"C:\Program Files\Maple 2015" **(1)**

```
> currentdir("C:/Users/slapl/Dropbox/repos/rationalSOS/");

################################################################################
# Load "Rational SOS" procedures
################################################################################
restart;
read("rationalSOS.mpl");
with(rationalSOS);

# Display tables of any size
interface(rtablesize=infinity);
```

"C:\Users\slapl\Dropbox\Repos\rationalsos"

"C:\Users\slapl\Dropbox\repos\rationalSOS"

"Opening connection with Matlab"

rationalSOS := **module**( ) ... **end module**

[*cancelDenominator, decompositionToMatrix, evalMat, evalSolution, exactSOS, getCoeffs, getDiag, getExtension, getVars, homogenize, isHomogeneous, linIndepRows, listSubsets, matrixToPoly, minorsDet, nonRatCoef, numericSolver, numericSolverSubmatrix, numericSolverSubmatrixMaxRank, numericSolverSubmatrixRoundBefore, polyToMatrix, polyToMatrixVars, primitiveMatrix, randomRank, reduceByLinearEquation, reduceByLinearEquationLinear, roundMat, roundMatToZero, roundToIntMatrix, roundVec, sedumiCall, smallToZero, solveSubmatrixGeneral, vectorTrace, zeroDetSRows, zeroDetSys, zeroRows*]

10 **(2)**

```
> ################################################################################
## Example 6.1
## Example of a polynomial in the border that is sum of less than
## n = 6 polynomials, and such that the polynomials in a SOS
## decomposition have a common complex zero.
################################################################################

# The 5 polynomials
p1 := x1^2 - x4^2;
p2 := x2^2 - x4^2;
p3 := x3^2 - x4^2;
p4 := -x1^2 - x1*x2 - x1*x3 + x1*x4 - x2*x3 + x2*x4 + x3*x4;
p5 := x5^2 + x6^2;

# f is the sum of squares of p1, ..., p5
f := p1^2 + p2^2 + p3^2 + p4^2 + p5^2;
f := expand(f);
```

$$p1 := x1^2 - x4^2$$

$$p2 := x2^2 - x4^2$$

$$p3 := x3^2 - x4^2$$

$$p4 := -x1^2 - x1\,x2 - x1\,x3 + x1\,x4 - x2\,x3 + x2\,x4 + x3\,x4$$

$$p5 := x5^2 + x6^2$$

$$f := \left(x1^2 - x4^2\right)^2 + \left(x2^2 - x4^2\right)^2 + \left(x3^2 - x4^2\right)^2 + \left(-x1^2 - x1\,x2 - x1\,x3 + x1\,x4 - x2\,x3 + x2\,x4 + x3\,x4\right)^2 + \left(x5^2 + x6^2\right)^2$$

$$f := 2\,x1^4 + 2\,x1^3\,x2 + 2\,x1^3\,x3 - 2\,x1^3\,x4 + x1^2\,x2^2 + 4\,x1^2\,x2\,x3 - 4\,x1^2\,x2\,x4 + x1^2\,x3^2 - 4\,x1^2\,x3\,x4 - x1^2\,x4^2 + 2\,x1\,x2^2\,x3 - 2\,x1\,x2^2\,x4 + 2\,x1\,x2\,x3^2 - 6\,x1\,x2\,x3\,x4 + 2\,x1\,x2\,x4^2 - 2\,x1\,x3^2\,x4 + 2\,x1\,x3\,x4^2 + x2^4 + x2^2\,x3^2 - 2\,x2^2\,x3\,x4 - x2^2\,x4^2 - 2\,x2\,x3^2\,x4 + 2\,x2\,x3\,x4^2 + x3^4 - x3^2\,x4^2 + 3\,x4^4 + x5^4 + 2\,x5^2\,x6^2 + x6^4 \quad (3)$$

```
> # We compute the common solutions of {p1, p2, p3, p4, p5}
  solve({p1, p2, p3, p4, p5})
```

$$\{x1 = 0, x2 = 0, x3 = 0, x4 = 0, x5 = x5, x6 = RootOf\left(\_Z^2 + 1\right)\,x5\} \quad (4)$$

```
> # We use SEDUMI to compute a SOS decomposition.
  # We do not perform facial reduction, since we are interested in the
  # solutions of maximum rank.
  out := exactSOS(f, facial = "no") :
```

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 21, " - Number of indeterminates: ", 105

"Matrix after facial reduction - Rank: ", 21, " - Number of indeterminates: ", 105

"Check 1 of random rank:", 21

"Check 2 of random rank:", 21

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 106, order n = 22, dim = 442, blocks = 2
nnz(A) = 231 + 0, nnz(ADA) = 11236, nnz(L) = 5671
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            1.94E+01 0.000
  1 :  -3.46E+00 6.23E+00 0.000 0.3211 0.9000 0.9000   0.55  1
1  1.4E+01
  2 :  -9.67E-01 2.07E+00 0.000 0.3319 0.9000 0.9000   2.70  1
1  2.3E+00
  3 :  -1.08E-01 6.85E-01 0.000 0.3313 0.9000 0.9000   2.79  1
1  7.4E-01
  4 :  -3.04E-02 2.27E-01 0.000 0.3312 0.9000 0.9000   1.31  1
1  4.5E-01
  5 :  -1.01E-02 8.70E-02 0.000 0.3837 0.9000 0.9000   1.14  1
1  3.4E-01
  6 :  -4.97E-03 3.36E-02 0.000 0.3859 0.9000 0.9000   1.06  1
1  3.5E-01
  7 :  -1.50E-03 1.29E-02 0.000 0.3847 0.9000 0.9000   1.05  1
```

```
1  3.1E-01
  8 :  -7.14E-04 4.56E-03 0.000 0.3527 0.9000 0.9000   1.02  1
1  2.2E-01
  9 :  -1.95E-04 1.03E-03 0.000 0.2250 0.9152 0.9000   1.02  1
1  6.3E-02
 10 :  -5.84E-05 3.16E-04 0.000 0.3083 0.9066 0.9000   1.01  1
1  2.1E-02
 11 :  -1.12E-05 1.15E-04 0.000 0.3649 0.9000 0.9271   1.01  1
1  5.8E-03
 12 :  -9.55E-06 5.02E-05 0.000 0.4349 0.9000 0.2155   1.00  1
1  2.7E-03
 13 :  -2.50E-06 7.57E-06 0.000 0.1510 0.9305 0.9000   1.01  1
1  8.2E-04
 14 :  -7.16E-07 2.30E-06 0.000 0.3037 0.9015 0.9000   1.00  2
2  2.5E-04
 15 :  -2.13E-07 7.60E-07 0.000 0.3303 0.9000 0.9054   1.00  3
3  8.1E-05
 16 :  -6.52E-08 2.64E-07 0.000 0.3476 0.9000 0.9126   1.00  3
3  2.7E-05
 17 :  -2.13E-08 9.61E-08 0.000 0.3639 0.9000 0.9164   1.00  3
3  9.0E-06
 18 :  -7.44E-09 3.58E-08 0.000 0.3729 0.9000 0.9163   1.00  3
3  3.1E-06
 19 :  -2.69E-09 1.34E-08 0.000 0.3726 0.9000 0.9130   1.00  3
3  1.1E-06
 20 :  -1.00E-09 4.87E-09 0.000 0.3649 0.9000 0.9067   1.00  3
3  3.8E-07
 21 :  -3.82E-10 1.72E-09 0.000 0.3526 0.9013 0.9000   1.00  1
3  1.4E-07
 22 :  -1.46E-10 5.62E-10 0.000 0.3275 0.9108 0.9000   1.00  3
3  4.7E-08
 23 :  -5.52E-11 1.67E-10 0.000 0.2976 0.9210 0.9000   1.00  3
3  1.6E-08
 24 :  -2.06E-11 4.42E-11 0.000 0.2638 0.9334 0.9000   1.00  3
3  5.7E-09
 25 :  -7.45E-12 1.37E-11 0.000 0.3092 0.9413 0.9000   1.00  9
9  2.1E-09
 26 :  -2.11E-12 4.08E-12 0.000 0.2990 0.9030 0.9000   1.00  9
9  6.5E-10
 27 :  -6.65E-13 1.24E-12 0.000 0.3028 0.9225 0.9000   1.00 12
12  2.0E-10
 28 :  -2.11E-13 3.84E-13 0.000 0.3108 0.9359 0.9000   1.00 12
12  6.3E-11
 29 :  -5.80E-14 1.10E-13 0.000 0.2869 0.9124 0.9000   1.00 31
32  1.8E-11
Run into numerical problems.

iter seconds digits       c*x               b*y
 29      0.3   9.2  5.8511270103e-14 -5.8036655397e-14
|Ax-b| =   6.6e-15, [Ay-c]_+ =   2.0E-14, |x|=  4.9e-01, |y|=
3.6e+00

Detailed timing (sec)
   Pre          IPM          Post
3.993E-03    1.870E-01    9.958E-04
Max-norms: ||b||=1, ||c|| = 6,
Cholesky |add|=9, |skip| = 0, ||L.L|| = 1.50757e+06.
```

"An exact positive definite solution could not be found for the reduced problem."
"matrixToPoly begins..."
"Computing decomposition..."
"Decomposition computed!" **(5)**

```
> # out[3] is a matrix in the spectrahedron of maximum rank.
  # We check the eigenvalues to determine the rank
  eig(out[3]);
```

$$\begin{bmatrix} -3.29278630064062\ 10^{-16} \\ -2.11135022737241\ 10^{-16} \\ -2.25169177191060\ 10^{-17} \\ -2.71570612502969\ 10^{-32} \\ -1.36969237233461\ 10^{-33} \\ -8.23778852492638\ 10^{-34} \\ -2.20374919543594\ 10^{-48} \\ 0. \\ 0. \\ 8.25323037263240\ 10^{-33} \\ 1.02463201235011\ 10^{-17} \\ 1.06108361618584\ 10^{-16} \\ 3.89671627465454\ 10^{-16} \\ 6.12948567374568\ 10^{-16} \\ 0.501791833000000 \\ 0.888960947926901 \\ 1.00000000000000 \\ 1.00358366600000 \\ 1.49820816700000 \\ 3.89989969879447 \\ 7.21113935327863 \end{bmatrix} \tag{6}$$

```
> # There are only 7 non-zero eigenvalues, the maximum rank in the
  # spectrahedron is 7. f is a polynomial in the border of the SOS cone.
```

```
> ##############################################################################
  ## Example 6.2
  ## Examples of polynomial in the border of the (6,4)-cone with
  ## different maximum ranks of the matrices in the spectrahedron.
  ##############################################################################
```

```
> # Maximum rank in the spectrahedron = 5

  # The 5 polynomials
  p1 := x1^2 - x4^2;
  p2 := x2^2 - x4^2;
  p3 := x3^2 - x4^2;
  p4 := -x1^2 - x1*x2 - x1*x3 + x1*x4 - x2*x3 + x2*x4 + x3*x4;
  p5 := x5^2 + x6^2- x4^2;
  f := p1^2 + p2^2 + p3^2 + p4^2 + p5^2;

