On Lebesgue Integration

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1 Introduction

1.1 Motivation

There are several ways of formally defining an integral. The most commonly used one is the Riemann integral, which is defined in terms of Riemann sums of functions with respect to tagged partitions of an interval in $\mathbb{R}$. The restricted domain of the function is the first major limitation of this definition.

Also, the Riemann integral is actually a modification of Cauchy's integral, which was crafted specifically to work on continuous functions, so the Riemann integral is heavily dependent on the continuity of the integrand.

Finally, the Riemann integral does not have nice limit properties, i.e. given a sequence of Riemann integrable functions f_n with the limit function $f = \lim_{n\to\infty} f_n$, f does not necessary need to be Riemann integrable.

All these flaws (and many others) were successfully resolved by Henri Léon Lebesgue, who presented an alternative, more efficient definition of the integral in his dissertation " Intégrale, longueur, aire" at the University of Nancy in 1902. (Complete Dictionary of Scientific Biography, 2008)

In this paper, we discuss a step-by-step construction of the Lebesgue integral, state important convergence theorems, and then show how it handles different situations where Riemann integral fails. But to start, we will discuss the development of the notion of the integral in time.

1.2 Historical background

In this section, we will examine different modifications and enrichments of the notion of the integral in time, closely following the story by Soo Bong Chai presented in the book Lebesgue Integration (1995).

Before Cauchy, no precise definition of the integral existed. The only thing one could say was which ares it was necessary to add or substract to obtain the intergal. Cauchy was the first to concern the rigor. He defined continuous functions and the integrals in almost the same way as we do it now, using the sums in the form $S = \sum f(\xi_i)(x_{i+1} - x_i)$ where $\xi_i \in [x_i, x_{i+1}]$, and passing it to the limit to get the intergal $\int_a^b f(x)dx$.

Although this definition seemed quite legitimate (at least, with respect to "adding areas" strategy), Cauchy had to prove that the sum S actually had a limit under some certain conditions. This was an extremely important step, as, for the first time, the experimental notion of the integral was transformed into the logical definition.

Note that before Cauchy, certain discontinuous functions were integrated; Cauchy's definition was still applicable here, but now it needed some extra investigation, with repsect to the formal definition, which is exactly what Riemann did.

He started by defining $\underline{S} = \sum \underline{f_i}(x_{i+1} - x_i)$ and $\overline{S} = \sum \overline{f_i}(x_{i+1} - x_i)$, where $\underline{f_i}$ and $\overline{f_i}$ are the lower and upper bounds of $f(x)$ on (x_i, x_{i+1}), respectively. Then, Riemann showed that the sufficient condition for Cauchy's definition to apply is that $\overline{S} - \underline{S} = \sum(\overline{f_i} - \underline{f_i})(x_{i+1} - x_i)$ tends toward zero, when the size of the partition of (a, b) gets smaller.

Darboux presented another condition: define the limits of $\overline{S}$ and $\underline{S}$ as $\overline{\int}_a^b f(x)\,dx$ and $\underline{\int}_a^b f(x)\,dx$, respectively; then, those integrals are equal only when the Cauchy-Riemann integral exists.

Indeed, decreasing the size of the partition of (a, b), i.e. making intervals (x_i, x_{i+1}) smaller, makes the differences $\overline{f_i} - \underline{f_i}$ smaller if $f(x)$ is continuous, so that $\overline{S} - \underline{S}$ tends towards zero even if there are some points of discontinuity. However, it's not the case for a function that is discontinuous everywhere.

Lebesgue used the same logic, but in alternative way: he proposed to partition the range of the function, rather than its domain, and then measure the area of the domain that corresponds to each interval in the partition. This approach turned out to be very powerful, as we will show later. However, "measuring" can be quite complicated, and that's where the measure theory steps in. This theory is a central concept used in the construction of the Lebesgue integral, and we will discuss its main ideas in the next section.

2 Intro to Measure Theory

2.1 Concept of Measure

Given an interval $E = [a, b]$ and a set S of subsets of E which is closed under countable unions, i.e. for all $(A_i)_{i\in\mathbb{N}}$ such that $A_i \in S$, $\bigcup_{i\in\mathbb{N}} A_i \in S$, we introduce the notion of measure as follows.

Definition Function $\mu : S \to \mathbb{R}$ is called a **measure** if the following properties hold:

- trivial case: $\mu(\emptyset) = 0$;
- μ is semipositive definite: $\mu(A) \in [0, b - a]$ for all $A \in S$;
- μ is monotonic: $\mu(A) \leq \mu(B)$ for all $A, B \in S : A \subset B$;
- μ preserves countable additivity: if $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in S$ for $n = 1, 2, \ldots$ and $A_n \cap A_m = \emptyset$ for $n \neq m$, then $\mu(A) = \sum_{n=1}^{\infty} A_n$.

