The Lebesgue Integral

Svyatoslav Glazyrin

Department of Mathematics
Simon Fraser University

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Outline

Introduction

- Riemann integral (1954)
- Shortcomings:
  1. small class of integrable functions (assumptions!)
  2. lack of nice limit properties
  3. other analysis problems
- Alternative integration theory by H.L. Lebesgue (1902, doctoral thesis at Sarbonne)

General Idea

- Intuitive way: counting rectangles!
- Partitioning the domain $\rightarrow$ partition the range
- Question: how to "measure" the domain?

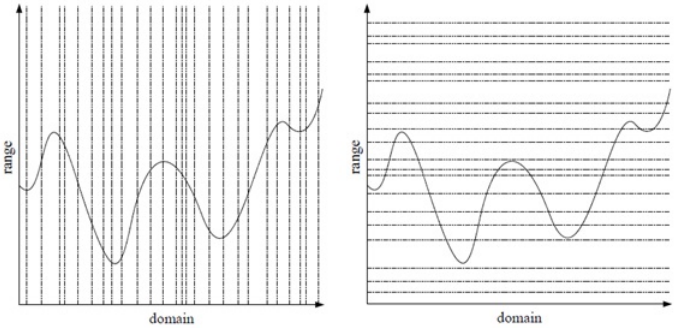


Figure: Henri Léon Lebesgue 1875-1941

Intro to Measure theory

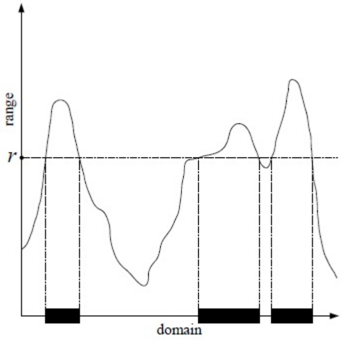
- E $= [a, b]$ and set S of subsets of E with σ-algebra on it (nonempty collection S of subsets of E that is closed under the complement and countable unions of its members and contains S itself).
- Function $\mu : S \to \Re$ is called a measure if the following properties hold: Semi-Positive-Definite, Trivial case, monotonicity, countable additivity.

Measurable sets

- The outer measure of any interval I on the real number line with endpoints $a < b$ is $b - a$ and is denoted as $m^*(I)$. extension to any subset!
- The inner measure of any set $A \subseteq E$, denoted $m_*(A)$, is defined as $m^*(E) - m^*(E/A)$.
- set $A \subseteq E$ is Lebesgue measurable, if $m_*(A) = m^*(A)$, in which case the measure of A is denoted by $m(A)$ and is given by $m(A) = m_*(A) = m^*(A)$.
- The measure for an unbounded set A is defined as $m(A) = \lim_{n \to +\infty}(A \cap [-n, n])$.

Measurable functions

- Let A be a bounded measurable subset of $\Re$ and $f : A \rightarrow \Re$. Then f is said to be measurable on A if $\{x \in A | f(x) > r\}$ is measurable set for $\forall r \in \Re$.



More of Measurable functions

- Step function $\rightarrow$ simple function
- Simple function $f : A \rightarrow \Re$ is a measurable function which takes on finitely many values.
- Theorem: A function $f : A \rightarrow \Re$ is measurable if and only if it is the pointwise limit of a sequence of simple functions.

Integrating Bounded Measurable Functions

- Let $f : A \to \Re$ be a bounded measurable function on a bounded measurable subset A of $\Re$. Let $l = glb\{f(x) \mid x \in A\}$ and $u > lub\{f(x) \mid x \in A\}$.
- The Lebesgue sum of f with respect to a partition P $= \{y_0, ..., y_n\}$ of the interval $[l, u]$ is given as $L(f, P) = \sum_{i=1}^{n} y_i^* m(\{x \in A \mid y_{i-1} \leq f(x) < y_i\})$
- A bounded measurable function $f : A \to \Re$, where A is a ,bounded measurable set, is Lebesgue integrable on A if there is a number $L \in \Re$ such that, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|L(f, P) - L| < \epsilon$ whenever $||P|| < \delta$. L is known as the Lebesgue integral of f on A and is denoted by $\int_A f(x)\, dm$.

Criteria for Integrability

- A bounded measurable function f is Lebesgue integrable on a bounded measurable set A if and only if, given $\epsilon > 0$, there exist simple functions f_l and f_u such that $f_l \leq f \leq f_u$ and $\int_A f_u\, dm - \int_A f_l\, dm < \epsilon$.
- Furthermore, $\int_A f\, dm = lub\{\int_A f_l\, dm \mid f_l$ is simple and $f_l \leq f\} = glb\{\int_A f_u\, dm \mid f_u$ is simple and $f \leq f_u\}$
- Ready to integrate!

Lebesgue integral of the Dirichlet function

- Dirichlet function is measurable and (obviously!) bounded $\Rightarrow$ Lebesgue integrable. Claim: $\int_{[0,1]} D\, dm = 0$.
- Proof by definition, i.e.show that, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|L(D, P)| < \epsilon$ whenever $||P|| < \delta$. Hint: $\delta = \epsilon/3$ works as a charm!

Pro et contra

- Advantages:
  1. works for wider class of functions (less strict assumptions)
  2. Rieman integrability $\rightarrow$ Lebesgue integrability
  3. Monotone convergency works
  4. allows to integrate over various structures
- Drawbacks:
  1. some functions are not Lebesgue integrable
  2. some improper Riemann integrals exist for functions that are not Lebesgue integrable

Thank you!