  # Numerical solution
  out := exactSOS(f, realPolynomials = [p1, p2, p3, p4, p5]) :
  eig(out[3]);
```

$$p1 := x1^2 - x4^2$$

$$p2 := x2^2 - x4^2$$

$$p3 := x3^2 - x4^2$$

$$p4 := -x1^2 - x1\,x2 - x1\,x3 + x1\,x4 - x2\,x3 + x2\,x4 + x3\,x4$$

$$p5 := -x4^2 + x5^2 + x6^2$$

$$f := \left(x1^2 - x4^2\right)^2 + \left(x2^2 - x4^2\right)^2 + \left(x3^2 - x4^2\right)^2 + \left(-x1^2 - x1\,x2 - x1\,x3 + x1\,x4 - x2\,x3 + x2\,x4 + x3\,x4\right)^2 + \left(-x4^2 + x5^2 + x6^2\right)^2$$

"Option traceEquations: yes - Only valid when looking for rational decompositions."

"no real roots in this solution, please check...", $RootOf\left(\_Z^2 + 1\right)$

"compute random solutions..."

"indetsCFEV", $\{x5\}$

"trueIndets", $\{x5\}$

"expectedRank", 2

"This will go on until i = ", 21

"DEBUG - randomSol", $RootOf\left(\_Z^2 + 1\right)$

"i = ", 1

"DEBUG - randomSol", $RootOf\left(\_Z^2 + 1\right)$

"i = ", 2

"DEBUG - randomSol", $RootOf\left(\_Z^2 + 1\right)$

"i = ", 3

"DEBUG - randomSol", $RootOf\left(\_Z^2 + 1\right)$

"i = ", 4

"DEBUG - randomSol", $RootOf\left(\_Z^2 + 1\right)$

"i = ", 5

"DEBUG - randomSol", $RootOf\left(\_Z^2 + 1\right)$

"i = ", 6

"DEBUG - randomSol", $RootOf\left(\_Z^2 + 1\right)$

"i = ", 7

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 8

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 9

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 10

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 11

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 12

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 13

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 14

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 15

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 16

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 17

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 18

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 19

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 20

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 21

"DEBUG - randomSol", $RootOf(\_Z^2+1)$

"i = ", 22

"number of solutions: ", 0

"No equations found. Check!"

"solve finished 2"

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 21, " - Number of indeterminates: ", 105

"Matrix after facial reduction - Rank: ", 21, " - Number of indeterminates: ", 105

"Check 1 of random rank:", 21

"Check 2 of random rank:", 21

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 106, order n = 22, dim = 442, blocks = 2
nnz(A) = 231 + 0, nnz(ADA) = 11236, nnz(L) = 5671
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            1.93E+01 0.000
  1 :  -3.89E+00 5.69E+00 0.000 0.2956 0.9000 0.9000   0.40  1
1  1.4E+01
  2 :  -1.12E+00 1.89E+00 0.000 0.3330 0.9000 0.9000   2.73  1
1  2.2E+00
  3 :  -1.21E-01 5.56E-01 0.000 0.2936 0.9000 0.9000   2.94  1
1  5.3E-01
  4 :  -3.04E-02 1.40E-01 0.000 0.2517 0.9000 0.9000   1.28  1
1  2.8E-01
  5 :  -6.25E-03 2.77E-02 0.000 0.1978 0.9000 0.9000   1.09  1
1  2.1E-01
  6 :  -2.65E-04 1.08E-03 0.000 0.0389 0.9900 0.9900   1.02  1
1  4.9E-02
  7 :  -6.64E-06 1.59E-05 0.042 0.0148 0.9901 0.9900   1.00  1
1  1.2E-03
  8 :  -5.22E-07 1.32E-06 0.481 0.0830 0.9900 0.9900   1.00  1
1  1.0E-04
  9 :  -1.55E-08 4.68E-08 0.260 0.0354 0.9900 0.9901   1.00  1
1  3.3E-06
 10 :  -3.91E-10 1.91E-09 0.000 0.0407 0.9900 0.9902   1.00  1
1  8.9E-08
 11 :  -3.06E-11 1.68E-10 0.428 0.0880 0.9900 0.9900   1.00  1
1  7.8E-09
 12 :  -1.37E-12 6.35E-12 0.098 0.0379 0.9901 0.9900   1.00  1
1  3.3E-10
 13 :  -2.92E-13 1.00E-12 0.000 0.1580 0.9042 0.9000   1.00  1
1  5.9E-11
 14 :  -2.61E-14 8.59E-14 0.370 0.0855 0.9900 0.9900   1.00  1
1  5.1E-12
 15 :  -7.24E-15 1.14E-14 0.000 0.1325 0.9091 0.9000   1.00  2
2  9.5E-13
 16 :  -4.50E-15 2.13E-15 0.000 0.1872 0.9146 0.9000   1.00  2
2  2.3E-13
 17 :  -3.95E-15 5.27E-16 0.000 0.2473 0.9000 0.9013   1.00  3
3  5.6E-14
 18 :  -3.84E-15 1.35E-16 0.000 0.2558 0.9000 0.9071   1.00  3
3  1.4E-14
 19 :  -3.82E-15 3.10E-17 0.000 0.2299 0.9000 0.9039   1.00  3
3  3.1E-15
 20 :  -3.81E-15 6.36E-18 0.000 0.2052 0.9000 0.9010   1.01  3
3  6.7E-16
 21 :  -3.81E-15 1.74E-18 0.000 0.2740 0.9000 0.9146   0.99  3
3  1.7E-16
Run into numerical problems.

iter seconds digits       c*x               b*y
 21      0.2  10.7 -5.8253355519e-16 -3.8077743791e-15
```

```
|Ax-b| =   1.2e-14, [Ay-c]_+ =   1.4E-14, |x|=  4.4e-01, |y|=
4.0e+00

Detailed timing (sec)
   Pre             IPM             Post
2.997E-03    7.400E-02    2.002E-03
Max-norms: ||b||=1, ||c|| = 6,
Cholesky |add|=0, |skip| = 0, ||L.L|| = 364.535.
```

"An exact positive definite solution could not be found for the reduced problem."

"matrixToPoly begins..."

"Computing decomposition..."

"Decomposition computed!"

$$\left[\begin{array}{c} -1.53154606192228\ 10^{-15} \\ -4.92732603348468\ 10^{-16} \\ -1.91786087875193\ 10^{-16} \\ -2.86646462579015\ 10^{-17} \\ -1.04633472507358\ 10^{-24} \\ -8.03279212322297\ 10^{-35} \\ -1.58360019936974\ 10^{-47} \\ 1.72563277638947\ 10^{-48} \\ 1.75373208200403\ 10^{-32} \\ 3.41149862314189\ 10^{-32} \\ 1.19672758059186\ 10^{-31} \\ 1.52368548694179\ 10^{-21} \\ 2.19483325284302\ 10^{-18} \\ 1.04695473377956\ 10^{-16} \\ 2.41625840744251\ 10^{-16} \\ 4.65131712447762\ 10^{-16} \\ 0.887306858117155 \\ 1. \\ 1.67794349070652 \\ 5.19617632925677 \\ 7.23857332191956 \end{array}\right] \quad \textbf{(7)}$$

> ##

Maximum rank in the spectrahedron = 6

```
# The 5 polynomials
p1 := x1^2 - x4^2;
p2 := x2^2 - x4^2;
p3 := x3^2 - x4^2;
p4 := -x1^2 - x1*x2 - x1*x3 + x1*x4 - x2*x3 + x2*x4 + x3*x4;
p5 := x5^2 - x4^2;
p6 := x6^2;
f := p1^2 + p2^2 + p3^2 + p4^2 + p5^2 + p6^2;

# Numerical solution
out := exactSOS(f, realPolynomials = [p1, p2, p3, p4, p5, p6]);
eig(out[3]);
```

$$p1 := x1^2 - x4^2$$

$$p2 := x2^2 - x4^2$$

$$p3 := x3^2 - x4^2$$

$$p4 := -x1^2 - x1\,x2 - x1\,x3 + x1\,x4 - x2\,x3 + x2\,x4 + x3\,x4$$

$$p5 := -x4^2 + x5^2$$

$$p6 := x6^2$$

$$f := \left(x1^2 - x4^2\right)^2 + \left(x2^2 - x4^2\right)^2 + \left(x3^2 - x4^2\right)^2 + \left(-x1^2 - x1\,x2 - x1\,x3 + x1\,x4 - x2\,x3 + x2\,x4 + x3\,x4\right)^2 + \left(-x4^2 + x5^2\right)^2 + x6^4$$

"Option traceEquations: yes - Only valid when looking for rational decompositions."

"No algebraic extension in this branch. Check: ", $\{x1 = 0, x2 = 0, x3 = 0, x4 = 0, x5 = 0, x6 = 0\}$

"compute random solutions..."

"indetsCFEV", { }

"trueIndets", { }

"number of solutions: ", 1

"No equations found. Check!"

"solve finished 2"

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 21, " - Number of indeterminates: ", 105

"Matrix after facial reduction - Rank: ", 21, " - Number of indeterminates: ", 105

"Check 1 of random rank:", 21

"Check 2 of random rank:", 21

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 106, order n = 22, dim = 442, blocks = 2
nnz(A) = 231 + 0, nnz(ADA) = 11236, nnz(L) = 5671
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
```

```
cg  prec
   0 :             1.84E+01 0.000
   1 :  -3.78E+00 5.58E+00 0.000 0.3034 0.9000 0.9000   0.43  1
1  1.4E+01
   2 :  -1.09E+00 1.88E+00 0.000 0.3362 0.9000 0.9000   2.73  1
1  2.2E+00
   3 :  -8.81E-02 5.33E-01 0.000 0.2836 0.9000 0.9000   2.97  1
1  5.9E-01
   4 :  -1.92E-02 1.31E-01 0.000 0.2457 0.9000 0.9000   1.23  1
1  3.7E-01
   5 :  -5.88E-03 4.08E-02 0.000 0.3118 0.9000 0.9000   1.07  1
1  3.3E-01
   6 :  -2.27E-03 1.38E-02 0.000 0.3389 0.9000 0.9000   1.03  1
1  2.8E-01
   7 :  -7.90E-04 4.25E-03 0.000 0.3071 0.9038 0.9000   1.02  1
1  2.1E-01
   8 :  -2.27E-04 1.49E-03 0.000 0.3510 0.9000 0.9141   1.01  1
1  6.1E-02
   9 :  -8.14E-05 6.13E-04 0.000 0.4115 0.9000 0.7630   1.00  1
1  2.0E-02
  10 :  -2.79E-05 1.69E-04 0.000 0.2756 0.9098 0.9000   1.00  1
1  6.4E-03
  11 :  -8.51E-06 6.13E-05 0.000 0.3630 0.9000 0.9113   1.00  1
1  1.9E-03
  12 :  -3.72E-06 2.63E-05 0.000 0.4281 0.9000 0.6263   1.00  1
1  7.1E-04
  13 :  -1.21E-06 5.40E-06 0.000 0.2058 0.9188 0.9000   1.00  1
1  2.5E-04
  14 :  -3.96E-07 1.87E-06 0.000 0.3457 0.9001 0.9000   1.00  1
1  8.6E-05
  15 :  -9.23E-08 7.76E-07 0.000 0.4153 0.9000 0.8334   1.00  1
1  2.1E-05
  16 :  -3.27E-08 2.55E-07 0.000 0.3292 0.9024 0.9000   1.00  1
1  7.1E-06
  17 :  -1.43E-08 1.07E-07 0.000 0.4174 0.9000 0.6816   1.00  1
1  2.6E-06
  18 :  -4.79E-09 2.99E-08 0.000 0.2801 0.9099 0.9000   1.00  1
1  9.2E-07
  19 :  -1.42E-09 1.15E-08 0.000 0.3856 0.9000 0.8858   1.00  1
1  2.6E-07
  20 :  -6.90E-10 4.81E-09 0.000 0.4179 0.9000 0.5762   1.00  1
1  1.1E-07
  21 :  -2.24E-10 9.81E-10 0.000 0.2038 0.9191 0.9000   1.00  1
1  4.2E-08
  22 :  -7.18E-11 3.39E-10 0.000 0.3458 0.9000 0.9005   1.00  1
1  1.4E-08
  23 :  -1.68E-11 1.42E-10 0.000 0.4176 0.9000 0.8291   1.00  1
1  3.4E-09
  24 :  -5.85E-12 4.38E-11 0.000 0.3093 0.9064 0.9000   1.00  4
4  1.2E-09
  25 :  -2.34E-12 1.82E-11 0.000 0.4152 0.9000 0.7393   1.00 16
16  4.1E-10
  26 :  -7.99E-13 5.91E-12 0.000 0.3250 0.9032 0.9000   1.00 21
16  1.4E-10
  27 :  -3.36E-13 2.42E-12 0.000 0.4091 0.9000 0.6765   1.00 20
21  5.3E-11
  28 :  -1.13E-13 6.27E-13 0.000 0.2590 0.9130 0.9000   1.00 27
```

