```
>  # Uncomment and set the path to rationalSOS.mpl file
   #currentdir("C:/Users/User/rationalSOS");
>  read("rationalSOS.mpl");
   with(rationalSOS);
```

"Opening connection with Matlab"

$rationalSOS := \textbf{module}( )\ ...\ \textbf{end module}$

$[decompositionToMatrix, evalMat, getDiag, getExtension, matrixToPoly, nonRatCoef,$ **(1)**

   $numericSolver, numericSolverSubmatrix, polyToMatrix, primitiveMatrix, randomRank,$

   $ratSOS, reduceByLinearEquation, roundMat, roundVec, vectorTrace, zeroDetSRows,$

   $zeroRows]$

```
>  # Display tables of any size
   interface(rtablesize = infinity);
```

$$10 \qquad\qquad\textbf{(2)}$$

```
>  ##############################################################################
   # Example 3.3 in [1]
   ##############################################################################

   # Constructin of f. We begin with generic polynomials in Q(alpha),
   # alpha a root of Z^4-3.
   mp := t^4-3;
   p1 := d1*t^3 + c1*t^2 + b1*t + a1;
   p2 := d2*t^3 + c2*t^2 + b2*t + a2;
   p3 := d3*t^3 + c3*t^2 + b3*t + a3;
```

$$mp := t^4 - 3$$
$$p1 := d1\,t^3 + c1\,t^2 + b1\,t + a1$$
$$p2 := d2\,t^3 + c2\,t^2 + b2\,t + a2$$
$$p3 := d3\,t^3 + c3\,t^2 + b3\,t + a3 \qquad\qquad\textbf{(3)}$$

```
>  # We rename some variables and impose some relations between the coefficients
   # to decrease the dimension of the problem
   b1 := x; d2 := z; c3 := y;
   b2 := 3*x; c2 := x + c1; c1 := d1-x; b3 := 2*x; d1 := 2*z; d3 := d1 + x;
   fGeneric := p1^2 + p2^2 + p3^2;
   fGeneric := expand(fGeneric);
```

$$b1 := x$$
$$d2 := z$$
$$c3 := y$$
$$b2 := 3\,x$$
$$c2 := x + c1$$
$$c1 := d1 - x$$
$$b3 := 2\,x$$
$$d1 := 2\,z$$

$$d3 := 2z + x$$

$$fGeneric := \left(2zt^3 + (2z-x)t^2 + xt + a1\right)^2 + \left(t^3z + 2t^2z + 3tx + a2\right)^2 + \left((2z+x)t^3 + yt^2 + 2xt + a3\right)^2$$

$$\begin{aligned}
fGeneric := \; & t^6x^2 + 4t^6xz + 9t^6z^2 + 2t^5xy - 4t^5xz + 4t^5yz + 12t^5z^2 + 5t^4x^2 + 14t^4xz \\
& + t^4y^2 + 8t^4z^2 + 4a1t^3z + 2a2t^3z + 2a3t^3x + 4a3t^3z - 2t^3x^2 + 4t^3xy + 16t^3xz \\
& - 2a1t^2x + 4a1t^2z + 4a2t^2z + 2a3t^2y + 14t^2x^2 + 2a1tx + 6a2tx + 4a3tx + a1^2 \\
& + a2^2 + a3^2
\end{aligned} \tag{4}$$

> ```
> # We solve the coefficients a1 and a2 so that the polynomial is in Q,
> f2 := NormalForm( fGeneric, [mp], plex(a1, a2, a3, x, y, z, t) );
> f3 := collect( f2, t );
> lz := CoefficientList( f3, t );
> ss := solve( {lz[2], lz[3], lz[4]}, {a1, a2, a3} );
> ```

$$\begin{aligned}
f2 := \; & 4a1t^3z + 2a2t^3z + 2a3t^3x + 4a3t^3z - 2t^3x^2 + 4t^3xy + 16t^3xz - 2a1t^2x \\
& + 4a1t^2z + 4a2t^2z + 2a3t^2y + 17t^2x^2 + 12t^2xz + 27t^2z^2 + 2a1tx + 6a2tx \\
& + 4a3tx + 6txy - 12txz + 12tyz + 36tz^2 + a1^2 + a2^2 + a3^2 + 15x^2 + 42xz + 3y^2 \\
& + 24z^2
\end{aligned}$$

$$\begin{aligned}
f3 := \; & \left(4a1z + 2a2z + 2a3x + 4a3z - 2x^2 + 4xy + 16xz\right)t^3 + \left(-2a1x + 4a1z + 4a2z \right. \\
& \left. + 2a3y + 17x^2 + 12xz + 27z^2\right)t^2 + \left(2a1x + 6a2x + 4a3x + 6xy - 12xz + 12yz \right. \\
& \left. + 36z^2\right)t + a1^2 + a2^2 + a3^2 + 15x^2 + 42xz + 3y^2 + 24z^2
\end{aligned}$$

$$\begin{aligned}
lz := \; & \left[a1^2 + a2^2 + a3^2 + 15x^2 + 42xz + 3y^2 + 24z^2, \; 2a1x + 6a2x + 4a3x + 6xy - 12xz \right. \\
& + 12yz + 36z^2, \; -2a1x + 4a1z + 4a2z + 2a3y + 17x^2 + 12xz + 27z^2, \; 4a1z \\
& \left. + 2a2z + 2a3x + 4a3z - 2x^2 + 4xy + 16xz\right]
\end{aligned}$$

$$\begin{aligned}
ss := \; & \left\{ a1 = \frac{1}{2}\frac{1}{x\left(3x^2 + 5yz - 12z^2\right)}\left(51x^4 + 6x^3y + 96x^3z - 12x^2y^2 - 44x^2yz \right.\right. \tag{5} \\
& \left. + 217x^2z^2 + 6xy^2z - 60xyz^2 + 84xz^3 + 12y^2z^2 - 12yz^3 - 144z^4\right), \; a2 = \\
& -\frac{1}{2}\frac{1}{x\left(3x^2 + 5yz - 12z^2\right)}\left(21x^4 - 74x^3z - 4x^2y^2 + 12x^2yz + 103x^2z^2 + 12xy^2z \right. \\
& \left. - 48xyz^2 - 6xz^3 + 24y^2z^2 + 24yz^3 - 144z^4\right), \; a3 \\
& = \frac{1}{2}\frac{1}{x\left(3x^2 + 5yz - 12z^2\right)}\left(6x^4 - 12x^3y - 141x^3z + 22x^2yz - 8x^2z^2 \right. \\
& \left.\left. + 24xyz^2 - 123xz^3 + 24yz^3 + 72z^4\right) \right\}
\end{aligned}$$

