"C:\Program Files\Maple 2015" **(1)**

```
> currentdir("C:/Users/slapl/Dropbox/repos/rationalSOS");
```

"C:\Users\slapl\Dropbox\Repos\rationalsos" **(2)**

```
> ##########################################################################
  # Load "Rational SOS" procedures
  ##########################################################################
  read("rationalSOS.mpl");
  with(rationalSOS);
```

"Opening connection with Matlab"

rationalSOS := **module**( ) ... **end module**

[*cancelDenominator*, *decompositionToMatrix*, *evalMat*, *evalSolution*, *exactSOS*, *getCoeffs*, *getDiag*, *getExtension*, *getVars*, *homogenize*, *isHomogeneous*, *linIndepRows*, *listSubsets*, *matrixToPoly*, *minorsDet*, *nonRatCoef*, *numericSolver*, *numericSolverSubmatrix*, *numericSolverSubmatrixMaxRank*, *numericSolverSubmatrixRoundBefore*, *polyToMatrix*, *polyToMatrixVars*, *primitiveMatrix*, *randomRank*, *reduceByLinearEquation*, *reduceByLinearEquationLinear*, *roundMat*, *roundMatToZero*, *roundToIntMatrix*, *roundVec*, *sedumiCall*, *smallToZero*, *solveSubmatrixGeneral*, *vectorTrace*, *zeroDetSRows*, *zeroDetSys*, *zeroRows*] **(3)**

```
> # Display tables of any size
  interface(rtablesize=infinity);
```

10 **(4)**

```
> ##########################################################################
  ## Example 5.7
  ## Example in the border with unique solution
  ##########################################################################

  # The 4 even polynomials from Reznick paper
  p1 := x*((2-1/2)*x^2-(y^2+z^2+w^2));
  p2 := y*((2-1/2)*y^2-(x^2+z^2+w^2));
  p3 := z*((2-1/2)*z^2-(x^2+y^2+w^2));
  p4 := w*((2-1/2)*w^2-(x^2+y^2+z^2));
```

$$p1 := x\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)$$

$$p2 := y\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)$$

$$p3 := z\left(\frac{3}{2}z^2 - w^2 - x^2 - y^2\right)$$

$$p4 := w\left(\frac{3}{2}w^2 - x^2 - y^2 - z^2\right) \quad \textbf{(5)}$$

```
> # f is the sum of squares of p1, ..., p4
```

$f := p1^{\wedge}2 + p2^{\wedge}2 + p3^{\wedge}2 + p4^{\wedge}2;$
$f := expand(f);$

$$f := \left(\frac{3}{2}x^3 - w^2 x - x y^2 - x z^2\right)^2 + \left(\frac{3}{2}y^3 - w^2 y - x^2 y - y z^2\right)^2 + \left(\frac{3}{2}z^3 - w^2 z - x^2 z - y^2 z\right)^2 + \left(\frac{3}{2}w^3 - w x^2 - w y^2 - w z^2\right)^2$$

$$f := -2x^4w^2 - 2x^4y^2 - 2x^4z^2 - 2x^2w^4 - 2x^2y^4 - 2x^2z^4 - 2y^4w^2 - 2y^4z^2 - 2y^2w^4 - 2y^2z^4 - 2z^4w^2 - 2z^2w^4 + 6x^2w^2z^2 + 6y^2w^2z^2 + 6x^2y^2z^2 + 6x^2w^2y^2 + \frac{9}{4}x^6 + \frac{9}{4}y^6 + \frac{9}{4}z^6 + \frac{9}{4}w^6 \quad (6)$$

\> *# We use SEDUMI to compute a SOS decomposition.*
 # We do not perform facial reduction, since we are interested in the
 # solutions of maximum rank.
$out := exactSOS(f, facial = \text{"no"}):$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 20, " - Number of indeterminates: ", 126

"Matrix after facial reduction - Rank: ", 20, " - Number of indeterminates: ", 126

"Check 1 of random rank:", 20

"Check 2 of random rank:", 20

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 127, order n = 21, dim = 401, blocks = 2
nnz(A) = 272 + 0, nnz(ADA) = 16129, nnz(L) = 8128
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            2.26E+01 0.000
  1 :  -4.50E+00 5.63E+00 0.000 0.2487 0.9000 0.9000   0.40  1
1  1.2E+01
  2 :  -1.06E+00 1.37E+00 0.000 0.2428 0.9000 0.9000   2.55  1
1  1.6E+00
  3 :  -1.62E-01 3.53E-01 0.000 0.2580 0.9000 0.9000   2.26  1
1  4.1E-01
  4 :  -3.46E-03 8.76E-03 0.000 0.0249 0.9900 0.9900   1.43  1
1  1.7E-01
  5 :  -7.80E-06 4.43E-05 0.000 0.0051 0.9990 0.9990   1.01  1
1  2.9E-03
  6 :  -2.57E-06 8.77E-06 0.000 0.1977 0.9000 0.9037   1.00  1
1  5.9E-04
  7 :  -1.57E-07 7.61E-07 0.344 0.0868 0.9900 0.9900   1.00  1
1  5.1E-05
  8 :  -1.28E-08 3.87E-08 0.000 0.0508 0.9900 0.9904   1.00  1
1  1.8E-06
  9 :  -1.87E-11 5.37E-11 0.000 0.0014 0.9990 0.9990   1.00  1
```

```
1  4.4E-09
 10 :  -3.97E-12 8.87E-12 0.102 0.1650 0.9173 0.9000   1.00  1
1  1.2E-09
 11 :  -3.63E-13 9.03E-13 0.000 0.1018 0.9450 0.9464   1.00  1
1  1.1E-10
 12 :  -1.52E-14 4.00E-14 0.000 0.0443 0.9900 0.9905   1.00  1
2  3.1E-12
 13 :  -3.01E-15 6.11E-15 0.000 0.1527 0.9144 0.9000   1.00  2
2  8.0E-13
 14 :  -1.34E-15 1.96E-15 0.000 0.3204 0.9161 0.9000   1.00  3
3  2.8E-13
 15 :  -7.44E-16 4.87E-16 0.000 0.2487 0.9156 0.9000   1.00  2
3  7.8E-14
 16 :  -5.54E-16 9.76E-17 0.000 0.2005 0.9047 0.9000   0.98  3
3  1.6E-14
 17 :  -5.13E-16 2.26E-17 0.000 0.2314 0.9000 0.9161   1.07  3
3  3.4E-15
 18 :  -5.08E-16 5.05E-18 0.000 0.2234 0.9000 0.9206   1.08  3
3  9.5E-16
 19 :  -5.07E-16 3.04E-18 0.000 0.6027 0.9000 0.9000   0.74  3
3  5.8E-16
 20 :  -5.07E-16 2.49E-18 0.000 0.8202 0.9000 0.9000   0.81  3
3  4.4E-16
 21 :  -5.07E-16 1.33E-18 0.000 0.5319 0.9000 0.9000   1.18  3
3  2.2E-16
 22 :  -5.07E-16 8.20E-19 0.000 0.6178 0.9000 0.9000   1.14  3
3  1.4E-16
Run into numerical problems.

iter seconds digits       c*x               b*y
 22      0.3  10.2  9.5845136075e-15 -5.0671088489e-16
|Ax-b| =   2.4e-14, [Ay-c]_+ =   1.2E-14, |x|=  3.9e-01, |y|=
5.6e+00

