ECE 6553: Optimal Control Notes

Klaus Z. Okkelberg

Spring 2017

Contents

Chapter 1

Parameter Optimization

1.1 What is Optimal Control?

Optimal Maximize/minimize cost (subject to constraints): $\min_u g(u)$
With constraints,

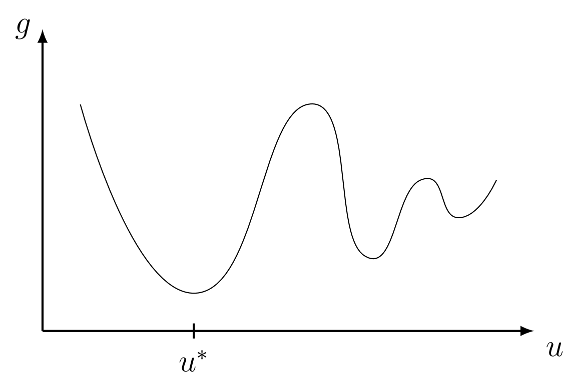
$$\min_u \; g(u)$$
$$\text{s.t.} \; \begin{cases} h_1(u) = 0 \\ h_2(u) \leq 0 \end{cases}$$

First-order necessary condition (FONC):

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Optimality can be

- local vs global
- max vs min



Control control design: pick u such that specifications are satisfied:

$$\dot{x} = f(x, u), \qquad \dot{x} = Ax + Bu,$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, and $f(\cdot)$ is the dynamics.

Actually, x and u are signals:

$$x : [0, T] \to \mathbb{R}^n, \qquad u : [0, T] \to \mathbb{R}^m$$

Optimal control find the "best" u!

For "best" to mean anything, we need a cost. The big/deep question is

$$\frac{\partial \text{“cost”}}{\partial u} = 0$$

Example

Suppose we have a car with position p. Its acceleration $\ddot{p}$ is controlled by the gas/brake input u ($\ddot{p} = u$). In order to express the dynamics of the system in the form $\dot{x} = f(x, u)$, we introduce state variables:

$$\begin{aligned} x_1 &= p \\ x_2 &= \dot{p} \end{aligned} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

The task is to move the car from its initial position to a stop at a distance c away.

Minimum energy problem

$$\begin{aligned} \min_u \quad & \int_0^T u^2(t)\, \mathrm{d}t \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0,\ x_2(0) = 0 \\ & x_1(T) = c,\ x_2(T) = 0 \end{aligned}$$

Minimum time problem

$$\begin{aligned} \min_{u,T} \quad & T = \int_0^T \mathrm{d}t \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0,\ x_2(0) = 0 \\ & x_1(T) = c,\ x_2(T) = 0 \\ & u(t) \in [u_{\min}, u_{\max}] \end{aligned}$$

The general optimal control problem we will solve will look like

$$
\begin{aligned}
\min_{u,T} \quad & \int_0^T L(x(t), u(t), t)\,\mathrm{d}t + \Psi(x(T)) \\
\text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t), t),\ t \in [0, T] \\
& x(0) = x_0 \\
& x(T) \in S \\
& u(t) \in \Omega,\ t \in [0, T]
\end{aligned}
$$

where $\Psi(\cdot)$ is the terminal cost and S is the terminal manifold. This is a so-called **Bolza Problem**.

What tools do we need to solve this?

1. optimality conditions $\partial\text{cost}/\partial u = 0$
2. some way of representing the optimal signal $u^*(x, t)$
3. some way of actually finding/computing the optimal controllers

1.2 Unconstrained Optimization

Let the decision variable be $u = [u_1, \dots, u_m]^{\mathrm{T}} \in \mathbb{R}^m$. The cost is $g(u) \in C^1$ (C^k means k times continuously differentiable). The problem is

$$\min_u g(u), \quad g : \mathbb{R}^m \to \mathbb{R}$$

For u^* to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Definition. u^* is a (local) minimizer to g if $\exists \delta > 0$ s.t.

$$
\begin{gathered}
g(u^*) \le g(u) \quad \forall u \in B_\delta(u^*) \\
B_\delta(u^*) = \{u \mid \|u - u^*\| \le \delta\}
\end{gathered}
$$

Note:

- $\dfrac{\partial g}{\partial u}(u^*)\delta u \in \mathbb{R}$ and δu is $m \times 1$, so $\dfrac{\partial g}{\partial u}$ is a $1 \times m$ row vector. For the column vector,

$$\nabla g = \frac{\partial g}{\partial u}^{\mathrm{T}} \in \mathbb{R}^m$$

- $\dfrac{\partial g}{\partial u}\,\delta u$ is an inner product

$$\langle \nabla g, \delta u \rangle = \left\langle \frac{\partial g}{\partial u}^{\mathrm{T}}, \delta u \right\rangle$$

- $o(\varepsilon)$ encodes higher-order terms

$$\lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0 \qquad \text{“faster than linear”}$$