> ```
> # We plug in the solutions found for a1 and a2
> ssDen := denom(rhs(ss[1]));
> p1s := simplify(subs(ss, p1) * ssDen);
> p2s := simplify(subs(ss, p2) * ssDen);
> ```

$$p3s := simplify(subs(ss, p3) * ssDen);$$

$$p1ss := subs(\{t = RootOf(x^4 - 3)\}, p1s);$$
$$p2ss := subs(\{t = RootOf(x^4 - 3)\}, p2s);$$
$$p3ss := subs(\{t = RootOf(x^4 - 3)\}, p3s);$$

$$ssDen := 2\,x\,\left(3\,x^2 + 5\,y\,z - 12\,z^2\right)$$

$$p1s := 12\,t^3\,x^3\,z + 20\,t^3\,x\,y\,z^2 - 48\,t^3\,x\,z^3 - 6\,t^2\,x^4 + 12\,t^2\,x^3\,z - 10\,t^2\,x^2\,y\,z + 24\,t^2\,x^2\,z^2$$
$$+ 20\,t^2\,x\,y\,z^2 - 48\,t^2\,x\,z^3 + 6\,t\,x^4 + 10\,t\,x^2\,y\,z - 24\,t\,x^2\,z^2 + 51\,x^4 + 6\,x^3\,y + 96\,x^3\,z$$
$$- 12\,x^2\,y^2 - 44\,x^2\,y\,z + 217\,x^2\,z^2 + 6\,x\,y^2\,z - 60\,x\,y\,z^2 + 84\,x\,z^3 + 12\,y^2\,z^2 - 12\,y\,z^3$$
$$- 144\,z^4$$

$$p2s := 6\,t^3\,x^3\,z + 10\,t^3\,x\,y\,z^2 - 24\,t^3\,x\,z^3 + 12\,t^2\,x^3\,z + 20\,t^2\,x\,y\,z^2 - 48\,t^2\,x\,z^3 + 18\,t\,x^4$$
$$+ 30\,t\,x^2\,y\,z - 72\,t\,x^2\,z^2 - 21\,x^4 + 74\,x^3\,z + 4\,x^2\,y^2 - 12\,x^2\,y\,z - 103\,x^2\,z^2 - 12\,x\,y^2\,z$$
$$+ 48\,x\,y\,z^2 + 6\,x\,z^3 - 24\,y^2\,z^2 - 24\,y\,z^3 + 144\,z^4$$

$$p3s := 6\,t^3\,x^4 + 12\,t^3\,x^3\,z + 10\,t^3\,x^2\,y\,z - 24\,t^3\,x^2\,z^2 + 20\,t^3\,x\,y\,z^2 - 48\,t^3\,x\,z^3 + 6\,t^2\,x^3\,y$$
$$+ 10\,t^2\,x\,y^2\,z - 24\,t^2\,x\,y\,z^2 + 12\,t\,x^4 + 20\,t\,x^2\,y\,z - 48\,t\,x^2\,z^2 + 6\,x^4 - 12\,x^3\,y - 141\,x^3\,z$$
$$+ 22\,x^2\,y\,z - 8\,x^2\,z^2 + 24\,x\,y\,z^2 - 123\,x\,z^3 + 24\,y\,z^3 + 72\,z^4$$

$$p1ss := 12\,RootOf(\_Z^4 - 3)^3\,x^3\,z + 20\,RootOf(\_Z^4 - 3)^3\,x\,y\,z^2 - 48\,RootOf(\_Z^4 - 3)^3\,x\,z^3$$
$$- 6\,RootOf(\_Z^4 - 3)^2\,x^4 + 12\,RootOf(\_Z^4 - 3)^2\,x^3\,z - 10\,RootOf(\_Z^4 - 3)^2\,x^2\,y\,z$$
$$+ 24\,RootOf(\_Z^4 - 3)^2\,x^2\,z^2 + 20\,RootOf(\_Z^4 - 3)^2\,x\,y\,z^2 - 48\,RootOf(\_Z^4 - 3)^2\,x\,z^3$$
$$+ 6\,RootOf(\_Z^4 - 3)\,x^4 + 10\,RootOf(\_Z^4 - 3)\,x^2\,y\,z - 24\,RootOf(\_Z^4 - 3)\,x^2\,z^2 + 51\,x^4$$
$$+ 6\,x^3\,y + 96\,x^3\,z - 12\,x^2\,y^2 - 44\,x^2\,y\,z + 217\,x^2\,z^2 + 6\,x\,y^2\,z - 60\,x\,y\,z^2 + 84\,x\,z^3 + 12\,y^2\,z^2$$
$$- 12\,y\,z^3 - 144\,z^4$$

$$p2ss := 6\,RootOf(\_Z^4 - 3)^3\,x^3\,z + 10\,RootOf(\_Z^4 - 3)^3\,x\,y\,z^2 - 24\,RootOf(\_Z^4 - 3)^3\,x\,z^3$$
$$+ 12\,RootOf(\_Z^4 - 3)^2\,x^3\,z + 20\,RootOf(\_Z^4 - 3)^2\,x\,y\,z^2 - 48\,RootOf(\_Z^4 - 3)^2\,x\,z^3$$
$$+ 18\,RootOf(\_Z^4 - 3)\,x^4 + 30\,RootOf(\_Z^4 - 3)\,x^2\,y\,z - 72\,RootOf(\_Z^4 - 3)\,x^2\,z^2$$
$$- 21\,x^4 + 74\,x^3\,z + 4\,x^2\,y^2 - 12\,x^2\,y\,z - 103\,x^2\,z^2 - 12\,x\,y^2\,z + 48\,x\,y\,z^2 + 6\,x\,z^3 - 24\,y^2\,z^2$$
$$- 24\,y\,z^3 + 144\,z^4$$

$$p3ss := 6\,RootOf(\_Z^4 - 3)^3\,x^4 + 12\,RootOf(\_Z^4 - 3)^3\,x^3\,z + 10\,RootOf(\_Z^4 - 3)^3\,x^2\,y\,z \tag{6}$$
$$- 24\,RootOf(\_Z^4 - 3)^3\,x^2\,z^2 + 20\,RootOf(\_Z^4 - 3)^3\,x\,y\,z^2 - 48\,RootOf(\_Z^4 - 3)^3\,x\,z^3$$
$$+ 6\,RootOf(\_Z^4 - 3)^2\,x^3\,y + 10\,RootOf(\_Z^4 - 3)^2\,x\,y^2\,z - 24\,RootOf(\_Z^4 - 3)^2\,x\,y\,z^2$$
$$+ 12\,RootOf(\_Z^4 - 3)\,x^4 + 20\,RootOf(\_Z^4 - 3)\,x^2\,y\,z - 48\,RootOf(\_Z^4 - 3)\,x^2\,z^2 + 6\,x^4$$
$$- 12\,x^3\,y - 141\,x^3\,z + 22\,x^2\,y\,z - 8\,x^2\,z^2 + 24\,x\,y\,z^2 - 123\,x\,z^3 + 24\,y\,z^3 + 72\,z^4$$

```
> # Starting polynomial, it is a polynomial in Q[x,y,z]
  f := simplify(p1ss^2 + p2ss^2 + p3ss^2);
```

$$f := 3618\,x^8 + 468\,x^7\,y + 6504\,x^7\,z - 1104\,x^6\,y^2 + 2616\,x^6\,y\,z + 57481\,x^6\,z^2 - 144\,x^5\,y^3 \tag{7}$$

$$-1652\,x^5\,y^2\,z-16440\,x^5\,y\,z^2+23420\,x^5\,z^3+160\,x^4\,y^4+1392\,x^4\,y^3\,z-2520\,x^4\,y^2\,z^2$$
$$-28448\,x^4\,y\,z^3+91320\,x^4\,z^4-240\,x^3\,y^4\,z+1728\,x^3\,y^3\,z^2+10524\,x^3\,y^2\,z^3-85500\,x^3\,y\,z^4$$
$$+34740\,x^3\,z^5-3696\,x^2\,y^3\,z^3+28920\,x^2\,y^2\,z^4-15192\,x^2\,y\,z^5-57267\,x^2\,z^6+720\,x\,y^4\,z^3$$
$$-3312\,x\,y^3\,z^4-3168\,x\,y^2\,z^5+26352\,x\,y\,z^6-40176\,x\,z^7+720\,y^4\,z^4+864\,y^3\,z^5$$
$$-9072\,y^2\,z^6+46656\,z^8$$