Detailed timing (sec)
   Pre           IPM           Post
1.600E-02    1.280E-01    2.997E-03
Max-norms: ||b||=1, ||c|| = 6,
Cholesky |add|=0, |skip| = 0, ||L.L|| = 6.51825.
```

"An exact positive definite solution could not be found for the reduced problem."

"matrixToPoly begins..."

"Computing decomposition..."

"Decomposition computed!" **(7)**

```
> # out[3] is a matrix in the spectrahedron of maximum rank.
  # We check the eigenvalues to determine the rank
  eig(out[3]);
```

$$
\begin{bmatrix}
-5.01801722339590\ 10^{-16} \\
-4.85945915609490\ 10^{-16} \\
-2.22044604925031\ 10^{-16} \\
-9.29621348680002\ 10^{-17} \\
-7.39745345304484\ 10^{-17} \\
-6.23544661899498\ 10^{-17} \\
-1.26618689106921\ 10^{-17} \\
1.23872053991529\ 10^{-32} \\
6.09035442735944\ 10^{-32} \\
2.44557887167503\ 10^{-18} \\
2.29234663963519\ 10^{-17} \\
4.01703980437330\ 10^{-17} \\
8.77952357951753\ 10^{-17} \\
2.21581403321475\ 10^{-16} \\
3.45021786970384\ 10^{-16} \\
5.48030462973794\ 10^{-16} \\
5.25000000000000 \\
5.25000000000000 \\
5.25000000000000 \\
5.25000000000000
\end{bmatrix} \tag{8}
$$

```
> # We compare the matrix obtained by SEDUMI with the matrix corresponding
  # to the original decomposition p1^2+p2^2+p3^2+p4^2.
  v := convert(out[5], list);  # The monomials indexing the columns of the Gram Matrix
  A1 := decompositionToMatrix([p1, p2, p3, p4], v);
  A2 := out[3];
  Norm(A1 - A2);
```

$v := [w^3, w^2 x, w^2 y, w^2 z, w x^2, w x y, w x z, w y^2, w y z, w z^2, x^3, x^2 y, x^2 z, x y^2, x y z, x z^2, y^3, y^2 z, y z^2, z^3]$

$$\frac{3}{2} x^3 - w^2 x - x y^2 - x z^2$$

$[w^3, w^2 x, w^2 y, w^2 z, w x^2, w x y, w x z, w y^2, w y z, w z^2, x^3, x^2 y, x^2 z, x y^2, x y z, x z^2, y^3, y^2 z, y z^2, z^3]$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$11$$

$$2$$

$$14$$

$$16$$

$$\frac{3}{2} y^3 - w^2 y - x^2 y - y z^2$$

$[w^3, w^2 x, w^2 y, w^2 z, w x^2, w x y, w x z, w y^2, w y z, w z^2, x^3, x^2 y, x^2 z, x y^2, x y z, x z^2, y^3, y^2 z, y z^2, z^3]$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$17$$

$$3$$

$$12$$

$$19$$

$$\frac{3}{2} z^3 - w^2 z - x^2 z - y^2 z$$

$$[w^3, w^2 x, w^2 y, w^2 z, w x^2, w x y, w x z, w y^2, w y z, w z^2, x^3, x^2 y, x^2 z, x y^2, x y z, x z^2, y^3, y^2 z, y z^2, z^3]$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$20$$

$$4$$

$$13$$

$$18$$

$$\frac{3}{2}\, w^3 - w\, x^2 - w\, y^2 - w\, z^2$$

$$[w^3, w^2\, x, w^2\, y, w^2\, z, w\, x^2, w\, x\, y, w\, x\, z, w\, y^2, w\, y\, z, w\, z^2, x^3, x^2\, y, x^2\, z, x\, y^2, x\, y\, z, x\, z^2, y^3, y^2\, z, y\, z^2, z^3]$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1$$

$$5$$

$$8$$

$$10$$

$$A1 := \left[\left[\frac{9}{4}, 0, 0, 0, -\frac{3}{2}, 0, 0, -\frac{3}{2}, 0, -\frac{3}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right], \right.$$

$$\left[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}, 0, 0, 1, 0, 1, 0, 0, 0, 0 \right],$$

$$\left[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -\frac{3}{2}, 0, 1, 0 \right],$$

$$\left[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, -\frac{3}{2} \right],$$

$$\left[-\frac{3}{2}, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right],$$

$$\left[-\frac{3}{2}, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[-\frac{3}{2}, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, -\frac{3}{2}, 0, 0, 0, 0, 0, 0, 0, 0, \frac{9}{4}, 0, 0, -\frac{3}{2}, 0, -\frac{3}{2}, 0, 0, 0, 0\right],$$

$$\left[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -\frac{3}{2}, 0, 1, 0\right],$$

$$\left[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, -\frac{3}{2}\right],$$

$$\left[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}, 0, 0, 1, 0, 1, 0, 0, 0, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}, 0, 0, 1, 0, 1, 0, 0, 0, 0\right],$$

$$\left[0, 0, -\frac{3}{2}, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}, 0, 0, 0, 0, \frac{9}{4}, 0, -\frac{3}{2}, 0\right],$$

$$\left[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, -\frac{3}{2}\right],$$

$$\left[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -\frac{3}{2}, 0, 1, 0\right],$$

$$\left[0, 0, 0, -\frac{3}{2}, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}, 0, 0, 0, 0, -\frac{3}{2}, 0, \frac{9}{4}\right]\Big]$$

$$A2 := \Big[\left[\frac{9}{4}, 0, 0, 0, -\frac{3}{2}, 0, 0, -\frac{3}{2}, 0, -\frac{3}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}, 0, 0, 1, 0, 1, 0, 0, 0, 0\right],$$

$$\left[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -\frac{3}{2}, 0, 1, 0\right],$$

$$\left[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, -\frac{3}{2}\right],$$

$$\left[-\frac{3}{2}, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[-\frac{3}{2}, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[-\frac{3}{2}, 0, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$
$$\left[0, -\frac{3}{2}, 0, 0, 0, 0, 0, 0, 0, 0, \frac{9}{4}, 0, 0, -\frac{3}{2}, 0, -\frac{3}{2}, 0, 0, 0, 0\right],$$
$$\left[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -\frac{3}{2}, 0, 1, 0\right],$$
$$\left[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, -\frac{3}{2}\right],$$
$$\left[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}, 0, 0, 1, 0, 1, 0, 0, 0, 0\right],$$
$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$
$$\left[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}, 0, 0, 1, 0, 1, 0, 0, 0, 0\right],$$
$$\left[0, 0, -\frac{3}{2}, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}, 0, 0, 0, 0, \frac{9}{4}, 0, -\frac{3}{2}, 0\right],$$
$$\left[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, -\frac{3}{2}\right],$$
$$\left[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -\frac{3}{2}, 0, 1, 0\right],$$
$$\left[0, 0, 0, -\frac{3}{2}, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}, 0, 0, 0, 0, -\frac{3}{2}, 0, \frac{9}{4}\right]\Bigg]$$

$$0 \qquad (9)$$

```
> # We see that both matrices are the same.
  # This gives strong numerical evidence that this is the unique matrix
  # in the spectrahedron of f.
```

```
> ########################################################################
  ## Proposition 5.8
  ## Computational proof of the uniquenes of the SOS decomposition
  ## of Example 5.7
  ########################################################################
```

