Parallel implementation of the ellipsoids method for optimization problems of large dimension

Bezborodov V.A.

Scientific Director:
candidate of physical and mathematical Sciences,
associate Professor
Golodov V.A.

Chelyabinsk, 2016

Table of contents

The purposes

The purposes of the work are:

1. to develop parallel implementation of the ellipsoids method that support arbitrary precision arithmetic;
2. to use developed implementation for finding solution of optimization problem of large dimension.

The objectives

The objectives are:

- to examine computational complexity of the algorithm;
- to develop a software implementation of the algorithm where the most time-consuming operations are parallel;
- to provide support for the arbitrary precision arithmetic;
- to demonstrate how to use developed software for solving optimization problems of large dimension;
- to inspect and test the code.

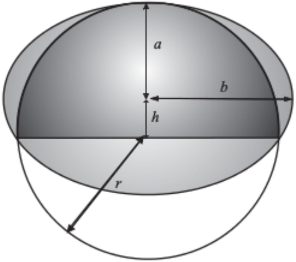
Table of contents

EM proposed

- in 1976 by **Iudin and Nemirovsky** as a method of centered cross-sections.
- in 1977 by **Shor** as a method with space dilation in the direction of the subgradient.
- in 1979 by **Khachiyan** as a polynomial algorithm for linear programming.

Table of contents

Properties of the 1-d ellipsoid



Ellipsoid ε_n, containing the bubble shape in E^n, has parameters

$$b = \left(\alpha + \frac{1}{\alpha}\right)\frac{r}{2}, \quad h = \left(1 - \frac{1}{\alpha^2}\right)\frac{r}{2},$$

where $\alpha = \dfrac{b}{a}$ and r is the radius of the ball S_n.

If the space «to stretch» with factor α in the direction of the semiaxes a, the ellipsoid ε_n will be a ball in the new space.

The ratio of the volume of the ellipsoid ε_n to the volume of a ball S_n is

$$q(n) = \frac{vol(\varepsilon_n)}{vol(S_n)} = \frac{1}{\alpha}\left(\frac{b}{r}\right)^n = \frac{1}{\alpha}\left(\frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)\right)^n.$$

Table of contents

Optimization in Economics

Applied convex programming problems:

- production problem;
- consumption problem;
- diet problem;
- nomenclature planning problem;
- transportation problem;
- traveling salesman problem.

Table of contents

EM algorithm

Choose $x_k := x_0 \in E^n$ and radius R, such that $||x_0 - x^*|| \leq R$. Set $h_k = \frac{R}{n+1}$, $B_k := E$, where E – identity matrix. To move to $(k+1)$-st iteration do:

Step 1. Calculate $g(x_k)$. If $g(x_k) = 0$, then **STOP**$(x^* = x_k)$.

Step 2. Calculate current point $x_{k+1} = x_k - h_k B_k \xi_k$, where

$$\xi_k = \frac{B_k^T g(x_k)}{||B_k^T g(x_k)||}.$$

Step 3. Calculate step $h_{k+1} = h_k r$ and matrix B_{k+1}

$$B_{k+1} = B_k + (\beta - 1)(B_k \xi_k)\xi_k^T, \quad \beta = \sqrt{\frac{n-1}{n+1}}.$$

Step 4. Move to $(k+1)$-st iteration with x_{k+1}, h_{k+1} and B_{k+1}.

The convergence

Theorem (The rate of convergence)

For all iterations of the EM factor reducing the volume of the ellipsoid, confining x^*, is constant and equal to

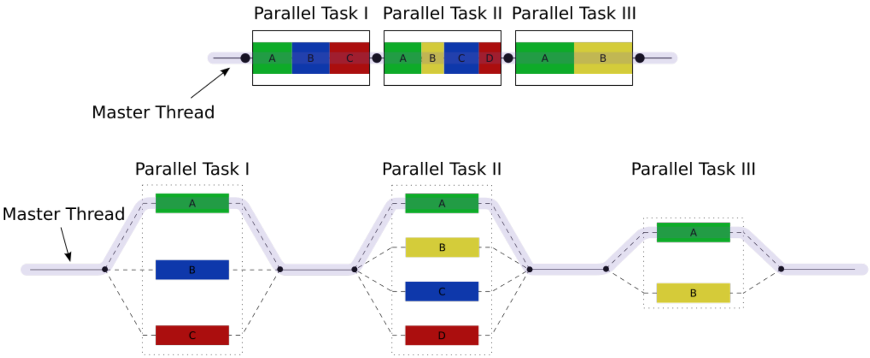
$$q(n) = \frac{vol(\varepsilon_{k+1})}{vol(\varepsilon_k)} = \frac{1}{\alpha}\left(\frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right)\right)^n < 1, \quad k = 0, 1, 2, \ldots$$

