```
> # Construction of a strictly positive polyonmial in the boundary
  # of Sigma_{3, 10} that is the sum of 8 squares.
```

```
> ############################################################################
  # Load "Rational SOS" procedures
  ############################################################################
  read("rationalSOS.mpl") :
  with(rationalSOS) :
  with(LinearAlgebra) :

  # Display tables of any size
  interface(rtablesize = infinity);
```

"Opening connection with Matlab"

$$10 \tag{1}$$

```
> ############################################################################

  # We can define two polynomials with 25 common roots
  # p1 := x1*(x1-x0)*(x1+x0)*(x1-2*x0)*(x1+2*x0);
  # p2 := x2*(x2-x0)*(x2+x0)*(x2-2*x0)*(x2+2*x0);

  # Here we use directly a polynomial of degree 5 and a polynomial
  # of degree 3, as suggested in Section 4.5
  p1 := (x1) * (x1-x0) * (x1 + x0);
  p2 := (x2) * (x2-x0) * (x2-2 * x0) * (x2 + x0) * (x2 + 2 * x0);

  # The list of common roots.
  sols := solve({p1, p2, x0-1});
  print("Number of solutions: ", nops([sols]));
```

$$p1 := x1\ (x1 - x0)\ (x1 + x0)$$

$$p2 := x2\ (x2 - x0)\ (x2 - 2\ x0)\ (x2 + x0)\ (x2 + 2\ x0)$$

$sols := \{x0 = 1, x1 = 0, x2 = 0\}, \{x0 = 1, x1 = 1, x2 = 0\}, \{x0 = 1, x1 = -1, x2 = 0\}, \{x0 = 1, x1 = 0, x2 = 1\}, \{x0 = 1, x1 = 0, x2 = 2\}, \{x0 = 1, x1 = 0, x2 = -1\}, \{x0 = 1, x1 = 0, x2 = -2\}, \{x0 = 1, x1 = 1, x2 = 1\}, \{x0 = 1, x1 = -1, x2 = 1\}, \{x0 = 1, x1 = -1, x2 = 2\}, \{x0 = 1, x1 = -1, x2 = -1\}, \{x0 = 1, x1 = -1, x2 = -2\}, \{x0 = 1, x1 = 1, x2 = 2\}, \{x0 = 1, x1 = 1, x2 = -1\}, \{x0 = 1, x1 = 1, x2 = -2\}$

"Number of solutions: ", 15 **(2)**

```
> ############################################################################
  # We compute the linear relation between the points
  ############################################################################
```

```
> # We define the set of monomials of degree d (stored in ctd) and
  # the set of monomials of degree 2d (stored in ct2d), for d = 5.
  d := 5 :
  polVars := [x0, x1, x2] :
  varSum := add(polVars[i], i = 1 ..nops(polVars)) :
  md := expand((varSum)^d) :
```

```
cfs := coeffs(md, polVars, 'ctd') :
print("Monomials of degree d: ", ctd);
m2d := expand(varSum^(2 * d)) :
cfs := coeffs(m2d, polVars, 'ct2d') :
print("Monomials of degree 2d: ", ct2d);
```

"Monomials of degree d: ", $x0^5, x0^4\,x1, x0^4\,x2, x0^3\,x1^2, x0^3\,x1\,x2, x0^3\,x2^2, x0^2\,x1^3, x0^2\,x1^2\,x2, x0^2\,x1\,x2^2, x0^2\,x2^3, x0\,x1^4, x0\,x1^3\,x2, x0\,x1^2\,x2^2, x0\,x1\,x2^3, x0\,x2^4, x1^5, x1^4\,x2, x1^3\,x2^2, x1^2\,x2^3, x1\,x2^4, x2^5$