```
31  2.0E-11
 29 :  -3.17E-14 2.35E-13 0.000 0.3743 0.9000 0.9142   1.00 27
27  5.3E-12
 30 :  -1.75E-14 1.06E-13 0.000 0.4526 0.9000 0.5668   1.00 36
28  2.3E-12
 31 :  -6.86E-15 1.35E-14 0.000 0.1270 0.9264 0.9000   1.00 31
32  9.7E-13
 32 :  -3.90E-15 4.35E-15 0.000 0.3230 0.9123 0.9000   1.00 38
38  3.6E-13
Run into numerical problems.

iter seconds digits       c*x               b*y
 32      0.5  10.5  5.2113885429e-16 -3.9015068473e-15
|Ax-b| =   7.0e-15, [Ay-c]_+ =   9.1E-15, |x|=  4.3e-01, |y|=
4.0e+00

Detailed timing (sec)
   Pre          IPM          Post
2.997E-03    1.790E-01    1.006E-03
Max-norms: ||b||=1, ||c|| = 6,
Cholesky |add|=12, |skip| = 0, ||L.L|| = 4.9571e+06.
```

"An exact positive definite solution could not be found for the reduced problem."

"matrixToPoly begins..."

"The computed matrix is not positive semidefinite (non-zero entries below a zero element in the diagonal). SOS decomposition may not exist."

$$out := \left[0, 0, \left[\left[2, 1, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 0, 0, -1, 0, 0, 0, 0, -\frac{8}{473213959} \right], \right. \right.$$

$$\left[1, 1, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, -\frac{3}{273294490}\right],$$

$$\left[1, 1, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, -\frac{2}{242311169}\right],$$

$$\left[-1, -1, -1, 1, 0, 0, 0, -1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, \frac{5}{507074234}\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{134710632}\right],$$

$$\left[0, 0, 0, 0, 0, \frac{16}{473213959}, 0, 0, 0, 0, \frac{3}{273294490}, 0, 0, 0, \frac{2}{242311169}, 0, 0, -\frac{5}{507074234}, 0, \frac{1}{134710632}, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, -\frac{2}{109442409}\right],$$

$$\left[1, 1, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, -\frac{2}{229480663}\right],$$

$$\left[-1, -1, -1, 1, 0, 0, 0, -1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, \frac{3}{267707686}\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{2}{166049583}\right],$$

$$\left[0, 0, 0, 0, 0, \frac{3}{273294490}, 0, 0, 0, 0, \frac{4}{109442409}, 0, 0, 0, \frac{2}{229480663}, 0, 0, -\frac{3}{267707686}, 0, \frac{2}{166049583}, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, 0, 0, 0, 0, -\frac{8}{499993537}\right],$$

$$\left[-1, -1, -1, 1, 0, 0, 0, -1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, \frac{3}{360948242}\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{379069411}\right],$$

$$\left[0, 0, 0, 0, 0, \frac{2}{242311169}, 0, 0, 0, 0, \frac{2}{229480663}, 0, 0, 0, \frac{16}{499993537}, 0, 0, -\frac{3}{360948242}, 0, \frac{1}{379069411}, 0\right],$$

$$\left[-1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, 4, 0, 0, -1, 0, -\frac{8}{469801495}\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{3}{378177200}\right],$$

$$\left[0, 0, 0, 0, 0, -\frac{5}{507074234}, 0, 0, 0, 0, -\frac{3}{267707686}, 0, 0, 0, -\frac{3}{360948242}, 0, 0, \frac{16}{469801495}, 0, -\frac{3}{378177200}, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 1, 0, -\frac{3}{182627984}\right],$$

$$\left[0, 0, 0, 0, 0, \frac{1}{134710632}, 0, 0, 0, 0, \frac{2}{166049583}, 0, 0, 0, \frac{1}{379069411}, 0, 0, -\frac{3}{378177200}, 0, \frac{3}{91313992}, 0\right],$$

$$\left[-\frac{8}{473213959}, -\frac{3}{273294490}, -\frac{2}{242311169}, \frac{5}{507074234}, -\frac{1}{134710632}, 0,\right.$$

$$-\frac{2}{109442409}, -\frac{2}{229480663}, \frac{3}{267707686}, -\frac{2}{166049583}, 0, -\frac{8}{499993537},$$

$$\frac{3}{360948242}, -\frac{1}{379069411}, 0, -\frac{8}{469801495}, \frac{3}{378177200}, 0, -\frac{3}{182627984}, 0, 1\Big]\Big],$$

$$\Big[\Big[2, 1, 1, -1, 0, 0, \frac{1}{2} - \frac{1}{2}\_t6_{7}, 2 - \_t6_{8}, -2 - \_t6_{9}, -\_t6_{10}, -\_t6_{11}, \frac{1}{2} - \frac{1}{2}\_t6_{24},$$

$$-2 - \_t6_{25}, -\_t6_{26}, -\_t6_{27}, \_t6_{1}, \_t6_{2}, \_t6_{3}, \_t6_{4}, \_t6_{5}, \_t6_{6}\Big],$$

$$\Big[1, \_t6_{7}, \_t6_{8}, \_t6_{9}, \_t6_{10}, \_t6_{11}, 0, 1 - \_t6_{28}, \_t6_{12}, \_t6_{13}, \_t6_{14}, 1 - \_t6_{29}, \_t6_{15}, \_t6_{16},$$

$\_t6_{17}, \_t6_{18}, \_t6_{19}, \_t6_{20}, \_t6_{21}, \_t6_{22}, \_t6_{23}\Big],$

$\big[1, \_t6_8, \_t6_{24}, \_t6_{25}, \_t6_{26}, \_t6_{27}, \_t6_{28}, \_t6_{29}, -3 - \_t6_{39} - \_t6_{15}, \_t6_{30}, \_t6_{31}, 0, -1 - \_t6_{42}, \_t6_{32}, \_t6_{33}, 1 - \_t6_{43}, \_t6_{34}, \_t6_{35}, \_t6_{36}, \_t6_{37}, \_t6_{38}\big],$

$\Big[-1, \_t6_9, \_t6_{25}, -1 - 2\ \_t6_1, -\_t6_2, -\_t6_3, -1 - \_t6_{12}, \_t6_{39}, 1 - \_t6_{18}, \_t6_{40}, \_t6_{41}, \_t6_{42}, \_t6_{43}, \_t6_{44}, \_t6_{45}, 0, -\_t6_{51}, \_t6_{46}, -\_t6_{52}, \_t6_{47}, \_t6_{48}\Big],$

$\Big[0, \_t6_{10}, \_t6_{26}, -\_t6_2, -2\ \_t6_4, -\_t6_5, -\_t6_{13}, -\_t6_{16} - \_t6_{30}, -\_t6_{40} - \_t6_{19}, -\_t6_{21}, \_t6_{49}, -\_t6_{32}, -\_t6_{34} - \_t6_{44}, -\_t6_{36}, \_t6_{50}, \_t6_{51}, \_t6_{52}, \_t6_{53}, 0, \_t6_{54}, \_t6_{55}\Big],$

$\big[0, \_t6_{11}, \_t6_{27}, -\_t6_3, -\_t6_5, -2\ \_t6_6, -\_t6_{14}, -\_t6_{17} - \_t6_{31}, -\_t6_{41} - \_t6_{20}, -\_t6_{49} - \_t6_{22}, -\_t6_{23}, -\_t6_{33}, -\_t6_{35} - \_t6_{45}, -\_t6_{50} - \_t6_{37}, -\_t6_{38}, -\_t6_{46}, -\_t6_{47} - \_t6_{53}, -\_t6_{48}, -\_t6_{54}, -\_t6_{55}, 0\big],$

$\Big[\frac{1}{2} - \frac{1}{2}\ \_t6_7, 0, \_t6_{28}, -1 - \_t6_{12}, -\_t6_{13}, -\_t6_{14}, 1, 0, 0, 0, 0, \frac{1}{2} - \frac{1}{2}\ \_t6_{57}, -1 - \_t6_{58}, -\_t6_{59}, -\_t6_{60}, -\frac{1}{2} - \frac{1}{2}\ \_t6_{67}, -\_t6_{68}, -\_t6_{69}, -\frac{1}{2}\ \_t6_{79}, -\_t6_{80}, \_t6_{56}\Big],$

$\Big[2 - \_t6_8, 1 - \_t6_{28}, \_t6_{29}, \_t6_{39}, -\_t6_{16} - \_t6_{30}, -\_t6_{17} - \_t6_{31}, 0, \_t6_{57}, \_t6_{58}, \_t6_{59}, \_t6_{60}, 0, -1 - \_t6_{70}, -\_t6_{81}, \_t6_{61}, 1 - \_t6_{71}, \_t6_{62}, \_t6_{63}, \_t6_{64}, \_t6_{65}, \_t6_{66}\Big],$

$\Big[-2 - \_t6_9, \_t6_{12}, -3 - \_t6_{39} - \_t6_{15}, 1 - \_t6_{18}, -\_t6_{40} - \_t6_{19}, -\_t6_{41} - \_t6_{20}, 0, \_t6_{58}, \_t6_{67}, \_t6_{68}, \_t6_{69}, \_t6_{70}, \_t6_{71}, \_t6_{72}, \_t6_{73}, 0, \_t6_{74}, \_t6_{75}, \_t6_{76}, \_t6_{77}, \_t6_{78}\Big],$

$\big[-\_t6_{10}, \_t6_{13}, \_t6_{30}, \_t6_{40}, -\_t6_{21}, -\_t6_{49} - \_t6_{22}, 0, \_t6_{59}, \_t6_{68}, \_t6_{79}, \_t6_{80}, \_t6_{81}, -\_t6_{62} - \_t6_{72}, -\_t6_{64}, -\_t6_{82} - \_t6_{65}, -\_t6_{74}, -\_t6_{76}, -\_t6_{83} - \_t6_{77}, 0, -\_t6_{84}, -\_t6_{85}\big],$

$\big[-\_t6_{11}, \_t6_{14}, \_t6_{31}, \_t6_{41}, \_t6_{49}, -\_t6_{23}, 0, \_t6_{60}, \_t6_{69}, \_t6_{80}, -2\ \_t6_{56}, -\_t6_{61}, -\_t6_{63} - \_t6_{73}, \_t6_{82}, -\_t6_{66}, -\_t6_{75}, \_t6_{83}, -\_t6_{78}, \_t6_{84}, \_t6_{85}, 0\big],$

$\Big[\frac{1}{2} - \frac{1}{2}\ \_t6_{24}, 1 - \_t6_{29}, 0, \_t6_{42}, -\_t6_{32}, -\_t6_{33}, \frac{1}{2} - \frac{1}{2}\ \_t6_{57}, 0, \_t6_{70}, \_t6_{81}, -\_t6_{61}, 1, 0, 0, 0, -\frac{1}{2} - \frac{1}{2}\ \_t6_{89}, -\_t6_{90}, -\_t6_{91}, \_t6_{86}, \_t6_{87}, \_t6_{88}\Big],$

$\big[-2 - \_t6_{25}, \_t6_{15}, -1 - \_t6_{42}, \_t6_{43}, -\_t6_{34} - \_t6_{44}, -\_t6_{35} - \_t6_{45}, -1 - \_t6_{58}, -1 - \_t6_{70}, \_t6_{71}, -\_t6_{62} - \_t6_{72}, -\_t6_{63} - \_t6_{73}, 0, \_t6_{89}, \_t6_{90}, \_t6_{91}, 0, -\_t6_{92}, -\_t6_{95}, -\_t6_{93}, -\_t6_{94} - \_t6_{96}, -\_t6_{97}\big],$

$\Big[-\_t6_{26}, \_t6_{16}, \_t6_{32}, \_t6_{44}, -\_t6_{36}, -\_t6_{50} - \_t6_{37}, -\_t6_{59}, -\_t6_{81}, \_t6_{72}, -\_t6_{64}, \_t6_{82}, 0,$

$\_t6_{90}, -2\ \_t6_{86}, -\_t6_{87}, \_t6_{92}, \_t6_{93}, \_t6_{94}, 0, -\_t6_{98}, -\_t6_{99}\Big],$

$\Big[-\_t6_{27}, \_t6_{17}, \_t6_{33}, \_t6_{45}, \_t6_{50}, -\_t6_{38}, -\_t6_{60}, \_t6_{61}, \_t6_{73}, -\_t6_{82} - \_t6_{65}, -\_t6_{66}, 0,$

$\_t6_{91}, -\_t6_{87}, -2\ \_t6_{88}, \_t6_{95}, \_t6_{96}, \_t6_{97}, \_t6_{98}, \_t6_{99}, 0\Big],$

$\Big[\_t6_{1}, \_t6_{18}, 1 - \_t6_{43}, 0, \_t6_{51}, -\_t6_{46}, -\frac{1}{2} - \frac{1}{2}\ \_t6_{67}, 1 - \_t6_{71}, 0, -\_t6_{74}, -\_t6_{75}, -\frac{1}{2}$

$- \frac{1}{2}\ \_t6_{89}, 0, \_t6_{92}, \_t6_{95}, 4, 0, 0, -1 - \frac{1}{2}\ \_t6_{101}, -\_t6_{102}, \_t6_{100}\Big],$

$[\_t6_{2}, \_t6_{19}, \_t6_{34}, -\_t6_{51}, \_t6_{52}, -\_t6_{47} - \_t6_{53}, -\_t6_{68}, \_t6_{62}, \_t6_{74}, -\_t6_{76}, \_t6_{83}, -\_t6_{90},$
$-\_t6_{92}, \_t6_{93}, \_t6_{96}, 0, \_t6_{101}, \_t6_{102}, 0, -\_t6_{104}, \_t6_{103}],$

$[\_t6_{3}, \_t6_{20}, \_t6_{35}, \_t6_{46}, \_t6_{53}, -\_t6_{48}, -\_t6_{69}, \_t6_{63}, \_t6_{75}, -\_t6_{83} - \_t6_{77}, -\_t6_{78}, -\_t6_{91},$
$-\_t6_{95}, \_t6_{94}, \_t6_{97}, 0, \_t6_{102}, -2\ \_t6_{100}, \_t6_{104}, -\_t6_{103}, 0],$

$\Big[\_t6_{4}, \_t6_{21}, \_t6_{36}, -\_t6_{52}, 0, -\_t6_{54}, -\frac{1}{2}\ \_t6_{79}, \_t6_{64}, \_t6_{76}, 0, \_t6_{84}, \_t6_{86}, -\_t6_{93}, 0,$

$\_t6_{98}, -1 - \frac{1}{2}\ \_t6_{101}, 0, \_t6_{104}, 1, 0, -\frac{1}{2}\ \_t6_{105}\Big],$

$[\_t6_{5}, \_t6_{22}, \_t6_{37}, \_t6_{47}, \_t6_{54}, -\_t6_{55}, -\_t6_{80}, \_t6_{65}, \_t6_{77}, -\_t6_{84}, \_t6_{85}, \_t6_{87}, -\_t6_{94}$
$- \_t6_{96}, -\_t6_{98}, \_t6_{99}, -\_t6_{102}, -\_t6_{104}, -\_t6_{103}, 0, \_t6_{105}, 0],$

$\left[\_t6_{6}, \_t6_{23}, \_t6_{38}, \_t6_{48}, \_t6_{55}, 0, \_t6_{56}, \_t6_{66}, \_t6_{78}, -\_t6_{85}, 0, \_t6_{88}, -\_t6_{97}, -\_t6_{99}, 0,\right.$

$$\_t6_{100}, \_t6_{103}, 0, -\frac{1}{2}\ \_t6_{105}, 0, 1\Big]\Big], \left[\left[\begin{array}{c} x1^2 \\ x1\ x2 \\ x1\ x3 \\ x1\ x4 \\ x1\ x5 \\ x1\ x6 \\ x2^2 \\ x2\ x3 \\ x2\ x4 \\ x2\ x5 \\ x2\ x6 \\ x3^2 \\ x3\ x4 \\ x3\ x5 \\ x3\ x6 \\ x4^2 \\ x4\ x5 \\ x4\ x6 \\ x5^2 \\ x5\ x6 \\ x6^2 \end{array}\right]\right]\right] \tag{8}$$

$$\begin{bmatrix} -1.50293612020083\ 10^{-15} \\ -9.45353174874660\ 10^{-17} \\ -5.70454401781955\ 10^{-17} \\ -2.07522872009834\ 10^{-17} \\ -7.49723986165755\ 10^{-18} \\ 5.84647098516838\ 10^{-17} \\ 7.04706904308364\ 10^{-17} \\ 1.24885567739551\ 10^{-16} \\ 1.57054232299321\ 10^{-16} \\ 2.39000850834945\ 10^{-16} \\ 2.08717717817661\ 10^{-8} \\ 2.39879440453773\ 10^{-8} \\ 2.40838625957560\ 10^{-8} \\ 3.05238243139267\ 10^{-8} \\ 6.98038682541110\ 10^{-8} \\ 0.876300380842109 \\ 0.999999998290433 \\ 0.999999999999999 \\ 1.00000000170957 \\ 4.89136883691186 \\ 7.23233078224603 \end{bmatrix} \quad (8)$$

