```
> currentdir("../rationalSOS");
      "C:\Users\Santiago\Documents\GitHub\sums-of-squares\positive-nonRationalSOS"     (1)
> ##########################################################################
  # Example of a strictly positive polynomial that is a SOS over R
  # but not over Q
  ##########################################################################

> ##########################################################################
  # Load "Rational SOS" procedures
  ##########################################################################
  read("rationalSOS.mpl");
  with(rationalSOS);

  # Display tables of any size
  interface(rtablesize=infinity);

                     rationalSOS := module( ) ... end module
[cancelDenominator, decompositionToMatrix, dimSimplex, evalMat, evalSolution, exactSOS,
   getCoeffs, getDiag, getExtension, getVars, homogenize, isHomogeneous, linIndepRows,
   listSubsets, matrixToPoly, minorsDet, nonRatCoef, numericSolver,
   numericSolverSubmatrix, numericSolverSubmatrixMaxRank,
   numericSolverSubmatrixRoundBefore, polyToMatrix, polyToMatrixVars, primitiveMatrix,
   randomRank, reduceByLinearEquation, reduceByLinearEquationLinear, roundAbs,
   roundMat, roundMatFloat, roundMatToZero, roundToIntMatrix, roundVec, rround,
   rrounde, sedumiCallMaxSpectralNorm, sedumiCallMaxSpectralNormSDP,
   sedumiCallObjective, smallToZero, smallToZeroMatrix, solveSubmatrixGeneral,
   solveSubset, solveToZero, vectorTrace, zeroDetSRows, zeroDetSys, zeroRows]
                                      10                                          (2)
> matlabConnection := 1;
      ## Use this option if you have a working connection with Matlab. Matlab is used only for
      numerical experiments.
  #matlabConnection := 0;    ## Use this option if you don't have a working connection with
      Matlab.

                            matlabConnection := 1                                 (3)
> ##########################################################################
  # Construction of the example
  ##########################################################################

  ##########################################################################
  # 1) A polynomial of degree 4 in 4 variables with no rational
  #    decomposition. This example was constructed by J. Capco,
  #    S. Laplagne and C. Scheiderer.
  ##########################################################################

  # We define a polynomial as the sum of three squares in an algebraic
```

```
# extension of degree 3 with generic coefficients.

mp := t^3-2;
p1 := c1*t^2 + b1*t + a1;
p2 := c2*t^2 + b2*t + a2;
p3 := c3*t^2 + b3*t + a3;

fGeneric := p1^2 + p2^2 + p3^2;
fGeneric := expand(fGeneric);
```

$$mp := t^3 - 2$$
$$p1 := -t^2\,x0 + t\,x0 + a1$$
$$p2 := -t^2\,x0 - t\,x0 + a2$$
$$p3 := t^2\,x3 + t\,x1 + x2$$
$$fGeneric := \left(-t^2\,x0 + t\,x0 + a1\right)^2 + \left(-t^2\,x0 - t\,x0 + a2\right)^2 + \left(t^2\,x3 + t\,x1 + x2\right)^2$$
$$fGeneric := 2\,t^4\,x0^2 + t^4\,x3^2 + 2\,t^3\,x1\,x3 - 2\,a1\,t^2\,x0 - 2\,a2\,t^2\,x0 + 2\,t^2\,x0^2 + t^2\,x1^2 + 2\,t^2\,x2\,x3 + 2\,a1\,t\,x0 - 2\,a2\,t\,x0 + 2\,t\,x1\,x2 + a1^2 + a2^2 + x2^2 \quad \textbf{(4)}$$

```
> # We choose arbitrary parameters so that all terms in the resulting
  # expressions for a1 and a2 are multiples of x0, and the degree
  # is reduced after cancellation.
  b2 := -b1; c2 := b2; c1 := b2;
  b1 := x0; b3 := x1; a3 := x2; c3 := x3;
```

$$b2 := -x0$$
$$c2 := -x0$$
$$c1 := -x0$$
$$b1 := x0$$
$$b3 := x1$$
$$a3 := x2$$
$$c3 := x3 \quad \textbf{(5)}$$

```
> # We solve the coefficients a1 and a2 so that the polynomial is in Q,
  f2 := NormalForm(fGeneric, [mp], plex(a1, a2, x0, x1, x2, x3, t));
  f3 := collect(f2, t);
  lf := CoefficientList(f3, t);
  ss := solve({lf[2], lf[3]}, {a1, a2});
```

$$f2 := -2\,a1\,t^2\,x0 - 2\,a2\,t^2\,x0 + 2\,t^2\,x0^2 + t^2\,x1^2 + 2\,t^2\,x2\,x3 + 2\,a1\,t\,x0 - 2\,a2\,t\,x0 + 4\,t\,x0^2 + 2\,t\,x1\,x2 + 2\,t\,x3^2 + a1^2 + a2^2 + 4\,x1\,x3 + x2^2$$
$$f3 := \left(-2\,a1\,x0 - 2\,a2\,x0 + 2\,x0^2 + x1^2 + 2\,x2\,x3\right)t^2 + \left(2\,a1\,x0 - 2\,a2\,x0 + 4\,x0^2 + 2\,x1\,x2 + 2\,x3^2\right)t + a1^2 + a2^2 + 4\,x1\,x3 + x2^2$$
$$lf := \left[a1^2 + a2^2 + 4\,x1\,x3 + x2^2,\ 2\,a1\,x0 - 2\,a2\,x0 + 4\,x0^2 + 2\,x1\,x2 + 2\,x3^2,\ -2\,a1\,x0 - 2\,a2\,x0 + 2\,x0^2 + x1^2 + 2\,x2\,x3\right]$$

$$ss := \left\{a1 = -\frac{1}{4}\,\frac{2\,x0^2 - x1^2 + 2\,x1\,x2 - 2\,x2\,x3 + 2\,x3^2}{x0},\ a2 = \frac{1}{4}\,\frac{6\,x0^2 + x1^2 + 2\,x1\,x2 + 2\,x2\,x3 + 2\,x3^2}{x0}\right\} \quad \textbf{(6)}$$

```
> # We plug in the solutions found for a1 and a2 and compute the resulting polynomial
ssDen := denom(rhs(ss[1]));
p1s := simplify(subs(ss, p1) * ssDen);
p2s := simplify(subs(ss, p2) * ssDen);
p3s := simplify(subs(ss, p3) * ssDen);
```

$$ssDen := 4\,x0$$

$$p1s := -4\,t^2\,x0^2 + 4\,t\,x0^2 - 2\,x0^2 + x1^2 - 2\,x1\,x2 + 2\,x2\,x3 - 2\,x3^2$$

$$p2s := -4\,t^2\,x0^2 - 4\,t\,x0^2 + 6\,x0^2 + x1^2 + 2\,x1\,x2 + 2\,x2\,x3 + 2\,x3^2$$

