```
> # Uncommment and set the path to rationalSOS.mpl file
  #currentdir("C:/Users/User/rationalSOS");

  ###############################################################################
  # Load "Rational SOS" procedures
  ###############################################################################
  read("rationalSOS.mpl");
  with(rationalSOS);

  # Display tables of any size
  interface(rtablesize = infinity);
```

"Opening connection with Matlab"

rationalSOS := **module**( ) ... **end module**

[*decompositionToMatrix*, *evalMat*, *evalSolution*, *getDiag*, *getExtension*, *matrixToPoly*, *nonRatCoef*, *numericSolver*, *numericSolverSubmatrix*, *polyToMatrix*, *primitiveMatrix*, *randomRank*, *ratSOS*, *reduceByLinearEquation*, *roundMat*, *roundVec*, *vectorTrace*, *zeroDetSRows*, *zeroRows*]

10 **(1)**

```
> ###############################################################################
  # Theorem 5.1
  ###############################################################################

  # We define a polynomial z as the sum of three squares in an algebraic
  # extension of degree 3 with generic coefficients.

  mp := t^3-2;
  p1 := c1*t^2+b1*t+a1;
  p2 := c2*t^2+b2*t+a2;
  p3 := c3*t^2+b3*t+a3;
```

$$mp := t^3 - 2$$
$$p1 := c1\,t^2 + b1\,t + a1$$
$$p2 := c2\,t^2 + b2\,t + a2$$
$$p3 := c3\,t^2 + b3\,t + a3 \quad \textbf{(2)}$$

```
> # We impose some relations between the coefficients to decrease
  # the dimension of the problem and rename the remaining variables
  b2 := 3*b1;  c2 := b1+7*c1;  a3 := 3*c2-b2;
  b1 := x; b3 := y; c1 := z; c3 := w;
```

$$b2 := 3\,b1$$
$$c2 := b1 + 7\,c1$$
$$a3 := 21\,c1$$
$$b1 := x$$
$$b3 := y$$

$$c1 := z$$
$$c3 := w \quad \textbf{(3)}$$

```
> fGeneric := p1^2 + p2^2 + p3^2;
  fGeneric := expand(fGeneric);
```

$$fGeneric := \left(t^2 z + t x + a1\right)^2 + \left((x + 7z)\, t^2 + 3 x t + a2\right)^2 + \left(t^2 w + t y + 21 z\right)^2$$

$$fGeneric := t^4 w^2 + t^4 x^2 + 14 t^4 x z + 50 t^4 z^2 + 2 t^3 w y + 6 t^3 x^2 + 44 t^3 x z + 2 a1 t^2 z + 2 a2 t^2 x + 14 a2 t^2 z + 42 t^2 w z + 10 t^2 x^2 + t^2 y^2 + 2 a1 t x + 6 a2 t x + 42 t y z + a1^2 + a2^2 + 441 z^2 \quad \textbf{(4)}$$

```
> # We solve the coefficients a1 and a2 so that the polynomial is in Q,
  f2 := NormalForm(fGeneric, [mp], plex(a1, a2, x, y, z, w, t));
  f3 := collect(f2, t);
  lf := CoefficientList(f3, t);
  ss := solve({lf[2], lf[3]}, {a1, a2});
```

$$f2 := 2 a1 t^2 z + 2 a2 t^2 x + 14 a2 t^2 z + 42 t^2 w z + 10 t^2 x^2 + t^2 y^2 + 2 a1 t x + 6 a2 t x + 2 t w^2 + 2 t x^2 + 28 t x z + 42 t y z + 100 t z^2 + a1^2 + a2^2 + 4 w y + 12 x^2 + 88 x z + 441 z^2$$

$$f3 := \left(2 a1 z + 2 a2 x + 14 a2 z + 42 w z + 10 x^2 + y^2\right) t^2 + \left(2 a1 x + 6 a2 x + 2 w^2 + 2 x^2 + 28 x z + 42 y z + 100 z^2\right) t + a1^2 + a2^2 + 4 w y + 12 x^2 + 88 x z + 441 z^2$$

$$lf := \left[a1^2 + a2^2 + 4 w y + 12 x^2 + 88 x z + 441 z^2,\ 2 a1 x + 6 a2 x + 2 w^2 + 2 x^2 + 28 x z + 42 y z + 100 z^2,\ 2 a1 z + 2 a2 x + 14 a2 z + 42 w z + 10 x^2 + y^2\right]$$

$$ss := \left\{a1 = -\frac{1}{2}\,\frac{1}{x\,(x + 4z)}\left(2 w^2 x + 14 w^2 z - 126 w x z - 28 x^3 + 42 x^2 z - 3 x y^2 + 42 x y z + 296 x z^2 + 294 y z^2 + 700 z^3\right),\ a2 = \frac{1}{2}\,\frac{2 w^2 z - 42 w x z - 10 x^3 + 2 x^2 z - x y^2 + 28 x z^2 + 42 y z^2 + 100 z^3}{x\,(x + 4z)}\right\} \quad \textbf{(5)}$$

```
> # We plug in the solutions found for a1 and a2 and compute the resulting polynomial
  ssDen := denom(rhs(ss[1]));
  p1s := simplify(subs(ss, p1) * ssDen);
  p2s := simplify(subs(ss, p2) * ssDen);
  p3s := simplify(subs(ss, p3) * ssDen);

  p1ss := subs({t = RootOf(x^3 - 2)}, p1s);
  p2ss := subs({t = RootOf(x^3 - 2)}, p2s);
  p3ss := subs({t = RootOf(x^3 - 2)}, p3s);

  f := simplify(p1ss^2 + p2ss^2 + p3ss^2);
```