> # Compute the matrix Q associated to the problem
> $Q, QVars, v := polyToMatrix(f);$

$Q, QVars, v := \big[\,[3618, 234, 3252, a\_0_{1,\,4}, a\_0_{1,\,5}, a\_0_{1,\,6}, a\_0_{1,\,7}, a\_0_{1,\,8}, -8220 - a\_0_{2,\,6}$ **(8)**

$\quad -a\_0_{3,\,5}, 11710 - a\_0_{3,\,6}, a\_0_{1,\,11}, a\_0_{1,\,12}, a\_0_{1,\,13}, a\_0_{1,\,14}, a\_0_{1,\,15}],$

$\quad\big[234, -1104 - 2\,a\_0_{1,\,4}, 1308 - a\_0_{1,\,5}, -72 - a\_0_{1,\,7}, -826 - a\_0_{3,\,4} - a\_0_{1,\,8}, a\_0_{2,\,6},$

$\quad a\_0_{2,\,7}, a\_0_{2,\,8}, a\_0_{2,\,9}, a\_0_{2,\,10}, a\_0_{2,\,11}, a\_0_{2,\,12}, a\_0_{2,\,13}, a\_0_{2,\,14}, a\_0_{2,\,15}\big],$

$\quad\big[3252, 1308 - a\_0_{1,\,5}, 57481 - 2\,a\_0_{1,\,6}, a\_0_{3,\,4}, a\_0_{3,\,5}, a\_0_{3,\,6}, a\_0_{3,\,7}, a\_0_{3,\,8}, a\_0_{3,\,9},$

$\quad a\_0_{3,\,10}, a\_0_{3,\,11}, a\_0_{3,\,12}, a\_0_{3,\,13}, a\_0_{3,\,14}, a\_0_{3,\,15}\big],$

$\quad\big[a\_0_{1,\,4}, -72 - a\_0_{1,\,7}, a\_0_{3,\,4}, 160 - 2\,a\_0_{1,\,11} - 2\,a\_0_{2,\,7}, 696 - a\_0_{1,\,12} - a\_0_{2,\,8}$

$\quad -a\_0_{3,\,7}, -1260 - \frac{1}{2}\,a\_0_{5,\,5} - a\_0_{1,\,13} - a\_0_{2,\,9} - a\_0_{3,\,8}, -a\_0_{2,\,11}, -120 - a\_0_{5,\,7}$

$\quad -a\_0_{2,\,12} - a\_0_{3,\,11}, 864 - a\_0_{5,\,8} - a\_0_{6,\,7} - a\_0_{2,\,13} - a\_0_{3,\,12}, 5262 - a\_0_{5,\,9} - a\_0_{6,\,8}$

$\quad -a\_0_{2,\,14} - a\_0_{3,\,13}, -\frac{1}{2}\,a\_0_{7,\,7}, a\_0_{4,\,12}, a\_0_{4,\,13}, a\_0_{4,\,14}, a\_0_{4,\,15}\big],$

$\quad\big[a\_0_{1,\,5}, -826 - a\_0_{3,\,4} - a\_0_{1,\,8}, a\_0_{3,\,5}, 696 - a\_0_{1,\,12} - a\_0_{2,\,8} - a\_0_{3,\,7}, a\_0_{5,\,5}, -14224$

$\quad -a\_0_{1,\,14} - a\_0_{2,\,10} - a\_0_{3,\,9}, a\_0_{5,\,7}, a\_0_{5,\,8}, a\_0_{5,\,9}, -42750 - a\_0_{6,\,9} - a\_0_{2,\,15}$

$\quad -a\_0_{3,\,14}, a\_0_{5,\,11}, a\_0_{5,\,12}, a\_0_{5,\,13}, a\_0_{5,\,14}, a\_0_{5,\,15}\big],$

$\quad\big[a\_0_{1,\,6}, a\_0_{2,\,6}, a\_0_{3,\,6}, -1260 - \frac{1}{2}\,a\_0_{5,\,5} - a\_0_{1,\,13} - a\_0_{2,\,9} - a\_0_{3,\,8}, -14224$

$\quad -a\_0_{1,\,14} - a\_0_{2,\,10} - a\_0_{3,\,9}, 91320 - 2\,a\_0_{1,\,15} - 2\,a\_0_{3,\,10}, a\_0_{6,\,7}, a\_0_{6,\,8}, a\_0_{6,\,9},$

$\quad 17370 - a\_0_{3,\,15}, a\_0_{6,\,11}, a\_0_{6,\,12}, a\_0_{6,\,13}, a\_0_{6,\,14}, a\_0_{6,\,15}\big],$

$\quad\big[a\_0_{1,\,7}, a\_0_{2,\,7}, a\_0_{3,\,7}, -a\_0_{2,\,11}, a\_0_{5,\,7}, a\_0_{6,\,7}, a\_0_{7,\,7}, -a\_0_{4,\,12} - a\_0_{5,\,11}, -\frac{1}{2}\,a\_0_{8,\,8}$

$\quad -a\_0_{4,\,13} - a\_0_{5,\,12} - a\_0_{6,\,11}, -1848 - a\_0_{8,\,9} - a\_0_{4,\,14} - a\_0_{5,\,13} - a\_0_{6,\,12}, 0,$

$\quad -a\_0_{8,\,11}, -a\_0_{8,\,12} - a\_0_{9,\,11}, 360 - a\_0_{8,\,13} - a\_0_{9,\,12} - a\_0_{10,\,11}, -1656 - a\_0_{8,\,14}$