$$p3s := 4\,\left(t^2\,x3 + t\,x1 + x2\right)\,x0 \quad \textbf{(7)}$$

```
> # We replace t by the root of X^3-2
p1ss := subs({t=RootOf(X^3-2)}, p1s);
p2ss := subs({t=RootOf(X^3-2)}, p2s);
p3ss := subs({t=RootOf(X^3-2)}, p3s);
```

$$p1ss := -4\,x0^2\,RootOf(\_Z^3 - 2)^2 + 4\,x0^2\,RootOf(\_Z^3 - 2) - 2\,x0^2 + x1^2 - 2\,x1\,x2 + 2\,x2\,x3 - 2\,x3^2$$

$$p2ss := -4\,x0^2\,RootOf(\_Z^3 - 2)^2 - 4\,x0^2\,RootOf(\_Z^3 - 2) + 6\,x0^2 + x1^2 + 2\,x1\,x2 + 2\,x2\,x3 + 2\,x3^2$$

$$p3ss := 4\,\left(RootOf(\_Z^3 - 2)^2\,x3 + RootOf(\_Z^3 - 2)\,x1 + x2\right)\,x0 \quad \textbf{(8)}$$

```
> # We compute f and verify that it is has rational coefficients
f := p1ss^2 + p2ss^2 + p3ss^2;
f := simplify(f);
```

$$f := \left(-4\,x0^2\,RootOf(\_Z^3 - 2)^2 + 4\,x0^2\,RootOf(\_Z^3 - 2) - 2\,x0^2 + x1^2 - 2\,x1\,x2 + 2\,x2\,x3 - 2\,x3^2\right)^2 + \left(-4\,x0^2\,RootOf(\_Z^3 - 2)^2 - 4\,x0^2\,RootOf(\_Z^3 - 2) + 6\,x0^2 + x1^2 + 2\,x1\,x2 + 2\,x2\,x3 + 2\,x3^2\right)^2 + 16\,\left(RootOf(\_Z^3 - 2)^2\,x3 + RootOf(\_Z^3 - 2)\,x1 + x2\right)^2\,x0^2$$

$$f := 40\,x0^4 + 8\,x0^2\,x1^2 + 32\,x0^2\,x1\,x2 + 64\,x0^2\,x1\,x3 + 16\,x0^2\,x2^2 + 16\,x0^2\,x2\,x3 + 32\,x0^2\,x3^2 + 2\,x1^4 + 8\,x1^2\,x2^2 + 8\,x1^2\,x2\,x3 + 16\,x1\,x2\,x3^2 + 8\,x2^2\,x3^2 + 8\,x3^4 \quad \textbf{(9)}$$

```
> # We verify that there is no solution with x2 = 0
sols := solve({p1ss, p2ss, p3ss, x2});
```

$$sols := \{x0 = 0, x1 = 0, x2 = 0, x3 = 0\} \quad \textbf{(10)}$$

```
> # Computation of a quadratic form associated to a linear functional in
# the dual of the cone of SOS polynomials that vanishes at f
```

```
# with kernel of small dimension.

# We construct all monomials of degree 2 and 4
d := 2;
polVarsX := [x0, x1, x2, x3];
varSum := add(polVarsX[i], i = 1 ..nops(polVarsX)) :
md := expand((varSum)^d) :
cfs := coeffs(md, polVarsX, 'ctdX') :
print("Monomials of degree d: ", ctdX);
m2d := expand(varSum^(2 * d)) :
cfs := coeffs(m2d, polVarsX, 'ct2dX') :
print("Monomials of degree 2d: ", ct2dX);
```

$$d := 2$$

$$polVarsX := [x0, x1, x2, x3]$$

$$\text{"Monomials of degree d: "}, x0^2, x1\,x0, x0\,x2, x3\,x0, x1^2, x1\,x2, x1\,x3, x2^2, x2\,x3, x3^2$$

$$\text{"Monomials of degree 2d: "}, x0^4, x0^3\,x1, x0^3\,x2, x0^3\,x3, x0^2\,x1^2, x0^2\,x1\,x2, x0^2\,x1\,x3, x0^2\,x2^2, x0^2\,x2\,x3, x0^2\,x3^2, x0\,x1^3, x0\,x1^2\,x2, x0\,x1^2\,x3, x0\,x1\,x2^2, x0\,x1\,x2\,x3, x0\,x1\,x3^2, x0\,x2^3, x0\,x2^2\,x3, x0\,x2\,x3^2, x0\,x3^3, x1^4, x1^3\,x2, x1^3\,x3, x1^2\,x2^2, x1^2\,x2\,x3, x3^2\,x1^2, x1\,x2^3, x1\,x2^2\,x3, x1\,x2\,x3^2, x1\,x3^3, x2^4, x2^3\,x3, x2^2\,x3^2, x2\,x3^3, x3^4 \tag{11}$$

```
> # We construct the linear form that vanishes at all products piss * h
pListX := [p1ss, p2ss, p3ss] :
MX := [ ];
for j from 1 to nops(pListX) do
 for i from 1 to nops([ctdX]) do
  p1tX := expand(pListX[j] * ctdX[i]);
  vec := LinearAlgebra[Transpose](getCoeffs(p1tX, [ct2dX]));
  if (nops(MX) = 0) then
   MX := ⟨vec⟩;
  else
   MX := ⟨MX, vec⟩;
  end if;
 end do:
end do:
rc := [Dimension(MX)];
nr := rc[1];
B := Vector(nr) :
sX := LinearAlgebra[LinearSolve](MX, B) :
varssX := indets(sX);
nops(varssX); # 8 indeterminates left to solve
```

$$MX := [\ ]$$

$$rc := [30, 35]$$

$$nr := 30$$

$$varssX := \{\_t1_{2}, \_t1_{7}, \_t1_{10}, \_t1_{20}, \_t1_{31}, \_t1_{33}, \_t1_{34}, \_t1_{35}\}$$

$$8 \tag{12}$$

```
> # We define a generic polynomial of degree d with coefficientes q_i.
  qIndX := [seq(q[i], i = 1 ..nops([ctdX]))]:
  psaX := add(q[i] * ctdX[i], i = 1 ..nops([ctdX]));
```

$$psaX := x0^2 q_1 + x0\,x1\,q_2 + x0\,x2\,q_3 + x0\,x3\,q_4 + x1^2 q_5 + x1\,x2\,q_6 + x1\,x3\,q_7 + x2^2 q_8 + x2\,x3\,q_9 + x3^2 q_{10} \tag{13}$$

```
> # The square of psaX and the coefficients
  ps2X := expand(psaX * psaX):
  aaX := getCoeffs(expand(ps2X), [ct2dX]):
```

```
> # We compute the form and look for a PSD matrix using SEDUMI
  ewX := LinearAlgebra[Transpose](sX) . aaX:
  ooX := polyToMatrixVars(expand(ewX), qIndX):
```

```
> #####
  ## This block can be skipped if Matlab is not connected.
   ####
   if (matlabConnection = 1) then
     # Numerical optimization using SEDUMI.
     # Numerically, the matrix has kernel of dimension 6.
     oEval := ooX[1]:
     outA := numericSolverSubmatrixMaxRank(evalf(oEval), "eig"):
     QEval := eval(oEval, Equate([op(outA[2])], outA[3][1 ..nops([op(outA[2])])])):
     smallToZeroMatrix(evalf(QEval), 8);
     print(eig(QEval));
   end if:
```

