```
> ##########################################################################
  # Example of a strictly positive polynomial that is a SOS over R
  # but not over Q
  ##########################################################################
```

```
> ##########################################################################
  # Load "Rational SOS" procedures
  ##########################################################################
  read("rationalSOS.mpl");
  with(rationalSOS);

  # Display tables of any size
  interface(rtablesize = infinity);
```

"Opening connection with Matlab"

rationalSOS := **module**( ) ... **end module**

[*cancelDenominator*, *decompositionToMatrix*, *dimSimplex*, *evalMat*, *evalSolution*, *exactSOS*, *getCoeffs*, *getDiag*, *getExtension*, *getVars*, *homogenize*, *isHomogeneous*, *linIndepRows*, *listSubsets*, *matrixToPoly*, *minorsDet*, *nonRatCoef*, *numericSolver*, *numericSolverSubmatrix*, *numericSolverSubmatrixMaxRank*, *numericSolverSubmatrixRoundBefore*, *polyToMatrix*, *polyToMatrixVars*, *primitiveMatrix*, *randomRank*, *reduceByLinearEquation*, *reduceByLinearEquationLinear*, *roundAbs*, *roundMat*, *roundMatFloat*, *roundMatToZero*, *roundToIntMatrix*, *roundVec*, *rround*, *rrounde*, *sedumiCallMaxSpectralNorm*, *sedumiCallMaxSpectralNormSDP*, *sedumiCallObjective*, *smallToZero*, *smallToZeroMatrix*, *solveSubmatrixGeneral*, *solveSubset*, *solveToZero*, *vectorTrace*, *zeroDetSRows*, *zeroDetSys*, *zeroRows*]

$$10 \tag{1}$$

```
> ##########################################################################
  # Construction of the example
  ##########################################################################

  ##########################################################################
  # 1) A polynomial of degree 4 in 4 variables with no rational
  #    decomposition. This example was constructed by J. Capco,
  #    S. Laplagne and C. Scheiderer.
  ##########################################################################

  # We define a polynomial as the sum of three squares in an algebraic
  # extension of degree 3 with generic coefficients.

  mp := t^3 - 2;
  p1 := c1 * t^2 + b1 * t + a1;
  p2 := c2 * t^2 + b2 * t + a2;
  p3 := c3 * t^2 + b3 * t + a3;

  fGeneric := p1^2 + p2^2 + p3^2;
  fGeneric := expand(fGeneric);
```

$$mp := t^3 - 2$$
$$p1 := c1\,t^2 + b1\,t + a1$$
$$p2 := c2\,t^2 + b2\,t + a2$$
$$p3 := c3\,t^2 + b3\,t + a3$$
$$fGeneric := (c1\,t^2 + b1\,t + a1)^2 + (c2\,t^2 + b2\,t + a2)^2 + (c3\,t^2 + b3\,t + a3)^2$$
$$fGeneric := c1^2\,t^4 + c2^2\,t^4 + c3^2\,t^4 + 2\,b1\,c1\,t^3 + 2\,b2\,c2\,t^3 + 2\,b3\,c3\,t^3 + 2\,a1\,c1\,t^2 + 2\,a2\,c2\,t^2 + 2\,a3\,c3\,t^2 + b1^2\,t^2 + b2^2\,t^2 + b3^2\,t^2 + 2\,a1\,b1\,t + 2\,a2\,b2\,t + 2\,a3\,b3\,t + a1^2 + a2^2 + a3^2 \quad \textbf{(2)}$$

```
> # We choose arbitrary parameters so that all terms in the resulting
  # expressions for a1 and a2 are multiples of x0, and the degree
  # is reduced after cancellation.
  b2 := -b1; c2 := b2; c1 := b2;
  b1 := x0; b3 := x1; a3 := x2; c3 := x3;
```

$$b2 := -b1$$
$$c2 := -b1$$
$$c1 := -b1$$
$$b1 := x0$$
$$b3 := x1$$
$$a3 := x2$$
$$c3 := x3 \quad \textbf{(3)}$$

```
> # We solve the coefficients a1 and a2 so that the polynomial is in Q,
  f2 := NormalForm(fGeneric, [mp], plex(a1, a2, x0, x1, x2, x3, t));
  f3 := collect(f2, t);
  lf := CoefficientList(f3, t);
  ss := solve({lf[2], lf[3]}, {a1, a2});
```

$$f2 := -2\,a1\,t^2\,x0 - 2\,a2\,t^2\,x0 + 2\,t^2\,x0^2 + t^2\,x1^2 + 2\,t^2\,x2\,x3 + 2\,a1\,t\,x0 - 2\,a2\,t\,x0 + 4\,t\,x0^2 + 2\,t\,x1\,x2 + 2\,t\,x3^2 + a1^2 + a2^2 + 4\,x1\,x3 + x2^2$$
$$f3 := (-2\,a1\,x0 - 2\,a2\,x0 + 2\,x0^2 + x1^2 + 2\,x2\,x3)\,t^2 + (2\,a1\,x0 - 2\,a2\,x0 + 4\,x0^2 + 2\,x1\,x2 + 2\,x3^2)\,t + a1^2 + a2^2 + 4\,x1\,x3 + x2^2$$
$$lf := [a1^2 + a2^2 + 4\,x1\,x3 + x2^2, 2\,a1\,x0 - 2\,a2\,x0 + 4\,x0^2 + 2\,x1\,x2 + 2\,x3^2, -2\,a1\,x0 - 2\,a2\,x0 + 2\,x0^2 + x1^2 + 2\,x2\,x3]$$
$$ss := \left\{a1 = -\frac{1}{4}\,\frac{2\,x0^2 - x1^2 + 2\,x1\,x2 - 2\,x2\,x3 + 2\,x3^2}{x0}, a2 = \frac{1}{4}\,\frac{6\,x0^2 + x1^2 + 2\,x1\,x2 + 2\,x2\,x3 + 2\,x3^2}{x0}\right\} \quad \textbf{(4)}$$

```
> # We plug in the solutions found for a1 and a2 and compute the resulting polynomial
  ssDen := denom(rhs(ss[1]));
  p1s := simplify(subs(ss, p1) * ssDen);
  p2s := simplify(subs(ss, p2) * ssDen);
```

```
p3s := simplify(subs(ss, p3) * ssDen);
```

$$ssDen := 4\, x0$$

$$p1s := -4\, t^2\, x0^2 + 4\, t\, x0^2 - 2\, x0^2 + x1^2 - 2\, x1\, x2 + 2\, x2\, x3 - 2\, x3^2$$

$$p2s := -4\, t^2\, x0^2 - 4\, t\, x0^2 + 6\, x0^2 + x1^2 + 2\, x1\, x2 + 2\, x2\, x3 + 2\, x3^2$$

$$p3s := 4\, \left(t^2\, x3 + t\, x1 + x2\right)\, x0 \qquad (5)$$

```
> # We replace t by the root of X^3-2
  p1ss := subs({t=RootOf(X^3-2)}, p1s);
  p2ss := subs({t=RootOf(X^3-2)}, p2s);
  p3ss := subs({t=RootOf(X^3-2)}, p3s);
```

