"C:\Program Files\Maple 2015" (1)

```
> ################################################################################
# Load "Rational SOS" procedures
################################################################################
read("rationalSOS.mpl");
with(rationalSOS);

# Display tables of any size
interface(rtablesize = infinity);
```

"Opening connection with Matlab"

rationalSOS := **module**( ) ... **end module**

[cancelDenominator, decompositionToMatrix, evalMat, evalSolution, exactSOS, getDiag, getExtension, getVars, matrixToPoly, nonRatCoef, numericSolver, numericSolverSubmatrix, polyToMatrix, primitiveMatrix, randomRank, reduceByLinearEquation, reduceByLinearEquationLinear, roundMat, roundVec, vectorTrace, zeroDetSRows, zeroRows]

10 (2)

```
> ################################################################################
# Construction of the example
################################################################################

# We define a polynomial z as the sum of three squares in an algebraic
# extension of degree 3 with generic coefficients.

mp := t^3-2;
p1 := c1*t^2 + b1*t + a1;
p2 := c2*t^2 + b2*t + a2;
p3 := c3*t^2 + b3*t + a3;

fGeneric := p1^2 + p2^2 + p3^2;
fGeneric := expand(fGeneric);

# We choose the parameters so that all terms are multiple by x, and we can reduce the degree.
b2 := -b1; c2 := b2; c1 := b2;
b1 := x0; b3 := x1; a3 := x2; c3 := x3;

# We solve the coefficients a1 and a2 so that the polynomial is in Q,
f2 := NormalForm(fGeneric, [mp], plex(a1, a2, x0, x1, x2, x3, t));
f3 := collect(f2, t);
lf := CoefficientList(f3, t);
ss := solve({lf[2], lf[3]}, {a1, a2});

# We plug in the solutions found for a1 and a2 and compute the resulting polynomial
ssDen := denom(rhs(ss[1]));
p1s := simplify(subs(ss, p1)*ssDen);
p2s := simplify(subs(ss, p2)*ssDen);
```

```
p3s := simplify(subs(ss, p3) * ssDen);

p1ss := subs( {t=RootOf(x^3-2)}, p1s);
p2ss := subs( {t=RootOf(x^3-2)}, p2s);
p3ss := subs( {t=RootOf(x^3-2)}, p3s);

f := simplify( p1ss^2 + p2ss^2 + p3ss^2 );

   # f := 40*x0^4+8*x0^2*x1^2+32*x0^2*x1*x2+64*x0^2*x1*x3+16*x0^2*x2^2+16*x0^2*
   x2*x3+32*x0^2*x3^2+2*x1^4+8*x1^2*x2^2+8*x1^2*x2*x3+16*x1*x2*x3^2+8*x2^2*
   x3^2+8*x3^4
```

$$mp := t^3 - 2$$

$$p1 := c1\, t^2 + b1\, t + a1$$

$$p2 := c2\, t^2 + b2\, t + a2$$

$$p3 := c3\, t^2 + b3\, t + a3$$

$$fGeneric := \left(c1\, t^2 + b1\, t + a1\right)^2 + \left(c2\, t^2 + b2\, t + a2\right)^2 + \left(c3\, t^2 + b3\, t + a3\right)^2$$

$$fGeneric := c1^2\, t^4 + c2^2\, t^4 + c3^2\, t^4 + 2\, b1\, c1\, t^3 + 2\, b2\, c2\, t^3 + 2\, b3\, c3\, t^3 + 2\, a1\, c1\, t^2 + 2\, a2\, c2\, t^2 + 2\, a3\, c3\, t^2 + b1^2\, t^2 + b2^2\, t^2 + b3^2\, t^2 + 2\, a1\, b1\, t + 2\, a2\, b2\, t + 2\, a3\, b3\, t + a1^2 + a2^2 + a3^2$$

$$b2 := -b1$$

$$c2 := -b1$$

$$c1 := -b1$$

$$b1 := x0$$

$$b3 := x1$$

$$a3 := x2$$

$$c3 := x3$$

$$f2 := -2\, a1\, t^2\, x0 - 2\, a2\, t^2\, x0 + 2\, t^2\, x0^2 + t^2\, x1^2 + 2\, t^2\, x2\, x3 + 2\, a1\, t\, x0 - 2\, a2\, t\, x0 + 4\, t\, x0^2 + 2\, t\, x1\, x2 + 2\, t\, x3^2 + a1^2 + a2^2 + 4\, x1\, x3 + x2^2$$