  This is opposed to big-O notation:

$$\lim_{\varepsilon \to 0} \frac{\mathcal{O}(\varepsilon)}{\varepsilon} = c$$

- δu has direction and scale so we could write it as

$$\delta u = \varepsilon v, \quad \varepsilon \in \mathbb{R},\ v \in \mathbb{R}^m$$

Theorem. *For u^* to be a minimizer, we need*

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or, equivalently,

$$\frac{\partial g}{\partial u}(u^*)v = 0 \quad \forall v \in \mathbb{R}^m$$

Proof. Let u^* be a minimizer. Evaluating the cost $g(u)$ in the ball and using Taylor's expansion,

$$g(u^* + \delta u) = g(u^*) + \frac{\partial g}{\partial u}(u^*)\delta u + o(\|\delta u\|) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$

Assume that $\frac{\partial g}{\partial u} \neq 0$. Then we could pick $v = -\frac{\partial g^{\mathrm{T}}}{\partial u}(u^*)$, i.e.

$$g(u^* + \varepsilon v) = g(u^*) - \varepsilon \left\| \frac{\partial g^{\mathrm{T}}}{\partial u}(u^*) \right\|^2 + o(\varepsilon)$$

Note that the second term is negative per our assumptions. So, for ε sufficiently small, we have

$$g\Big(u^* - \varepsilon \frac{\partial g^{\mathrm{T}}}{\partial u}(u^*)\Big) < g(u^*)$$

This contradicts u^* being a minimizer. ⚔ (crossed swords) □

Definition (Positive definite)**.** $M = M^{\mathrm{T}} \succ 0$ if

$$z^{\mathrm{T}} M z > 0 \quad \forall z \neq 0,\ z \in \mathbb{R}^m$$
$$\iff M \text{ has real and positive eigenvalues}$$

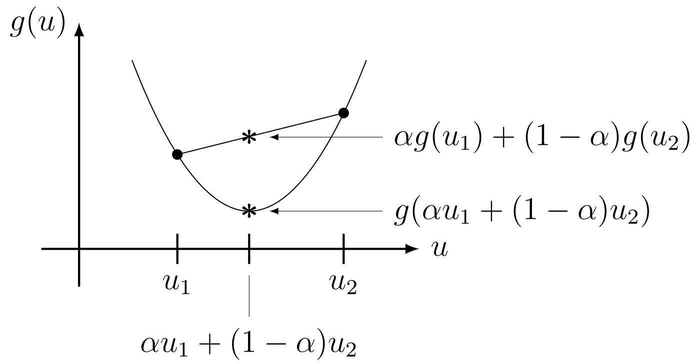
Theorem. *If $g \in C^2$, then a **sufficient** condition for u^* to be a (local) minimizer is*

1. $\dfrac{\partial g}{\partial u}(u^*) = 0$

2. $\dfrac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ *(the Hessian is positive definite)*

Definition. $g : \mathbb{R}^m \to \mathbb{R}$ is convex if

$$g(\alpha u_1 + (1-\alpha)u_2) \leq \alpha g(u_1) + (1-\alpha)g(u_2) \quad \forall \alpha \in [0,1],\ u_1,u_2 \in \mathbb{R}^m$$



Theorem. *If* $\frac{\partial^2 g}{\partial u^2}(u) \succeq 0\ \forall u \in \mathbb{R}^m$*, then* g *is convex. (*$\Longleftrightarrow$ *for* $g \in C^2$*)*

Example $\min\limits_u u^{\mathrm{T}}Qu - b^{\mathrm{T}}u$ where $Q = Q^{\mathrm{T}} \succ 0$ (positive definite matrix)

$$\begin{aligned}\frac{\partial g}{\partial u} &= \frac{\partial}{\partial u}(u^{\mathrm{T}}Qu - b^{\mathrm{T}}u) \\ &= u^{\mathrm{T}}Q^{\mathrm{T}} + u^{\mathrm{T}}Q - b^{\mathrm{T}} \\ &= 2u^{\mathrm{T}}Q - b^{\mathrm{T}} \\ \frac{\partial^2 g}{\partial u^2} &= 2Q\end{aligned} \qquad \frac{\partial^2 g}{\partial u^2} = \begin{bmatrix} \frac{\partial^2 g}{\partial u_1^2} & \cdots & \frac{\partial^2 g}{\partial u_1 \partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 g}{\partial u_m^2} \end{bmatrix}$$

From $\frac{\partial g}{\partial u} = 2u^{\mathrm{T}}Q - b^{\mathrm{T}} = 0$,

$$u = \frac{1}{2}Q^{-1}b$$

To see whether this is a minimizer, consider the Hessian. Since $Q \succ 0$, it follows that $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ and $u^* = \frac{1}{2}Q^{-1}b$ is a (local) minimizer. Additionally, since $\frac{\partial^2 g}{\partial u^2} \succ 0$, g is convex and u^* is a global minimizer. In fact, since we have strict convexity ($\succ 0$ rather than $\succeq 0$), it is the unique global minimizer.

In optimal control, *local* is typically all we can ask for. In optimization, we can do better!

But wait, just because we know $\frac{\partial g}{\partial u} = 0$, it doesn't follow that we can actually find u^*...

