FORMULA FOR THE DELTA INVARIANT (AND MILNOR NUMBER) FROM THE CHARACTERISTIC EXPONENTS IN THE PUISEUX EXPANSIONS

We give formulae to compute the delta invariant of singularities. The delta invariant is equal to the sum of the exponents in the denominator of an integral basis. (See for example Maple help for command IntegralBasis: `https://www.maplesoft.com/support/help/Maple/view.aspx?path=examples/algcurve&cid=889`).

For example, for $f = (x^3 - y^5) + y^6$, the integral basis is

$$\left\{1, y, y^2, \frac{y^3 - y^2}{x}, \frac{y^4 - y^3}{x}, \frac{y^5 - y^4}{x^2}\right\}$$

and hence the delta invariant at the origin is $1 + 1 + 2 = 4$.

We will show in next section how to compute the delta invariant for a branch. For the case of of singularities with more than one branch, we can compute the delta invariant by the formula

$$\delta(C) = \sum_i \delta(C_i) + \sum_{i<j} (C_i, C_j)$$

where (C_i, C_j) is the intersection multiplicity between branch C_i and branch C_j.

The intersection multiplicity can be computed from the Puiseux expansions using the formula in [2, Definition 3.29]:

$$(C_i, C_j) = \mathrm{ord}(f(\phi_1(t), \phi_2(t))$$

where $C_1 = V(f)$ with minimal $f \in \mathbb{C}\{x, y\}$ and C_2 is irreducible with good parametrization $\Phi = (\phi_1, \phi_2)$.

1. Formulae for the delta invariant

Let $f \in K[X][Y]$ monic with an irreducible singularity at the origin, that is, only one class of conjugated Puiseux expansions at the origin. We show how to compute the delta invariant of f at the origin.

1.1. **One characteristic exponent.**

Let $f = (Y - \gamma_1) \cdots (Y - \gamma_s)$. We assume first that f has only one characteristic exponent, hence the Puiseux expansions of f at the origin are

$$\gamma_i = \text{rational part} + \alpha_i X^{\frac{a}{s}} + \ldots, \quad 1 \le i \le s,$$

where $\alpha_i \ne \alpha_j$ for $i \ne j$.

For $\gamma \in K\{\{X\}\}$, we define the valuation

$$\upsilon : K\{\{X\}\} \setminus \{0\} \to \mathbb{Q}, \quad \upsilon(\gamma) = \mathrm{ord}_X(\gamma),$$

the smallest exponent appearing in a term of γ.

Hence, in our case,

$$\upsilon(\gamma_i - \gamma_j) = \frac{a}{s},$$

for $i \ne j$.

The numerator of the element of degree d in the integral basis is

$$p_d := \prod_{i=1}^{d} (Y - \overline{\gamma_i}^{<a/s}),$$

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where we use the notation $\overline{\gamma}^{<d}$ for truncation of all the terms of γ of order greater than or equal to d. The element p_d has order at $X = 0$ equal to

$$o_d = v_f(p_d) = \min_{1 \le i \le s} v_{\gamma_i}(b_d) = d\frac{a}{s}$$

and hence the denominator of the element of degree d in the integral basis is x^{e_d}, where

$$e_d = [o_d] = \left[d\frac{a}{s}\right].$$

The delta invariant is therefore

$$\delta(f) = \sum_{d=0}^{s-1} e_d = \sum_{d=0}^{s-1} \left[d\frac{a}{s}\right].$$

1.2. **Two characteristic exponents.** We consider now the case of two characteristic exponents in the expansions of f. The Puiseux expansions of f at the origin are

$$\gamma_i = \text{rational part} + \alpha_i X^{\frac{a_1}{s_1}} + \cdots + \beta_i X^{\frac{a_2}{s_2}} + \ldots, \quad 1 \le i \le s_2 = s.$$

For example, $\gamma_i = X + \alpha_i X^{3/2} + c_i X^{5/2} + \beta_i X^{13/4}$.