```
> ###############################################################################

# Maximum rank in the spectrahedron = 11

# The 7 polynomials
p1 := x1^2 - x4^2;
p2 := x2^2 - x4^2;
p3 := x3^2 - x4^2;
p4 := -x1^2 - x1 * x2 - x1 * x3 + x1 * x4 - x2 * x3 + x2 * x4 + x3 * x4;
p5 := x5^2;
p6 := x6^2;
p7 := x5 * x6 + x1 * x5;
f := p1^2 + p2^2 + p3^2 + p4^2 + p5^2 + p6^2 + p7^2;

# Numerical solution of rank 5, there seems to be unique solution.
```

```
out := exactSOS(f, realPolynomials = [p1, p2, p3, p4, p5, p6, p7],
      computePolynomialDecomposition = "no");
eig(out[3]);
```

$$p1 := x1^2 - x4^2$$

$$p2 := x2^2 - x4^2$$

$$p3 := x3^2 - x4^2$$

$$p4 := -x1^2 - x1\,x2 - x1\,x3 + x1\,x4 - x2\,x3 + x2\,x4 + x3\,x4$$

$$p5 := x5^2$$

$$p6 := x6^2$$

$$p7 := x1\,x5 + x5\,x6$$

$$f := \left(x1^2 - x4^2\right)^2 + \left(x2^2 - x4^2\right)^2 + \left(x3^2 - x4^2\right)^2 + \left(-x1^2 - x1\,x2 - x1\,x3 + x1\,x4 - x2\,x3 + x2\,x4 + x3\,x4\right)^2 + x5^4 + x6^4 + \left(x1\,x5 + x5\,x6\right)^2$$

"Option traceEquations: yes - Only valid when looking for rational decompositions."

"No algebraic extension in this branch. Check: ", $\{x1 = 0, x2 = 0, x3 = 0, x4 = 0, x5 = 0, x6 = 0\}$

"compute random solutions..."

"indetsCFEV", { }

"trueIndets", { }

"number of solutions: ", 1

"No equations found. Check!"

"solve finished 2"

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 21, " - Number of indeterminates: ", 105

"Matrix after facial reduction - Rank: ", 21, " - Number of indeterminates: ", 105

"Check 1 of random rank:", 21

"Check 2 of random rank:", 21

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 106, order n = 22, dim = 442, blocks = 2
nnz(A) = 231 + 0, nnz(ADA) = 11236, nnz(L) = 5671
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            1.94E+01 0.000
  1 :  -3.49E+00 6.19E+00 0.000 0.3193 0.9000 0.9000   0.54  1
1  1.4E+01
  2 :  -9.75E-01 2.05E+00 0.000 0.3310 0.9000 0.9000   2.70  1
1  2.3E+00
  3 :  -1.02E-01 6.64E-01 0.000 0.3241 0.9000 0.9000   2.78  1
1  7.2E-01
  4 :  -2.18E-02 1.82E-01 0.000 0.2746 0.9000 0.9000   1.28  1
1  4.0E-01
```

```
  5 :  -5.92E-03 6.04E-02 0.000 0.3311 0.9000 0.9000   1.08  1
1  3.1E-01
  6 :  -2.17E-03 1.94E-02 0.000 0.3217 0.9000 0.9000   1.02  1
1  3.1E-01
  7 :  -6.09E-04 4.74E-03 0.000 0.2439 0.9077 0.9000   1.00  1
1  2.8E-01
  8 :  -1.33E-04 1.57E-03 0.000 0.3302 0.9000 0.9151   1.00  1
1  8.3E-02
  9 :  -6.10E-05 6.18E-04 0.000 0.3947 0.9000 0.7283   1.00  1
1  3.0E-02
 10 :  -1.74E-05 1.45E-04 0.000 0.2339 0.9112 0.9000   1.00  1
1  8.0E-03
 11 :  -5.13E-06 4.93E-05 0.000 0.3415 0.9000 0.9094   1.00  1
1  2.5E-03
 12 :  -1.29E-06 1.99E-05 0.000 0.4029 0.9000 0.9043   1.00  1
1  7.6E-04
 13 :  -4.49E-07 5.86E-06 0.000 0.2945 0.9012 0.9000   0.99  1
2  2.2E-04
 14 :  -1.29E-07 2.16E-06 0.000 0.3689 0.9000 0.9254   1.00  1
3  6.2E-05
 15 :  -4.21E-08 8.65E-07 0.000 0.4006 0.9000 0.7514   1.01  1
3  1.7E-05
 16 :  -1.23E-08 2.02E-07 0.000 0.2334 0.9070 0.9000   1.01  1
3  4.4E-06
 17 :  -3.86E-09 6.19E-08 0.000 0.3063 0.9000 0.9025   1.00  1
3  1.2E-06
 18 :  -1.75E-09 2.19E-08 0.000 0.3546 0.9000 0.6416   1.00  1
3  4.4E-07
 19 :  -5.24E-10 3.83E-09 0.000 0.1746 0.9180 0.9000   0.99  1
4  1.3E-07
 20 :  -1.69E-10 1.19E-09 0.000 0.3115 0.9060 0.9000   0.99  1
4  4.5E-08
 21 :  -2.24E-11 4.70E-10 0.000 0.3940 0.9000 0.9244   1.00  1
4  9.4E-09
 22 :  -1.70E-11 2.17E-10 0.000 0.4622 0.9000 0.2696   1.00  2
5  4.3E-09
 23 :  -4.77E-12 2.30E-11 0.000 0.1057 0.9248 0.9000   1.00  3
8  1.2E-09
 24 :  -1.47E-12 7.04E-12 0.000 0.3066 0.9067 0.9000   1.00  5
15  4.0E-10
 25 :  -3.86E-13 2.61E-12 0.000 0.3708 0.9000 0.9159   1.00 11
22  1.3E-10
 26 :  -6.65E-14 1.08E-12 0.000 0.4155 0.9000 0.9407   1.00 20
38  3.3E-11
 27 :  -2.55E-14 3.81E-13 0.000 0.3513 0.9000 0.9107   1.00 31
71  1.1E-11
 28 :  -7.36E-15 1.53E-13 0.000 0.4024 0.9000 0.9334   1.01 44
75  3.0E-12
Run into numerical problems.

iter seconds digits       c*x               b*y
 28      0.5  10.1  4.9675488643e-15 -7.3570867760e-15
|Ax-b| =   5.3e-14, [Ay-c]_+ =   5.4E-15, |x|=  4.9e-01, |y|=
4.7e+00

Detailed timing (sec)
   Pre          IPM          Post
```

```
2.997E-03    1.700E-01    1.006E-03
Max-norms: ||b||=1, ||c|| = 6,
Cholesky |add|=16, |skip| = 3, ||L.L|| = 1.18417e+07.
```

"An exact positive definite solution could not be found for the reduced problem."