1.3 Numerical Methods

Idea: $u_{k+1} = u_k + \text{step}_k$. What should step_k be? For small $\text{step}_k = \gamma_k v_k$,

$$g(u_k \cdot \text{step}_k) = g(u_k) + \frac{\partial g}{\partial u}(u_k) \cdot \text{step}_k + o(\|\text{step}_k\|) = g(u_k) + \gamma_k \frac{\partial g}{\partial u}(u_k) v_k + o(\gamma_k)$$

A perfectly reasonable choice of step direction is

$$v_k = -\frac{\partial g^{\mathrm{T}}}{\partial u}(u_k),$$

known as the *steepest descend* direction. This produces

$$g\Big(u_k - \gamma_k \frac{\partial g}{\partial u}(u_k)\Big) = g(u_k) - \gamma_k \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 + o(\gamma_k)$$

Steepest descent

$$u_{k+1} = u_k - \gamma_k \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)$$

Note:

- What should γ_k be?
- This method "pretends" that $g(u)$ is linear. If we pretend $g(u)$ is quadratic, we get

  $$u_{k+1} = u_k - \left(\frac{\partial^2 g}{\partial u^2}(u_k)\right)^{-1} \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k),$$

  i.e. Newton's Method

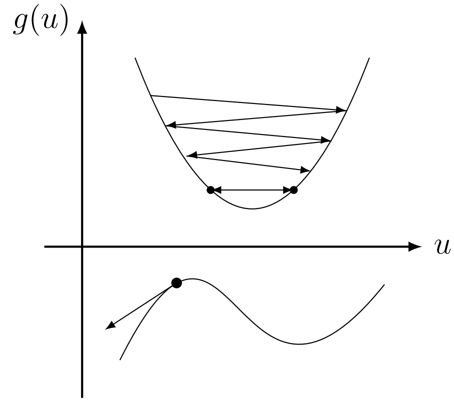
This course: steepest descent

Step-size selection?

- Choice 1: $\gamma_k = \gamma$ "small" $\forall k$; will get close to a minimizer if u_0 is close enough and γ small enough

  Problems:

  - You may not converge! (but you'll get close)
  - You may go off to infinity (diverge)



- Choice 2: Reduce γ_k as a function of k; will get close to a minimizer if u_0 is close enough

  Problem: slow

  Theorem. *If u_0 is close enough to u^* and γ_k satisfies*

  - $\sum_{k=0}^{\infty} \gamma_k = \infty$
  - $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$

  e.g. $\gamma_k = c/k$, then $u_k \to u^*$ as $k \to \infty$.

- Choice 3: **Armijo step-size:** Take as big a step as possible, but no larger

  Pick $\alpha \in (0,1)$, $\beta \in (0,1)$. Let i be the smallest non-negative integer such that

$$g\left(u_k - \beta^i \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)\right) - g(u_k) < -\alpha\beta^i \left\|\frac{\partial g}{\partial u}(u_k)\right\|^2$$

$$u_{k+1} = u_k - \beta^i \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)$$

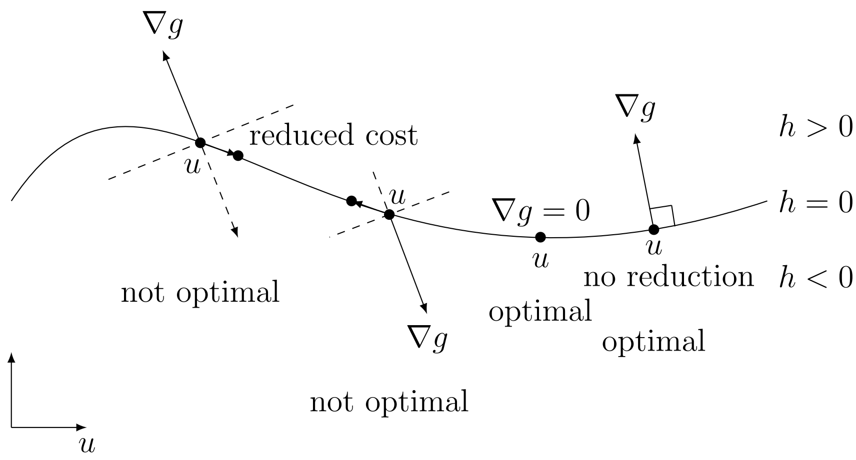
  This will get to a minimizer blazingly fast if u_0 is close enough.

1.4 Constrained Optimization

Equality constraints:

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \ & g(u) \\ \text{s.t. } & h(u) = \mathbf{0} \end{aligned}$$

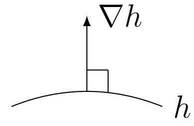
Consider $u \in \mathbb{R}^2$, $h : \mathbb{R}^2 \to \mathbb{R}$



So u is (locally) optimal if $\nabla g \parallel$ (is parallel to) the normal vector to tangent plane to h.