$\quad -a\_0_{9,\,13} - a\_0_{10,\,12}\big],$

$$\Big[a\_0_{1,\,8},\ a\_0_{2,\,8},\ a\_0_{3,\,8},\ -120 - a\_0_{5,\,7} - a\_0_{2,\,12} - a\_0_{3,\,11},\ a\_0_{5,\,8},\ a\_0_{6,\,8},\ -a\_0_{4,\,12}$$
$$- a\_0_{5,\,11},\ a\_0_{8,\,8},\ a\_0_{8,\,9},\ 14460 - \frac{1}{2}\,a\_0_{9,\,9} - a\_0_{4,\,15} - a\_0_{5,\,14} - a\_0_{6,\,13},\ a\_0_{8,\,11},$$
$$a\_0_{8,\,12},\ a\_0_{8,\,13},\ a\_0_{8,\,14},\ -1584 - a\_0_{9,\,14} - a\_0_{10,\,13}\Big],$$
$$\Big[-8220 - a\_0_{2,\,6} - a\_0_{3,\,5},\ a\_0_{2,\,9},\ a\_0_{3,\,9},\ 864 - a\_0_{5,\,8} - a\_0_{6,\,7} - a\_0_{2,\,13} - a\_0_{3,\,12},$$
$$a\_0_{5,\,9},\ a\_0_{6,\,9},\ -\frac{1}{2}\,a\_0_{8,\,8} - a\_0_{4,\,13} - a\_0_{5,\,12} - a\_0_{6,\,11},\ a\_0_{8,\,9},\ a\_0_{9,\,9},\ -7596 - a\_0_{5,\,15}$$
$$- a\_0_{6,\,14},\ a\_0_{9,\,11},\ a\_0_{9,\,12},\ a\_0_{9,\,13},\ a\_0_{9,\,14},\ 13176 - a\_0_{10,\,14}\Big],$$
$$\Big[11710 - a\_0_{3,\,6},\ a\_0_{2,\,10},\ a\_0_{3,\,10},\ 5262 - a\_0_{5,\,9} - a\_0_{6,\,8} - a\_0_{2,\,14} - a\_0_{3,\,13},\ -42750$$
$$- a\_0_{6,\,9} - a\_0_{2,\,15} - a\_0_{3,\,14},\ 17370 - a\_0_{3,\,15},\ -1848 - a\_0_{8,\,9} - a\_0_{4,\,14} - a\_0_{5,\,13}$$
$$- a\_0_{6,\,12},\ 14460 - \frac{1}{2}\,a\_0_{9,\,9} - a\_0_{4,\,15} - a\_0_{5,\,14} - a\_0_{6,\,13},\ -7596 - a\_0_{5,\,15}$$
$$- a\_0_{6,\,14},\ -57267 - 2\,a\_0_{6,\,15},\ a\_0_{10,\,11},\ a\_0_{10,\,12},\ a\_0_{10,\,13},\ a\_0_{10,\,14},\ -20088\Big],$$
$$\Big[a\_0_{1,\,11},\ a\_0_{2,\,11},\ a\_0_{3,\,11},\ -\frac{1}{2}\,a\_0_{7,\,7},\ a\_0_{5,\,11},\ a\_0_{6,\,11},\ 0,\ a\_0_{8,\,11},\ a\_0_{9,\,11},\ a\_0_{10,\,11},\ 0,\ 0,$$
$$a\_0_{11,\,13},\ a\_0_{11,\,14},\ a\_0_{11,\,15}\Big],$$
$$\Big[a\_0_{1,\,12},\ a\_0_{2,\,12},\ a\_0_{3,\,12},\ a\_0_{4,\,12},\ a\_0_{5,\,12},\ a\_0_{6,\,12},\ -a\_0_{8,\,11},\ a\_0_{8,\,12},\ a\_0_{9,\,12},\ a\_0_{10,\,12},$$
$$0,\ -2\,a\_0_{11,\,13},\ -a\_0_{11,\,14},\ 360 - \frac{1}{2}\,a\_0_{13,\,13} - a\_0_{11,\,15},\ 432 - a\_0_{13,\,14}\Big],$$
$$\Big[a\_0_{1,\,13},\ a\_0_{2,\,13},\ a\_0_{3,\,13},\ a\_0_{4,\,13},\ a\_0_{5,\,13},\ a\_0_{6,\,13},\ -a\_0_{8,\,12} - a\_0_{9,\,11},\ a\_0_{8,\,13},\ a\_0_{9,\,13},$$
$$a\_0_{10,\,13},\ a\_0_{11,\,13},\ -a\_0_{11,\,14},\ a\_0_{13,\,13},\ a\_0_{13,\,14},\ -4536 - \frac{1}{2}\,a\_0_{14,\,14}\Big],$$
$$\Big[a\_0_{1,\,14},\ a\_0_{2,\,14},\ a\_0_{3,\,14},\ a\_0_{4,\,14},\ a\_0_{5,\,14},\ a\_0_{6,\,14},\ 360 - a\_0_{8,\,13} - a\_0_{9,\,12} - a\_0_{10,\,11},$$
$$a\_0_{8,\,14},\ a\_0_{9,\,14},\ a\_0_{10,\,14},\ a\_0_{11,\,14},\ 360 - \frac{1}{2}\,a\_0_{13,\,13} - a\_0_{11,\,15},\ a\_0_{13,\,14},\ a\_0_{14,\,14},\ 0\Big],$$
$$\Big[a\_0_{1,\,15},\ a\_0_{2,\,15},\ a\_0_{3,\,15},\ a\_0_{4,\,15},\ a\_0_{5,\,15},\ a\_0_{6,\,15},\ -1656 - a\_0_{8,\,14} - a\_0_{9,\,13}$$
$$- a\_0_{10,\,12},\ -1584 - a\_0_{9,\,14} - a\_0_{10,\,13},\ 13176 - a\_0_{10,\,14},\ -20088,\ a\_0_{11,\,15},\ 432$$
$$- a\_0_{13,\,14},\ -4536 - \frac{1}{2}\,a\_0_{14,\,14},\ 0,\ 46656\Big]\Big],\ \{a\_0_{1,\,4},\ a\_0_{1,\,5},\ a\_0_{1,\,6},\ a\_0_{1,\,7},\ a\_0_{1,\,8},$$
$$a\_0_{1,\,11},\ a\_0_{1,\,12},\ a\_0_{1,\,13},\ a\_0_{1,\,14},\ a\_0_{1,\,15},\ a\_0_{2,\,6},\ a\_0_{2,\,7},\ a\_0_{2,\,8},\ a\_0_{2,\,9},\ a\_0_{2,\,10},\ a\_0_{2,\,11},$$

$a\_0_{2,\,12}, a\_0_{2,\,13}, a\_0_{2,\,14}, a\_0_{2,\,15}, a\_0_{3,\,4}, a\_0_{3,\,5}, a\_0_{3,\,6}, a\_0_{3,\,7}, a\_0_{3,\,8}, a\_0_{3,\,9}, a\_0_{3,\,10},$

$a\_0_{3,\,11}, a\_0_{3,\,12}, a\_0_{3,\,13}, a\_0_{3,\,14}, a\_0_{3,\,15}, a\_0_{4,\,12}, a\_0_{4,\,13}, a\_0_{4,\,14}, a\_0_{4,\,15}, a\_0_{5,\,5},$

$a\_0_{5,\,7}, a\_0_{5,\,8}, a\_0_{5,\,9}, a\_0_{5,\,11}, a\_0_{5,\,12}, a\_0_{5,\,13}, a\_0_{5,\,14}, a\_0_{5,\,15}, a\_0_{6,\,7}, a\_0_{6,\,8}, a\_0_{6,\,9},$

$a\_0_{6,\,11}, a\_0_{6,\,12}, a\_0_{6,\,13}, a\_0_{6,\,14}, a\_0_{6,\,15}, a\_0_{7,\,7}, a\_0_{8,\,8}, a\_0_{8,\,9}, a\_0_{8,\,11}, a\_0_{8,\,12},$

$a\_0_{8,\,13}, a\_0_{8,\,14}, a\_0_{9,\,9}, a\_0_{9,\,11}, a\_0_{9,\,12}, a\_0_{9,\,13}, a\_0_{9,\,14}, a\_0_{10,\,11}, a\_0_{10,\,12}, a\_0_{10,\,13},$