"SEDUMI CALL - eig"

"Opening connection with Matlab"

$$\begin{bmatrix} -1.20089925866854\ 10^{-8} \\ -2.52574800579448\ 10^{-11} \\ -5.78923235550882\ 10^{-12} \\ 8.03840592758803\ 10^{-11} \\ 1.41951864243287\ 10^{-9} \\ 6.58154553229688\ 10^{-9} \\ 0.335198431470853 \\ 0.480960966002649 \\ 0.535333846973967 \\ 2.00597125966693 \end{bmatrix} \tag{14}$$

```
> # Based on the numerical solution, we compute exact values of the
  # unknowns to construct an exact PSD matrix.
```

```
# The values obtained by SEDUMI are:
# t_1 = -0.0000, t_2 = -0.1996, t_3 = 0.1272, t_4 = 0.0000,
# t_5 =  0.4435, t_6 =  0.0724, t_7 = 0.0080, t_8 = 0.2442

## We fix some variables to 0
oEvalX := eval(ooX[1], {varssX[1] = 0, varssX[4] = 0, varssX[7] = 0}) :

## We fix other variables with positive values
oEvalX := eval(oEvalX, {varssX[5] = 1, varssX[8] = 1, varssX[3] = 1/2}) :
```

```
> ## We compute exact values for the remaining two variables.
```

```
> #####
## This block can be skipped if Matlab is not connected.
## ##
# We observe that in the numerical solution, there are two singular
 # principal matrices.
if (matlabConnection = 1) then
  evalf(Determinant(QEval[5 .. 7, 5 .. 7]));
  evalf(Determinant(QEval[2 .. 3, 2 .. 3]));
 end if;
```

$$-4.40990888250781\ 10^{-9}$$

$$1.53690277171492\ 10^{-9} \tag{15}$$

```
> # We compute symbolically the values of the unknowns that make the matrices singular.
S_23 := solve(Determinant(oEvalX[2 .. 3, 2 .. 3]), indets(oEvalX[2 .. 3, 2 .. 3])) :
S_23_1 := simplify(S_23[1]);
```

$$S\_23\_1 := \left\{\_t1_7 = -\frac{1}{4}\left(\mathrm{RootOf}(\_Z^3-2)^2 + \sqrt{-\mathrm{RootOf}(\_Z^3-2)^2+2\,\mathrm{RootOf}(\_Z^3-2)+1}-1\right)\mathrm{RootOf}(\_Z^3-2)^2\right\} \tag{16}$$

```
> oX_567 := eval(oEvalX[5 .. 7, 5 .. 7], S_23_1) :
S_567 := solve(Determinant(oX_567), indets(oX_567)) :
S_567_2 := simplify(S_567[2]);
```

$$S\_567\_2 := \left\{\_t1_{33} = \frac{1}{2}\,\mathrm{RootOf}(\_Z^3-2)^2\left(\sqrt{-\mathrm{RootOf}(\_Z^3-2)^2+2\,\mathrm{RootOf}(\_Z^3-2)+1}-1\right)\right\} \tag{17}$$

```
> # We apply these substitutions (using simplified expressions equivalent to the results by Maple)
rA := RootOf(_Z^3-2) :
rB := RootOf(_Z*rA^2-rA^2 + _Z^2 + 1) :
rC := -(1/2)*rA-(1/2)*rB :
```

```
> oEvalX := eval(oEvalX, {varssX[2] = rC}) :
oEvalX := eval(oEvalX, {varssX[6] = rB}) :
```

```
> # No indetermiantes in the resulting oEvalX
evalf(oEvalX);
```

[[0.3728809420, 0., 0., 0., 1.231761842, -0.3061607886, -0.7847808106, 0.07609785017, 0.1950613368, 0.5000000000], **(18)**
[0., 1.231761842, -0.3061607886, -0.7847808106, 0., 0., 0., 0., 0., 0.],
[0., -0.3061607886, 0.07609785017, 0.1950613368, 0., 0., 0., 0., 0., 0.],
[0., -0.7847808106, 0.1950613368, 0.5000000000, 0., 0., 0., 0., 0., 0.],
[1.231761842, 0., 0., 0., 4.899359550, -1.534480445, -1.933328015, 0.580923683, 0.229158469, 2.174802104],
[-0.3061607886, 0., 0., 0., -1.534480445, 0.580923683, 0.229158469, -0.2700817033, 0.101401094, -0.740078950],
[-0.7847808106, 0., 0., 0., -1.933328015, 0.229158469, 2.174802104, 0.101401094, -0.740078950, -0.6371205733],
[0.07609785017, 0., 0., 0., 0.580923683, -0.2700817033, 0.101401094, 1., -0.124964077, 0.3096405713],
[0.1950613368, 0., 0., 0., 0.229158469, 0.101401094, -0.740078950, -0.124964077, 0.3096405713, 0.],
[0.5000000000, 0., 0., 0., 2.174802104, -0.740078950, -0.6371205733, 0.3096405713, 0., 1.]]

```
> # The matrix has kernel of rank 6
  evalf(Eigenvalues(oEvalX));
```

$$\left[\begin{array}{c} 0.773176732402680 \\ 1.95800566763583 \\ 7.60642445121149 \\ 1.807859692 \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{array}\right] \quad \mathbf{(19)}$$

```
> # The kernel of the resulting form
  LX := NullSpace(oEvalX) :
  nops(LX);
```