"Monomials of degree 2d: ", $x0^{10}, x0^9\,x1, x0^9\,x2, x0^8\,x1^2, x0^8\,x1\,x2, x0^8\,x2^2, x0^7\,x1^3, x0^7\,x1^2\,x2, x0^7\,x1\,x2^2, x0^7\,x2^3, x0^6\,x1^4, x0^6\,x1^3\,x2, x0^6\,x1^2\,x2^2, x0^6\,x1\,x2^3, x0^6\,x2^4, x0^5\,x1^5, x0^5\,x1^4\,x2, x0^5\,x1^3\,x2^2, x0^5\,x1^2\,x2^3, x0^5\,x1\,x2^4, x0^5\,x2^5, x0^4\,x1^6, x0^4\,x1^5\,x2, x0^4\,x1^4\,x2^2, x0^4\,x1^3\,x2^3, x0^4\,x1^2\,x2^4, x0^4\,x1\,x2^5, x0^4\,x2^6, x0^3\,x1^7, x0^3\,x1^6\,x2, x0^3\,x1^5\,x2^2, x0^3\,x1^4\,x2^3, x0^3\,x1^3\,x2^4, x0^3\,x1^2\,x2^5, x0^3\,x1\,x2^6, x0^3\,x2^7, x0^2\,x1^8, x0^2\,x1^7\,x2, x0^2\,x1^6\,x2^2, x0^2\,x1^5\,x2^3, x0^2\,x1^4\,x2^4, x0^2\,x1^3\,x2^5, x0^2\,x1^2\,x2^6, x0^2\,x1\,x2^7, x0^2\,x2^8, x0\,x1^9, x0\,x1^8\,x2, x0\,x1^7\,x2^2, x0\,x1^6\,x2^3, x0\,x1^5\,x2^4, x0\,x1^4\,x2^5, x0\,x1^3\,x2^6, x0\,x1^2\,x2^7, x0\,x1\,x2^8, x0\,x2^9, x1^{10}, x1^9\,x2, x1^8\,x2^2, x1^7\,x2^3, x1^6\,x2^4, x1^5\,x2^5, x1^4\,x2^6, x1^3\,x2^7, x1^2\,x2^8, x1\,x2^9, x2^{10}$ **(3)**

```
> # We define a generic polynomial of degree d with coefficientes h_i.
hCoeff := [h[1]] :
for i from 2 to nops([ctd]) do
  hCoeff := [op(hCoeff), h[i]] :
end do:
hd := add(hCoeff[i] * ctd[i], i = 1 ..nops(hCoeff)) :
print("Generic polynomial h of degree d: ", hd);
```

"Generic polynomial h of degree d: ", $h_1\,x0^5 + h_2\,x0^4\,x1 + h_3\,x0^4\,x2 + h_4\,x0^3\,x1^2 + h_5\,x0^3\,x1\,x2 + h_6\,x0^3\,x2^2 + h_7\,x0^2\,x1^3 + h_8\,x0^2\,x1^2\,x2 + h_9\,x0^2\,x1\,x2^2 + h_{10}\,x0^2\,x2^3 + h_{11}\,x0\,x1^4 + h_{12}\,x0\,x1^3\,x2 + h_{13}\,x0\,x1^2\,x2^2 + h_{14}\,x0\,x1\,x2^3 + h_{15}\,x0\,x2^4 + h_{16}\,x1^5 + h_{17}\,x1^4\,x2 + h_{18}\,x1^3\,x2^2 + h_{19}\,x1^2\,x2^3 + h_{20}\,x1\,x2^4 + h_{21}\,x2^5$ **(4)**

```
> # We consider the set of 15 roots
# (1:a:b) with a in {-1, 0, 1} and b in {0, 1, 2, 3, 4}.
> # Evaluating the generic polynomial in the 15 points, we get 15 linear
# forms, and we look for a linear relation among these forms.
nRoots := 15;
nCoeff := 21;
alphaSeq := seq(eval(hd, sols[i]), i = 1 ..nRoots) :
```

$$nRoots := 15$$
$$nCoeff := 21 \qquad \textbf{(5)}$$

```
> # There is an unique relationship with all nonzero coefficients
MEval := Matrix(nRoots, nCoeff) :
```

```
for i from 1 to nRoots do:
  aaC := getCoeffs(expand(alphaSeq[i]), hCoeff);
  MEval[i, 1 ..nCoeff] := aaC :
end:
u := NullSpace(Transpose(MEval));
uVec := u[1]:
```

$$u := \left\{ \begin{bmatrix} -12 \\ 6 \\ 6 \\ 8 \\ -2 \\ 8 \\ -2 \\ -4 \\ -4 \\ 1 \\ -4 \\ 1 \\ 1 \\ -4 \\ 1 \end{bmatrix} \right\} \tag{6}$$

```
> ##############################################################################
# We construct quadratic form Q and the associated matrix.
##############################################################################
```

```
> # The coefficients of the linear form l.
# We will leave the last coefficient a_15 as indeterminate and
# compute it using Maple to verify the theoretical formulas.
a := [seq(uVec[i]^2, i=1 ..nRoots)]:
a[nRoots] := cc:
print(a);
```

