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Part II Algebraic Curves

2

Computing with Plane Algebraic Curves and Riemann Surfaces: The Algorithms of the Maple Package “Algcurves”

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2.1 Introduction

In this chapter, we present an overview of different algorithms for computing with compact connected Riemann surfaces, obtained from desingularized and compactified plane algebraic curves. As mentioned in Chap. 1 [Bob11], all compact connected Riemann surfaces may be represented this way. The Maple package “algcurves”, largely developed by the authors and Mark van Hoeij contains implementations of these algorithms. A few recent additions to the “algcurves” package are not due to the authors or Mark van Hoeij. The algorithm behind those commands are not discussed here as they have no bearing on anything associated with Riemann surfaces.

Because some of the algorithms presented here are algebraic in nature, they rely on exact arithmetic, which implies that the coefficients of the algebraic curves are required to have an exact representation. Most importantly, floating point numbers are not allowed as coefficients for these algorithms. The reason for not allowing floating point numbers is that the geometry of the Riemann surface is highly dependent on the accuracy of the coefficients in its algebraic curve representation. If an algebraic curve has singularities, then, almost surely, the nature of these singularities will be affected by inaccuracies in the coefficients of the curve. This may affect the algebraic algorithms discussed below, such as those for the calculation of the genus, homology and holomorphic 1-forms on the Riemann surface. Users of the “algcurves” package can consider floating point coefficients, but these need to be converted to a different form (rational, for instance), before the programs will accept the input. Furthermore, since we are using algebraic curves to represent Riemann surfaces, the algebraic curves are always considered over the

A.I. Bobenko and C. Klein (eds.), *Computational Approach to Riemann Surfaces*, Lecture Notes in Mathematics 2013, DOI 10.1007/978-3-642-17413-1_2,

complex numbers. Throughout this chapter, “calculation” is used when exact results are obtained, whereas “computation” is used for numerical results.

Apart from the restriction to an exact representation, all of the algorithms discussed in detail in this chapter are general in the sense that they apply to all compact connected Riemann surfaces. An appendix is presented discussing the use of a few algorithms that apply to a restricted class of algebraic curves and Riemann surfaces, such as elliptic and hyperelliptic surfaces. This appendix contains many examples, but no detailed explanation of the specifics of the algorithms.

All of the descriptions of the algorithms of the main body of this chapter are preceded by the next section which outlines the connection between plane algebraic curves and Riemann surfaces with a level of detail appropriate for what follows. The work reviewed here may be found in [DvH01, DP07a, Pat07, vH95, vH94]. The examples of the implementations use commands available in Maple 11 (Released Spring 2007). A few commands are used that are not available in Maple 11 yet. They are identified as such in the text.

2.2 Relationship Between Plane Algebraic Curves and Riemann Surfaces

In this section, some required background from the theory of Riemann surfaces is introduced. More details can be found in such standard references as [FK92, Spr57]. Excellent places to read up on Riemann surfaces and how they relate to plane algebraic curves that do not require an extensive background are the monographs by Brieskorn and Knörrer [BK86] and Griffiths [Gri89].

Consider a plane algebraic curve, defined over the complex numbers $\mathbb{C}$, i.e., consider the subset of $\mathbb{C}^2$ consisting of all points (x, y) satisfying a polynomial relation in two variables x and y with complex coefficients:

$$f(x,y) = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \ldots + a_1(x)y + a_0(x) = 0 \ . \tag{2.1}$$

Here $a_j(x)$, $j = 0, \ldots, n$ are polynomials in x. Write $a_j(x) = \sum_i a_{ij}x^i$, where the coefficients a_{ij} are complex numbers. Assuming $a_n(x)$ does not vanish identically, n is the degree of $f(x, y)$ considered as a polynomial in y. We only consider irreducible algebraic curves, so $f(x, y)$ cannot be written as the product of two non-constant polynomials with complex coefficients.

Let d denote the degree of $f(x, y)$ as a polynomial in x and y, i.e., d is the largest $i + j$ for which the coefficient a_{ij} of x^iy^j in $f(x, y)$ is non-zero. The behavior at infinity for both x and y is examined by homogenizing $f(x, y) = 0$ by letting

$$x = X/Z \ , \quad y = Y/Z \ , \tag{2.2}$$

and investigating

$$F(X, Y, Z) = Z^d f(X/Z, Y/Z) = 0 \ . \tag{2.3}$$

Here $F(X, Y, Z)$ is a homogeneous polynomial equation of degree d. Finite points $(x, y) \in \mathbb{C}^2$ on the algebraic curve correspond to triples $(X : Y : Z)$ with $Z \neq 0$. Since for these points $(X : Y : Z) = (X/Z : Y/Z : 1)$, we may equate $Z = 1$, so finite points can be denoted unambiguously by (x, y) instead of $(X/Z, Y/Z)$. Points at infinity correspond to triples $(X : Y : Z)$, with $Z = 0$. Hence, at a point at infinity, at least one of the two coordinate functions x or y is infinite. Because $F(X, Y, 0)$ is a homogeneous polynomial of degree d, there are at most d points at infinity.