Optimal space dilation factor

$$\alpha = \sqrt{\frac{n+1}{n-1}} \Rightarrow q(n) = \sqrt{\frac{n-1}{n+1}}\left(\frac{n}{\sqrt{n^2-1}}\right)^n < 1.$$

Table of contents

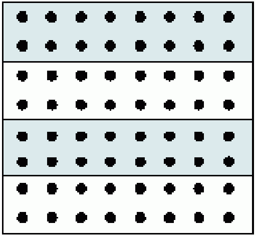
Fork–Join model



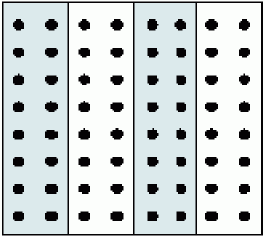
Matrix decomposition

The acceleration matrix operations

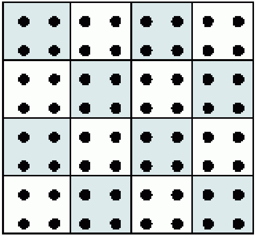
A subset of the matrix is allocated for each thread for processing. A subset type is defined by the partition.



Horizontal

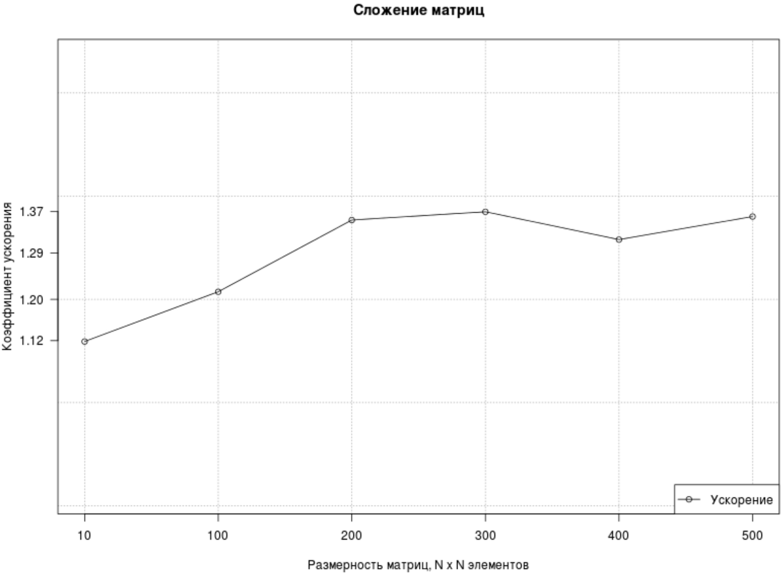


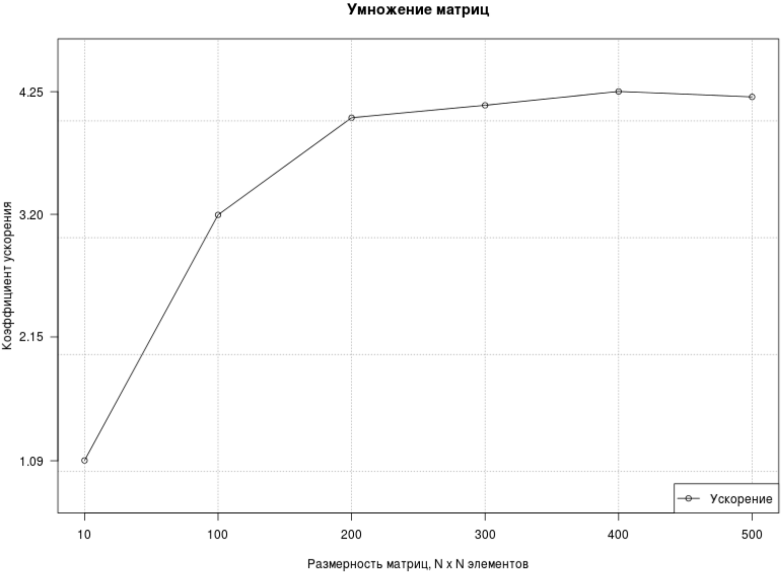
Vertical



Modular

Table of contents


Сложение матриц
Коэффициент ускорения
1.37
1.29
1.20
1.12
10
100
200
300
400
500
Размерность матриц, N x N элементов
Ускорение