$$[144, 36, 36, 64, 4, 64, 4, 16, 16, 1, 16, 1, 1, 16, cc] \tag{7}$$

```
> # We define the quadratic form Q(h), as a linear combination of
# evaluations of h^2 in the 15 points defined above, with
# coefficients a_i
hd_square := expand(hd^2) :
QForm := add(a[i]*eval(hd_square, sols[i]), i=1 ..nRoots) :
```

```
> # We construct the matrix associated to Q.
```

```
# We define it as a 15x15 matrix with indeterminate entries, and
# compute the entries so that c' Qmatrix c = Qform, where
# c are the monomials of degree d.
mSize := nCoeff:
MM := Matrix(mSize) :
for i to mSize do
  for j from i to mSize do
    MM[i, j] := c[i, j];
    MM[j, i] := c[i, j];
  end do:
end do:
```

```
> # Vector q of monomials of degree d, with generic coefficients
  hCoeffVector := Vector(hCoeff) :
```

```
> # We compute h' * MM * h
  hCoeffVector_T := Transpose(hCoeffVector) :
  hTMh := expand(hCoeffVector_T . MM . hCoeffVector) :
```

```
> # Finally we equate the coefficients of h'*MM*h and those of QForm
  # and compute the coefficients of MM.
  eqs := {coeffs(collect(hTMh-QForm, hCoeff,'distributed'), hCoeff) } :
  sol := solve(eqs) :
```

```
> # We replace the coefficients by the values obtained
  MMC := eval(MM, sol);
```

$$MMC := [[419 + cc, -1 + cc, 2 - 2\,cc, 139 + cc, 2 - 2\,cc, 236 + 4\,cc, -1 + cc, 2 - 2\,cc, -4 + 4\,cc, 8 - 8\,cc, 139 + cc, 2 - 2\,cc, 76 + 4\,cc, 8 - 8\,cc, 368 + 16\,cc, -1 + cc, 2 - 2\,cc, -4 + 4\,cc, 8 - 8\,cc, -16 + 16\,cc, 32 - 32\,cc],$$
$$[-1 + cc, 139 + cc, 2 - 2\,cc, -1 + cc, 2 - 2\,cc, -4 + 4\,cc, 139 + cc, 2 - 2\,cc, 76 + 4\,cc, 8 - 8\,cc, -1 + cc, 2 - 2\,cc, -4 + 4\,cc, 8 - 8\,cc, -16 + 16\,cc, 139 + cc, 2 - 2\,cc, 76 + 4\,cc, 8 - 8\,cc, 112 + 16\,cc, 32 - 32\,cc],$$
$$[2 - 2\,cc, 2 - 2\,cc, 236 + 4\,cc, 2 - 2\,cc, -4 + 4\,cc, 8 - 8\,cc, 2 - 2\,cc, 76 + 4\,cc, 8 - 8\,cc, 368 + 16\,cc, 2 - 2\,cc, -4 + 4\,cc, 8 - 8\,cc, -16 + 16\,cc, 32 - 32\,cc, 2 - 2\,cc, 76 + 4\,cc, 8 - 8\,cc, 112 + 16\,cc, 32 - 32\,cc, 896 + 64\,cc],$$
$$[139 + cc, -1 + cc, 2 - 2\,cc, 139 + cc, 2 - 2\,cc, 76 + 4\,cc, -1 + cc, 2 - 2\,cc, -4 + 4\,cc, 8 - 8\,cc, 139 + cc, 2 - 2\,cc, 76 + 4\,cc, 8 - 8\,cc, 112 + 16\,cc, -1 + cc, 2 - 2\,cc, -4 + 4\,cc, 8 - 8\,cc, -16 + 16\,cc, 32 - 32\,cc],$$
$$[2 - 2\,cc, 2 - 2\,cc, -4 + 4\,cc, 2 - 2\,cc, 76 + 4\,cc, 8 - 8\,cc, 2 - 2\,cc, -4 + 4\,cc, 8 - 8\,cc, -16 + 16\,cc, 2 - 2\,cc, 76 + 4\,cc, 8 - 8\,cc, 112 + 16\,cc, 32 - 32\,cc, 2 - 2\,cc, -4 + 4\,cc, 8 - 8\,cc, -16 + 16\,cc, 32 - 32\,cc, -64 + 64\,cc],$$
$$[236 + 4\,cc, -4 + 4\,cc, 8 - 8\,cc, 76 + 4\,cc, 8 - 8\,cc, 368 + 16\,cc, -4 + 4\,cc, 8 - 8\,cc, -16 + 16\,cc, 32 - 32\,cc, 76 + 4\,cc, 8 - 8\,cc, 112 + 16\,cc, 32 - 32\,cc, 896 + 64\,cc, -4 + 4\,cc, 8 - 8\,cc, -16 + 16\,cc, 32 - 32\,cc, -64 + 64\,cc, 128 - 128\,cc],$$
$$[-1 + cc, 139 + cc, 2 - 2\,cc, -1 + cc, 2 - 2\,cc, -4 + 4\,cc, 139 + cc, 2 - 2\,cc, 76 + 4\,cc, 8 \tag{8}$$