6 **(20)**

```
> # The polynoials in the kernel
  LinearAlgebra[Transpose](simplify(LX[01]));
  LinearAlgebra[Transpose](simplify(LX[02]));
  LinearAlgebra[Transpose](simplify(LX[03]));
  LinearAlgebra[Transpose](simplify(LX[04]));
```

```
LinearAlgebra[Transpose](simplify(LX[05]));
LinearAlgebra[Transpose](simplify(LX[06]));
```

$$\Big[-\mathrm{RootOf}(\_Z^3-2)+2, 0, 0, 0, \frac{1}{2}\big(-2+2\,\mathrm{RootOf}(\_Z^3-2)^2-\mathrm{RootOf}(\_Z^3-2)\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)\big)\Big/\big(\mathrm{RootOf}(\_Z^3-2)^2+\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)-2\,\mathrm{RootOf}(\_Z^3-2)\big), 0, 0, 0, 0, 1\Big]$$

$$\begin{bmatrix} -2\,\mathrm{RootOf}(\_Z^3-2)^2+1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Big[\mathrm{RootOf}(\_Z^3-2)^2+2\,\mathrm{RootOf}(\_Z^3-2)+2\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big), 0, 0, 0, -\frac{1}{2}\big(\mathrm{RootOf}(\_Z^3-2)^2\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)-4+2\,\mathrm{RootOf}(\_Z^3-2)\big)\Big/\big(\mathrm{RootOf}(\_Z^3-2)^2+\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)-2\,\mathrm{RootOf}(\_Z^3-2)\big), 0, 1, 0, 0, 0\Big]$$

$$\Big[-\mathrm{RootOf}(\_Z^3-2), 0, 0, 0, -\frac{1}{2}\big(-2+2\,\mathrm{RootOf}(\_Z^3-2)^2-\mathrm{RootOf}(\_Z^3-2)\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)\big)\Big/\big(\mathrm{RootOf}(\_Z^3-2)^2+\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)-2\,\mathrm{RootOf}(\_Z^3-2)\big), 1, 0, 0, 0, 0\Big]$$

$$\Big[0, -\big(\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)+\mathrm{RootOf}(\_Z^3-2)\big)\Big/\big(\mathrm{RootOf}(\_Z^3-2)^2\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)-2\,\mathrm{RootOf}(\_Z^3-2)\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)-2\,\mathrm{RootOf}(\_Z^3-2)^2+1\big), 0, 1, 0, 0, 0, 0, 0, 0\Big]$$

$$\Big[0, -\big(\mathrm{RootOf}(\_Z^3-2)^2\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)+2-2\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)-\mathrm{RootOf}(\_Z^3-2)\big)\Big/\big(\mathrm{RootOf}(\_Z^3-2)^2\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)-2\,\mathrm{RootOf}(\_Z^3-2)\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big)-2\,\mathrm{RootOf}(\_Z^3-2)^2+1\big), 1, 0, 0, 0, 0, 0, 0, 0\Big] \tag{21}$$

```
> ##############################################################################
# 2) We add the second block from an example in the strictly
 #    positive border of Sigma(4,4)
##############################################################################

# g is the sum of 4 squares
```

```
g1 := y0^2-y3^2;
g2 := y1^2-y3^2;
g3 := y2^2-y3^2;
g4 := -y0^2 - y0*y1 -y0*y2 + y0*y3 - y1*y2 + y1*y3 + y2*y3;
g  :=  g1^2 + g2^2 + g3^2 + g4^2;
```

$$g1 := y0^2 - y3^2$$
$$g2 := y1^2 - y3^2$$
$$g3 := y2^2 - y3^2$$
$$g4 := -y0^2 - y0\,y1 - y0\,y2 + y0\,y3 - y1\,y2 + y1\,y3 + y2\,y3$$
$$g := \left(y0^2 - y3^2\right)^2 + \left(y1^2 - y3^2\right)^2 + \left(y2^2 - y3^2\right)^2 + \left(-y0^2 - y0\,y1 - y0\,y2 + y0\,y3 - y1\,y2 + y1\,y3 + y2\,y3\right)^2 \tag{22}$$

```
> # We look for a form in the y-monomials that vanishes in the g_i
d := 2;
polVarsY := [y0, y1, y2, y3];
varSumY := add(polVarsY[i], i=1..nops(polVarsY));
mdY := expand((varSumY)^d);
cfsY := coeffs(mdY, polVarsY, 'ctdY');
m2dY := expand(varSumY^(2*d));
cfsY := coeffs(m2dY, polVarsY, 'ct2dY');
```

$$d := 2$$
$$polVarsY := [y0, y1, y2, y3]$$
$$varSumY := y0 + y1 + y2 + y3$$
$$mdY := y0^2 + 2\,y0\,y1 + 2\,y0\,y2 + 2\,y0\,y3 + y1^2 + 2\,y1\,y2 + 2\,y1\,y3 + y2^2 + 2\,y2\,y3 + y3^2$$
$$cfsY := 1, 2, 2, 2, 1, 2, 2, 1, 2, 1$$
$$m2dY := y0^4 + 4\,y0^3\,y1 + 4\,y0^3\,y2 + 4\,y0^3\,y3 + 6\,y0^2\,y1^2 + 12\,y0^2\,y1\,y2 + 12\,y0^2\,y1\,y3 + 6\,y0^2\,y2^2 + 12\,y0^2\,y2\,y3 + 6\,y0^2\,y3^2 + 4\,y0\,y1^3 + 12\,y0\,y1^2\,y2 + 12\,y0\,y1^2\,y3 + 12\,y0\,y1\,y2^2 + 24\,y0\,y1\,y2\,y3 + 12\,y0\,y1\,y3^2 + 4\,y0\,y2^3 + 12\,y0\,y2^2\,y3 + 12\,y0\,y2\,y3^2 + 4\,y0\,y3^3 + y1^4 + 4\,y1^3\,y2 + 4\,y1^3\,y3 + 6\,y1^2\,y2^2 + 12\,y1^2\,y2\,y3 + 6\,y1^2\,y3^2 + 4\,y1\,y2^3 + 12\,y1\,y2^2\,y3 + 12\,y1\,y2\,y3^2 + 4\,y1\,y3^3 + y2^4 + 4\,y2^3\,y3 + 6\,y2^2\,y3^2 + 4\,y2\,y3^3 + y3^4$$
$$cfsY := 1, 4, 4, 4, 6, 12, 12, 6, 12, 6, 4, 12, 12, 12, 24, 12, 4, 12, 12, 4, 1, 4, 4, 6, 12, 6, 4, 12, 12, 4, 1, 4, 6, 4, 1 \tag{23}$$

```
> # We compute all the restrictions to phi: A4 -> R given
# by the five polynomials. There are 20 restrictions for each polynomial

pListY := [g1, g2, g3, g4];
MY := [ ]:
for j from 1 to nops(pListY) do
 for i from 1 to nops([ctdY]) do
  p1t := expand(pListY[j] * ctdY[i]);
  vec := LinearAlgebra[Transpose](getCoeffs(p1t, [ct2dY]));
  if (nops(MY) = 0) then
```

```
      MY := ⟨vec⟩;
    else
      MY := ⟨MY, vec⟩;
    end if;
  end do:
end do:
```