Умножение матриц
Коэффициент ускорения
4.25
3.20
2.15
1.09
10
100
200
300
400
500
Размерность матриц, N x N элементов
Ускорение

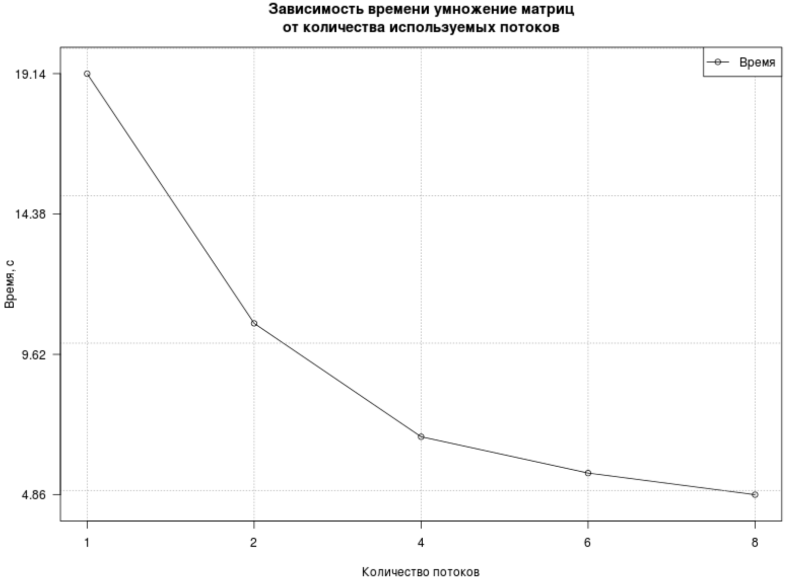

Зависимость времени умножение матриц
от количества используемых потоков
Время
Время, с
19.14
14.38
9.62
4.86
1
2
4
6
8
Количество потоков

Table of contents

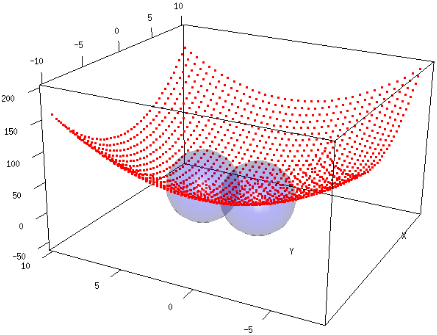
Example 1

The minimization problem

$$x_1^2 + (x_2 - 2)^2 \to \min .$$

Restrictions:

$$\begin{cases} x_1^2 + x_2^2 - 9; \\ x_1^2 + (x_2 - 4)^2 - 9. \end{cases}$$



Optimum

$$x^* = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

EM: 160 iterations, accuracy – up to 9 decimal places

$$x^* = \begin{pmatrix} 0.000000000011922745523 \\ 2.000000000033459867651 \end{pmatrix}$$

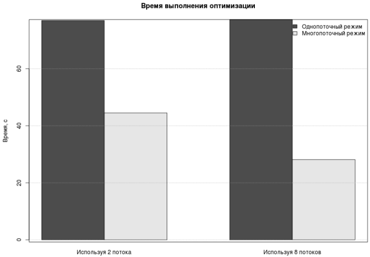
Example 2

The problem of large dimension

$$f_0 = \sum_{i=1}^{n} x_i^2 \to \min .$$

Restrictions:

$$f_m = \sum_{i=1, i \neq m}^{n} x_i^2 + (x_m - \alpha/2)^2 - \alpha^2 .$$



For $n = 100$, $m = \overline{1, n}$, $\alpha = 1$

The solution was found in 403 iterations, accuracy – up to 9 decimal places.

1 thread: 78 s. Time rate:

2 threads: $T_2 = 44.89$ s.
8 threads: $T_8 = 28.14$ s.

Conclusion

The following **problems** solved:

- the computational complexity of the EM algorithm is examined;
- the software implementation of the EM algorithm is developed; the most time-consuming operations are parallel;
- the support for arbitrary precision arithmetic is provided;
- the developed software is used to find solution for optimization problem of large dimension;
- the developed software is inspected and tested.

Questions?

Ellipsoids method