```
> # Some preparation (computing all monomials of degree 3 and 6 in 4 variables)
  p3 := expand( (x + y + z + w)^3);
  cfs := coeffs(p3, [x, y, z, w], 'ct3');
  p6 := expand( (x + y + z + w)^6);
  cfs := coeffs(p6, [x, y, z, w], 'ct6');
  psa := a1 * w^3 + a2 * w^2 * x + a3 * w^2 * y + a4 * w^2 * z + a5 * w * x^2 + a6 * w * x * y
       + a7 * w * x * z + a8 * w * y^2 + a9 * w * y * z + a10 * w * z^2 + a11 * x^3 + a12 * x^2 * y
       + a13 * x^2 * z + a14 * x * y^2 + a15 * x * y * z + a16 * x * z^2 + a17 * y^3 + a18 * y^2 * z
       + a19 * y * z^2 + a20 * z^3;
  ps2 := psa * psa;
  aa := getCoeffs(expand(ps2), [ct6]) :
```

$$p3 := w^3 + 3\,w^2\,x + 3\,w^2\,y + 3\,w^2\,z + 3\,w\,x^2 + 6\,w\,x\,y + 6\,w\,x\,z + 3\,w\,y^2 + 6\,w\,y\,z + 3\,w\,z^2$$

$$+x^3+3x^2y+3x^2z+3xy^2+6xyz+3xz^2+y^3+3y^2z+3yz^2+z^3$$

$$cfs := 1, 3, 3, 3, 3, 6, 6, 3, 6, 3, 1, 3, 3, 3, 6, 3, 1, 3, 3, 1$$

$$p6 := w^6+6w^5x+6w^5y+6w^5z+15w^4x^2+30w^4xy+30w^4xz+15w^4y^2+30w^4yz+15w^4z^2+20w^3x^3+60w^3x^2y+60w^3x^2z+60w^3xy^2+120w^3xyz+60w^3xz^2+20w^3y^3+60w^3y^2z+60w^3yz^2+20w^3z^3+15w^2x^4+60w^2x^3y+60w^2x^3z+90w^2x^2y^2+180w^2x^2yz+90w^2x^2z^2+60w^2xy^3+180w^2xy^2z+180w^2xyz^2+60w^2xz^3+15w^2y^4+60w^2y^3z+90w^2y^2z^2+60w^2yz^3+15w^2z^4+6wx^5+30wx^4y+30wx^4z+60wx^3y^2+120wx^3yz+60wx^3z^2+60wx^2y^3+180wx^2y^2z+180wx^2yz^2+60wx^2z^3+30wxy^4+120wxy^3z+180wxy^2z^2+120wxyz^3+30wxz^4+6wy^5+30wy^4z+60wy^3z^2+60wy^2z^3+30wyz^4+6wz^5+x^6+6x^5y+6x^5z+15x^4y^2+30x^4yz+15x^4z^2+20x^3y^3+60x^3y^2z+60x^3yz^2+20x^3z^3+15x^2y^4+60x^2y^3z+90x^2y^2z^2+60x^2yz^3+15x^2z^4+6xy^5+30xy^4z+60xy^3z^2+60xy^2z^3+30xyz^4+6xz^5+y^6+6y^5z+15y^4z^2+20y^3z^3+15y^2z^4+6yz^5+z^6$$

$$cfs := 1, 6, 6, 6, 15, 30, 30, 15, 30, 15, 20, 60, 60, 60, 120, 60, 20, 60, 60, 20, 15, 60, 60, 90, 180, 90, 60, 180, 180, 60, 15, 60, 90, 60, 15, 6, 30, 30, 60, 120, 60, 60, 180, 180, 60, 30, 120, 180, 120, 30, 6, 30, 60, 60, 30, 6, 1, 6, 6, 15, 30, 15, 20, 60, 60, 20, 15, 60, 90, 60, 15, 6, 30, 60, 60, 30, 6, 1, 6, 15, 20, 15, 6, 1$$

$$psa := a1\,w^3+a10\,wz^2+a11\,x^3+a12\,x^2y+a13\,x^2z+a14\,xy^2+a15\,xyz+a16\,xz^2+a17\,y^3+a18\,y^2z+a19\,yz^2+a2\,w^2x+a20\,z^3+a3\,w^2y+a4\,w^2z+a5\,wx^2+a6\,wxy+a7\,wxz+a8\,wy^2+a9\,wyz$$

$$ps2 := \left(a1\,w^3+a10\,wz^2+a11\,x^3+a12\,x^2y+a13\,x^2z+a14\,xy^2+a15\,xyz+a16\,xz^2+a17\,y^3+a18\,y^2z+a19\,yz^2+a2\,w^2x+a20\,z^3+a3\,w^2y+a4\,w^2z+a5\,wx^2+a6\,wxy+a7\,wxz+a8\,wy^2+a9\,wyz\right)^2 \tag{10}$$

```
> # The 4 even polynomials from Reznick paper
p1 := x*((2-1/2)*x^2-(y^2+z^2+w^2));
p2 := y*((2-1/2)*y^2-(x^2+z^2+w^2));
p3 := z*((2-1/2)*z^2-(x^2+y^2+w^2));
p4 := w*((2-1/2)*w^2-(x^2+y^2+z^2));
```

$$p1 := x\left(\frac{3}{2}x^2-w^2-y^2-z^2\right)$$

$$p2 := y\left(\frac{3}{2}y^2-w^2-x^2-z^2\right)$$

$$p3 := z\left(\frac{3}{2}z^2-w^2-x^2-y^2\right)$$

$$p4 := w\left(\frac{3}{2}w^2-x^2-y^2-z^2\right) \tag{11}$$

```
> # In order to prove that the given decomposition is unique, we need to
# find a PSD form whose kernel is only these 4 polynomials
```

```
> # We compute all the restrictions to phi: A6 -> R given
  # by the four polynomials. There are 20 restrictions for each polynomial

> for i from 1 to nops([ct3]) do
   p1t := expand(p1 * ct3[i]);
   vec := LinearAlgebra[Transpose](getCoeffs(p1t, [ct6]));
   if (i = 1) then
     M := <vec>;
   else
     M := <M, vec>;
   end if;
  end do:
  for i from 1 to nops([ct3]) do
   p2t := expand(p2 * ct3[i]);
   vec := LinearAlgebra[Transpose](getCoeffs(p2t, [ct6]));
   M := <M, vec>;
  end do:
  for i from 1 to nops([ct3]) do
   p3t := expand(p3 * ct3[i]);
   vec := LinearAlgebra[Transpose](getCoeffs(p3t, [ct6]));
   M := <M, vec>;
  end do:
  for i from 1 to nops([ct3]) do
   p4t := expand(p4 * ct3[i]);
   vec := LinearAlgebra[Transpose](getCoeffs(p4t, [ct6]));
   M := <M, vec>;
  end do:

> # We solve the system using only these 80 restriction
  B := Vector(80) :
  s := LinearAlgebra[LinearSolve](M, B) :
  varss := indets(s);
  nops(varss); # 10 indeterminates left to solve
```

$$varss := \{\_t_{24}, \_t_{25}, \_t_{26}, \_t_{33}, \_t_{41}, \_t_{53}, \_t_{54}, \_t_{69}, \_t_{74}, \_t_{75}\}$$

$$10 \tag{12}$$

```
> # This is the expected number of indeterminates.
  # The original space has dimension 84, and the restrictions
  # 20 + 19 + 18 + 17 = 74 (because pi*pj=pj*pi give the same restriction)

> # To construct the desired form we add a new polynomial in the kernel.
  # We will find different psd forms and then add them so that
  # the kernel is generated by just the 4 polynomials

> ###############
  ## p5 := x^3;

