```
> currentdir("C:/Users/slapl/Dropbox/Repos/rationalsos");
```

"C:\Users\slapl\Dropbox\Repos\rationalsos" **(1)**

```
> ##############################################################################
  # Load "Rational SOS" procedures
  ##############################################################################
  read("rationalSOS.mpl");
  with(rationalSOS);

  # Display tables of any size
  interface(rtablesize = infinity);
```

"Opening connection with Matlab"

rationalSOS := **module**( ) ... **end module**

[*decompositionToMatrix*, *evalMat*, *evalSolution*, *exactSOS*, *getDiag*, *getExtension*, *getVars*, *matrixToPoly*, *nonRatCoef*, *numericSolver*, *numericSolverSubmatrix*, *polyToMatrix*, *primitiveMatrix*, *randomRank*, *reduceByLinearEquation*, *roundMat*, *roundVec*, *vectorTrace*, *zeroDetSRows*, *zeroRows*]

10 **(2)**

```
> ##############################################################################
  # Construction of Theorem 5.1 using a totally real extension of Q
  ##############################################################################

  # In this worksheet we do the same construction as in Theorem 5.1
  # for a sum of three squares in an algebraic extension of degree 3,
  # but instead of using Q(cubic root of 2), we use a totally real
  # extension. By Hillar results, this polynomial must be decomposable
  # as the sum of squares with rational coefficients.

  # We define a polynomial z as the sum of three squares in an algebraic
  # extension of degree 3 with generic coefficients.

  mp := t^3-7*t^2+2*t+10;
  p1 := c1*t^2+b1*t+a1;
  p2 := c2*t^2+b2*t+a2;
  p3 := c3*t^2+b3*t+a3;
```

$$mp := t^3 - 7\,t^2 + 2\,t + 10$$
$$p1 := c1\,t^2 + b1\,t + a1$$
$$p2 := c2\,t^2 + b2\,t + a2$$
$$p3 := c3\,t^2 + b3\,t + a3$$ **(3)**

```
> # We impose some relations between the coefficients to decrease
  # the dimension of the problem and rename the remaining variables
  b2 := 3*b1;  c2 := b1+7*c1;  a3 := 3*c2-b2;
  b1 := x; b3 := y; c1 := z; c3 := w;

  fGeneric := p1^2+p2^2+p3^2;
```

```
fGeneric := expand( fGeneric);
```

$$b2 := 3\ b1$$
$$c2 := b1 + 7\ c1$$
$$a3 := 21\ c1$$
$$b1 := x$$
$$b3 := y$$
$$c1 := z$$
$$c3 := w$$

$$fGeneric := \left(t^2 z + t x + a1\right)^2 + \left((x + 7 z)\ t^2 + 3 x t + a2\right)^2 + \left(t^2 w + t y + 21 z\right)^2$$

$$fGeneric := t^4 w^2 + t^4 x^2 + 14\ t^4 x z + 50\ t^4 z^2 + 2\ t^3 w y + 6\ t^3 x^2 + 44\ t^3 x z + 2\ a1\ t^2 z + 2\ a2\ t^2 x + 14\ a2\ t^2 z + 42\ t^2 w z + 10\ t^2 x^2 + t^2 y^2 + 2\ a1\ t x + 6\ a2\ t x + 42\ t y z + a1^2 + a2^2 + 441\ z^2 \quad \textbf{(4)}$$

```
> # We solve the coefficients a1 and a2 so that the polynomial is in Q,
  f2 := NormalForm( fGeneric, [mp], plex(a1, a2, x, y, z, w, t) );
  f3 := collect( f2, t);
  lf := CoefficientList( f3, t);
  ss := solve( {lf[2], lf[3]}, {a1, a2});
```

$$f2 := 2\ a1\ t^2 z + 2\ a2\ t^2 x + 14\ a2\ t^2 z + 47\ t^2 w^2 + 14\ t^2 w y + 42\ t^2 w z + 99\ t^2 x^2 + 966\ t^2 x z + t^2 y^2 + 2350\ t^2 z^2 + 2\ a1\ t x + 6\ a2\ t x - 24\ t w^2 - 4\ t w y - 36\ t x^2 - 424\ t x z + 42\ t y z - 1200\ t z^2 + a1^2 + a2^2 - 70\ w^2 - 20\ w y - 130\ x^2 - 1420\ x z - 3059\ z^2$$

$$f3 := \left(2\ a1\ z + 2\ a2\ x + 14\ a2\ z + 47\ w^2 + 14\ w y + 42\ w z + 99\ x^2 + 966\ x z + y^2 + 2350\ z^2\right) t^2 + \left(2\ a1\ x + 6\ a2\ x - 24\ w^2 - 4\ w y - 36\ x^2 - 424\ x z + 42\ y z - 1200\ z^2\right) t + a1^2 + a2^2 - 70\ w^2 - 20\ w y - 130\ x^2 - 1420\ x z - 3059\ z^2$$