$$pListY := [y0^2 - y3^2, y1^2 - y3^2, y2^2 - y3^2, -y0^2 - y0\,y1 - y0\,y2 + y0\,y3 - y1\,y2 + y1\,y3 + y2\,y3] \tag{24}$$

```
> # We solve the system using only these restrictions
  rc := [Dimension(MY)];
  nr := rc[1];
  B := Vector(nr) :
  s := LinearAlgebra[LinearSolve](MY, B) :
  varssY := indets(s);
  nops(varssY); # Onlt 1 indeterminate left to solve
```

$$rc := [40, 35]$$

$$nr := 40$$

$$varssY := \{\_t2_9\}$$

$$1 \tag{25}$$

```
> # We define a generic polynomial of degree d with coefficientes q_i.
  qIndY := [seq(qY[i], i = 1 ..nops([ctdY]))] :
  psaY := add(qIndY[i] * ctdY[i], i = 1 ..nops([ctdY])) :
  ps2Y := expand(psaY*psaY) :
  aaY := getCoeffs(expand(ps2Y), [ct2dY]) :
```

```
> # We compute the associated quadratic form
  ewY := LinearAlgebra[Transpose](s) . aaY:
  ooY := polyToMatrixVars(expand(ewY), qIndY) :
  ooMY := ooY[1] :
```

```
> # We give value 1 to the only indeterminate and verify that the resulting form is positive definite
  oEvalY := eval(ooMY, {varssY[1] = 1});
```

$$oEvalY := \begin{bmatrix} 6 & -1 & -1 & 1 & 6 & -1 & 1 & 6 & 1 & 6 \\ -1 & 6 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 6 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 6 & 1 & 1 & -1 & 1 & -1 & 1 \\ 6 & -1 & -1 & 1 & 6 & -1 & 1 & 6 & 1 & 6 \\ -1 & -1 & -1 & 1 & -1 & 6 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 6 & 1 & -1 & 1 \\ 6 & -1 & -1 & 1 & 6 & -1 & 1 & 6 & 1 & 6 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 6 & 1 \\ 6 & -1 & -1 & 1 & 6 & -1 & 1 & 6 & 1 & 6 \end{bmatrix} \tag{26}$$

```
> Eigenvalues(oEvalY);
```

$$\begin{bmatrix} 25 \\ 0 \\ 0 \\ 0 \\ 0 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \end{bmatrix} \quad (27)$$

```
> ###############################################################################
  # 3) Finally, we add the two examples plus a third polynomial to
  # ensure that there are no real roots.
  ###############################################################################

  h := (- x2^2 + y2^2);
  p := f + g1^2 + g2^2 + g3^2 + g4^2 + h^2 :
  expand(p);
```

$$h := -x2^2 + y2^2$$

$$40\,x0^4 + 8\,x0^2\,x1^2 + 32\,x0^2\,x1\,x2 + 64\,x0^2\,x1\,x3 + 16\,x0^2\,x2^2 + 16\,x0^2\,x2\,x3 + 32\,x0^2\,x3^2 + 2\,x1^4 + 8\,x1^2\,x2^2 + 8\,x1^2\,x2\,x3 + 16\,x1\,x2\,x3^2 + x2^4 + 8\,x2^2\,x3^2 - 2\,x2^2\,y2^2 + 8\,x3^4 + 2\,y0^4 + 2\,y0^3\,y1 + 2\,y0^3\,y2 - 2\,y0^3\,y3 + y0^2\,y1^2 + 4\,y0^2\,y1\,y2 - 4\,y0^2\,y1\,y3 + y0^2\,y2^2 - 4\,y0^2\,y2\,y3 - y0^2\,y3^2 + 2\,y0\,y1^2\,y2 - 2\,y0\,y1^2\,y3 + 2\,y0\,y1\,y2^2 - 6\,y0\,y1\,y2\,y3 + 2\,y0\,y1\,y3^2 - 2\,y0\,y2^2\,y3 + 2\,y0\,y2\,y3^2 + y1^4 + y1^2\,y2^2 - 2\,y1^2\,y2\,y3 - y1^2\,y3^2 - 2\,y1\,y2^2\,y3 + 2\,y1\,y2\,y3^2 + 2\,y2^4 - y2^2\,y3^2 + 3\,y3^4 \quad (28)$$

```
> # We construct a linear form in the dual of the SOS cone that
  # vanishes in all the 9 polynomials such that the associated
  # quadratic form has kernel of minimal rank.

  d := 2;
  polVarsXY := [x0, x1, x2, x3, y0, y1, y2, y3];
  varSumXY := add(polVarsXY[i], i = 1 ..nops(polVarsXY));
```

$$d := 2$$
$$polVarsXY := [x0, x1, x2, x3, y0, y1, y2, y3]$$
$$varSumXY := x0 + x1 + x2 + x3 + y0 + y1 + y2 + y3 \quad (29)$$

```
> # We use a block ordering for the monomials
  mdXY := [x0^2, x0*x1, x0*x2, x0*x3, x1^2, x1*x2, x1*x3, x2^2, x2*x3, x3^2, y0^2, y0*y1,
     y0*y2, y0*y3, y1^2, y1*y2, y1*y3, y2^2, y2*y3, y3^2, x0*y0, x0*y1, x0*y2, x0*y3, x1
     *y0, x1*y1, x1*y2, x1*y3, x2*y0, x2*y1, x2*y2, x2*y3, x3*y0, x3*y1, x3*y2, x3*y3]
```

```
:
ctdXY := op(mdXY) :
print("Monomials of degree d: ", ctdXY);

m2dXY := expand(varSumXY^(2*d)) :
cfs := coeffs(m2dXY, polVarsXY, 'ct2dXY') :
print("Monomials of degree 2d: ", ct2dXY);
```

"Monomials of degree d: ", $x0^2, x1\,x0, x0\,x2, x3\,x0, x1^2, x1\,x2, x1\,x3, x2^2, x2\,x3, x3^2, y0^2, y0\,y1, y0\,y2, y0\,y3, y1^2, y1\,y2, y1\,y3, y2^2, y2\,y3, y3^2, x0\,y0, x0\,y1, x0\,y2, x0\,y3, x1\,y0, x1\,y1, x1\,y2, x1\,y3, x2\,y0, x2\,y1, x2\,y2, x2\,y3, x3\,y0, x3\,y1, x3\,y2, x3\,y3$