  M2 := M:
```

```
p5x := x^3;
for i from 1 to nops([ct3]) do
  pst := expand(p5x * ct3[i]);
  vec := LinearAlgebra[Transpose](getCoeffs(pst, [ct6]));
  M2 := ⟨M2, vec⟩;
end do:
B := Vector(100) :
s := LinearAlgebra[LinearSolve](M2, B) :
varss := indets(s); # 1 -> We got a unique PSD form

# We compute the form and verify it is PSD
s1 := eval(s, {varss[1]=1}) :
ex := LinearAlgebra[Transpose](s1) . aa :
outx := exactSOS(ex, facial="no") :
eig(outx[3]); # 7 positive eigenvalues and 3 null eigenvalues

# Note that this also proofs that the sum p1^2+p2^2+p3^2+p4^2+p5^2 is
# in the border, because we have a psd form that vanishes in this
# five polynomials and it is not null.
```

$$p5x := x^3$$

$$varss := \{\_t0_{33}\}$$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 7, " - Number of indeterminates: ", 0

"Matrix after facial reduction - Rank: ", 7, " - Number of indeterminates: ", 0

"Check 1 of random rank:", 7

"Check 2 of random rank:", 7

"An exact solution was found without calling the numerical solver. The solution matrix is unique under the specified conditions."

"matrixToPoly begins..."

"Computing decomposition..."

"Decomposition computed!"

"Extension succedeed. An exact SOS decomposition has been found for the input polynomial."

(13)

$$\begin{bmatrix} 1.76469949764169\ 10^{-10} \\ 1.76471122437238\ 10^{-10} \\ 1.76471171009496\ 10^{-10} \\ 0.999999999999999 \\ 1.00000000000000 \\ 1. \\ 1.00000000000000 \\ 5.66666666682353 \\ 5.66666666682353 \\ 5.66666666682353 \end{bmatrix} \tag{13}$$

```
> ###############
  ## p5 := y^3;

  M2 := M:
  p5y := y^3;
  for i from 1 to nops([ct3]) do
   pst := expand(p5y * ct3[i]);
   vec := LinearAlgebra[Transpose](getCoeffs(pst, [ct6]));
   M2 := ⟨M2, vec⟩;
  end do:
  B := Vector(100) :
  s := LinearAlgebra[LinearSolve](M2, B) :
  varss := indets(s); # 1 -> We got a unique PSD form

  # We compute the form and verify it is PSD
  s1 := eval(s, {varss[1] = 1}) :
  ey := LinearAlgebra[Transpose](s1) . aa :
  outy := exactSOS(ey, facial = "no") :
  eig(outy[3]); # 7 positive eigenvalues and 3 null eigenvalues
```

$$p5y := y^3$$

$$varss := \{\_t1_{26}\}$$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 7, " - Number of indeterminates: ", 0

"Matrix after facial reduction - Rank: ", 7, " - Number of indeterminates: ", 0

"Check 1 of random rank:", 7

"Check 2 of random rank:", 7

"An exact solution was found without calling the numerical solver. The solution matrix is unique under the specified conditions."

"matrixToPoly begins..."

"Computing decomposition..."

"Decomposition computed!"

"Extension succedeed. An exact SOS decomposition has been found for the input polynomial."

$$\begin{bmatrix} 1.76470528704292\ 10^{-10} \\ 1.76471107733873\ 10^{-10} \\ 1.76471784200522\ 10^{-10} \\ 0.999999999999999 \\ 0.999999999999999 \\ 1.00000000000000 \\ 1.00000000000000 \\ 5.66666666682353 \\ 5.66666666682353 \\ 5.66666666682353 \end{bmatrix} \quad (14)$$

```
> ###############
  ## p5 := z^3;

  M2 := M:
  p5z := z^3;
  for i from 1 to nops([ct3]) do
   pst := expand(p5z * ct3[i]);
   vec := LinearAlgebra[Transpose](getCoeffs(pst, [ct6]));
   M2 := ⟨M2, vec⟩;
  end do:
  B := Vector(100) :
  s := LinearAlgebra[LinearSolve](M2, B) :
  varss := indets(s); # 1 -> We got a unique PSD form

  # We compute the form and verify it is PSD
  s1 := eval(s, {varss[1] = 1}) :
  ez := LinearAlgebra[Transpose](s1) . aa :
  outz := exactSOS(ez, facial = "no") :
  eig(outz[3]); # 7 positive eigenvalues and 3 null eigenvalues
```

$$p5z := z^3$$

$$varss := \{\_t2_{24}\}$$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 7, " - Number of indeterminates: ", 0

"Matrix after facial reduction - Rank: ", 7, " - Number of indeterminates: ", 0

"Check 1 of random rank:", 7

"Check 2 of random rank:", 7

"An exact solution was found without calling the numerical solver. The solution matrix is unique under the specified conditions."

"matrixToPoly begins..."

"Computing decomposition..."

"Decomposition computed!"

"Extension succedeed. An exact SOS decomposition has been found for the input polynomial."

$$\begin{bmatrix} 1.76469980683759\ 10^{-10} \\ 1.76470137744996\ 10^{-10} \\ 1.76470504875680\ 10^{-10} \\ 1.00000000000000 \\ 1. \\ 1.00000000000000 \\ 1.00000000000000 \\ 5.66666666682353 \\ 5.66666666682353 \\ 5.66666666682353 \end{bmatrix} \tag{15}$$

```
> ###############
## p5 := w^3;

M2 := M:
p5w := w^3;
for i from 1 to nops([ct3]) do
  pst := expand(p5w * ct3[i]);
  vec := LinearAlgebra[Transpose](getCoeffs(pst, [ct6]));
  M2 := ⟨M2, vec⟩;
end do:
B := Vector(100) :
s := LinearAlgebra[LinearSolve](M2, B) :
varss := indets(s); # 1 -> We got a unique PSD form

# We compute the form and verify it is PSD
s1 := eval(s, {varss[1] = 1}) :
ew := LinearAlgebra[Transpose](s1) . aa :
outw := exactSOS(ew, facial = "no") :
eig(outw[3]); # 7 positive eigenvalues and 3 null eigenvalues
```

$$p5w := w^3$$

$$varss := \{\_t3_{69}\}$$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 7, " - Number of indeterminates: ", 0

"Matrix after facial reduction - Rank: ", 7, " - Number of indeterminates: ", 0

"Check 1 of random rank:", 7

"Check 2 of random rank:", 7

"An exact solution was found without calling the numerical solver. The solution matrix is unique under the specified conditions."

"matrixToPoly begins..."

"Computing decomposition..."

"Decomposition computed!"

"Extension succedeed. An exact SOS decomposition has been found for the input polynomial."

$$\begin{bmatrix} 1.76469366764802\ 10^{-10} \\ 1.76470033163174\ 10^{-10} \\ 1.76470643653559\ 10^{-10} \\ 1. \\ 1. \\ 1. \\ 1. \\ 5.66666666682353 \\ 5.66666666682353 \\ 5.66666666682353 \end{bmatrix} \quad \textbf{(16)}$$

> *### The desidered form is the sum of all the rank 7 forms:*
> *eall* := *ex* + *ey* + *ez* + *ew*;
> *outall* := *exactSOS*(*eall*, *facial* = "no") :
> *Eigenvalues*(*outall*[3]);

$$eall := 8\,a1^2 + 8\,a1\,a10 + 8\,a1\,a5 + 8\,a1\,a8 + 4\,a10^2 + 2\,a10\,a5 + 2\,a10\,a8 + 8\,a11^2 + 8\,a11\,a14 + 8\,a11\,a16 + 8\,a11\,a2 + 4\,a12^2 + 8\,a12\,a17 + 2\,a12\,a19 + 2\,a12\,a3 + 4\,a13^2 + 2\,a13\,a18 + 8\,a13\,a20 + 2\,a13\,a4 + 4\,a14^2 + 2\,a14\,a16 + 2\,a14\,a2 + a15^2 + 4\,a16^2 + 2\,a16\,a2 + 8\,a17^2 + 8\,a17\,a19 + 8\,a17\,a3 + 4\,a18^2 + 8\,a18\,a20 + 2\,a18\,a4 + 4\,a19^2 + 2\,a19\,a3 + 4\,a2^2 + 8\,a20^2 + 8\,a20\,a4 + 4\,a3^2 + 4\,a4^2 + 4\,a5^2 + 2\,a5\,a8 + a6^2 + a7^2 + 4\,a8^2 + a9^2$$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 16, " - Number of indeterminates: ", 0

"Matrix after facial reduction - Rank: ", 16, " - Number of indeterminates: ", 0

"Check 1 of random rank:", 16

"Check 2 of random rank:", 16

"An exact solution was found without calling the numerical solver. The solution matrix is unique under the specified conditions."

"matrixToPoly begins..."

"Computing decomposition..."

"Decomposition computed!"

"Extension succedeed. An exact SOS decomposition has been found for the input polynomial."

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 14 \\ 14 \\ 14 \\ 14 \\ 1 \\ 1 \\ 1 \\ 1 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \tag{17}$$