$$out := \left[0, 0, \left[\left[2, 1, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 0, 0, -1, 0, 0, \frac{29427}{148550}, -\frac{2}{491218221}, -\frac{7}{128912663}\right],\right.\right.$$

$$\left[1, 1, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, -\frac{5871}{256852}, -\frac{790715777}{534206042086151937}, -\frac{8}{293789551}\right],$$

$$\left[1, 1, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, -\frac{5871}{256852}, -\frac{1}{675604913}, -\frac{4}{146894775}\right],$$

$$\left[-1, -1, -1, 1, 0, 0, 0, -1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, \frac{5871}{256852}, \frac{1}{675600887}, \frac{3}{110174990}\right],$$

$$\left[0, 0, 0, 0, \frac{44848}{74275}, \frac{2}{491218221}, 0, 0, 0, \frac{5871}{256852}, \frac{2}{711312769}, 0, 0, \frac{5871}{256852},\right.$$
$$\left.-\frac{1}{1845798300}, 0, -\frac{5871}{256852}, -\frac{1}{853996198}, 0, \frac{1222606}{1222605}, -\frac{1}{168860333}\right],$$
$$\left[0, 0, 0, 0, \frac{2}{491218221}, \frac{14}{128912663}, 0, 0, 0, -\frac{1}{751014273}, \frac{8}{293789551}, 0, 0,\right.$$

$\frac{2521403213}{1247030399887047900}, \frac{4}{146894775}, 0, -\frac{178395311}{5769605888863427626}, -\frac{3}{110174990},$
$-\frac{1}{1222605}, \frac{1}{168860333}, 0\Big],$

$\Big[0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, -\frac{4921}{65838}, \frac{1}{1309081939}, -\frac{17}{312145498}\Big],$

$\Big[1, 1, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 0, 0, 0, 0, 0, -\frac{11052}{483517}, -\frac{1}{635579553},$

$-\frac{5}{183340958}\Big]$,

$\Big[-1, -1, -1, 1, 0, 0, 0, -1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, \frac{1727}{75555}, \frac{1}{675598566}, \frac{4}{146663583}\Big]$,

$\Big[0, 0, 0, 0, \frac{5871}{256852}, -\frac{1}{751014273}, 0, 0, 0, \frac{4921}{32919}, -\frac{1}{1309081939}, 0, 0, \frac{11052}{483517}, 0, 0,$
$-\frac{1727}{75555}, -\frac{1}{3746301953}, 0, \frac{5}{275441102}, -\frac{2}{447842051}\Big]$,

$$\left[0, 0, 0, 0, \frac{2}{711312769}, \frac{8}{293789551}, 0, 0, 0, -\frac{1}{1309081939}, \frac{17}{156072749}, 0, 0, \frac{1}{635579553}, \frac{5}{183340958}, 0, -\frac{3070703387}{2530996227249799398}, -\frac{4}{146663583}, -\frac{5}{275441102}, \frac{2}{447842051}, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -1, 0, 0, -\frac{4402}{59021}, \frac{2}{710835271}, -\frac{8}{146891987}\right],$$

$$\left[-1, -1, -1, 1, 0, 0, 0, -1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, \frac{1727}{75555}, \frac{2705223077}{1827655881491887800}, \frac{10}{366658941}\right],$$

$$\left[0, 0, 0, 0, \frac{5871}{256852}, \frac{2521403213}{1247030399887047900}, 0, 0, 0, \frac{11052}{483517}, \frac{1}{635579553}, 0, 0, \frac{8804}{59021}, -\frac{2}{710835271}, 0, -\frac{1727}{75555}, -\frac{1}{1396224525}, 0, \frac{13}{854995584}, \frac{2}{337729935}\right],$$

$$\left[0, 0, 0, 0, -\frac{1}{1845798300}, \frac{4}{146894775}, 0, 0, 0, 0, \frac{5}{183340958}, 0, 0, -\frac{2}{710835271}, \frac{16}{146891987}, 0, -\frac{1}{1308998552}, -\frac{10}{366658941}, -\frac{13}{854995584}, -\frac{2}{337729935}, 0\right],$$

$$\left[-1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, -1, 0, 0, 0, 3, 0, 0, -\frac{9368}{130793}, -\frac{1}{1014054312}, -\frac{11}{202025570}\right],$$

$$\left[0, 0, 0, 0, -\frac{5871}{256852}, -\frac{178395311}{576960588863427626}, 0, 0, 0, -\frac{1727}{75555}, -\frac{3070703387}{2530996227249799398}, 0, 0, -\frac{1727}{75555}, -\frac{1}{1308998552}, 0, \frac{18736}{130793}, \frac{1}{1014054312}, 0, -\frac{2}{93177471}, \frac{1}{988892198}\right],$$

$$\left[0, 0, 0, 0, -\frac{1}{853996198}, -\frac{3}{110174990}, 0, 0, 0, -\frac{1}{3746301953}, -\frac{4}{146663583}, 0, 0, -\frac{1}{1396224525}, -\frac{10}{366658941}, 0, \frac{1}{1014054312}, \frac{11}{101012785}, \frac{2}{93177471}, -\frac{1}{988892198}, 0\right],$$

$$\left[\frac{29427}{148550}, -\frac{5871}{256852}, -\frac{5871}{256852}, \frac{5871}{256852}, 0, -\frac{1}{1222605}, -\frac{4921}{65838}, -\frac{11052}{483517}, \frac{1727}{75555}, 0, -\frac{5}{275441102}, -\frac{4402}{59021}, \frac{1727}{75555}, 0, -\frac{13}{854995584}, -\frac{9368}{130793}, 0, \frac{2}{93177471}, 1, 0, -\frac{40759}{65996}\right],$$

$$\left[-\frac{2}{491218221}, -\frac{790715777}{534206042086151937}, -\frac{1}{675604913}, \frac{1}{675600887}, \frac{1222606}{1222605}, \frac{1}{168860333}, \frac{1}{1309081939}, -\frac{1}{635579553}, \frac{1}{675598566}, \frac{5}{275441102}, \frac{2}{447842051}, \frac{2}{710835271}, \frac{2705223077}{1827655881491887800}, \frac{13}{854995584}, -\frac{2}{337729935}, -\frac{1}{1014054312}, -\frac{2}{93177471}, -\frac{1}{988892198}, 0, \frac{73757}{32998}, 0\right],$$

$$\left[-\frac{7}{128912663}, -\frac{8}{293789551}, -\frac{4}{146894775}, \frac{3}{110174990}, -\frac{1}{168860333}, 0, \right.$$

$-\frac{17}{312145498}, -\frac{5}{183340958}, \frac{4}{146663583}, -\frac{2}{447842051}, 0, -\frac{8}{146891987},$

$\frac{10}{366658941}, \frac{2}{337729935}, 0, -\frac{11}{202025570}, \frac{1}{988892198}, 0, -\frac{40759}{65996}, 0, 1\Big]\Big], \Big[$

$\Big[2, 1, 1, -1, 0, 0, \frac{1}{2} - \frac{1}{2}\ \_t14_{11}, 2 - \_t14_{12}, -2 - \_t14_{13}, -\_t14_{14}, -\_t14_{15}, \_t14_{1}, \_t14_{2},$

$\_t14_{3}, \_t14_{4}, \_t14_{5}, \_t14_{6}, \_t14_{7}, \_t14_{8}, \_t14_{9}, \_t14_{10}\Big],$

$$\left[1, \_t14_{11}, \_t14_{12}, \_t14_{13}, \_t14_{14}, \_t14_{15}, 0, \_t14_{16}, \_t14_{17}, \_t14_{18}, \_t14_{19}, \_t14_{20}, \_t14_{21}, \_t14_{22}, -\_t14_{23} - \_t14_{51}, 1 - \_t14_{28}, -\_t14_{29} - \_t14_{39}, -\_t14_{30} - \_t14_{52}, -\_t14_{40}, -\_t14_{41} - \_t14_{53}, -\_t14_{54}\right],$$

$$\left[1, \_t14_{12}, 1 - 2\ \_t14_{1}, -2 - \_t14_{2}, -\_t14_{3}, -\_t14_{4}, 1 - \_t14_{16}, 1 - \_t14_{20}, -3 - \_t14_{27} - \_t14_{21}, -\_t14_{38} - \_t14_{22}, \_t14_{23}, 0, -1 - \_t14_{31}, -\_t14_{42}, -\_t14_{55}, 1 - \_t14_{32}, -\_t14_{33} - \_t14_{43}, \_t14_{24}, -\_t14_{44}, \_t14_{25}, \_t14_{26}\right],$$

$$\left[-1, \_t14_{13}, -2 - \_t14_{2}, -1 - 2\ \_t14_{5}, -\_t14_{6}, -\_t14_{7}, -1 - \_t14_{17}, \_t14_{27}, \_t14_{28}, \_t14_{29}, \_t14_{30}, \_t14_{31}, \_t14_{32}, \_t14_{33}, \_t14_{34}, 0, -\_t14_{46}, \_t14_{35}, -\_t14_{47}, \_t14_{36}, \_t14_{37}\right],$$

$$\left[0, \_t14_{14}, -\_t14_{3}, -\_t14_{6}, 1 - 2\ \_t14_{8}, -\_t14_{9}, -\_t14_{18}, \_t14_{38}, \_t14_{39}, \_t14_{40}, \_t14_{41}, \_t14_{42}, \_t14_{43}, \_t14_{44}, \_t14_{45}, \_t14_{46}, \_t14_{47}, \_t14_{48}, 0, \_t14_{49}, \_t14_{50}\right],$$

$$\left[0, \_t14_{15}, -\_t14_{4}, -\_t14_{7}, -\_t14_{9}, -2\ \_t14_{10}, -\_t14_{19}, \_t14_{51}, \_t14_{52}, \_t14_{53}, \_t14_{54}, \_t14_{55}, -\_t14_{24} - \_t14_{34}, -\_t14_{45} - \_t14_{25}, -\_t14_{26}, -\_t14_{35}, -\_t14_{36} - \_t14_{48}, -\_t14_{37}, 1 - \_t14_{49}, -\_t14_{50}, 0\right],$$

$$\left[\frac{1}{2} - \frac{1}{2}\ \_t14_{11}, 0, 1 - \_t14_{16}, -1 - \_t14_{17}, -\_t14_{18}, -\_t14_{19}, 1, 0, 0, 0, 0, \frac{1}{2} - \frac{1}{2}\ \_t14_{56}, -1 - \_t14_{57}, -\_t14_{58}, -\_t14_{59}, -\frac{1}{2} - \frac{1}{2}\ \_t14_{60}, -\_t14_{61}, -\_t14_{62}, -\frac{1}{2}\ \_t14_{70}, -\_t14_{71}, -\frac{1}{2}\ \_t14_{81}\right],$$

$$\left[2 - \_t14_{12}, \_t14_{16}, 1 - \_t14_{20}, \_t14_{27}, \_t14_{38}, \_t14_{51}, 0, \_t14_{56}, \_t14_{57}, \_t14_{58}, \_t14_{59}, 0, -1 - \_t14_{63}, -\_t14_{72}, -\_t14_{82}, 1 - \_t14_{64}, -\_t14_{65} - \_t14_{73}, -\_t14_{66} - \_t14_{83}, -\_t14_{74}, -\_t14_{75} - \_t14_{84}, -\_t14_{85}\right],$$

$$\left[-2 - \_t14_{13}, \_t14_{17}, -3 - \_t14_{27} - \_t14_{21}, \_t14_{28}, \_t14_{39}, \_t14_{52}, 0, \_t14_{57}, \_t14_{60}, \_t14_{61}, \_t14_{62}, \_t14_{63}, \_t14_{64}, \_t14_{65}, \_t14_{66}, 0, -\_t14_{76}, \_t14_{67}, -\_t14_{77}, \_t14_{68}, \_t14_{69}\right],$$