$$lf := [a1^2 + a2^2 - 70\ w^2 - 20\ w y - 130\ x^2 - 1420\ x z - 3059\ z^2,\ 2\ a1\ x + 6\ a2\ x - 24\ w^2 - 4\ w y - 36\ x^2 - 424\ x z + 42\ y z - 1200\ z^2,\ 2\ a1\ z + 2\ a2\ x + 14\ a2\ z + 47\ w^2 + 14\ w y + 42\ w z + 99\ x^2 + 966\ x z + y^2 + 2350\ z^2]$$

$$ss := \left\{a1 = \frac{1}{2}\ \frac{1}{x\ (x + 4 z)}\left(165\ w^2 x + 168\ w^2 z + 46\ w x y + 126\ w x z + 28\ w y z + 333\ x^3 + 3574\ x^2 z + 3\ x y^2 - 42\ x y z + 11218\ x z^2 - 294\ y z^2 + 8400\ z^3\right),\ a2 = -\frac{1}{2}\ \frac{1}{x\ (x + 4 z)}\left(47\ w^2 x + 24\ w^2 z + 14\ w x y + 42\ w x z + 4\ w y z + 99\ x^3 + 1002\ x^2 z + x y^2 + 2774\ x z^2 - 42\ y z^2 + 1200\ z^3\right)\right\} \quad \textbf{(5)}$$

```
> # We plug in the solutions found for a1 and a2 and compute the resulting polynomial
  ssDen := denom(rhs(ss[1]));
  p1s := simplify(subs(ss, p1) * ssDen);
  p2s := simplify(subs(ss, p2) * ssDen);
  p3s := simplify(subs(ss, p3) * ssDen);
```

```
p1ss := subs({t = RootOf(x^3 - 7*x^2 + 2*x + 10)}, p1s);
p2ss := subs({t = RootOf(x^3 - 7*x^2 + 2*x + 10)}, p2s);
p3ss := subs({t = RootOf(x^3 - 7*x^2 + 2*x + 10)}, p3s);
```

$$ssDen := 2x\,(x + 4z)$$

$$p1s := 2t^2x^2z + 8t^2xz^2 + 2tx^3 + 8tx^2z + 165w^2x + 168w^2z + 46wxy + 126wxz + 28wyz + 333x^3 + 3574x^2z + 3xy^2 - 42xyz + 11218xz^2 - 294yz^2 + 8400z^3$$

$$p2s := 2t^2x^3 + 22t^2x^2z + 56t^2xz^2 + 6tx^3 + 24tx^2z - 47w^2x - 24w^2z - 14wxy - 42wxz - 4wyz - 99x^3 - 1002x^2z - xy^2 - 2774xz^2 + 42yz^2 - 1200z^3$$

$$p3s := 2\,(t^2w + ty + 21z)\,x\,(x + 4z)$$

$$p1ss := 2\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10)^2x^2z + 8\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10)^2xz^2 + 2x^3\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10) + 8x^2\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10)z + 165w^2x + 168w^2z + 46wxy + 126wxz + 28wyz + 333x^3 + 3574x^2z + 3xy^2 - 42yzx + 11218xz^2 - 294yz^2 + 8400z^3$$

$$p2ss := 2\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10)^2x^3 + 22\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10)^2x^2z + 56\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10)^2xz^2 + 6x^3\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10) + 24x^2\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10)z - 47w^2x - 24w^2z - 14wxy - 42wxz - 4wyz - 99x^3 - 1002x^2z - xy^2 - 2774xz^2 + 42yz^2 - 1200z^3$$

$$p3ss := 2\,(w\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10)^2 + y\,RootOf(\_Z^3 - 7\_Z^2 + 2\_Z + 10) + 21z)\,x\,(x + 4z) \qquad \textbf{(6)}$$

```
> f := simplify(p1ss^2 + p2ss^2 + p3ss^2);
```

$$f := 29434w^4x^2 + 57696w^4xz + 28800w^4z^2 + 16496w^3x^2y + 45528w^3x^2z + 25744w^3xyz + 44352w^3xz^2 + 9600w^3yz^2 + 118916w^2x^4 + 1388008w^2x^3z + 3396w^2x^2y^2 - 1092w^2x^2yz + 5224816w^2x^2z^2 + 3744w^2xy^2z - 107688w^2xyz^2 + 6787200w^2xz^3 + 800w^2y^2z^2 - 100800w^2yz^3 + 2880000w^2z^4 + 33328wx^4y + 92232wx^4z + 375664wx^3yz + 984816wx^3z^2 + 304wx^2y^3 - 3024wx^2y^2z + 1306024wx^2yz^2 + 3059952wx^2z^3 + 176wxy^3z - 30576wxy^2z^2 + 1379184wxyz^3 + 2217600wxz^4 - 16800wy^2z^3 + 480000wyz^4 + 120170x^6 + 2568840x^5z + 2196x^4y^2 - 27972x^4yz + 21731924x^4z^2 + 23448x^3y^2z - 504336x^3yz^2 + 91388592x^3z^3 + 10x^2y^4 - 252x^2y^3z + 74620x^2y^2z^2 - 3127992x^2yz^3 + 195790824x^2z^4 - 1848xy^3z^2 + 77496xy^2z^3 - 7534800xyz^4 + 195120000xz^5 + 88200y^2z^4 - 5040000yz^5 + 72000000z^6 \qquad \textbf{(7)}$$

```
> # Real solutions - In this problem, the system obtained by equating f and the partial derivatives
  # to 0 is too complicated for Maple solver.
  # If we call exactSOS(f), it will not finish.
  #
  # We need to compute some solutions manually.
  #Using the starting polynomials we get a system that # is easier to solve.
  sSym := solve({p1ss = 0, p2ss = 0, p3ss = 0});
```

