_2) = \deg_y(\tilde{f}_1) \cdot \deg_y(\tilde{f}_2) \cdot \nu(\phi_2) \geq \deg_y(\tilde{g}_1) \cdot \deg_y(\tilde{g}_2) \cdot \nu(\gamma_2)$$
$$> \deg_y(\tilde{g}_1) \cdot \deg_y(\tilde{g}_2) \cdot \nu(\gamma_1) = i(\tilde{g}_1, \tilde{g}_2).$$

This again contradicts our assumption, and we conclude that $\tilde{g}$ is non-degenerate.

$\square$

Corollary 4.18. *Under the same assumptions as in Proposition 4.17, suppose that f is equivalent to the germ $\tilde{f}$ with a non-degenerate Newton boundary. Then g is equivalent to a germ with non-degenerate Newton boundary which has the same Newton polygon as $\tilde{f}$.*

Proof. By Proposition 4.17, g is equivalent to a germ with non-degenerate Newton boundary. Then the claim follows directly from Proposition 4.11: If the facet with factor g_1 is smooth, then the slope of $\Gamma(g_1)$ can be changed by a right-equivalence (which does not change any terms on any facet of the Newton polygon of f except that with factor g_1) to any rational number $-\frac{1}{a}$ which is smaller than the slope of $\Gamma(g_2)$, see Remark 4.10. If the facet containing g_s is smooth, we can argue analogously. $\square$

5. A Classification Algorithm for Isolated Singularities with a non-degenerate Newton Boundary

On the algorithmic side, our aim is to find for a given input polynomial in $\mathbb{Q}[x, y]$, that over $\mathbb{C}$ is right-equivalent to a germ with a non-degenerate Newton boundary, an explicit right-equivalence to a normal form equation as in Theorem 3.20. The first step in the process is to transform the input polynomial to a germ with a non-degenerate Newton boundary. In the present section, we describe an algorithm for this purpose. Our algorithm is based on the following observations (the first two being obvious and the third implied by Remark 3.17):

Remark 5.1. A (weighted) homogeneous polynomial $f \in \mathbb{C}[x, y]$ is degenerate (that is, its Milnor number is infinite, equivalently, it has a non-isolated singularity at zero) if and only if one of its irreducible factors has multiplicity larger than one.

Remark 5.2. A germ f has a non-degenerate Newton boundary if the Milnor number of the saturation of $\mathrm{jet}(f, \Delta)$ for each facet Δ of the Newton polygon of f is finite.

Remark 5.3. Suppose f is in the right-equivalence class of a germ with non-degenerate Newton boundary. Then f has a non-degenerate Newton boundary if and only if it has maximal Newton number (within the right-equivalence class).

The basic idea of the classification algorithm in this section is now as follows: Let g be a weighted homogeneous polynomial (think of the jet of the input polynomial with regard to a facet of its Newton polygon). If g has no factor of multiplicity larger than 1, then g is non-degenerate. The number, degree or multiplicity of factors of g is invariant under automorphisms. Hence, if g has more than two different linear homogeneous factors of multiplicity larger than 1 or if it has a non-linear homogeneous factor of multiplicity larger than 1, it cannot be mapped by a right-equivalence to a polynomial with a non-degenerate saturation.

We first consider the case that g is homogeneous with respect to the standard grading. If g has exactly one linear homogeneous factor with multiplicity bigger than 1, we can map this factor to x. If g has exactly two linear homogeneous factors with multiplicity bigger than 1, we can map these factors to x and y, respectively. In both cases, after the linear transformation, the polynomial is non-degenerate after saturation. We thus obtain Algorithm 2.

Algorithm 2 Transform the standard homogeneous d-jet of a polynomial, where d is the maximal filtration of the polynomial, to a polynomial with a non-degenerate saturation.

Input: A polynomial $f \in \mathfrak{m}^3$, $f \in \mathbb{Q}[x, y]$ with maximal filtration d.
Output: A polynomial $h \in \mathbb{K}[x, y]$ defined over an extension field $\mathbb{Q} \subset \mathbb{K} \subset \overline{\mathbb{K}}$ such that h is
    right-equivalent to f, and $\mathrm{sat}(\mathrm{jet}(f, d))$ is a non-degenerate polynomial (possibly term), if
    such an h exists, and false otherwise.
1: $g := \mathrm{jet}(f, d)$.
2: Factorize $g = c \cdot g_1^{l_1} \cdot \ldots \cdot g_n^{l_n}$, where $l_1 \geq l_2 \geq \cdots \geq l_n$, $c \in \mathbb{Q}$, $g_1 \in \mathbb{K}[x, y]$ with $\mathbb{K} = \mathbb{Q}$ in
    case $n = 1$, and $g_1, \ldots, g_n \in \overline{\mathbb{K}}[x, y]$ linear homogeneous and coprime, $g_1, g_2 \in \mathbb{K}[x, y]$ with
    $[\mathbb{K} : \mathbb{Q}]$ minimal (among all admissible choices of g_1, g_2), in case $n \geq 2$.
3: **if** $n = 1$ **then**
4:     Apply a linear automorphism to f sending $g_1 \mapsto x$.
5: **else**
6:     Apply $g_1 \mapsto x$, $g_2 \mapsto y$ to f.
7: **if** the saturation of $\mathrm{jet}(f, d)$ is degenerate **then**
8:     **return false**
9: **return** f

Proof. Correctness of the algorithm follows directly from Remark 5.1 and the above discussion.
$\square$

Remark 5.4. To match the actual implementation, the input polynomial in Algorithm 2 is assumed to be defined over $\mathbb{Q}$. Note that the transformations that are applied in lines 4 and 6 of Algorithm 2 may require to pass to an algebraic extension of the base field.

Algorithm 3 Transform a non-standard weighted homogeneous d-jet of a polynomial, where d is the maximum weighted filtration of the polynomial, to a polynomial with a non-degenerate saturation.

Input: A polynomial $f \in \mathfrak{m}^3$, $f \in \mathbb{K}[x,y]$, where $\mathbb{K}$ is an extension field of $\mathbb{Q}$, and a weight $w = (w_1, w_2)$ with $w_1 \neq w_2$ the weight of one of the facets of the Newton boundary of f.

Output: A polynomial $h \in \mathbb{L}[x,y]$ defined over an extension field $\mathbb{K} \subset \mathbb{L} \subset \overline{\mathbb{K}}$, such that h is right-equivalent to f, where $\mathrm{jet}(h,w)$ is a polynomial (possibly term) with a non-degenerate saturation, if such an h exists, and false otherwise.

1: **if** $w_1 < w_2$ **then**
2:     $g := \mathrm{sat}(\mathrm{jet}(f,w), x)$.
3: **if** $w_1 > w_2$ **then**
4:     $g := \mathrm{sat}(\mathrm{jet}(f,w), y)$.
5: Factorize $g = c g_1^{l_1} \cdots g_n^{l_n} \widetilde{g}$, where $c \in \mathbb{K}$, $l_1 \geq l_2 \geq \cdots \geq l_n$ and $g_1 \ldots, g_n \in \overline{\mathbb{K}}[x,y]$ weighted linear homogeneous and coprime with n maximal, $\widetilde{g} \in \mathbb{K}[x,y]$ a product of non-associated irreducible polynomials in $\overline{\mathbb{K}}[x,y]$, and $g_1 \in \mathbb{L}[x,y]$ with $[\mathbb{L} : \mathbb{K}]$ minimal.
6: **if** $l_2 > 1$ **then**
7:     **return** false
8: **if** $\widetilde{g}$ is degenerate **then**
9:     **return** false
10: **if** $w_1 < w_2$ **and** $n \geq 1$ **then**
11:     Apply right equivalence to f which sends $g_1 \mapsto y, x \mapsto x$.
12: **if** $w_1 > w_2$ **and** $n \geq 1$ **then**
13:     Apply right equivalence to f which sends $g_1 \mapsto x, y \mapsto y$.
14: **return** f

A similar approach can be used to decide whether a non-standard weighted homogeneous polynomial can be transformed by a weighted linear homogeneous transformation to a polynomial with non-degenerate saturation. Again, if g has exactly one linear homogeneous factor with multiplicity bigger than 1, we can map this factor to x or y, in case $w_1 > w_2$ or $w_2 > w_1$, respectively. After this weighted linear transformation, the polynomial is non-degenerate after saturation. If g has two (or more) linear homogeneous factors with multiplicity bigger than one, it cannot be mapped by a right-equivalence to a polynomial with a non-degenerate saturation. The approach is summarized in Algorithm 3.

Proof. Correctness of the algorithm follows directly from Remark 5.1 and the above discussion. $\square$

Remark 5.5. In Algorithm 3, the transformations in line 11 to 13 are defined over the ground field $\mathbb{K}$, except if all homogeneous linear factors appear with exponent one: Consider a factorization

$$g = c \cdot g_1^{l_1} \cdot \ldots \cdot g_n^{l_n} \cdot \widetilde{g},$$

where $c \in \mathbb{K}$, $l_1 \geq l_2 \geq \ldots \geq l_n$ and $g_1, \ldots, g_n \in \overline{\mathbb{K}}[x,y]$ are linear homogeneous and coprime, with n maximal, and $\widetilde{g} \in \overline{\mathbb{K}}[x,y]$. If $l_1 > 1$ and g_1 has a conjugate, then g cannot be transformed to a germ with non-degenerate Newton boundary. Otherwise g_1 can be chosen with coefficients in $\mathbb{K}$, which implies that the corresponding weighted linear transformation in the algorithm is defined over $\mathbb{K}$.

Remark 5.6. Note that the if-clauses in line 10 and 12 of Algorithm 3 ensure that the transformations in lines 11 and 13 are of w-degree 0, and hence do not create terms below the Newton polygon.

Using Algorithms 2 and 3, we now state Algorithm 4, which, for a germ which is equivalent to a germ with non-degenerate Newton boundary, determines a right-equivalent germ with normalized non-degenerate Newton boundary.

Remark 5.7. More generally, given a germ $f \in \mathbb{Q}[x_1 \ldots, x_n]$ of corank ≤ 2 with finite Milnor number, we can obtain a germ which has a non-degenerate Newton boundary and is stable equivalent to f by first applying the Splitting Lemma to f. In case $\mathrm{corank}(f) = 2$, we continue with Algorithm 4 applied to the degenerate part of f. If $\mathrm{corank}(f) = 1$, after applying the splitting lemma, we can just return $\mathrm{jet}(f, d)$, where d is the maximal filtration of f with respect to the standard grading. Note that then the Newton polygon is given by the vertex $\mu(f) + 1$. If $\mathrm{corank}(f) = 0$, that is, f has an A_1-singularity, we just return 0.

Remark 5.8. Using the notation of Remark 5.5, the case $l_1 = 1$ can only occur in the last call of Algorithm 3 from Algorithm 4 with regard to facets with $w_2 > w_1$, and similarly in the last call with regard to facets with $w_1 > w_2$. Hence, the ordering in which Algorithm 4 considers the facets of the Newton polygon can be chosen in a way that these (one or two) calls, and hence towers of field extensions, occur only in the last steps of the algorithm. This improves the performance and simplifies the implementation.

Remark 5.9. Composition of all right-equivalences applied in the course of Algorithm 4 yields an explicit polynomial right-equivalence from the input germ to a germ with normalized non-degenerate Newton boundary. Addition of the monomials x^a and y^a in lines 24 and 25, can also be realized by a polynomial right-equivalence (creating higher order terms). Note that, after each right-equivalence in the algorithm, we may truncate above the determinacy. These truncations amount to non-polynomial right-equivalences and thus are not explicitly known. However, a lower bound (depended on the truncation order) for the filtration of the truncation transformations can be specified.

Proof of Algorithm 4. We show that the algorithm will either transform f in finitely many steps to a germ with a non-degenerate Newton boundary which is normalized according to the conditions in Definition 3.24, or, if the germ is not right-equivalent to a germ with a non-degenerate Newton boundary, will return false. We write T for the type associated to this Newton boundary, which we will denote by $\Gamma(T)$.

It is sufficient to prove that if f can be transformed to a germ with these properties, the algorithm finds such a transformation in finitely many steps. So assume that there is a right-equivalence ψ such that $\tilde{f} = \psi(f)$ satisfies the above properties. We will iteratively change ψ by composing it with suitable right-equivalences until $\psi_0^{w(T)_i} = \mathrm{id}$ for all i. We will do this by considering jets of f with regard to a suitable sequence w_j of weights. In the i-th iteration we find a right-equivalence ϕ such that the $w_j\text{-}\mathrm{jet}(\phi(f), d(f)) = w_j\text{-}\mathrm{jet}(\psi(f), d(f))$ for all $j \leq i$. Note that, before the i-th iteration, w_i will be an entry of the current $w(f)$, and any entry of $w(T)$ will occur among the w_j's. We then replace f by $\phi(f)$ and ψ by $\psi \circ \phi^{-1}$ and iterate (denoting in every iteration, for simplicity of notation, the current germ by f, and the current transformation to $\tilde{f}$ by ψ).

We start out by considering $\mathrm{jet}(f, d)$, where d is the maximal filtration of f with respect to the standard grading. Let

$$\mathrm{jet}(f, d) = c \cdot g_1^{l_1} \cdot \ldots \cdot g_n^{l_n}$$

be the factorization of this jet as in Algorithm 2, where $l_1 \geq l_2 \geq \ldots \geq l_n$, $g_1, \ldots, g_n$ are linear homogeneous. There are the following cases:

- If more than two of the l_i are larger than 1, then f cannot be transformed to a germ with non-degenerate Newton boundary.

Algorithm 4 Transform a germ to a germ with a non-degenerate, convenient Newton boundary.

Input: A polynomial germ $f \in \mathbb{Q}[x, y]$, $f \in \mathfrak{m}^3$, of corank 2 with finite Milnor number.

Output: A polynomial g with a non-degenerate Newton boundary which is right-equivalent to f, if such g exists, and **false** otherwise. Moreover, g is convenient and satisfies the normalization condition in Definition 3.24.