$$ssDen := 2 x\,(x + 4 z)$$

$$p1s := 2 t^2 x^2 z + 8 t^2 x z^2 + 2 t x^3 + 8 t x^2 z - 2 w^2 x - 14 w^2 z + 126 w x z + 28 x^3 - 42 x^2 z + 3 x y^2 - 42 x y z - 296 x z^2 - 294 y z^2 - 700 z^3$$

$$p2s := 2\,t^2 x^3 + 22\,t^2 x^2 z + 56\,t^2 x z^2 + 6\,t x^3 + 24\,t x^2 z + 2\,w^2 z - 42\,w x z - 10\,x^3 + 2\,x^2 z - x y^2 + 28\,x z^2 + 42\,y z^2 + 100\,z^3$$

$$p3s := 2\,(t^2 w + t y + 21\,z)\,x\,(x + 4\,z)$$

$$p1ss := 2\,RootOf(\_Z^3 - 2)^2 z x^2 + 8\,RootOf(\_Z^3 - 2)^2 z^2 x + 2\,x^3\,RootOf(\_Z^3 - 2) + 8\,x^2\,RootOf(\_Z^3 - 2)\,z - 2\,w^2 x - 14\,w^2 z + 126\,w x z + 28\,x^3 - 42\,x^2 z + 3\,x y^2 - 42\,x y z - 296\,x z^2 - 294\,y z^2 - 700\,z^3$$

$$p2ss := 2\,RootOf(\_Z^3 - 2)^2 x^3 + 22\,RootOf(\_Z^3 - 2)^2 z x^2 + 56\,RootOf(\_Z^3 - 2)^2 z^2 x + 6\,x^3\,RootOf(\_Z^3 - 2) + 24\,x^2\,RootOf(\_Z^3 - 2)\,z + 2\,w^2 z - 42\,w x z - 10\,x^3 + 2\,x^2 z - x y^2 + 28\,x z^2 + 42\,y z^2 + 100\,z^3$$

$$p3ss := 2\,(w\,RootOf(\_Z^3 - 2)^2 + y\,RootOf(\_Z^3 - 2) + 21\,z)\,x\,(x + 4\,z)$$

$$f := 4\,w^4 x^2 + 56\,w^4 x z + 200\,w^4 z^2 - 504\,w^3 x^2 z - 3696\,w^3 x z^2 - 112\,w^2 x^4 - 656\,w^2 x^3 z - 12\,w^2 x^2 y^2 + 168\,w^2 x^2 y z + 20008\,w^2 x^2 z^2 - 88\,w^2 x y^2 z + 2352\,w^2 x y z^2 + 11200\,w^2 x z^3 + 8400\,w^2 y z^3 + 20000\,w^2 z^4 + 16\,w x^4 y + 7896\,w x^4 z + 128\,w x^3 y z - 10752\,w x^3 z^2 + 840\,w x^2 y^2 z - 10328\,w x^2 y z^2 - 76944\,w x^2 z^3 - 77616\,w x y z^3 - 184800\,w x z^4 + 932\,x^6 - 1656\,x^5 z + 188\,x^4 y^2 - 2352\,x^4 y z - 10020\,x^4 z^2 - 256\,x^3 y^2 z - 13776\,x^3 y z^2 + 3520\,x^3 z^3 + 10\,x^2 y^4 - 252\,x^2 y^3 z - 68\,x^2 y^2 z^2 + 49728\,x^2 y z^3 + 175824\,x^2 z^4 - 1848\,x y^3 z^2 + 20296\,x y^2 z^3 + 235200\,x y z^4 + 420000\,x z^5 + 88200\,y^2 z^4 + 420000\,y z^5 + 500000\,z^6 \quad \textbf{(6)}$$

```
> # Matrix Q associated to the problem (parametrization of the space L)
  Q, QVars, v := polyToMatrix( f ) :
```

```
> # Matrix associated to the original decomposition
  MNEW := decompositionToMatrix( [p1ss, p2ss, p3ss], v) :
```

```
> #---------------------------------

  # We start from Q and go step by step.
  nops(indets(Q));
  randomRank(Q);
```

$$126$$

$$20 \quad \textbf{(7)}$$

```
> # Real solutions - In this problem, the system obtained by equating f and the partial derivatives
  # to 0 is too complicated for Maple solver. Using the starting polynomials we get a system that
  # is easier to solve.

      #sSym := solve({f=0, diff(f, x)=0, diff(f,y)=0, diff(f,z)=0, diff(f,w)=0, p1ss = 0, p2ss = 0, p3ss
      = 0});
  sSym := solve( {p1ss = 0, p2ss = 0, p3ss = 0} );
```

