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A1)

a)

A	B	$\overbrace{A \wedge B}^{C}$	$\neg C$	$A \mid B$
w	w	w	f	f
w	f	f	w	w
f	w	f	w	w
f	f	f	w	w

■

b) z.z. $\neg A \Leftrightarrow A \mid A$

aus a): $\underline{A \mid A} = \underline{\neg(A \wedge A)}$

$A \wedge A = A \Rightarrow \underline{\neg(A \wedge A)} = \underline{\neg A}$ ■

c.1) $A \wedge B$

aus a): $\neg(A \wedge B) = A \mid B$ ~~$\Rightarrow A \wedge B = \neg(A \mid B)$~~

$\Rightarrow \neg(\neg(A \wedge B)) = \neg(A \mid B)$ nach b)

$\Rightarrow A \wedge B = \neg(A \mid B)$ ■ $\Rightarrow (A \mid B) \mid (A \mid B)$ ■

c.2) $A \vee B$

aus b): $A = \neg(A \mid A)$ UND $B = \neg(B \mid B)$

$\Rightarrow A \vee B = \neg(A \mid A) \vee \neg(B \mid B)$

de Morg $\Rightarrow A \vee B = \neg((A \mid A) \wedge (B \mid B))$ nach b) $= \neg((\neg A) \wedge (\neg B))$

nach a) $\Rightarrow \neg((\neg A) \wedge (\neg B)) = (\neg A) \mid (\neg B)$ nach b) $\Rightarrow (A \mid A) \mid (B \mid B)$ ■

c.3) $A \Rightarrow B$

$$A \Rightarrow B \overset{S.1.1.6\ iii)}{=} \neg(A \wedge (\neg B))$$

nach a): $\neg(A \wedge (\neg B)) = A \mid (\neg B)$

$$\Rightarrow A \Rightarrow B = A \mid (\neg B) \overset{\text{nach } b)}{\Rightarrow} A \mid (B \mid B)$$

c.4) $A \Leftrightarrow B$

Fall $A \Rightarrow B$ – Folgt aus c.3 ✓

Fall $B \Rightarrow A$:

$$B \Rightarrow A \overset{S.1.1.6.\ iii)}{=} \neg(B \wedge (\neg A))$$

nach a): $\neg(B \wedge (\neg A)) = B \mid (\neg A)$

$$\Rightarrow B \Rightarrow A = B \mid (\neg A) \overset{\text{nach } b)}{\Rightarrow} B \mid (A \mid A)$$ ✓

~~Def. für "$\Leftrightarrow$"~~

$\cancel{= (A \mid (B \mid B)) \wedge (B \mid (A \mid A))}$

$$A \Leftrightarrow B = (A \Rightarrow B) \wedge (B \Rightarrow A)$$

$$\overset{\text{nach } c.1}{\Rightarrow} ((A \Rightarrow B) \mid (B \Rightarrow A)) \mid ((A \Rightarrow B) \mid (B \Rightarrow A))$$

$$= ((A \mid (B \mid B)) \mid (B \mid (A \mid A))) \mid ((A \mid (B \mid B)) \mid (B \mid (A \mid A)))$$ ∎

A2) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

a) Sei $x \in ((A \setminus B) \cap (A \setminus C))$

$\Rightarrow \{ x \in (A \setminus B) \wedge x \in (A \setminus C) \}$

$\Downarrow \qquad \Downarrow$

$(x \in A \wedge x \notin B) \wedge ((x \in A) \wedge (x \notin C))$

$x \in A \wedge \underbrace{((x \notin B) \wedge (x \notin C))}_{\neg(x \in B \vee x \in C)}$ De Morgan

$\underbrace{\neg(x \in (B \cup C))}_{x \notin (B \cup C)}$

$x \in (A \setminus (B \cup C))$

b) $A \times \bigcap_{i \in I} M_i = \bigcap_{i \in I} (A \times M_i)$, $M_i \subseteq D$

$\cancel{\bigcap_{i \in I} M_i \quad A \times \bigcap_{i \in I} M_i = (x,y) \mid x \in A,\ y \in \bigcap_{i \in I} M_i}$

$\cancel{\{ y \in D \mid \forall i \in I : y \in M_i \}}$

$\cancel{(x,y) \mid x \in A,\ \forall i \in I : y \in M_i}$

$\forall i \in I : x \in (A \times M_i) \qquad (x,y) = z$

$\forall i \in I : (x,y) \mid x \in A,\ y \in M_i$

$\overset{(x,y)}{\Rightarrow} x \in A,\ \forall i \in I : y \in M_i \Rightarrow x \in A,\ y \in \bigcap_{i \in I} M_i$

$\Rightarrow z \in (A \times \bigcap_{i \in I} M_i)$

c) $\bigcup_{i\in I} \mathcal{P}(M_i) \subseteq \mathcal{P}\left(\bigcup_{i\in I} M_i\right)$

z.z. $N \in \bigcup_{i\in I} \mathcal{P}(M_i) \Rightarrow N \in \mathcal{P}\left(\bigcup_{i\in I} M_i\right)$

Annahme: $N \in \bigcup_{i\in I} \mathcal{P}(M_i)$

z.z. $N \in \mathcal{P}\left(\bigcup_{i\in I} M_i\right)$

$\exists i\in I : N \in \mathcal{P} M_i$

$N \subseteq \bigcup_{i\in I} M_i$

$\forall n: n\in N \Rightarrow n \in \bigcup_{i\in I} M_i$

$N \subseteq M_i$

~~$\forall i\in I: N\subseteq M_i$~~

$\exists i\in I$

~~$\forall n: n\in N \Rightarrow n\in M_i$~~

~~Annahme~~

$\forall n\in N \Rightarrow \exists i\in I: n\in M_i$

$\Rightarrow n\in \bigcup_{i\in I} M_i \Rightarrow N \subseteq \bigcup_{i\in I} M_i \Rightarrow N \in \mathcal{P}\left(\bigcup_{i\in I} M_i\right)$ ■

d) $\mathcal{P}\left(\bigcup_{i\in I} M_i\right) \subseteq \bigcup_{i\in I} \mathcal{P}(M_i)$?