$$f3 := \left(-2\, a1\, x0 - 2\, a2\, x0 + 2\, x0^2 + x1^2 + 2\, x2\, x3\right) t^2 + \left(2\, a1\, x0 - 2\, a2\, x0 + 4\, x0^2 + 2\, x1\, x2 + 2\, x3^2\right) t + a1^2 + a2^2 + 4\, x1\, x3 + x2^2$$

$$lf := \left[a1^2 + a2^2 + 4\, x1\, x3 + x2^2,\ 2\, a1\, x0 - 2\, a2\, x0 + 4\, x0^2 + 2\, x1\, x2 + 2\, x3^2,\ -2\, a1\, x0 - 2\, a2\, x0 + 2\, x0^2 + x1^2 + 2\, x2\, x3\right]$$

$$ss := \left\{a1 = -\frac{1}{4}\, \frac{2\, x0^2 - x1^2 + 2\, x1\, x2 - 2\, x2\, x3 + 2\, x3^2}{x0},\ a2 = \frac{1}{4}\, \frac{6\, x0^2 + x1^2 + 2\, x1\, x2 + 2\, x2\, x3 + 2\, x3^2}{x0}\right\}$$

$$ssDen := 4\, x0$$

$$p1s := -4\, t^2\, x0^2 + 4\, t\, x0^2 - 2\, x0^2 + x1^2 - 2\, x1\, x2 + 2\, x2\, x3 - 2\, x3^2$$

$$p2s := -4\, t^2\, x0^2 - 4\, t\, x0^2 + 6\, x0^2 + x1^2 + 2\, x1\, x2 + 2\, x2\, x3 + 2\, x3^2$$

$$p3s := 4\, \left(t^2\, x3 + t\, x1 + x2\right) x0$$

$$p1ss := -4\,x0^2\,\mathit{RootOf}(\_Z^3-2)^2+4\,x0^2\,\mathit{RootOf}(\_Z^3-2)-2\,x0^2+x1^2-2\,x1\,x2+2\,x2\,x3-2\,x3^2$$

$$p2ss := -4\,x0^2\,\mathit{RootOf}(\_Z^3-2)^2-4\,x0^2\,\mathit{RootOf}(\_Z^3-2)+6\,x0^2+x1^2+2\,x1\,x2+2\,x2\,x3+2\,x3^2$$

$$p3ss := 4\left(\mathit{RootOf}(\_Z^3-2)^2\,x3+\mathit{RootOf}(\_Z^3-2)\,x1+x2\right)x0$$

$$f := 40\,x0^4+8\,x0^2\,x1^2+32\,x0^2\,x1\,x2+64\,x0^2\,x1\,x3+16\,x0^2\,x2^2+16\,x0^2\,x2\,x3+32\,x0^2\,x3^2+2\,x1^4+8\,x1^2\,x2^2+8\,x1^2\,x2\,x3+16\,x1\,x2\,x3^2+8\,x2^2\,x3^2+8\,x3^4 \quad (3)$$

```
> # We verify that the polynomial is absolutely irreducible
  evala(AIrreduc(f));
  ## -> true

  # Matrix Q associated to the problem (parametrization of the space L)
  Q, QVars, v := polyToMatrix(f):

  # Matrix associated to the original decomposition (for verifications)
  MNEW := decompositionToMatrix([p1ss, p2ss, p3ss], v):


  # We start from Q and go step by step.
  nops(indets(Q)); # 20
  randomRank(Q);   # 10
```

$$true$$
$$20$$
$$10 \quad (4)$$

```
> # Real solutions
  sSym := solve({p1ss = 0, p2ss = 0, p3ss = 0});
  # Alternative:
  #sSym := solve({f=0, diff(f, x0)=0, diff(f,x1)=0, diff(f,x2)=0, diff(f,x3)=0});
```