Fact: (HW# 1)

$$\nabla h \perp Th \quad \text{(tangent plane to h)}$$



We need $\nabla g \parallel \nabla h$ at u^* for optimality, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \alpha \frac{\partial h}{\partial u}(u^*), \quad \text{for some } \alpha \in \mathbb{R}$$

or $(\lambda = -\alpha)$,

$$\frac{\partial g}{\partial u}(u^*) + \lambda \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}\big(g(u^*) + \lambda h(u^*)\big) = 0, \quad \text{for some } \lambda \in \mathbb{R}$$

More generally,

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \ & g(u) \\ \text{s.t. } & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k \end{aligned}$$

Note that $h(u) = [h_1(u), \ldots, h_k(u)]^{\mathrm{T}}$.

We need $\frac{\partial g}{\partial u}(u^*)$ to be a linear combination of $\frac{\partial h_i}{\partial u}(u^*)$, $i = 1, \ldots, k$, for exactly the same reasons, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \sum_{i=1}^{k} \alpha_i \frac{\partial h_i}{\partial u}(u^*)$$

or $(\lambda = -[\alpha_1, \ldots, \alpha_k]^{\mathrm{T}})$

$$\frac{\partial g}{\partial u}(u^*) + \lambda^{\mathrm{T}} \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}\big(g(u^*) + \lambda^{\mathrm{T}} h(u^*)\big) = 0, \quad \text{for some } \lambda \in \mathbb{R}^k$$

Theorem. *If u^* is a minimizer to*

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \ & g(u) \\ s.t. \ & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k \end{aligned}$$

then $\exists \lambda \in \mathbb{R}^k$ s.t.

$$\begin{cases} \dfrac{\partial L}{\partial u}(u^*, \lambda) = 0 \\ \dfrac{\partial L}{\partial \lambda}(u^*, \lambda) = 0 \end{cases}$$

where the Lagrangian L is given by

$$L(u, \lambda) = g(u) + \lambda^T h(u)$$

Note:

- λ are the Lagrange multipliers
- $\frac{\partial L}{\partial \lambda} = 0$ is fancy speak for $h(u^*) = 0$

Example

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \ & \frac{1}{2}\|u\|^2 \\ \text{s.t.} \ & Au = b \end{aligned}$$

where A is $k \times m$, $k \leq m$. Assume $(AA^{\mathrm{T}})^{-1}$ exists (constraints are linearly independent, none of the constraints are "duplicates", all the constraints are essential).

$$\begin{aligned} L &= \frac{1}{2}u^{\mathrm{T}}u + \lambda^{\mathrm{T}}(Au - b) \\ \frac{\partial L}{\partial u} &= u^{\mathrm{T}} + \lambda^{\mathrm{T}}A = 0 \\ u^* &= -A^{\mathrm{T}}\lambda \end{aligned}$$

Using the equality constraint,

$$\begin{aligned} Au^* &= b \\ -AA^{\mathrm{T}}\lambda &= b \\ \lambda &= -(AA^{\mathrm{T}})^{-1}b \\ u^* &= A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b \end{aligned}$$

Example

$$\begin{aligned} \min \ & u_1u_2 + u_2u_3 + u_1u_3 \\ \text{s.t.} \ & u_1 + u_2 + u_3 = 3 \end{aligned}$$

$$L = u_1u_2 + u_2u_3 + u_1u_3 + \lambda(u_1 + u_2 + u_3 - 3)$$

$$\begin{cases} \frac{\partial L}{\partial u_1} = u_2 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_2} = u_1 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_3} = u_2 + u_1 + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = u_1 + u_2 + u_3 = 3 \end{cases} \implies \begin{cases} u_1^* = 1 \\ u_2^* = 1 \\ u_3^* = 1 \\ \lambda = -2 \end{cases} \quad \text{optimal solution}$$

Note: This was actually the worst we can do—maximize! Even weirder: no local minimizer!

1.4.1 Equality Constraints

$$\min_{u \in \mathcal{R}^m} \; g(u)$$
$$\text{s.t. } h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k$$

Theorem. *If u^* is a minimizer/maximizer then $\exists \lambda \in \mathbb{R}^k$ s.t.*

$$\frac{\partial L}{\partial u}(u^*, \lambda) = 0$$
$$\frac{\partial L}{\partial \lambda}(u^*, \lambda) = 0 \qquad (\Longleftrightarrow h(u^*) = 0)$$

where $L(u, \lambda) = g(u) + \lambda^T h(u)$.