$$sSym := \left\{ w = w,\ x = x,\ y = RootOf\left(8\ \_Z^2 + (104\ w - 21\ x)\ \_Z + 328\ w^2 - 84\ w\ x + 25\ x^2\right),\ z = \right. \tag{8}$$

$$-\frac{1}{4}\ x\Big\},\ \left\{w = w,\ x = 0,\ y = -\frac{12\ (w^2 + 50\ z^2)}{2\ w - 21\ z},\ z = z\right\},\ \{w = w,\ x = 0,\ y = y,\ z = 0\},\ \Big\{w =$$

$$-\frac{1}{420}\ \Big(x\ \Big(488291\ RootOf\Big(55495088990330083695116023\ \_Z^4$$

$$+ 6536473418007600185344\ RootOf\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2$$

$$- 35621304306714433558528\ RootOf\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)$$

$$+ 97631054059229026696960 + \Big(14234952898514621430579 2\ RootOf\left(\_Z^3 - 7\ \_Z^2\right.$$

$$\left. + 2\ \_Z + 10\right)^2 - 775751015207672352648704\ RootOf\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)$$

$$+ 8438454967649945102092 80\Big)\ \_Z + \Big(49516752849498274368896 0\ RootOf\left(\_Z^3\right.$$

$$\left. - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 2698475475587555107663520\ RootOf\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z\right.$$

$$\left. + 10\right) + 6451864814418505482149200\Big)\ \_Z^2$$

$$+ \Big(1434568870226452574174464\ RootOf\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2$$

$$- 7817856970778027151839968\ RootOf\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)$$

$$+ 2336928129948832603162 2960\Big)\ \_Z^3\Big)^2\ RootOf\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)$$

$$- 768920\ RootOf\Big(55495088990330083695116023\ \_Z^4$$

$$+ 6536473418007600185344\ RootOf\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2$$

$$
\begin{aligned}
&- 35621304306714433558528\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) \\
&+ 9763105405922902669696 0 + \left(14234952898514621430579 2\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2\right.\right. \\
&\left.\left. + 2\ \_Z + 10\right)^2 - 7757510152076723526487 04\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)\right. \\
&\left. + 8438454967649945102092 80\right)\ \_Z + \left(49516752849498274368896 0\ \mathit{RootOf}\left(\_Z^3\right.\right. \\
&\left.\left. - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 2698475475587555107663520\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z\right.\right. \\
&\left.\left. + 10\right) + 6451864814418505482149200\right)\ \_Z^2 \\
&+ \left(1434568870226452574174464\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2\right. \\
&- 7817856970778027151839968\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) \\
&\left. + 2336928129948832603162296 0\right)\ \_Z^3\Big)^2 \\
&+ 414312\ \mathit{RootOf}\left(55495088990330083695116023\ \_Z^4\right. \\
&+ 6536473418007600185344\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 \\
&- 35621304306714433558528\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) \\
&+ 9763105405922902669696 0 + \left(14234952898514621430579 2\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2\right.\right. \\
&\left.\left. + 2\ \_Z + 10\right)^2 - 7757510152076723526487 04\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)\right.
\end{aligned}
$$

$$+ 84384549676499451020928\,0)\,\_Z + (49516752849498274368896\,0\,RootOf(\_Z^3$$

$$- 7\,\_Z^2 + 2\,\_Z + 10)^2 - 26984754755875551076635\,20\,RootOf(\_Z^3 - 7\,\_Z^2 + 2\,\_Z$$

$$+ 10) + 64518648144185054821492\,00)\,\_Z^2$$

$$+ (14345688702264525741744\,64\,RootOf(\_Z^3 - 7\,\_Z^2 + 2\,\_Z + 10)^2$$

$$- 78178569707780271518399\,68\,RootOf(\_Z^3 - 7\,\_Z^2 + 2\,\_Z + 10)$$

$$+ 23369281299488326031622960)\,\_Z^3)\,RootOf(\_Z^3 - 7\,\_Z^2 + 2\,\_Z + 10)$$

$$- 642240\,RootOf(55495088990330083695116023\,\_Z^4$$

$$+ 6536473418007600185344\,RootOf(\_Z^3 - 7\,\_Z^2 + 2\,\_Z + 10)^2$$

$$- 35621304306714433558528\,RootOf(\_Z^3 - 7\,\_Z^2 + 2\,\_Z + 10)$$

$$+ 9763105405922902669696\,0 + (14234952898514621430579\,20\,RootOf(\_Z^3 - 7\,\_Z^2$$

$$+ 2\,\_Z + 10)^2 - 77575101520767235264870\,4\,RootOf(\_Z^3 - 7\,\_Z^2 + 2\,\_Z + 10)$$