$a\_0_{10,\,14}, a\_0_{11,\,13}, a\_0_{11,\,14}, a\_0_{11,\,15}, a\_0_{13,\,13}, a\_0_{13,\,14}, a\_0_{14,\,14}\}, [x^4, x^3 y, x^3 z, x^2 y^2,$

$x^2 y z, x^2 z^2, x y^3, x y^2 z, x y z^2, x z^3, y^4, y^3 z, y^2 z^2, y z^3, z^4]$

> *# Dimension and rank of Q*
> *nops(indets(Q));*
> *randomRank(Q);*

$$75$$
$$15 \tag{9}$$

> # Computes numerically a SDP solution using SEDUMI
> xVars, xSol := numericSolver(Q) :

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 76, order n = 16, dim = 226, blocks = 2
nnz(A) = 165 + 0, nnz(ADA) = 5776, nnz(L) = 2926
 it :      b*y         gap    delta   rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :             7.38E+04 0.000
  1 :  -4.91E+04 2.00E+04 0.000 0.2709 0.9000 0.9000   0.50  1
1  1.0E+01
  2 :  -9.84E+03 5.78E+03 0.000 0.2893 0.9000 0.9000   2.33  1
1  2.1E+00
  3 :  -1.42E+03 1.60E+03 0.000 0.2768 0.9000 0.9000   1.81  1
1  9.0E-01
  4 :  -2.46E+02 3.28E+02 0.000 0.2050 0.9000 0.9000   1.33  1
1  5.9E-01
  5 :  -5.37E+01 6.94E+01 0.000 0.2112 0.9000 0.9000   1.16  1
1  5.6E-01
  6 :  -1.03E+01 1.43E+01 0.000 0.2068 0.9000 0.9000   1.12  1
1  5.1E-01
  7 :  -1.80E+00 3.18E+00 0.000 0.2217 0.9000 0.9000   1.07  1
1  4.2E-01
  8 :  -3.67E-01 7.32E-01 0.000 0.2304 0.9000 0.9000   1.03  1
1  3.9E-01
  9 :  -8.67E-02 1.59E-01 0.000 0.2167 0.9000 0.9023   1.01  1
1  4.1E-01
 10 :  -2.05E-02 3.48E-02 0.000 0.2193 0.9000 0.9013   1.01  1
1  4.2E-01
 11 :  -5.00E-03 8.28E-03 0.000 0.2377 0.9001 0.9000   1.01  4
3  4.3E-01
 12 :  -1.31E-03 2.15E-03 0.000 0.2602 0.9000 0.9028   1.01  6
```

```
6   4.3E-01
 13 :   -3.69E-04 5.69E-04 0.000 0.2645 0.9000 0.9015   1.00 14
13   2.0E-01
 14 :   -1.06E-04 1.57E-04 0.000 0.2750 0.9000 0.9022   1.00 27
27   5.7E-02
 15 :   -3.02E-05 4.35E-05 0.000 0.2778 0.9000 0.9006   1.00 37
43   1.6E-02
 16 :   -8.64E-06 1.22E-05 0.000 0.2795 0.9011 0.9000   1.00 88
99   4.3E-03
 17 :   -2.41E-06 3.42E-06 0.000 0.2813 0.9008 0.9000   1.02 99
99   1.2E-03
 18 :   -5.88E-07 1.15E-06 0.000 0.3354 0.9000 0.9324   1.01 99
99   4.5E-04
Run into numerical problems.

iter seconds digits       c*x             b*y
 18      0.4   3.8   1.2318375999e-06 -5.8815564959e-07
|Ax-b| =   2.4e-10, [Ay-c]_+ =   1.5E-07, |x|=  8.8e-01, |y|=
6.0e+04

Detailed timing (sec)
   Pre          IPM          Post
1.250E-01    3.750E-01    3.100E-02
Max-norms: ||b||=1, ||c|| = 91320,
Cholesky |add|=8, |skip| = 6, ||L.L|| = 1.01686e+08.
```

> *# Solution matrix and eigenvalues*
> $Qsol := evalMat(Q, QVars, xSol)$:
> $eig(Qsol)$;

$$
\begin{bmatrix}
-7.41172293266850 \cdot 10^{-7} \\
7.83327799407880 \cdot 10^{-7} \\
0.00000313961279746914 \\
0.0000157236680403937 \\
0.0000244300369499428 \\
0.252460060329230 \\
0.613749216572089 \\
289.360052430707 \\
2269.36066325846 \\
3419.23305298117 \\
4263.45724618975 \\
14039.3037468656 \\
23202.5947259418 \\
39861.4243642085 \\
1.37034301234899 \cdot 10^{5}
\end{bmatrix}
$$

(10)

```
> # Ten positive eigenvalues and five approximate zeros
> ##################################################################################

  # Matrix associated to the original decomposition
  MNEW := decompositionToMatrix( [p1ss, p2ss, p3ss], v) :

> #############################################
  ##  STEP BY STEP
  #############################################

> # Real solutions
  sSym := solve( { f = 0, diff ( f, x) = 0, diff ( f, y) = 0, diff ( f, z) = 0 });
```

$$sSym := \{x = 0, y = y, z = 0\}, \{x = 0, y = -3\,z, z = z\}, \Big\{x = x, y = \frac{1}{60}\,x\,\big(648\,RootOf(648\,\_Z^5 \tag{11}$$

$$-327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36)^4 - 327\,RootOf(648\,\_Z^5 - 327\,\_Z^4$$

$$+ 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36)^3 + 152\,RootOf(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3$$

$$- 921\,\_Z^2 - 36\,\_Z + 36)^2 - 777\,RootOf(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z$$

$$+ 36) - 36\big), z = RootOf(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36)\,x\Big\}$$