Example [Entropy Maximization]

Given $S = \{x_1, \ldots, x_n\}$ and a distribution over S such that it takes the value x_j with probability p_j. The entropy is

$$E(p) = \sum_{j=1}^{n} (-p_j \ln p_j).$$

The mean is

$$m = \sum_{j=1}^{n} p_j x_j.$$

Problem: Given m, find p such that E is maximized.

$$\begin{aligned}
\min_p \quad & -\sum_{j=1}^{n} p_j \ln p_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} p_j x_j = m \\
& \sum_{j=1}^{n} p_j = 1 \\
& p_j \geq 0, \; j = 1, \ldots, n \quad \text{(ignore this...)}
\end{aligned}$$

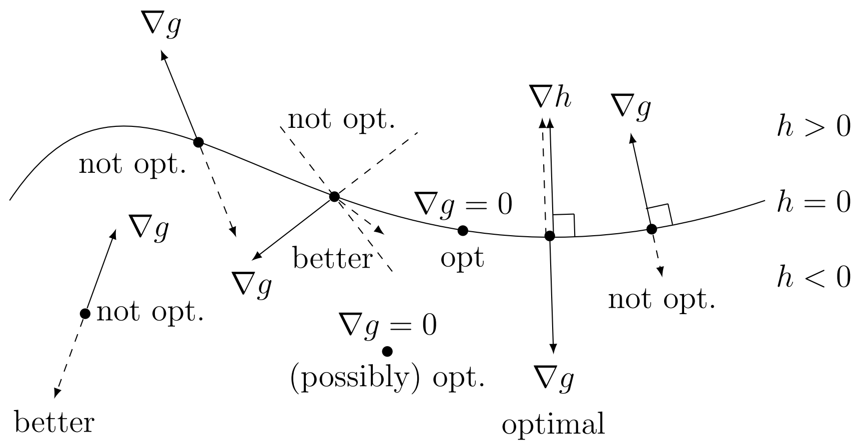
$$\begin{aligned}
L &= -\sum p_j \ln p_j + \lambda_1 \left[\sum p_j x_j - m\right] + \lambda_2 \left[\sum p_j - 1\right] \\
\frac{\partial L}{\partial p_j} &= -\ln p_j - 1 + \lambda_1 x_j + \lambda_2 = 0 \\
p_j &= e^{\lambda_2 - 1 + \lambda_1 x_j}, \quad j = 1, \ldots, n \qquad (p_j \geq 0 \text{ so we're ok with ignoring that})
\end{aligned}$$

$$\begin{aligned}
\sum e^{\lambda_2 - 1 + \lambda_1 x_j} x_j &= m \qquad && n+2 \text{ equations and} \\
\sum e^{\lambda_2 - 1 + \lambda_1 x_j} &= 1 \qquad && n+2 \text{ unknowns...}
\end{aligned}$$

No analytical solution, but numerically "solvable"

1.4.2 Inequality Constraints

$$\min_{u \in \mathcal{R}^m} \; g(u)$$
$$\text{s.t. } h(u) \leq \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k$$



We need:

- if $h(u^*) < 0$ then $\frac{\partial g}{\partial u}(u^*) = 0$
- if $h(u^*) = 0$ then we need either

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or

$$\frac{\partial g}{\partial u}(u^*) = -\lambda \frac{\partial h}{\partial u}(u^*) \quad \text{for } \lambda > 0$$

Or, even better,

$$\frac{\partial}{\partial u}\left(g(u^*) + \lambda h(u^*)\right) = 0 \quad \text{for } \lambda \geq 0,$$

where $\lambda h(u^*) = 0$. ($h < 0 \to \lambda = 0$, $h = 0 \to \lambda \geq 0$)

In general, if $u \in \mathbb{R}^m$ and $h : \mathbb{R}^m \to \mathbb{R}^k$, we have that u^*, if optimal, has to satisfy

$$\frac{\partial}{\partial u} L(u^*, \lambda) = 0$$
$$h(u^*) \leq \mathbf{0}$$
$$\lambda^{\mathrm{T}} h(u^*) = 0$$
$$\lambda \geq \mathbf{0}$$

where the Lagrangian is $L(u, \lambda) = g(u) + \lambda^{\mathrm{T}} h(u)$. Note that if we're maximizing, the same holds except we need $\lambda \leq 0$.

Example

$$\min\ 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2$$
$$\text{s.t.}\ \begin{cases} u_1^2 + u_2^2 \le 5 \\ 3u_1 + u_2 \le 6 \end{cases}$$

$$L = 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 + \lambda_1(u_1^2 + u_2^2 - 5) + \lambda_2(3u_1 + u_2 - 6)$$

FONC:

i) $\partial L/\partial u_1 = 4u_1 + 2u_2 - 10 + 2\lambda_1 u_1 + 3\lambda_2$

ii) $\partial L/\partial u_2 = 2u_1 + 2u_2 - 10 + 2\lambda_1 u_2 + \lambda_2$

iii) $u_1^2 + u_2^2 \le 5$

iv) $3u_1 + u_2 \le 6$

v) $\lambda_1(u_1^2 + u_2^2 - 5) = 0$

vi) $\lambda_2(3u_1 + u_2 - 6) = 0$

vii) $\lambda_1 \ge 0$

viii) $\lambda_2 \ge 0$

To solve, assume different constraints are active/inactive:

1. Both constraints are inactive ($u_1^2 + u_2^2 < 5$, $3u_1 + u_2 < 6$) $\implies \lambda_1 = \lambda_2 = 0$

$$\begin{cases} 4u_1 + 2u_2 - 10 = 0 \\ 2u_1 + 2u_2 - 10 = 0 \end{cases} \implies \begin{cases} u_1 = 0 \\ u_2 = 5 \end{cases}$$