z.z. $N\in \mathcal{P}\left(\bigcup_{i\in I} M_i\right) \Rightarrow N \in \bigcup_{i\in I}\mathcal{P}(M_i)$

Annahme: $N\in \mathcal{P}\left(\bigcup_{i\in I} M_i\right)$

$\forall n\, \exists i\in I: n\in N\in \mathcal{P}(M_i)$

$\forall n: n\in N \in \bigcup_{i\in I}\mathcal{P}(M_i)$

$N \subseteq \bigcup_{i\in I} M_i$

$\forall n: n\in N \Rightarrow n\in \bigcup_{i\in I} M_i$

$n\in \{x \mid \exists i\in I: x\in M_i\}$

$\forall n$ $\exists i\in I: n\in M_i$

$\exists i\in I: (n\in N \Rightarrow n\in M_i) \neq \forall n\, \exists i\in I: N\subseteq M_i$

falls inj: $\forall m_1, m_2 \in M \; (f(m_1) = f(m_2)) \Rightarrow m_1 = m_2$

A3)

a) $f_1: \mathbb{Z} \to \mathbb{Z}$
$z \mapsto 2z$

z.z. $(f_1(z_1) = f_1(z_2)) \Rightarrow (z_1 = z_2)$

Seien $z_1, z_2 \in \mathbb{Z}$

$f_1(z_1) = z_1/2$
$f_1(z_2) = z_2/2$

Annahme: $f_1(z_1) = f_1(z_2)$

$z_1/2 = z_2/2 \Rightarrow z_1 = z_2 \Rightarrow$ injektiv

z.z. $f_1(\mathbb{Z}) = \mathbb{Z}$

$f_1(\mathbb{Z}) = \{z' \in \mathbb{Z} \mid \exists z \in \mathbb{Z}: f_1(z) = z'\}$

$z' \in f_1(\mathbb{Z}) \Leftrightarrow z' \in \mathbb{Z}$

Annahme:

Fall "$\Rightarrow$"

$z' \in f_1(\mathbb{Z})$

~~$\Rightarrow \exists z \in \mathbb{Z}: f_1(z) = z'$~~

$f_1(\mathbb{Z}) \subseteq \mathbb{Z} \Rightarrow z' \in \mathbb{Z}$ (per Definition)

Fall "$\Leftarrow$"

Annahme:

$z' \in \mathbb{Z}$

$z/2 \in \mathbb{Z}$

Sei z' ungerade

$\Rightarrow (\nexists z \in \mathbb{Z}: 2z = z') \Rightarrow$!surjektiv

$\Rightarrow \underline{z \notin f_1(\mathbb{Z})}$ ↯

$\lfloor x \rfloor$ = größte Zahl z, $z \leq x$ $\Rightarrow z+1 > x$

b) $f_2 : \mathbb{Z} \to \mathbb{Z}$ $\quad\forall z_1, z_2 \in \mathbb{Z} \left(f_2(z_1) = f_2(z_2) \Rightarrow z_1 = z_2 \right)$

$z \mapsto \left\lfloor \frac{z}{2} \right\rfloor$

Annahme: ~~$f_2(z_1) = f_2(z_2)$~~ $z_1 \neq z_2$

~~$f_2(z_1) \neq f_2(z_2)$~~ Sei $z_1 = z_2 + 1$

~~$f_2(z_2) \neq z_2$~~ $f_2(z_1) = \left\lfloor \frac{z_2+1}{2} \right\rfloor =$

$f_2(z_2) = \left\lfloor \frac{z_2}{2} \right\rfloor$

~~$f_2(z_1) + 1 > \frac{z_2+1}{2} = \frac{z_2}{2} + \frac{1}{2}$~~

~~$f_2(z_2) + 1 > \frac{z_2}{2}$~~

$f_2(z_1) \leq \frac{z_2+1}{2} = \frac{z_2}{2} + \frac{1}{2}$ ~~$f_2(z_1)$~~

$f_2(z_2) \leq \frac{z_2}{2} -$

$\Rightarrow f_2(z_2) < \frac{z_2}{2} + \frac{1}{2}$

$\Rightarrow$ Falls $f_2(z_1) = \frac{z_2}{2} + \frac{1}{2}$

$f_2(z_1) > f_2(z_2) \Rightarrow \exists\, z_1, z_2 \in \mathbb{Z} : f_2(z_1) \neq f_2(z_2)$ ↯

$\Rightarrow$! injektiv

$f_2(z) = z'$

Fall "=>" z.z. $z' \in f_2(\mathbb{Z}) \iff z' \in \mathbb{Z}$

$f_2(\mathbb{Z}) \subseteq \mathbb{Z}$ (per Definition)

$\Rightarrow z' \in f_2(\mathbb{Z}) \Rightarrow z' \in \mathbb{Z}$

Fall "<=" z.z. $z' \in \mathbb{Z} \Rightarrow z' \in f_2(\mathbb{Z})$

Annahme $z' \in \mathbb{Z}$

$f_2(\mathbb{Z}) = \{ z' \in \mathbb{Z} \mid \exists z \in \mathbb{Z} : f_2(z) = z' \}$

$z' = \lfloor \frac{z}{2} \rfloor \in \mathbb{Z}$

$f_2(z) \leq \frac{z}{2}$