```
> #############################################

  ## sSym[1] plain equation
  v1 := eval(Vector(v), sSym[1]);
  v11 := eval(v1, {y = 1}) :
```

$$\tag{12}$$

$$v1 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ y^4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(12)

> *# We verify that it satisfies the condition vt.Q.v = 0, which must always*
  # be satisfied for real solutions
  simplify(LinearAlgebra[Transpose](v11).Q.v11);

$$0$$

(13)

> *# We reduce the dimension*
  Q1 := reduceByLinearEquation(Q, v11) :
  nops(indets(Q1));
  randomRank(Q1); # 14

$$63$$
$$14$$

(14)

> *###*

  ## sSym[2] plain equation
  v2 := eval(Vector(v), sSym[2]);
  v21 := eval(v2, {z = 1}) :

$$v2 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 81\,z^4 \\ -27\,z^4 \\ 9\,z^4 \\ -3\,z^4 \\ z^4 \end{bmatrix}$$

(15)

> *# We verify that it satisfies the condition vt.Q.v = 0, which must always*
> *# be satisfied for real solutions*
> $simplify(LinearAlgebra[\,Transpose\,](v21).Q.v21);$

$$0$$

(16)

> *# We reduce the dimension*
> $Q2 := reduceByLinearEquation(Q1, v21) :$
> $nops(indets(Q2));$
> $randomRank(Q2); \text{ # 13}$

$$52$$
$$13$$

(17)

> ##

sSym[3] trace equation
$v3 := eval(Vector(v), sSym[3]);$
$v31 := eval(v3, \{x = 1\}) :$
$v31t := vectorTrace(v31);$

$$v3 := \left[\left[x^4\right],\right.$$

$$\left[\frac{1}{60}\,x^4 \left(648\,RootOf\!\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^4\right.\right.$$

$$\left.- 327\,RootOf\!\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^3\right.$$

$$+ 152\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^2$$

$$- 777\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right) - 36\Big)\Big],$$

$$\left[x^4\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)\right],$$

$$\left[\frac{1}{3600}\, x^4 \left(648\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^4\right.\right.$$

$$- 327\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^3$$

$$+ 152\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^2$$

$$- 777\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right) - 36\Big)^2\Big],$$

$$\left[\frac{1}{60}\, x^4 \left(648\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^4\right.\right.$$

$$- 327\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^3$$

$$+ 152\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^2$$

$$- 777\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right) - 36\Big)$$

$$RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)\Big],$$

$$\left[x^4\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^2\right],$$

$$\left[\frac{1}{216000}\, x^4 \left(648\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^4\right.\right.$$

$$- 327\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^3$$

$$+ 152\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^2$$

$$- 777\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right) - 36\Big)^3\Big],$$

$$\left[\frac{1}{3600}\, x^4 \left(648\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^4\right.\right.$$

$$- 327\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^3$$

$$+ 152\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^2$$

$$- 777\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right) - 36\Big)$$

$$^2\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)\Big],$$

$$\left[\frac{1}{60}\, x^4 \left(648\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^4\right.\right.$$

$$- 327\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^3$$

$$+ 152\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right)^2$$

$$- 777\, RootOf\left(648\, \_Z^5 - 327\, \_Z^4 + 152\, \_Z^3 - 921\, \_Z^2 - 36\, \_Z + 36\right) - 36\Big)$$

$$RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^2\Big],$$

$$\left[x^4\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^3\right],$$

$$\left[\frac{1}{12960000}\,x^4\left(648\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^4\right.\right.$$

$$- 327\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^3$$

$$+ 152\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^2$$

$$\left.- 777\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right) - 36\right)^4\Big],$$

$$\left[\frac{1}{216000}\,x^4\left(648\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^4\right.\right.$$

$$- 327\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^3$$

$$+ 152\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^2$$

$$\left.- 777\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right) - 36\right)$$

$$^3\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)\Big],$$

$$\left[\frac{1}{3600}\,x^4\left(648\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^4\right.\right.$$

$$- 327\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^3$$

$$+ 152\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^2$$

$$\left.- 777\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right) - 36\right)$$

$$^2\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^2\Big],$$

$$\left[\frac{1}{60}\,x^4\left(648\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^4\right.\right.$$

$$- 327\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^3$$

$$+ 152\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^2$$

$$\left.- 777\,RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right) - 36\right)$$

$$RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^3\Big],$$

$$\left[RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right)^4 x^4\right]\Big]$$

$$v31t := \begin{bmatrix} 5 \\[4pt] \dfrac{11}{18} \\[8pt] \dfrac{109}{216} \\[8pt] \dfrac{25471}{8100} \\[8pt] -\dfrac{68327}{19440} \\[8pt] -\dfrac{10007}{46656} \\[8pt] \dfrac{1140547}{29160} \\[8pt] \dfrac{38773621}{1749600} \\[8pt] \dfrac{39415141}{4199040} \\[8pt] \dfrac{40686517}{10077696} \\[8pt] \dfrac{16215066883}{65610000} \\[8pt] \dfrac{7205920129}{157464000} \\[8pt] \dfrac{7503013873}{377913600} \\[8pt] \dfrac{1341209549}{181398528} \\[8pt] \dfrac{6589326097}{2176782336} \end{bmatrix}$$

(18)

> $Q3 := reduceByLinearEquation(Q2, v31t) :$
> $nops(indets(Q3));$
> $randomRank(Q3); \# 12$

$$\frac{39}{12}$$

(19)

> ##

> *# Equations from zeros in the diagonal*
> $Q4 := zeroRows(Q3) :$
> $nops(indets(Q4));$

randomRank(*Q4*); # *10*

> ###

MMT, *tVars*, *vSol* := *numericSolverSubmatrix*(*Q4*, 10) :
vSolR := *roundVec*(*vSol*, 15) :
QSol := *evalMat*(*Q4*, *tVars*, *vSolR*) :
evalf(*LinearAlgebra*[*Eigenvalues*](*QSol*));

"SEDUMI CALL"
```
The coefficient matrix is not full row rank, numerical problems
may occur.
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 24, order n = 11, dim = 101, blocks = 2
nnz(A) = 428 + 0, nnz(ADA) = 576, nnz(L) = 300
 it :      b*y         gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :             7.93E+02 0.000
  1 :   -7.03E+08 2.68E+02 0.000 0.3377 0.9000 0.9000    1.87  1
1  1.3E+06
  2 :   -7.21E+07 1.09E+02 0.000 0.4068 0.9000 0.9000    3.52  2
1  2.3E+05
  3 :   -1.67E+07 2.79E+01 0.000 0.2561 0.9000 0.9000    1.70  2
2  4.3E+04
  4 :   -3.19E+06 4.19E+00 0.000 0.1501 0.8227 0.9000    1.33  2
2  6.5E+03
  5 :   -2.74E+05 2.49E-01 0.000 0.0594 0.9900 0.9900    1.19  2
1  3.4E+02
  6 :   -6.12E+03 9.46E-03 0.000 0.0380 0.9900 0.9900    1.03  2
2  1.3E+01
  7 :   -2.67E+03 3.87E-03 0.000 0.4093 0.9000 0.9000    1.00  2
2  5.2E+00
  8 :   -3.06E+02 8.25E-04 0.000 0.2130 0.9000 0.9000    1.00  1
1  1.1E+00
  9 :   -2.49E+01 2.01E-04 0.000 0.2432 0.9000 0.9000    1.00  1
1  2.7E-01
 10 :   -2.55E+00 4.29E-05 0.000 0.2141 0.9000 0.9000    1.00  1
1  5.8E-02
 11 :   -5.42E-04 4.21E-06 0.140 0.0980 0.9900 0.9900    1.00  1
1  5.7E-03
 12 :   -1.27E-02 8.47E-07 0.000 0.2013 0.9000 0.9000    1.00  4
4  1.1E-03
 13 :   -1.78E-02 5.68E-08 0.182 0.0670 0.9900 0.9900    1.00  7
7  7.7E-05
 14 :   -1.90E-02 2.32E-08 0.000 0.4086 0.9000 0.9000    1.00 15
15  3.1E-05
 15 :   -1.90E-02 7.26E-09 0.139 0.3130 0.9000 0.0000    1.00 19
19  1.3E-05
 16 :   -1.73E-02 2.89E-09 0.000 0.3982 0.5795 0.9000    1.00 29
29  6.4E-06
```

```
 17 :  -1.73E-02 2.45E-10 0.000 0.0849 0.9000 0.0000   1.00 25
25  1.3E-06
 18 :  -2.25E-02 1.66E-11 0.072 0.0676 0.9900 0.9900   1.00 33
33  8.5E-08
 19 :  -2.23E-02 1.32E-11 0.211 0.7934 0.0000 0.9000  -8.16 99
99  7.9E-08
Run into numerical problems.