(8)

$$sSym := \left\{w = w, x = x, y = \frac{1}{2}\ RootOf\left(4\ \_Z^2 - 21\ \_Z x + 8\ w^2 - 168\ w x + 165\ x^2\right), z = -\frac{1}{4}\ x\right\}, \quad \textbf{(8)}$$

$$\{w = RootOf\left(\_Z^2 + 21\ y z + 50\ z^2\right), x = 0, y = y, z = z\}, \{w = w, x = 0, y = y, z = 0\}, \left\{w\right.$$

$$= \frac{1}{84}\ \left(x\ \left(100\ RootOf\left(203063181099439307\ \_Z^4 + 9972349183004\ RootOf\left(\_Z^3\right.\right.\right.$$

$$\left.\left.\left. - 2\right)^2 - 72643630017800\ RootOf\left(\_Z^3 - 2\right) + 22338188120720 + \right.\right.$$

$$-23075699835888\ RootOf\left(\_Z^3 - 2\right)^2 - 60460295392000\ RootOf\left(\_Z^3 - 2\right)$$

$$\left.\left. + 394712269021280\right)\ \_Z + \left(2865014874818964\ RootOf\left(\_Z^3 - 2\right)^2\right.\right.$$

$$\left.\left. - 478608060541564\ RootOf\left(\_Z^3 - 2\right) + 7876919179391632\right)\ \_Z^2 + \right.$$

$$-1068782164884000\ RootOf\left(\_Z^3 - 2\right)^2 + 340609155970104\ RootOf\left(\_Z^3 - 2\right)$$

$$\left.\left. + 54362206908700768\right)\ \_Z^3\right)^2\ RootOf\left(\_Z^3 - 2\right)^2$$

$$- 441\ RootOf\left(203063181099439307\ \_Z^4 + 9972349183004\ RootOf\left(\_Z^3 - 2\right)^2\right.$$

$$- 72643630017800\ RootOf\left(\_Z^3 - 2\right) + 22338188120720 + \left.\right.$$

$$-23075699835888\ RootOf\left(\_Z^3 - 2\right)^2 - 60460295392000\ RootOf\left(\_Z^3 - 2\right)$$

$$\left.\left. + 394712269021280\right)\ \_Z + \left(2865014874818964\ RootOf\left(\_Z^3 - 2\right)^2\right.\right.$$

$$\left.\left. - 478608060541564\ RootOf\left(\_Z^3 - 2\right) + 7876919179391632\right)\ \_Z^2 + \right.$$

$$-1068782164884000\ RootOf\left(\_Z^3 - 2\right)^2 + 340609155970104\ RootOf\left(\_Z^3 - 2\right)$$

$$+ 54362206908700768\Big)\,\_Z^3\Big)^2\,RootOf\left(\_Z^3-2\right)+28\,RootOf\Big(203063181099439307\,\_Z^4$$

$$+ 9972349183004\,RootOf\left(\_Z^3-2\right)^2-72643630017800\,RootOf\left(\_Z^3-2\right)$$

$$+ 22338188120720+\Big(-23075699835888\,RootOf\left(\_Z^3-2\right)^2$$

$$- 60460295392000\,RootOf\left(\_Z^3-2\right)+394712269021280\Big)\,\_Z$$

$$+\Big(2865014874818964\,RootOf\left(\_Z^3-2\right)^2-478608060541564\,RootOf\left(\_Z^3-2\right)$$

$$+ 7876919179391632\Big)\,\_Z^2+\Big(-1068782164884000\,RootOf\left(\_Z^3-2\right)^2$$

$$+ 340609155970104\,RootOf\left(\_Z^3-2\right)+54362206908700768\Big)\,\_Z^3\Big)\,RootOf\left(\_Z^3-2\right)^2$$

$$+ 2\,RootOf\left(\_Z^3-2\right)^2-20\Big)\Big)\Big/\Big(RootOf\Big(203063181099439307\,\_Z^4$$

$$+ 9972349183004\,RootOf\left(\_Z^3-2\right)^2-72643630017800\,RootOf\left(\_Z^3-2\right)$$

$$+ 22338188120720+\Big(-23075699835888\,RootOf\left(\_Z^3-2\right)^2$$

$$- 60460295392000\,RootOf\left(\_Z^3-2\right)+394712269021280\Big)\,\_Z$$

$$+\Big(2865014874818964\,RootOf\left(\_Z^3-2\right)^2-478608060541564\,RootOf\left(\_Z^3-2\right)$$

$$+ 7876919179391632\Big)\,\_Z^2+\Big(-1068782164884000\,RootOf\left(\_Z^3-2\right)^2$$

$$+ 340609155970104\,RootOf\left(\_Z^3-2\right)+54362206908700768\Big)\,\_Z^3\Big)\Big),\,x=x,\,y=$$

$$-\frac{1}{84}\,\Big(x\,\Big(100\,RootOf\Big(203063181099439307\,\_Z^4+9972349183004\,RootOf\left(\_Z^3-2\right)^2$$