```
> # [0, 0, 0, 0, 14, 14, 14, 14, 1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 3]
  # (we used Eigenvalues to compute the exact values)
  # We get a psd form of rank four and p1, p2, p3, p4 are in the kernel
  # so this is form we were looking for.
```

```
> ###########################################################################
  ## Example 5.9
  ## Sum of 4 squares with unique SOS decomposition.
  ## Arbitrary perturbation of Reznick example.
  ###########################################################################
```

```
> p1b := x*((2-1/2)*x^2-(y^2+z^2+w^2));
  p2b := y*((2-1/2)*y^2-(x^2+z^2+w^2));
  p3b := z*((2-1/2)*z^2-(y^2+w^2));
  p4b := w*((1)*w^2-(x^2+z^2));

  # f is the sum of squares of p1b, ..., p4b
  f := p1b^2+p2b^2+p3b^2+p4b^2;
  f := expand(f);
```

$$p1b := x\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)$$

$$p2b := y\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)$$

$$p3b := z\left(\frac{3}{2}z^2 - w^2 - y^2\right)$$

$$p4b := w\left(w^2 - x^2 - z^2\right)$$

$$f := x^2\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)^2 + y^2\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)^2 + z^2\left(\frac{3}{2}z^2 - w^2 - y^2\right)^2 + w^2\left(w^2 - x^2 - z^2\right)^2$$

$$f := -2x^4w^2 - 2x^4y^2 - 3x^4z^2 - x^2w^4 - 2x^2y^4 + x^2z^4 - 3y^4w^2 - 2y^4z^2 + y^2w^4 - 2y^2z^4 - 2z^4w^2 - z^2w^4 + 4x^2w^2z^2 + 4y^2w^2z^2 + 4x^2y^2z^2 + 4x^2w^2y^2 + \frac{9}{4}x^6 + \frac{9}{4}y^6 + \frac{9}{4}z^6 + w^6 \quad \textbf{(18)}$$

> $out := exactSOS(f, facial = \text{"no"})$:
> $eig(out[3]);$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 20, " - Number of indeterminates: ", 126

"Matrix after facial reduction - Rank: ", 20, " - Number of indeterminates: ", 126

"Check 1 of random rank:", 20

"Check 2 of random rank:", 20

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 127, order n = 21, dim = 401, blocks = 2
nnz(A) = 272 + 0, nnz(ADA) = 16129, nnz(L) = 8128
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            1.48E+01 0.000
  1 :  -3.81E+00 2.89E+00 0.000 0.1959 0.9000 0.9000   0.16  1
1  9.7E+00
  2 :  -6.72E-01 7.19E-01 0.000 0.2486 0.9000 0.9000   2.75  1
1  1.1E+00
  3 :  -9.99E-02 1.70E-01 0.000 0.2367 0.9000 0.9000   2.18  1
1  2.1E-01
  4 :  -1.84E-02 3.47E-02 0.000 0.2039 0.9000 0.9000   1.32  1
1  8.9E-02
  5 :  -1.66E-03 3.01E-03 0.000 0.0867 0.9900 0.9900   1.06  1
1  6.4E-02
  6 :  -8.90E-05 1.76E-04 0.325 0.0585 0.9900 0.9900   1.01  1
1  6.2E-03
  7 :  -1.58E-05 2.69E-05 0.000 0.1526 0.9076 0.9000   1.00  1
1  2.4E-03
  8 :  -1.07E-06 2.04E-06 0.464 0.0760 0.9900 0.9900   1.00  1
```

```
1  1.8E-04
  9 :  -4.62E-08 8.95E-08 0.000 0.0438 0.9900 0.9902   1.00  1
1  5.9E-06
 10 :  -2.03E-09 3.45E-09 0.098 0.0386 0.9903 0.9900   1.00  1
1  4.4E-07
 11 :  -4.21E-10 6.61E-10 0.000 0.1916 0.9056 0.9000   1.00  1
1  9.5E-08
 12 :  -2.40E-11 4.27E-11 0.350 0.0646 0.9900 0.9900   1.00  1
1  6.2E-09
 13 :  -1.45E-12 2.23E-12 0.439 0.0522 0.9903 0.9900   1.00  1
1  3.8E-10
 14 :  -3.93E-14 7.24E-14 0.000 0.0325 0.9900 0.9903   1.00  1
2  1.0E-11
 15 :  -9.87E-15 1.12E-14 0.000 0.1543 0.9092 0.9000   1.00  2
2  1.8E-12
 16 :  -6.26E-15 2.54E-15 0.000 0.2273 0.9224 0.9000   1.00  3
3  4.7E-13
 17 :  -5.41E-15 7.20E-16 0.000 0.2836 0.9000 0.9241   0.99  3
3  1.3E-13
 18 :  -5.11E-15 1.78E-16 0.000 0.2475 0.9000 0.9002   0.99  3
3  3.0E-14
 19 :  -5.05E-15 5.11E-17 0.000 0.2870 0.9000 0.9313   1.04  3
3  7.6E-15
 20 :  -5.05E-15 3.15E-17 0.000 0.6162 0.9000 0.9000   0.92  3
3  4.0E-15
 21 :  -5.04E-15 1.07E-17 0.000 0.3409 0.9000 0.9334   1.12  3
3  1.4E-15
Run into numerical problems.

iter seconds digits       c*x               b*y
 21      0.1  10.6 -1.8302058242e-15 -5.0427597253e-15
|Ax-b| =   1.7e-14, [Ay-c]_+ =   9.7E-15, |x|=  4.5e-01, |y|=
4.2e+00

Detailed timing (sec)
   Pre          IPM          Post
1.100E-02    7.700E-02    2.002E-03
Max-norms: ||b||=1, ||c|| = 4,
Cholesky |add|=0, |skip| = 0, ||L.L|| = 113.32.
```

"An exact positive definite solution could not be found for the reduced problem."

"matrixToPoly begins..."

"Computing decomposition..."