"Monomials of degree 2d: ", $x0^4, x0^3\,x1, x0^3\,x2, x0^3\,x3, x0^3\,y0, x0^3\,y1, x0^3\,y2, x0^3\,y3, x0^2\,x1^2, x0^2\,x1\,x2, x0^2\,x1\,x3, x0^2\,x1\,y0, x0^2\,x1\,y1, x0^2\,x1\,y2, x0^2\,x1\,y3, x0^2\,x2^2, x0^2\,x2\,x3, x0^2\,x2\,y0, x0^2\,x2\,y1, x0^2\,x2\,y2, x0^2\,x2\,y3, x0^2\,x3^2, x0^2\,x3\,y0, x0^2\,x3\,y1, x0^2\,x3\,y2, x0^2\,x3\,y3, x0^2\,y0^2, x0^2\,y0\,y1, x0^2\,y0\,y2, x0^2\,y0\,y3, x0^2\,y1^2, x0^2\,y1\,y2, x0^2\,y1\,y3, x0^2\,y2^2, x0^2\,y2\,y3, x0^2\,y3^2, x0\,x1^3, x0\,x1^2\,x2, x0\,x1^2\,x3, x0\,x1^2\,y0, x0\,x1^2\,y1, x0\,x1^2\,y2, x0\,x1^2\,y3, x0\,x1\,x2^2, x0\,x1\,x2\,x3, x0\,x1\,x2\,y0, x0\,x1\,x2\,y1, x0\,x1\,x2\,y2, x0\,x1\,x2\,y3, x0\,x1\,x3^2, x0\,x1\,x3\,y0, x0\,x1\,x3\,y1, x0\,x1\,x3\,y2, x0\,x1\,x3\,y3, x0\,x1\,y0^2, x0\,x1\,y0\,y1, x0\,x1\,y0\,y2, x0\,x1\,y0\,y3, x0\,x1\,y1^2, x0\,x1\,y1\,y2, x0\,x1\,y1\,y3, x0\,x1\,y2^2, x0\,x1\,y2\,y3, x0\,x1\,y3^2, x0\,x2^3, x0\,x2^2\,x3, x0\,x2^2\,y0, x0\,x2^2\,y1, x0\,x2^2\,y2, x0\,x2^2\,y3, x0\,x2\,x3^2, x0\,x2\,x3\,y0, x0\,x2\,x3\,y1, x0\,x2\,x3\,y2, x0\,x2\,x3\,y3, x0\,x2\,y0^2, x0\,x2\,y0\,y1, x0\,x2\,y0\,y2, x0\,x2\,y0\,y3, x0\,x2\,y1^2, x0\,x2\,y1\,y2, x0\,x2\,y1\,y3, x0\,x2\,y2^2, x0\,x2\,y2\,y3, x0\,x2\,y3^2, x0\,x3^3, x0\,x3^2\,y0, x0\,x3^2\,y1, x0\,x3^2\,y2, x0\,x3^2\,y3, x0\,x3\,y0^2, x0\,x3\,y0\,y1, x0\,x3\,y0\,y2, x0\,x3\,y0\,y3, x0\,x3\,y1^2, x0\,x3\,y1\,y2, x0\,x3\,y1\,y3, x0\,x3\,y2^2, x0\,x3\,y2\,y3, x0\,x3\,y3^2, x0\,y0^3, x0\,y0^2\,y1, x0\,y0^2\,y2, x0\,y0^2\,y3, x0\,y0\,y1^2, x0\,y0\,y1\,y2, x0\,y0\,y1\,y3, x0\,y0\,y2^2, x0\,y0\,y2\,y3, x0\,y0\,y3^2, x0\,y1^3, x0\,y1^2\,y2, x0\,y1^2\,y3, x0\,y1\,y2^2, x0\,y1\,y2\,y3, x0\,y1\,y3^2, x0\,y2^3, x0\,y2^2\,y3, x0\,y2\,y3^2, x0\,y3^3, x1^4, x1^3\,x2, x1^3\,x3, x1^3\,y0, x1^3\,y1, x1^3\,y2, x1^3\,y3, x1^2\,x2^2, x1^2\,x2\,x3, x1^2\,x2\,y0, x1^2\,x2\,y1, x1^2\,x2\,y2, x1^2\,x2\,y3, x3^2\,x1^2, x1^2\,x3\,y0, x1^2\,x3\,y1, x1^2\,x3\,y2, x1^2\,x3\,y3, x1^2\,y0^2, x1^2\,y0\,y1, x1^2\,y0\,y2, x1^2\,y0\,y3, x1^2\,y1^2, x1^2\,y1\,y2, x1^2\,y1\,y3, x1^2\,y2^2, x1^2\,y2\,y3, x1^2\,y3^2, x1\,x2^3, x1\,x2^2\,x3, x1\,x2^2\,y0, x1\,x2^2\,y1, x1\,x2^2\,y2, x1\,x2^2\,y3, x1\,x2\,x3^2, x1\,x2\,x3\,y0, x1\,x2\,x3\,y1, x1\,x2\,x3\,y2, x1\,x2\,x3\,y3, x1\,x2\,y0^2, x1\,x2\,y0\,y1, x1\,x2\,y0\,y2, x1\,x2\,y0\,y3, x1\,x2\,y1^2, x1\,x2\,y1\,y2, x1\,x2\,y1\,y3, x1\,x2\,y2^2, x1\,x2\,y2\,y3, x1\,x2\,y3^2, x1\,x3^3, x1\,x3^2\,y0, x1\,x3^2\,y1, x1\,x3^2\,y2, x1\,x3^2\,y3, x1\,x3\,y0^2, x1\,x3\,y0\,y1, x1\,x3\,y0\,y2, x1\,x3\,y0\,y3, x1\,x3\,y1^2, x1\,x3\,y1\,y2, x1\,x3\,y1\,y3, x1\,x3\,y2^2, x1\,x3\,y2\,y3, x1\,x3\,y3^2, x1\,y0^3, x1\,y0^2\,y1, x1\,y0^2\,y2, x1\,y0^2\,y3, x1\,y0\,y1^2, x1\,y0\,y1\,y2, x1\,y0\,y1\,y3, x1\,y0\,y2^2, x1\,y0\,y2\,y3, x1\,y0\,y3^2, x1\,y1^3, x1\,y1^2\,y2, x1\,y1^2\,y3, x1\,y1\,y2^2, x1\,y1\,y2\,y3, x1\,y1\,y3^2, x1\,y2^3, x1\,y2^2\,y3, x1\,y2\,y3^2, x1\,y3^3, x2^4, x2^3\,x3, x2^3\,y0, x2^3\,y1, x2^3\,y2, x2^3\,y3, x2^2\,x3^2, x2^2\,x3\,y0, x2^2\,x3\,y1, x2^2\,x3\,y2, x2^2\,x3\,y3, x2^2\,y0^2, x2^2\,y0\,y1, x2^2\,y0\,y2, x2^2\,y0\,y3, x2^2\,y1^2, x2^2\,y1\,y2, x2^2\,y1\,y3, x2^2\,y2^2, x2^2\,y2\,y3, x2^2\,y3^2, x2\,x3^3, x2\,x3^2\,y0, x2\,x3^2\,y1, x2\,x3^2\,y2, x2\,x3^2\,y3, x2\,x3\,y0^2,$ **(30)**