$$p1ss := -4\, x0^2\, \mathrm{RootOf}\left(\_Z^3 - 2\right)^2 + 4\, x0^2\, \mathrm{RootOf}\left(\_Z^3 - 2\right) - 2\, x0^2 + x1^2 - 2\, x1\, x2 + 2\, x2\, x3 - 2\, x3^2$$

$$p2ss := -4\, x0^2\, \mathrm{RootOf}\left(\_Z^3 - 2\right)^2 - 4\, x0^2\, \mathrm{RootOf}\left(\_Z^3 - 2\right) + 6\, x0^2 + x1^2 + 2\, x1\, x2 + 2\, x2\, x3 + 2\, x3^2$$

$$p3ss := 4\, \left(\mathrm{RootOf}\left(\_Z^3 - 2\right)^2\, x3 + \mathrm{RootOf}\left(\_Z^3 - 2\right)\, x1 + x2\right)\, x0 \qquad (6)$$

```
> # We compute f and verify that it is has rational coefficients
  f := p1ss^2 + p2ss^2 + p3ss^2;
  f := simplify(f);
```

$$f := \left(-4\, x0^2\, \mathrm{RootOf}\left(\_Z^3 - 2\right)^2 + 4\, x0^2\, \mathrm{RootOf}\left(\_Z^3 - 2\right) - 2\, x0^2 + x1^2 - 2\, x1\, x2 + 2\, x2\, x3 - 2\, x3^2\right)^2 + \left(-4\, x0^2\, \mathrm{RootOf}\left(\_Z^3 - 2\right)^2 - 4\, x0^2\, \mathrm{RootOf}\left(\_Z^3 - 2\right) + 6\, x0^2 + x1^2 + 2\, x1\, x2 + 2\, x2\, x3 + 2\, x3^2\right)^2 + 16\, \left(\mathrm{RootOf}\left(\_Z^3 - 2\right)^2\, x3 + \mathrm{RootOf}\left(\_Z^3 - 2\right)\, x1 + x2\right)^2\, x0^2$$

$$f := 40\, x0^4 + 8\, x0^2\, x1^2 + 32\, x0^2\, x1\, x2 + 64\, x0^2\, x1\, x3 + 16\, x0^2\, x2^2 + 16\, x0^2\, x2\, x3 + 32\, x0^2\, x3^2 + 2\, x1^4 + 8\, x1^2\, x2^2 + 8\, x1^2\, x2\, x3 + 16\, x1\, x2\, x3^2 + 8\, x2^2\, x3^2 + 8\, x3^4 \qquad (7)$$

```
> # We verify that there is no solution with x2 = 0
  sols := solve({p1ss, p2ss, p3ss, x2});
```

$$sols := \{x0 = 0, x1 = 0, x2 = 0, x3 = 0\} \qquad (8)$$

```
> # Computation of a quadratic form associated to a linear functional in
  # the dual of the cone of SOS polynomials that vanishes at f
  # with kernel of small dimension.

  # We construct all monomials of degree 2 and 4
  d := 2;
  polVarsX := [x0, x1, x2, x3];
  varSum := add(polVarsX[i], i=1..nops(polVarsX)) :
  md := expand((varSum)^d) :
  cfs := coeffs(md, polVarsX, 'ctdX') :
  print("Monomials of degree d: ", ctdX);
```

```
m2d := expand(varSum^(2*d)) :
cfs := coeffs(m2d, polVarsX, 'ct2dX') :
print("Monomials of degree 2d: ", ct2dX);
```

$$d := 2$$

$$polVarsX := [x0, x1, x2, x3]$$

$$\text{"Monomials of degree d: ", } x0^2, x1\,x0, x0\,x2, x3\,x0, x1^2, x1\,x2, x1\,x3, x2^2, x2\,x3, x3^2$$

$$\text{"Monomials of degree 2d: ", } x0^4, x0^3\,x1, x0^3\,x2, x0^3\,x3, x0^2\,x1^2, x0^2\,x1\,x2, x0^2\,x1\,x3, x0^2\,x2^2, x0^2\,x2\,x3, x0^2\,x3^2, x0\,x1^3, x0\,x1^2\,x2, x0\,x1^2\,x3, x0\,x1\,x2^2, x0\,x1\,x2\,x3, x0\,x1\,x3^2, x0\,x2^3, x0\,x2^2\,x3, x0\,x2\,x3^2, x0\,x3^3, x1^4, x1^3\,x2, x1^3\,x3, x1^2\,x2^2, x1^2\,x2\,x3, x3^2\,x1^2, x1\,x2^3, x1\,x2^2\,x3, x1\,x2\,x3^2, x1\,x3^3, x2^4, x2^3\,x3, x2^2\,x3^2, x2\,x3^3, x3^4 \tag{9}$$

```
> # We construct the linear form that vanishes at all products piss * h
pListX := [p1ss, p2ss, p3ss] :
MX := [ ];
for j from 1 to nops(pListX) do
  for i from 1 to nops([ctdX]) do
   p1tX := expand(pListX[j] * ctdX[i]);
   vec := LinearAlgebra[Transpose](getCoeffs(p1tX, [ct2dX]));
   if (nops(MX) = 0) then
     MX := ⟨vec⟩;
   else
     MX := ⟨MX, vec⟩;
   end if;
  end do:
end do:
rc := [Dimension(MX)];
nr := rc[1];
B := Vector(nr) :
sX := LinearAlgebra[LinearSolve](MX, B) :
varssX := indets(sX);
nops(varssX); # 8 indeterminates left to solve
```

$$MX := [\ ]$$

$$rc := [30, 35]$$

$$nr := 30$$

$$varssX := \{\_t_2, \_t_7, \_t_{10}, \_t_{20}, \_t_{31}, \_t_{33}, \_t_{34}, \_t_{35}\}$$

$$8 \tag{10}$$

```
> # We define a generic polynomial of degree d with coefficientes q_i.
qIndX := [seq(q[i], i = 1 ..nops([ctdX]))] :
psaX := add(q[i] * ctdX[i], i = 1 ..nops([ctdX]));
```