"Decomposition computed!"

$$
\begin{bmatrix}
-4.76286774002109\ 10^{-16} \\
-2.26183111190056\ 10^{-16} \\
-1.00910161886404\ 10^{-16} \\
-8.30796213210682\ 10^{-17} \\
-1.94253450998455\ 10^{-17} \\
-3.92916377554329\ 10^{-19} \\
-1.44271702430309\ 10^{-32} \\
0. \\
4.32673798585461\ 10^{-48} \\
2.73547699964761\ 10^{-32} \\
1.19991074734557\ 10^{-16} \\
2.11919137050047\ 10^{-16} \\
2.22044604925031\ 10^{-16} \\
3.15102349627848\ 10^{-16} \\
3.46008175630129\ 10^{-16} \\
5.34081322472003\ 10^{-16} \\
3. \\
4.25000000000000 \\
5.25000000000000 \\
5.25000000000000
\end{bmatrix} \tag{19}
$$

```
> # There are only 4 non-zero eigenvalues, the maximum rank in the
  # spectrahedron is 4.
```

```
> # We compare the matrix obtained by SEDUMI with the matrix corresponding
  # to the original decomposition p1b^2+p2b^2+p3b^2+p4b^2.
  v := convert(out[5], list) : # The monomials indexing the columns of the Gram Matrix
  A1 := decompositionToMatrix([p1b, p2b, p3b, p4b], v) :
  A2 := out[3] :
  Norm(A1 - A2);
```

$$0 \tag{20}$$

```
> # We see that both matrices are the same.
  # This gives strong numerical evidence that this is the unique matrix
  # in the spectrahedron of f.
```

```
> ##########################################################################
```

Example 5.10
Sum of 5 squares with unique SOS decomposition.
##

```
p1c := x*((2-1/2)*x^2-(y^2+z^2+w^2));
p2c := y*((2-1/2)*y^2-(x^2+z^2+w^2));
p3c := z*((2-1/2)*z^2-(x^2+y^2+w^2));
p4c := w*((2-1/2)*w^2-(x^2+y^2+z^2));
p5c := w*y*z;
f := p1c^2+p2c^2+p3c^2+p4c^2+p5c^2;
out := exactSOS(f, facial = "no") :
eig(out[3]);
```

$$p1c := x\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)$$

$$p2c := y\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)$$

$$p3c := z\left(\frac{3}{2}z^2 - w^2 - x^2 - y^2\right)$$

$$p4c := w\left(\frac{3}{2}w^2 - x^2 - y^2 - z^2\right)$$

$$p5c := w\,y\,z$$

$$f := x^2\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)^2 + y^2\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)^2 + z^2\left(\frac{3}{2}z^2 - w^2 - x^2 - y^2\right)^2 + w^2\left(\frac{3}{2}w^2 - x^2 - y^2 - z^2\right)^2 + y^2 w^2 z^2$$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 20, " - Number of indeterminates: ", 126

"Matrix after facial reduction - Rank: ", 20, " - Number of indeterminates: ", 126

"Check 1 of random rank:", 20

"Check 2 of random rank:", 20

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 127, order n = 21, dim = 401, blocks = 2
nnz(A) = 272 + 0, nnz(ADA) = 16129, nnz(L) = 8128
 it :     b*y       gap    delta  rate   t/tP*  t/tD*    feas cg
cg  prec
  0 :            7.89E+00 0.000
  1 :  -4.59E+00 2.19E+00 0.000 0.2773 0.9000 0.9000   0.57  1
1  1.1E+01
  2 :  -1.12E+00 5.31E-01 0.000 0.2424 0.9000 0.9000   2.58  1
1  1.3E+00
  3 :  -1.75E-01 1.42E-01 0.000 0.2678 0.9000 0.9000   2.24  1
```

```
1  2.2E-01
  4 :  -9.43E-03 1.25E-02 0.000 0.0883 0.9900 0.9900   1.44  1
1  3.8E-02
  5 :  -2.23E-03 2.47E-03 0.000 0.1973 0.9000 0.9000   1.02  1
1  2.4E-02
  6 :  -3.83E-05 5.41E-05 0.000 0.0219 0.9894 0.9900   1.00  1
1  5.1E-03
  7 :  -6.29E-06 9.67E-06 0.000 0.1786 0.9086 0.9000   1.00  1
1  1.2E-03
  8 :  -1.05E-06 1.62E-06 0.000 0.1677 0.9069 0.9000   1.00  1
1  2.4E-04
  9 :  -2.93E-08 5.10E-08 0.344 0.0315 0.9900 0.9900   1.00  1
1  8.5E-06
 10 :  -4.93E-09 7.24E-09 0.000 0.1420 0.9000 0.9090   1.00  1
1  1.1E-06
 11 :  -1.68E-10 2.68E-10 0.236 0.0371 0.9903 0.9900   1.00  1
1  4.9E-08
 12 :  -1.68E-11 2.19E-11 0.015 0.0816 0.9900 0.9902   1.00  1
1  3.9E-09
 13 :  -2.90E-12 3.85E-12 0.000 0.1759 0.9087 0.9000   1.00  1
1  7.2E-10
 14 :  -5.47E-13 7.10E-13 0.000 0.1845 0.9120 0.9000   1.00  1
1  1.4E-10
 15 :  -3.03E-14 5.04E-14 0.072 0.0710 0.9900 0.9900   1.00  2
2  9.9E-12
 16 :  -3.96E-15 4.12E-15 0.000 0.0817 0.9902 0.9900   1.00  8
22  8.1E-13
Run into numerical problems.

iter seconds digits       c*x               b*y
 16      0.3   11.0 -5.7504882645e-15 -3.9591905269e-15
|Ax-b| =   1.2e-14, [Ay-c]_+ =   1.0E-14, |x|=  3.7e-01, |y|=
5.3e+00

Detailed timing (sec)
   Pre          IPM          Post
1.300E-02    1.190E-01    2.002E-03
Max-norms: ||b||=1, ||c|| = 7,
Cholesky |add|=0, |skip| = 9, ||L.L|| = 10.311.
```

"An exact positive definite solution could not be found for the reduced problem."

"matrixToPoly begins..."

"Computing decomposition..."

"Decomposition computed!"

$$\begin{bmatrix} -4.71297667200202\ 10^{-16} \\ -4.63686454086492\ 10^{-16} \\ -2.22044604925031\ 10^{-16} \\ -1.00126674685203\ 10^{-16} \\ -5.52127389071324\ 10^{-17} \\ -4.27402309541227\ 10^{-17} \\ -1.34157724023131\ 10^{-17} \\ 4.33410145398643\ 10^{-18} \\ 1.55323759989331\ 10^{-17} \\ 3.93095198347363\ 10^{-17} \\ 5.40278097455944\ 10^{-17} \\ 8.52460020300183\ 10^{-17} \\ 2.31283731507712\ 10^{-16} \\ 3.59274101249545\ 10^{-16} \\ 5.77398408421529\ 10^{-16} \\ 1.00000000000000 \\ 5.25000000000000 \\ 5.25000000000000 \\ 5.25000000000000 \\ 5.25000000000000 \end{bmatrix} \quad (21)$$

```
> # There are only 5 non-zero eigenvalues, the maximum rank in the
  # spectrahedron is 5.
```

```
> # We compare the matrix obtained by SEDUMI with the matrix corresponding
  # to the original decomposition p1c^2+p2c^2+p3c^2+p4c^2+p5c^2.
  v := convert(out[5], list) : # The monomials indexing the columns of the Gram Matrix
  A1 := decompositionToMatrix([p1c, p2c, p3c, p4c, p5c], v) :
  A2 := out[3] :
  Norm(A1 - A2);
```

$$0 \quad (22)$$

```
> ##########################################################################
  ## Example 5.11
  ## Sum of 5 squares with maximum rank 8 in the spectrahedron.
  ##########################################################################
```