The algebraic curve can have singular points. An algorithm to efficiently calculate the singular points of an algebraic curve is discussed in Sect. 2.5. Here we briefly discuss singularities as they need to be dealt with in order to construct the Riemann surface from an algebraic curve. Finite singular points on the algebraic curve specified by $f(x, y) = 0$ satisfy $f(x, y) = 0 = \partial_x f(x, y) = \partial_y f(x, y)$. Points at infinity can also be singular. Singular points at infinity satisfy $\partial_X F(X, Y, Z) = \partial_Y F(X, Y, Z) = \partial_Z F(X, Y, Z) = 0$ (then also $F(X, Y, Z) = 0$, by Euler's theorem for homogeneous functions). Desingularizing the algebraic curve results in a Riemann surface, i.e., a one-dimensional complex-analytic manifold (so it is two-dimensional over the real numbers; it is a surface). There are various ways of desingularizing algebraic curves. Our methods use Puiseux series, as detailed in Sect. 2.3. Each nonsingular point on the algebraic curve corresponds to one place[1] on the Riemann surface, whereas a singular point on the algebraic curve can correspond to multiple places on the Riemann surface.

In what follows, Γ is used to denote the Riemann surface obtained by desingularizing and compactifying (by adding the places at infinity) the algebraic curve represented by $f(x, y) = 0$. All Riemann surfaces obtained this way are connected (because $f(x, y)$ is irreducible) and compact (because the points at infinity are included). Conversely, as stated in Chap. 1 it is known that every compact connected Riemann surface can be obtained as described above [BBE$^+$94, Spr57]. From here on out, all Riemann surfaces considered are understood to be connected and compact. We use $\hat{\Gamma}$ to denote the compactified, not desingularized algebraic curve.

2.3 Puiseux Series

Many of the algorithms presented in the next sections make use of local coordinates on an algebraic curve. For our algorithms, this local behavior is understood using Puiseux expansions. These expansions allow us to distinguish between regular points, branch points and singular points. Further, in addition to determine the nature of singular points, Puiseux expansions characterize

[1] We use the term "point" to denote a value in the complex x-plane. On the other hand, "place" is used to denote a location on the Riemann surface Γ, or, unambiguously, a location on the desingularized plane algebraic curve.

the topology of an algebraic curve near branch points. One way to look at this is that Puiseux series are our way to desingularize the algebraic curves we are working with. There are other ways of doing so, all with their advantages and disadvantages. A popular alternative is the use of quadratic transformations to "lift" the singular plane algebraic curve to a higher-dimensional nonsingular curve [Abh90].

Newton's Theorem, which we summarize below, completely describes the local behavior of a plane algebraic curve. Over the neighborhoods of a regular point $x = \alpha$ the coordinate function y is given locally as a series in ascending powers of $x - \alpha$ [Bli66]. Near a branch point however, y is given necessarily by a series with ascending fractional powers in $x - \alpha$. Such series are known as *Puiseux series*. It is common to choose a local parameter, say t, such that x and y are written as Laurent series in that local parameter.[2] That is, if α is a branch point of order r, then $t^r = x - \alpha$, and y is written as a Laurent series in t. A pair $(x(t), y(t))$ is referred to as a Puiseux expansion as it is equivalent to a Puiseux series.

2.3.1 Newton's Theorem

In a lift of the neighborhood of $x = \alpha$, the n y-roots of (2.1) are determined by a number $\leq n$ of pairs of expansions of the form

$$P_j = \left(x = \alpha + t^{r_j}, y = \beta_j t^{s_j} + \beta_j' t^{s_j'} + \cdots \right) \tag{2.4}$$

with $r_j, s_j, s_j', \ldots \in \mathbb{Z}$, $t \in \mathbb{C}$, $\alpha, \beta_j, \beta_j', \ldots \in \mathbb{C}$. Here $|r_j|$ is the number (the *branching number*) of y-roots that merge at place P_j. If $|r_j| > 1$ for one of the P_j, then α is a branch point and P_j is a branch place. For $|t| > 0$, a place P_j represents $|r_j|$ distinct y-values and $\sum_j |r_j| = n$. The coefficients $\beta, \beta', \ldots$ are all non zero; only a finite number of the integer exponents $s_j < s_j' < \ldots$ are negative; and $|r_j|$ is coprime with at least one of the $s_j, s_j', \ldots$. For places over $x = \infty$, one has $\alpha = 0$ and $r_j < 0$, i.e., $x = 1/t^{-r_j}$. A Puiseux expansion evaluated at $t = 0$ is called a *center*.

The algorithm to compute Puiseux expansions implemented in the "algcurves" package is essentially that described by Newton in letters to Oldenburg and Leibniz [BK86]. A treatment of the algorithm with much detail may be found, for instance, in [Wal62]. The paper by van Hoeij [vH94] discusses a modern implementation of the algorithm, most importantly providing a method allowing one to determine how many terms of the expansions are required to guarantee distinct branches are recognized as such.

[2] For finite points, only Taylor series in the local parameter are required. Laurent series with a finite number of singular terms are necessary to encompass points at infinity.