1: $d :=$ maximal filtration of f w.r.t. the standard grading.
2: **if** $\mu(\mathrm{sat}(\mathrm{jet}(f, d))) = \infty$ or the normalization condition is not satisfied **then**
3:      apply Algorithm 2 to f.
4: **if** $f = $ **false then**
5:      **return false** (the germ has a degenerate Newton boundary)
6: $S_0 :=$ the set of monomials of $\mathrm{jet}(f, d)$ that lie on the vertices of $\Gamma(f)$.
7: $S_1 := \emptyset$
8: **while true do**
9:      **if** $\Gamma(f)$ does not intersect the x-axis **then** $f = f + x^a$, where $a = \mu(f) + 2$.
10:      **if** $\Gamma(f)$ does not intersect the y-axis **then** $f = f + y^a$, where $a := \mu(f) + 2$.
11:      Let $\Delta_1, \Delta_2, \ldots, \Delta_v$ be the facets of $\Gamma(f)$ ordered by increasing slope.
12:      **if** $\mu(\mathrm{sat}(\mathrm{jet}(f, \Delta_i))) < \infty \ \forall i = 1, \ldots, v$ and the normalization condition is satisfied **then**
13:          **return** $(f, \Gamma(f))$
14:      **else**
15:          Let m be an element of S_0.
16:          Let $\delta_1, \ldots, \delta_r$ $(r \leq 2)$ be the facets of $\Gamma(f)$, ordered by increasing slope, adjacent to m.
17:          **for** i **from** 1 **to** r **do**
18:              $h := \mathrm{jet}(f, \delta_i)$, $w :=$ the weight defined by h.
19:              **while** $\mu(\mathrm{sat}(h)) = \infty$ or the normalization condition is not satisfied **do**
20:                  $f :=$ output of Algorithm 3 applied to f and w.
21:                  **if** $f = $ **false then**
22:                      **return false** (the germ has a degenerate Newton boundary)
23:                  Let $\delta_1, \ldots, \delta_r$, ordered by increasing slope, adjacent to m.
24:                  $h := \mathrm{jet}(f, \delta_i)$, $w :=$ the weight defined by h.
25:          Let $\eta_1, \ldots, \eta_r$ $(r \leq 2)$ be the facets of $\Gamma(f)$, ordered by increasing slope, containing m.
26:          $S_1 := S_1 \cup \{m\}$
27:          $f_1 := \mathrm{jet}(f, \eta_1 \cap \ldots \cap \eta_r))$.
28:          $S_0 :=$ monomials in $(S_0 \cup \mathrm{supp}(f_1)) \setminus S_1$ lying on the vertices of $\Gamma(f)$.
29: **if** $\mathrm{sat}(\mathrm{jet}(f, \Delta_1))$ is smooth **then**
30:      **for** $d \leq l \leq \mu + 2$ **do**
31:          **if** Δ_1 cuts the x-axis at x^l **then**
32:              Write $\mathrm{jet}(f, \Delta_1)$ as $\mathrm{jet}(f, \Delta_1) = c_1 x^l + c_2 x^s y$
33:              Apply $y \mapsto y - \frac{c_1}{c_2} x^{l-s}$, $x \mapsto x$
34:              Recalculate Δ_1
35: **if** Δ_1 does not cut the x-axis at x^l, where $l = \mu + 2$ **then** $f = f + x^l$
36: **if** $\mathrm{sat}(\mathrm{jet}(f, \Delta_v))$ is smooth **then**
37:      **for** $d \leq l \leq \mu + 2$ **do**
38:          **if** Δ_v cuts the y-axis at x^l **then**
39:              Write $\mathrm{jet}(f, \Delta_v)$ as $\mathrm{jet}(f, \Delta_v) = c_1 y^l + c_2 y^s x$
40:              Apply $y \mapsto x - \frac{c_1}{c_2} y^{l-s}$, $y \mapsto y$
41:              Recalculate Δ_v
42: **if** Δ_v does not cut the y-axis at y^l, where $l = \mu + 2$ **then** $f = f + y^l$
43: **return** f

- Suppose $l_1 \geq l_2 > 1$, and the remaining $l_i = 1$ then, up to permutation and scaling of the variables, the only way to obtain a non-degenerate jet is to map $g_1 \mapsto x$ and $g_2 \mapsto y$.
- Suppose that $l_1 = \ldots = l_n = 1$. Note that this is the case of an ordinary n-fold point. If $n = 1$ then f is a smooth germ. If $n \geq 2$, the lowest jet $\mathrm{jet}(f,d)$ is already non-degenerate. Algorithm 4 then maps $g_i \mapsto x$ and $g_j \mapsto y$ for some choice of $i \neq j$, adds terms to create smooth facets towards the coordinate axes to satisfy our normalization condition, if necessary, and then finishes without any further iteration of the main loop. Hence our choice of i and j, and of the normalization, will not influence subsequent iterations of the algorithm. The choice of i and j also does not influence the face of the Newton polygon corresponding to $\mathrm{jet}(f,d)$ after the transformation (and any further facets have a smooth saturation and cut one of the coordinate axis, so are unique according to our normalization condition). Note that, after the transformation, $\mathrm{jet}(f,d)$ is a term if and only if $n = 2$.
- Lastly, suppose that $l_1 > 1$ and $l_2 = \ldots = l_n = 1$. To obtain a non-degenerate lowest jet, we have to map g_1 to a non-zero multiple of a variable. For what follows, we fix the choice $g_1 \mapsto x$. This determines one vertex of the face corresponding to $\mathrm{jet}(f,d)$ and makes the saturation of the face non-degenerate (but not necessarily normalized). If $n = 1$, the algorithm maps $y \mapsto y$. If $n \geq 2$, in order to satisfy our normalization condition, the algorithm chooses an $i \neq 1$, maps $g_i \mapsto y$ and, if necessary, adds a smooth facet towards the coordinate axis.

  We will show that if for one of these choices of transformation $\rho_0 : g_1 \mapsto x, g_i \mapsto y$, the algorithm determines in subsequent steps that there is no transformation to a germ with a non-degenerate Newton boundary, then this is also the case for all other choices.

  For this we prove in the case $n \geq 2$, that if g is *any* linear divisor of $\mathrm{jet}(f,d)$ with $g \neq g_1 = x$, then there is a right-equivalence ρ with $\rho_0(g) = y$ such that $\rho(f)$ is a germ with non-degenerate Newton boundary: By assumption, for $g_0 := (\psi^{-1})_0(y)$, we have $\psi_0(g_0) = y$ and $\psi(f)$ has a non-degenerate and normalized Newton boundary. Let $w(\psi(f)) = (w_1, \ldots, w_s)$ be the associated piecewise weight, and let $\Delta_1, \ldots, \Delta_s$ be the corresponding facets of $\Gamma(\psi(f))$. Note that $\mathrm{jet}(\psi(f), \Delta_s)$ has smooth saturation since $\mathrm{jet}(\psi(f), d)$ has a factor y, but none of the other factors is divisible by y, that is, $\mathrm{jet}(\psi(f), d)$ has a monomial of the form yx^l, $l > 1$. If g is any linear divisor of $\mathrm{jet}(f,d)$ with $g \neq x$, and λ is the linear transformation $\lambda = (x \mapsto x, g \mapsto y) \circ (x \mapsto x, y \mapsto g_0)$, then $\rho = \lambda \circ \psi$ maps g to y, and $\mathrm{jet}(\rho(f), \Delta_i)$ and $\mathrm{jet}(\psi(f), \Delta_i)$ agree for $i = 1, \ldots, s-2$ up to scaling of y. To see this, note that, since for all the weights $w_1, \ldots, w_s$ we have $w_i\text{-}\deg(y) \geq w_i\text{-}\deg(x)$, a linear transformation will only create terms of $(w_1, \ldots, w_{s-1})$-weight bigger than or equal to $d(\psi(f))$. Moreover, since we choose g to be a linear factor of $\mathrm{jet}(f,d) = \mathrm{jet}(f, \Delta_{s-1})$, the support of the d-jet does not change under λ. The facet Δ_s with smooth saturation may still change. If this facet is eliminated, we modify ρ by a right-equivalence of higher filtration to make the Newton polygon convenient again, thus satisfying the normalization condition. In the case $n = 1$, we have to prove that if g is *any* linear form which is transversal to $l_1 = x$ then there is a right-equivalence ρ with $\rho_0(g) = y$ such that $\rho(f)$ is a germ with normalized, non-degenerate Newton boundary. For this we can use a similar argument as above with the only difference that there is no smooth facet towards the x-axis, and $\mathrm{jet}(\rho(f), \Delta_i)$ and $\mathrm{jet}(\psi(f), \Delta_i)$ agree for $i = 1, \ldots, s$ up to scaling of y.

In any of the above cases (except of course the first one, where the algorithm finishes with `false`), we can now assume without loss of generality that the algorithm has chosen the linear transformation as ψ_0^{-1}, that is, we may assume that ψ_0 is the identity. In particular, we can assume that $\mathrm{jet}(f,d)$ has non-degenerate saturation and (if it is not just a term) satisfies the normalization condition.

If f has a normalized non-degenerate Newton boundary we are finished. If the Newton boundary is non-degenerate and all facets except the smooth facets are normalized, we can normalize these facets and are also finished. So suppose this is not the case. Starting with the face corresponding to the lowest standard graded jet, we find $\Gamma(T)$ by inductively determining a facet of the Newton polygon adjacent to the faces already coinciding with those of $\Gamma(T)$. In every iteration, if f does not have terms on both coordinate axes, we add terms as described in lines 24

and 25 of Algorithm 4 such that $\Gamma(f)$ does not have infinite faces. In the following, suppose m is a vertex monomial of $\Gamma(T)$ and δ is a facet of $\Gamma(f)$ adjacent to m which has not been considered so far in the sense of verifying whether δ is a facet of $\Gamma(T)$.[14] As in Algorithm 3, write

$$g := \mathrm{jet}(f, \delta) = c g_1^{l_1} \cdot \ldots \cdot g_n^{l_n} \widetilde{g}$$

where $l_1 \geq l_2 \geq \ldots \geq l_n$ and $g_1, \ldots, g_n$ are (weighted) linear homogeneous and coprime, and n is maximal. Note that if $n = 0$, then g is non-degenerate and satisfies the normalization condition.

The next step is to find a transformation ϕ such that $\widetilde{w}\text{-}\mathrm{jet}(\phi(f), k) = \widetilde{w}\text{-}\mathrm{jet}(\psi(f), k)$, where $\widetilde{w} = (\widetilde{w}_1, \widetilde{w}_2)$ is a weight associated to δ and $k = \widetilde{w}\text{-}\deg(g)$. We have the following cases:

- If more than one of the l_i is bigger than 1, then (since $\widetilde{w}$ is not the standard weight) f cannot be transformed to a germ with a non-degenerate Newton boundary. In this case, Algorithm 3 returns `false` (see line 22).
- If g is non-degenerate and the normalization condition is satisfied we pass on to the next facet.
- Suppose now $l_1 > l_2 = 1$. Without loss of generality, we may assume that $\widetilde{w}_1 < \widetilde{w}_2$ and $g_1 = c_1 y + h$, where c_1 is a constant and the standard degree of h is bigger than 1. Algorithm 3 applies the transformation $\phi : g_1 \mapsto y$, $x \mapsto x$ to f, which maps g to a polynomial with a non-degenerate saturation which, in case it is not a term, satisfies the normalization condition. If g is transformed to a term m, a new adjacent facet of m appears with a new associated weight $\widetilde{w}$ and we repeat the process via the while-loop (line 19 to line 24). On the other hand, if g is not transformed to a term, the slope of g is preserved, and we pass to the next facet.
- Finally, suppose $l_1 = l_2 = \ldots = l_n = 1$. In this case, g is non-degenerate, but does not necessarily satisfy the normalization condition. Without loss of generality we may assume that $\widetilde{w}_1 > \widetilde{w}_2$. Then Algorithm 3 maps $g_i \mapsto x$ for some choice of i and $y \mapsto y$. Similar to the standard weighted case, we will show that, if for the choice $\rho_0^{\widetilde{w}} : g_i \mapsto x, y \mapsto y$ the algorithm determines in subsequent steps that there is no transformation to a germ with a non-degenerate Newton boundary, then this is also the case for all other choices of i. For this we prove that, if h is any $\widetilde{w}$-weighted linear divisor of g, then there is a right-equivalence ρ with $\rho_0^{\widetilde{w}}(h) = x$ such that $\rho(f)$ is a germ with a non-degenerate Newton boundary: By assumption, for $h := (\psi^{-1})_0^{\widetilde{w}}(x)$, we have $\psi_0^{\widetilde{w}}(h) = x$ and $\psi(f)$ has a non-degenerate Newton boundary. Let $w(\psi(f)) = (w_1, \ldots, w_s)$ be the associated piecewise weight, and let $\Delta_1, \ldots, \Delta_s$ be the corresponding facets of $\Gamma(\psi(f))$. Note that $\mathrm{jet}(\psi(f), \Delta_1)$ has a smooth saturation since $\psi(g)$ has a monomial of the form xy^l, $l > 1$. We can now use a similar argument as in the standard weighted case to prove that ρ transforms f to a germ with a non-degenerate normalized Newton boundary.

Note that a facet with a non-degenerate saturation which satisfies the normalization condition cannot be transformed by a weighted linear transformation with regard to the weight of the facet to a term or a different facet. Hence, in Algorithm 4, the vertex points of $\mathrm{jet}(f, \mathrm{span}(\delta_1, \ldots, \delta_r))$ lie on $\Gamma(T)$ after each iteration of the while-loop (line 8 to line 28). Lines 30 to 41 ensure that the facets which have smooth saturation and cut the coordinate axis satisfy the normalization condition.

We now prove that the algorithm terminates after finitely many iterations. For that we have to show that the while-loop in lines 8 to 28 and the while loop in lines 19 to 24 terminates. After a finite number of iterations of the loop in lines 19 to 24, the loop either terminates or the highest x-power or the highest y-power below $\Gamma(f)$ increases. If x^{m_1} and y^{m_2} are the highest powers of x and y below $\Gamma(f)$, then $1, x, \ldots, x^{m_1-1}, y, \ldots, y^{m_2-1}$ represent different equivalence classes in the local algebra. Hence, since $\mu(f)$ is finite, the process must stop after finitely many iterations. In each iteration of the while-loop in lines 8 to 28 the process may either stop (in case f is not equivalent to a germ with non-degenerate Newton boundary), or create at least one

---

[14]For book-keeping purposes in this iteration the algorithm uses two sets of monomials S_0 and S_1. The set S_0 contains all monomials that correspond to vertices of $\Gamma(T)$ and have at least one adjacent facet that is not, or has not yet been verified to be a facet of $\Gamma(T)$. The set S_1 consists of all monomials in S_0 for which we have verified that for its adjacent facets $\delta_1, \ldots, \delta_r$ ($r \leq 2$), $\mathrm{jet}(f, \mathrm{span}(\delta_1, \ldots, \delta_r))$ has non-degenerate saturation and is normalized (hence, is a facet of $\Gamma(T)$).

new non-degenerate facet of $\Gamma(f)$. Since the number of facets of a Newton polygon with a fixed minimum degree monomial is bounded, the algorithm terminates.