```
p1c := x*((2-1/2)*x^2-(y^2+z^2+w^2));
p2c := y*((2-1/2)*y^2-(x^2+z^2+w^2));
p3c := z*((2-1/2)*z^2-(x^2+y^2+w^2));
p4c := w*((2-1/2)*w^2-(x^2+y^2+z^2));
p5c := y^2*z;
f := p1c^2+p2c^2+p3c^2+p4c^2+p5c^2;
out := exactSOS(f, facial="no", computePolynomialDecomposition="no") :
eig(out[3]);
```

$$p1c := x\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)$$

$$p2c := y\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)$$

$$p3c := z\left(\frac{3}{2}z^2 - w^2 - x^2 - y^2\right)$$

$$p4c := w\left(\frac{3}{2}w^2 - x^2 - y^2 - z^2\right)$$

$$p5c := y^2 z$$

$$f := x^2\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)^2 + y^2\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)^2 + z^2\left(\frac{3}{2}z^2 - w^2 - x^2 - y^2\right)^2 + w^2\left(\frac{3}{2}w^2 - x^2 - y^2 - z^2\right)^2 + y^4 z^2$$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 20, " - Number of indeterminates: ", 126

"Matrix after facial reduction - Rank: ", 20, " - Number of indeterminates: ", 126

"Check 1 of random rank:", 20

"Check 2 of random rank:", 20

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 127, order n = 21, dim = 401, blocks = 2
nnz(A) = 272 + 0, nnz(ADA) = 16129, nnz(L) = 8128
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            2.26E+01 0.000
  1 :  -4.41E+00 5.70E+00 0.000 0.2519 0.9000 0.9000   0.42  1
1  1.2E+01
  2 :  -1.02E+00 1.39E+00 0.000 0.2438 0.9000 0.9000   2.55  1
1  1.6E+00
  3 :  -1.45E-01 3.80E-01 0.000 0.2736 0.9000 0.9000   2.23  1
1  4.9E-01
  4 :  -2.16E-02 8.74E-02 0.000 0.2298 0.9000 0.9000   1.41  1
1  3.7E-01
  5 :  -1.77E-03 4.65E-03 0.000 0.0532 0.9900 0.9900   1.06  1
1  2.1E-01
```

```
  6 :  -5.25E-05 1.36E-04 0.075 0.0292 0.9900 0.9900   1.01  1
1  1.1E-02
  7 :  -1.20E-06 2.39E-06 0.000 0.0176 0.9901 0.9900   1.00  1
1  3.0E-04
  8 :  -1.96E-07 3.59E-07 0.000 0.1498 0.9068 0.9000   1.00  3
3  5.2E-05
  9 :  -3.58E-09 1.09E-08 0.234 0.0304 0.9900 0.9900   1.00  4
1  1.7E-06
 10 :  -2.63E-10 7.82E-10 0.085 0.0717 0.9900 0.9906   1.00  5
5  1.1E-07
 11 :  -1.57E-11 3.20E-11 0.372 0.0409 0.9904 0.9900   1.00  7
7  5.2E-09
 12 :  -1.07E-12 2.96E-12 0.299 0.0924 0.9900 0.9900   1.00 11
11  4.8E-10
 13 :  -8.73E-14 2.21E-13 0.000 0.0748 0.9900 0.9905   1.01  8
28  3.1E-11
Run into numerical problems.

iter seconds digits       c*x               b*y
 13      0.3   9.0  5.4990717310e-14 -8.7320040180e-14
|Ax-b| =   2.7e-14, [Ay-c]_+ =   0.0E+00, |x|=  4.0e-01, |y|=
5.2e+00

Detailed timing (sec)
   Pre          IPM          Post
2.997E-03    1.000E-01    9.958E-04
Max-norms: ||b||=1, ||c|| = 6,
Cholesky |add|=2, |skip| = 19, ||L.L|| = 1.3199e+06.
```

"An exact positive definite solution could not be found for the reduced problem."

$$\begin{bmatrix} -1.29360835670273\ 10^{-14} \\ -6.47262787581414\ 10^{-15} \\ -1.07818028515066\ 10^{-15} \\ -9.55194332996441\ 10^{-16} \\ -2.24753157063483\ 10^{-16} \\ -1.81090000405085\ 10^{-16} \\ -1.69388900528030\ 10^{-17} \\ 1.32352612697247\ 10^{-17} \\ 4.40124374398257\ 10^{-17} \\ 1.58323249767909\ 10^{-16} \\ 3.72598391667753\ 10^{-16} \\ 5.03769865935563\ 10^{-16} \\ 0.168836836133502 \\ 0.171218574059929 \\ 0.948552901300000 \\ 0.948552901300000 \\ 4.69569891456651 \\ 5.04141657244008 \\ 5.25000000000000 \\ 5.25000000000000 \end{bmatrix} \tag{23}$$

```
> ##########################################################################
## Example 5.12
## First example.
## Sum of 5 squares with maximum rank 13 in the spectrahedron.
##########################################################################

p1d := x * ((2 - 1/2) * x^2 - (y^2 + z^2 + w^2));
p2d := y * ((2 - 1/2) * y^2 - (x^2 + z^2 + w^2));
p3d := z * ((2 - 1/2) * z^2 - (x^2 + y^2 + w^2));
p4d := w * ((2 - 1/2) * w^2 - (x^2 + y^2 + z^2));
p5d := z^3;
f := p1d^2 + p2d^2 + p3d^2 + p4d^2 + p5d^2;
out := exactSOS(f, facial = "no", computePolynomialDecomposition = "no") :
eig(out[3]);
```

$$p1d := x\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)$$

$$p2d := y\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)$$

$$p3d := z\left(\frac{3}{2}z^2 - w^2 - x^2 - y^2\right)$$

$$p4d := w\left(\frac{3}{2}w^2 - x^2 - y^2 - z^2\right)$$

$$p5d := z^3$$

$$f := x^2\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)^2 + y^2\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)^2 + z^2\left(\frac{3}{2}z^2 - w^2 - x^2 - y^2\right)^2 + w^2\left(\frac{3}{2}w^2 - x^2 - y^2 - z^2\right)^2 + z^6$$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 20, " - Number of indeterminates: ", 126

"Matrix after facial reduction - Rank: ", 20, " - Number of indeterminates: ", 126

"Check 1 of random rank:", 20

"Check 2 of random rank:", 20

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 127, order n = 21, dim = 401, blocks = 2
nnz(A) = 272 + 0, nnz(ADA) = 16129, nnz(L) = 8128
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            1.37E+01 0.000
  1 :  -4.59E+00 3.29E+00 0.000 0.2396 0.9000 0.9000   0.34  1
1  1.1E+01
  2 :  -1.00E+00 8.12E-01 0.000 0.2465 0.9000 0.9000   2.56  1
1  1.4E+00
  3 :  -1.18E-01 2.29E-01 0.000 0.2820 0.9000 0.9000   2.25  1
1  3.7E-01
  4 :  -2.25E-02 8.34E-02 0.000 0.3644 0.9000 0.9000   1.39  1
1  3.3E-01
  5 :  -1.87E-03 3.39E-03 0.000 0.0407 0.9900 0.9900   1.09  1
1  1.3E-01
  6 :  -1.48E-05 2.30E-05 0.198 0.0068 0.9990 0.9990   1.01  1
1  1.6E-03
  7 :  -6.28E-08 2.39E-07 0.000 0.0104 0.9990 0.9990   1.00  1
1  1.8E-05
  8 :  -7.94E-09 2.14E-08 0.368 0.0897 0.9471 0.9450   0.99  7
9  2.1E-06
  9 :  -1.15E-09 3.40E-09 0.216 0.1583 0.9079 0.9000   1.00 10
10  4.1E-07
 10 :  -1.66E-10 5.31E-10 0.000 0.1564 0.9067 0.9000   1.00 26
27  7.2E-08
```