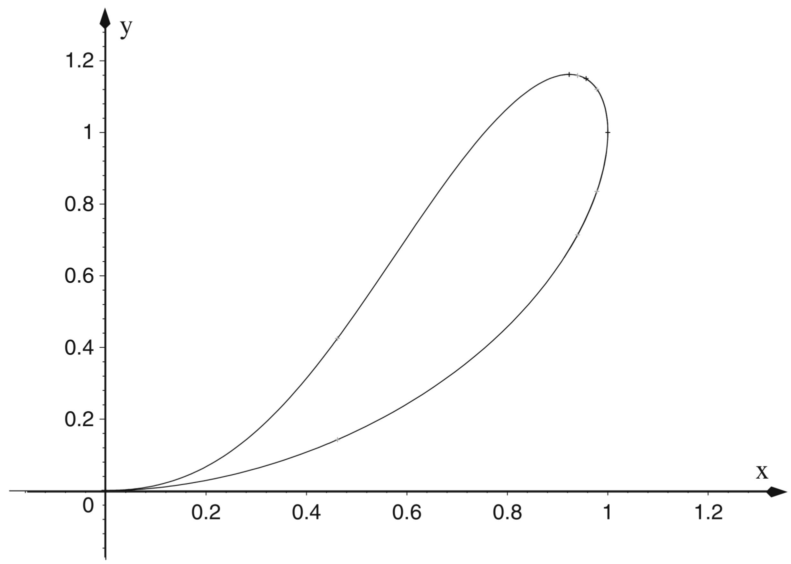

Fig. 2.1. The real graph of the Ramphoid cusp curve

Example. Consider the curve

$$f(x, y) = (x^2 - x + 1)y^2 - 2x^2y + x^4 = 0\ . \tag{2.5}$$

Note that for this example $a_n(x) = x^2 - x + 1$ is not constant, a fact complicating the calculation of Puiseux expansions [Bli66]. Nevertheless, such computations are still possible. The curve (2.5) is known as a "ramphoid cusp" because of the structure of the real part of the graph at the origin [Wal62], as shown in Fig. 2.1. The graph was produced using the command `plot_real_curve` of the "algcurves" package. We compute the local structure of this curve at two different x-values. First we compute the expansions over $x = 0$, which is not a zero of a_n, and second, the expansions over one of the roots of $a_n(x) = 0$.

The command `puiseux` used below computes the y-expansions of $f(x, y) = 0$ over $x = 0$. To give zero as the fourth argument of this command implies that the procedure calculates as many terms as are necessary to distinguish separate expansions.

```
># read in the package
>with(algcurves):
># define the algebraic curve
>f:=x^4+x^2*y^2-2*x^2*y-y^2*x+y^2:
># compute the Puiseux expansions over x=0
>puiseux(f,x=0,y,0);
```

$$\left\{x^2 + x^{5/2}\right\}$$

Note that $x = 0$ is a branch point by virtue of the fractional power in the second term. This one expansion represents two distinct places for $|x| > 0$. To recover both roots near $x = 0$ one 'conjugates' the series using the r-th roots of unity, where r is the greatest common denominator of the exponents of the series. Thus, one makes the r substitutions $x \mapsto e^{2\pi ij/r}x$, $j = 1, \ldots, r$. In this case, the $r = 2$ different y-roots are

$$y = \left\{x^2 + x^{5/2} + \cdots, x^2 - x^{5/2} + \cdots\right\}.$$

If, instead of 0, the fourth argument is specified to be a positive integer value M, then the expansions are computed up to x^M. For example, if $M = 4$, then the $x^{7/2}$ term is included, but terms of order x^4 and higher are not.

```
>puiseux(f,x=0,y,4);
```

$$\left\{x^2 + x^{5/2} + x^3 + \frac{1}{2}x^{7/2}\right\}$$

```
>puiseux(f,x=0,y,5);
```

$$\left\{x^2 + x^{5/2} + x^3 + \frac{1}{2}x^{7/2} - \frac{5}{8}x^{9/2}\right\}$$

Including a fifth argument t in the call to `puiseux` changes the output. In this case the output is a pair of expansions in the local parameter t.

```
>f:=x^4+x^2*y^2-2*x^2*y-y^2*x+y^2:
># compute the Puiseux expansions using local coordinate t
>puiseux(f,x=0,y,0,t);
```

$$\{[x = t^2, y = t^4 + t^5]\}$$

2.4 Integral Basis

We start by explaining the concept of an integral basis. Integral bases are used in all aspects of algebra, but they have become particularly important with the rise of computer algebra systems, as their construction often allows convenient calculation of many other derived quantities. An example of this in our chapter is the use of the integral basis for the calculation of the holomorphic 1-forms on a Riemann surface, see Sect. 2.9. Another example is their use in the algorithmic integration in finite terms of algebraic functions.

Consider the coordinate functions x and y on the Riemann surface Γ. These two functions are algebraically dependent, by the defining equation $f(x, y) = 0$. Denote by $A(\Gamma)$ the part of the Riemann surface where both x