$\square$

Note that from the proof of Algorithm 4 we in particular obtain uniqueness of the Newton polygon determined by the algorithm. This follows also from Corollary 4.12 using the invariance of the charateristic exponents and intersection numbers under right-equivalence.

Corollary 5.10. *Let* $f \in \mathbb{C}[x, y]$ *be a germ which is right-equivalent to a germ with non-degenerate Newton boundary* Γ. *If* Γ *is assumed to satisfy the normalization condition, then* Γ *is uniquely determined by* f *(up to permutation of the variables).*

The above algorithms are implemented in the Singular library `arnold.lib`. We illustrate the implementation at the following example:

Example 5.11. We start out with a germ with non-degenerate Newton boundary and apply the automorphism ϕ. The result is a rather generic polynomial and has a degenerate Newton boundary. We then use our algorithm to transform this germ to a germ with non-degenerate Newton boundary. Note that our implementation is based (both for input and output) on the Singular data type `Poly` which (in contrast to `poly`) allows us to handle polynomials that are not in the current active ring. In particular, this allows us to handle extensions of the coefficient field in a transparent manner.

```
> LIB "Arnold.lib";
> ring R = 0,(x,y),ds;
> Poly g = 4*x^2*y^4+x^3*y^3+y^2*x^4+5*x^10+y^13;
> Poly phix = x+y^2+x^2+x*y+x^2*y+x*y^3;
> Poly phiy = y+y^2+2*x^2+x*y+y*x^2+y^2*x+x*y^4;
> RingHom phi;
> phi.source = R;
> phi.target = R;
> phi.images = list(phix,phiy);
> Poly f = phi(g);
> newtonPolygon(f);
Polygon with
Vertices: [0,6], [6,0]
Facets:   [[0,6], [6,0]]
Normals:  [1,1]
> def (F, P) = classifyNewtonBoundary(G);
```

This function call returns a germ F and a Newton polygon P, where F which is right-equivalent to f and has a non-degenerate normalized Newton boundary P. Here, P is represented by the data type `Polygon`, which stores the vertices, one dimensional facets and their normal vectors:

```
> F;
 4*x4y2+x3y3+x2y4+24*x5y2+30*x4y3+12*x3y4+6*x2y5+28*x6y2+86*x5y3+25*x4y4-4*x3y5-5*x2y6
-4*xy7-40*x7y2-104*x6y3-151*x5y4-223*x4y5-159*x3y6-78*x2y7-16*xy8-32*x9y-48*x8y2
-263*x7y3-277*x6y4+397*x5y5+396*x4y6+90*x3y7+18*x2y8+12*xy9+9*y10-128*x10y-312*x9y2
+226*x8y3+544*x7y4+2016*x6y5+3202*x5y6+554*x4y7-366*x3y8-6*x2y9+102*xy10+58*y11
+24*x11y-387*x10y2-195*x9y3-2150*x8y4-10*x7y5-2795*x6y6-5869*x5y7-8957*x4y8-3989*x3y9
-462*x2y10+544*xy11+129*y12+x13+248*x12y+1853*x11y2+2163*x10y3+3042*x9y4+10302*x8y5
-2426*x7y6-6092*x6y7+7112*x5y8+8012*x4y9+8600*x3y10+6128*x2y11+2834*xy12-368*y13
+77*x14-131*x13y-713*x12y2+833*x11y3-17096*x10y4-28769*x9y5-2038*x8y6-47695*x7y7
-14989*x6y8+33021*x5y9-5681*x4y10-6632*x3y11+7616*x2y12-164*xy13-3138*y14+193*x15
+1565*x14y+429*x13y2-7032*x12y3-98*x11y4-49796*x10y5+7939*x9y6+130879*x8y7
-59808*x7y8+48755*x6y9+39932*x5y10-99485*x4y11-13597*x3y12+28520*x2y13-21596*xy14
-5924*y15
> P;
Polygon with
```

```
vertices: [0,10], [2,4], [4,2], [13,0]
facets:   [[0,10], [2,4]], [[2,4], [4,2]], [[4,2], [13,0]]
normals:  [3,1], [1,1], [2,9]
```

6. Computing the Modality and a Regular Basis for a Germ with a non-degenerate Newton Boundary

In (Kouchnirenko, 1976, Proposition 7.2), a method to determine the (inner) modality of a germ with a non-degenerate Newton boundary has been described:

Theorem 6.1. *Let $f \in \mathbb{C}\{x,y\}$ be a convenient series with a non-degenerate principle part. Construct two half lines in $\mathbb{R}^2_+$ originating at the point $(2,2)$ parallel to the x-axis and y-axis. Let D be the polygon limited by a piece of the Newton polygon and the two half lines originating at the point $(2,2)$. The modality of f is the number of points of $\mathbb{N}^2$ inside and on the boundary of D.*

We summarize Kouchnirenko's approach in Algorithm 5 in form of an elementary implementation.[15]

Algorithm 5 Modality of a germ with a non-degenerate Newton boundary

---

Input: A polynomial germ $f \in \mathfrak{m}^2$ with a (convenient) non-degenerate Newton boundary.
Output: The modality of f.

1: Let h be the maximal standard degree of a monomial with exponent vector on $\Gamma(f)$.
2: $m := 0$.
3: **for** $2 \le i \le h$ **do**
4:     **for** $2 \le j \le h$ **do**
5:         **if** $w\text{-}\deg(x^i y^j) \le d(f)$ **then**
6:             $m := m + 1$.
7: **return** m

Algorithm 5 in combination with Algorithm 4 allows us to compute the modality of any germ that can be transformed to a germ with a non-degenerate Newton boundary. This approach is implemented in the command `modality` of the Singular library `arnold.lib`.

Example 6.2. Applying the Singular function to the polynomial F of Example 5.11 we obtain:
```
> modality(F);
6
```

In order to compute a normal form for a given germ with non-degenerate Newton boundary via Theorem 3.20, we have to determine a regular basis for the local algebra of f_0 as defined in the theorem. In Algorithm 6 we describe a method to do this (based on the ideal membership problem, which can be decided via Gröbner bases). Correctness and termination is obvious.

The computation of a regular basis of a germ with a non-degenerate Newton boundary is implemented in the function `regularBasis` of the library `arnold.lib`.

Example 6.3. Continuing Example 5.11, we determine a regular basis for F by computing a regular basis for f_0. Note that this is a particular choice of a regular basis of F. We also determine a the basis elements on and above the Newton boundary. Note that the size of the latter, by Theorem 3.20, agrees with the modality as computed in Example 6.2.
```
> vertexRegularBasis(F);
[ x3y5, x3y4, x4y3, x2y4, y10, x3y3, x12, y9, x11, x2y3, x3y2, y8, x10, y7, x9, x2y2, x8,
xy3, y6, x3y, x7, xy2, y5,  x2y, x6, y4, x5, xy, x4, y3, x3, y2, x2, y, x]

> vertexRegularBasisOnAndAboveNewtonPolygon(F);
[ x3y5, x3y4, x4y3, x2y4, y10, x3y3 ]
```

Algorithm 6 Monomial regular basis of the local algebra of a germ with a non-degenerate Newton boundary

---

Input: A polynomial germ $f \in \mathfrak{m}^2$ with a (convenient) non-degenerate Newton boundary, and an integer b.

Output: A monomial regular basis for the local algebra of f in $w(f)$-degree $\geq b$. Note that, for $b = 0$ the algorithm computes a regular basis, and for $b = d(f)$ a regular basis above and on the Newton boundary.

1: Let h be the highest w-weight of a monomial lying on a determinacy bound for f, where w is the piecewise weight associated to $\Gamma(f)$.
2: $J := \mathrm{Jac}(f)$
3: $B := \emptyset$
4: **for** d **from** h **to** b **by** -1 **do**
5:     Let L be the set of all monomials of piecewise degree d.
6:     **for** $x^i y^j$ **in** L **do**
7:         **if** $x^i y^j \notin J$ **then**
8:             $J := J + \langle x^i y^j \rangle$
9:             $B := B \cup \{x^i y^j\}$
10: **return** B

An algorithm to determine a normal form which in addition explicitly finds the exceptional hypersurface in the parameter space will be stated in the next section. We also discuss how to find the admissible values of the moduli parameters of the normal form family.

7. Computing a Normal Form for a germ with a non-degenerate Newton Boundary.

We first describe an algorithm to compute a normal form for a given input polynomial that can be transformed to a germ with a non-degenerate Newton boundary: We use Algorithm 4 to determine a polynomial with non-degenerate Newton boundary in the same right-equivalence class, we use Algorithm 6 to find a regular basis for f_0 on and above the Newton polygon, and then apply Theorem 3.20.

Remark 7.1. If $f \in \mathbb{C}[x,y]$ be a convenient weighted homogeneous polynomial with $w = w(f) = (a,b)$, then $f \in \mathbb{C}[x^b, y^a]$. Furthermore, defining $\phi : \mathbb{C}[x^b, y^a] \to \mathbb{C}[t_1, t_2]$, by $x^b \mapsto t_1$ and $y^a \mapsto t_2$, the polynomial $\phi(f)$ is standard homogeneous.

Proof. Since f is convenient, it contains monomials $x^{b'}$ and $y^{a'}$. Since $w(f)$ is in lowest terms, $b \cdot \gcd(a,b) = b'$ and $a \cdot \gcd(a,b) = a'$. Suppose now f has a mixed monomial $x^c y^d$, $c, d > 1$. Then, since f is weighted homogeneous of weight $ab' = a'b = \gcd(a', b')ab$, it follows that $ca + db = \gcd(a', b')ab$. Therefore $b|c$ and $a|d$, implying that $f \in \mathbb{C}[x^b, y^a]$. Furthermore, since $w(x^a) = w(y^b)$, it follows that $\phi(f)$ is standard homogeneous. $\square$

Algorithm 7 summarizes this approach. In the algorithm we also effectively determine the valid parameter values in the family.

Proof. Termination of the algorithm is clear. Correctness follows directly from the proof of Theorem 3.20 and Remark 7.1. Note that the discriminant can be computed in terms of a resultant. $\square$

Example 7.2. By applying Algorithm 7 to the polynomial f of Example 5.11, we determine a normal form for the μ-constant stratum of f.[16]

---

[15]Note that for large examples we can improve the performance using algorithms for computing lattice points of convex hulls. For this purpose, we rely on the SINGULAR library `gfanlib.so`.

[16]All relevant data computed in the course of the algorithm is collected in the SINGULAR data type `NormalForm` and can be accessed through the respective keys. For more detail, please refer to the documentation of the library.

```
> NormalForm G = normalForm(f);
> G;
corank = 2
Milnor number = 36
modality = 6
Newton polygon:
vertices: [0,10], [2,4], [4,2], [13,0]
facets:   [[0,10], [2,4]], [[2,4], [4,2]], [[4,2], [13,0]]
normals:  [3,1], [1,1], [2,9]
parameter monomials = [ x3y5, x3y4, x4y3, x2y4, y10, x3y3 ]
normal form =  x4y2 + x2y4 + y10 + x13
+ a(3,3)x3y3 +  a(2,4)x2y4 + a(4,3)x4y3 + a(3,4)x3y4 + a(3,5)x3y5  + a(0,10)y10
exceptional hypersurface = (1+a(2,4))*(1+a(0,10))
```

Algorithm 7 Normal form of a germ which is equivalent to a germ with a non-degenerate Newton boundary

---

Input: A polynomial germ $f \in \mathbb{Q}[x,y]$, $f \in \mathfrak{m}^3$ of corank 2 with finite Milnor number.

Output: A normal form for f, if f is equivalent to a germ with non-degenerate Newton bound-
ary, and `false` otherwise. We return a polynomial $F \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m][x,y]$ representing the
normal form, and a principal ideal $J \subset \mathbb{C}[\alpha_1, \ldots, \alpha_m]$ such that any germ $F(\alpha)(x,y)$ with
$\alpha \notin V(J)$ is in the μ-constant stratum of f, and for any germ $g \in \mathbb{C}[x,y]$ in the μ-constant
stratum of f, there is an $\alpha \notin V(J)$ with g right equivalent to $F(\alpha)(x,y)$, and there exist only
finitely many such α.

1: Replace f by the output of Algorithm 4 applied to f.
2: Let f_0 be the sum of monomials corresponding to the vertices of $\Gamma(f)$.
3: Let B be a regular basis for f_0 in $w(f)$-degree $\geq d(f)$, as given by Algorithm 6.
4: $F := f_0 + \sum_{i=1}^{m} \alpha_i \cdot b_i$
5: $J := \langle 1 \rangle$
6: **for** $1 \leq i \leq m$ **do**
7:     **if** α_i is the coefficient of a monomial of F corresponding to a vertex of $\Gamma(f)$ **then**
8:         $J = J \cap \langle \alpha_i \rangle$.
9: Let $\Delta_1, \ldots, \Delta_n$ be the facets of $\Gamma(f)$.
10: **for** $1 \leq j \leq n$ **do**
11:     Let (a,b) be the normal vector of Δ_j in lowest terms.
12:     Define $\Phi : \mathbb{Q}[x^b, y^a] \to \mathbb{Q}[t]$, $x^b \mapsto 1, y^a \mapsto t$.
13:     $g := \Phi(\text{sat}(\text{jet}(F, \Delta_j)))$
14:     $J := J \cap \langle \text{disc}_t g \rangle$, where disc_t denotes the discriminant with respect to the variable t.
15: **return** F, J

---

8. Determining the Values of the Moduli Parameters in the Normal From of a germ with a Non-Degenerate Newton Boundary

After finding the normal form, the values of the moduli parameters can, in theory, be computed
via an Ansatz for the right equivalence (making use of finite determinacy). This is, however, not
practicable except for very small examples. In this section, we discuss an efficient algorithm for
this problem.

Now, let $f \in \mathfrak{m}^2$ be a polynomial with a normalized non-degenerate Newton boundary, with
$w(f) = (w_1, \ldots, w_n)$ the induced weight on the Newton polygon. Let f_0 be the sum of the vertex
terms of $\text{jet}(f, \Gamma(f))$ and write $f_0' = w(f) \text{-} \text{jet}(f, d(f))$ for the sum of the terms of f on $\Gamma(f)$.