   Note: iii) $0^2 + 5^2 \not\le 5$

   Not feasible

2. Assume constraint 1 is active and constraint 2 is inactive ($u_1^2 + u_2^2 = 5$, $\lambda_2 = 0$)

$$\begin{cases} 4u_1 + 2u_2 - 10 + 2\lambda_1 u_1 = 0 \\ 2u_1 + 2u_2 - 10 + 2\lambda_1 u_2 = 0 \\ u_1^2 + u_2^2 = 5 \end{cases} \implies \begin{cases} u_1 = 1 \\ u_2 = 2 \\ \lambda_1 = 1 \end{cases}$$

   - ☑ $\lambda_1 \ge 0$
   - ☑ $3 \cdot 1 + 2 \le 6$

   This is a local minimizer

3. Assume constraint 2 is active and constraint 1 is inactive

4. Assume both constraints are active

Kuhn-Tucker Conditions (KKT conditions, Karush-Kuhn-Tucker)

Problem:

$$\begin{aligned} &\min_{u\in\mathbb{R}^m} \; g(u) \\ &\text{s.t.} \; \begin{cases} h_1(u) = 0, & h_1 : \mathbb{R}^m \to \mathbb{R}^p \\ h_2(u) \le 0, & h_2 : \mathbb{R}^m \to \mathbb{R}^k \end{cases} \end{aligned} \tag{1.1}$$

Theorem. *Let u^* be feasible ($h_1 = 0$, $h_2 \le 0$). If u^* is a minimizer to (1.1) than there exists vectors $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^k$ with $\mu \ge \mathbf{0}$ such that*

$$\begin{cases} \dfrac{\partial g}{\partial u}(u^*) + \lambda^T \dfrac{\partial h_1}{\partial u}(u^*) + \mu^T \dfrac{\partial h_2}{\partial u}(u^*) = 0 \\ \mu^T h_2(u^*) = 0 \end{cases}$$

Looking ahead: $\min \operatorname{cost}(u(\cdot))$ s.t. $\dot{x} = f(x,u)$ (dynamics), where u is a function. Note the equality constraint.

Question: How do we go from $u \in \mathbb{R}^m$ to $u \in \mathcal{U}$ (function space)?

Note: Function space is a set of functions of a given kind from a set X to a set Y

1. linear function
2. square-integrable functions: $L_2[0,T] : \int_0^T \|u(t)\|^2 \, \mathrm{d}t < \infty$
3. $C^\infty(\mathbb{R})$

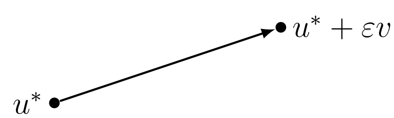
What would ∂"cost"$/\partial u$ mean?

Chapter 2

Calculus of Variations

2.1 Directional Derivatives

Recall: To minimize $g(u)$, let u^* be a candidate minimizer and pitch a perturbation on u^* of εv, where ε is the scale and v is the direction. Taking Taylor's expansion at the perturbation produces



$$g(u^* + \varepsilon v) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$
$$\text{FONC: } \frac{\partial g}{\partial u}(u^*) = 0$$

Note: $\frac{\partial g}{\partial u}(u^*)v$ tells us how much $g(u)$ increases/decreases in the direction of v.

Definition. The directional (Gateaux) derivative is given by

$$\delta g(u; v) = \lim_{\varepsilon \to 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon}$$

Example

$$g(u) = \frac{1}{2}u_1^2 - u_1 + 2u_2, \quad g : \mathbb{R}^2 \to \mathbb{R}$$

Let's consider $e_1 = [1\ 0]^{\mathrm{T}}$, $e_2 = [0\ 1]^{\mathrm{T}}$. What is $\delta g(u; e_i)$, $i = 1, 2$?

$$\begin{aligned}
\delta g(u; v) &= \lim_{\varepsilon \to 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{g(u) + \varepsilon \frac{\partial g}{\partial u}(u)v + o(\varepsilon) - g(u)}{\varepsilon} \\
&= \frac{\partial g}{\partial u}(u)v
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial g}{\partial u}(u) &= [u_1 - 1\ 2] \\
\delta g(u; e_1) &= [u_1 - 1\ 2]e_1 = u_1 - 1 \\
\delta g(u; e_2) &= [u_1 - 1\ 2]e_2 = 2
\end{aligned}
$$

But the beauty of directional derivatives is that they generalize beyond vectors, $u \in \mathbb{R}^m$, to function spaces ($\mathcal{U}$) or other "objects" like matrices.