$-8\,cc, -1+cc, 2-2\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 139+cc, 2-2\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc],$

$[2-2\,cc, 2-2\,cc, 76+4\,cc, 2-2\,cc, -4+4\,cc, 8-8\,cc, 2-2\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 2-2\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, 2-2\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc],$

$[-4+4\,cc, 76+4\,cc, 8-8\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc, 128-128\,cc],$

$[8-8\,cc, 8-8\,cc, 368+16\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 896+64\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 128-128\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc, 128-128\,cc, 3008+256\,cc],$

$[139+cc, -1+cc, 2-2\,cc, 139+cc, 2-2\,cc, 76+4\,cc, -1+cc, 2-2\,cc, -4+4\,cc, 8-8\,cc, 139+cc, 2-2\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, -1+cc, 2-2\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc],$

$[2-2\,cc, 2-2\,cc, -4+4\,cc, 2-2\,cc, 76+4\,cc, 8-8\,cc, 2-2\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 2-2\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 2-2\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc],$

$[76+4\,cc, -4+4\,cc, 8-8\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 128-128\,cc],$

$[8-8\,cc, 8-8\,cc, -16+16\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc, 128-128\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 128-128\,cc, -256+256\,cc],$

$[368+16\,cc, -16+16\,cc, 32-32\,cc, 112+16\,cc, 32-32\,cc, 896+64\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 128-128\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc, 128-128\,cc, 3008+256\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 128-128\,cc, -256+256\,cc, 512-512\,cc],$

$[-1+cc, 139+cc, 2-2\,cc, -1+cc, 2-2\,cc, -4+4\,cc, 139+cc, 2-2\,cc, 76+4\,cc, 8-8\,cc, -1+cc, 2-2\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 139+cc, 2-2\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc],$

$[2-2\,cc, 2-2\,cc, 76+4\,cc, 2-2\,cc, -4+4\,cc, 8-8\,cc, 2-2\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 2-2\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, 2-2\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc],$

$[-4+4\,cc, 76+4\,cc, 8-8\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, -4+4\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 76+4\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc, 128-128\,cc],$

$[8-8\,cc, 8-8\,cc, 112+16\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc, 8-8\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 128-128\,cc, 8-8\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc, 128-128\,cc, 832+256\,cc],$

$[-16+16\,cc, 112+16\,cc, 32-32\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc, 128-128\,cc, -16+16\,cc, 32-32\,cc, -64+64\,cc, 128-128\,cc, -256+256\,cc, 112+16\,cc, 32-32\,cc, 256+64\,cc, 128-128\,cc, 832$

$$
\begin{aligned}
&+256\,cc,\ 512-512\,cc],\\
&[32-32\,cc,\ 32-32\,cc,\ 896+64\,cc,\ 32-32\,cc,\ -64+64\,cc,\ 128-128\,cc,\ 32-32\,cc,\\
&256+64\,cc,\ 128-128\,cc,\ 3008+256\,cc,\ 32-32\,cc,\ -64+64\,cc,\ 128-128\,cc,\ -256\\
&+256\,cc,\ 512-512\,cc,\ 32-32\,cc,\ 256+64\,cc,\ 128-128\,cc,\ 832+256\,cc,\ 512\\
&-512\,cc,\ 11456+1024\,cc]]
\end{aligned}
$$

```
> ################################################################################
  # We compute the value of cc so that the kernel has dimension 8
  ################################################################################
> # The first seven eigenvalues are 0.
  ev := Eigenvalues(MMC);
```