$$
\begin{aligned}
&-72643630017800\,\mathit{RootOf}\left(\_Z^3-2\right)+22338188120720+\left(\right.\\
&-23075699835888\,\mathit{RootOf}\left(\_Z^3-2\right)^2-60460295392000\,\mathit{RootOf}\left(\_Z^3-2\right)\\
&\left.+394712269021280\right)\_Z+\left(2865014874818964\,\mathit{RootOf}\left(\_Z^3-2\right)^2\right.\\
&\left.-478608060541564\,\mathit{RootOf}\left(\_Z^3-2\right)+7876919179391632\right)\_Z^2+\left(\right.\\
&-1068782164884000\,\mathit{RootOf}\left(\_Z^3-2\right)^2+340609155970104\,\mathit{RootOf}\left(\_Z^3-2\right)\\
&\left.+54362206908700768\right)\_Z^3\Big)^2\,\mathit{RootOf}\left(\_Z^3-2\right)^4\\
&-441\,\mathit{RootOf}\left(203063181099439307\,\_Z^4+9972349183004\,\mathit{RootOf}\left(\_Z^3-2\right)^2\right.\\
&-72643630017800\,\mathit{RootOf}\left(\_Z^3-2\right)+22338188120720+\left(\right.\\
&-23075699835888\,\mathit{RootOf}\left(\_Z^3-2\right)^2-60460295392000\,\mathit{RootOf}\left(\_Z^3-2\right)\\
&\left.+394712269021280\right)\_Z+\left(2865014874818964\,\mathit{RootOf}\left(\_Z^3-2\right)^2\right.\\
&\left.-478608060541564\,\mathit{RootOf}\left(\_Z^3-2\right)+7876919179391632\right)\_Z^2+\left(\right.\\
&-1068782164884000\,\mathit{RootOf}\left(\_Z^3-2\right)^2+340609155970104\,\mathit{RootOf}\left(\_Z^3-2\right)\\
&\left.+54362206908700768\right)\_Z^3\Big)^2\,\mathit{RootOf}\left(\_Z^3-2\right)^3\\
&+28\,\mathit{RootOf}\left(203063181099439307\,\_Z^4+9972349183004\,\mathit{RootOf}\left(\_Z^3-2\right)^2\right.\\
&-72643630017800\,\mathit{RootOf}\left(\_Z^3-2\right)+22338188120720+\left(\right.\\
&-23075699835888\,\mathit{RootOf}\left(\_Z^3-2\right)^2-60460295392000\,\mathit{RootOf}\left(\_Z^3-2\right)\\
&\left.+394712269021280\right)\_Z+\left(2865014874818964\,\mathit{RootOf}\left(\_Z^3-2\right)^2\right.\\
&\left.-478608060541564\,\mathit{RootOf}\left(\_Z^3-2\right)+7876919179391632\right)\_Z^2+\left(\right.\\
&-1068782164884000\,\mathit{RootOf}\left(\_Z^3-2\right)^2+340609155970104\,\mathit{RootOf}\left(\_Z^3-2\right)\\
&\left.+54362206908700768\right)\_Z^3\Big)\,\mathit{RootOf}\left(\_Z^3-2\right)^4+2\,\mathit{RootOf}\left(\_Z^3-2\right)^4\\
&+1764\,\mathit{RootOf}\left(203063181099439307\,\_Z^4+9972349183004\,\mathit{RootOf}\left(\_Z^3-2\right)^2\right.\\
&-72643630017800\,\mathit{RootOf}\left(\_Z^3-2\right)+22338188120720+\left(\right.\\
&-23075699835888\,\mathit{RootOf}\left(\_Z^3-2\right)^2-60460295392000\,\mathit{RootOf}\left(\_Z^3-2\right)\\
&\left.+394712269021280\right)\_Z+\left(2865014874818964\,\mathit{RootOf}\left(\_Z^3-2\right)^2\right.\\
&\left.-478608060541564\,\mathit{RootOf}\left(\_Z^3-2\right)+7876919179391632\right)\_Z^2+\left(\right.\\
&-1068782164884000\,\mathit{RootOf}\left(\_Z^3-2\right)^2+340609155970104\,\mathit{RootOf}\left(\_Z^3-2\right)\\
&\left.+54362206908700768\right)\_Z^3\Big)^2-20\,\mathit{RootOf}\left(\_Z^3-2\right)^2\Big)\Big)\Big/
\end{aligned}
$$

$$\left(RootOf\left(203063181099439307\,\_Z^4+9972349183004\,RootOf\left(\_Z^3-2\right)^2-72643630017800\,RootOf\left(\_Z^3-2\right)+22338188120720+\left(-23075699835888\,RootOf\left(\_Z^3-2\right)^2-60460295392000\,RootOf\left(\_Z^3-2\right)+394712269021280\right)\_Z+\left(2865014874818964\,RootOf\left(\_Z^3-2\right)^2-478608060541564\,RootOf\left(\_Z^3-2\right)+7876919179391632\right)\_Z^2+\left(-1068782164884000\,RootOf\left(\_Z^3-2\right)^2+340609155970104\,RootOf\left(\_Z^3-2\right)+54362206908700768\right)\_Z^3\right)RootOf\left(\_Z^3-2\right)\right),\ z=RootOf\left(203063181099439307\,\_Z^4+9972349183004\,RootOf\left(\_Z^3-2\right)^2-72643630017800\,RootOf\left(\_Z^3-2\right)+22338188120720+\left(-23075699835888\,RootOf\left(\_Z^3-2\right)^2-60460295392000\,RootOf\left(\_Z^3-2\right)+394712269021280\right)\_Z+\left(2865014874818964\,RootOf\left(\_Z^3-2\right)^2-478608060541564\,RootOf\left(\_Z^3-2\right)+7876919179391632\right)\_Z^2+\left(-1068782164884000\,RootOf\left(\_Z^3-2\right)^2+340609155970104\,RootOf\left(\_Z^3-2\right)+54362206908700768\right)\_Z^3\right)x\right\}$$

```
> ## sSym[3] plain equations - reduction to 71 variables and rank 16
  v3 := eval(Vector(v), sSym[3]);
  v31 := eval(v3, {y = 1, w = 1}) :
  simplify(LinearAlgebra[Transpose](v31).Q.v31);
  Q1 := reduceByLinearEquation(Q, v31) :
  nops(indets(Q1));
  randomRank(Q1);
```

$$v3 := \begin{bmatrix} w^3 \\ 0 \\ w^2 y \\ 0 \\ 0 \\ 0 \\ 0 \\ w y^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ y^3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 0 \\ 110 \\ 19 \end{matrix} \quad \textbf{(9)}$$

```
> v32 := eval(v3, {y = 1, w = 0}) :
  simplify(LinearAlgebra[Transpose](v32).Q1.v32);
  Q2 := reduceByLinearEquation(Q1, v32) :
  nops(indets(Q2));
  randomRank(Q2);
```