$$psaX := x0^2\,q_1 + x0\,x1\,q_2 + x0\,x2\,q_3 + x0\,x3\,q_4 + x1^2\,q_5 + x1\,x2\,q_6 + x1\,x3\,q_7 + x2^2\,q_8 + x2\,x3\,q_9 + x3^2\,q_{10} \tag{11}$$

```
> # The square of psaX and the coefficients
ps2X := expand(psaX*psaX) :
```

```
aaX := getCoeffs(expand(ps2X), [ct2dX]) :
```

```
> # We compute the form and look for a PSD matrix using SEDUMI
  ewX := LinearAlgebra[Transpose](sX) . aaX:
  ooX := polyToMatrixVars(expand(ewX), qIndX) :
```

```
> # Numerical optimization using SEDUMI.
  # Numerically, the matrix has kernel of dimension 6.
  oEval := ooX[1] :
  outA := numericSolverSubmatrixMaxRank(evalf(oEval), "eig") :
  QEval := eval(oEval, Equate([op(outA[2])], outA[3][1 ..nops([op(outA[2])])])) :
  smallToZeroMatrix(evalf(QEval), 8);
  print(eig(QEval));
```

```
"rndRank", 7
[3, 4, 6, 7, 8, 9, 10]
"SEDUMI CALL - eig"
[[0.09482924, 0, 0, 0, 0.31325558, -0.07786130, -0.19958157, 0.01935283, 0.04960703,
  0.12715753],
  [0, 0.31325558, -0.07786130, -0.19958157, 0, 0, 0, 0, 0, 0],
  [0, -0.07786130, 0.01935283, 0.04960703, 0, 0, 0, 0, 0, 0],
  [0, -0.19958157, 0.04960703, 0.12715753, 0, 0, 0, 0, 0, 0],
  [0.31325558, 0, 0, 0, 1.22052476, -0.37420513, -0.51187895, 0.13763537, 0.07100650,
  0.53704856],
  [-0.07786130, 0, 0, 0, -0.37420513, 0.13763537, 0.07100650, -0.06232181, 0.01776965,
  -0.17811095],
  [-0.19958157, 0, 0, 0, -0.51187895, 0.07100650, 0.53704856, 0.01776965, -0.17811095,
  -0.17475741],
  [0.01935283, 0, 0, 0, 0.13763537, -0.06232181, 0.01776965, 0.44352360, -0.02672910,
  0.07238223],
  [0.04960703, 0, 0, 0, 0.07100650, 0.01776965, -0.17811095, -0.02672910, 0.07238223,
  0.00801817],
  [0.12715753, 0, 0, 0, 0.53704856, -0.17811095, -0.17475741, 0.07238223, 0.00801817,
  0.24421278]]
```

(12)

$$\begin{bmatrix} -8.15611288952737\ 10^{-9} \\ -2.40785649813595\ 10^{-11} \\ -5.54200099950070\ 10^{-12} \\ 7.66214592535787\ 10^{-11} \\ 9.20239493165339\ 10^{-10} \\ 4.43807228701980\ 10^{-9} \\ 0.320666707904826 \\ 0.459765930092981 \\ 0.512004830764087 \\ 1.91748500909260 \end{bmatrix} \quad \textbf{(12)}$$

```
> # Based on the numerical solution, we compute exact values of the
  # unknowns to construct an exact PSD matrix.

  # The values obtained by SEDUMI are:
  # t_1 = -0.0000, t_2 = -0.1996, t_3 = 0.1272, t_4 = 0.0000,
  # t_5 =  0.4435, t_6 =  0.0724, t_7 = 0.0080, t_8 = 0.2442

  ## We fix some variables to 0
  oEvalX := eval(ooX[1], {varssX[1] = 0, varssX[4] = 0, varssX[7] = 0}) :

  ## We fix other variables with positive values
  oEvalX := eval(oEvalX, {varssX[5] = 1, varssX[8] = 1, varssX[3] = 1/2}) :
```

```
> ## We compute exact values for the remaining two variables.

  # We observe that in the numerical solution, there are two singular
  # principal matrices.
  evalf(Determinant(QEval[5 .. 7, 5 .. 7]));
  evalf(Determinant(QEval[2 .. 3, 2 .. 3]));
```

$$-2.82782586236152\ 10^{-9}$$
$$9.83991388370242\ 10^{-10} \quad \textbf{(13)}$$

```
> # We compute symbolically the values of the unknowns that make the matrices singular.
  S_23 := solve(Determinant(oEvalX[2 .. 3, 2 .. 3]), indets(oEvalX[2 .. 3, 2 .. 3])) :
  S_23_1 := simplify(S_23[1]);
```

$$S\_23\_1 := \left\{ \_t_7 = -\frac{1}{4}\left(\mathit{RootOf}(\_Z^3-2)^2 + \sqrt{-\mathit{RootOf}(\_Z^3-2)^2 + 2\,\mathit{RootOf}(\_Z^3-2) + 1} - 1\right)\mathit{RootOf}(\_Z^3-2)^2 \right\} \quad \textbf{(14)}$$

```
> oX_567 := eval(oEvalX[5 .. 7, 5 .. 7], S_23_1) :
  S_567 := solve(Determinant(oX_567), indets(oX_567)) :
```

```
S_567_2 := simplify(S_567[2]);
```

$$S\_567\_2 := \left\{ \_t_{33} = \frac{1}{2}\, RootOf(\_Z^3 - 2)^2 \left(\sqrt{-RootOf(\_Z^3-2)^2 + 2\, RootOf(\_Z^3-2) + 1} - 1 \right) \right\} \tag{15}$$

```
> # We apply these substitutions (using simplified expressions equivalent to the results by Maple)
  rA := RootOf(_Z^3-2) :
  rB := RootOf(_Z*rA^2-rA^2 + _Z^2 + 1) :
  rC := -(1/2)*rA-(1/2)*rB :
> oEvalX := eval(oEvalX, {varssX[2]=rC}) :
  oEvalX := eval(oEvalX, {varssX[6]=rB}) :
> # No indetermiantes in the resulting oEvalX
  evalf(oEvalX);
```

[[0.3728809420, 0., 0., 0., 1.231761842, -0.3061607886, -0.7847808106, 0.07609785017, 0.1950613368, 0.5000000000], **(16)**
[0., 1.231761842, -0.3061607886, -0.7847808106, 0., 0., 0., 0., 0., 0.],
[0., -0.3061607886, 0.07609785017, 0.1950613368, 0., 0., 0., 0., 0., 0.],
[0., -0.7847808106, 0.1950613368, 0.5000000000, 0., 0., 0., 0., 0., 0.],
[1.231761842, 0., 0., 0., 4.899359550, -1.534480445, -1.933328015, 0.580923683, 0.229158469, 2.174802104],
[-0.3061607886, 0., 0., 0., -1.534480445, 0.580923683, 0.229158469, -0.2700817033, 0.101401094, -0.740078950],
[-0.7847808106, 0., 0., 0., -1.933328015, 0.229158469, 2.174802104, 0.101401094, -0.740078950, -0.6371205733],
[0.07609785017, 0., 0., 0., 0.580923683, -0.2700817033, 0.101401094, 1., -0.124964077, 0.3096405713],
[0.1950613368, 0., 0., 0., 0.229158469, 0.101401094, -0.740078950, -0.124964077, 0.3096405713, 0.],
[0.5000000000, 0., 0., 0., 2.174802104, -0.740078950, -0.6371205733, 0.3096405713, 0., 1.]]

```
> # The matrix has kernel of rank 6
  evalf(Eigenvalues(oEvalX));
```