$$\left[-\_t14_{14}, \_t14_{18}, -\_t14_{38} - \_t14_{22}, \_t14_{29}, \_t14_{40}, \_t14_{53}, 0, \_t14_{58}, \_t14_{61}, \_t14_{70}, \_t14_{71}, \_t14_{72}, \_t14_{73}, \_t14_{74}, \_t14_{75}, \_t14_{76}, \_t14_{77}, \_t14_{78}, 0, \_t14_{79}, \_t14_{80}\right],$$

$$\left[-\_t14_{15}, \_t14_{19}, \_t14_{23}, \_t14_{30}, \_t14_{41}, \_t14_{54}, 0, \_t14_{59}, \_t14_{62}, \_t14_{71}, \_t14_{81}, \_t14_{82}, \_t14_{83}, \_t14_{84}, \_t14_{85}, -\_t14_{67}, -\_t14_{68} - \_t14_{78}, -\_t14_{69}, -\_t14_{79}, -\_t14_{80}, 0\right],$$

$$\left[\_t14_{1}, \_t14_{20}, 0, \_t14_{31}, \_t14_{42}, \_t14_{55}, \frac{1}{2} - \frac{1}{2}\_t14_{56}, 0, \_t14_{63}, \_t14_{72}, \_t14_{82}, 1, 0, 0, 0, -\frac{1}{2} - \frac{1}{2}\_t14_{86}, -\_t14_{87}, -\_t14_{88}, -\frac{1}{2}\_t14_{89}, -\_t14_{90}, -\frac{1}{2}\_t14_{94}\right],$$

$$\left[\_t14_{2}, \_t14_{21}, -1 - \_t14_{31}, \_t14_{32}, \_t14_{43}, -\_t14_{24} - \_t14_{34}, -1 - \_t14_{57}, -1 - \_t14_{63}, \_t14_{64}, \_t14_{73}, \_t14_{83}, 0, \_t14_{86}, \_t14_{87}, \_t14_{88}, 0, -\_t14_{91}, -\_t14_{95}, -\_t14_{92}, -\_t14_{93} - \_t14_{96}, -\_t14_{97}\right],$$

$$\left[\_t14_{3}, \_t14_{22}, -\_t14_{42}, \_t14_{33}, \_t14_{44}, -\_t14_{45} - \_t14_{25}, -\_t14_{58}, -\_t14_{72}, \_t14_{65}, \_t14_{74}, \_t14_{84}, 0, \_t14_{87}, \_t14_{89}, \_t14_{90}, \_t14_{91}, \_t14_{92}, \_t14_{93}, 0, -\_t14_{98}, -\_t14_{99}\right],$$

$$\left[\_t14_{4}, -\_t14_{23} - \_t14_{51}, -\_t14_{55}, \_t14_{34}, \_t14_{45}, -\_t14_{26}, -\_t14_{59}, -\_t14_{82}, \_t14_{66}, \_t14_{75}, \_t14_{85}, 0, \_t14_{88}, \_t14_{90}, \_t14_{94}, \_t14_{95}, \_t14_{96}, \_t14_{97}, \_t14_{98}, \_t14_{99}, 0\right],$$

$$\left[\_t14_{5}, 1 - \_t14_{28}, 1 - \_t14_{32}, 0, \_t14_{46}, -\_t14_{35}, -\frac{1}{2} - \frac{1}{2}\_t14_{60}, 1 - \_t14_{64}, 0, \_t14_{76}, -\_t14_{67}, -\frac{1}{2} - \frac{1}{2}\_t14_{86}, 0, \_t14_{91}, \_t14_{95}, 3, 0, 0, -\frac{1}{2}\_t14_{100}, -\_t14_{101}, -\frac{1}{2}\_t14_{102}\right],$$

$$\left[\_t14_{6}, -\_t14_{29} - \_t14_{39}, -\_t14_{33} - \_t14_{43}, -\_t14_{46}, \_t14_{47}, -\_t14_{36} - \_t14_{48}, -\_t14_{61}, -\_t14_{65} - \_t14_{73}, -\_t14_{76}, \_t14_{77}, -\_t14_{68} - \_t14_{78}, -\_t14_{87}, -\_t14_{91}, \_t14_{92}, \_t14_{96}, 0, \_t14_{100}, \_t14_{101}, 0, -\_t14_{103}, -\_t14_{104}\right],$$

$$\left[\_t14_{7}, -\_t14_{30} - \_t14_{52}, \_t14_{24}, \_t14_{35}, \_t14_{48}, -\_t14_{37}, -\_t14_{62}, -\_t14_{66} - \_t14_{83}, \_t14_{67}, \_t14_{78}, -\_t14_{69}, -\_t14_{88}, -\_t14_{95}, \_t14_{93}, \_t14_{97}, 0, \_t14_{101}, \_t14_{102}, \_t14_{103}, \_t14_{104}, 0\right],$$

$$\left[\_t14_{8}, -\_t14_{40}, -\_t14_{44}, -\_t14_{47}, 0, 1 - \_t14_{49}, -\frac{1}{2}\_t14_{70}, -\_t14_{74}, -\_t14_{77}, 0, -\_t14_{79}, -\frac{1}{2}\_t14_{89}, -\_t14_{92}, 0, \_t14_{98}, -\frac{1}{2}\_t14_{100}, 0, \_t14_{103}, 1, 0, \frac{1}{2} - \frac{1}{2}\_t14_{105}\right],$$

$$\left[\_t14_{9}, -\_t14_{41} - \_t14_{53}, \_t14_{25}, \_t14_{36}, \_t14_{49}, -\_t14_{50}, -\_t14_{71}, -\_t14_{75} - \_t14_{84}, \_t14_{68}, \_t14_{79}, -\_t14_{80}, -\_t14_{90}, -\_t14_{93} - \_t14_{96}, -\_t14_{98}, \_t14_{99}, -\_t14_{101}, -\_t14_{103}, \_t14_{104}, 0, \_t14_{105}, 0\right],$$

$$\left[\_t14_{10}, -\_t14_{54}, \_t14_{26}, \_t14_{37}, \_t14_{50}, 0, -\frac{1}{2}\,\_t14_{81}, -\_t14_{85}, \_t14_{69}, \_t14_{80}, 0,\right.$$

$$-\frac{1}{2}\_t14_{94}, -\_t14_{97}, -\_t14_{99}, 0, -\frac{1}{2}\_t14_{102}, -\_t14_{104}, 0, \frac{1}{2}-\frac{1}{2}\_t14_{105}, 0, 1\Big]\Big],$$

$$\left.\begin{bmatrix} x1^{2} \\ x1\,x2 \\ x1\,x3 \\ x1\,x4 \\ x1\,x5 \\ x1\,x6 \\ x2^{2} \\ x2\,x3 \\ x2\,x4 \\ x2\,x5 \\ x2\,x6 \\ x3^{2} \\ x3\,x4 \\ x3\,x5 \\ x3\,x6 \\ x4^{2} \\ x4\,x5 \\ x4\,x6 \\ x5^{2} \\ x5\,x6 \\ x6^{2} \end{bmatrix}\right] \tag{9}$$

$$\begin{bmatrix} -8.86803987920611\ 10^{-15} \\ -2.86061226323615\ 10^{-16} \\ -3.28775760309768\ 10^{-17} \\ -1.50231504747582\ 10^{-17} \\ 5.36519489091238\ 10^{-17} \\ 3.55995698749453\ 10^{-16} \\ 8.14691679052609\ 10^{-8} \\ 8.16322162519331\ 10^{-8} \\ 8.16519748777477\ 10^{-8} \\ 1.90590339731452\ 10^{-7} \\ 0.112862852768371 \\ 0.122481114773471 \\ 0.126472375795705 \\ 0.208989865455605 \\ 0.332561376999138 \\ 0.902708249653447 \\ 1.00000000000000 \\ 1.65185100333032 \\ 2.71010466770685 \\ 3.90144248266687 \\ 7.21143688735144 \end{bmatrix} \quad (9)$$

```
> ##########################################################################

# Maximum rank in the spectrahedron = 15

# The 8 polynomials
p1 := x1^2 - x4^2;
p2 := x2^2 - x4^2;
p3 := x3^2 - x4^2;
p4 := -x1^2 - x1*x2 - x1*x3 + x1*x4 - x2*x3 + x2*x4 + x3*x4;
p5 := x5^2;
p6 := x6^2;
p7 := x5*x6 + x1*x5;
p8 := x2*x6;
f := p1^2 + p2^2 + p3^2 + p4^2 + p5^2 + p6^2 + p7^2 + p8^2;

# Numerical solution of rank 5, there seems to be unique solution.
```

```
out := exactSOS(f, realPolynomials = [p1, p2, p3, p4, p5, p6, p7, p8],
    computePolynomialDecomposition = "no");
eig(out[3]);
```

$$p1 := x1^2 - x4^2$$
$$p2 := x2^2 - x4^2$$
$$p3 := x3^2 - x4^2$$
$$p4 := -x1^2 - x1\,x2 - x1\,x3 + x1\,x4 - x2\,x3 + x2\,x4 + x3\,x4$$
$$p5 := x5^2$$
$$p6 := x6^2$$
$$p7 := x1\,x5 + x5\,x6$$
$$p8 := x2\,x6$$
$$f := (x1^2 - x4^2)^2 + (x2^2 - x4^2)^2 + (x3^2 - x4^2)^2 + (-x1^2 - x1\,x2 - x1\,x3 + x1\,x4 - x2\,x3 + x2\,x4 + x3\,x4)^2 + x5^4 + x6^4 + (x1\,x5 + x5\,x6)^2 + x2^2\,x6^2$$

"Option traceEquations: yes - Only valid when looking for rational decompositions."

"No algebraic extension in this branch. Check: ", $\{x1 = 0, x2 = 0, x3 = 0, x4 = 0, x5 = 0, x6 = 0\}$

"compute random solutions..."

"indetsCFEV", { }

"trueIndets", { }

"number of solutions: ", 1

"No equations found. Check!"

"solve finished 2"

"-----"

"Facial reduction results:"

"Original matrix - Rank: ", 21, " - Number of indeterminates: ", 105

"Matrix after facial reduction - Rank: ", 21, " - Number of indeterminates: ", 105

"Check 1 of random rank:", 21

"Check 2 of random rank:", 21

"Calling numerical solver SEDUMI to find the values of the remaining indeterminates..."

"SEDUMI CALL"

```
SeDuMi 1.3 by AdvOL, 2005-2008 and Jos F. Sturm, 1998-2003.
Alg = 2: xz-corrector, Adaptive Step-Differentiation, theta =
0.250, beta = 0.500
eqs m = 106, order n = 22, dim = 442, blocks = 2
nnz(A) = 231 + 0, nnz(ADA) = 11236, nnz(L) = 5671
 it :     b*y       gap    delta  rate   t/tP*  t/tD*   feas cg
cg  prec
  0 :            1.94E+01 0.000
  1 :  -3.46E+00 6.19E+00 0.000 0.3194 0.9000 0.9000   0.55  1
1  1.4E+01
  2 :  -9.42E-01 2.05E+00 0.000 0.3304 0.9000 0.9000   2.70  1
1  2.3E+00
  3 :  -7.80E-02 6.53E-01 0.000 0.3192 0.9000 0.9000   2.75  1
1  6.5E-01
```

```
  4 :  -1.19E-02 1.86E-01 0.000 0.2841 0.9000 0.9000   1.26  1
1  3.3E-01
  5 :  -2.33E-03 4.94E-02 0.000 0.2664 0.9000 0.9000   1.07  1
1  2.5E-01
  6 :  -3.61E-04 2.28E-03 0.000 0.0461 0.9900 0.9900   1.01  1
1  1.4E-01
  7 :  -2.41E-06 1.40E-05 0.458 0.0061 0.9990 0.9990   1.00  1
1  1.1E-03
  8 :  -3.70E-07 2.72E-06 0.000 0.1936 0.9210 0.9000   0.99  2
2  2.8E-04
  9 :  -4.48E-08 4.99E-07 0.000 0.1837 0.9229 0.9000   1.00  3
3  6.2E-05
 10 :   4.09E-09 7.70E-08 0.000 0.1545 0.9137 0.9000   1.00  4
4  1.0E-05
 11 :   3.19E-09 1.15E-08 0.000 0.1487 0.9062 0.9000   1.00  7
7  1.5E-06
 12 :   4.94E-10 8.73E-10 0.372 0.0763 0.9900 0.9900   1.00  4
8  1.3E-07
 13 :   3.42E-11 5.90E-11 0.000 0.0676 0.9901 0.9900   1.00 15
9  8.6E-09
 14 :   8.05E-12 6.76E-12 0.000 0.1145 0.9078 0.9000   1.00 17
21  7.8E-10
 15 :   1.86E-12 7.03E-13 0.000 0.1041 0.9065 0.9000   1.00 61
60  1.8E-11
 16 :   7.31E-13 1.38E-13 0.000 0.1964 0.9000 0.7249   0.99 82
59  3.9E-13
Run into numerical problems.

iter seconds digits       c*x               b*y
 16      0.3   9.1  6.1124500922e-13  7.3054965525e-13
|Ax-b| =   7.1e-14, [Ay-c]_+ =   1.2E-12, |x|=  4.9e-01, |y|=
4.5e+00

Detailed timing (sec)
   Pre          IPM          Post
3.007E-03    1.290E-01    9.958E-04
Max-norms: ||b||=1, ||c|| = 6,
Cholesky |add|=15, |skip| = 0, ||L.L|| = 2.00201e+10.
```