Lemma 8.1. *Any regular basis of f_0 is also a regular basis for every germ with a non-degenerate Newton boundary in the μ-constant stratum of f.*

Proof. The proof is similar to that of Theorem 3.20. Let g be a germ with a non-degenerate Newton boundary in the μ-constant stratum of f, and let B be a regular basis for f_0. By Proposition 3.12, B is also a regular basis for every germ with the same Newton boundary in a sufficiently small neighbourhood M of f_0 in the μ-constant stratum. As a linear combination of all monomials on and above the Newton polygon of f_0, we obtain a germ $\tilde{g} \in M$ with the same monomials as g and with regular basis B. By varying the coefficients of the terms on and above the Newton polygon, we can construct a path τ in the germspace connecting g to $\tilde{g}$.

Since being degenerate is a Zariski closed condition, there are only finitely many points on the paths such that the Newton boundary is degenerate. Moreover, if we denote by $\text{reg}(h)$ the set of all regular bases of the germ h, then for all germs h in a Zariski open subsets $U_1 \subset \tau_1$, and $U_2 \subset \tau_2$, the set $\text{reg}(h)$ is the same. Since, by Proposition 3.12, $\text{reg}(h)$ is constant on a Euclidean open neighbourhoods of f and f_0', we have $f_0' \in U_1$ and $f \in U_2$. Hence, the sets $A_1 := \{h \in \tau_1 \mid h \text{ Newton non-degenerate, } \text{reg}(h) = \text{reg}(f_0')\} \subset \tau_1$ and $A_2 := \{h \in \tau_2 \mid h \text{ Newton non-degenerate, } \text{reg}(h) = \text{reg}(f)\} \subset \tau_2$ is Zariski open. By construction, $\tau_1 \cap M$ and $\tau_2 \cap M$ are non-empty and contained in the subspace M that consists out of germs with regular basis B. This implies that the intersection $A_1 \cap \tau_1 \cap M$ and $A_2 \cap \tau_2 \cap M$ are both non-empty, so $B \in \text{reg}(f_0')$ and $B \in \text{reg}(f)$. $\qquad\square$

This implies:

Remark 8.2. Any regular basis for f_0 is also a regular basis for f_0' and for f.

Suppose B is a regular basis for f_0. Assume that f has a term above $\Gamma(f)$ not in B, and let d' be the lowest $w(f)$-degree occurring among these terms. Let t be a term of piecewise degree d' in f. Note that by the properties of a regular basis we can write

$$t = g\frac{\partial f}{\partial x} + h\frac{\partial f}{\partial y} + \text{terms of } w(f)\text{-degree } d' \text{ in } B + \text{terms of } w(f)\text{-degree} > d', \qquad (3)$$

where $g, h \in \mathbb{C}[x, y]$.

We now define the map $\phi : \mathbb{C}[[x, y]] \to \mathbb{C}[[x, y]]$ by

$$\phi(x) = x - g, \ \phi(y) = y - h, \qquad (4)$$

where g, h are defined as in (3).

Note that

$$\phi(f) = \underbrace{\frac{\partial f}{\partial x}g + \frac{\partial f}{\partial y}h}_{\text{first order of the binomial expansion of } \phi(f)} + \underbrace{\frac{\partial f}{\partial x \partial y}gh + \frac{\partial f}{(\partial x)^2}g^2 + \cdots}_{\text{higher order of the binomial expansion of } \phi(f)} \qquad (5)$$

A germ with a non-degenerate Newton boundary does not necessarily satisfy Condition A. Hence we cannot be certain that terms in the higher-order terms in the binomial expansion of $\phi(f)$ are of $w(f)$-degree larger than d', and thus the method introduced in Böhm, Marais, and Pfister (2016a) for finding a normalform equation cannot be applied. Algorithm 8 gives us an approach which works for any germ with non-degenerate Newton boundary. To formulate the algorithm we need the following lemma.

We fix a sequence Γ' of connected facets of the Newton boundary of f.

Lemma 8.3. *If there exist $0 \neq g, h \in \mathbb{C}[x, y]$, $g \neq \frac{\partial f_{\Gamma'}}{\partial y}, h \neq \frac{\partial f_{\Gamma'}}{\partial x}$ such that*

$$g \cdot \frac{\partial f_{\Gamma'}}{\partial x} + h \cdot \frac{\partial f_{\Gamma'}}{\partial y} = 0,$$

then

$$g \cdot \frac{\partial f_{\Gamma'}}{\partial x} + h \cdot \frac{\partial f_{\Gamma'}}{\partial y} = x^{a'}y^{b'}\left(\frac{g}{x^s} \cdot (x^i \cdot \text{sat}(\frac{\partial f_{\Gamma'}}{\partial x})) + \frac{h}{y^t} \cdot (y^\nu \cdot \text{sat}(\frac{\partial f_{\Gamma'}}{\partial y}))\right),$$

where we write $f_{\Gamma'} = x^{a'}y^{b'} \operatorname{sat}(f_{\Gamma'})$, and s, i, t, ν are defined as follows:

$$s = \begin{cases} \max\{\alpha \mid x^{\alpha} \; divides \; \frac{\partial f_{\Gamma'}}{\partial y}\} & if \; a = 0, \\ 1 & if \; a \neq 0, \end{cases} \qquad i = \begin{cases} \max\{\alpha \mid x^{\alpha} \; divides \; \frac{\partial f_{\Gamma'}}{\partial x}\} & if \; a = 0, \\ 0 & if \; a \neq 0, \end{cases}$$

$$t = \begin{cases} \max\{\beta \mid y^{\beta} \; divides \; \frac{\partial f_{\Gamma'}}{\partial x}\} & if \; b = 0, \\ 1 & if \; b \neq 0, \end{cases} \qquad \nu = \begin{cases} \max\{\beta \mid y^{\beta} \; divides \; \frac{\partial f_{\Gamma'}}{\partial y}\} & if \; b = 0, \\ 0 & if \; b \neq 0, \end{cases}$$

Moreover, the vector $\left(\frac{g}{x^s}, \frac{h}{y^t}\right)$ is a polynomial multiple of the Koszul syzygy of

$$\left(x^i \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial x}\right), y^{\nu} \cdot \operatorname{sat}\left(\frac{\partial f_{\Gamma'}}{\partial y}\right)\right).$$

Proof. Let a and b be maximal, such that $x^a y^b$ divides both $f_x := \frac{\partial f_{\Gamma'}}{\partial x}$ and $f_y := \frac{\partial f_{\Gamma'}}{\partial y}$. We say that a term of f_x and a term of f_y are in correspondence if they come from the same term of $f_{\Gamma'}$.

- If $a \neq 0$, then x^a is the largest power of x dividing f_x, and x^{a+1} is the largest power of x dividing f_y. To see this, we first note that, excluding a pure x-power term in f_x, all terms of f_x and f_y are in one-to one correspondence. If f_x would be divisible by x^{a+1} then, since all terms of f_y are in correspondence to a term in f_x, all terms of f_y would be divisible by x^{a+2}, contradicting the minimal choice of a. Thus x^a is the largest power of x dividing f_x, and f_y is divisible by x^{a+1}. If x^{a+2} would divide f_y, then x^{a+1} would also divide every corresponding term of f_x. Since also a possible x^{α}-term of f_x is then divisible by x^{a+2} (since it is the term with the largest x-exponent), this again contradicts the minimal choice of x. Moreover the argument above implies that x^{a+1} is the highest power of x dividing f.
- If $a = 0$, then $x \nmid f_y$ or $x \nmid f_x$.
  - If $x \nmid f_y$ then $f_{\Gamma'}$ has a term that is a pure y-power. Hence $x \nmid f$.
  - If $x | f_y$, then $x \nmid f_x$, which implies that $f_{\Gamma'}$ has a term of the form xy^l with $l \geq 1$. Hence, in this case, if x^s divides f_y (and hence f), $s = 1$.

Now, let $x^{a'}y^{b'}$ be the product of the highest power of x and the highest power of y dividing $\operatorname{jet}(f, \Gamma')$.

- Then $a' = a + 1$, if $a \neq 0$.
- If $a = 0$, we have two cases:
  - If $x \nmid f_y$, then $a' = 0$.
  - If $x | f_y$, then x is the highest power of x dividing $f_{\Gamma'}$. Hence $a' = a + 1$.

Similarly $b' = b + 1$, if $b \neq 0$. If $b = 0$ and $y \nmid f_x$, then $b' = 0$. If $b = 0$, $y | f_x$ (implying that $y \nmid f_y$, then y is the biggest power of y dividing $f_{\Gamma'}$.

We assume that

$$g \cdot f_x + h \cdot f_y = 0.$$

Then

$$x^a y^b (g \cdot y^t \cdot x^i \cdot \operatorname{sat}(f_x) + h \cdot x^s \cdot y^{\nu} \cdot \operatorname{sat}(f_y)) = 0,$$

where s, t, i and ν are defined as above. To see this, consider the case $a \neq 0$, $b \neq 0$, then

$$x^a y^b (g \cdot y \cdot \operatorname{sat}(f_x) + h \cdot x \cdot \operatorname{sat}(f_y)) = x^{a'} y^{b'} (g \cdot \operatorname{sat}(f_x) + h \cdot \operatorname{sat}(f_y)) = 0,$$

which can be verified by the arguments above. The other cases can be similarly verified. If $a \neq 0$, or if $a = 0$, and $x | f_y$, then $s = 1$ and $a' = a + 1$. In both these cases $i = 0$. If $a = 0$ and $x \nmid f_y$, then $a' = a = 0$ and $s = 0$. Therfore we have in general.

$$x^{a'} y^{b'} (\frac{g}{x^s} \cdot y^t \cdot x^i \cdot \operatorname{sat}(f_x) + h \cdot y^{\nu} \cdot \operatorname{sat}(f_y)) = 0.$$

Similarly, if $b \neq 0$, or if $b = 0$, and $y | f_x$, then $\nu = 1$ and $b' = b + 1$. In both these cases $t = 0$. If $b = 0$ and $y \nmid f_x$, then $b' = b = 0$ and $t = 0$. Therfore we have in general.

$$x^{a'} y^{b'} (\frac{g}{x^s} \cdot x^i \cdot \operatorname{sat}(f_x) + \frac{h}{y^t} \cdot y^{\nu} \cdot \operatorname{sat}(f_y)) = 0.$$

Therefore
$$\frac{g}{x^s} \cdot x^i \cdot \mathrm{sat}(f_x) + \frac{h}{y^t} \cdot y^\nu \cdot \mathrm{sat}(f_y) = 0.$$

Hence $x|g$, if $a \neq 0$, or if $a = 0$ and $x|f_y$, and $y|h$, if $b \neq 0$, or if $b = 0$ and $y|f_x$.

We now show that $x^i \cdot \mathrm{sat}(f_x)$ and $y^\nu \cdot \mathrm{sat}(f_y)$ has no common factor. To do that, first note that $\mathrm{sat}(f_{\Gamma'})$ is nondegenerate by definition. This implies that $\mathrm{sat}(f_{\Gamma'})$ does not have a multiple factor, hence $\mathrm{sat}(f_{\Gamma'})$ and $\frac{\partial \,\mathrm{sat}\, f_{\Gamma'}}{\partial x}$ have no common factor, and $\mathrm{sat}(f_{\Gamma'})$ and $\frac{\partial \,\mathrm{sat}\, f_{\Gamma'}}{\partial y}$ have no common factor. Thus, $x^{a'} y^{b'}$ accounts for the only multiple factors in $f_{\Gamma'}$.

Consider the following equations for the partial derivatives of $f_{\Gamma'}$:

$$f_x = a' x^{a'-1} y^{b'} (\mathrm{sat}(f_{\Gamma'})) + x^{a'} y^{b'} \left(\frac{\partial \,\mathrm{sat}\, f_{\Gamma'}}{\partial x} \right)$$

$$f_y = b' x^{a'} y^{b'-1} (\mathrm{sat}(f_{\Gamma'})) + x^{a'} y^{b'} \left(\frac{\partial \,\mathrm{sat}\, f_{\Gamma'}}{\partial y} \right).$$

If the saturations of the partial derivatives of $f_{\Gamma'}$ with respect to x and y share a common factor, then this factor would also be a factor of $\mathrm{sat}(f_{\Gamma'})$. However, this contradicts the previous equations. Furthermore $y \nmid \mathrm{sat}(f_x)$ if $\nu > 0$ and $x \nmid \mathrm{sat}(f_y)$ if $i > 0$.

Therefore, the polynomial expression $(\frac{g}{x^s} \cdot x^i \cdot \mathrm{sat}(f_x) + \frac{h}{y^t} \cdot y^\nu \cdot \mathrm{sat}(f_y)) = 0$ must be a multiple of the Koszul syzygy of $\left(\frac{g}{x^s}, \frac{h}{y^t} \right)$. This completes the proof.

$\square$

Definition 8.4. *Let $f \in \mathbb{C}[x,y]$ and $m = \frac{m_1}{m_2} \in \mathrm{Quot}(\mathbb{C}[x,y])$, $m_1, m_2 \in \mathbb{C}[x,y]$ monomials. Then $\overline{f}^m$ is the sum of all terms t of f such that $mt \in \mathbb{C}[x,y]$.*

Theorem 8.5. *Let (a,b) be an arbitrary syzygy of $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ over $\mathbb{C}[x,y]/I$, where I is the ideal generated by all the monomials of w-degree d', $d' \geq d(f)$, then $a - \sum_{j=1}^k z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \in \mathrm{ann}(\frac{\partial f}{\partial x})$ and $b - \sum_{j=1}^k z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \in \mathrm{ann}(\frac{\partial f}{\partial y})$ over $\mathbb{C}[x,y]/I$, where z are of the form $\frac{m_1}{m_2} \in \mathrm{Quot}(\mathbb{C}[x,y])$, m_1, m_2 monomials in $\mathbb{C}[x,y]$. Furthermore if $y^\alpha \notin \sum_j \overline{\frac{\partial f}{\partial y}}^{z_j}$, then $x \mid \sum_{j=1}^k \sum_j z_j \overline{\frac{\partial f}{\partial y}}^{z_j}$ and, if $x^\beta \notin \sum_j \overline{\frac{\partial f}{\partial x}}^{z_j}$, then $y \mid \sum_{j=1}^k z_j \overline{\frac{\partial f}{\partial x}}^{z_j}$, where $\alpha, \beta > 0$.*

Proof. Let (a,b) be a syzygy of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ in $\mathbb{C}[x,y]/I$, where I is the ideal generated by all the monomials of w-degree d'. If the syzygy equation $g_1 = a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y}$ has no cancellation underneath w-degree d' then we are finished. Now suppose g has cancellation underneath degree d'. Suppose in particular g has cancellation underneath w_1-degree d'. Let l_1 be $\min \left(w_1\text{-}\mathrm{ord}(a \frac{\partial f}{\partial x}), w_1\text{-}\mathrm{ord}(b \frac{\partial f}{\partial y}) \right)$. Then

$$w_1\text{-}\mathrm{jet}(g_1, l_1) = m^{(1)} \left(x^{s_1} y^{\nu_1} \mathrm{sat} \left(\frac{\partial \,\mathrm{jet}(f, \Delta_1)}{\partial y} \right) \frac{\partial \,\mathrm{jet}(f, \Delta_1)}{\partial x} - y^{t_1} x^{i_1} \mathrm{sat} \left(\frac{\partial \,\mathrm{jet}(f, \Delta_1)}{\partial x} \right) \frac{\partial \,\mathrm{jet}(f, \Delta_1)}{\partial y} \right),$$

where $m^{(1)}$ is a polynomial, Δ_1 is the face with the smallest slope of $\Gamma(f)$, and i_1, ν_1, s_1 and t_1 is as in Lemma 8.3.