$x2\,x3\,y0\,y1, x2\,x3\,y0\,y2, x2\,x3\,y0\,y3, x2\,x3\,y1^2, x2\,x3\,y1\,y2, x2\,x3\,y1\,y3, x2\,x3\,y2^2,$
$x2\,x3\,y2\,y3, x2\,x3\,y3^2, x2\,y0^3, x2\,y0^2\,y1, x2\,y0^2\,y2, x2\,y0^2\,y3, x2\,y0\,y1^2, x2\,y0\,y1\,y2,$
$x2\,y0\,y1\,y3, x2\,y0\,y2^2, x2\,y0\,y2\,y3, x2\,y0\,y3^2, x2\,y1^3, x2\,y1^2\,y2, x2\,y1^2\,y3, x2\,y1\,y2^2,$
$x2\,y1\,y2\,y3, x2\,y1\,y3^2, x2\,y2^3, x2\,y2^2\,y3, x2\,y2\,y3^2, x2\,y3^3, x3^4, x3^3\,y0, x3^3\,y1, x3^3\,y2, x3^3\,y3,$
$x3^2\,y0^2, x3^2\,y0\,y1, x3^2\,y0\,y2, x3^2\,y0\,y3, x3^2\,y1^2, x3^2\,y1\,y2, x3^2\,y1\,y3, x3^2\,y2^2, x3^2\,y2\,y3,$
$x3^2\,y3^2, x3\,y0^3, x3\,y0^2\,y1, x3\,y0^2\,y2, x3\,y0^2\,y3, x3\,y0\,y1^2, x3\,y0\,y1\,y2, x3\,y0\,y1\,y3, x3\,y0\,y2^2,$
$x3\,y0\,y2\,y3, x3\,y0\,y3^2, x3\,y1^3, x3\,y1^2\,y2, x3\,y1^2\,y3, x3\,y1\,y2^2, x3\,y1\,y2\,y3, x3\,y1\,y3^2, x3\,y2^3,$
$x3\,y2^2\,y3, x3\,y2\,y3^2, x3\,y3^3, y0^4, y0^3\,y1, y0^3\,y2, y0^3\,y3, y0^2\,y1^2, y0^2\,y1\,y2, y0^2\,y1\,y3, y0^2\,y2^2,$
$y0^2\,y2\,y3, y0^2\,y3^2, y0\,y1^3, y0\,y1^2\,y2, y0\,y1^2\,y3, y0\,y1\,y2^2, y0\,y1\,y2\,y3, y0\,y1\,y3^2, y0\,y2^3,$
$y0\,y2^2\,y3, y0\,y2\,y3^2, y0\,y3^3, y1^4, y1^3\,y2, y1^3\,y3, y1^2\,y2^2, y1^2\,y2\,y3, y1^2\,y3^2, y1\,y2^3, y1\,y2^2\,y3,$
$y1\,y2\,y3^2, y1\,y3^3, y2^4, y2^3\,y3, y2^2\,y3^2, y2\,y3^3, y3^4$

```
> # We compute all the restrictions to phi: A4 -> R given
  # by the five polynomials. There are 20 restrictions for each polynomial

  pList := [p1ss, p2ss, p3ss, g1, g2, g3, g4, h];
  MXY := [ ]:
  for j from 1 to nops(pList) do
   for i from 1 to nops([ctdXY]) do
    p1t := expand(pList[j] * ctdXY[i]);
    vec := LinearAlgebra[Transpose](getCoeffs(p1t, [ct2dXY]));
    if (nops(MXY) = 0) then
     MXY := <vec>;
    else
     MXY := <MXY, vec>;
    end if;
   end do:
  end do:
```

$$pList := \left[-4\,x0^2\,RootOf(\_Z^3-2)^2+4\,x0^2\,RootOf(\_Z^3-2)-2\,x0^2+x1^2-2\,x1\,x2+2\,x2\,x3-2\,x3^2, -4\,x0^2\,RootOf(\_Z^3-2)^2-4\,x0^2\,RootOf(\_Z^3-2)+6\,x0^2+x1^2+2\,x1\,x2+2\,x2\,x3+2\,x3^2, 4\left(RootOf(\_Z^3-2)^2\,x3+RootOf(\_Z^3-2)\,x1+x2\right)x0, y0^2-y3^2, y1^2-y3^2, y2^2-y3^2, -y0^2-y0\,y1-y0\,y2+y0\,y3-y1\,y2+y1\,y3+y2\,y3, -x2^2+y2^2\right] \tag{31}$$

```
> # We solve the system using only these restrictions
  rc := [Dimension(MXY)];
  nr := rc[1];
  B := Vector(nr):
  sXY := LinearAlgebra[LinearSolve](MXY, B):
  varssXY := indets(sXY);
  nops(varssXY); # 70 indeterminates left to solve
```

$$rc := [288, 330]$$
$$nr := 288$$

$varssXY := \{\_t3_{2}, \_t3_{5}, \_t3_{6}, \_t3_{7}, \_t3_{8}, \_t3_{11}, \_t3_{12}, \_t3_{13}, \_t3_{14}, \_t3_{15}, \_t3_{22}, \_t3_{23}, \_t3_{24}, \_t3_{25}, \_t3_{26}, \_t3_{29}, \_t3_{30}, \_t3_{32}, \_t3_{33}, \_t3_{35}, \_t3_{52}, \_t3_{53}, \_t3_{54}, \_t3_{57}, \_t3_{58}, \_t3_{60}, \_t3_{61}, \_t3_{63}, \_t3_{86}, \_t3_{93}, \_t3_{94}, \_t3_{96}, \_t3_{97}, \_t3_{99}, \_t3_{115}, \_t3_{177}, \_t3_{178}, \_t3_{180}, \_t3_{181}, \_t3_{183}, \_t3_{199}, \_t3_{226}, \_t3_{228}, \_t3_{229}, \_t3_{230}, \_t3_{233}, \_t3_{234}, \_t3_{236}, \_t3_{237}, \_t3_{239}, \_t3_{255}, \_t3_{256}, \_t3_{259}, \_t3_{260}, \_t3_{261}, \_t3_{262}, \_t3_{263}, \_t3_{264}, \_t3_{265}, \_t3_{268}, \_t3_{269}, \_t3_{271}, \_t3_{272}, \_t3_{274}, \_t3_{275}, \_t3_{290}, \_t3_{291}, \_t3_{294}, \_t3_{295}, \_t3_{330}\}$

