```
> with(Groebner):
  with(PolynomialTools):
  # Display tables of any size
  interface(rtablesize = infinity);
```

$$\infty \tag{1}$$

```
> ##############################################################################
  # Example 4.4 in [1]
  ##############################################################################

  # Constructin of f. We begin with generic polynomials in Q(alpha),
  # alpha a root of Z^3-2.

  mp := t^3-2;
  p1 := c1*t^2+b1*t+a1;
  p2 := c2*t^2+b2*t+a2;
  b1 := x; b2 := y; c1 := z;

  c2 := b1;
  fGeneric := p1^2+p2^2;
  fGeneric := expand(fGeneric);
```

$$mp := t^3 - 2$$
$$p1 := t^2 z + t x + a1$$
$$p2 := t^2 x + t y + a2$$
$$b1 := x$$
$$b2 := y$$
$$c1 := z$$
$$c2 := x$$
$$fGeneric := \left(t^2 z + t x + a1\right)^2 + \left(t^2 x + t y + a2\right)^2$$

$$fGeneric := t^4 x^2 + t^4 z^2 + 2 t^3 x y + 2 t^3 x z + 2 a1 t^2 z + 2 a2 t^2 x + t^2 x^2 + t^2 y^2 + 2 a1 t x + 2 a2 t y + a1^2 + a2^2 \tag{2}$$

```
> # We solve the coefficients a1 and a2 so that the polynomial is in Q,
  f2 := NormalForm(fGeneric, [mp], plex(a1, a2, b1, b2, c1, t));
  f3 := collect(f2, t);
  lf := CoefficientList(f3, t);
  ss := solve({lf[2], lf[3]}, {a1, a2});
```

$$f2 := 2 a1 t^2 z + 2 a2 t^2 x + t^2 x^2 + t^2 y^2 + 2 a1 t x + 2 a2 t y + 2 t x^2 + 2 t z^2 + a1^2 + a2^2 + 4 x y + 4 x z$$
$$f3 := \left(2 a1 z + 2 a2 x + x^2 + y^2\right) t^2 + \left(2 a1 x + 2 a2 y + 2 x^2 + 2 z^2\right) t + a1^2 + a2^2 + 4 x y + 4 x z$$
$$lf := \left[a1^2 + a2^2 + 4 x y + 4 x z,\ 2 a1 x + 2 a2 y + 2 x^2 + 2 z^2,\ 2 a1 z + 2 a2 x + x^2 + y^2\right]$$
$$ss := \left\{a1 = -\frac{1}{2}\ \frac{2 x^3 - x^2 y + 2 x z^2 - y^3}{x^2 - y z},\ a2 = -\frac{1}{2}\ \frac{x^3 - 2 x^2 z + x y^2 - 2 z^3}{x^2 - y z}\right\} \tag{3}$$

```
> # We plug in the solutions found for a1 and a2
```

```
ssDen := denom(rhs(ss[1]));
p1s := simplify(subs(ss, p1) * ssDen);
p2s := simplify(subs(ss, p2) * ssDen);

p1ss := subs({t = RootOf(Z^3 - 2)}, p1s);
p2ss := subs({t = RootOf(Z^3 - 2)}, p2s);
```

$$ssDen := 2x^2 - 2yz$$

$$p1s := 2t^2x^2z - 2t^2yz^2 + 2tx^3 - 2txyz - 2x^3 + x^2y - 2xz^2 + y^3$$

$$p2s := 2t^2x^3 - 2t^2xyz + 2tx^2y - 2ty^2z - x^3 + 2x^2z - xy^2 + 2z^3$$

$$p1ss := 2\,RootOf(\_Z^3 - 2)^2 x^2 z - 2\,RootOf(\_Z^3 - 2)^2 y z^2 + 2\,RootOf(\_Z^3 - 2)\,x^3 - 2\,RootOf(\_Z^3 - 2)\,xyz - 2x^3 + x^2y - 2xz^2 + y^3$$

$$p2ss := 2\,RootOf(\_Z^3 - 2)^2 x^3 - 2\,RootOf(\_Z^3 - 2)^2 xyz + 2\,RootOf(\_Z^3 - 2)\,x^2y - 2\,RootOf(\_Z^3 - 2)\,y^2z - x^3 + 2x^2z - xy^2 + 2z^3 \quad \textbf{(4)}$$

```
# Verification
f := simplify(p1ss^2 + p2ss^2);
```