Now consider the syzygy $\left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \right)$ of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ in $\mathbb{C}[x,y]$, with syzygy equation $g_0 = \frac{\partial f}{\partial y} \frac{\partial f}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial f}{\partial y}$. Let $l_0 = w_1\text{-}\mathrm{ord}(\frac{\partial f}{\partial y} \frac{\partial f}{\partial x})$, then

$$w_1\text{-}\mathrm{jet}(g_0, l_0) = m_{w_1} \left(x^{s_1} y^{\nu_1} \mathrm{sat} \left(\frac{\partial \,\mathrm{jet}(f, \Delta_1)}{\partial y} \right) \frac{\partial \,\mathrm{jet}(f, \Delta_1)}{\partial x} - y^{t_1} x^{i_1} \mathrm{sat} \left(\frac{\partial \,\mathrm{jet}(f, \Delta_1)}{\partial x} \right) \frac{\partial \,\mathrm{jet}(f, \Delta_1)}{\partial y} \right),$$

where $m_{w_1} = \gcd \left(\frac{\partial \,\mathrm{jet}(f, \Delta_1)}{\partial x}, \frac{\partial \,\mathrm{jet}(f, \Delta_1)}{\partial y} \right)$. Let $n^{(1)} = \mathrm{gcm}(m_{w_1}, m^{(1)})$. Then the lowest nonzero w_1-jet of $\frac{n^{(1)}}{m^{(1)}}(a,b)$ and $\frac{n^{(1)}}{m^{(1)}} \left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \right)$ coincide.

Then

$$(a^{(1)}, b^{(1)}) = \left(\frac{n^{(1)}}{m^{(1)}} a - \frac{n^{(1)}}{m_{w_1}} \frac{\partial f}{\partial y}, \frac{n^{(1)}}{m^{(1)}} b - \frac{n^{(1)}}{m_{w_1}} \frac{\partial f}{\partial x} \right)$$

is a syzygy of $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ in $\mathbb{C}[x,y]/I^{(1)}$, where $I^{(1)} = \frac{n^{(1)}}{m^{(1)}} I$.

Let $d_2' = d' \left(w_1 \text{-} \deg \left(\frac{n^{(1)}}{m^{(1)}} \right) \right)$. Now, if the equation $g_2 = a^{(1)} \frac{\partial f}{\partial x} + b^{(1)} \frac{\partial f}{\partial y}$ has any terms of w_1-degree less than d_2', then the lowest non-zero w_1-jet of $(a^{(1)}, b^{(1)})$ is

$$m^{(2)} \left(x^{s_2} y^{\nu_2} \operatorname{sat} \left(\frac{\partial \operatorname{jet}(f, \Delta_1)}{\partial y} \right), y^{t_2} x^{i_2} \operatorname{sat} \left(\frac{\partial \operatorname{jet}(f, \Delta_1)}{\partial x} \right) \right).$$

Let $n^{(2)} = \operatorname{gcm} \left(m^{(2)}, m_{w_1} \right)$, then the lowest non-zero w_1-jet of the syzygys $\frac{n^{(2)}}{m^{(2)}} \left(a^{(1)}, b^{(1)} \right)$ and $\frac{n^{(2)}}{m_{w_1}} \left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \right)$ coincide. Similarly, as before, we can create the syzygy

$$\left(a^{(2)}, b^{(2)} \right) = \left(\frac{n^{(2)}}{m^{(2)}} a^{(1)} - \frac{n^{(2)}}{m_{w_1}} \frac{\partial f}{\partial y}, \frac{n^{(2)}}{m^{(2)}} b^{(1)} - \frac{n^{(2)}}{m_{w_1}} \frac{\partial f}{\partial x} \right)$$

in $\mathbb{C}[x,y]/I^{(2)}$, where $I^{(2)} = \frac{n^{(2)}}{m^{(2)}} I^{(1)}$.

We can go on with this process until the syzygy equation g_k has no terms of w_1-degree less than w_1-degree d_k'. We show now that this will eventually happen. Now, $n^{(j)} = \gcd(c, d, e, h)$ and $m^{(j+1)} = \gcd(c - d, e - h)$, where $c = \frac{n^{(j+1)}}{m^{(j+1)}} a^{(j)}$, $d = \frac{n^{(j+1)}}{m_{w_1}} \frac{\partial f}{\partial y}$, $e = \frac{n^{(j+1)}}{m^{(j+1)}} b^{(j)}$ and $h = \frac{n^{(j+1)}}{m_{w_1}} \frac{\partial f}{\partial x}$. Since the lowest non-zero w_1-jet of c and d, and e and h, cancel, respectively, in $c - d$ and $e - h$, it follows that $w_1 \text{-} \deg \left(m^{(j+1)} \right) > w_1 \text{-} \deg \left(n^{(j)} \right)$. Now,

$$
\begin{aligned}
w_1 \text{-} \deg \left(\frac{n^{(j+1)}}{m_{w_1}} \right) w_1 \text{-} \deg \left(\frac{\partial f}{\partial y} \right) - d_{j+1}' &= w_1 \text{-} \deg \left(\frac{n^{(j+1)}}{m_{w_1}} \right) w_1 \text{-} \deg \left(\frac{\partial f}{\partial y} \right) - \frac{(n^{(j+1)})}{m^{(j+1)}} d_j' \\
&= w_1 \text{-} \deg \left(\frac{m^{(j+1)}}{n^{(j+1)}} \frac{n^{(j+1)}}{m_{w_1}} \right) w_1 \text{-} \deg \left(\frac{\partial f}{\partial y} \right) - d_j' \\
&= w_1 \text{-} \deg \left(\frac{m^{(j+1)}}{m_{w_1}} \right) w_1 \text{-} \deg \left(\frac{\partial f}{\partial y} \right) - d_j' \\
&< w_1 \text{-} \deg \left(\frac{n^{(j)}}{m_{w_1}} \right) w_1 \text{-} \deg \left(\frac{\partial f}{\partial y} \right) - d_j'.
\end{aligned}
$$

This implies that the difference in the degree of the lowest non-zero w_1-jet of $(a^{(j)}, b^{(j)})$ and the degree of the lowest order elements in d_k' becomes smaller. Hence eventually the w_1-degree of $(a^{(j)}, b^{(j)})$ will be $\geq d_j'$.

Suppose now that $(a^{(j-1)}, b^{(j-1)})$ is such that g_j has no terms underneath w_1-degree d_{j-1}'. We now consider the w_2-degree of $(a^{(j-1)}, b^{(j-1)})$. We follow the same strategy. Suppose g_j has terms of w_2-degree less than $d_{j-1}^{(2)}$, where $d_k^{(r)} = w_r \text{-} \deg(\frac{n^{(1)} n^{(2)} \dots n^{(k)}}{m^{(1)} m^{(2)} \dots m^{(k)}}) d'$, then let l_j be $\min \left(w_2 \text{-} \operatorname{ord}(a^{(j-1)} \frac{\partial f}{\partial x}), w_2 \text{-} \operatorname{ord}(b^{(j-1)} \frac{\partial f}{\partial y}) \right)$. Then

$$w_2 \text{-} \operatorname{jet}(g_j, l_j) = m^{(j)} \left(x^{s_j} y^{\nu_j} \operatorname{sat} \left(\frac{\partial \operatorname{jet}(f, \Delta_2)}{\partial y} \right) \frac{\partial \operatorname{jet}(f, \Delta_2)}{\partial x} - y^{t_j} x^{i_j} \operatorname{sat} \left(\frac{\partial \operatorname{jet}(f, \Delta_2)}{\partial x} \right) \frac{\partial \operatorname{jet}(f, \Delta_2)}{\partial y} \right),$$

where $m^{(j)}$ is a monomial, Δ_2 is the face with the smallest slope of $\Gamma(f)$, and s_j, ν_j, i_j and t_j is as in Lemma 8.3. Furthermore

$$w_2 \text{-} \operatorname{jet}(g_0, l_0) = m_{w_2} \left(x^{s_j} y^{\nu_j} \operatorname{sat} \left(\frac{\partial \operatorname{jet}(f, \Delta_2)}{\partial y} \right) \frac{\partial \operatorname{jet}(f, \Delta_2)}{\partial x} - y^{t_j} x^{i_j} \operatorname{sat} \left(\frac{\partial \operatorname{jet}(f, \Delta_2)}{\partial x} \right) \frac{\partial \operatorname{jet}(f, \Delta_2)}{\partial y} \right),$$

where $m_{w_2} = \gcd \left(\frac{\partial \operatorname{jet}(f, \Delta_2)}{\partial x}, \frac{\partial \operatorname{jet}(f, \Delta_2)}{\partial y} \right)$. Let $n^{(j)} = \operatorname{gcm}(m_{w_2}, m^{(j)})$. Then the lowest nonzero w_2-jet of $\frac{n^{(j)}}{m^{(j)}} \left(a^{(j-1)}, b^{(j-1)} \right)$ and $\frac{n^{(j)}}{m_{w_2}} \left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \right)$ coincide. Similarly, as before, we can create the

syzygy

$$\left(a^{(j)}, b^{(j)}\right) = \left(\frac{n^{(j)}}{m^{(j)}} a^{(j-1)} - \frac{n^{(j)}}{m_{w_j}} \frac{\partial f}{\partial y}, \frac{n^{(j)}}{m^{(j)}} b^{(j-1)} - \frac{n^{(j)}}{m_{w_j}} \frac{\partial f}{\partial x}\right)$$

in $\mathbb{C}[x,y]/I^{(j)}$, where $I^{(j)} = \frac{n^{(j)}}{m^{(j)}} I^{(j-1)}$.

Note that the syzygy equation $g_{(j)}$ has no terms underneath w_2-degree $d'_j = w_2\text{-}\deg(\frac{n^{(j)}}{m^{(j)}})d^{(2)}_{j-1}$ and w_1-degree $(w_1\text{-}\deg(\frac{n^{(j)}}{m^{(j)}})) \cdot d'_{j-1}$. If this was not the case, it would mean that $\frac{n^{(j)}}{m_{w_2}} \frac{\partial f}{\partial y}\frac{\partial f}{\partial x} - \frac{n^{(j)}}{m_{w_2}} \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$ has terms underneath w_1-degree $(w_1\text{-}\deg(\frac{n^{(j)}}{m^{(j)}})) \cdot d'_{j-1}$. Let $l'_{(x,w_1)}$ and $l'_{(y,w_1)}$ be the lowest w_1-order of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. Let $l'_{(x,w_2)}$ and $l'_{(y,w_2)}$ be the lowest w_2-order of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. We have that $w_1\text{-}\mathrm{jet}(\frac{\partial f}{\partial x}, l'_{(x,w_1)})$ and $w_2\text{-}\mathrm{jet}(\frac{\partial f}{\partial x}, l'_{(x,w_2)})$, and $w_1\text{-}\mathrm{jet}(\frac{\partial f}{\partial y}, l'_{(y,w_1)})$ and $w_2\text{-}\mathrm{jet}(\frac{\partial f}{\partial y}, l'_{(y,w_2)})$ coincide at a vertex monomial, respectively, and all the terms of $w_1\text{-}\mathrm{jet}(\frac{\partial f}{\partial x}, l'_{(x,w_1)})$, and $w_1\text{-}\mathrm{jet}(\frac{\partial f}{\partial y}, l_{(y,w_1)})$, respectively, have the same w_1-degree, and all terms of $w_2\text{-}\mathrm{jet}(\frac{\partial f}{\partial x}, l'_{(x,w_2)})$, and $w_2\text{-}\mathrm{jet}(\frac{\partial f}{\partial y}, l'_{(y,w_2)})$, respectively, have the same w_2-degree. Hence it follows that $w_2\text{-}\mathrm{jet}(\frac{n^{(j)}}{m_{w_2}} \frac{\partial f}{\partial y}\frac{\partial f}{\partial x} - \frac{n^{(j)}}{m_{w_2}} \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}, l_j)$ also has terms underneath w_1-degree $(w_1\text{-}\deg(\frac{n^{(j)}}{m^{(j)}})) \cdot d'_{j-1}$, which means that $w_2\text{-}\mathrm{jet}(\frac{n^{(j)}}{m^{(j)}} a^{(j-1)}\frac{\partial f}{\partial x} - \frac{n^{(j)}}{m^{(j)}} b^{(j-1)}\frac{\partial f}{\partial y}, l_j)$ has terms underneath w_1 degree $(w_1\text{-}\deg(\frac{n^{(j)}}{m^{(j)}})) \cdot d'_{j-1}$. This again implies that $w_2\text{-}\mathrm{jet}(a^{(j-1)}\frac{\partial f}{\partial x} - b^{(j-1)}\frac{\partial f}{\partial y}, l_{j-1})$ has terms underneath w_1-degree d'_{j-1}. Which implies that $a^{(j-1)}\frac{\partial f}{\partial x} - b^{(j-1)}\frac{\partial f}{\partial y}$ has terms underneath w_1 degree d'_{j-1} Therefore we have a contradiction.