$$\begin{matrix} 0 \\ 95 \\ 18 \end{matrix} \quad \textbf{(10)}$$

```
> v33 := eval(v3, {y = 0, w = 1}) :
  simplify(LinearAlgebra[Transpose](v33).Q2.v33);
  Q3 := reduceByLinearEquation(Q2, v33) :
  nops(indets(Q3));
  randomRank(Q3);
```

$$0$$
$$81$$
$$17 \tag{11}$$

```
> v34 := eval(v3, {y=-1, w=1}) :
  simplify(LinearAlgebra[Transpose](v34).Q3.v34);
  Q4 := reduceByLinearEquation(Q3, v34) :
  nops(indets(Q4));
  randomRank(Q4);
```

$$0$$
$$71$$
$$16 \tag{12}$$

```
> ## Determinant equations - reduction to 29 variables and rank 12

  Q5 := zeroRows(Q4) :
  nops(indets(Q5));
  randomRank(Q5);

  Q6 := zeroRows(Q5) :
  nops(indets(Q6));
  randomRank(Q6);

  Q7 := zeroDetSRows(Q6, 2) :
  nops(indets(Q7));
  randomRank(Q7);
```

$$48$$
$$14$$
$$48$$
$$14$$
$$29$$
$$12 \tag{13}$$

```
> ## sSym[1] plain equations - reduction to 8 variables and rank 9

   v1 := eval(Vector(v), sSym[1]) :
  v11 := eval(v1, {x=1, w=1}) :
  evalf(allvalues(v11));
  simplify(LinearAlgebra[Transpose](v11).Q.v11);
  Q8 := reduceByLinearEquation(Q7, v11) :
  nops(indets(Q8));
  randomRank(Q8);
```

$$\left[\begin{array}{c} 1. \\ 1. \\ 0.1250000000 \\ -0.2500000000 \\ 1. \\ 0.1250000000 \\ -0.2500000000 \\ 0.01562500000 \\ -0.03125000000 \\ 0.06250000000 \\ 1. \\ 0.1250000000 \\ -0.2500000000 \\ 0.01562500000 \\ -0.03125000000 \\ 0.06250000000 \\ 0.001953125000 \\ -0.003906250000 \\ 0.007812500000 \\ -0.01562500000 \end{array}\right], \left[\begin{array}{c} 1. \\ 1. \\ 2.500000000 \\ -0.2500000000 \\ 1. \\ 2.500000000 \\ -0.2500000000 \\ 6.250000000 \\ -0.6250000000 \\ 0.06250000000 \\ 1. \\ 2.500000000 \\ -0.2500000000 \\ 6.250000000 \\ -0.6250000000 \\ 0.06250000000 \\ 15.62500000 \\ -1.562500000 \\ 0.1562500000 \\ -0.01562500000 \end{array}\right]$$

$$\begin{array}{c} 0 \\ 21 \\ 11 \end{array} \qquad \textbf{(14)}$$

```
> v12 := eval(v1, {x = 1, w = 2}) :
  evalf(allvalues(v12));
  simplify(LinearAlgebra[Transpose](v12).Q.v12);
  Q9 := reduceByLinearEquation(Q8, v12) :
  nops(indets(Q9));
  randomRank(Q9);
```

$$\begin{bmatrix} 8. \\ 4. \\ 18.15590950 \\ -1. \\ 2. \\ 9.077954750 \\ -0.5000000000 \\ 41.20463122 \\ -2.269488688 \\ 0.1250000000 \\ 1. \\ 4.538977375 \\ -0.2500000000 \\ 20.60231561 \\ -1.134744344 \\ 0.06250000000 \\ 93.51344442 \\ -5.150578902 \\ 0.2836860859 \\ -0.01562500000 \end{bmatrix}, \begin{bmatrix} 8. \\ 4. \\ -7.655909500 \\ -1. \\ 2. \\ -3.827954750 \\ -0.5000000000 \\ 7.326618785 \\ 0.9569886880 \\ 0.1250000000 \\ 1. \\ -1.913977375 \\ -0.2500000000 \\ 3.663309392 \\ 0.4784943438 \\ 0.06250000000 \\ -7.011491294 \\ -0.9158273481 \\ -0.1196235859 \\ -0.01562500000 \end{bmatrix}$$

$$\begin{matrix} 0 \\ 14 \\ 10 \end{matrix} \quad \textbf{(15)}$$

```
> v13 := eval(v1, {x = 1, w = 3}) :
  evalf(allvalues(v13));
  simplify(LinearAlgebra[Transpose](v13).Q.v13);
  Q10 := reduceByLinearEquation(Q9, v13) :
  nops(indets(Q10));
  randomRank(Q10);
```

$$\left[\begin{array}{c} 27. \\ 9. \\ 50.42885219 \\ -2.250000000 \\ 3. \\ 16.80961740 \\ -0.7500000000 \\ 94.18774568 \\ -4.202404349 \\ 0.1875000000 \\ 1. \\ 5.603205799 \\ -0.2500000000 \\ 31.39591522 \\ -1.400801450 \\ 0.06250000000 \\ 175.9177744 \\ -7.848978806 \\ 0.3502003624 \\ -0.01562500000 \end{array}\right], \left[\begin{array}{c} 27. \\ 9. \\ -26.80385219 \\ -2.250000000 \\ 3. \\ -8.934617400 \\ -0.7500000000 \\ 26.60912934 \\ 2.233654349 \\ 0.1875000000 \\ 1. \\ -2.978205799 \\ -0.2500000000 \\ 8.869709780 \\ 0.7445514500 \\ 0.06250000000 \\ -26.41582110 \\ -2.217427445 \\ -0.1861378624 \\ -0.01562500000 \end{array}\right]$$

