AN ALGORITHM DETERMINING THE CONDUCTOR OF AN IRREDUCIBLE CURVE THAT CAN BE REPRESENTED BY A GERM WITH A NON-DEGENERATE NEWTON BOUNDARY

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Algorithm 1 Algorithm to determine the local conductor of a irreducible curve that is right-equivalent to a germ with a non-degenerate Newton boundary

Input: A polynomial $f \in \mathbb{K}[x, y]$ in $\langle x, y\rangle^2$ that is equivalent to a germ in $\mathbb{K}[[x, y]]$ with a non-degenerate Newton boundary
Output: The conductor $\mathcal{C}(f) = \mathcal{C}_{A_P}$, where $A = \mathbb{K}[x, y]/f$ and $P = \langle x, y\rangle$.
1: find polynomial right-equivalences $\phi_0, \dots, \phi_r$ such that $(\phi_r \circ \cdots \circ \phi_0)(f)$, has a non-degenerate Newton boundary, using Algorithm **??**.
2: compute an upper bound D for the determinacy of f, using Theorem A.9.6 **?**.
3: $D_0 := D$.
4: let $\tilde{f} := \operatorname{jet}((\phi_r \circ \cdots \circ \phi_1)(f), D_0) \in \mathbb{K}[x, y]$.
5: determine a reduced Gröbner basis $\{g_1, \dots, g_n\}$ of the conductor $\mathcal{C}(\tilde{f})$ of $\tilde{f}$.
6: let d be the degree of the lowest jet of f.
7: $b = D_0 - d$.
8: let c be the smallest order of any element of the generating system of $\mathcal{C}(\tilde{f})$.
9: let j be minimal such that $\langle x, y\rangle^j \subset \mathcal{C}(\tilde{f})$.
10: **if** $c + b - 1 < j$ **then**
11:     $D_0 := j - c + d + 1$.
12:     $\tilde{f} := \operatorname{jet}((\phi_r \circ \cdots \circ \phi_1)(f), D_0)$.
13:     determine a reduced Gröbner basis $\{g_1, \dots, g_n\}$ of the conductor $\mathcal{C}(\tilde{f})$ of $\tilde{f}$.
14: delete all elements of degree j in $\{g_1, \dots, g_n\}$.
15: **for** $i = r \dots 0$ **do**
16:     for each g_l, set $g_l := \operatorname{jet}(\phi_i^{-1}(g_l), j)$.
17: **return** $\langle g_1, \dots, g_n\rangle + \langle x, y\rangle^j$

Proof. In line 8 the smallest order of any element in the conductor will not change in line 13 working with $\tilde{f}$ instead of f, since there is and automorphism between the conductors. □

Proof. (Asssuming that the Newton boundary of $\tilde{f}$ is non-degenerate): Since the conductor contains a power of x we may take that power as the denominator. Then the corresponding ideal U contains that power of x, but also a power of y, by a Puiseux series argument (take the largest slope of a face of the Newton polygon and see where it cuts the x-axis). It follows that U contains a power of the maximal ideal. □

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Algorithm 2

Input: A polynomial $f \in \mathbb{K}[x,y]$ in $\langle x,y\rangle^2$ that is equivalent to a germ in $\mathbb{K}[[x,y]]$ with a non-degenerate Newton boundary

Output: Generators of $\overline{A}_P$ as an A_p-module, where $A = \mathbb{K}[x,y]/f$ and $P = \langle x,y\rangle$.

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using Algorithm ??, find polynomial right-equivalences φ_0, ..., φ_r such that (φ_r ∘ ··· ∘ φ_0)(f)
  has a non-degenerate (convenient) Newton boundary, φ_0 is a linear transformation, and for
  each i = 1, ..., r either φ_i(x) = x or φ_i(y) = y.
compute an upper bound D for the determinacy of f, using Theorem A.9.6 ?.
D_0 := D.
let f̃ := jet((φ_r ∘ ··· ∘ φ_1)(f), D_0) ∈ K[x,y], and B = K[x,y]/f̃.
determine a reduced Gröbner basis {g_1, ..., g_n} of an ideal U ⊂ B with √U = ⟨x,y⟩, and an
  element d ∈ U such that B̄_P = U_P/d.
let d be the degree of the lowest jet of f.
b = D_0 − d.
let c be the smallest order of any element of the generating system of U.
let j be minimal such that ⟨x,y⟩^j ⊂ U.
if c + b − 1 < j then
    D_0 := j − c + d + 1.
    f̃ := jet((φ_r ∘ ··· ∘ φ_1)(f), D_0).
    determine an ideal U ⊂ B with √U = ⟨x,y⟩, and an element d ∈ U such that B̄_P = U_P/d.
d_0 := d
for i = r ... 1 do
    if φ_i(x) = x then
        set z := x.
    else
        set z := y.
    compute the minimal h such that z^h ∈ C_B.
    find an ideal U' ⊂ B with √U' = ⟨x,y⟩, such that U_P/d = U'_P/z^h, using Algorithm ??.
    compute a reduced Gröbner basis of {g_1, ..., g_n} of U' in B_P.
    let j be minimal such that ⟨x,y⟩^j ⊂ U'.
    delete all elements of degree j in {g_1, ..., g_n}.
    for each g_l, set g_l := jet(φ_i^{-1}(g_l), j).
return ⟨g_1, ..., g_n⟩ + ⟨x,y⟩^j
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