iter seconds digits       c*x                b*y
 19     0.3   7.4 -2.2659811770e-02 -2.2255902919e-02
|Ax-b| =   1.0e-07, [Ay-c]_+ =   6.2E-04, |x|=  1.0e+00, |y|=
2.8e+04

Detailed timing (sec)
   Pre           IPM         Post
0.000E+00    3.100E-01    9.958E-04
Max-norms: ||b||=1, ||c|| = 2445982320,
Cholesky |add|=0, |skip| = 9, ||L.L|| = 2153.2.
```

$$
\begin{bmatrix}
0. \\
0. \\
0. \\
0. \\
0. \\
0.02683300392 \\
0.1741592093 \\
98.02871739 \\
423.6837629 \\
2378.576492 \\
15539.13615 \\
46723.27310 \\
1.431728899 \ 10^5 \\
-0.01863828153 \\
-0.05308948233
\end{bmatrix}
$$

(21)

> *# 5 exact null eigenvalues and 4 approximate null eigenvalues*
> *###*

 # sSym[3] plain equations - reduction to rank 9
Q5 := reduceByLinearEquation(Q4, v31) :
nops(indets(Q5));
randomRank(Q5); # 9

$$16$$

$$9$$

(22)

> `#############################################`

   # We force the entries of Q5 to be rational expressions
   Q5Prim := primitiveMatrix(Q5) :
   L := getExtension(Q5Prim);
   Q6 := nonRatCoef(Q5Prim, 15, L) :

$$L := RootOf\left(648\,\_Z^5 - 327\,\_Z^4 + 152\,\_Z^3 - 921\,\_Z^2 - 36\,\_Z + 36\right) \tag{23}$$

> *nops(indets(Q6))*;
   randomRank(Q6); *# 6*

$$4$$
$$6 \tag{24}$$

> *MMT, tVars, vSol := numericSolverSubmatrix(Q6, 6)*;
   vSolR := roundVec(vSol, 1) :
   Q6Sol := evalMat(Q6, tVars, vSolR) :
   eigval := LinearAlgebra[Eigenvalues](Q6Sol) :
   evalf(eigval);

<div align="center">"SEDUMI CALL"</div>

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 5, order n = 7, dim = 37, blocks = 2
nnz(A) = 59 + 0, nnz(ADA) = 25, nnz(L) = 15
 it :      b*y        gap    delta   rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            1.30E+04 0.000
  1 :  -1.52E+05 4.18E+03 0.000 0.3219 0.9000 0.9000   1.84  1
1  9.5E+01
  2 :  -7.81E+02 9.86E+02 0.000 0.2357 0.9000 0.9000   2.90  1
1  1.1E+01
  3 :   5.51E+02 1.92E+02 0.000 0.1951 0.9000 0.9000   1.13  1
1  2.0E+00
  4 :   1.05E+02 3.63E+01 0.000 0.1885 0.9000 0.9000   1.11  1
1  3.5E-01
  5 :   5.36E+01 8.38E+00 0.000 0.2310 0.9000 0.9000   1.08  1
1  7.7E-02
  6 :   5.56E+00 1.52E+00 0.000 0.1820 0.9000 0.9000   1.02  1
1  1.4E-02
  7 :   3.89E-02 3.14E-03 0.000 0.0021 0.9990 0.9990   1.00  1
1  1.2E-04
  8 :   8.25E-03 2.61E-04 0.329 0.0832 0.9900 0.9900   0.98  1
1  1.0E-05
  9 :   6.07E-03 1.31E-05 0.000 0.0502 0.9000 0.9287   0.98  3
3  1.9E-05
 10 :   6.53E-03 3.32E-10 0.000 0.0000 0.9000 0.8968   1.00  3
3  8.0E-07
 11 :   6.71E-03 9.52E-11 0.000 0.2871 0.9000 0.9000   1.00  3
3  2.3E-07
 12 :   6.82E-03 1.82E-11 0.000 0.1911 0.9000 0.9000   1.00  4
4  4.4E-08
```

```
 13 :   6.83E-03 1.38E-12 0.238 0.0760 0.9900 0.9900   1.00   4
4  3.3E-09

iter seconds digits        c*x                b*y
 13      0.0    5.4   6.8318450548e-03   6.8325593786e-03
|Ax-b| =   4.0e-09, [Ay-c]_+ =   0.0E+00, |x|=  1.0e+00, |y|=
2.4e+04

Detailed timing (sec)
   Pre            IPM            Post
3.993E-03    4.801E-02    9.958E-04
Max-norms: ||b||=1, ||c|| = 5.693186e+05,
Cholesky |add|=1, |skip| = 0, ||L.L|| = 1.75001e+06.
```

$$MMT, tVars, vSol := \left[\left[3618, 234, 3252, -\frac{35}{123}\_tl_5 - \frac{301}{328}\_tl_6 - \frac{5}{82}\_tl_{13} - \frac{122933}{41}, \right. \right.$$

$$\frac{221}{41}\_tl_5 + \frac{5829}{820}\_tl_6 + \frac{21}{41}\_tl_{13} - \frac{3}{5}\_tl_7 + \frac{769122}{41}, \frac{1433}{41}\_tl_5 + \frac{16993}{205}\_tl_6$$

$$\left. + \frac{228}{41}\_tl_{13} - \frac{9}{5}\_tl_7 + \frac{9142646}{41} \right],$$

$$\left[234, \frac{70}{123}\_tl_5 + \frac{301}{164}\_tl_6 + \frac{5}{41}\_tl_{13} + \frac{200602}{41}, -\frac{221}{41}\_tl_5 - \frac{5829}{820}\_tl_6 \right.$$

$$- \frac{21}{41}\_tl_{13} + \frac{3}{5}\_tl_7 - \frac{715494}{41}, -72, -\frac{116045}{41} - \frac{301}{328}\_tl_6 - \frac{35}{123}\_tl_5$$

$$\left. - \frac{5}{82}\_tl_{13}, -\frac{1105}{369}\_tl_5 - \frac{2107}{492}\_tl_6 - \frac{35}{123}\_tl_{13} - \frac{1223404}{123} \right],$$

$$\left[3252, -\frac{221}{41}\_tl_5 - \frac{5829}{820}\_tl_6 - \frac{21}{41}\_tl_{13} + \frac{3}{5}\_tl_7 - \frac{715494}{41}, -\frac{2866}{41}\_tl_5 \right.$$

$$- \frac{33986}{205}\_tl_6 - \frac{456}{41}\_tl_{13} + \frac{18}{5}\_tl_7 - \frac{15928571}{41}, \_tl_6, \frac{1343}{369}\_tl_5 + \frac{8321}{492}\_tl_6$$

$$\left. + \frac{52}{123}\_tl_{13} + \frac{2292704}{123}, \frac{238}{123}\_tl_5 - \frac{2305}{82}\_tl_6 + \frac{17}{41}\_tl_{13} - \frac{99774}{41} \right],$$

$$\left[-\frac{35}{123}\_tl_5 - \frac{301}{328}\_tl_6 - \frac{5}{82}\_tl_{13} - \frac{122933}{41}, -72, \_tl_6, 160, -\frac{476905}{41} - \frac{380}{123}\_tl_5 \right.$$