Now, we start by defining the outer measure of an interval $I \subset \mathbb{R}$, and then extend it to more general cases of open set and arbitrary set, consequently.

Definition Given an interval $I \subset \mathbb{R}$ with endpoints $a < b$, define the **outer measure** of I as $m^*(I) = b - a$.

Recall, from set theory, that every non-empty open set $G \in \mathbb{R}$ can be uniquely expressed as a finite or countably infinite union of pairwise disjoint open intervals. Then, the definition of the outer measure can be extended to opensets in $\mathbb{R}$.

Definition Given an open set $G \subset \mathbb{R}$ and its (unique) decomposition $\{I_i\}$ into a finite or countably infinite union of pairwise disjoint open intervals, define the **outer measure** of G as $m^*(G) = \sum_i m^*(I_i)$.

Finally, we are fully set to define the outer measure for any set.

Definition Given an an arbitrary set $A \subset \mathbb{R}$, define the **outer measure** of A as $m^*(A) = \text{glb}(m^*(G)|A \in G, G$ is open in $\mathbb{R})$.

Now, we define another important concept, the inner measure, as follows.

Definition Given an arbitrary set $A \subset E$, define the **inner measure** of A as $m_*(A) = m^*(E) - m^*(E\backslash A)$, where $E\backslash A$ is the compliment of A with respect to E.

The inner and outer measures exhibit some nice properties, such as:

- for $A \subset B \subset E$, $m^*(A) \leq m^*(B)$ and $m_*(A) \leq m_*(B)$;
- for $A \subset E$, $m_*(A) \leq m^*(A)$;
- for $\{A_n\}_{n=1}^{\infty}$, $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$.

Note that the outer measure is not countably additive on some sets, e.g. set of all subsets of E (can be proved by construction). Fortunately, the property holds for a wide class of sets, which we call measurable and define in the beginning of the next subsetion.

2.2 Measurable sets

Definition An arbitrary set $A \subset E$ is **Lebesgue measurable**, or simply **measurable**, if $m_*(A) = m^*(A)$. In this case, the **measure** of A is defined as $m(A) = m_*(A) = m^*(A)$.

Measurable sets exhibit some usefull properties, such as:

- $A \subset E$ is measurable if and only if $E\backslash A$ is measurable;
- $A \subset E$ is measurable if and only if $m^*(A) + m^*(E\backslash A) = b - a$.

And now, we are ready to present the first notable result:

Theorem 2.1. *The outer measure is countably additive on the set of all measurable subsets of E.*

The proof of this statement is left for the enrichment of the reader. And now, we can proceed to the concept of measurable functions, which we define in the next subsection.

2.3 Measurable functions

Definition For $A \subset \mathbb{R}$, function $f : A \to \mathbb{R}$ is said to be **measurable on *A***, if $\{x \in A | f(x) > r\}$ is measurable for all $r \in \mathbb{R}$.

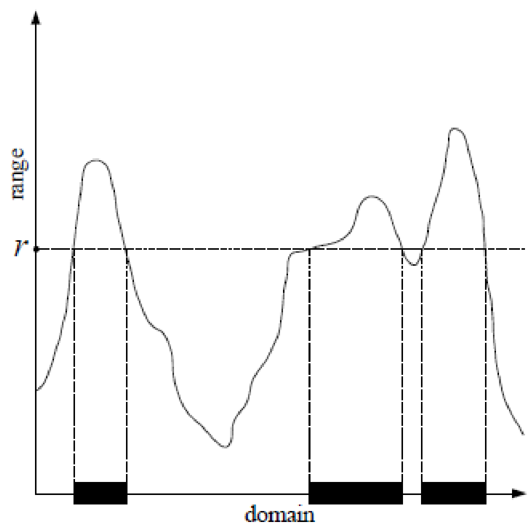

Figure 1: The shaded region must be measurable $\forall r \in \mathbb{R}$ for f to be measurable. (Lifton, 2004)

It turns out that measurable functions have a very nice limit property, which is formulated in the following lemma.

Lemma 2.2. *If a sequence $\{f_n\}$ of functions that are measurable on set A converges to f pointwise, then f is measurable on A.*

The last concept we need is a simple function, which is analogue of step function for Riemann integral.

Definition A measurable function $f : A \to \mathbb{R}$ is called **simple**, if it takes on finitely many values.

The following theorem (given w/o proof for brevity) describes an important connection between simple functions and measurability.