(17)

$$\begin{bmatrix} 0.773176735490537 \\ 1.95800566390025 \\ 7.60642445185921 \\ 1.807859692 \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{bmatrix} \quad (17)$$

```
> # The kernel of the resulting form
  LX := NullSpace(oEvalX) :
  nops(LX);
```

$$6 \quad (18)$$

```
> # The polynoials in the kernel
  LinearAlgebra[Transpose](simplify(LX[01]));
  LinearAlgebra[Transpose](simplify(LX[02]));
  LinearAlgebra[Transpose](simplify(LX[03]));
  LinearAlgebra[Transpose](simplify(LX[04]));
  LinearAlgebra[Transpose](simplify(LX[05]));
  LinearAlgebra[Transpose](simplify(LX[06]));
```

$$\Big[-RootOf(\_Z^3-2)+2, 0, 0, 0, -\frac{1}{2}\Big(2+RootOf(\_Z^3-2)\,RootOf\big(\_Z\,RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2+1\big) - 2\,RootOf(\_Z^3-2)^2\Big)\Big/\Big(RootOf(\_Z^3-2)^2 + RootOf\big(\_Z\,RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2+1\big) - 2\,RootOf(\_Z^3-2)\Big), 0, 0, 0, 0, 1\Big]$$

$$\begin{bmatrix} -2\,RootOf(\_Z^3-2)^2+1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Big[RootOf(\_Z^3-2)^2+2\,RootOf(\_Z^3-2)+2\,RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2 - RootOf(\_Z^3-2)^2+1\big), 0, 0, 0, -\frac{1}{2}\Big(RootOf(\_Z^3-2)^2\,RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2 - RootOf(\_Z^3-2)^2+1\big) - 4 + 2\,RootOf(\_Z^3-2)\Big)\Big/\Big(RootOf(\_Z^3-2)^2 + RootOf\big(\_Z\,RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2+1\big) - 2\,RootOf(\_Z^3-2)\Big), 0, 1, 0, 0, 0\Big]$$

$$\Big[-RootOf(\_Z^3-2), 0, 0, 0, \frac{1}{2}\Big(2+RootOf(\_Z^3-2)\,RootOf\big(\_Z\,RootOf(\_Z^3-2)^2+\_Z^2$$

$- RootOf(\_Z^3-2)^2+1) - 2\, RootOf(\_Z^3-2)^2) \Big/ (RootOf(\_Z^3-2)^2$
$+ RootOf(\_Z\, RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2+1) - 2\, RootOf(\_Z^3$
$-2)), 1, 0, 0, 0, 0]$

$[0, -(RootOf(\_Z\, RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2+1) + RootOf(\_Z^3-2)) \Big/$
$(RootOf(\_Z^3-2)^2\, RootOf(\_Z\, RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2+1)$
$- 2\, RootOf(\_Z^3-2)\, RootOf(\_Z\, RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2+1)$
$- 2\, RootOf(\_Z^3-2)^2+1), 0, 1, 0, 0, 0, 0, 0, 0]$

$[0, -(RootOf(\_Z^3-2)^2\, RootOf(\_Z\, RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2+1) + 2$ **(19)**
$- 2\, RootOf(\_Z\, RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2+1) - RootOf(\_Z^3$
$-2)) \Big/ (RootOf(\_Z^3-2)^2\, RootOf(\_Z\, RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2$
$+1) - 2\, RootOf(\_Z^3-2)\, RootOf(\_Z\, RootOf(\_Z^3-2)^2 + \_Z^2 - RootOf(\_Z^3-2)^2$
$+1) - 2\, RootOf(\_Z^3-2)^2+1), 1, 0, 0, 0, 0, 0, 0, 0]$

```
> ##########################################################################
  # 2) We add the second block from an example in the strictly
  #    positive border of (4,4)
  ##########################################################################

  # g is the sum of 4 squares
  g1 := y0^2-y3^2;
  g2 := y1^2-y3^2;
  g3 := y2^2-y3^2;
  g4 := -y0^2 - y0*y1 -y0*y2 + y0*y3 - y1*y2 + y1*y3 + y2*y3;
  g  :=  g1^2 + g2^2 + g3^2 + g4^2;
```

$$g1 := y0^2 - y3^2$$
$$g2 := y1^2 - y3^2$$
$$g3 := y2^2 - y3^2$$
$$g4 := -y0^2 - y0\,y1 - y0\,y2 + y0\,y3 - y1\,y2 + y1\,y3 + y2\,y3$$
$$g := (y0^2-y3^2)^2 + (y1^2-y3^2)^2 + (y2^2-y3^2)^2 + (-y0^2 - y0\,y1 - y0\,y2 + y0\,y3 - y1\,y2 + y1\,y3 + y2\,y3)^2 \quad \textbf{(20)}$$

```
> # We look for a form in the y-monomials that vanishes in the g_i
  d := 2;
  polVarsY := [y0, y1, y2, y3];
  varSumY := add(polVarsY[i], i=1..nops(polVarsY));
  mdY := expand((varSumY)^d);
  cfsY := coeffs(mdY, polVarsY, 'ctdY');
  m2dY := expand(varSumY^(2*d));
  cfsY := coeffs(m2dY, polVarsY, 'ct2dY');
```

$$d := 2$$

$$polVarsY := [y0, y1, y2, y3]$$

$$varSumY := y0 + y1 + y2 + y3$$

$$mdY := y0^2 + 2\,y0\,y1 + 2\,y0\,y2 + 2\,y0\,y3 + y1^2 + 2\,y1\,y2 + 2\,y1\,y3 + y2^2 + 2\,y2\,y3 + y3^2$$

$$cfsY := 1, 2, 2, 2, 1, 2, 2, 1, 2, 1$$

$$m2dY := y0^4 + 4\,y0^3\,y1 + 4\,y0^3\,y2 + 4\,y0^3\,y3 + 6\,y0^2\,y1^2 + 12\,y0^2\,y1\,y2 + 12\,y0^2\,y1\,y3 + 6\,y0^2\,y2^2 + 12\,y0^2\,y2\,y3 + 6\,y0^2\,y3^2 + 4\,y0\,y1^3 + 12\,y0\,y1^2\,y2 + 12\,y0\,y1^2\,y3 + 12\,y0\,y1\,y2^2 + 24\,y0\,y1\,y2\,y3 + 12\,y0\,y1\,y3^2 + 4\,y0\,y2^3 + 12\,y0\,y2^2\,y3 + 12\,y0\,y2\,y3^2 + 4\,y0\,y3^3 + y1^4 + 4\,y1^3\,y2 + 4\,y1^3\,y3 + 6\,y1^2\,y2^2 + 12\,y1^2\,y2\,y3 + 6\,y1^2\,y3^2 + 4\,y1\,y2^3 + 12\,y1\,y2^2\,y3 + 12\,y1\,y2\,y3^2 + 4\,y1\,y3^3 + y2^4 + 4\,y2^3\,y3 + 6\,y2^2\,y3^2 + 4\,y2\,y3^3 + y3^4$$

$$cfsY := 1, 4, 4, 4, 6, 12, 12, 6, 12, 6, 4, 12, 12, 12, 24, 12, 4, 12, 12, 4, 1, 4, 4, 6, 12, 6, 4, 12, 12, 4, 1, 4, 6, 4, 1 \quad \textbf{(21)}$$

```
> # We compute all the restrictions to phi: A4 -> R given
  # by the five polynomials. There are 20 restrictions for each polynomial

  pListY := [g1, g2, g3, g4];
  MY := [ ]:
  for j from 1 to nops(pListY) do
   for i from 1 to nops([ctdY]) do
    p1t := expand(pListY[j] * ctdY[i]);
    vec := LinearAlgebra[Transpose](getCoeffs(p1t, [ct2dY]));
    if (nops(MY) = 0) then
     MY := ⟨vec⟩;
    else
     MY := ⟨MY, vec⟩;
    end if;
   end do:
  end do:
```