$$70 \qquad \textbf{(32)}$$

```
> # We give values to the unknowns so that the form is PSD
  qIndXY := [seq(qXY[i], i=1..nops([ctdXY]))]:
  psaXY := add(qIndXY[i] * ctdXY[i], i=1..nops([ctdXY])) :
  ps2XY := expand(psaXY* psaXY) :
  aaXY := getCoeffs(expand(ps2XY), [ct2dXY]) :
```

```
> # We compute the associated form
  ewXY := LinearAlgebra[Transpose](sXY) . aaXY :
  ooXY := polyToMatrixVars(expand(ewXY), qIndXY) :
  ooMXY := ooXY[1]:
```

```
> # We copy Q_x and Q_y in Q_xy
  s1 := solve(Equate(ooMXY[1..10, 1..10], oEvalX)) :
  oEvalXY := eval(ooMXY, {ooMXY[20, 20]=6}) :
  oEvalXY := eval(oEvalXY, s1) :
```

```
> # We replace all the remaining variables by 0
  s2 := solve(Equate([op(indets(oEvalXY))], ZeroVector(nops(indets(oEvalXY))))) :
  oEvalXY := eval(oEvalXY, s2) :
```

```
> # We verify that the matrix is positive semidefinite and has kernel
  # of rank 14.
  evalf(Eigenvalues(oEvalXY));
```

$$\begin{bmatrix} 0.551105095682439 \\ 1.16935580263953 \\ 7.06066760967802 \\ 0.275755761932722 \\ 0.600977973484481 \\ 7.01383051958280 \\ 1.807859692 \\ 0.275620309240295 \\ 0.589700596013643 \\ 3.01830865108949 \\ 7.00693469865657 \\ 1.72628035929815 \\ 6.66336637618740 \\ 7.83746821087346 \\ 31.1104919036410 \\ 0.152195701 \\ 0.076097850 \\ 0.076097850 \\ 7. \\ 7. \\ 7. \\ 7. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{bmatrix} \tag{33}$$

> # The kernel is generated by 14 polynomials
L := NullSpace(oEvalXY) :
nops(L);

$$14 \tag{34}$$

> # The polynoials in the kernel
simplify(L[01]) . convert(mdXY, Vector);
simplify(L[02]) . convert(mdXY, Vector);
simplify(L[03]) . convert(mdXY, Vector);
simplify(L[04]) . convert(mdXY, Vector);
simplify(L[05]) . convert(mdXY, Vector);
simplify(L[06]) . convert(mdXY, Vector);
simplify(L[07]) . convert(mdXY, Vector);
simplify(L[08]) . convert(mdXY, Vector);
simplify(L[09]) . convert(mdXY, Vector);
simplify(L[10]) . convert(mdXY, Vector);
simplify(L[11]) . convert(mdXY, Vector);
simplify(L[12]) . convert(mdXY, Vector);
simplify(L[13]) . convert(mdXY, Vector);
simplify(L[14]) . convert(mdXY, Vector);

$$(-RootOf(\_Z^3-2)+2)\,x0^2+\frac{1}{2}\,\Big(\big(-2+2\,RootOf(\_Z^3-2)^2-RootOf(\_Z^3-2)\,RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2-RootOf(\_Z^3-2)^2+1\big)\big)\,x1^2\Big)\Big/\Big(RootOf(\_Z^3-2)^2+RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2-RootOf(\_Z^3-2)^2+1\big)-2\,RootOf(\_Z^3-2)\Big)+x3^2$$

$$(-2\,RootOf(\_Z^3-2)^2+1)\,x0^2+\frac{1}{2}\,x1^2+x2\,x3$$

$$\Big(RootOf(\_Z^3-2)^2+2\,RootOf(\_Z^3-2)+2\,RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2-RootOf(\_Z^3-2)^2+1\big)\Big)\,x0^2-\frac{1}{2}\,\Big(\big(RootOf(\_Z^3-2)^2\,RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2-RootOf(\_Z^3-2)^2+1\big)-4+2\,RootOf(\_Z^3-2)\big)\,x1^2\Big)\Big/\Big(RootOf(\_Z^3-2)^2+RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2-RootOf(\_Z^3-2)^2+1\big)-2\,RootOf(\_Z^3-2)\Big)+x1\,x3$$

$$-x0^2\,RootOf(\_Z^3-2)-\frac{1}{2}\,\Big(\big(-2+2\,RootOf(\_Z^3-2)^2-RootOf(\_Z^3-2)\,RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2-RootOf(\_Z^3-2)^2+1\big)\big)\,x1^2\Big)\Big/\Big(RootOf(\_Z^3-2)^2+RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2-RootOf(\_Z^3-2)^2+1\big)-2\,RootOf(\_Z^3-2)\Big)+x1\,x2$$

$$-\Big(\big(RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2-RootOf(\_Z^3-2)^2+1\big)+RootOf(\_Z^3-2)\big)\,x1\,x0\Big)\Big/\Big(RootOf(\_Z^3-2)^2\,RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2$$

$$- RootOf(\_Z^3 - 2)^2 + 1) - 2\, RootOf(\_Z^3 - 2)\, RootOf(\_Z\, RootOf(\_Z^3 - 2)^2 + \_Z^2$$
$$- RootOf(\_Z^3 - 2)^2 + 1) - 2\, RootOf(\_Z^3 - 2)^2 + 1) + x3\, x0$$
$$-((RootOf(\_Z^3 - 2)^2\, RootOf(\_Z\, RootOf(\_Z^3 - 2)^2 + \_Z^2 - RootOf(\_Z^3 - 2)^2 + 1) + 2$$
$$- 2\, RootOf(\_Z\, RootOf(\_Z^3 - 2)^2 + \_Z^2 - RootOf(\_Z^3 - 2)^2 + 1) - RootOf(\_Z^3$$
$$- 2))\, x1\, x0) / (RootOf(\_Z^3 - 2)^2\, RootOf(\_Z\, RootOf(\_Z^3 - 2)^2 + \_Z^2$$
$$- RootOf(\_Z^3 - 2)^2 + 1) - 2\, RootOf(\_Z^3 - 2)\, RootOf(\_Z\, RootOf(\_Z^3 - 2)^2 + \_Z^2$$
$$- RootOf(\_Z^3 - 2)^2 + 1) - 2\, RootOf(\_Z^3 - 2)^2 + 1) + x0\, x2$$
$$x3\, y0 + x3\, y1$$
$$x1\, y0 + x1\, y1$$
$$x0\, y0 + x0\, y1$$
$$-x2^2 + y3^2$$
$$-x2^2 - y0\, y1 - y0\, y2 + y0\, y3 - y1\, y2 + y1\, y3 + y2\, y3$$
$$-x2^2 + y2^2$$
$$-x2^2 + y1^2$$
$$-x2^2 + y0^2 \quad \textbf{(35)}$$

> *# We can verify that there is no polynomails in the kernel with rational coefficients that has non-zero coefficient in* $x0^2$.

>