$$f := 5x^6 + 12x^5y + 12x^5z + 3x^4y^2 + 12x^4z^2 - 4x^3y^3 - 36x^3y^2z - 36x^3yz^2 - 4x^3z^3 + 3x^2y^4 + 12x^2z^4 + 12xy^3z^2 + 12xy^2z^3 + y^6 + 4z^6 \quad \textbf{(5)}$$

```
# Conjugates of the original polynomial decomposition
pa1, pa2, pa3 := allvalues(p1ss);
pb1, pb2, pb3 := allvalues(p2ss);
```

$$pa1, pa2, pa3 := 2\,2^{2/3}x^2z - 2\,2^{2/3}yz^2 + 2\,2^{1/3}x^3 - 2\,2^{1/3}xyz - 2x^3 + x^2y - 2xz^2 + y^3,\ 2\left(-\frac{1}{2}2^{1/3} + \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)^2 x^2z - 2\left(-\frac{1}{2}2^{1/3} + \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)^2 yz^2 + 2\left(-\frac{1}{2}2^{1/3} + \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)x^3 - 2\left(-\frac{1}{2}2^{1/3} + \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)xyz - 2x^3 + x^2y - 2xz^2 + y^3,\ 2\left(-\frac{1}{2}2^{1/3} - \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)^2 x^2z - 2\left(-\frac{1}{2}2^{1/3} - \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)^2 yz^2 + 2\left(-\frac{1}{2}2^{1/3} - \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)x^3 - 2\left(-\frac{1}{2}2^{1/3} - \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)xyz - 2x^3 + x^2y - 2xz^2 + y^3$$

$$pb1, pb2, pb3 := 2\,2^{2/3}x^3 - 2\,2^{2/3}xyz + 2\,2^{1/3}x^2y - 2\,2^{1/3}y^2z - x^3 + 2x^2z - xy^2 + 2z^3,\ 2\left(-\frac{1}{2}2^{1/3} + \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)^2 x^3 - 2\left(-\frac{1}{2}2^{1/3} + \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)^2 xyz + 2\left(-\frac{1}{2}2^{1/3} + \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)x^2y - 2\left(-\frac{1}{2}2^{1/3} + \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)y^2z - x^3 + 2x^2z - xy^2 + 2z^3,\ 2\left(-\frac{1}{2}2^{1/3} - \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)^2 x^3 - 2\left(-\frac{1}{2}2^{1/3} - \frac{1}{2}I\sqrt{3}\,2^{1/3}\right)^2 xyz + 2\left(-\frac{1}{2}2^{1/3}\right. \quad \textbf{(6)}$$

$$-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\Big)x^2y-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)y^2z-x^3+2x^2z-xy^2+2z^3$$

```
> # f^3 = (P1 + IP2)*(P1 - IP2)
  P1 := pa1*pa2*pa3 - pb1*pb2*pa3 - pa1*pb2*pb3 - pb1*pa2*pb3;
  P2 := -pb1*pb2*pb3 + pa1*pb2*pa3 + pa1*pa2*pb3 + pb1*pa2*pa3;
  P1 := simplify(P1);
  P2 := simplify(P2);
  simplify(P1^2 + P2^2 - f^3);
```

$$P1 := \left(2\,2^{2/3}x^2z-2\,2^{2/3}yz^2+2\,2^{1/3}x^3-2\,2^{1/3}xyz-2x^3+x^2y-2xz^2+y^3\right)\left(2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^2z-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2yz^2+2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^3-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz-2x^3+x^2y-2xz^2+y^3\right)\left(2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^2z-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2yz^2+2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^3-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz-2x^3+x^2y-2xz^2+y^3\right)-\left(2\,2^{2/3}x^2z-2\,2^{2/3}yz^2+2\,2^{1/3}x^3-2\,2^{1/3}xyz-2x^3+x^2y-2xz^2+y^3\right)\left(2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^3-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz+2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^2y-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)y^2z-x^3+2x^2z-xy^2+2z^3\right)\left(2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^3-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz+2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^2y-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)y^2z-x^3+2x^2z-xy^2+2z^3\right)-\left(2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^2z-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2yz^2+2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^3-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz-2x^3+x^2y-2xz^2+y^3\right)\left(2\,2^{2/3}x^3-2\,2^{2/3}xyz+2\,2^{1/3}x^2y-2\,2^{1/3}y^2z-x^3+2x^2z-xy^2+2z^3\right)\left(2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^3-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz+2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^2y-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)y^2z-x^3+2x^2z-xy^2+2z^3\right)-\left(2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^2z-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2yz^2\right.$$

$$+2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^3-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz-2x^3+x^2y$$
$$-2xz^2+y^3\Big)\left(2\,2^{2/3}x^3-2\,2^{2/3}xyz+2\,2^{1/3}x^2y-2\,2^{1/3}y^2z-x^3+2x^2z-xy^2\right.$$
$$\left.+2z^3\right)\left(2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^3-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2xyz+2\left(\right.\right.$$
$$\left.-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^2y-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)y^2z-x^3+2x^2z-xy^2$$
$$\left.+2z^3\right)$$