$$sSym := \Bigg\{x0 = x0,\ x1 = -\frac{x0\left(\mathit{RootOf}(3\,\_Z^4-4+(-4\,\mathit{RootOf}(\_Z^3-2)+4)\,\_Z^2)^2+2\right)}{\mathit{RootOf}(\_Z^3-2)^2\,\mathit{RootOf}(3\,\_Z^4-4+(-4\,\mathit{RootOf}(\_Z^3-2)+4)\,\_Z^2)},\ x2 = -\left(x0\left(\mathit{RootOf}(\_Z^3-2)^3\,\mathit{RootOf}(3\,\_Z^4-4+(-4\,\mathit{RootOf}(\_Z^3-2)+4)\,\_Z^2)^2-\mathit{RootOf}(3\,\_Z^4-4+(-4\,\mathit{RootOf}(\_Z^3-2)+4)\,\_Z^2)^2-2\right)\right)\Big/\left(\mathit{RootOf}(\_Z^3-2)\,\mathit{RootOf}(3\,\_Z^4-4+(-4\,\mathit{RootOf}(\_Z^3-2)+4)\,\_Z^2)\right),\ x3 = \mathit{RootOf}(3\,\_Z^4-4+(-4\,\mathit{RootOf}(\_Z^3-2)+4)\,\_Z^2)\,x0\Bigg\},\ \Bigg\{x0 = 0,\ x1 = \quad (5)$$

$$- \frac{2\,x2\,\left(RootOf(\_Z^3+2)^2 - RootOf(\_Z^3+2)+1\right)}{RootOf(\_Z^3+2)^2 - RootOf(\_Z^3+2)+2}, x2=x2, x3=RootOf(\_Z^3+2)\,x2\Big\}$$

$, \{x0=0, x1=0, x2=x2, x3=0\}$

```
> # We replace the RootOf(Z^3+2) by -RootOf(Z^3-2) in the second branch,
  # so that we have all solutions in terms of that root
  alias(r1=RootOf(_Z^3-2));
  s2 := {x0=0, x1 = -2*x2*((-r1)^2-(-r1)+1)/((-r1)^2-(-r1)+2), x2=x2, x3=(
      -r1)*x2};
```

$$r1$$

$$s2 := \left\{x0=0, x1 = -\frac{2\,x2\,(r1^2+r1+1)}{r1^2+r1+2}, x2=x2, x3=-r1\,x2\right\} \tag{6}$$

```
> ## Plain reduction

  ## sSym[3] plain equations - reduction to 14 variables and rank 9
  v1 := eval(Vector(v), sSym[3]) :
  v11 := eval(v1, {x2=1}) :
  Q1 := reduceByLinearEquation(Q, v11) :
  nops(indets(Q1)); # 14
  randomRank(Q1); # 9
```

$$14$$

$$9 \tag{7}$$

```
> ## sSym[2] plain equations - reduction to 9 variables and rank 8
  v2 := eval(Vector(v), s2) :
  v21 := eval(v2, {x2=1}) :
  Q2 := reduceByLinearEquation(Q1, v21) :
  nops(indets(Q2)); # 9
  randomRank(Q2); #8
```

$$9$$

$$8 \tag{8}$$

```
> ## sSym[3] plain equations - reduction to 5 variables and rank 7
  v3 := eval(Vector(v), sSym[1]) :
  v31 := eval(v3, {x0=1}) :
  Q3 := reduceByLinearEquation(Q2, v31) :
  nops(indets(Q3)); # 5
  randomRank(Q3); # 7
```

$$5$$

$$7 \tag{9}$$

```
> ## sSym[3]b plain equations - reduction to 2 variables and rank 6
  # The first branch contain two real points. Here we take the second real point.
  sS3b := eval(sSym[1], {x0=1}) :
  v3b := eval(Vector(v), sS3b) :
```

```
v3b := eval(v3b, {x0=-1}) :
simplify(LinearAlgebra[Transpose](v3b).MNEW.v3b); # Verification
Q4 := reduceByLinearEquation(Q3, v3b) :
nops(indets(Q4)); # 2
randomRank(Q4); # 6
```

