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Aufgabe 1) $\varphi: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

a) Sei $\varphi(x) = \varphi(y)$

$$\Longleftrightarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} x_1 + 2x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 + 2y_2 \\ y_2 \end{pmatrix}$$

$$\Rightarrow \boxed{x_2 = y_2} \Longleftrightarrow 2x_2 = 2y_2$$

$$x_1 + 2x_2 = y_1 + 2y_2 \Rightarrow \boxed{x_1 = y_1}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \varphi \text{ ist injektiv } \blacksquare$$

b) Sei $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$

$$\Rightarrow \begin{pmatrix} x_1 + 2x_2 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \Rightarrow \boxed{x_2 \in \mathbb{R}}$$

$$\Rightarrow x_1 + 2x_2 \text{ auch } \in \mathbb{R} \Rightarrow \boxed{x_1 \in \mathbb{R}}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \Rightarrow \varphi \text{ ist surjektiv } \blacksquare$$

Aufgabe 2) $A := \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{3\times 3}$

a) $AB = 0_{3\times 3}$

Sei $B := \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\Rightarrow AB = \begin{pmatrix} (a+2d+3g) & (b+2e+3h) & (c+2f+3i) \\ (a+2d+3g) & (b+2e+3h) & (c+2f+3i) \\ (4a+5d+6g) & (4b+5e+6h) & (4c+5f+6i) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} a+2d+3g &= 0 \\ a+2d+3g &= 0 \\ 4a+5d+6g &= 0 \end{aligned}\right\} \begin{aligned} a &= -2d-3g \\ a+2d+3g &= 0 \\ -8d-12g+5d+6g &= 0 \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} a &= -2d-3g \\ a+2d+3g &= 0 \\ -3d-6g &= 0 \end{aligned}\right\} \left.\begin{aligned} a &= -2d-3g \\ a+2d+3g &= 0 \\ d+2g &= 0 \end{aligned}\right\} \begin{aligned} a &= -2d-3g \\ a+2d+3g &= 0 \\ d &= -2g \end{aligned}$$

Sei $g = 2 \Rightarrow d = -4 \Rightarrow a = 8-6 = 2$

Sei $B := \begin{pmatrix} 2 & 0 & 0 \\ -4 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \neq 0_{3\times 3}$

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ -4 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} (2-8+6) & 0 & 0 \\ (2-8+6) & 0 & 0 \\ (8-20+12) & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 0 & 0 \\ -4 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \text{ ist eine Lösung}$$

b) $CA = 0_{3\times3}$

Sei $C := \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\Rightarrow CA = \begin{pmatrix} (a+b+4c) & (2a+2b+5c) & (3a+3b+6c) \\ (d+e+4f) & (2d+2e+5f) & (3d+3e+6f) \\ (g+h+4i) & (2g+2h+5i) & (3g+3h+6i) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{matrix} a+b+4c = 0 \\ 2a+2b+5c = 0 \\ 3a+3b+6c = 0 \end{matrix}\right\} \left.\begin{matrix} a = -b-4c \\ -2b-8c+2b+5c = 0 \\ -3b-12c+3b+6c = 0 \end{matrix}\right\} \begin{matrix} a = -b-4c \\ -3c = 0 \\ -6c = 0 \end{matrix}$$

$$\Rightarrow c = 0 \Rightarrow a = -b$$

Sei $C := \begin{pmatrix} 2 & (-2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0_{3\times3}$

$$CA = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} (2-2+0) & (4-4+0) & (6-6+0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & (-2) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ ist eine Lösung

Aufgabe 3)

$A := (a_{i,j})_{i\leq n, j\leq n}$ $A^T := (a'_{i,j})_{i\leq n, j\leq n}$ mit $a_{i,j} = a'_{j,i}$

a)

$$AA^T = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \begin{pmatrix} a'_{1,1} & \dots & a'_{1,n} \\ a'_{2,1} & \dots & a'_{2,n} \\ \vdots & & \vdots \\ a'_{n,1} & \dots & a'_{n,n} \end{pmatrix} =$$

$$= \begin{pmatrix} (a_{1,1}a'_{1,1} + \dots + a_{1,n}a'_{n,1}) & \dots & (a_{1,1}a'_{1,n} + \dots + a_{1,n}a'_{n,n}) \\ (a_{2,1}a'_{1,1} + \dots + a_{2,n}a'_{n,1}) & \dots & (a_{2,1}a'_{1,n} + \dots + a_{2,n}a'_{n,n}) \\ \vdots & & \vdots \\ (a_{n,1}a'_{1,1} + \dots + a_{n,n}a'_{n,1}) & \dots & (a_{n,1}a'_{1,n} + \dots + a_{n,n}a'_{n,n}) \end{pmatrix}$$

$$AA^T := (aa'_{i,j}) := \left(\sum_{k=1}^{n} a_{i,k}a'_{k,j}\right)_{i\leq n, j\leq n}$$