Continuing as above, we can create a syzygy $(a^{(k)}, b^{(k)})$ of $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ in $\mathbb{C}[x,y]/I^{(k)}$, where the syzygy equation g_k has no terms underneath w_i-degree $d^{(i)}_k$, for $i = 1, 2, \ldots, n$. Let $r_0 = \frac{n^{(1)}\ldots n^{(k)}}{m^{(1)}\ldots m^{(k)}}$ and $r_j = \frac{n^{(j)}\ldots n^{(k)}}{m^{(j+1)}\ldots m^{(k)} m_{w_j}}$, for $j = \{1, \ldots, k-1\}$, and $r_k = \frac{n^{(k)}}{m_{w_{i_k}}}$. By construction

$$w_i\text{-}\mathrm{jet}\left(r_0 a, d^{(i)}_k - (w_i\text{-}\mathrm{ord}(\frac{\partial f}{\partial x}))\right) = w_i\text{-}\mathrm{jet}\left(\sum_{j=1}^{k} r_j \frac{\partial f}{\partial y}, d^{(i)}_k - (w_i\text{-}\mathrm{ord}(\frac{\partial f}{\partial x}))\right). \qquad (6)$$

Since r_0 divides all the terms on the right side of the equation, it follows that if $\frac{r_j}{r_0} t \notin \mathbb{C}[x,y]$, for some term t of $\frac{\partial f}{\partial y}$, then t either get cancelled on the righthand side of the equation or $r_j t$ is of w_i-degree higher than $d^{(i)}_k - (w_i\text{-}\mathrm{ord}(\frac{\partial f}{\partial x}))$ for all i. In other words $r_j t \frac{\partial f}{\partial x}$ is of w_i-order higher than $d^{(i)}$ for all i. Hence $r_j t \frac{\partial f}{\partial x}$ is contained in $I^{(k)}$. Therefore

$$w_i\text{-}\mathrm{jet}\left(r_0 a, d^{(i)}_k - (w_i\text{-}\mathrm{ord}(\frac{\partial f}{\partial x}))\right) = w_i\text{-}\mathrm{jet}\left(\sum_{j=1}^{k} r_j \overline{\frac{\partial f}{\partial y}}^{z_j}, d^{(i)}_k - (w_i\text{-}\mathrm{ord}(\frac{\partial f}{\partial x}))\right), \qquad (7)$$

where $z_j = \frac{m^{(1)}\ldots m^{(j)}}{n^{(1)}\ldots n^{(j-1)} m_{w_j}}$.

Therefore $a - \sum_{j=1}^{k} z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \in \mathrm{ann}(\frac{\partial f}{\partial x})$ and $b - \sum_{j=1}^{k} z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \in \mathrm{ann}(\frac{\partial f}{\partial y})$ over $\mathbb{C}[x,y]/I$.

We will now show that if $y^\alpha \notin \overline{\frac{\partial f}{\partial y}}^{z_j}$, then $x \mid \sum_{j=1}^{k} z_j \overline{\frac{\partial f}{\partial y}}^{z_j}$ and, if $x^\beta \notin \overline{\frac{\partial f}{\partial x}}^{z_j}$, then $y \mid \sum_{j=1}^{k} z_j \overline{\frac{\partial f}{\partial x}}^{z_j}$, where $\alpha, \beta > 0$. Hence, lets suppose first that $\overline{\frac{\partial f}{\partial y}}^{z_j}$ only has mixed terms. If $x^\alpha y^\beta$ is a monomial of $\overline{\frac{\partial f}{\partial y}}^{z_j}$, then $x^{\alpha-1} y^{\beta+1}$ is a monomial of $\frac{\partial f}{\partial x}$, $\alpha, \beta > 0$, and there exist a monomial $x^\eta y^\gamma$ of $\frac{\partial f}{\partial x}$ such that $x^\alpha y^\beta x^\eta y^\gamma \notin I$. But then $x^{\eta+1} y^{\gamma-1}$ is a monomial of $\frac{\partial f}{\partial y}$. Since $x^{\alpha-1} y^{\beta+1} x^{\eta+1} y^{\gamma-1} \notin I$, it follows that $x^{\alpha-1} y^{\beta+1}$ is a monomial of $\overline{\frac{\partial f}{\partial x}}^{z_j}$. On the other hand y^β is not a monomial of $\overline{\frac{\partial f}{\partial x}}^{z_j}$, for any $\beta > 0$, and, similarly, if $x^\alpha y^\beta$ is a monomial of $\overline{\frac{\partial f}{\partial x}}^{z_j}$, $\alpha, \beta > 0$, then $x^{\alpha+1} y^{\beta-1}$ is a monomial of $\overline{\frac{\partial f}{\partial y}}^{z_j}$. It can be similarly proven that if $x^\beta \notin \overline{\frac{\partial f}{\partial x}}^{z_j}$, then $y \mid \sum_{j=1}^{k} z_j \overline{\frac{\partial f}{\partial x}}^{z_j}$, $\beta > 0$. $\qquad \square$

In line 23 of Algorithm 8, we use Lemma 8.6 that state that every monomial corresponding to a lattice point of the Newton polygon of a germ with a normalized non-degenerate Newton boundary is a monomial of any normal form of the germ with this Newton polygon. Furthermore it states that only two vertex monomials, under the determinacy, of the Newton boundary are not parameter monomials in a corresponding normal form. This implies that the sum of the terms on the Newton boundary can be transformed to the sum of the terms on the Newton boundary of any normal form equation by scaling x and y and adjusting the terms above the determinacy. For clarity, we postpone the proof to the end of the chapter.

Lemma 8.6. *Let Γ be the Newton polygon of a germ with normalized, non-degenerate Newton boundary. Let f_0 be the sum of the monomials corresponding to the vertices of Γ, and let B be a regular basis for f_0. Then*

(1) any monomial corresponding to a lattice point of Γ is either a monomial of f_0 or an element of B, and

(2) at most two monomials of f_0 of degree $\leq \mathrm{dt}(f_0)$ are not in B.

We now proof Algorithm 8.

Proof of Algorithm 8. In the transformation ϕ in line 10 we know that terms coming from first partial dirivitives, as described in 5, cancel. We now consider terms coming from second order dirivitives in the transformation ϕ. Let $z_j = \frac{a_j}{b_j}$ in lowest terms. We systematically consider the terms coming from the higher order binomial expansions of $\phi_{z_j}(f)$, where $\phi_{z_j}(x) = x + z_j \overline{\frac{\partial f}{\partial y}}^{z_j}$, $\phi_{z_j}(y) = z_j \overline{\frac{\partial f}{\partial x}}^{z_j}$. We have to destinguish between the following cases:

(1) $b_j = x^\alpha y^\beta$, $\alpha, \beta > 0$;
(2) $b_j = y^\beta$, $\beta > 0$;
(3) $b_j = x^\alpha$, $\alpha > 0$;
(4) $b_j = 1$.

For each j the second order terms of the binomial expansion of ϕ_{z_j} are

$$\frac{\partial^2 f}{\partial x^2} \cdot (z_j \overline{\frac{\partial f}{\partial y}}^{z_j})^2 + \frac{\partial^2 f}{\partial x \partial y} \cdot (z_j \overline{\frac{\partial f}{\partial y}}^{z_j}) \cdot (z_j \overline{\frac{\partial f}{\partial x}}^{z_j}) + \frac{\partial^2 f}{\partial y^2} \cdot (z_j \overline{\frac{\partial f}{\partial x}}^{z_j})^2 + \text{terms in } J,$$

where J is the ideal generated by monomials of w-degree $d' + 1$.

(1): Adding terms in J, and using Theorem 8.5, the above can be written as

$$z_j \overline{(x \frac{\partial^2 f}{\partial x^2}}^{z_j}) \cdot \frac{z_j \overline{\frac{\partial f}{\partial y}}^{z_j}}{x} \cdot \frac{\partial f}{\partial y} + z_j \overline{(x \frac{\partial^2 f}{\partial x \partial y}}^{z_j}) \cdot \frac{z_j \overline{\frac{\partial f}{\partial y}}^{z_j}}{x} \cdot \frac{\partial f}{\partial x} + z_j \overline{(y \cdot \frac{\partial^2 f}{\partial y^2}}^{z_j}) \cdot \frac{z_j \overline{\frac{\partial f}{\partial x}}^{z_j}}{y} \cdot \frac{\partial f}{\partial x} + \text{terms in } J.$$

Note that $x \frac{\partial^2 f}{\partial x^2}$, $x \frac{\partial^2 f}{\partial x \partial y}$ and $y \cdot \frac{\partial^2 f}{\partial y^2}$ has a subset of the monomials of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively. Therfore $y \mid z_j (\overline{x \frac{\partial^2 f}{\partial x^2}}^{z_j})$ and $x \mid z_j (\overline{y \cdot \frac{\partial^2 f}{\partial y^2}})$. Hence we have:

$$q_{1,1} \cdot \frac{y}{x} \cdot (z_j \overline{\frac{\partial f}{\partial y}}^{z_j}) \cdot \frac{\partial f}{\partial y} + q_{2,1} \cdot \frac{y}{x} \cdot (z_j \cdot \overline{\frac{\partial f}{\partial y}}^{z_j}) \cdot \frac{\partial f}{\partial x} + q_{3,1} \cdot \frac{x}{y} \cdot (z_j \overline{\frac{\partial f}{\partial x}}^{z_j}) \cdot \frac{\partial f}{\partial x} + \text{terms in } J, \quad (8)$$

Note that $q_{1,1}, q_{2,1}, q_{3,1}$ can be chosen such that $q_{1,1}, q_{2,1}, q_{3,1} \in \langle x, y \rangle$. Since g and h are at least of order 2, it follows that $z_j (\overline{x \frac{\partial^2 f}{\partial x^2}}^{z_j})$ and $z_j (\overline{y \frac{\partial^2 f}{\partial y^2}}^{z_j})$ are at least of order two, which implies there is a solution for $q_{1,1}, q_{2,1}$ and $q_{3,1}$ in $\langle x, y \rangle$. The higher orders of the binomial expansion of $\phi_{z_j}(f)$ can be written in the same form.

(2): In a similar way, the higher orders of the binomial expansion of $\phi_{z_j}(f)$ can be written as:

$$q_{1,2} \cdot z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \cdot \frac{\partial f}{\partial y} + q_{2,2} \cdot z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \cdot \frac{\partial f}{\partial x} + q_{3,2} \cdot \frac{x}{y} \cdot z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \cdot \frac{\partial f}{\partial x} + \text{terms in } J, \quad (9)$$

Algorithm 8 Determining the Moduli Parameters in the Normal Form of a Germ with a Normalized Non-degenerate Newton Boundary

Input: A polynomial germ $f \in \mathbb{Q}[x,y]$, $f \in \mathfrak{m}^3$ of corank 2 with finite Milnor number.

Output: A normal form of f and a normal form equation equivalent to f such that the normal form equation is a member of the normal form.

1: replace f by the output of Algorithm 4 applied to f.

2: let d'' be a determinacy bound for f.

3: let $w := w(f) = (w_1, \ldots, w_n)$.

4: let f_0 be the sum of monomials corresponding to the vertices of $\Gamma(f)$.

5: let $B = \{b_1, \ldots, b_m\}$ be a regular basis for f_0 in $w(f)$-degree $\geq d(f)$, as given by Algorithm 6.

6: $F := f_0 + \sum_{i=1}^{m} \alpha_i \cdot b_i$

7: $S := \mathrm{mon}(f - w(f)\text{-}\mathrm{jet}(f, d(f)))$.

8: **while** $S \not\subset B$ **do**

9:     let d' be the lowest non-zero $w(f)$-degree above $w(f)$-degree $d(f)$.

10:     write the sum of the terms of $w(f)$-degree d' as

$$q = g\frac{\partial f}{\partial x} + h\frac{\partial f}{\partial y} + \text{terms in } B \text{ of } w(f)\text{-degree } d' + \text{terms of higher } w(f)\text{-degree than } d'.$$

11:     let $\phi : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$, $\phi(x) = x + g$, $\phi(y) = y + h$.

12:     let $q = \phi(f) - f + g\frac{\partial f}{\partial x} + h\frac{\partial f}{\partial y}$;

13:     let $q = w(f) - \mathrm{jet}(f, d')$.

14:     write $g = \sum_i z_i \overline{\frac{\partial f}{\partial y}}^{z_i}$ and $h = \sum_i z_i \overline{\frac{\partial f}{\partial x}}^{z_i}$.

15:     let $z_i = \frac{a_i}{b_i}$ in lowest form.

16:     let

$$Q_1 = \{i \mid b_i = x^\alpha y^\beta, \alpha, \beta > 0\}, \quad Q_2 = \{i \mid b_i = y^\beta, \beta > 0\},$$
$$Q_3 = \{i \mid b_i = x^\alpha, \alpha > 0\}, \qquad Q_4 = \{i \mid b_i = 1\}.$$

17:     replace f by $\mathrm{jet}(\phi(f), d'')$.

18:     **while** $q <> 0$ **do**

19:         write

$$
\begin{aligned}
q \;=\; & q_1 \sum_{i \in Q_1 \cup Q_3} \left(\frac{y}{x} \cdot z_j \cdot \overline{\frac{\partial f}{\partial y}}^{z_j} \right) \frac{\partial f}{\partial x} + q_2 \sum_{i \in Q_2 \cup Q_4} \left(z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \right) \frac{\partial f}{\partial x} \\
& + q_3 \sum_{i \in Q_1} \left(\frac{x}{y} \cdot z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \right) \frac{\partial f}{\partial x} + q_4 \sum_{i \in Q_2 \cup Q_3 \cup Q_4} \left(z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \right) \frac{\partial f}{\partial x}, \\
& + q_5 \sum_{i \in Q_1 \cup Q_3} \left(\frac{y}{x} \cdot z_i \overline{\frac{\partial f}{\partial y}}^{z_i} \right) \frac{\partial f}{\partial y} + q_6 \sum_{Q_2 \cup Q_4} \left(z_i \overline{\frac{\partial f}{\partial y}}^{z_i} \right) \frac{\partial f}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
\phi(x) \;=\; & x - q_1 \sum_{i \in Q_1 \cup Q_3} \left(\frac{y}{x} \cdot z_j \cdot \overline{\frac{\partial f}{\partial y}}^{z_j} \right) - q_2 \sum_{i \in Q_2 \cup Q_4} \left(z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \right) \\
& - q_3 \sum_{i \in Q_1 \cup Q_2} \left(\frac{x}{y} \cdot z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \right) - q_4 \sum_{i \in Q_3 \cup Q_4} \left(z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \right),
\end{aligned}
$$

$$
\phi(y) \;=\; y - q_5 \sum_{i \in Q_1 \cup Q_3} \left(\frac{y}{x} \cdot z_i \overline{\frac{\partial f}{\partial y}}^{z_i} \right) - q_6 \sum_{Q_2 \cup Q_4} \left(z_i \overline{\frac{\partial f}{\partial y}}^{z_i} \right),
$$

    $q_{i,j} \in \langle x, y \rangle$.

20:         let $q = \phi(f) - f + q$;

21:         replace f by $\mathrm{jet}(\phi(f), d'')$.

22:     $S := \mathrm{mon}(f - w(f)\text{-}\mathrm{jet}(f, d(f)))$.

23: normalize two terms of f of $w(f)$-degree $d(f)$, of degree $\leq d''$, and not in B to coefficient 1.