$$\begin{array}{c} 0 \\ 8 \\ 9 \end{array} \qquad \textbf{(16)}$$

```
> ## sSym[2] plain equations - reduction to 6 variables and rank 8

  v2 := eval(Vector(v), sSym[2]);
  v21 := eval(v2, {y = -3, z = 1});
  evalf(allvalues(v21));
  simplify(LinearAlgebra[Transpose](v21).Q.v21);
  Q11 := reduceByLinearEquation(Q10, v21) :
  nops(indets(Q11));
  randomRank(Q11);
```

$$v2 := \begin{bmatrix} RootOf\left(\_Z^2 + 21\,y\,z + 50\,z^2\right)^3 \\ 0 \\ RootOf\left(\_Z^2 + 21\,y\,z + 50\,z^2\right)^2 y \\ RootOf\left(\_Z^2 + 21\,y\,z + 50\,z^2\right)^2 z \\ 0 \\ 0 \\ 0 \\ RootOf\left(\_Z^2 + 21\,y\,z + 50\,z^2\right) y^2 \\ RootOf\left(\_Z^2 + 21\,y\,z + 50\,z^2\right) y\,z \\ RootOf\left(\_Z^2 + 21\,y\,z + 50\,z^2\right) z^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ y^3 \\ y^2 z \\ y\,z^2 \\ z^3 \end{bmatrix}$$

$$v21 := \begin{bmatrix} RootOf(\_Z^2 - 13)^3 \\ 0 \\ -3\ RootOf(\_Z^2 - 13)^2 \\ RootOf(\_Z^2 - 13)^2 \\ 0 \\ 0 \\ 0 \\ 9\ RootOf(\_Z^2 - 13) \\ -3\ RootOf(\_Z^2 - 13) \\ RootOf(\_Z^2 - 13) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -27 \\ 9 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 46.87216658 \\ 0. \\ -39. \\ 13. \\ 0. \\ 0. \\ 0. \\ 32.44996148 \\ -10.81665382 \\ 3.605551275 \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ -27. \\ 9. \\ -3. \\ 1. \end{bmatrix}, \begin{bmatrix} -46.87216658 \\ 0. \\ -39. \\ 13. \\ 0. \\ 0. \\ 0. \\ -32.44996148 \\ 10.81665382 \\ -3.605551275 \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ -27. \\ 9. \\ -3. \\ 1. \end{bmatrix}$$

$$\begin{matrix} 0 \\ 6 \\ 8 \end{matrix} \quad \textbf{(17)}$$

```
> # CHECK IF MNEW IS SOLUTION - passed
  eqs := Equate(Q11, MNEW) :
  solve(eqs);
```

$$\left\{ _t3_{10} = 16\ RootOf\left(_Z^3 - 2\right)^2 - 64\ RootOf\left(_Z^3 - 2\right) + 592,\ _t3_{16} = 0,\ _t3_{17} = 0,\ _t3_{35} = 4\ RootOf\left(_Z^3 - 2\right)^2,\ _t3_{36} = 84\ RootOf\left(_Z^3 - 2\right),\ _t3_{44} = 1764 + 64\ RootOf\left(_Z^3 - 2\right)^2 \right\} \quad \textbf{(18)}$$

```
> ## Determinant equations - reduction to 3 variables and rank 7

  Q12 := zeroRows(Q11) :
  nops(indets(Q12));
  randomRank(Q12);
```

$$3$$

$$\textbf{(19)}$$

$$7 \tag{19}$$

```
# CHECK IF MNEW IS SOLUTION - passed
eqs := Equate(Q12, MNEW) :
solve(eqs);
```

$$\left\{ \_t5_1 = 16\,RootOf\left(\_Z^3 - 2\right)^2 - 64\,RootOf\left(\_Z^3 - 2\right) + 592,\ \_t5_2 = 4\,RootOf\left(\_Z^3 - 2\right)^2,\ \_t5_3 = 84\,RootOf\left(\_Z^3 - 2\right) \right\} \tag{20}$$

```
## sSym[4] plain equations
v4 := eval(Vector(v), sSym[4]) :
v41 := eval(v4, {x = 1}) :
evalf(allvalues(v41));
simplify(LinearAlgebra[Transpose](v41).Q.v41);
Q13 := reduceByLinearEquation(Q12, v41) :
Q13 := simplify(Q13) :
nops(indets(Q13));
randomRank(Q13);
```