The numerators in the integral basis are the product of two different factors (see [1, Algorithm 6]):

$$f_0 = Y - \overline{\gamma_i}^{<a_1/s_1}) = Y - \text{rational part}$$
$$f_1 = \prod_{i=1}^{s_1} (Y - \overline{\gamma_i}^{<a_2/s_2})$$

Since f_1 and f_2 are elements in $K[X][Y]$, the valuation at an expansions of f is independent of the expansion chosen. We obtain

$$v_{\gamma_i}(f_0) = \frac{a_1}{s_1},$$
$$v_{\gamma_i}(f_1) = (s_1 - 1)\frac{a_1}{s_1} + \frac{a_2}{s_2} = a_1 + \left(\frac{a_2}{s_2} - \frac{a_1}{s_1}\right).$$

For a given d, the numerator in the integral basis is

$$p_d := f_1^{[d/s_1]} f_0^{d-[d/s_1]\cdot s_1}$$

and hence the exponent of the numerator is

$$e_d = [v_{\gamma_i}(p_d)] = \left[d\frac{a_1}{s_1} + \left[\frac{d}{s_1}\right]\left(\frac{a_2}{s_2} - \frac{a_1}{s_1}\right)\right]$$

and the delta invariant is $\sum_{d=0}^{s-1} e_d$.

1.3. **Any number of characteristic exponents.** Let the characteristic exponents of f be

$$\frac{a_1}{s_1}, \frac{a_2}{s_2}, \ldots, \frac{a_m}{s_m}$$

where $s_i \mid s_{i+1}$.

Now the factors appearing in the numerators of the integral basis are

$$\begin{aligned} f_0 &= Y - \overline{\gamma_i}^{<a_1/s_1} = Y - \text{rational part}, \\ f_1 &= \prod_{i=1}^{s_1} (Y - \overline{\gamma_i}^{<a_2/s_2}), \\ &\vdots \\ f_{m-1} &= \prod_{i=1}^{s_{m-1}} (Y - \overline{\gamma_i}^{<a_m/s_m}), \end{aligned}$$

where we always assume that the truncations are all different hence conjugated and hence the factors are all over the ground field.

The numerator of degree d in the integral basis is

$$p_d = f_{m-1}^{u_{m-1}} f_{m-2}^{u_{m-2}} \cdots f_0^{u_0}$$

where

$$u_{m-1} = [d/s_{m-1}], \quad u_{m-2} = \left[\frac{d - u_{m-1}}{s_{m-2}}\right], \quad \ldots, \quad u_i = \left[\frac{d - u_{i+1}}{s_i}\right], \quad \ldots$$

Noting that for a given expansion γ_i, for $j \geq 1$, in the factor f_k there will be exactly one expansion γ_j such that $\upsilon_{\gamma_i}(\gamma_j) = \frac{a_k}{s_k}$, we obtain that the element p_d has order

$$o_d = d \cdot \frac{a_1}{s_1} + \left[\frac{d}{s_1}\right]\left(\frac{a_2}{s_2} - \frac{a_1}{s_1}\right) + \left[\frac{d}{s_2}\right]\left(\frac{a_3}{s_3} - \frac{a_2}{s_2}\right) + \cdots + \left[\frac{d}{s_{m-1}}\right]\left(\frac{a_m}{s_m} - \frac{a_{m-1}}{s_{m-1}}\right)$$

and the delta invariant is

$$\delta(f) = \sum_{d=0}^{s-1} e_d = \sum_{d=0}^{s-1} [o_d].$$

References

[1] Janko Boehm, Wolfram Decker, Santiago Laplagne, and Gerhard Pfister. Computing integral bases via localization and hensel lifting. *Journal of Symbolic Computation*, 109:283–324, 2022.

[2] Daniel Matei. Topology of plane algebraic curves, 2007.