24: **if** $\Gamma(f)$ does not intersect the x-axis **then** $f = f + x^a$, where $a = \mu(f) + 2$.

25: **if** $\Gamma(f)$ does not intersect the y-axis **then** $f = f + y^a$, where $a := \mu(f) + 2$.

26: **return** F,f

(3): In this case the higher orders of the monomial expansion of $\phi_{z_j}(f)$ can be written as:

$$q_{1,3} \cdot \frac{y}{x} \cdot z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \cdot \frac{\partial f}{\partial y} + q_{2,3} \cdot z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \cdot \frac{\partial f}{\partial x} + q_{3,3} \cdot z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \cdot \frac{\partial f}{\partial x} + \text{terms in } J, \qquad (10)$$

(4): In this case the higher orders of the binomial expansion of $\phi_{z_j}(f)$ can be written as:

$$q_1 \cdot z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \cdot \frac{\partial f}{\partial y} + q_2 \cdot z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \cdot \frac{\partial f}{\partial x} + q_3 \cdot z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \cdot \frac{\partial f}{\partial x} + \text{terms in } J. \qquad (11)$$

Now, let $Q_k = \{i : z_i = \frac{a_i}{b_i} \text{ in lowest form , where } b_i \text{ is of case k}\}$. Then the higher orders of the binomial expansion of $\phi(f)$ of degree $\leq d'$ can be removed by the first order terms of the transformation:

$$
\begin{aligned}
\phi_{\text{new}}(x) &= x - \sum_{i \in Q_1 \cup Q_3} \left(q_{2,i} \cdot \frac{y}{x} \cdot z_j \cdot \overline{\frac{\partial f}{\partial y}}^{z_j} \right) - \sum_{i \in Q_2 \cup Q_4} \left(q_{2,i} \cdot z_j \overline{\frac{\partial f}{\partial y}}^{z_j} \right) \\
&\quad - \sum_{i \in Q_1 \cup Q_2} \left(q_{3,i} \cdot \frac{x}{y} \cdot z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \right) - \sum_{i \in Q_3 \cup Q_4} \left(q_{3,i} \cdot z_j \overline{\frac{\partial f}{\partial x}}^{z_j} \right), \\
\phi_{\text{new}}(y) &= y - \sum_{i \in Q_1 \cup Q_3} \left(q_{1,i} \cdot \frac{y}{x} \cdot z_i \overline{\frac{\partial f}{\partial y}}^{z_i} \right) - \sum_{Q_2 \cup Q_4} \left(q_{1,i} \cdot z_i \overline{\frac{\partial f}{\partial y}}^{z_i} \right).
\end{aligned}
$$

It can be proven, using similar methods, that the higher order terms of the binomial expansion of $\phi_{\text{new}}(\phi(f))$ can be written in terms of the exact same formulas as above. Moreover if a set $\{q_{i,j}\}$ was all set to zero, it can be done again.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In general transformations in line 10 can be chosen in such a way that the higher orders of the binomial expansion of $\phi(f)$ will be of degree higher than d'. Choosing the coefficients g and h in such a way, the While-loop from line 18 to 21 is adundant. In the next example we show that this is unfortunately not in general the case.

Example 8.7. Let $f = y^{28} + xy^7 + x^2y^3 + 11x^2y^4 + x^{22}$. Then $\frac{\partial f}{\partial x} = 2xy^3 + 22xy^4 + y^7 + 22x^{21}$ and $\frac{\partial f}{\partial y} = 3x^2y^2 + 44x^2y^3 + 7xy^6 + 28y^{27}$, and a regular basis for f is $x^{22}y, x^{21}y, x^{20}y, x^{19}y, x^{18}y, x^{17}y, x^{16}y, x^2y^3$. Note that x^2y^4 is the only monomial above the Newton Boundary, with $w(f)$-degree 1008. We can write $-11x^2y^4$ as follows in terms of the first partial derivitives.

$$-11x^2y^4 = -(7xy + 28y^{22})\frac{\partial f}{\partial x} + y^2\frac{\partial f}{\partial y}.$$

The corresponding transformation ϕ is given by

$$
\begin{aligned}
\phi(x) &= x - (7xy + 28y^{22}) = x - \frac{1}{y^5} \overline{\frac{\partial f}{\partial y}}^{\frac{1}{y^5}} + px \\
\phi(y) &= y + y^2 = y - \left(-\frac{1}{y^5} \overline{\frac{\partial f}{\partial x}}^{\frac{1}{y^5}} \right) + py,
\end{aligned}
$$

where $p_x \frac{\partial f}{\partial x}$ and $p_y \frac{\partial f}{\partial y}$ is of $w(f)$-degree 1008 or higher. Hence i has only the value 1 in the summands in line 14, and $z_1 = -\frac{1}{y^5}$. Note that in line 16 only $Q_2 = \{1\}$ is non-empty. Now let

$$q = w\text{-jet}(\phi(f) - f - \phi(x)\frac{\partial f}{\partial x} - \phi(y)\frac{\partial f}{\partial y}, 1008).$$

Then q is the contribution of the orders > 1 in the binomial expansion of ϕ in the 1008-jet of $\phi(f)$.

$$q = -28xy^9 + 182y^{30}$$

To write q as in line 19 in Algorithm 8, we are computing q_2, q_3 and q_6. We set $q_1 = 0, q_4 = 0$ and $q_6 = 0$. We do this by forming an ideal I generated by $g_2 = z_1 \overline{\frac{\partial f}{\partial y}}^{z_1} \cdot \frac{\partial f}{\partial x}, g_3 = \frac{x}{y} \cdot z_1 \overline{\frac{\partial f}{\partial x}}^{z_1} \cdot \frac{\partial f}{\partial x},$

$g_6 = z_1 \overline{\frac{\partial f}{\partial y}}^{z_1} \cdot \frac{\partial f}{\partial y}$ and all monomials of higher piecewise degree than 1008. We then write q in terms of g_2, g_3 and g_6, using a Gröbner basis of I with a global ordering, and then lifting the result back to the original generators using the command `liftstd()` in SINGULAR. By abuse of notation, let $f = \phi(f)$. We now form a new ϕ as described in line 19 of Algorithm 8. Note that

$$w\text{-}\mathrm{jet}((-q_6 \cdot z_1 \overline{\frac{\partial f}{\partial y}}^{z_1})\frac{\partial f}{\partial y} + (-q_2 \cdot z_1 \overline{\frac{\partial f}{\partial y}}^{z_1} - q_3\frac{x}{y} \cdot z_1 \overline{\frac{\partial f}{\partial x}}^{z_1})\frac{\partial f}{\partial x}, 1008) = q.$$

Hence the first order terms of the binomial expansion of $\phi(f)$ up to degree 1008 cancel q. In this case higher order terms of the binomial expansion of $\phi(f)$ does not have terms of $w(f)$-degree 1008 or lower. In fact $\phi(f) = y^{28} + xy^7 + x^2y^3 + x^{22}$.

Example 8.8. Let $f = x^2y^4 + x^4y^2 + x^{20} + y^{40} + 60x^{21}y^{14}$. Then $\frac{\partial f}{\partial x} = 4x^3y^2 + 2xy^4 + 20x^{19} + 1260x^{20}y^{14}$ and $\frac{\partial f}{\partial y} = 2x^4y + 4x^2y^3 + 840x^{21}y^{13} + 40y^{39}$, and a regular basis for f is $x^4y^4, x^4y^3, x^3y^4, x^4y^2, x^3y^3, x^2y^4$. Note that $x^{21}y^{14}$ is the only monomial above the Newton boundary with $w(f)$-degree 700. Now $60x^{21}y^{14}$ can be written as

$$60x^{21}y^{14} = (2x^4y^{12} + 4x^2y^{14} + 40y^{50} - 10x^{20}y^{10})\frac{\partial f}{\partial x} - (4x^3y^{13} + 2xy^{15})\frac{\partial f}{\partial y}$$

The corresponding transformation ϕ is given by

$$\phi(x) = x - (2x^4y^{12} + 4x^2y^{14} + 40y^{50} - 10x^{20}y^{10}) = x - y^{11}\overline{\frac{\partial f}{\partial y}}^{y^{11}} - 5x^{16}y^9\overline{\frac{\partial f}{\partial y}}^{x^{16}y^9} + p_x$$

$$\phi(y) = y + (4x^3y^{13} + 2xy^{15}) = y - (-y^{11}\overline{\frac{\partial f}{\partial x}}^{y^{11}} - 5x^{16}y^9\overline{\frac{\partial f}{\partial x}}^{x^{16}y^9}) + p_y,$$

where $p_x\frac{\partial f}{\partial x}$ and $p_y\frac{\partial f}{\partial y}$ is of w-degree 700 or higher. Hence i has value 1 and 2 in the summands in line 14, and $z_1 = \frac{a_1}{b_1} = y^{11}$ and $z_2 = \frac{a_2}{b_2} = 5x^{16}y^9$. Both Now let

$$q = w\text{-}\mathrm{jet}(\phi(f) - f - \phi(x)\frac{\partial f}{\partial x} - \phi(y)\frac{\partial f}{\partial y}, 700).$$

Then q is the contribution of the orders > 1 in the binomial expansion of ϕ in the 700-jet of $\phi(f)$.

$$\begin{aligned}
q = &-24x^{10}y^{26} - 12x^8y^{28} - 12x^6y^{30} - 24x^4y^{32} - 224x^7y^{44} - 32x^5y^{46} + 144x^6y^{60} + 12160x^6y^{64} \\
&+ 12640x^4y^{66} + 2800x^2y^{68} + 384x^7y^{74} + 952320x^7y^{78} + 472320x^5y^{80} \\
&+ 79680x^3y^{82} + 256x^8y^{88} + 11704320x^6y^{94} + 1467360x^4y^{96} + 9600x^2y^{102} + 1600y^{104} \\
&+ 21065216x^5y^{110} + 38400x^5y^{114} - 12800x^3y^{116} + 12800xy^{118} + 245661440x^6y^{124} \\
&+ 38400x^4y^{130} + 38400x^2y^{132} + 51200x^3y^{146} - 256000xy^{152} + 25600x^4y^{160} + 2560000y^{202}
\end{aligned}$$

Note that the order of q is 36.

To write q as in line (19) in Algorithm 8, we are computing q_2, q_4 and q_6. We do this by forming an ideal I generated by $g_2 = \left(\sum_{i=1,2} z_i \cdot \overline{\frac{\partial f}{\partial y}}^{z_i}\right) \cdot \frac{\partial f}{\partial y}$, $g_4 = \left(\sum_{i=1,2} z_i \cdot \overline{\frac{\partial f}{\partial x}}^{z_i}\right) \cdot \frac{\partial f}{\partial y}$, $g_6 = \left(\sum_{i=1,2} z_i \cdot \overline{\frac{\partial f}{\partial y}}^{z_i}\right) \frac{\partial f}{\partial x}$ and all monomials of higher piecewise degree than 700. We then write q in terms of g_2, g_4 and g_6, using a Groebner basis of I with a global ordering, and then lifting the result back to the original generators using the command `liftstd()`. We now form ϕ as described in line 19 of Algorithm 8. Note that

$$w\text{-}\mathrm{jet}\left(-q_6\left(\sum_{i=1,2} z_i \cdot \overline{\frac{\partial f}{\partial y}}^{z_i}\right)\frac{\partial f}{\partial x} - \left(q_2\left(\sum_{i=1,2} z_i \cdot \overline{\frac{\partial f}{\partial y}}^{z_i}\right) + q_4\left(\sum_{i=1,2} z_i \cdot \overline{\frac{\partial f}{\partial x}}^{z_i}\right)\right)\frac{\partial f}{\partial y}, 700\right) = q.$$

Hence the first order tems of the binomail expansion of $\phi(f)$ up to degree 700 cancel q. Unfortunately the higher order binomial expansions of $\phi(f)$ again contributes terms of w-degree ≤ 700.

Note that this time the w-order of these terms are $66 > 36$. Now let

$$q = w\text{-}\mathrm{jet}(\phi(f) - f - \phi(x)\frac{\partial f}{\partial x} - \phi(y)\frac{\partial f}{\partial y}, 700).$$

Then

$$
\begin{aligned}
q \;=\; & \frac{57088}{3}x^8y^{58} + 1008x^6y^{60} + \frac{484352}{9}x^9y^{72} + 1280x^7y^{74} - 18368x^8y^{88} - \frac{21816800}{3}x^6y^{94} \\
& -501760x^4y^{96} - \frac{2832270080}{9}x^7y^{108} - 17546240x^5y^{110} - 311015680x^6y^{124} \\
& -\frac{449953600}{3}x^4y^{130} - 13032000x^2y^{132} + \frac{28180582400}{9}x^5y^{144} - 674534400x^3y^{146} \\
& -19728697600x^4y^{160} - \frac{270208000}{3}x^2y^{166} - 5408000y^{168} + \frac{368310272000}{3}x^3y^180 \\
& -94208000xy^{182} + 33799296000x^2y^{196} + 36106240000xy^{216} + 13038080000y^{232}.
\end{aligned}
$$

Let $f = \phi(f)$. Repeating the process, we again write q in terms of g_2, g_4, and g_6 and all terms of piecewise degree higher than 700 to compute q_2, q_4 and q_6. Applying ϕ, formed as in line (19),

$$w\text{-}\mathrm{jet}(\phi(f), 700) = x^4y^2 + x^2y^4 + x^{20} + y^{40}.$$

We now proof Lemma 8.6:

For the proof we use the following observation:

Remark 8.9. With the notation of Lemma 8.6, let m be a monomial corresponding to a lattice point of Γ. Suppose g is a germ with non-degenerate Newton boundary $\Gamma(g) = \Gamma$ and with monomial m, such that the corresponding term cannot be removed from g by a right equivalence. Then Theorem 3.20 implies that m is a monomial of f_0 or $m \in B$.

We now prove Lemma 8.6:

Proof. We prove the first part of the lemma: It is sufficient to show that for any monomial m in the relative interior of a face Δ of Γ, there exists a germ g as in Remark 8.9. For any germ g with $\Gamma(g) = \Gamma$, we can write

$$\mathrm{jet}(g, \Delta) = x^a \cdot y^b \cdot g_1 \cdots g_n \cdot \widetilde{g}$$

where a,b are integers, $g_1, \ldots, g_n$ are linear homogeneous polynomials not associated to x or y, and $\widetilde{g}$ is a product of non-associated irreducible non-linear homogeneous polynomials. Note that a and b are the distances of Δ from the y- and x-coordinate axes. Note also that smooth faces meeting the coordinate axes do not contain interior lattice points. After possibly exchanging x and y, we may assume that for the weight w of Δ we have $w(x) \geq w(y)$. First consider the case that $w(x) > w(y)$:

(1.a) Suppose that $a \geq 2$. Let g be a germ non-degenerate Newton boundary Γ. Then any right-equivalence which does not act only as a rescaling of variables on $\mathrm{jet}(g, \Delta)$ generates terms on the coordinate axes below Γ, hence, changes the Newton polygon. The claim follows directly (choosing a non-degenerate $g = f_0 + c \cdot m$ with $c \in \mathbb{C}^*$).