$$+ 84384549676499451020928\,0)\,\_Z + (49516752849498274368896\,0\,RootOf(\_Z^3$$

$$- 7\,\_Z^2 + 2\,\_Z + 10)^2 - 26984754755875551076635\,20\,RootOf(\_Z^3 - 7\,\_Z^2 + 2\,\_Z$$

$$+ 10) + 64518648144185054821492\,00)\,\_Z^2$$

$$+\left(14345688702264525741744 64\ RootOf\left(\_Z^3-7\ \_Z^2+2\ \_Z+10\right)^2\right.$$

$$-781785697077802715183996 8\ RootOf\left(\_Z^3-7\ \_Z^2+2\ \_Z+10\right)$$

$$\left.+2336928129948832603162296 0\right)\ \_Z^3\Big)+65968\ RootOf\left(\_Z^3-7\ \_Z^2+2\ \_Z+10\right)$$

$$\left.-103360\right)\Big)\Big/\Big((79\ RootOf\left(\_Z^3-7\ \_Z^2+2\ \_Z+10\right)$$

$$-130)\ RootOf\left(554950889903300836951160 23\ \_Z^4\right.$$

$$+6536473418007600185344\ RootOf\left(\_Z^3-7\ \_Z^2+2\ \_Z+10\right)^2$$

$$-35621304306714433558528\ RootOf\left(\_Z^3-7\ \_Z^2+2\ \_Z+10\right)$$

$$+9763105405922902669696 0+\left(14234952898514621430579 2\ RootOf\left(\_Z^3-7\ \_Z^2\right.\right.$$

$$\left.+2\ \_Z+10\right)^2-7757510152076723526487 04\ RootOf\left(\_Z^3-7\ \_Z^2+2\ \_Z+10\right)$$

$$\left.+8438454967649945102092 80\right)\ \_Z+\left(4951675284949827436889 60\ RootOf\left(\_Z^3\right.\right.$$

$$\left.-7\ \_Z^2+2\ \_Z+10\right)^2-2698475475587555107663520\ RootOf\left(\_Z^3-7\ \_Z^2+2\ \_Z\right.$$

$$\left.+10\right)+6451864814418505482149200\Big)\ \_Z^2$$

$$+\left(14345688702264525741744 64\ RootOf\left(\_Z^3-7\ \_Z^2+2\ \_Z+10\right)^2\right.$$

$$-781785697077802715183996 8\ RootOf\left(\_Z^3-7\ \_Z^2+2\ \_Z+10\right)$$

$$+ 233692812994883260316229 60\Big)\ \_Z^3\Big)\Big), x = x, y$$

$$= \frac{1}{210}\ \Big(x\ \Big(1953164\ RootOf\big(5549508899033008369511602 3\ \_Z^4$$

$$+ 6536473418007600185344\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)^2$$

$$- 35621304306714433558528\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)$$

$$+ 9763105405922902669696 0 + \Big(1423495289851462143057 92\ RootOf\big(\_Z^3 - 7\ \_Z^2$$

$$+ 2\ \_Z + 10\big)^2 - 7757510152076723526487 04\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)$$

$$+ 8438454967649945102092 80\Big)\ \_Z + \Big(4951675284949827436889 60\ RootOf\big(\_Z^3$$

$$- 7\ \_Z^2 + 2\ \_Z + 10\big)^2 - 2698475475587555107663520\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z$$

$$+ 10\big) + 6451864814418505482149200\Big)\ \_Z^2$$

$$+ \Big(1434568870226452574174464\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)^2$$

$$- 7817856970778027151839968\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)$$

$$+ 233692812994883260316229 60\Big)\ \_Z^3\Big)^2\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)^2$$

$$- 5865525\ RootOf\big(5549508899033008369511602 3\ \_Z^4$$

$$+ 6536473418007600185344\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)^2$$

$$- 35621304306714433558528\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)$$

$$+ 9763105405922902669696 0 + \Big(1423495289851462143057 92\ RootOf\big(\_Z^3 - 7\ \_Z^2$$

$$+ 2\ \_Z + 10\big)^2 - 7757510152076723526487 04\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)$$

$$+ 8438454967649945102092 80\Big)\ \_Z + \Big(4951675284949827436889 60\ RootOf\big(\_Z^3$$

$$- 7\ \_Z^2 + 2\ \_Z + 10\big)^2 - 2698475475587555107663520\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z$$

$$+ 10\big) + 6451864814418505482149200\Big)\ \_Z^2$$

$$+ \Big(1434568870226452574174464\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)^2$$

$$- 7817856970778027151839968\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)$$

$$+ 233692812994883260316229 60\Big)\ \_Z^3\Big)^2\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)$$

$$+ 1657248\ RootOf\big(5549508899033008369511602 3\ \_Z^4$$

$$+ 6536473418007600185344\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)^2$$

$$- 35621304306714433558528\ RootOf\big(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\big)$$