$$P2:=\left(2\,2^{2/3}x^2z-2\,2^{2/3}yz^2+2\,2^{1/3}x^3-2\,2^{1/3}xyz-2x^3+x^2y-2xz^2+y^3\right)\left(2\left(\right.\right.$$
$$\left.-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^2z-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2yz^2+2\left(-\frac{1}{2}\,2^{1/3}\right.$$
$$\left.+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^3-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz-2x^3+x^2y-2xz^2+y^3\Big)$$
$$\left(2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^3-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2xyz+2\left(-\frac{1}{2}\,2^{1/3}\right.\right.$$
$$\left.-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^2y-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)y^2z-x^3+2x^2z-xy^2+2z^3\Big)$$
$$+\left(2\,2^{2/3}x^2z-2\,2^{2/3}yz^2+2\,2^{1/3}x^3-2\,2^{1/3}xyz-2x^3+x^2y-2xz^2+y^3\right)\left(2\left(\right.\right.$$
$$\left.-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^2z-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2yz^2+2\left(-\frac{1}{2}\,2^{1/3}\right.$$
$$\left.-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^3-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz-2x^3+x^2y-2xz^2+y^3\Big)$$
$$\left(2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^3-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2xyz+2\left(-\frac{1}{2}\,2^{1/3}\right.\right.$$
$$\left.+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^2y-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)y^2z-x^3+2x^2z-xy^2+2z^3\Big)$$
$$+\left(2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^2z-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2yz^2+2\left(\right.\right.$$
$$\left.-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^3-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz-2x^3+x^2y-2xz^2$$
$$\left.+y^3\right)\left(2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^2z-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2yz^2+2\left(\right.\right.$$
$$\left.-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^3-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)xyz-2x^3+x^2y-2xz^2$$
$$\left.+y^3\right)\left(2\,2^{2/3}x^3-2\,2^{2/3}xyz+2\,2^{1/3}x^2y-2\,2^{1/3}y^2z-x^3+2x^2z-xy^2+2z^3\right)$$

$$-\left(2\,2^{2/3}x^3-2\,2^{2/3}xyz+2\,2^{1/3}x^2y-2\,2^{1/3}y^2z-x^3+2x^2z-xy^2+2z^3\right)\left(2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^3-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2xyz+2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^2y-2\left(-\frac{1}{2}\,2^{1/3}+\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)y^2z-x^3+2x^2z-xy^2+2z^3\right)\left(2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2x^3-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)^2xyz+2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)x^2y-2\left(-\frac{1}{2}\,2^{1/3}-\frac{1}{2}\,I\sqrt{3}\,2^{1/3}\right)y^2z-x^3+2x^2z-xy^2+2z^3\right)$$

$$P1 := -10x^9-39x^8y-24x^8z-42x^7y^2-36x^7yz+6x^7z^2+4x^6y^3+72x^6y^2z+108x^6yz^2+80x^6z^3+18x^5y^4+120x^5y^3z+126x^5y^2z^2+12x^5yz^3+48x^5z^4-6x^4y^5-36x^4y^3z^2-240x^4y^2z^3-252x^4yz^4-24x^4z^5-6x^3y^6-36x^3y^5z-54x^3y^4z^2-40x^3y^3z^3+64x^3z^6+84x^2y^3z^4+72x^2y^2z^5-12x^2yz^6+18xy^6z^2+12xy^5z^3+24xz^8+y^9+4y^3z^6$$

$$P2 := 5x^9+12x^8y+42x^8z-12x^7y^2+72x^7yz+84x^7z^2-40x^6y^3-54x^6y^2z-36x^6yz^2+58x^6z^3-6x^5y^4-24x^5y^3z-252x^5y^2z^2-240x^5yz^3-36x^5z^4+12x^4y^5+126x^4y^4z+120x^4y^3z^2+18x^4y^2z^3+48x^4z^5-8x^3y^6+80x^3y^3z^3+108x^3y^2z^4+72x^3yz^5+12x^3z^6+6x^2y^6z-36x^2y^5z^2-42x^2y^4z^3-3xy^8-24xy^3z^5-36xy^2z^6-2y^6z^3-8z^9$$

$$0 \qquad \textbf{(7)}$$

```
> #f = p1^2 + p2^2
  p1 := simplify(P1 / f);
  p2 := simplify(P2 / f);
```

$$p1 := -2x^3-3x^2y+6xz^2+y^3$$
$$p2 := x^3+6x^2z-3xy^2-2z^3 \qquad \textbf{(8)}$$

```
>
```