(1.b) Consider now the case $a = 0$. Since Γ corresponds to a normalized germ with respect to Δ, we have $n = 0$ and $\widetilde{g} \neq 1$. This implies that $w(y) \nmid w(x)$, hence there does not exist a w-weighted homogeneous right-equivalence except rescaling of variables.

(1.c) Finally, suppose $a = 1$. If $w(y) \nmid w(x)$, then, as above, $n = 0$ and $\tilde{g} \neq 0$ which implies that rescalings are the only right-equivalences on the face. If $w(y) \mid w(x)$, then, writing $\tau = w(x)/w(y)$, any right-equivalence which does not create any terms of lower w-weight than that of Δ is of the form

$$x \mapsto c_1 x + c_2 y^\tau, \; y \mapsto c_3 y,$$

where $c_1, c_3 \in \mathbb{C}^*$ and $c_2 \in \mathbb{C}$, hence acts on y as a rescaling. We may therefore assume that $b = 0$. The vertices of Δ correspond to monomials of the form x^p and $xy^{(p-1)\cdot\tau}$ with $p \geq 2$. Any monomial in the interior of Δ is of the form $m = x^s y^{\tau\cdot(p-s)}$ with $0 < s < p$. For $g = f_0 + c \cdot m$ with $c \in \mathbb{C}^*$, the jet with regard to Δ is then $\mathrm{jet}(g, \Delta) = x^p + c \cdot m + xy^{(p-1)\cdot\tau}$.

We now show that there is no right-equivalence which keeps $\Gamma(g)$ and, hence, is of the above form, that removes m. Keeping the face Δ and removing m amounts to the conditions

$$\binom{p}{p-s} c_1^s c_2^{p-s} + c \cdot c_1^s c_3^{\tau \cdot (p-s)} = 0$$

$$c_2^p + c \cdot c_2^s c_3^{\tau \cdot (p-s)} + c_2 c_3^{(p-1) \cdot \tau} = 0$$

which correspond to the vanishing of the coefficients of m and $y^{p \cdot \tau}$. Using $c_1 \neq 0$ a solution of the first equation for c_2 is of the form

$$c_2 = \tilde{c} \cdot c_3^\tau$$

where $\tilde{c}^{p-s} = -c/\binom{p}{s}$. Inserting this into the second equation, leads to the equation

$$\begin{aligned}
0 &= \tilde{c}^p \cdot c_3^{\tau \cdot p} + c \cdot \tilde{c}^s c_3^{\tau \cdot p} + \tilde{c} \cdot c_3^{\tau \cdot p} \\
&= c_3^{\tau \cdot p} \cdot (\tilde{c}^p + c \cdot \tilde{c}^s + \tilde{c}) \\
&= c_3^{\tau \cdot p} \cdot \left(\tilde{c}^p - \binom{p}{s} \cdot \tilde{c}^p + \tilde{c} \right) \\
&= c_3^{\tau \cdot p} \cdot \tilde{c} \cdot \left(\left(1 - \binom{p}{s}\right) \cdot \tilde{c}^{p-1} + 1 \right)
\end{aligned}$$

Since $\binom{p}{s} \neq 1$, there is a Zariski open set of values of $\tilde{c}$, equivalently of c, such that the expression in the bracket does not vanish and g is non-degenerate. For such a choice of c and thus of g, it follows that $c_3 = 0$, a contradiction.

Now consider the case that $w(x) = w(y)$. Since $\mathrm{jet}(g, \Delta)$ is homogeneous, hence factorizes into linears, in this case we have $a \geq 1$, $b \geq 1$ and $\tilde{g} = 0$.

(1.d) If $a, b \geq 2$, and g is a germ with non-degenerate Newton boundary and Newton polygon Γ, then any right-equivalence which does not only act as a rescaling of variables on $\mathrm{jet}(g, \Delta)$ changes the Newton polygon (with the same argument as in the case $w(x) > w(y)$, $a \geq 2$).

(1.e) The case $a \geq 2$ and $b = 1$, can be handled in the same way as the case $w(x) > w(y)$, $a = 1$, $w(y) \mid w(x)$ above. An analoguos argument also applies to the case $b \geq 2$ and $a = 1$.

(1.f) In the case $a = b = 1$, a Gröbner basis calculation shows directly that any monomial corresponding to a lattice point of Δ is either a monomimal of f_0 or an element of B (note that smooth faces do not contain any interior lattice points): The germ $x^p y + x y^p$, $p \geq 2$, is right-equivalent to a germ h (applying for instance the right-equivalance $x \mapsto x + y$, $y \mapsto y + 2x$) with vertex monomials x^{p+1} and y^{p+1}, hence, by Theorem has Milnor number $\mu = (p+1)^2 - 2(p+1) + 1 = p^2$. The germ $g = x^p y + x y^p + x^{p^2+2} + y^{p^2+2}$ is right-equivalent to h, hence also has Milnor number $\mu = p^2$. This implies that $\langle x, y \rangle^{p^2+2} \subset \mathrm{Jac}(g)$. We determine a standard basis of

$$\mathrm{Jac}(g) = \langle g_x, g_y \rangle + \langle x, y \rangle^{p^2+2}$$

where

$$g_x = p \cdot x^{p-1} y + y^p + (p^2 + 2) \cdot x^{p^2+1}, \quad g_y = x^p + p \cdot x y^{p-1} + (p^2 + 2) \cdot y^{p^2+1}$$

with regard the local degree reverse lexicographic ordering. The S-polynomial of g_x and g_y leads to the standard basis element $x y^p$ after reducing the tail by $\langle x, y \rangle^{p^2+2}$. The S-polynomial of $x y^p$ with g_x leads to the standard basis element y^{2p-1} after reducing the tail by $\langle x, y \rangle^{p^2+2}$. The S-polynomial of $x y^p$ with g_y reduces to zero, while all remaining ones vanish. Hence, the classes of the monomials

$$y^{p+1}, x^2 y^{p-1}, \dots, x^{p-2} y^3,$$

form a basis of the Milnor algebra in degree $p + 1$. Using the relation g_x these monomials are equivalent to

$$x^2 y^{p-1}, \dots, x^{p-2} y^3, x^{p-1} y^2$$

which form a regular basis in degree $p + 1$. From the standard basis of the Jacobian it is clear that this is the only option for a regular basis, which proves the claim.

We now prove the second part of the lemma. Let $m_1, \ldots, m_n$ be the monomials of f_0 and let $g = \sum_i k_i m_i$. For a face Δ of Γ, we denote again its weight by w. In the first part of the proof we have seen that for all faces Δ, except those where $a = 1$, $w(x) \geq w(y)$, $w(y)|w(x)$, and $b \geq 2$ if $w(x) = w(y)$, or those where $b = 1$, $w(x) \leq w(y)$, $w(x)|w(y)$, and $b \geq 2$ if $w(x) = w(y)$, there does not exist a w-weighted homogeneous right-equivalence except rescaling of variables.[17] In order to describe in these two cases the w-homogeneous transformations on $\mathrm{jet}(g, \Delta)$ keeping the face, by symmetry, it is sufficient to consider the first of the two. Here, writing $\tau = w(x)/w(y)$, the jet is of the from $\mathrm{jet}(g, \Delta) = y^q \cdot \left(k_i x^p + k_j x y^{(p-1) \cdot \tau} \right)$, $k_i, k_j \neq 0$, $p \geq 2$. All homogeneous transformations are of the form $x \mapsto c_1 x + c_2 y^\tau$, $y \mapsto c_3 y$, and keeping Δ amounts to the condition

$$k_i c_2^p + k_j c_2 c_3^{(p-1) \cdot \tau} = 0,$$

which implies that either $c_2 = 0$ (which corresponds to a rescaling of variables), or that between c_2 and c_3 there is an algebraic relation $c_2 = k \cdot c_3^\tau$ (with $k^{p-1} = -k_j/k_i$).

We now show that there is a germ g with the same monomials as f_0 and non-degenerate Newton boundary, such that only two terms of g can be normalized to coefficient one by a right-equivalence keeping the Newton polygon. Since this g is right-equivalent to a germ in the normal form described in Theorem 3.20, this then implies that at most two monomials of f_0 are not in B.

In case there are only two monomials of f_0 of degree $\leq dt$, the claim is trivial choosing $g = f_0$.[18] Otherwise, let m_s, m_t and m_l be three distinct monomials of f_0 of degree $\leq dt$. We prove that there is a Zariski open set of germs g such that not all three monomials m_s, m_t, m_l can be normalized to coefficient one. To see this, it is enough to prove that, for any g with the same monomials as f_0, after choosing two monomials out of m_s, m_t, m_l and normalizing their coefficients to one, when restricting the action of the right-equivalence group to the Newton boundary and stabilizing the two normalized coefficients, the stabilizer acts as a finite group.

(2.a) If all three monomials lie on faces of Γ which permit only rescaling of variables, then the claim is obvious.

(2.b) If exactly two of the three monomials, say m_s, m_t, lie on faces of Γ which permit only rescaling of variables, then, without loss of generality, we may assume that m_l lies on a face with $a = 1$, $w(x) \geq w(y)$, $w(y)|w(x)$. If $w(x) = w(y)$, then it is sufficient to consider the case $b \geq 2$. After normalization of the coefficients of m_s and m_t to value one via rescaling of variables, stabilizing the two normalized coefficients together with the above condition $c_2 = 0$ or $c_2 = k \cdot c_3^\tau$ admits only finitely many solutions.

(2.c) Suppose that exactly one of the three monomials, say m_s, lies on a face of Γ which permits only rescaling of variables.

   If m_t and m_l lie on the same face Δ, we may assume that the jet of Δ is of the form $\mathrm{jet}(g, \Delta) = y^q \cdot \left(k_l x^p + k_t x y^{(p-1) \cdot \tau} \right)$. Then coefficients of the monomials m_s and m_l can only be changed via rescaling of variables. Then, similar to the previous case, normalization of the coefficients of m_s and m_l to value one together with the condition $c_2 = 0$ or $c_2 = k \cdot c_3^\tau$ admits only finitely many solutions.

   If m_t and m_l lie on different faces, we may assume that m_t lies on a face Δ with weight w and $a = 1$, $w(x) \geq w(y)$, $w(y)|w(x)$. Moreover, in case $w(x) = w(y)$, we can assume that $b \geq 2$, since the case $b = 1$ has already been discussed. Then $\mathrm{jet}(g, \Delta)$ is of the form

$$\mathrm{jet}(g, \Delta) = y^q \cdot \left(k_i x^p + k_j x y^{(p-1) \cdot \tau} \right),$$

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[17]Note that these cases correspond to the second part of (1.c) and to (1.e), which are the only settings where there may exist (and, in fact, exist) transformations on the jet of the respective face, which are not just rescalings of variables.

[18]Note that this includes the case where there is a face with $w(x) = w(y)$ and $a = b = 1$.

with $\tau = w(x)/w(y)$ and $t \in \{i, j\}$, and right-equivalences keeping the Newton polygon act on the jet as

$$x \mapsto c_1 x + c_2 y^\tau, \ y \mapsto c_3 y$$

satisfy the condition $c_2 = 0$ or $c_2 = k \cdot c_3^\tau$ with $k^{p-1} = -k_j/k_i$. Similarly, we may assume that m_l lies on a face Π with weight v with $b = 1$, $v(x) \le v(y)$, $v(x)|v(y)$. Again, in case $v(x) = v(y)$, we may assume that $a \ge 2$. Then $\text{jet}(g, \Pi)$ is of the form

$$\text{jet}(g, \Pi) = x^{q'} \cdot \left(k_r y^{p'} + k_s y x^{(p'-1) \cdot \tau'} \right),$$

with $\tau' = v(y)/v(x)$ and $l \in \{r, s\}$, and right-equivalences keeping the Newton polygon act on the jet as

$$x \mapsto c_1 x, \ y \mapsto c_3 y + c_2' x^{\tau'}$$

satisfying the condition $c_2' = 0$ or $c_2' = k' \cdot c_1^{\tau'}$ with $(k')^{p-1} = -k_s/k_r$. If m_t is the monomial of $\text{jet}(g, \Delta)$ of larger x-degree (that is, $m_t = y^q x^p$ and $t = i$), or if m_l is the monomial of $\text{jet}(g, \Pi)$ of larger y-degree (that is, $m_l = y^{q'} x^{p'}$ and $l = r$), then the coefficient of the respective monomial can only be changed via rescaling of variables, and we can argue as in previous case. Suppose now that m_t and m_l are the x-linear monomials of the respective jets. Normalizing the coefficients of m_t and m_l then amounts to the relations

$$c_1 c_3^q \cdot (k_t c_3^{(p-1)\cdot\tau} + k_i c_2^{p-1}) = 1$$
$$c_3 c_1^{q'} \cdot (k_l c_1^{(p'-1)\cdot\tau'} + k_r (c_2')^{p'-1}) = 1,$$

After inserting $c_2 = k \cdot c_3^\tau$ with $k^{p-1} = -k_t/k_i$ the first equation implies that $c_2 = 0$, a contradiction. Similarly, inserting $c_2' = k' \cdot c_1^{\tau'}$ with $(k')^{p-1} = -k_l/k_r$ into the second equation, yields a contradiction. Hence, $c_2 = c_2' = 0$, that is, right-equivalences keeping the Newton polygon act on the jets as rescaling of variables. So after normalizing the coeffcients of two monomials, the coefficient of the third one can take only finitely many values.

(2.d) If none of the three monomials m_s, m_t and m_l lies on a face of Γ which permits only rescaling of variables, then we may assume that m_s, m_t lie on a face with $a = 1$, $w(x) \ge w(y)$, $w(y)|w(x)$, and $b \ge 2$ if $w(x) = w(y)$, and m_l lies on a face with $b = 1$, $w(x) \le w(y)$, $w(x)|w(y)$, and $a \ge 2$ if $w(x) = w(y)$. We can thus argue as previous cases to obtain only finitely many solutions for the action of the right-equivalence group on the Newton boundary.

$\square$

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