$$+ 9763105405922902669696 0 + \left(1423495289851462143057 92\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 7757510152076723526487 04\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) + 8438454967649945102092 80\right)\ \_Z + \left(4951675284949827436889 60\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 2698475475587555107663520\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) + 6451864814418505482149200\right)\ \_Z^2 + \left(1434568870226452574174464\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 7817856970778027151839968\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) + 2336928129948832603162296 0\right)\ \_Z^3\right)\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 + 4417900\ \mathit{RootOf}\left(5549508899033008369511602 3\ \_Z^4 + 6536473418007600185344\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 3562130430671443355852 8\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) + 9763105405922902669696 0 + \left(1423495289851462143057 92\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 7757510152076723526487 04\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) + 8438454967649945102092 80\right)\ \_Z + \left(4951675284949827436889 60\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 2698475475587555107663520\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) + 6451864814418505482149200\right)\ \_Z^2 + \left(1434568870226452574174464\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 7817856970778027151839968\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) + 2336928129948832603162296 0\right)\ \_Z^3\right)^2 - 4640520\ \mathit{RootOf}\left(5549508899033008369511602 3\ \_Z^4 + 6536473418007600185344\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 3562130430671443355852 8\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) + 9763105405922902669696 0 + \left(1423495289851462143057 92\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 7757510152076723526487 04\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) + 8438454967649945102092 80\right)\ \_Z + \left(4951675284949827436889 60\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right)^2 - 2698475475587555107663520\ \mathit{RootOf}\left(\_Z^3 - 7\ \_Z^2 + 2\ \_Z + 10\right) + 6451864814418505482149200\right)\ \_Z^2$$

$$+\left(143456887022645257417446 4\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)^2\right.$$
$$-781785697077802715183996 8\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)$$
$$\left.+233692812994883260316229 60\right)\,\_Z^3\Big)\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)$$
$$+263872\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)^2$$
$$+3211200\,\mathrm{RootOf}\Big(554950889903300836951160 23\,\_Z^4$$
$$+6536473418007600185344\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)^2$$
$$-3562130430671443355852 8\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)$$
$$+9763105405922902669696 0+\left(1423495289851462143057 92\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2\right.\right.$$
$$\left.+2\,\_Z+10\right)^2-7757510152076723526487 04\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)$$
$$+8438454967649945102092 80\Big)\,\_Z+\left(4951675284949827436889 60\,\mathrm{RootOf}\left(\_Z^3\right.\right.$$
$$-7\,\_Z^2+2\,\_Z+10\left.\right)^2-2698475475587555107663 520\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z\right.$$
$$\left.\left.+10\right)+6451864814418505482149 200\right)\,\_Z^2$$
$$+\left(143456887022645257417446 4\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)^2\right.$$
$$-781785697077802715183996 8\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)$$
$$\left.+233692812994883260316229 60\right)\,\_Z^3\Big)-743280\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)$$
$$\left.+516800\right)\Big)\Big/\Big(\left(79\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)\right.$$
$$-130\Big)\,\mathrm{RootOf}\Big(554950889903300836951160 23\,\_Z^4$$
$$+6536473418007600185344\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)^2$$
$$-3562130430671443355852 8\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)$$
$$+9763105405922902669696 0+\left(1423495289851462143057 92\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2\right.\right.$$
$$\left.+2\,\_Z+10\right)^2-7757510152076723526487 04\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)$$
$$+8438454967649945102092 80\Big)\,\_Z+\left(4951675284949827436889 60\,\mathrm{RootOf}\left(\_Z^3\right.\right.$$
$$-7\,\_Z^2+2\,\_Z+10\left.\right)^2-2698475475587555107663 520\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z\right.$$
$$\left.\left.+10\right)+6451864814418505482149 200\right)\,\_Z^2$$
$$+\left(143456887022645257417446 4\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)^2\right.$$
$$-781785697077802715183996 8\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)$$
$$\left.+233692812994883260316229 60\right)\,\_Z^3\Big)\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)\Big),z$$
$$=\mathrm{RootOf}\Big(554950889903300836951160 23\,\_Z^4$$
$$+6536473418007600185344\,\mathrm{RootOf}\left(\_Z^3-7\,\_Z^2+2\,\_Z+10\right)^2$$