$$
\begin{aligned}
ev := [&[0],\\
&[0],\\
&[0],\\
&[0],\\
&[0],\\
&[0],\\
&[0],\\
&[\mathit{RootOf}(\_Z^{14} + (-19062 - 1818\,cc)\,\_Z^{13} + (21133296\,cc + 97129152)\,\_Z^{12} + (\\
&-76003679808\,cc - 203413906368)\,\_Z^{11} + (121198027189248\,cc\\
&+ 221353076892672)\,\_Z^{10} + (-104836091743567872\,cc - 140741258092609536)\,\_Z^{9}\\
&+ (54105540844150849536\,cc + 55441920659174719488)\,\_Z^{8} + (\\
&-17407890060547719168000\,cc - 13881059969366753280000)\,\_Z^{7}\\
&+ (3546613107855109103026176\,cc + 2218732010619420587065344)\,\_Z^{6} + (\\
&-457041790840223371408441344\,cc - 224114187791046641555865600)\,\_Z^{5}\\
&+ (36860385511737044389381275648\,cc + 14005002266874623908191928320)\,\_Z^{4} + (\\
&-1830611364752422331633425711104\,cc - 524399060721108317960977514496)\,\_Z^{3}\\
&+ (53970697666219706647640480415744\,cc\\
&+ 11090273922988758627972689166336)\,\_Z^{2} + (\\
&-862472135508234301898147802120192\,cc - 115926019343223934799524675977216)\\
&\_Z + 5737442244447861473624580931190784\,cc\\
&+ 4098173031748472481160414950850 56)]]
\end{aligned}
\tag{9}
$$

```
> # There are 7 null eigenvalues and the remaining 14 are roots of a
  # polynomial of degree 14.
  # We choose cc so that this polynomial has a root equal to 0.
  e8 := op(ev[8]) :
  e80 := eval(e8, {_Z=0}) :
  fac := factors(e80) :
  rr := solve(fac[2][1][1]);
```

$$rr := -\frac{1}{14} \tag{10}$$

```
> # We recover the value for a_15 predicted by the theoretical formula.
> ################################################################################
  # The eight polynomials in the kernel
  ################################################################################
> MMC2 := eval(MMC, {cc = rr}) :
  nspace := NullSpace(MMC2);
```

$$nspace := \left\{ \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{4}{15} \\ 0 \\ 0 \\ -\frac{4}{5} \\ 2 \\ -1 \\ 0 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 2 \\ -2 \\ \frac{1}{3} \\ 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{11}$$

```
> # Polynomials
  w1 := LinearAlgebra[DotProduct](nspace[1], Vector([ctd]));
  w2 := LinearAlgebra[DotProduct](nspace[2], Vector([ctd]));
  w3 := LinearAlgebra[DotProduct](nspace[3], Vector([ctd]));
```

$w4 := LinearAlgebra[DotProduct](nspace[4], Vector([ctd]));$
$w5 := LinearAlgebra[DotProduct](nspace[5], Vector([ctd]));$
$w6 := LinearAlgebra[DotProduct](nspace[6], Vector([ctd]));$
$w7 := LinearAlgebra[DotProduct](nspace[7], Vector([ctd]));$
$w8 := LinearAlgebra[DotProduct](nspace[8], Vector([ctd]));$

$$w1 := 4\,x0^4\,x2 - 5\,x0^2\,x2^3 + x2^5$$

$$w2 := \frac{4}{15}\,x0^5 - \frac{4}{5}\,x0^3\,x1^2 + 2\,x0^3\,x1\,x2 - x0^3\,x2^2 + 2\,x0^2\,x1^2\,x2 - x0^2\,x1\,x2^2 + 2\,x0\,x1^2\,x2^2 - 2\,x0\,x1\,x2^3 + \frac{1}{3}\,x0\,x2^4 - 2\,x1^2\,x2^3 + x1\,x2^4$$

$$w3 := -x0^2\,x1\,x2^2 + x1^3\,x2^2$$

$$w4 := -x0^2\,x1^2\,x2 + x1^4\,x2$$

$$w5 := -x0^4\,x1 + x1^5$$

$$w6 := -x0^3\,x1\,x2 + x0\,x1^3\,x2$$

$$w7 := -x0^3\,x1^2 + x0\,x1^4$$

$$w8 := -x0^4\,x1 + x0^2\,x1^3 \qquad \textbf{(12)}$$

> $solve(\{w1, w2, w3, w4, w5, w6, w7, w8\});$

$$\{x0 = 0, x1 = 0, x2 = 0\} \qquad \textbf{(13)}$$

> *# We construct the sum of squares*
pp := w1^2 + w2^2 + w3^2 + w4^2 + w5^2 + w6^2 + w7^2 + w8^2;