```
 11 :  -7.13E-13 1.71E-11 0.199 0.0323 0.9900 0.9902   1.00 29
31  2.6E-09
 12 :  -1.15E-12 3.04E-12 0.000 0.1775 0.9000 0.8853   1.00 66
90  5.3E-10
 13 :  -1.45E-14 5.59E-14 0.000 0.0184 0.9901 0.9900   1.00 37
74  1.1E-11
Run into numerical problems.

iter seconds digits       c*x                  b*y
 13      0.2   9.5  2.4419065113e-14 -1.4525132481e-14
|Ax-b| =   2.4e-14, [Ay-c]_+ =   0.0E+00, |x|=  4.8e-01, |y|=
4.9e+00

Detailed timing (sec)
   Pre           IPM           Post
3.007E-03    1.420E-01    1.006E-03
Max-norms: ||b||=1, ||c|| = 6,
Cholesky |add|=17, |skip| = 20, ||L.L|| = 1.46367e+09.
```

"An exact positive definite solution could not be found for the reduced problem."

$$\begin{bmatrix} -4.46315749723804\ 10^{-16} \\ -3.22975941478770\ 10^{-16} \\ -1.20768177311148\ 10^{-24} \\ 1.63328315096677\ 10^{-16} \\ 2.89529762440302\ 10^{-16} \\ 4.68749637818417\ 10^{-16} \\ 1.29920332281201\ 10^{-15} \\ 0.0569237061920757 \\ 0.0618848402667536 \\ 0.0619241996890027 \\ 0.0959242427880142 \\ 0.310415758871800 \\ 0.326218088018794 \\ 0.657503739223340 \\ 0.673398884975089 \\ 0.675677438073741 \\ 5.24591882096061 \\ 5.26314645003701 \\ 5.26455293364701 \\ 5.48844228095675 \end{bmatrix} \tag{24}$$

> *# There are 13 non-zero eigenvalues, the maximum rank in the spectrahedron is 13.*

```
> ##########################################################################
## Example 5.12
## Second example.
## Sum of 5 squares with maximum rank 13 in the spectrahedron.
##########################################################################

p1d := x * ((2-1/2) * x^2 - (y^2 + z^2 + w^2));
p2d := y * ((2-1/2) * y^2 - (x^2 + z^2 + w^2));
p3d := z * ((2-1/2) * z^2 - (x^2 + y^2 + w^2));
p4d := w * ((2-1/2) * w^2 - (x^2 + y^2 + z^2));
p5d := y^2 * z - z^3;
f := p1d^2 + p2d^2 + p3d^2 + p4d^2 + p5d^2;
out := exactSOS(f, facial = "no", computePolynomialDecomposition = "no") :
eig(out[3]);
```

$$p1d := x\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)$$

$$p2d := y\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)$$

$$p3d := z\left(\frac{3}{2}z^2 - w^2 - x^2 - y^2\right)$$

$$p4d := w\left(\frac{3}{2}w^2 - x^2 - y^2 - z^2\right)$$

$$p5d := y^2 z - z^3$$

$$f := x^2\left(\frac{3}{2}x^2 - w^2 - y^2 - z^2\right)^2 + y^2\left(\frac{3}{2}y^2 - w^2 - x^2 - z^2\right)^2 + z^2\left(\frac{3}{2}z^2 - w^2 - x^2 - y^2\right)^2 + w^2\left(\frac{3}{2}w^2 - x^2 - y^2 - z^2\right)^2 + \left(y^2 z - z^3\right)^2$$

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 20, " - Number of indeterminates: ", 126

"Matrix after facial reduction - Rank: ", 20, " - Number of indeterminates: ", 126

"Check 1 of random rank:", 20

"Check 2 of random rank:", 20

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 127, order n = 21, dim = 401, blocks = 2
nnz(A) = 272 + 0, nnz(ADA) = 16129, nnz(L) = 8128
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            2.61E+01 0.000
  1 :  -4.76E+00 6.03E+00 0.000 0.2313 0.9000 0.9000   0.30  1
1  1.2E+01
```

```
  2 :  -1.06E+00 1.53E+00 0.000 0.2539 0.9000 0.9000   2.58  1
1  1.7E+00
  3 :  -1.59E-01 4.32E-01 0.000 0.2821 0.9000 0.9000   2.24  1
1  5.3E-01
  4 :  -2.03E-02 9.69E-02 0.000 0.2242 0.9000 0.9000   1.43  1
1  4.5E-01
  5 :  -3.02E-03 2.72E-02 0.000 0.2811 0.9000 0.9000   1.07  1
1  3.4E-01
  6 :  -2.50E-04 9.55E-04 0.000 0.0351 0.9900 0.9900   1.01  1
1  8.2E-02
  7 :  -4.41E-06 1.72E-05 0.066 0.0180 0.9900 0.9900   1.00  1
1  1.3E-03
  8 :  -2.33E-07 8.88E-07 0.479 0.0516 0.9675 0.9675   1.00  1
1  7.1E-05
  9 :  -1.46E-08 6.58E-08 0.149 0.0741 0.9900 0.9900   1.00  7
7  5.3E-06
 10 :  -4.17E-10 3.87E-09 0.000 0.0588 0.9901 0.9900   1.00  5
12  2.8E-07
 11 :  -1.21E-11 1.86E-10 0.000 0.0480 0.9900 0.9874   1.01 19
19  1.2E-08
 12 :  -1.62E-12 1.86E-11 0.000 0.1000 0.9073 0.9000   1.01 13
90  1.5E-09
 13 :  -2.61E-13 2.13E-12 0.000 0.1146 0.9119 0.9000   1.01 99
99  2.1E-10
Run into numerical problems.

iter seconds digits       c*x               b*y
 13      0.1   8.2  6.3582292460e-13 -2.6114242107e-13
|Ax-b| =   6.4e-13, [Ay-c]_+ =   1.6E-13, |x|=  4.8e-01, |y|=
5.1e+00

Detailed timing (sec)
   Pre          IPM          Post
2.997E-03    1.230E-01    0.000E+00
Max-norms: ||b||=1, ||c|| = 6,
Cholesky |add|=19, |skip| = 0, ||L.L|| = 3.18247e+08.
```

"An exact positive definite solution could not be found for the reduced problem."

$$
\left[\begin{array}{c}
-2.04675747978441\ 10^{-14} \\
-1.99986968072215\ 10^{-14} \\
-6.62465337564953\ 10^{-15} \\
-2.28911133297461\ 10^{-16} \\
-1.24017472987280\ 10^{-25} \\
3.02761144163167\ 10^{-16} \\
4.84646747011860\ 10^{-16} \\
0.0103108711176932 \\
0.0163864904641396 \\
0.0173213312995627 \\
0.0174518751306393 \\
0.0779426901946921 \\
0.104368970323972 \\
0.104639674310470 \\
0.116294881056991 \\
1.13361911444835 \\
5.06716380703805 \\
5.06755134942765 \\
5.22426820696161 \\
6.41425103832623
\end{array}\right] \tag{25}
$$

> *# There are 13 non-zero eigenvalues, the maximum rank in the*
> *# spectrahedron is 8.*

>