$$\begin{bmatrix} -48.46676706 \\ 13.29319476 \\ 49.78487571 \\ 0.6767230530 \\ -3.645983371 \\ -13.65471826 \\ -0.1856078276 \\ -51.13883187 \\ -0.6951273034 \\ -0.009448826857 \\ 1. \\ 3.745140024 \\ 0.05090748051 \\ 14.02607379 \\ 0.1906556428 \\ 0.002591571572 \\ 52.52961035 \\ 0.7140320783 \\ 0.009705798422 \\ 0.0001319303793 \end{bmatrix}, \begin{bmatrix} 2.839738216 - 1.219246169\,I \\ 2.044596719 - 0.5666670190\,I \\ -1.834459011 - 6.434546906\,I \\ -0.1045966623 + 0.4782120832\,I \\ 1.443306547 - 0.1963086148\,I \\ -0.6525657377 - 4.546955871\,I \\ -0.1154006513 + 0.3156349162\,I \\ -14.32382101 + 2.163425205\,I \\ 0.9980431320 + 0.3565934086\,I \\ -0.05197231898 - 0.05754256692\,I \\ 1. \\ -0.02321085392 - 3.153531291\,I \\ -0.1077077376 + 0.2040390932\,I \\ -9.944220861 + 0.1463923104\,I \\ 0.6459436535 + 0.3349237992\,I \\ -0.03003099481 - 0.04395317822\,I \\ 0.6924665901 + 31.35601376\,I \\ 1.041199777 - 2.044777392\,I \\ -0.1379106778 + 0.09572387267\,I \\ 0.01220273714 - 0.001393399563\,I \end{bmatrix}, \begin{bmatrix} 28.45735347 \\ 9.321010677 \\ -20.79013880 \\ -0.9037761585 \\ 3.053033027 \\ -6.809667171 \\ -0.2960256735 \\ 15.18868829 \\ 0.6602733383 \\ 0.02870299746 \\ 1. \\ -2.230459712 \\ -0.09696117619 \\ 4.974950528 \\ 0.2162679971 \\ 0.009401469688 \\ -11.09642672 \\ -0.4823770547 \\ -0.02096959938 \\ -0.0009115775589 \end{bmatrix},$$

$$\left[\begin{array}{c} 2.839738216 + 1.219246169\,I \\ 2.044596719 + 0.5666670190\,I \\ -1.834459011 + 6.434546906\,I \\ -0.1045966623 - 0.4782120832\,I \\ 1.443306547 + 0.1963086148\,I \\ -0.6525657377 + 4.546955871\,I \\ -0.1154006513 - 0.3156349162\,I \\ -14.32382101 - 2.163425205\,I \\ 0.9980431320 - 0.3565934086\,I \\ -0.05197231898 + 0.05754256692\,I \\ 1. \\ -0.02321085392 + 3.153531291\,I \\ -0.1077077376 - 0.2040390932\,I \\ -9.944220861 - 0.1463923104\,I \\ 0.6459436535 - 0.3349237992\,I \\ -0.03003099481 + 0.04395317822\,I \\ 0.6924665901 - 31.35601376\,I \\ 1.041199777 + 2.044777392\,I \\ -0.1379106778 - 0.09572387267\,I \\ 0.01220273714 + 0.001393399563\,I \end{array}\right], \left[\begin{array}{c} 3.089978560 - 0.2372078107\,I \\ -1.155414168 - 1.784185917\,I \\ 5.701924193 - 3.535238334\,I \\ 0.1196771081 - 0.2089977007\,I \\ -0.6964967522 + 1.280828596\,I \\ -3.998535752 - 2.277398421\,I \\ -0.1651486908 - 0.00363169698\,I \\ 7.434848356 - 12.47637067\,I \\ 0.02171500914 - 0.5209179108\,I \\ -0.008209672830 - 0.01681949512\,I \\ 1. \\ -0.06209447645 + 3.155601275\,I \\ 0.05192529514 + 0.1007026948\,I \\ -9.953963688 - 0.3918908196\,I \\ -0.3210018262 + 0.1576024465\,I \\ -0.007444796465 + 0.01045803430\,I \\ 1.854737331 - 31.38640627\,I \\ -0.4773980408 - 1.022740014\,I \\ -0.03253910563 - 0.02414219537\,I \\ -0.001439725490 - 0.0002066745487\,I \end{array}\right],$$

$$\left[\begin{array}{c} -12.63386320 + 0.8272995743\,I \\ -2.508511597 + 4.818331484\,I \\ -0.3032874984 + 2.435015894\,I \\ -0.4142368028 - 0.7069823810\,I \\ 1.209070383 + 1.992576921\,I \\ 0.8256781780 + 0.6532197151\,I \\ -0.3515248540 - 0.00541082602\,I \\ 0.4524419217 + 0.1465367116\,I \\ -0.1496808449 + 0.05306558822\,I \\ 0.02889227416 - 0.04446887445\,I \\ 1. \\ 0.4233805164 - 0.1574751426\,I \\ -0.08022507555 + 0.1277376489\,I \\ 0.1544526411 - 0.1333438144\,I \\ -0.01385022943 + 0.06671508696\,I \\ -0.009880844199 - 0.02049552507\,I \\ 0.04439390282 - 0.08077762477\,I \\ 0.004642050543 + 0.03042693483\,I \\ -0.007410892651 - 0.007121418634\,I \\ 0.003410741657 + 0.0003820992398\,I \end{array}\right], \left[\begin{array}{c} 0.8680264766 + 1.198884212\,I \\ 0.1371773755 - 1.291509950\,I \\ -4.082971091 - 0.4292539574\,I \\ -0.1904603971 + 0.3014033316\,I \\ -0.8473346511 + 0.7621014602\,I \\ 2.411894797 + 2.675876057\,I \\ 0.3011173758 - 0.08487949812\,I \\ 8.450377668 - 7.633132810\,I \\ -0.2693254967 - 0.9515552199\,I \\ -0.08444978251 - 0.01562773243\,I \\ 1. \\ -0.003382409486 - 3.161034181\,I \\ -0.2462582532 - 0.1213148503\,I \\ -9.992125651 + 0.02138382053\,I \\ -0.3826474421 + 0.7788410920\,I \\ 0.04592583437 + 0.05974956624\,I \\ 0.1013924540 + 31.58537843\,I \\ 2.463237585 + 1.206927285\,I \\ 0.1887150812 - 0.1453752297\,I \\ -0.004061106063 - 0.02028530953\,I \end{array}\right],$$