$$pp := \left(4\,x0^4\,x2 - 5\,x0^2\,x2^3 + x2^5\right)^2 + \left(\frac{4}{15}\,x0^5 - \frac{4}{5}\,x0^3\,x1^2 + 2\,x0^3\,x1\,x2 - x0^3\,x2^2 + 2\,x0^2\,x1^2\,x2 - x0^2\,x1\,x2^2 + 2\,x0\,x1^2\,x2^2 - 2\,x0\,x1\,x2^3 + \frac{1}{3}\,x0\,x2^4 - 2\,x1^2\,x2^3 + x1\,x2^4\right)^2 + \left(-x0^2\,x1\,x2^2 + x1^3\,x2^2\right)^2 + \left(-x0^2\,x1^2\,x2 + x1^4\,x2\right)^2 + \left(-x0^4\,x1 + x1^5\right)^2 + \left(-x0^3\,x1\,x2 + x0\,x1^3\,x2\right)^2 + \left(-x0^3\,x1^2 + x0\,x1^4\right)^2 + \left(-x0^4\,x1 + x0^2\,x1^3\right)^2 \qquad \textbf{(14)}$$

> *# And verify using SEDUMI that this polynomial is in the border of*
 # the SOS cone.
$out := exactSOS(pp, facial = $"no"$, objFunction = $"eig"$)$:
$eig(out[3]);$

"Number of indeterminates: ", 165
"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."
"SEDUMI CALL - eig"
"An exact positive definite solution could not be found for the reduced problem."
"Computing Cholesky decomposition..."
"Cholesky decomposition failed. Matrix is not positive definite. We use LDLt decomposition."

$$
\begin{bmatrix}
-3.89467146750081\ 10^{-9} \\
-1.23946296778899\ 10^{-9} \\
-6.11652326046839\ 10^{-10} \\
-5.46877295330366\ 10^{-10} \\
-3.62605718345171\ 10^{-10} \\
-3.65089203461806\ 10^{-11} \\
1.30645775087807\ 10^{-10} \\
4.61612174929171\ 10^{-10} \\
7.15108747955539\ 10^{-10} \\
1.95261091756736\ 10^{-9} \\
2.91409116581574\ 10^{-9} \\
4.31859542456449\ 10^{-9} \\
4.54424744921654\ 10^{-9} \\
1.06089993349089 \\
2.03107734192738 \\
2.50111174694548 \\
3.10207100870197 \\
3.49691677255360 \\
3.93397866195548 \\
24.4383294560612 \\
42.0000000002288
\end{bmatrix} \tag{15}
$$

> *# We obtain a sum of 8 squares in the border (the matrix has rank 8).*

> *# In this case the decomposition is not unique.*
 # Minimizing another function we obtain a different matrix, with
 # smaller rank.
out := *exactSOS*(*pp*, *facial* = "no", *objFunction* = "random") :
eig(*out*[3]);

"Number of indeterminates: ", 165

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL - random"

"An exact positive definite solution could not be found for the reduced problem."

"Computing Cholesky decomposition..."

"Cholesky decomposition failed. Matrix is not positive definite. We use LDLt decomposition."

$$\begin{bmatrix} -2.28352354571579\ 10^{-8} \\ -2.23705789167117\ 10^{-8} \\ -6.17453602485539\ 10^{-9} \\ -4.10419576984005\ 10^{-9} \\ 1.29041331469062\ 10^{-8} \\ 2.77463300430008\ 10^{-8} \\ 3.30316580387789\ 10^{-8} \\ 7.25546044022952\ 10^{-8} \\ 8.76171778188704\ 10^{-8} \\ 2.57481394999824\ 10^{-7} \\ 4.80054668057677\ 10^{-7} \\ 5.42813007495058\ 10^{-7} \\ 8.70102741377469\ 10^{-7} \\ 0.000418821455678908 \\ 0.00734231191582154 \\ 1.35393166560430 \\ 2.28666761845657 \\ 2.93077951334188 \\ 3.07821953951601 \\ 24.3494506273425 \\ 42.0000010702561 \end{bmatrix} \tag{16}$$

>