$$pListY := [y0^2 - y3^2, y1^2 - y3^2, y2^2 - y3^2, -y0^2 - y0\,y1 - y0\,y2 + y0\,y3 - y1\,y2 + y1\,y3 + y2\,y3] \quad \textbf{(22)}$$

```
> # We solve the system using only these restrictions
  rc := [Dimension(MY)];
  nr := rc[1];
  B := Vector(nr):
  s := LinearAlgebra[LinearSolve](MY, B):
  varssY := indets(s);
  nops(varssY); # Onlt 1 indeterminate left to solve
```

$$rc := [40, 35]$$

$$nr := 40$$

$$varssY := \{\_t0_9\}$$

$$1 \quad \textbf{(23)}$$

```
> # We give values to the unknowns so that the form is PSD
  qIndY := [seq(qY[i], i = 1 ..nops([ctdY]))];
```

```
psaY := add(qIndY[i] * ctdY[i], i = 1 ..nops([ctdY])) :
ps2Y := expand(psaY* psaY) :
aaY := getCoeffs(expand(ps2Y), [ct2dY]) :
```

$$qIndY := \left[qY_1, qY_2, qY_3, qY_4, qY_5, qY_6, qY_7, qY_8, qY_9, qY_{10}\right] \tag{24}$$

```
> # We compute the form and verify it is PSD
ewY := LinearAlgebra[Transpose](s) . aaY:
ooY := polyToMatrixVars(expand(ewY), qIndY) :
ooMY := ooY[1] :
```

```
> # We give value 1 to the only indeterminate
oEvalY := eval(ooMY, {varssY[1] = 1});
```

$$oEvalY := \begin{bmatrix} 6 & -1 & -1 & 1 & 6 & -1 & 1 & 6 & 1 & 6 \\ -1 & 6 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 6 & 1 & -1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 6 & 1 & 1 & -1 & 1 & -1 & 1 \\ 6 & -1 & -1 & 1 & 6 & -1 & 1 & 6 & 1 & 6 \\ -1 & -1 & -1 & 1 & -1 & 6 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 6 & 1 & -1 & 1 \\ 6 & -1 & -1 & 1 & 6 & -1 & 1 & 6 & 1 & 6 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 6 & 1 \\ 6 & -1 & -1 & 1 & 6 & -1 & 1 & 6 & 1 & 6 \end{bmatrix} \tag{25}$$

```
> ##########################################################################
# 3) Finally, we add the two examples plus a third polynomial to
# ensure that there are no real roots.
##########################################################################

h := (- x2^2 + y2^2);
p := f + g1^2 + g2^2 + g3^2 + g4^2 + h^2 :
expand(p);
```

$$h := -x2^2 + y2^2$$

$$\begin{aligned} &40\, x0^4 + 8\, x0^2\, x1^2 + 32\, x0^2\, x1\, x2 + 64\, x0^2\, x1\, x3 + 16\, x0^2\, x2^2 + 16\, x0^2\, x2\, x3 + 32\, x0^2\, x3^2 \\ &\quad + 2\, x1^4 + 8\, x1^2\, x2^2 + 8\, x1^2\, x2\, x3 + 16\, x1\, x2\, x3^2 + x2^4 + 8\, x2^2\, x3^2 - 2\, x2^2\, y2^2 + 8\, x3^4 \\ &\quad + 2\, y0^4 + 2\, y0^3\, y1 + 2\, y0^3\, y2 - 2\, y0^3\, y3 + y0^2\, y1^2 + 4\, y0^2\, y1\, y2 - 4\, y0^2\, y1\, y3 + y0^2\, y2^2 \\ &\quad - 4\, y0^2\, y2\, y3 - y0^2\, y3^2 + 2\, y0\, y1^2\, y2 - 2\, y0\, y1^2\, y3 + 2\, y0\, y1\, y2^2 - 6\, y0\, y1\, y2\, y3 \\ &\quad + 2\, y0\, y1\, y3^2 - 2\, y0\, y2^2\, y3 + 2\, y0\, y2\, y3^2 + y1^4 + y1^2\, y2^2 - 2\, y1^2\, y2\, y3 - y1^2\, y3^2 \\ &\quad - 2\, y1\, y2^2\, y3 + 2\, y1\, y2\, y3^2 + 2\, y2^4 - y2^2\, y3^2 + 3\, y3^4 \end{aligned} \tag{26}$$

```
> # We construct a linear form in the dual of the SOS cone that
# vanishes in all the 9 polynomials such that the associated
# quadratic form has kernel of minimal rank.

d := 2;
```

```
polVarsXY := [x0, x1, x2, x3, y0, y1, y2, y3];
varSumXY := add(polVarsXY[i], i = 1 ..nops(polVarsXY));
```

$$d := 2$$
$$polVarsXY := [x0, x1, x2, x3, y0, y1, y2, y3]$$
$$varSumXY := x0 + x1 + x2 + x3 + y0 + y1 + y2 + y3 \tag{27}$$

```
> # We use a block ordering for the monomials
mdXY := [x0^2, x0*x1, x0*x2, x0*x3, x1^2, x1*x2, x1*x3, x2^2, x2*x3, x3^2, y0^2, y0*y1,
    y0*y2, y0*y3, y1^2, y1*y2, y1*y3, y2^2, y2*y3, y3^2, x0*y0, x0*y1, x0*y2, x0*y3, x1
    *y0, x1*y1, x1*y2, x1*y3, x2*y0, x2*y1, x2*y2, x2*y3, x3*y0, x3*y1, x3*y2, x3*y3]
    :
ctdXY := op(mdXY) :
print("Monomials of degree d: ", ctdXY);

m2dXY := expand(varSumXY^(2*d)) :
cfs := coeffs(m2dXY, polVarsXY, 'ct2dXY') :
print("Monomials of degree 2d: ", ct2dXY);
```

"Monomials of degree d: ", $x0^2, x1\,x0, x0\,x2, x3\,x0, x1^2, x1\,x2, x1\,x3, x2^2, x2\,x3, x3^2, y0^2, y0\,y1, y0\,y2, y0\,y3, y1^2, y1\,y2, y1\,y3, y2^2, y2\,y3, y3^2, x0\,y0, x0\,y1, x0\,y2, x0\,y3, x1\,y0, x1\,y1, x1\,y2, x1\,y3, x2\,y0, x2\,y1, x2\,y2, x2\,y3, x3\,y0, x3\,y1, x3\,y2, x3\,y3$