$$0$$
$$2$$
$$6 \tag{10}$$

```
> # Using the real points, we reduce to only two parameters, but we still don't get the uniqueness of
     the Gram matrix.

  # We now look for ghost solutions


     # We try to find some principal submatrix Q4S3 of Q4 for which there is a vector v with real
     coordiantes such that vt . Q4S3 . v = 0.
  # This will imply that also vt . Q4 . v = 0 (filling v with zeros in the other entries).
  # and hence Q4 . v = 0, a new linear restriction.

  vv := [3, 4, 7]; # The indexes are chosen so that the system we obtain has real solutions
  # I have tried other triplets and this was the first one I found containing real points.

  Q4S3 := SubMatrix(Q4, vv, vv) :
  indets(Q4S3); # {a_0[4, 6], a_0[7, 9]}
  vx := Vector([z0, z1, z2]);
  vxt := LinearAlgebra[Transpose](vx);
  ff := expand(vxt . Q4S3 . vx);
```

$$vv := [3, 4, 7]$$
$$\{a\_0_{3,5}, a\_0_{7,9}\}$$
$$vx := \begin{bmatrix} z0 \\ z1 \\ z2 \end{bmatrix}$$
$$vxt := \begin{bmatrix} z0 & z1 & z2 \end{bmatrix}$$

$$\begin{aligned} ff := {} & -2\, z1^2\, a\_0_{7,9} + 32\, z1^2\, r1 + 16\, z0^2 + 12\, z0\, z1\, \mathrm{RootOf}(3\, \_Z^4 - 4 + (-4\, r1 \\ & + 4)\, \_Z^2)\, a\_0_{3,5} + 2\, z0\, z1\, r1^2\, a\_0_{7,9} - 6\, z0\, z1\, a\_0_{7,9}\, r1 - 2\, z0\, z2\, r1^2\, a\_0_{3,5} \\ & + z0\, z2\, r1\, a\_0_{3,5} - 3\, z1^2\, r1^5\, \mathrm{RootOf}(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)\, a\_0_{3,5} \\ & + \frac{3}{2}\, z1^2\, r1^3\, \mathrm{RootOf}(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2\, a\_0_{7,9} + 6\, z1^2\, \mathrm{RootOf}(3\, \_Z^4 - 4 + (\\ & -4\, r1 + 4)\, \_Z^2)\, r1^2\, a\_0_{3,5} - 4\, z1\, z2\, r1\, a\_0_{3,5} + 3\, z1\, z2\, \mathrm{RootOf}(3\, \_Z^4 - 4 + (-4\, r1 \\ & + 4)\, \_Z^2)^2\, a\_0_{3,5} - 3\, z0\, z1\, a\_0_{7,9}\, r1\, \mathrm{RootOf}(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2 \end{aligned} \tag{11}$$

$$- 6\, z0\, z1\, r1^3\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)\, a\_0_{3,5} + \frac{3}{2}\, z0\, z2\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2\, r1\, a\_0_{3,5} + z1^2\, r1^3\, a\_0_{7,9} + z1^2\, r1^4\, a\_0_{7,9} - 3\, z1^2\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2\, a\_0_{7,9} - 4\, z1^2\, a\_0_{7,9}\, r1 + 2\, z1\, z2\, a\_0_{3,5} + z2^2\, a\_0_{7,9}\, r1 + 32\, z0\, z1\, r1^2$$

```
> # We must find the values of (z0, z1, z2) for which ff is 0 for any choice of the
  # two parameters {a_0[4, 6], a_0[7, 9]} of the matrix Q4.
  # For this, we compute the coefficient of each of the two parameters and the independent term.

  fco := coeffs(ff, indets(Q4));
  # We get a sytem of the polynomial equations in (z0, z1, z2) and we solve it using command solve.
  newS := solve(Equate([fco], [0, 0, 0]));
```

$$fco := 32\, z1^2\, r1 + 16\, z0^2 + 32\, z0\, z1\, r1^2,\ -2\, z1^2 + 2\, z0\, z1\, r1^2 - 6\, z0\, z1\, r1 + \frac{3}{2}\, z1^2\, r1^3\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2 - 3\, z0\, z1\, r1\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2 + z1^2\, r1^3 + z1^2\, r1^4 - 3\, z1^2\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2 - 4\, z1^2\, r1 + z2^2\, r1,\ 12\, z0\, z1\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2) - 2\, z0\, z2\, r1^2 + z0\, z2\, r1 - 3\, z1^2\, r1^5\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2) + 6\, z1^2\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)\, r1^2 - 4\, z1\, z2\, r1 + 3\, z1\, z2\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2 - 6\, z0\, z1\, r1^3\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2) + \frac{3}{2}\, z0\, z2\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2\, r1 + 2\, z1\, z2$$