$$\left[\begin{array}{c} 23.25098517 + 19.17629902\,I \\ -8.061170511 + 5.367439566\,I \\ 26.70921668 + 3.251672392\,I \\ 0.5936105406 + 0.9566392684\,I \\ -0.9009578673 - 2.978740604\,I \\ -3.484882749 + 7.912566850\,I \\ -0.3494611132 + 0.09358344298\,I \\ 21.65701668 - 10.39105141\,I \\ 0.9063853673 + 0.4343986122\,I \\ 0.009571344737 + 0.04095307492\,I \\ 1. \\ -2.109504575 - 1.807964575\,I \\ 0.00372641730 - 0.1161913084\,I \\ 1.181273651 + 7.627819095\,I \\ -0.2179306638 + 0.2383688662\,I \\ -0.01348653396 - 0.0008659546035\,I \\ 11.29892454 - 18.22662021\,I \\ 0.8906881993 - 0.1088292941\,I \\ 0.02688428985 + 0.02620991085\,I \\ -0.0001508728519 + 0.001563791119\,I \end{array}\right], \left[\begin{array}{c} 3.089978560 + 0.2372078107\,I \\ -1.155414168 + 1.784185917\,I \\ 5.701924193 + 3.535238334\,I \\ 0.1196771081 + 0.2089977007\,I \\ -0.6964967522 - 1.280828596\,I \\ -3.998535752 + 2.277398421\,I \\ -0.1651486908 + 0.00363169698\,I \\ 7.434848356 + 12.47637067\,I \\ 0.02171500914 + 0.5209179108\,I \\ -0.008209672830 + 0.01681949512\,I \\ 1. \\ -0.06209447645 - 3.155601275\,I \\ 0.05192529514 - 0.1007026948\,I \\ -9.953963688 + 0.3918908196\,I \\ -0.3210018262 - 0.1576024465\,I \\ -0.007444796465 - 0.01045803430\,I \\ 1.854737331 + 31.38640627\,I \\ -0.4773980408 + 1.022740014\,I \\ -0.03253910563 + 0.02414219537\,I \\ -0.001439725490 + 0.0002066745487\,I \end{array}\right],$$

$$\left[\begin{array}{c} 23.25098517 - 19.17629902\,I \\ -8.061170511 - 5.367439566\,I \\ 26.70921668 - 3.251672392\,I \\ 0.5936105406 - 0.9566392684\,I \\ -0.9009578673 + 2.978740604\,I \\ -3.484882749 - 7.912566850\,I \\ -0.3494611132 - 0.09358344298\,I \\ 21.65701668 + 10.39105141\,I \\ 0.9063853673 - 0.4343986122\,I \\ 0.009571344737 - 0.04095307492\,I \\ 1. \\ -2.109504575 + 1.807964575\,I \\ 0.00372641730 + 0.1161913084\,I \\ 1.181273651 - 7.627819095\,I \\ -0.2179306638 - 0.2383688662\,I \\ -0.01348653396 + 0.0008659546035\,I \\ 11.29892454 + 18.22662021\,I \\ 0.8906881993 + 0.1088292941\,I \\ 0.02688428985 - 0.02620991085\,I \\ -0.0001508728519 - 0.001563791119\,I \end{array}\right], \left[\begin{array}{c} 0.8680264766 - 1.198884212\,I \\ 0.1371773755 + 1.291509950\,I \\ -4.082971091 + 0.4292539574\,I \\ -0.1904603971 - 0.3014033316\,I \\ -0.8473346511 - 0.7621014602\,I \\ 2.411894797 - 2.675876057\,I \\ 0.3011173758 + 0.08487949812\,I \\ 8.450377668 + 7.633132810\,I \\ -0.2693254967 + 0.9515552199\,I \\ -0.08444978251 + 0.01562773243\,I \\ 1. \\ -0.003382409486 + 3.161034181\,I \\ -0.2462582532 + 0.1213148503\,I \\ -9.992125651 - 0.02138382053\,I \\ -0.3826474421 - 0.7788410920\,I \\ 0.04592583437 - 0.05974956624\,I \\ 0.1013924540 - 31.58537843\,I \\ 2.463237585 - 1.206927285\,I \\ 0.1887150812 + 0.1453752297\,I \\ -0.004061106063 + 0.02028530953\,I \end{array}\right],$$

$$\begin{bmatrix} -12.63386320 - 0.8272995743\,I \\ -2.508511597 - 4.818331484\,I \\ -0.3032874984 - 2.435015894\,I \\ -0.4142368028 + 0.7069823810\,I \\ 1.209070383 - 1.992576921\,I \\ 0.8256781780 - 0.6532197151\,I \\ -0.3515248540 + 0.00541082602\,I \\ 0.4524419217 - 0.1465367116\,I \\ -0.1496808449 - 0.05306558822\,I \\ 0.02889227416 + 0.04446887445\,I \\ 1. \\ 0.4233805164 + 0.1574751426\,I \\ -0.08022507555 - 0.1277376489\,I \\ 0.1544526411 + 0.1333438144\,I \\ -0.01385022943 - 0.06671508696\,I \\ -0.009880844199 + 0.02049552507\,I \\ 0.04439390282 + 0.08077762477\,I \\ 0.004642050543 - 0.03042693483\,I \\ -0.007410892651 + 0.007121418634\,I \\ 0.003410741657 - 0.0003820992398\,I \end{bmatrix} \quad \begin{matrix} 0 \\ 0 \\ 3 \end{matrix} \qquad \textbf{(21)}$$

```
> simplify(MNEW-Q13);
```

(22)

$$
\left[\begin{array}{cccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \qquad \textbf{(22)}
$$

>