"Monomials of degree 2d: ", $x0^4, x0^3\,x1, x0^3\,x2, x0^3\,x3, x0^3\,y0, x0^3\,y1, x0^3\,y2, x0^3\,y3, x0^2\,x1^2, x0^2\,x1\,x2, x0^2\,x1\,x3, x0^2\,x1\,y0, x0^2\,x1\,y1, x0^2\,x1\,y2, x0^2\,x1\,y3, x0^2\,x2^2, x0^2\,x2\,x3, x0^2\,x2\,y0, x0^2\,x2\,y1, x0^2\,x2\,y2, x0^2\,x2\,y3, x0^2\,x3^2, x0^2\,x3\,y0, x0^2\,x3\,y1, x0^2\,x3\,y2, x0^2\,x3\,y3, x0^2\,y0^2, x0^2\,y0\,y1, x0^2\,y0\,y2, x0^2\,y0\,y3, x0^2\,y1^2, x0^2\,y1\,y2, x0^2\,y1\,y3, x0^2\,y2^2, x0^2\,y2\,y3, x0^2\,y3^2, x0\,x1^3, x0\,x1^2\,x2, x0\,x1^2\,x3, x0\,x1^2\,y0, x0\,x1^2\,y1, x0\,x1^2\,y2, x0\,x1^2\,y3, x0\,x1\,x2^2, x0\,x1\,x2\,x3, x0\,x1\,x2\,y0, x0\,x1\,x2\,y1, x0\,x1\,x2\,y2, x0\,x1\,x2\,y3, x0\,x1\,x3^2, x0\,x1\,x3\,y0, x0\,x1\,x3\,y1, x0\,x1\,x3\,y2, x0\,x1\,x3\,y3, x0\,x1\,y0^2, x0\,x1\,y0\,y1, x0\,x1\,y0\,y2, x0\,x1\,y0\,y3, x0\,x1\,y1^2, x0\,x1\,y1\,y2, x0\,x1\,y1\,y3, x0\,x1\,y2^2, x0\,x1\,y2\,y3, x0\,x1\,y3^2, x0\,x2^3, x0\,x2^2\,x3, x0\,x2^2\,y0, x0\,x2^2\,y1, x0\,x2^2\,y2, x0\,x2^2\,y3, x0\,x2\,x3^2, x0\,x2\,x3\,y0, x0\,x2\,x3\,y1, x0\,x2\,x3\,y2, x0\,x2\,x3\,y3, x0\,x2\,y0^2, x0\,x2\,y0\,y1, x0\,x2\,y0\,y2, x0\,x2\,y0\,y3, x0\,x2\,y1^2, x0\,x2\,y1\,y2, x0\,x2\,y1\,y3, x0\,x2\,y2^2, x0\,x2\,y2\,y3, x0\,x2\,y3^2, x0\,x3^3, x0\,x3^2\,y0, x0\,x3^2\,y1, x0\,x3^2\,y2, x0\,x3^2\,y3, x0\,x3\,y0^2, x0\,x3\,y0\,y1, x0\,x3\,y0\,y2, x0\,x3\,y0\,y3, x0\,x3\,y1^2, x0\,x3\,y1\,y2, x0\,x3\,y1\,y3, x0\,x3\,y2^2, x0\,x3\,y2\,y3, x0\,x3\,y3^2, x0\,y0^3, x0\,y0^2\,y1, x0\,y0^2\,y2, x0\,y0^2\,y3, x0\,y0\,y1^2, x0\,y0\,y1\,y2, x0\,y0\,y1\,y3, x0\,y0\,y2^2, x0\,y0\,y2\,y3, x0\,y0\,y3^2, x0\,y1^3, x0\,y1^2\,y2, x0\,y1^2\,y3, x0\,y1\,y2^2, x0\,y1\,y2\,y3, x0\,y1\,y3^2, x0\,y2^3, x0\,y2^2\,y3, x0\,y2\,y3^2, x0\,y3^3, x1^4, x1^3\,x2, x1^3\,x3, x1^3\,y0, x1^3\,y1, x1^3\,y2, x1^3\,y3, x1^2\,x2^2, x1^2\,x2\,x3, x1^2\,x2\,y0, x1^2\,x2\,y1, x1^2\,x2\,y2, x1^2\,x2\,y3, x3^2\,x1^2, x1^2\,x3\,y0, x1^2\,x3\,y1, x1^2\,x3\,y2, x1^2\,x3\,y3, x1^2\,y0^2, x1^2\,y0\,y1, x1^2\,y0\,y2, x1^2\,y0\,y3, x1^2\,y1^2, x1^2\,y1\,y2, x1^2\,y1\,y3, x1^2\,y2^2, x1^2\,y2\,y3, x1^2\,y3^2, x1\,x2^3, x1\,x2^2\,x3, x1\,x2^2\,y0, x1\,x2^2\,y1, x1\,x2^2\,y2, x1\,x2^2\,y3, x1\,x2\,x3^2, x1\,x2\,x3\,y0, x1\,x2\,x3\,y1, x1\,x2\,x3\,y2, x1\,x2\,x3\,y3, x1\,x2\,y0^2, x1\,x2\,y0\,y1,$ **(28)**