"An exact positive definite solution could not be found for the reduced problem."

$$out := \left[0, 0, \left[\left[2, 1, 1, -1, 0, 0, 0, 1, -1, 0, \frac{8}{318113289}, 0, -1, 0, \frac{14}{176762067}, -1, 0, -\frac{2}{105325103}, \frac{5389}{20351}, \frac{79}{5772941}, -\frac{14404}{75689}\right],\right.\right.$$
$$\left[1, 1, 1, -1, 0, -\frac{8}{318113289}, 0, 1, -1, 0, -\frac{25}{129848296}, 0, -1, 0, \frac{16}{154063471}, 0, 0, -\frac{7}{59368447}, -\frac{15678}{439261}, \frac{10}{3866989}, -\frac{10725}{158633}\right],$$
$$\left[1, 1, 1, -1, 0, -\frac{14}{176762067}, 0, 1, -1, 0, -\frac{2257075715}{26183723640775643}, 0, -1, 0, \frac{9}{166224458}, 0, \frac{1}{9279181990}, -\frac{3604942655}{26366433970608792}, -\frac{4651}{130310}, \frac{1228303}{475011991284}, -\frac{8845}{130826}\right],$$

$$\left[-1, -1, -1, 1, 0, \frac{2}{105325103}, 0, -1, 1, 0, \frac{3750132577}{39358390661210722}, 0, 1, -\frac{1}{9279181990}, -\frac{11}{104170632}, 0, 0, \frac{1}{270582600}, \frac{4576}{128209}, -\frac{229283832}{88667835119215}, \frac{57540}{851071}\right],$$

$$\left[0, 0, 0, 0, \frac{9573}{20351}, -\frac{79}{5772941}, 0, 0, 0, \frac{15678}{439261}, -\frac{89331876}{792117893749}, 0, 0, \frac{4651}{130310}, \frac{289}{3386436}, 0, -\frac{4576}{128209}, \frac{419}{16347155}, 0, \frac{89838}{126929}, \frac{125}{982312}\right],$$

$$\left[0, -\frac{8}{318113289}, -\frac{14}{176762067}, \frac{2}{105325103}, -\frac{79}{5772941}, \frac{28808}{75689}, \frac{25}{129848296}, -\frac{3}{169954133}, \frac{15}{662951326}, \frac{158}{1433887}, \frac{10725}{158633}, -\frac{9}{166224458}, \frac{184}{759324393}, -\frac{37}{420807}, \frac{8845}{130826}, -\frac{1}{270582600}, -\frac{125}{5424053}, -\frac{57540}{851071}, \frac{37091}{126929}, -\frac{125}{982312}, 0\right],$$

$$\left[0, 0, 0, 0, 0, \frac{25}{129848296}, 1, 0, 0, 0, 0, 0, 0, 0, \frac{7}{58158288}, -1, 0, -\frac{17}{122743239}, -\frac{8410}{81063}, -\frac{155}{4871491}, \frac{10520}{27757}\right],$$

$$\left[1, 1, 1, -1, 0, -\frac{3}{169954133}, 0, 1, -1, 0, -\frac{7}{58158288}, 0, -1, 0, \frac{4}{91994077}, 0, 0, -\frac{19}{180690272}, -\frac{3839}{107560}, \frac{60}{23203217}, -\frac{27553}{407535}\right],$$

$$\left[-1, -1, -1, 1, 0, \frac{15}{662951326}, 0, -1, 1, 0, \frac{17}{122743239}, 0, 1, 0, -\frac{3416305}{1471082079806304}, 0, 0, \frac{9}{76504318}, \frac{8070}{226103}, -\frac{121020638}{46801219694719}, \frac{5147}{76129}\right],$$

$$\left[0, 0, 0, 0, \frac{15678}{439261}, \frac{158}{1433887}, 0, 0, 0, \frac{16820}{81063}, \frac{155}{4871491}, 0, 0, \frac{3839}{107560}, \frac{7293090785}{95889127306643}, 0, -\frac{8070}{226103}, -\frac{43}{5161093}, 0, -\frac{2812}{117171}, \frac{71}{383343}\right],$$

$$\left[\frac{8}{318113289}, -\frac{25}{129848296}, -\frac{2257075715}{26183723640775643}, \frac{3750132577}{39358390661210722}, -\frac{89331876}{792117893749}, \frac{10725}{158633}, 0, -\frac{7}{58158288}, \frac{17}{122743239}, \frac{155}{4871491}, \frac{6717}{27757}, -\frac{4}{91994077}, \frac{7}{65131656}, -\frac{325}{4132579}, \frac{27553}{407535}, -\frac{9}{76504318}, \frac{99}{9068083}, -\frac{5147}{76129}, \frac{2812}{117171}, -\frac{71}{383343}, 0\right],$$

$$\left[0, 0, 0, 0, 0, -\frac{9}{166224458}, 0, 0, 0, 0, -\frac{4}{91994077}, 1, 0, 0, 0, -1, 0, \frac{11}{119285875}, -\frac{15437}{160900}, \frac{174}{5233781}, -\frac{20362}{146915}\right],$$

$$\left[-1, -1, -1, 1, 0, \frac{184}{759324393}, 0, -1, 1, 0, \frac{7}{65131656}, 0, 1, 0, -\frac{11}{119285875}, 0, 0,\right.$$

$$\frac{17}{110693650}, \frac{3307}{92654}, -\frac{101099230}{39096835392749}, \frac{5281}{78111}\Bigg],$$

$$\Bigg[0, 0, 0, -\frac{1}{9279181990}, \frac{4651}{130310}, -\frac{37}{420807}, 0, 0, 0, \frac{3839}{107560}, -\frac{325}{4132579}, 0, 0,$$

$$\frac{15437}{80450}, -\frac{174}{5233781}, 0, -\frac{3307}{92654}, \frac{339}{3411763}, 0, -\frac{3075}{307303}, -\frac{124}{6035293}\Bigg],$$

$$\Bigg[\frac{14}{176762067}, \frac{16}{154063471}, \frac{9}{166224458}, -\frac{11}{104170632}, \frac{289}{3386436}, \frac{8845}{130826},$$

$$\frac{7}{58158288}, \frac{4}{91994077}, -\frac{3416305}{1471082079806304}, \frac{7293090785}{95889127306643}, \frac{27553}{407535}, 0,$$

$$-\frac{11}{119285875}, -\frac{174}{5233781}, \frac{40724}{146915}, -\frac{17}{110693650}, -\frac{1109}{11459423}, -\frac{5281}{78111}, \frac{3075}{307303},$$

$$\frac{124}{6035293}, 0\Bigg],$$

$$\Bigg[-1, 0, 0, 0, 0, -\frac{1}{270582600}, -1, 0, 0, 0, -\frac{9}{76504318}, -1, 0, 0, -\frac{17}{110693650}, 3, 0, 0,$$

$$-\frac{6689}{66355}, -\frac{336}{26823337}, -\frac{10633}{90332}\Bigg],$$

$$\Bigg[0, 0, \frac{1}{9279181990}, 0, -\frac{4576}{128209}, -\frac{125}{5424053}, 0, 0, 0, -\frac{8070}{226103}, \frac{99}{9068083}, 0, 0,$$

$$-\frac{3307}{92654}, -\frac{1109}{11459423}, 0, \frac{13378}{66355}, \frac{336}{26823337}, 0, \frac{3939}{225260}, -\frac{1588}{27414111}\Bigg],$$

$$\Bigg[-\frac{2}{105325103}, -\frac{7}{59368447}, -\frac{3604942655}{2636643397060879 2}, \frac{1}{270582600}, \frac{419}{16347155},$$

$$-\frac{57540}{851071}, -\frac{17}{122743239}, -\frac{19}{180690272}, \frac{9}{76504318}, -\frac{43}{5161093}, -\frac{5147}{76129},$$

$$\frac{11}{119285875}, \frac{17}{110693650}, \frac{339}{3411763}, -\frac{5281}{78111}, 0, \frac{336}{26823337}, \frac{10633}{45166}, -\frac{3939}{225260},$$

$$\frac{1588}{27414111}, 0\Bigg],$$

$$\Bigg[\frac{5389}{20351}, -\frac{15678}{439261}, -\frac{4651}{130310}, \frac{4576}{128209}, 0, \frac{37091}{126929}, -\frac{8410}{81063}, -\frac{3839}{107560}, \frac{8070}{226103},$$

$$0, \frac{2812}{117171}, -\frac{15437}{160900}, \frac{3307}{92654}, 0, \frac{3075}{307303}, -\frac{6689}{66355}, 0, -\frac{3939}{225260}, 1, 0, -\frac{37466}{73941}\Bigg],$$

$$\Bigg[\frac{79}{5772941}, \frac{10}{3866989}, \frac{1228303}{475011991284}, -\frac{229283832}{88667835119215}, \frac{89838}{126929}, -\frac{125}{982312},$$

$$-\frac{155}{4871491}, \frac{60}{23203217}, -\frac{121020638}{46801219694719}, -\frac{2812}{117171}, -\frac{71}{383343}, \frac{174}{5233781},$$

$$-\frac{101099230}{39096835392749}, -\frac{3075}{307303}, \frac{124}{6035293}, -\frac{336}{26823337}, \frac{3939}{225260}, \frac{1588}{27414111}, 0,$$

$$\frac{148873}{73941}, 0\Bigg],$$

$$\left[-\frac{14404}{75689}, -\frac{10725}{158633}, -\frac{8845}{130826}, \frac{57540}{851071}, \frac{125}{982312}, 0, \frac{10520}{27757}, -\frac{27553}{407535}, \frac{5147}{76129},\right.$$

$$\frac{71}{383343}, 0, -\frac{20362}{146915}, \frac{5281}{78111}, -\frac{124}{6035293}, 0, -\frac{10633}{90332}, -\frac{1588}{27414111}, 0, -\frac{37466}{73941}, 0, 1$$

$$\Big]\Big], \Big[\Big[2, 1, 1, -1, 0, 0, \frac{1}{2} - \frac{1}{2}\,\_t22_8, 2 - \_t22_9, -2 - \_t22_{10}, -\_t22_{11}, -\_t22_{12}, \frac{1}{2}$$
$$- \frac{1}{2}\,\_t22_{27}, -2 - \_t22_{28}, -\_t22_{29}, \_t22_1, \_t22_2, \_t22_3, \_t22_4, \_t22_5, \_t22_6, \_t22_7\Big],$$
$$\Big[1, \_t22_8, \_t22_9, \_t22_{10}, \_t22_{11}, \_t22_{12}, 0, \_t22_{13}, \_t22_{14}, \_t22_{15}, \_t22_{16}, \_t22_{17}, \_t22_{18},$$