Theorem 2.3. *Function $f : A \to \mathbb{R}$ is measurable if and only if it is the pointwise limit of a sequence of simple functions.*

Example Is Dirichlet function $D(x)$ measurable?

Solution. The answer is yes, it is, because $D(X) = \lim_{n\to\infty} d_n(x)$, where $d_n(x)$ takes only values 0 and 1.

Now we are fully set to construct a Lebesgue integral!

3 Lebesgue Integral

We start with construction of Lebesgue integral for bounded measurable functions. The general case, i.e. Lebesgue integral of an arbitrary measurable functions, will not be discussed here. However, it's worth to mention that most theorems stated in this section have rather straight-forward analogues for the general case, by allowing the value of the integral to be infinite. (Whyburn, 1932)

3.1 Definition and properties

First, we define the Lebesgue sum, which is an analogue of the Riemann sum. Note that we partion the range, not the domain.

Definition Let $f : A \to \mathbb{R}$ be a bounded measurable function on a bounded measurable subset $A \subset \mathbb{R}$. Let $l = glb\{f(x) \mid x \in A\}$ and $u > lub\{f(x) \mid x \in A\}$. The **Lebesgue sum** of f with respect to partition $P = \{y_0, ..., y_n\}$ of the interval $[l, u]$ is defined as $L(f, P) = \sum_{i=1}^{n} y_i^* m(\{x \in A \mid y_{i-1} \leq f(x) < y_i\})$, where $y_i^* \in [y_{i-1}, y_i]$ for $i = 1, \dots, n$.

The straight-forward interpretation is virtually the same as for Riemann sum: we count rectangles; but here y_i^* serves as height and $m(\{x \in A \mid y_{i-1} \leq f(x) < y_i\}$ as width of the rectangle. Shrinking the mesh of the partition, we define the Lebesgue integral using standard $\delta - \epsilon$ notation:

Definition A bounded measurable function $f : A \to \mathbb{R}$, where A is a bounded measurable set, is **Lebesgue integrable** on A, if there is a number $L \in \mathbb{R}$ such that, given $\epsilon > 0$, there exists $\delta > 0$, such that $|L(f, P) - L| < \epsilon$, whenever $||P|| < \delta$. L is known as the **Lebesgue integral** of f on A and is denoted by $\int_A f(x)\, dm$.

Clearly, the Lebesgue integral have many common properties with Riemann integral, such as:

- *linearity*: for bounded measurable function f, g on a measurable set A $\int_A (f+g)dm = \int_A fdm + \int_A gdm$ and $\int_A c \cdot fdm = c \cdot \int_A fdm$ for $c \in \mathbb{R}$;
- *monotonicity*: for $f \leq g$, $\int_A fdm \leq \int_A gdm$;
- $|\int_A fdm| \leq \int_A |f|dm$;
- $\forall l, u \in \mathbb{R}$, such that $l \leq f \leq u$: $l \cdot m(A) \leq \int_A fdm \leq u \cdot m(A)$;
- for any two disjoint bounded measurable sets A, B and bounded measurable function $f : A \cup B \to \mathbb{R}$: $\int_{A\cup B} fdm = \int_A fdm + \int_B fdm$; moreover, the Lebesgue integral preserves *countable additivity*, i.e. for $A = \cup_{i=1}^{\infty} A_i$, where A_i are pairwise disjoint measurable sets: $\int_A fdm = \sum_{i=1}^{\infty} fdm$.

Another important property, which is worth to distinguish, is based on the definition of notion "almost everywhere":

Definition functions f and g are said to be equal **almost everywhere**, if $m(\{x \in E : f(x) \neq g(x)\}) = 0$ (E is often called a *set of measure* 0).

The property is, then, formulated as follows.

Theorem 3.1. *If f, g are measurable functions defined on a measurable set A, such that $f = g$ almost everywhere, then $\int_A fdm = \int_A gdm$.*

Therefore, we conclude that the Lebesgue integral does not distinguish between functions which differ only on a set of measure zero, e.g. any countable set.

Until now, two integrals seem to be very similar. The next subsection, however, reveals the first major difference.

3.2 Criterion for Integrability

Recall the integrability criterion for the Riemann integral: A function on a compact interval $[a, b]$ is Riemann integrable if and only if it is bounded and continuous almost everywhere (also known as Lebesgue's criterion for Riemann integrability). Now, we present the integrability criterion for the Lebesgue integral along with the extremely important corollary.