$$-356213043067144335585 28\ RootOf(\_Z^3-7\_Z^2+2\_Z+10)+976310540592290266969 60+(142349528985146214305792\ RootOf(\_Z^3-7\_Z^2+2\_Z+10)^2-775751015207672352648704\ RootOf(\_Z^3-7\_Z^2+2\_Z+10)+843845496764994510209280)\_Z+(495167528494982743688960\ RootOf(\_Z^3-7\_Z^2+2\_Z+10)^2-269847547558755107663520\ RootOf(\_Z^3-7\_Z^2+2\_Z+10)+645186481441850548214920 0)\_Z^2+(143456887022645257417446 4\ RootOf(\_Z^3-7\_Z^2+2\_Z+10)^2-781785697077802715183996 8\ RootOf(\_Z^3-7\_Z^2+2\_Z+10)+233692812994883260316229 60)\_Z^3)\ x\},\left\{w=-\frac{4}{105}\left(x\left(183\ RootOf(1091493\_Z^4+316290\_Z^3+70385\_Z^2-1600\_Z+800)^2-5\ RootOf(1091493\_Z^4+316290\_Z^3+70385\_Z^2-1600\_Z+800)+5\right)\right)/\left(RootOf(1091493\_Z^4+316290\_Z^3+70385\_Z^2-1600\_Z+800)\right),x=x,y=\frac{1}{105}\left(x\left(1473\ RootOf(1091493\_Z^4+316290\_Z^3+70385\_Z^2-1600\_Z+800)^2+20\ RootOf(1091493\_Z^4+316290\_Z^3+70385\_Z^2-1600\_Z+800)-20\right)\right)/\left(RootOf(1091493\_Z^4+316290\_Z^3+70385\_Z^2-1600\_Z+800)\right),z=RootOf(1091493\_Z^4+316290\_Z^3+70385\_Z^2-1600\_Z+800)\ x\right\}$$

```
> # Maple founds 4 branches of solutions.
  # We use the first branch:

    # {w = w, x = x, y = RootOf(8*_Z^2+(104*w-21*x)*_Z+328*w^2-84*w*x+25*x^2), z = -(1/4)*x}

  # We need to compute real points in these branch. We give different values to x and w:
  # (we use procedure evalSolution from rationalSOS package to evaluate the solution)
  s1 := evalSolution(sSym[1], {x = 1, w = -1});
  s2 := evalSolution(sSym[1], {x = 1, w = -2});
  s3 := evalSolution(sSym[1], {x = 1, w = -3});
```

$$s1:=\left\{w=-1,x=1,y=RootOf(8\_Z^2-125\_Z+437),z=-\frac{1}{4}\right\}$$

$$s2:=\left\{w=-2,x=1,y=RootOf(8\_Z^2-229\_Z+1505),z=-\frac{1}{4}\right\}$$

$$s3 := \left\{ w = -3,\ x = 1,\ y = RootOf\left(8\_Z^2 - 333\_Z + 3229\right),\ z = -\frac{1}{4} \right\} \quad \textbf{(9)}$$

```
> # We verify that these solutions contain real points:
  evalf(allvalues(s1));
  evalf(allvalues(s2));
  evalf(allvalues(s3));
```

$$\{w = -1.,\ x = 1.,\ y = 5.280671362,\ z = -0.2500000000\},\ \{w = -1.,\ x = 1.,\ y = 10.34432864,\ z = -0.2500000000\}$$
$$\{w = -2.,\ x = 1.,\ y = 10.22316555,\ z = -0.2500000000\},\ \{w = -2.,\ x = 1.,\ y = 18.40183445,\ z = -0.2500000000\}$$
$$\{w = -3.,\ x = 1.,\ y = 15.37787432,\ z = -0.2500000000\},\ \{w = -3.,\ x = 1.,\ y = 26.24712568,\ z = -0.2500000000\} \quad \textbf{(10)}$$

```
> # We now call procedure exactSOS using only these 3 points as solutions.
  out := exactSOS( f, zeros = {s1, s2, s3});
```

"Option traceEquations: yes - Only valid when looking for rational decompositions."

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 20, " - Number of indeterminates: ", 126

"Matrix after facial reduction - Rank: ", 6, " - Number of indeterminates: ", 2

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 3, order n = 7, dim = 37, blocks = 2
nnz(A) = 22 + 0, nnz(ADA) = 9, nnz(L) = 6
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            1.81E+07 0.000
  1 :  -4.85E+06 5.81E+06 0.000 0.3214 0.9000 0.9000   1.65  1
1  6.7E+01
  2 :  -5.52E+05 1.45E+06 0.000 0.2499 0.9000 0.9000   2.79  1
1  8.8E+00
  3 :  -1.72E+05 3.80E+05 0.000 0.2616 0.9000 0.9000   1.07  1
1  2.7E+00
  4 :  -2.97E+04 9.30E+04 0.000 0.2447 0.9000 0.9000   0.86  1
1  1.2E+00
  5 :  -8.20E+03 1.93E+04 0.000 0.2078 0.9000 0.9000   1.04  1
1  7.9E-01
  6 :  -3.06E+03 5.01E+03 0.000 0.2595 0.9000 0.9192   1.04  1
1  5.4E-01
  7 :   1.08E+01 2.64E+02 0.000 0.0526 0.9906 0.9900   1.04  1
1  3.3E-01
  8 :   1.08E+01 7.02E+01 0.000 0.2665 0.9000 0.0000   1.00  1
1  2.8E-01
  9 :   9.75E+00 2.01E+01 0.000 0.2858 0.9000 0.9174   1.00  1
1  2.2E-01
 10 :   9.75E+00 4.72E+00 0.000 0.2349 0.9000 0.0000   1.00  1
```

```
1  8.8E-03
 11 :   5.69E+00 3.04E-01 0.000 0.0645 0.9902 0.9900   1.00  1
1  3.2E-02
 12 :   5.30E+00 4.04E-14 0.000 0.0000 0.9145 0.9000   1.00  1
1  9.9E-08
 13 :   5.10E+00 2.26E-15 0.000 0.0559 0.9900 0.9900   1.00  1
1  5.6E-09

iter seconds digits       c*x                b*y
 13      0.0   3.5  5.0965550245e+00  5.0982914600e+00
|Ax-b| =   3.8e-09, [Ay-c]_+ =   1.1E-02, |x|=  1.0e+00, |y|=
3.8e+04