$$\left. - \frac{1505}{328}\_tl_6 - \frac{25}{82}\_tl_{13}, -\frac{275}{41}\_tl_5 - \frac{903}{164}\_tl_6 - \frac{15}{41}\_tl_{13} - \frac{763838}{41} \right],$$

$$\left[\frac{221}{41}\_tl_5 + \frac{5829}{820}\_tl_6 + \frac{21}{41}\_tl_{13} - \frac{3}{5}\_tl_7 + \frac{769122}{41}, -\frac{116045}{41} - \frac{301}{328}\_tl_6 \right.$$

$$- \frac{35}{123}\_tl_5 - \frac{5}{82}\_tl_{13}, \frac{1343}{369}\_tl_5 + \frac{8321}{492}\_tl_6 + \frac{52}{123}\_tl_{13} + \frac{2292704}{123}, -\frac{476905}{41}$$

$$- \frac{380}{123}\_tl_5 - \frac{1505}{328}\_tl_6 - \frac{25}{82}\_tl_{13}, -\frac{27463}{738}\_tl_5 - \frac{40195}{984}\_tl_6 - \frac{635}{246}\_tl_{13}$$

$$- 2\_tl_7 - \frac{12890255}{123}, \frac{10763}{246}\_tl_5 + \frac{580459}{4920}\_tl_6 + \frac{637}{82}\_tl_{13} - \frac{3}{5}\_tl_7$$

$$\left. + \frac{35013091}{123} \right],$$

$$\left[\frac{1433}{41}\_t1_5 + \frac{16993}{205}\_t1_6 + \frac{228}{41}\_t1_{13} - \frac{9}{5}\_t1_7 + \frac{9142646}{41}, -\frac{1105}{369}\_t1_5 \right.$$

$$-\frac{2107}{492}\_t1_6 - \frac{35}{123}\_t1_{13} - \frac{1223404}{123}, \frac{238}{123}\_t1_5 - \frac{2305}{82}\_t1_6 + \frac{17}{41}\_t1_{13}$$

$$-\frac{99774}{41}, -\frac{275}{41}\_t1_5 - \frac{903}{164}\_t1_6 - \frac{15}{41}\_t1_{13} - \frac{763838}{41}, \frac{10763}{246}\_t1_5 + \frac{580459}{4920}\_t1_6$$

$$+\frac{637}{82}\_t1_{13} - \frac{3}{5}\_t1_7 + \frac{35013091}{123}, \frac{21269}{82}\_t1_5 - \frac{53617}{1640}\_t1_6 - \frac{213}{82}\_t1_{13}$$

$$\left. +\frac{72}{5}\_t1_7 + \frac{3117329}{41}\right]\right], \{\_t1_5, \_t1_6, \_t1_7, \_t1_{13}\}, \begin{bmatrix} 190.971040616505 \\ 934.503025558974 \\ 1694.55649627063 \\ 24274.7383582838 \\ -0.00683255937863813 \end{bmatrix}$$

$$\begin{bmatrix} 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 383.5376545 \\ 3463.279519 \\ 8144.903009 \\ 41263.53537 \\ 52762.88867 \\ 1.663858888 \ 10^5 \end{bmatrix}$$

(25)

```
> # Verification
  L, DD, Lt, fSol, a, p := matrixToPoly(Q6Sol, v) :
  fSol;
```

$$\frac{186668504897587}{9628765950}\left(x^3 z + \frac{17435189552945}{336003308156566} x^2 y^2 - \frac{90942931196749}{1120011029385522} x^2 y z \right.$$

(26)

$$\left. -\frac{259657938304291}{560005514692761} x^2 z^2 - \frac{214348225496155}{1680016544078283} x y^2 z + \frac{15609223220747815}{40320397057878792} x y z^2 \right.$$

$$+ \frac{3221005067103799}{13440132352626264} x z^3 - \frac{4200265730203955}{40320397057878792} y^2 z^2 - \frac{39635312689433}{189297638769384} y z^3$$

$$+ \frac{346539632313553}{1120011029385522} z^4 \Big)^2 + 3618 \left(x^4 + \frac{13}{201} x^3 y + \frac{542}{603} x^3 z - \frac{298675}{1780056} x^2 y^2 \right.$$

$$- \frac{131639}{1483380} x^2 y z + \frac{716572}{370845} x^2 z^2 + \frac{269155}{1780056} x y^2 z - \frac{604979}{1335042} x y z^2 - \frac{206459}{445014} x z^3$$

$$+ \frac{3182689}{21360672} y^2 z^2 - \frac{2130911}{35601120} y z^3 - \frac{4511089}{2966760} z^4 \Big)^2 + \frac{1565653}{16482} \left(x^3 y \right.$$

$$+ \frac{116918559}{7828265} x^3 z - \frac{3237449}{9393918} x^2 y^2 - \frac{68943751}{15656530} x^2 y z + \frac{266403973}{23484795} x^2 z^2$$

$$+ \frac{952636}{4696959} x y^2 z + \frac{116308433}{112727016} x y z^2 + \frac{6766981}{187878360} x z^3 + \frac{59641211}{112727016} y^2 z^2$$

$$- \frac{348087757}{187878360} y z^3 - \frac{161573453}{15656530} z^4 \Big)^2$$

$$+ \frac{9166346794467568181242571 4071}{1251608934091982428071288450} \left(x^2 y z \right.$$

$$+ \frac{5136293090944600248906324 0151}{18332693588935136362485142 8142} x^2 z^2$$

$$+ \frac{3635149857359406597949104 4275}{18332693588935136362485142 8142} x y^2 z$$

$$+ \frac{1534033239805669584334522 068405}{7333077435574054544994057 12568} x y z^2$$

$$- \frac{4994695904011824665582069 483385}{7333077435574054544994057 12568} x z^3$$

$$+ \frac{8075801837259746201857840 07885}{5499808076680540908745542 84426} y^2 z^2$$

$$+ \frac{2051378350335078032125460 54707}{18332693588935136362485142 8142} y z^3 - \frac{9036635230387002254985692 9767}{9166346794467568181242571 4071} z^4 \Big)$$

$$^2 + \frac{1256257085307620624 3815}{27111434981717699 7408} \left(x^2 y^2 + \frac{3672152317226165952 11726}{3768771255922861873 1445} x^2 y z \right.$$

$$- \frac{4721091286311756426 49024}{1256257085307620624 3815} x^2 z^2 - \frac{47357736046271609 4971}{25125141706152412 48763} x y^2 z$$

$$+ \frac{4080872320075553080 45526}{3768771255922861873 1445} x y z^2 - \frac{26476191534277799 2705834}{12562570853076206 243815} x z^3$$

$$- \frac{1949622682744722351 56587}{9045051014214868495 5468} y^2 z^2 - \frac{48158817943490424 083563}{15075085023691447 4925780} y z^3$$

$$+ \frac{2316631308572176879 24843}{1256257085307620624 3815} z^4 \Big)^2$$

$$+ \frac{23963940563781293336406067 535615395}{81177167211804783813084212 3812776} \left(x^2 z^2 - x y^2 z - \frac{211}{20} x y z^2 \right.$$

$$+ \frac{687}{20} x z^3 + y^2 z^2 - \frac{3}{5} y z^3 - \frac{54}{5} z^4 \Big)^2$$

> *simplify*(*f* − *fSol*);