Theorem 3.2. *A bounded measurable function f is Lebesgue integrable on a bounded measurable set A if and only if, given $\epsilon > 0$, there exist simple functions f_l and f_u such that $f_l \leq f \leq f_u$, and $\int_A f_u \, dm - \int_A f_l \, dm < \epsilon$. Then, $\int_A f \, dm$ is the Lebesgue integral of f, and $\int_A f \, dm = \int_A f_u \, dm = \int_A f_l \, dm$.*

Corollary 3.3. *If f is a bounded measurable function on a bounded measurable set A, then f is Lebesgue integrable on A.*

Example Is Dirichlet function $D(x)$ Lebesgue integrable?

Solution. The answer is yes, it is, because $D(x)$ is a measurable function, as shown in section 2.3, it's obviously bounded, and its domain, $[0, 1]$ is measurable. Moreover, the are of the domain, on which $D(x)$ equals to 1, is a zero-measure set, so the integral $\int_{[0,1]} D(x) \, dm = 0$.

This is a moment of glory. The above example suggests that the class of Lebesgue integrable functions can be potentially wider than Riemann's.

3.3 Convergence Theorems

The following two theorems, in Lifton's formulation (2004), describe the fundamental convergence properties.

Theorem 3.4. ***Monotone Convergence Theorem*** *Let $A \subset \mathbb{R}$ be a bounded measurable set and $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of non-negative measurable functions on A with a pointwise limit f, such that $f_k(x) \leq f_{k+1}(x)$ for all $x \in A$ and for all $k \in \mathbb{N}$. Then f is Lebesgue integrable and $\lim\limits_{k \to \infty} \int_A f_k dm = \int_A f dm$.*

Theorem 3.5. ***Dominated Convergence Theorem*** *Let $A \subset \mathbb{R}$ be a bounded measurable set and $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of non-negative measurable functions on A with a pointwise limit f. If there exists a Lebesgue integrable function g, such that $|f_k(x)| \leq g(x)$ for all $x \in A$ and for all $k \in \mathbb{N}$, then f is Lebesgue integrable and $\lim\limits_{k \to \infty} \int_A f_k dm = \int_A f dm$.*

Recall that for Riemann integral the limit and integral are interchangeble only if the onvergency is uniform, i.e. given $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that $|f_n(x) - f(x)| < \epsilon$, whenever $n < N$, for all $x \in [a, b]$. For Lebesgue integral, though, the conditions are more weak, which suggests that the last one, indeed, has better limit properties!

The next result establishes the fundamental connection between the Riemann and Lebesgue integrals.

Theorem 3.6. *If f is Riemann integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$, and $\int_a^b f(x)dx = \int_{[a,b]} f dm$.*

In fact, there are plenty of examples which show that the reverse is not true, e.g. Dirichlet function. Thus, we conclude that the class of Lebesgue integrable functions is larger than the class of Riemann integrable functions.

Example Find the Lebesgue integral of $xcos(x)$ over the interval $[-1, 1]$.

Solution. Function $xcos(x)$ is continuous and bounded over the region $[-1, 1]$, so it is Riemann integrable. By the previous theorem, it is also Lebesgue integrable, and the values of these integrals are the same:
$\int_{[-1,1]} xcos(x)\, dm = \int_{-1}^{1} xcos(x)\, dx = 1{\cdot}sin(1) - (-1){\cdot}sin(-1) - \int_{-1}^{1} sin(x)\, dx =$
$= sin(1) + sin(-1) + cos(1) - cos(-1) = 0.$

Example Find, if possible, the Riemann and Lebesgue integrals of the constant function over the Cantor set.

Solution. Recall that the Cantor set C is created by repeatedly deleting the open middle thirds of a set of line segments, starting from the interval $[0, 1]$. One can easily prove that this is a zero-measure set (idea: C s an intersection of the sequence of collections of disjoint closed intervals, its measure $m(C)$ is a limit of measures of these collection: $m(C) = \lim\limits_{n\to\infty} = (\frac{2}{3})^n = 0$). Then, the Lebesgue integral is equal to 0, and the Riemann integral is undefined, because the Cantor set is not an interval.

4 Conclusion

In the previous section, we have shown that the Lebesgue integral possesses some important limit properties, which the Riemann integral fails to satisfy. This makes the Lebesgue integral more suitable for more advanced analysis. Also, we have shown that the class of Lebesgue integrable functions is strictly larger than Riemann's.

Note that the core structure of Lebesgue integration, namely measure, can be defined in various ways and is not restricted to the Euclidian space, while the Riemann integral can only be defined on the real line.

Finally, there are some other useful features of the Lebesgue integral than the Riemann integral does not have, such as completness of L_p spaces (Franks, 2009); these features are not discussed in this paper, but can be found in the referenced sources.

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