Detailed timing (sec)
   Pre           IPM           Post
0.000E+00    3.400E-02    9.958E-04
Max-norms: ||b||=1, ||c|| = 15419879,
Cholesky |add|=0, |skip| = 0, ||L.L|| = 215.716.
```

"Problem solved. Positive definite matrix found for the reduced problem."

"---"

"SOS a_1 f_1^2 + ... + a_s f_s^2"

"Coefficients a_i: "

$$\left[29434,\ \frac{6866723727459505923763473494437037116 1}{2803000670512264043696654433218 8224},\ \frac{7746048}{14717},\ \frac{243204542597383}{606867038208},\ \frac{24108487}{137712},\ \frac{5364819753799}{1021547065152}\right]$$

"Polynomials f_i: "

$$\left[w^2 x+\frac{14424}{14717} w^2 z+\frac{4124}{14717} y w x+\frac{11382}{14717} w z x+\frac{2404}{14717} y w z+\frac{16352051705}{8106830016} x^3+\frac{21862710473}{1013353752} z x^2+\frac{271}{14717} y^2 x-\frac{3465}{14717} y z x+\frac{225478391}{3355476} z^2 x-\frac{25242}{14717} y z^2+\frac{721200}{14717} z^3, x^2 z+4 x z^2, w^2 z-\frac{37}{984} y w x-\frac{21}{82} w z x+\frac{1}{6} y w z+\frac{40392767}{240904192} x^3+\frac{989953351}{271017216} z x^2-\frac{1}{164} y^2 x-\frac{329}{656} y z x+\frac{21648449}{897408} z^2 x-\frac{7}{4} y z^2+50 z^3, x^3+\frac{1943822092094648}{243204542597383} z x^2+\frac{3884015686820464}{243204542597383} z^2 x, x^2 w+4 w z x+\frac{646543}{3708998} x^2 y+\frac{30571611}{24108487} z x^2+\frac{1293086}{1854499} y z x+\frac{122286444}{24108487} z^2 x, x^2 y-\frac{2181824874201}{5364819753799} z x^2+4 y z x-\frac{8727299496804}{5364819753799} z^2 x\right]$$

$$out := \left[29434,\ \frac{6866723727459505923763473494437037116 1}{28030006705122640436966544332188224},\ \frac{7746048}{14717},\right. \tag{11}$$

$$\frac{243204542597383}{606867038208}, \frac{24108487}{137712}, \frac{5364819753799}{1021547065152}\Big], \Big[w^2 x + \frac{14424}{14717}\, w^2 z + \frac{4124}{14717}\, y\, w\, x$$

$$+\frac{11382}{14717}\,w\,z\,x+\frac{2404}{14717}\,y\,w\,z+\frac{16352051705}{8106830016}\,x^3+\frac{21862710473}{1013353752}\,z\,x^2+\frac{271}{14717}\,y^2\,x$$

$$-\frac{3465}{14717}\,y\,z\,x+\frac{225478391}{3355476}\,z^2\,x-\frac{25242}{14717}\,y\,z^2+\frac{721200}{14717}\,z^3,x^2\,z+4\,x\,z^2,w^2\,z$$

$$-\frac{37}{984}\,y\,w\,x-\frac{21}{82}\,w\,z\,x+\frac{1}{6}\,y\,w\,z+\frac{40392767}{240904192}\,x^{3}+\frac{989953351}{271017216}\,z\,x^{2}-\frac{1}{164}\,y^{2}\,x$$

$$-\frac{329}{656}\,y\,z\,x+\frac{21648449}{897408}\,z^2\,x-\frac{7}{4}\,y\,z^2+50\,z^3,x^3+\frac{1943822092094648}{243204542597383}\,z\,x^2$$

$$+\frac{3884015686820464}{243204542597383}\,z^2\,x,x^2\,w+4\,w\,z\,x+\frac{646543}{3708998}\,x^2\,y+\frac{30571611}{24108487}\,z\,x^2$$

$$+\frac{1293086}{1854499}\,y\,z\,x+\frac{122286444}{24108487}\,z^2\,x,\,x^2\,y-\frac{2181824874201}{5364819753799}\,z\,x^2+4\,y\,z\,x$$

$$-\frac{8727299496804}{5364819753799}\,z^2\,x\Big],\,\Big[\Big[\Big[0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0\Big],$$

$$\Big[0,\,29434,\,0,\,28848,\,0,\,8248,\,22764,\,0,\,4808,\,0,\,\frac{16352051705}{275424},\,0,\,\frac{21862710473}{34428},\,542,$$

$$-6930,\ \frac{225478391}{114},\ 0,\ 0,\ -50484,\ 1442400\Big],$$

$$\Big[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big],$$

$$\Big[0,\ 28848,\ 0,\ 28800,\ 0,\ 8064,\ 22176,\ 0,\ 4800,\ 0,\ \frac{8799784}{151},\ 0,\ \frac{94270336}{151},\ 528,\ -7056,$$