Example $M \in \mathbb{R}^{n \times n}$, $F(M) = M^2$
What is $\frac{\partial F}{\partial M}$? (ponder at home...)
We can easily compute $\delta F(M; N)$!

$$
\begin{aligned}
F(M + \varepsilon N) = (M + \varepsilon N)(M + \varepsilon N) &= M^2 + \varepsilon MN + \varepsilon NM + \varepsilon^2 N^2 \\
\delta F(M; N) = \lim_{\varepsilon \to 0} \frac{F(M + \varepsilon N) - F(M)}{\varepsilon} & \\
= \lim_{\varepsilon \to 0} \frac{\varepsilon MN + \varepsilon NM + \varepsilon^2 N^2}{\varepsilon} &= MN + NM
\end{aligned}
$$

Infinite Dimensional Optimization Let $u \in \mathcal{U}$ (function space) and let $J(u)$ be the cost:

$$
\min_{u \in \mathcal{U}} J(u)
$$

Theorem. *If $u^* \in \mathcal{U}$ is a (local) minimizer then*

$$
\delta J(u^*; v) = 0, \quad \forall v \in \mathcal{U}
$$

Example Find minimizer u^* to

$$
J(u) = \int_0^T L(u(t))\, \mathrm{d}t
$$

$$
\begin{aligned}
J(u + \varepsilon v) - J(u) &= \int_0^T L(u(t) + \varepsilon v(t))\, \mathrm{d}t - \int_0^T L(u(t))\, \mathrm{d}t, \quad u, v \in \mathcal{U} \\
&= \int_0^T \left[L(u(t)) + \varepsilon \frac{\partial L}{\partial u}(u(t)) v(t) + o(\varepsilon) - L(u(t)) \right] \mathrm{d}t \\
\delta J(u^*; v) &= \lim_{\varepsilon \to 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\int_0^T \varepsilon \frac{\partial L}{\partial u}(u(t)) v(t)\, \mathrm{d}t + o(\varepsilon)}{\varepsilon} \\
&= \int_0^T \frac{\partial L}{\partial u}(u(t)) v(t)\, \mathrm{d}t
\end{aligned}
$$

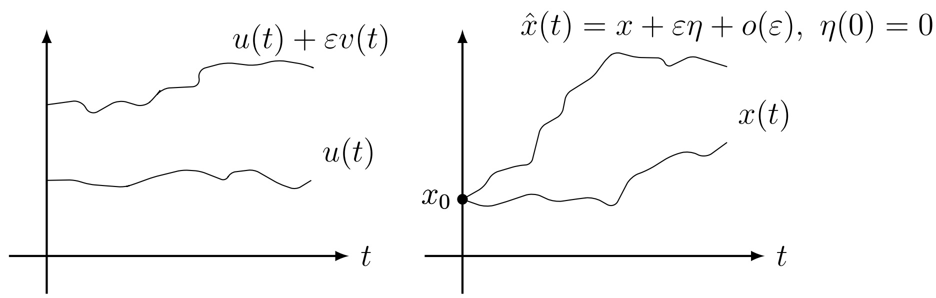


Figure 2.1: Variation in u causes a variation in x.

u^* optimizer:

$$\delta J(u^*; v) = \int_0^T \frac{\partial L}{\partial u}(u(t))v(t)\,\mathrm{d}t = 0 \quad \forall v \in \mathcal{U}$$
$$\Updownarrow$$
$$\frac{\partial L}{\partial u}(u(t)) = 0 \quad \forall t \in [0, T]$$

But, we want *optimal control*! We want our cost to look like

$$\int_0^T L(x(t), u(t))\,\mathrm{d}t$$
$$\dot{x} = f(x, u)$$

2.2 Calculus of Variations

What happens to $x(t)$ when $u(t)$ changes to $u(t) + \varepsilon v(t)$? Let the system be given by

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

After perturbation of u, the new system is

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}, u + \varepsilon v) \\ x(0) = x_0 \end{cases}$$

Consider

$$\tilde{x} = x + \varepsilon\eta,$$

where

$$\begin{aligned} \dot{x} &= f(x, u), & x(0) &= x_0 \\ \dot{\eta} &= \frac{\partial f}{\partial x}(x, u)\eta + \frac{\partial f}{\partial u}(x, u)v, & \eta(0) &= 0 \end{aligned}$$

Theorem. *If f is continuously differentiable in x and u then*

$$\hat{x}(t) = \tilde{x}(t) + o(\varepsilon)$$

Proof.

i) Initial conditions:

$$\begin{aligned}
\hat{x}(0) &= x_0 \\
\tilde{x}(0) &= x(0) + \varepsilon\eta(0) = x_0
\end{aligned}$$

ii) Dynamics:

$$\begin{aligned}
\dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v) \\
\dot{\tilde{x}} &= \dot{x} + \varepsilon\dot{\eta} = f(x,u) + \varepsilon\frac{\partial f}{\partial x}(x,u)\eta + \varepsilon\frac{\partial f}{\partial u}(x,u)v \\
&= f(x + \varepsilon\eta, u + \varepsilon v) + o(\varepsilon) \\
&= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon)
\end{aligned}$$

We can see that the dynamics of $\hat{x}(t)$ are equal to those of $\tilde{x}(t)$ plus higher order terms:

$$\begin{aligned}
\dot{\tilde{x}} &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon) \\
\dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v)
\end{aligned}$$

Therefore, if our perturbation is small enough, we can model $\hat{x}(t)$ as $\tilde{x}(t)$.