$$x1\,x2\,y0\,y2,\ x1\,x2\,y0\,y3,\ x1\,x2\,y1^2,\ x1\,x2\,y1\,y2,\ x1\,x2\,y1\,y3,\ x1\,x2\,y2^2,\ x1\,x2\,y2\,y3,\ x1\,x2\,y3^2,$$
$$x1\,x3^3,\ x1\,x3^2\,y0,\ x1\,x3^2\,y1,\ x1\,x3^2\,y2,\ x1\,x3^2\,y3,\ x1\,x3\,y0^2,\ x1\,x3\,y0\,y1,\ x1\,x3\,y0\,y2,$$
$$x1\,x3\,y0\,y3,\ x1\,x3\,y1^2,\ x1\,x3\,y1\,y2,\ x1\,x3\,y1\,y3,\ x1\,x3\,y2^2,\ x1\,x3\,y2\,y3,\ x1\,x3\,y3^2,\ x1\,y0^3,$$
$$x1\,y0^2\,y1,\ x1\,y0^2\,y2,\ x1\,y0^2\,y3,\ x1\,y0\,y1^2,\ x1\,y0\,y1\,y2,\ x1\,y0\,y1\,y3,\ x1\,y0\,y2^2,\ x1\,y0\,y2\,y3,$$
$$x1\,y0\,y3^2,\ x1\,y1^3,\ x1\,y1^2\,y2,\ x1\,y1^2\,y3,\ x1\,y1\,y2^2,\ x1\,y1\,y2\,y3,\ x1\,y1\,y3^2,\ x1\,y2^3,\ x1\,y2^2\,y3,$$
$$x1\,y2\,y3^2,\ x1\,y3^3,\ x2^4,\ x2^3\,x3,\ x2^3\,y0,\ x2^3\,y1,\ x2^3\,y2,\ x2^3\,y3,\ x2^2\,x3^2,\ x2^2\,x3\,y0,\ x2^2\,x3\,y1,$$
$$x2^2\,x3\,y2,\ x2^2\,x3\,y3,\ x2^2\,y0^2,\ x2^2\,y0\,y1,\ x2^2\,y0\,y2,\ x2^2\,y0\,y3,\ x2^2\,y1^2,\ x2^2\,y1\,y2,\ x2^2\,y1\,y3,$$
$$x2^2\,y2^2,\ x2^2\,y2\,y3,\ x2^2\,y3^2,\ x2\,x3^3,\ x2\,x3^2\,y0,\ x2\,x3^2\,y1,\ x2\,x3^2\,y2,\ x2\,x3^2\,y3,\ x2\,x3\,y0^2,$$
$$x2\,x3\,y0\,y1,\ x2\,x3\,y0\,y2,\ x2\,x3\,y0\,y3,\ x2\,x3\,y1^2,\ x2\,x3\,y1\,y2,\ x2\,x3\,y1\,y3,\ x2\,x3\,y2^2,$$
$$x2\,x3\,y2\,y3,\ x2\,x3\,y3^2,\ x2\,y0^3,\ x2\,y0^2\,y1,\ x2\,y0^2\,y2,\ x2\,y0^2\,y3,\ x2\,y0\,y1^2,\ x2\,y0\,y1\,y2,$$
$$x2\,y0\,y1\,y3,\ x2\,y0\,y2^2,\ x2\,y0\,y2\,y3,\ x2\,y0\,y3^2,\ x2\,y1^3,\ x2\,y1^2\,y2,\ x2\,y1^2\,y3,\ x2\,y1\,y2^2,$$
$$x2\,y1\,y2\,y3,\ x2\,y1\,y3^2,\ x2\,y2^3,\ x2\,y2^2\,y3,\ x2\,y2\,y3^2,\ x2\,y3^3,\ x3^4,\ x3^3\,y0,\ x3^3\,y1,\ x3^3\,y2,\ x3^3\,y3,$$
$$x3^2\,y0^2,\ x3^2\,y0\,y1,\ x3^2\,y0\,y2,\ x3^2\,y0\,y3,\ x3^2\,y1^2,\ x3^2\,y1\,y2,\ x3^2\,y1\,y3,\ x3^2\,y2^2,\ x3^2\,y2\,y3,$$
$$x3^2\,y3^2,\ x3\,y0^3,\ x3\,y0^2\,y1,\ x3\,y0^2\,y2,\ x3\,y0^2\,y3,\ x3\,y0\,y1^2,\ x3\,y0\,y1\,y2,\ x3\,y0\,y1\,y3,\ x3\,y0\,y2^2,$$
$$x3\,y0\,y2\,y3,\ x3\,y0\,y3^2,\ x3\,y1^3,\ x3\,y1^2\,y2,\ x3\,y1^2\,y3,\ x3\,y1\,y2^2,\ x3\,y1\,y2\,y3,\ x3\,y1\,y3^2,\ x3\,y2^3,$$
$$x3\,y2^2\,y3,\ x3\,y2\,y3^2,\ x3\,y3^3,\ y0^4,\ y0^3\,y1,\ y0^3\,y2,\ y0^3\,y3,\ y0^2\,y1^2,\ y0^2\,y1\,y2,\ y0^2\,y1\,y3,\ y0^2\,y2^2,$$
$$y0^2\,y2\,y3,\ y0^2\,y3^2,\ y0\,y1^3,\ y0\,y1^2\,y2,\ y0\,y1^2\,y3,\ y0\,y1\,y2^2,\ y0\,y1\,y2\,y3,\ y0\,y1\,y3^2,\ y0\,y2^3,$$
$$y0\,y2^2\,y3,\ y0\,y2\,y3^2,\ y0\,y3^3,\ y1^4,\ y1^3\,y2,\ y1^3\,y3,\ y1^2\,y2^2,\ y1^2\,y2\,y3,\ y1^2\,y3^2,\ y1\,y2^3,\ y1\,y2^2\,y3,$$
$$y1\,y2\,y3^2,\ y1\,y3^3,\ y2^4,\ y2^3\,y3,\ y2^2\,y3^2,\ y2\,y3^3,\ y3^4$$

```
> # We compute all the restrictions to phi: A4 -> R given
  # by the five polynomials. There are 20 restrictions for each polynomial

  pList := [p1ss, p2ss, p3ss, g1, g2, g3, g4, h];
  MXY := [ ]:
  for j from 1 to nops(pList) do
   for i from 1 to nops([ctdXY]) do
    p1t := expand(pList[j] * ctdXY[i]);
    vec := LinearAlgebra[Transpose](getCoeffs(p1t, [ct2dXY]));
    if (nops(MXY) = 0) then
     MXY := ⟨vec⟩;
    else
     MXY := ⟨MXY, vec⟩;
    end if;
   end do:
  end do:
```

$$pList := \Big[-4\,x0^2\,\mathrm{RootOf}(\_Z^3-2)^2+4\,x0^2\,\mathrm{RootOf}(\_Z^3-2)-2\,x0^2+x1^2-2\,x1\,x2 \qquad \textbf{(29)}$$
$$+2\,x2\,x3-2\,x3^2,\ -4\,x0^2\,\mathrm{RootOf}(\_Z^3-2)^2-4\,x0^2\,\mathrm{RootOf}(\_Z^3-2)+6\,x0^2+x1^2$$
$$+2\,x1\,x2+2\,x2\,x3+2\,x3^2,\ 4\,\big(\mathrm{RootOf}(\_Z^3-2)^2\,x3+\mathrm{RootOf}(\_Z^3-2)\,x1+x2\big)\,x0,$$
$$y0^2-y3^2,\ y1^2-y3^2,\ y2^2-y3^2,\ -y0^2-y0\,y1-y0\,y2+y0\,y3-y1\,y2+y1\,y3+y2\,y3,\ -x2^2$$

$+ y2^2]$

```
> # We solve the system using only these restrictions
rc := [Dimension(MXY) ];
nr := rc[1];
B := Vector(nr) :
sXY := LinearAlgebra[LinearSolve](MXY, B) :
varssXY := indets(sXY);
nops(varssXY); # 70 indeterminates left to solve
```

$$rc := [288, 330]$$

$$nr := 288$$

$varssXY := \{\_t1_{2}, \_t1_{5}, \_t1_{6}, \_t1_{7}, \_t1_{8}, \_t1_{11}, \_t1_{12}, \_t1_{13}, \_t1_{14}, \_t1_{15}, \_t1_{22}, \_t1_{23}, \_t1_{24}, \_t1_{25}, \_t1_{26}, \_t1_{29}, \_t1_{30}, \_t1_{32}, \_t1_{33}, \_t1_{35}, \_t1_{52}, \_t1_{53}, \_t1_{54}, \_t1_{57}, \_t1_{58}, \_t1_{60}, \_t1_{61}, \_t1_{63}, \_t1_{86}, \_t1_{93}, \_t1_{94}, \_t1_{96}, \_t1_{97}, \_t1_{99}, \_t1_{115}, \_t1_{177}, \_t1_{178}, \_t1_{180}, \_t1_{181}, \_t1_{183}, \_t1_{199}, \_t1_{226}, \_t1_{228}, \_t1_{229}, \_t1_{230}, \_t1_{233}, \_t1_{234}, \_t1_{236}, \_t1_{237}, \_t1_{239}, \_t1_{255}, \_t1_{256}, \_t1_{259}, \_t1_{260}, \_t1_{261}, \_t1_{262}, \_t1_{263}, \_t1_{264}, \_t1_{265}, \_t1_{268}, \_t1_{269}, \_t1_{271}, \_t1_{272}, \_t1_{274}, \_t1_{275}, \_t1_{290}, \_t1_{291}, \_t1_{294}, \_t1_{295}, \_t1_{330}\}$