$$1951200,\ 0,\ 0,\ -50400,\ 1440000\Big],$$

$$\Big[0,\ 0,\ 0,\ 0,\ \frac{24108487}{137712},\ 0,\ \frac{24108487}{34428},\ 0,\ 0,\ 0,\ 0,\ \frac{8405059}{275424},\ \frac{67487}{304},\ 0,\ \frac{8405059}{68856},$$

$$\frac{67487}{76},\ 0,\ 0,\ 0,\ 0\Big],$$

$$\Big[0,\ 8248,\ 0,\ 8064,\ 0,\ 2312,\ 6384,\ 0,\ 1344,\ 0,\ \frac{4581260477}{275424},\ 0,\ \frac{2041294415}{11476},\ 152,\ -1932,$$

$$\frac{63129163}{114},\ 0,\ 0,\ -14112,\ 403200\Big],$$

$$\Big[0,\ 22764,\ 0,\ 22176,\ \frac{24108487}{34428},\ 6384,\ \frac{175935967}{8607},\ 0,\ 3696,\ 0,\ \frac{13951777}{304},\ \frac{8405059}{68856},$$

$$\frac{37355521}{76},\ 420,\ -\frac{82691429}{17214},\ 1529976,\ 0,\ 0,\ -38808,\ 1108800\Big],$$

$$\Big[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big],$$

$$\Big[0,\ 4808,\ 0,\ 4800,\ 0,\ 1344,\ 3696,\ 0,\ 800,\ 0,\ \frac{4399892}{453},\ 0,\ \frac{47135168}{453},\ 88,\ -1176,\ 325200,$$

$$0,\ 0,\ -8400,\ 240000\Big],$$

$$\Big[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\Big],$$

$$\Big[0,\ \frac{16352051705}{275424},\ 0,\ \frac{8799784}{151},\ 0,\ \frac{4581260477}{275424},\ \frac{13951777}{304},\ 0,\ \frac{4399892}{453},\ 0,\ 120170,$$

$$0,\ 1284420,\ \frac{1993111}{1824},\ -\frac{2574766761}{183616},\ \frac{550580103145}{137712},\ 0,\ 0,\ -\frac{15399622}{151},\ \frac{439989200}{151}$$

$$\Big],$$

$$\Big[0,\ 0,\ 0,\ 0,\ \frac{8405059}{275424},\ 0,\ \frac{8405059}{68856},\ 0,\ 0,\ 0,\ 0,\ \frac{9641}{912},\ \frac{6713385}{183616},\ 0,\ \frac{9641}{228},\ \frac{6713385}{45904},\ 0,$$

$$0,\ 0,\ 0\Big],$$

$$\Big[0,\ \frac{21862710473}{34428},\ 0,\ \frac{94270336}{151},\ \frac{67487}{304},\ \frac{2041294415}{11476},\ \frac{37355521}{76},\ 0,\ \frac{47135168}{453},\ 0,$$

$$1284420,\ \frac{6713385}{183616},\ \frac{945793255799}{68856},\ \frac{2663431}{228},\ -\frac{45700319}{304},\ \frac{6459849496}{151},\ 0,\ 0,$$

$$-\frac{164973088}{151},\ \frac{4713516800}{151}\Big],$$

$$\left[0, 542, 0, 528, 0, 152, 420, 0, 88, 0, \frac{1993111}{1824}, 0, \frac{2663431}{228}, 10, -126, \frac{4143151}{114}, 0, 0, -924, 26400\right],$$

$$\left[0, -6930, 0, -7056, \frac{8405059}{68856}, -1932, -\frac{82691429}{17214}, 0, -1176, 0, -\frac{2574766761}{183616}, \frac{9641}{228}, -\frac{45700319}{304}, -126, \frac{110189}{57}, -\frac{71190308}{151}, 0, 0, 12348, -352800\right],$$

$$\left[0, \frac{225478391}{114}, 0, 1951200, \frac{67487}{76}, \frac{63129163}{114}, 1529976, 0, 325200, 0, \frac{550580103145}{137712}, \frac{6713385}{45904}, \frac{6459849496}{151}, \frac{4143151}{114}, -\frac{71190308}{151}, \frac{20137380824}{151}, 0, 0, -3414600, 97560000\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, -50484, 0, -50400, 0, -14112, -38808, 0, -8400, 0, -\frac{15399622}{151}, 0, -\frac{164973088}{151}, -924, 12348, -3414600, 0, 0, 88200, -2520000\right],$$

$$\left[0, 1442400, 0, 1440000, 0, 403200, 1108800, 0, 240000, 0, \frac{439989200}{151}, 0, \frac{4713516800}{151}, 26400, -352800, 97560000, 0, 0, -2520000, 72000000\right]\Bigg]\Bigg]$$

```
> # The problem is solved. We verify the solution
  p := 0 :
  for i from 1 to 6 do p := p + out[1][i] * out[2][i]^2 end:
  simplify(f-p);
```

$$0 \qquad \textbf{(12)}$$

```
>
```