$$newS := \Big\{ z0 = -\frac{27}{64}\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^{10}\, z1 - \frac{135}{64}\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^8\, z1 - \frac{9}{4}\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^6\, z1 + \frac{35}{8}\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^4\, z1 + \frac{19}{4}\, z1\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2 - \frac{7}{4}\, z1,\ z1 = z1,\ z2 = \frac{1}{8}\, RootOf\big(-384 + 384\, r1^2 + 192\, r1^2\, RootOf(3\, \_Z^4 - 4 + (-4\, r1 + 4)\, \_Z^2)^2 + \_Z^2\big)\, z1 \Big\} \tag{12}$$

```
> # The system has unique solution (up to conjugacy) and one of the conjugated points is real.
  # We can use it as a new restriction.

  v10 := [0, 0, z0, z1, 0, 0, z2, 0, 0, 0]:
  v10s := eval(v10, newS) :
  v10s1 := eval(v10s, {z1 = 1}) :
  v10sv := Vector(v10s1) :
```

> *# We verify that this vector contains real points*
> *evalf(allvalues(v10sv));*

$$\left[\begin{array}{c} 0. \\ 0. \\ -1.587401044 \\ 1. \\ 0. \\ 0. \\ 3.148030321\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ -1.587401044 \\ 1. \\ 0. \\ 0. \\ -3.148030321\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ -1.587401052 \\ 1. \\ 0. \\ 0. \\ 1.100402874 \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ -1.587401052 \\ 1. \\ 0. \\ 0. \\ -1.100402874 \\ 0. \\ 0. \\ 0. \end{array}\right], \tag{13}$$

$$\left[\begin{array}{c} 0. \\ 0. \\ -1.587401044 \\ 1. \\ 0. \\ 0. \\ 3.148030321\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ -1.587401044 \\ 1. \\ 0. \\ 0. \\ -3.148030321\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ -1.587401052 \\ 1. \\ 0. \\ 0. \\ 1.100402874 \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ -1.587401052 \\ 1. \\ 0. \\ 0. \\ -1.100402874 \\ 0. \\ 0. \\ 0. \end{array}\right],$$

$$\left[\begin{array}{c} 0. \\ 0. \\ 0.793700525 + 1.374729637\,\mathrm{I} \\ 1. \\ 0. \\ 0. \\ 3.608288995 + 1.447928726\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ 0.793700525 + 1.374729637\,\mathrm{I} \\ 1. \\ 0. \\ 0. \\ -3.608288995 - 1.447928726\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right],$$

$$\left[\begin{array}{c} 0. \\ 0. \\ 0.793700532 + 1.374729686\,\mathrm{I} \\ 1. \\ 0. \\ 0. \\ 0.3318133293 + 0.8268904100\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ 0.793700532 + 1.374729686\,\mathrm{I} \\ 1. \\ 0. \\ 0. \\ -0.3318133293 - 0.8268904100\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right],$$

$$\left[\begin{array}{c} 0. \\ 0. \\ 0.793700525 + 1.374729637\,\mathrm{I} \\ 1. \\ 0. \\ 0. \\ 3.608288995 + 1.447928726\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ 0.793700525 + 1.374729637\,\mathrm{I} \\ 1. \\ 0. \\ 0. \\ -3.608288995 - 1.447928726\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right],$$

$$\left[\begin{array}{c} 0. \\ 0. \\ 0.793700532 + 1.374729686\,\mathrm{I} \\ 1. \\ 0. \\ 0. \\ 0.3318133293 + 0.8268904100\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ 0.793700532 + 1.374729686\,\mathrm{I} \\ 1. \\ 0. \\ 0. \\ -0.3318133293 - 0.8268904100\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right],$$

$$\begin{bmatrix} 0. \\ 0. \\ 0.793700525 - 1.374729637\,I \\ 1. \\ 0. \\ 0. \\ 3.608288995 - 1.447928726\,I \\ 0. \\ 0. \\ 0. \end{bmatrix}, \begin{bmatrix} 0. \\ 0. \\ 0.793700525 - 1.374729637\,I \\ 1. \\ 0. \\ 0. \\ -3.608288995 + 1.447928726\,I \\ 0. \\ 0. \\ 0. \end{bmatrix},$$