$\_t22_{19}, \_t22_{20}, \_t22_{21}, \_t22_{22}, \_t22_{23}, \_t22_{24}, \_t22_{25}, \_t22_{26}\Big],$

$\big[1, \_t22_{9}, \_t22_{27}, \_t22_{28}, \_t22_{29}, -\_t22_{1}, 1-\_t22_{13}, 1-\_t22_{17}, -3-\_t22_{30}-\_t22_{18},$
$-\_t22_{35}-\_t22_{19}, -\_t22_{44}-\_t22_{20}, 0, -1-\_t22_{31}, -\_t22_{37}, -\_t22_{47}, 1-\_t22_{32}, -\_t22_{33}$
$-\_t22_{38}, -\_t22_{34}-\_t22_{48}, -\_t22_{39}, -\_t22_{40}-\_t22_{49}, -\_t22_{50}\big],$

$\big[-1, \_t22_{10}, \_t22_{28}, -1-2\ \_t22_{2}, -\_t22_{3}, -\_t22_{4}, -1-\_t22_{14}, \_t22_{30}, 1-\_t22_{21}, -\_t22_{36}$
$-\_t22_{22}, -\_t22_{45}-\_t22_{23}, \_t22_{31}, \_t22_{32}, \_t22_{33}, \_t22_{34}, 0, -\_t22_{41}, -\_t22_{51}, -\_t22_{42},$
$-\_t22_{43}-\_t22_{52}, -\_t22_{53}\big],$

$\big[0, \_t22_{11}, \_t22_{29}, -\_t22_{3}, 1-2\ \_t22_{5}, -\_t22_{6}, -\_t22_{15}, \_t22_{35}, \_t22_{36}, -\_t22_{24}, -\_t22_{46}$
$-\_t22_{25}, \_t22_{37}, \_t22_{38}, \_t22_{39}, \_t22_{40}, \_t22_{41}, \_t22_{42}, \_t22_{43}, 0, 1-\_t22_{54}, -\_t22_{55}\big],$

$\Big[0, \_t22_{12}, -\_t22_{1}, -\_t22_{4}, -\_t22_{6}, -2\ \_t22_{7}, -\_t22_{16}, \_t22_{44}, \_t22_{45}, \_t22_{46}, -\_t22_{26},$

$\_t22_{47}, \_t22_{48}, \_t22_{49}, \_t22_{50}, \_t22_{51}, \_t22_{52}, \_t22_{53}, \_t22_{54}, \_t22_{55}, 0\Big],$

$\Big[\frac{1}{2}-\frac{1}{2}\ \_t22_{8}, 0, 1-\_t22_{13}, -1-\_t22_{14}, -\_t22_{15}, -\_t22_{16}, 1, 0, 0, 0, 0, \frac{1}{2}$

$-\frac{1}{2}\ \_t22_{59}, -1-\_t22_{60}, -\_t22_{61}, -\_t22_{62}, -\frac{1}{2}-\frac{1}{2}\ \_t22_{70}, -\_t22_{71}, -\_t22_{72}, \_t22_{56},$

$\_t22_{57}, \_t22_{58}\Big],$

$\big[2-\_t22_{9}, \_t22_{13}, 1-\_t22_{17}, \_t22_{30}, \_t22_{35}, \_t22_{44}, 0, \_t22_{59}, \_t22_{60}, \_t22_{61}, \_t22_{62}, 0, -1$
$-\_t22_{73}, \_t22_{63}, \_t22_{64}, 1-\_t22_{74}, \_t22_{65}, \_t22_{66}, \_t22_{67}, \_t22_{68}, \_t22_{69}\big],$

$\Big[-2-\_t22_{10}, \_t22_{14}, -3-\_t22_{30}-\_t22_{18}, 1-\_t22_{21}, \_t22_{36}, \_t22_{45}, 0, \_t22_{60}, \_t22_{70},$

$\_t22_{71}, \_t22_{72}, \_t22_{73}, \_t22_{74}, \_t22_{75}, -\_t22_{79}-\_t22_{66}, 0, -\_t22_{76}, -\_t22_{81}, -\_t22_{77},$
$-\_t22_{78}-\_t22_{82}, -\_t22_{83}\big],$

$\big[-\_t22_{11}, \_t22_{15}, -\_t22_{35}-\_t22_{19}, -\_t22_{36}-\_t22_{22}, -\_t22_{24}, \_t22_{46}, 0, \_t22_{61}, \_t22_{71},$
$-2\ \_t22_{56}, -\_t22_{57}, -\_t22_{63}, -\_t22_{65}-\_t22_{75}, -\_t22_{67}, -\_t22_{80}-\_t22_{68}, \_t22_{76}, \_t22_{77},$
$\_t22_{78}, 0, -\_t22_{84}, -\_t22_{85}\big],$

$\Big[-\_t22_{12}, \_t22_{16}, -\_t22_{44}-\_t22_{20}, -\_t22_{45}-\_t22_{23}, -\_t22_{46}-\_t22_{25}, -\_t22_{26}, 0, \_t22_{62},$

$\_t22_{72}, -\_t22_{57}, 1-2\ \_t22_{58}, -\_t22_{64}, \_t22_{79}, \_t22_{80}, -\_t22_{69}, \_t22_{81}, \_t22_{82}, \_t22_{83},$

$\_t22_{84}, \_t22_{85}, 0\Big],$

$\Big[\frac{1}{2}-\frac{1}{2}\ \_t22_{27}, \_t22_{17}, 0, \_t22_{31}, \_t22_{37}, \_t22_{47}, \frac{1}{2}-\frac{1}{2}\ \_t22_{59}, 0, \_t22_{73}, -\_t22_{63},$

$-\_t22_{64}, 1, 0, 0, 0, -\frac{1}{2}-\frac{1}{2}\ \_t22_{86}, -\_t22_{87}, -\_t22_{88}, -\frac{1}{2}\ \_t22_{89}, -\_t22_{90}, -\frac{1}{2}\ \_t22_{94}\Big],$

$$\left[-2-\_t22_{28}, \_t22_{18}, -1-\_t22_{31}, \_t22_{32}, \_t22_{38}, \_t22_{48}, -1-\_t22_{60}, -1-\_t22_{73}, \_t22_{74}, -\_t22_{65}-\_t22_{75}, \_t22_{79}, 0, \_t22_{86}, \_t22_{87}, \_t22_{88}, 0, -\_t22_{91}, -\_t22_{95}, -\_t22_{92}, -\_t22_{93}-\_t22_{96}, -\_t22_{97}\right],$$

$$\left[-\_t22_{29}, \_t22_{19}, -\_t22_{37}, \_t22_{33}, \_t22_{39}, \_t22_{49}, -\_t22_{61}, \_t22_{63}, \_t22_{75}, -\_t22_{67}, \_t22_{80}, 0, \_t22_{87}, \_t22_{89}, \_t22_{90}, \_t22_{91}, \_t22_{92}, \_t22_{93}, 0, -\_t22_{98}, -\_t22_{99}\right],$$

$$\left[\_t22_{1}, \_t22_{20}, -\_t22_{47}, \_t22_{34}, \_t22_{40}, \_t22_{50}, -\_t22_{62}, \_t22_{64}, -\_t22_{79}-\_t22_{66}, -\_t22_{80}-\_t22_{68}, -\_t22_{69}, 0, \_t22_{88}, \_t22_{90}, \_t22_{94}, \_t22_{95}, \_t22_{96}, \_t22_{97}, \_t22_{98}, \_t22_{99}, 0\right],$$

$$\left[\_t22_{2}, \_t22_{21}, 1-\_t22_{32}, 0, \_t22_{41}, \_t22_{51}, -\frac{1}{2}-\frac{1}{2}\_t22_{70}, 1-\_t22_{74}, 0, \_t22_{76}, \_t22_{81}, -\frac{1}{2}-\frac{1}{2}\_t22_{86}, 0, \_t22_{91}, \_t22_{95}, 3, 0, 0, -\frac{1}{2}\_t22_{100}, -\_t22_{101}, -\frac{1}{2}\_t22_{102}\right],$$

$$\left[\_t22_{3}, \_t22_{22}, -\_t22_{33}-\_t22_{38}, -\_t22_{41}, \_t22_{42}, \_t22_{52}, -\_t22_{71}, \_t22_{65}, -\_t22_{76}, \_t22_{77}, \_t22_{82}, -\_t22_{87}, -\_t22_{91}, \_t22_{92}, \_t22_{96}, 0, \_t22_{100}, \_t22_{101}, 0, -\_t22_{103}, -\_t22_{104}\right],$$

$$\left[\_t22_{4}, \_t22_{23}, -\_t22_{34}-\_t22_{48}, -\_t22_{51}, \_t22_{43}, \_t22_{53}, -\_t22_{72}, \_t22_{66}, -\_t22_{81}, \_t22_{78}, \_t22_{83}, -\_t22_{88}, -\_t22_{95}, \_t22_{93}, \_t22_{97}, 0, \_t22_{101}, \_t22_{102}, \_t22_{103}, \_t22_{104}, 0\right],$$

$$\left[\_t22_{5}, \_t22_{24}, -\_t22_{39}, -\_t22_{42}, 0, \_t22_{54}, \_t22_{56}, \_t22_{67}, -\_t22_{77}, 0, \_t22_{84}, -\frac{1}{2}\_t22_{89}, -\_t22_{92}, 0, \_t22_{98}, -\frac{1}{2}\_t22_{100}, 0, \_t22_{103}, 1, 0, \frac{1}{2}-\frac{1}{2}\_t22_{105}\right],$$

$$\left[\_t22_{6}, \_t22_{25}, -\_t22_{40}-\_t22_{49}, -\_t22_{43}-\_t22_{52}, 1-\_t22_{54}, \_t22_{55}, \_t22_{57}, \_t22_{68}, -\_t22_{78}-\_t22_{82}, -\_t22_{84}, \_t22_{85}, -\_t22_{90}, -\_t22_{93}-\_t22_{96}, -\_t22_{98}, \_t22_{99}, -\_t22_{101}, -\_t22_{103}, \_t22_{104}, 0, \_t22_{105}, 0\right],$$

$\left[\_t22_7, \_t22_{26}, -\_t22_{50}, -\_t22_{53}, -\_t22_{55}, 0, \_t22_{58}, \_t22_{69}, -\_t22_{83}, -\_t22_{85}, 0,\right.$

$$-\frac{1}{2}\_t22_{94}, -\_t22_{97}, -\_t22_{99}, 0, -\frac{1}{2}\_t22_{102}, -\_t22_{104}, 0, \frac{1}{2}-\frac{1}{2}\_t22_{105}, 0, 1\big]\big],$$

$$\left.\begin{bmatrix} x1^2 \\ x1\,x2 \\ x1\,x3 \\ x1\,x4 \\ x1\,x5 \\ x1\,x6 \\ x2^2 \\ x2\,x3 \\ x2\,x4 \\ x2\,x5 \\ x2\,x6 \\ x3^2 \\ x3\,x4 \\ x3\,x5 \\ x3\,x6 \\ x4^2 \\ x4\,x5 \\ x4\,x6 \\ x5^2 \\ x5\,x6 \\ x6^2 \end{bmatrix}\right] \tag{10}$$

$$\begin{bmatrix} -1.97438314472190\ 10^{-13} \\ -7.74453423829609\ 10^{-14} \\ -4.55258899080516\ 10^{-14} \\ -5.44911073466646\ 10^{-15} \\ -7.78284570602484\ 10^{-16} \\ -3.19207941293114\ 10^{-16} \\ 0.143231334802181 \\ 0.154855377655251 \\ 0.159833493330887 \\ 0.169060403176407 \\ 0.171251956674729 \\ 0.194835305928768 \\ 0.312100683217818 \\ 0.363928955464681 \\ 0.536902335368795 \\ 0.731792265926532 \\ 1.10404726543883 \\ 1.75821983423874 \\ 2.28893553057623 \\ 3.91182748699246 \\ 7.21918144190802 \end{bmatrix} \tag{10}$$

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