Dann $(AA^T)^T := (aa'_{j,i}) := \left(\sum_{k=1}^{n} a_{j,k}a'_{k,i}\right)_{i\leq n, j\leq n}$

Nach Definition:

$a_{i,k} = a'_{k,i}$ und $a'_{k,j} = a_{j,k}$

$$\Rightarrow AA^T := \left(\sum_{k=1}^{n} a_{i,k} a'_{k,j}\right) = \sum_{k=1}^{n} a'_{k,i} a_{j,k} = \sum_{k=1}^{n} a_{j,k} a'_{k,i} =: \left(AA^T\right)^T$$

$$\Rightarrow \boxed{AA^T \text{ ist symmetrisch}} \quad \blacksquare$$

Analog zu oben:

$$A^TA := \left(a'a_{i,j}\right) \qquad i \leq n, j \leq n$$

$$\left(A^TA\right)^T := \left(a'a_{j,i}\right) := \left(\sum_{k=1}^{n} a'_{j,k} a_{k,i}\right) \quad i \leq n, j \leq n$$

Da $a'_{i,k} = a_{k,i}$ und $a_{k,j} = a'_{j,k}$

$$\Rightarrow A^TA := \left(\sum_{k=1}^{n} a'_{i,k} a_{k,j}\right) = \sum_{k=1}^{n} a_{k,i} a'_{j,k} = \sum_{k=1}^{n} a'_{j,k} a_{k,i} =: \left(A^TA\right)^T$$

$$\Rightarrow \boxed{A^TA \text{ ist symmetrisch}} \quad \blacksquare$$

b) $A + A^T := \left(a_{i,j} + a'_{i,j}\right)_{i \leq n, j \leq n}$

$$\Rightarrow \left(A + A^T\right)^T := \left(a_{j,i} + a'_{j,i}\right)_{i \leq n, j \leq n}$$

Nach Definition:

$a_{i,j} = a'_{j,i}$ und $a'_{i,j} = a_{j,i}$

$\Rightarrow A + A^T := (a_{i,j} + a'_{i,j}) = a'_{j,i} + a_{j,i} = a_{j,i} + a'_{j,i} =: (A + A^T)^T$

$\Rightarrow A + A^T$ ist symmetrisch ∎

Aufgabe 4)

a) $\forall i \in I: U_i \subseteq \mathbb{R}^n$ UVR $\quad \underline{u \in \bigcap_{i\in I} U_i \Leftrightarrow u \in U_i \;\; \forall i \in I}$

Sei $i \in I$ beliebig

Da U_i UVR $\Rightarrow 0 \in U_i \xRightarrow{i \text{ beliebig}} 0 \in U_i \;\forall i \in I \xRightarrow{\text{Def. } \bigcap_{i\in I} U_i} \boxed{0 \in \bigcap_{i\in I} U_i}$

$u, v \in \bigcap_{i\in I} U_i \Leftrightarrow u, v \in U_i \;\; \forall i \in I$

U_i UVR $\Rightarrow (u, v \in U_i \Rightarrow (u+v) \in U_i) \xRightarrow{i \text{ beliebig}} (u, v \in U_i \Rightarrow (u+v) \in U_i) \;\forall i \in I$

$\xRightarrow{\text{Def. } \bigcap_{i\in I} U_i} \boxed{u, v \in \bigcap_{i\in I} U_i \Rightarrow (u+v) \in \bigcap_{i\in I} U_i}$

U_i UVR $\Rightarrow (u \in U_i, \lambda \in \mathbb{R} \Rightarrow \lambda u \in U_i)$

$\xRightarrow{i \text{ beliebig}} (u \in U_i, \lambda \in \mathbb{R} \Rightarrow \lambda u \in U_i) \;\; \forall i \in I$

$\xRightarrow{\text{Def. } \bigcap_{i\in I} U_i} \boxed{(u \in \bigcap_{i\in I} U_i, \lambda \in \mathbb{R} \Rightarrow \lambda u \in \bigcap_{i\in I} U_i)}$

Per Def. $\bigcap_{i\in I} U_i \subseteq U_i \;\forall i \in I \Rightarrow \bigcap_{i\in I} U_i$ UVR von $\mathbb{R}^n$ ∎

b) $U_1 := \left\{ \begin{pmatrix} u_1' \\ u_1'' \end{pmatrix} \middle| \; u_1' = 0 \right\}$

$U_2 := \left\{ \begin{pmatrix} u_2' \\ u_2'' \end{pmatrix} \middle| \; u_2'' = 0 \right\}$

Seien $u_1 \in U_1$ und $u_2 \in U_2 \Rightarrow u_1, u_2 \in U_1 \cup U_2$

$u_1 + u_2 = \begin{pmatrix} 0 \\ u_1'' \end{pmatrix} + \begin{pmatrix} u_2' \\ 0 \end{pmatrix} = \begin{pmatrix} u_2' \\ u_1'' \end{pmatrix}$

Seien $u_2', u_1'' \neq 0$, so gilt $u_1 + u_2 \notin U_1 \cup U_2$ ■