$$70 \tag{30}$$

```
> # We give values to the unknowns so that the form is PSD
qIndXY := [seq(qXY[i], i = 1 ..nops([ctdXY]) ) ] :
psaXY := add(qIndXY[i] * ctdXY[i], i = 1 ..nops([ctdXY]) ) :
ps2XY := expand(psaXY* psaXY) :
aaXY := getCoeffs(expand(ps2XY), [ct2dXY]) :
```

```
> # We copy Q_x and Q_y in Q_xy
s1 := solve(Equate(ooMXY[1 ..10, 1 ..10], oEvalX) ) :
oEvalXY := eval(ooMXY, {ooMXY[20, 20] = 6}) :
oEvalXY := eval(oEvalXY, s1) :
```

```
> # We replace all the remaining variables by 0
s2 := solve(Equate([op(indets(oEvalXY) ) ], ZeroVector(nops(indets(oEvalXY) ) ) ) ) :
oEvalXY := eval(oEvalXY, s2) :
```

```
> # We verify that the matrix is positive semidefinite and has kernel
# of rank 14.
evalf(Eigenvalues(oEvalXY) );
```

$$\begin{bmatrix} 0.275755770741317 \\ 0.600977953313586 \\ 7.01383053094510 \\ 0.152195701 \\ 1.72628036257243 \\ 6.66336623908240 \\ 7.83746836150327 \\ 31.1104918868419 \\ 0.275620309502316 \\ 0.589700594592797 \\ 3.01830865794753 \\ 7.00693469295736 \\ 1.807859692 \\ 0.551105092943717 \\ 1.16935580906050 \\ 7.06066760599579 \\ 0.076097850 \\ 0.076097850 \\ 7. \\ 7. \\ 7. \\ 7. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \\ 0. \end{bmatrix} \tag{31}$$

```
> # The kernel is generated by 14 polynomials
  L := NullSpace(oEvalXY) :
  nops(L);
```

$$14 \tag{32}$$

```
> # The polynoials in the kernel
  simplify(L[01]) . convert(mdXY, Vector);
  simplify(L[02]) . convert(mdXY, Vector);
  simplify(L[03]) . convert(mdXY, Vector);
  simplify(L[04]) . convert(mdXY, Vector);
  simplify(L[05]) . convert(mdXY, Vector);
  simplify(L[06]) . convert(mdXY, Vector);
  simplify(L[07]) . convert(mdXY, Vector);
  simplify(L[08]) . convert(mdXY, Vector);
  simplify(L[09]) . convert(mdXY, Vector);
  simplify(L[10]) . convert(mdXY, Vector);
  simplify(L[11]) . convert(mdXY, Vector);
  simplify(L[12]) . convert(mdXY, Vector);
  simplify(L[13]) . convert(mdXY, Vector);
  simplify(L[14]) . convert(mdXY, Vector);
```

$$-x2^2 + y3^2$$

$$-x2^2 - y0\,y1 - y0\,y2 + y0\,y3 - y1\,y2 + y1\,y3 + y2\,y3$$

$$-x2^2 + y2^2$$

$$-x2^2 + y1^2$$

$$-x2^2 + y0^2$$

$$\left(-\mathit{RootOf}\left(\_Z^3 - 2\right) + 2\right) x0^2 - \frac{1}{2}\left(\left(2 + \mathit{RootOf}\left(\_Z^3 - 2\right)\mathit{RootOf}\left(\_Z\,\mathit{RootOf}\left(\_Z^3 - 2\right)^2 + \_Z^2 - \mathit{RootOf}\left(\_Z^3 - 2\right)^2 + 1\right) - 2\,\mathit{RootOf}\left(\_Z^3 - 2\right)^2\right) x1^2\right) \Big/ \left(\mathit{RootOf}\left(\_Z^3 - 2\right)^2 + \mathit{RootOf}\left(\_Z\,\mathit{RootOf}\left(\_Z^3 - 2\right)^2 + \_Z^2 - \mathit{RootOf}\left(\_Z^3 - 2\right)^2 + 1\right) - 2\,\mathit{RootOf}\left(\_Z^3 - 2\right)\right) + x3^2$$

$$\left(-2\,\mathit{RootOf}\left(\_Z^3 - 2\right)^2 + 1\right) x0^2 + \frac{1}{2}\,x1^2 + x2\,x3$$

$$\left(\mathit{RootOf}\left(\_Z^3 - 2\right)^2 + 2\,\mathit{RootOf}\left(\_Z^3 - 2\right) + 2\,\mathit{RootOf}\left(\_Z\,\mathit{RootOf}\left(\_Z^3 - 2\right)^2 + \_Z^2 - \mathit{RootOf}\left(\_Z^3 - 2\right)^2 + 1\right)\right) x0^2 - \frac{1}{2}\left(\left(\mathit{RootOf}\left(\_Z^3 - 2\right)^2 \mathit{RootOf}\left(\_Z\,\mathit{RootOf}\left(\_Z^3 - 2\right)^2 + \_Z^2 - \mathit{RootOf}\left(\_Z^3 - 2\right)^2 + 1\right) - 4 + 2\,\mathit{RootOf}\left(\_Z^3 - 2\right)\right) x1^2\right) \Big/ \left(\mathit{RootOf}\left(\_Z^3 - 2\right)^2 + \mathit{RootOf}\left(\_Z\,\mathit{RootOf}\left(\_Z^3 - 2\right)^2 + \_Z^2 - \mathit{RootOf}\left(\_Z^3 - 2\right)^2 + 1\right) - 2\,\mathit{RootOf}\left(\_Z^3 - 2\right)\right) + x1\,x3$$

$$-x0^2\,\mathit{RootOf}\left(\_Z^3 - 2\right) + \frac{1}{2}\left(\left(2 + \mathit{RootOf}\left(\_Z^3 - 2\right)\mathit{RootOf}\left(\_Z\,\mathit{RootOf}\left(\_Z^3 - 2\right)^2 + \_Z^2\right.\right.\right.$$

$$- \mathrm{RootOf}(\_Z^3-2)^2+1\big) - 2\,\mathrm{RootOf}(\_Z^3-2)^2\big)\,x1^2\big) \big/ \big(\mathrm{RootOf}(\_Z^3-2)^2 + \mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big) - 2\,\mathrm{RootOf}(\_Z^3-2)\big) + x1\,x2$$

$$-\big(\big(\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big) + \mathrm{RootOf}(\_Z^3-2)\big)\,x1\,x0\big) \big/ \big(\mathrm{RootOf}(\_Z^3-2)^2\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big) - 2\,\mathrm{RootOf}(\_Z^3-2)\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big) - 2\,\mathrm{RootOf}(\_Z^3-2)^2+1\big) + x3\,x0$$

$$-\big(\big(\mathrm{RootOf}(\_Z^3-2)^2\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big) + 2 - 2\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big) - \mathrm{RootOf}(\_Z^3-2)\big)\,x1\,x0\big) \big/ \big(\mathrm{RootOf}(\_Z^3-2)^2\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big) - 2\,\mathrm{RootOf}(\_Z^3-2)\,\mathrm{RootOf}\big(\_Z\,\mathrm{RootOf}(\_Z^3-2)^2+\_Z^2-\mathrm{RootOf}(\_Z^3-2)^2+1\big) - 2\,\mathrm{RootOf}(\_Z^3-2)^2+1\big) + x0\,x2$$

$$x3\,y0 + x3\,y1$$

$$x1\,y0 + x1\,y1$$

$$x0\,y0 + x0\,y1 \qquad \mathbf{(33)}$$

> *# There is no polynomails in the kernel with rational coefficients that has non-zero coefficient in $x0^2$.*

>