$$\begin{bmatrix} 0. \\ 0. \\ 0.793700532 - 1.374729686\,I \\ 1. \\ 0. \\ 0. \\ 0.3318133293 - 0.8268904100\,I \\ 0. \\ 0. \\ 0. \end{bmatrix}, \begin{bmatrix} 0. \\ 0. \\ 0.793700532 - 1.374729686\,I \\ 1. \\ 0. \\ 0. \\ -0.3318133293 + 0.8268904100\,I \\ 0. \\ 0. \\ 0. \end{bmatrix},$$

$$\begin{bmatrix} 0. \\ 0. \\ 0.793700525 - 1.374729637\,I \\ 1. \\ 0. \\ 0. \\ 3.608288995 - 1.447928726\,I \\ 0. \\ 0. \\ 0. \end{bmatrix}, \begin{bmatrix} 0. \\ 0. \\ 0.793700525 - 1.374729637\,I \\ 1. \\ 0. \\ 0. \\ -3.608288995 + 1.447928726\,I \\ 0. \\ 0. \\ 0. \end{bmatrix},$$

$$\left[\begin{array}{c} 0. \\ 0. \\ 0.793700532 - 1.374729686\,\mathrm{I} \\ 1. \\ 0. \\ 0. \\ 0.3318133293 - 0.8268904100\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right], \left[\begin{array}{c} 0. \\ 0. \\ 0.793700532 - 1.374729686\,\mathrm{I} \\ 1. \\ 0. \\ 0. \\ -0.3318133293 + 0.8268904100\,\mathrm{I} \\ 0. \\ 0. \\ 0. \end{array}\right]$$

```
> # We verify that indeed vt . Q4 . v = 0
  v10svt := LinearAlgebra[Transpose](v10sv) :
  ff := expand(v10svt . Q4 . v10sv) :
  simplify(ff);
  # 0 -> We found a new vector that must be in the kernel
```

$$0 \tag{14}$$

```
> # We add this restriction
  Q5 := reduceByLinearEquation(Q4, v10sv) :
  nops(indets(Q5)); # 0
  randomRank(Q5); # 3
```

$$0$$
$$3 \tag{15}$$

```
> # We proved unique solution!

  Q5;   # The matrix found
  MNEW; # The original matrix corresponding to the real SOS decomposition

  # The matrix obtained corresponds to the real SOS decomposition.
```

$[[40, 0, 0, 0, -8\,r1^2 + 4, -16\,r1 + 16, 0, 0, -16\,r1^2 + 8, -16\,r1 + 16],$
$[0, 16\,r1^2, 16\,r1, 32, 0, 0, 0, 0, 0, 0],$
$[0, 16\,r1, 16, 16\,r1^2, 0, 0, 0, 0, 0, 0],$
$[0, 32, 16\,r1^2, 32\,r1, 0, 0, 0, 0, 0, 0],$
$[-8\,r1^2 + 4, 0, 0, 0, 2, 0, 0, 0, 4, 0],$
$[-16\,r1 + 16, 0, 0, 0, 0, 8, 0, 0, 0, 8],$
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0],$
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0],$
$[-16\,r1^2 + 8, 0, 0, 0, 4, 0, 0, 0, 8, 0],$

$[-16\, r1 + 16, 0, 0, 0, 0, 8, 0, 0, 0, 8]]$

$$[[40, 0, 0, 0, -8\, r1^2 + 4, -16\, r1 + 16, 0, 0, -16\, r1^2 + 8, -16\, r1 + 16], \tag{16}$$

$[0, 16\, r1^2, 16\, r1, 32, 0, 0, 0, 0, 0, 0],$

$[0, 16\, r1, 16, 16\, r1^2, 0, 0, 0, 0, 0, 0],$

$[0, 32, 16\, r1^2, 32\, r1, 0, 0, 0, 0, 0, 0],$

$[-8\, r1^2 + 4, 0, 0, 0, 2, 0, 0, 0, 4, 0],$

$[-16\, r1 + 16, 0, 0, 0, 0, 8, 0, 0, 0, 8],$

$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0],$

$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0],$

$[-16\, r1^2 + 8, 0, 0, 0, 4, 0, 0, 0, 8, 0],$

$[-16\, r1 + 16, 0, 0, 0, 0, 8, 0, 0, 0, 8]]$

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