$\square$

Note: Taylor expansion with two elements is

$$\begin{aligned}
h(w + \varepsilon v, z + \varepsilon y) &= h(w, z + \varepsilon y) + \frac{\partial h}{\partial w}(w, z + \varepsilon y)\varepsilon v + o(\varepsilon) \\
&= \left\{ h(w,z) + \frac{\partial h}{\partial z}(w,z)\varepsilon y + o(\varepsilon) \right\} \\
&\quad + \left\{ \frac{\partial h}{\partial w}(w,z)\varepsilon v + \underbrace{\frac{\partial^2 h}{\partial z \partial w}\varepsilon v \odot \varepsilon y}_{o(\varepsilon)} + o(\varepsilon) \right\} \\
&= h(w,z) + \frac{\partial h}{\partial z}\varepsilon y + \frac{\partial h}{\partial w}\varepsilon v + o(\varepsilon)
\end{aligned}$$

Last class:

1. $u \in \mathcal{U}$ (space of functions), $J : \mathcal{U} \to \mathbb{R}$ (cost).
   FONC: If u^* is optimal, then
   $$\delta J(u; \nu) = 0 \quad \forall \nu \in \mathcal{U},$$
   where the directional derivative is given by
   $$\delta J(u; \nu) = \lim_{\varepsilon \to 0} \frac{J(u + \varepsilon\nu) - J(u)}{\varepsilon}.$$
2. If
   $$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$
   then a variation in u:
   $$u \longmapsto u + \varepsilon\nu$$
   results in a variation in x:
   $$x \longmapsto x + \varepsilon\eta + o(\varepsilon)$$
   See Figure 2.1. Note $\eta(0) = 0$.

2.2.1 An (Almost) Optimal Control Problem

Let $\dot{x} = f(x)$, $x(0) = x_0$. Note we get to pick the initial condition!

Problem

$$\min_{x_0 \in \mathbb{R}^m} J(x_0) = \int_0^T L(x(t))\, \mathrm{d}t$$
$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = f(x(t)) & \text{the } \textit{constraint}\text{! (equality)} \\ x(0) = x_0 \end{cases}$$

Note every constraint needs a Lagrange multiplier. We have infinitely many constraints:

$$\dot{x}(t) = f(x(t)) \quad \forall t \in [0, T]$$

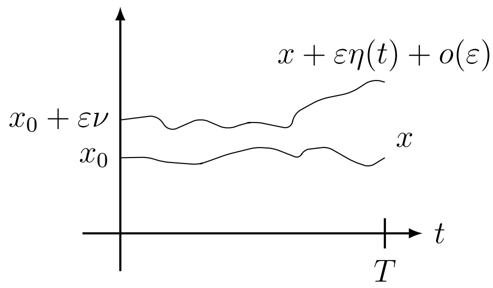
We need $\lambda(t)$ as a function of t. Also, the sum in the Lagrangian has to become an integral. The continuous-time Lagrangian thus becomes

$$\tilde{J}(x_0, \lambda) = \int_0^T \Big[L(x(t)) + \lambda^{\mathrm{T}}(t)(f(x(t)) - \dot{x}(t)) \Big]\, \mathrm{d}t$$

The task is to perturb x_0 as $x_0 \longmapsto x_0 + \varepsilon\nu$, $\nu \in \mathbb{R}^m$ and compute

$$\delta\tilde{J}(x_0; \nu) = \lim_{\varepsilon \to 0} \frac{\tilde{J}(x_0 + \varepsilon\nu) - \tilde{J}(x_0)}{\varepsilon}$$

and make this equal to 0 $\forall \nu \in \mathbb{R}^m$. The variation in x is



Note:

x_0	decision variable
ν	variation in x_0
$x(t)$	trajectory starting at x_0
$\eta(t)$	change in trajectory resulting from ν-variation in x_0
$\lambda(t)$	time-varying Lagrange multiplier

$$\begin{aligned}
\tilde{J}(x_0+\varepsilon\nu) &= \int_0^T \Big\{ L(x(t)) + \lambda^{\mathrm{T}}(t)[f(x(t)+\varepsilon\eta(t)) - \dot{x}(t) - \varepsilon\dot{\eta}(t)] \Big\}\,\mathrm{d}t + o(\varepsilon) \\
&= \int_0^T \left[L(x) + \varepsilon\frac{\partial L}{\partial x}(x)\eta + \lambda^{\mathrm{T}}\left(f(x) + \varepsilon\frac{\partial f}{\partial x}(x)\eta - \dot{x} - \varepsilon\dot{\eta} \right) \right]\mathrm{d}t + o(\varepsilon) \\
\tilde{J}(x_0+\varepsilon\nu) - \tilde{J}(x_0) &= \int_0^T \left[\varepsilon\frac{\partial L}{\partial x}(x)\eta + \lambda^{\mathrm{T}}\left(\varepsilon\frac{\partial f}{\partial x}\eta - \varepsilon\dot{\eta} \right) \right]\mathrm{d}t + o(\varepsilon) \\
\delta\tilde{J}(x_0;\nu) &= \int_0^T \left[\frac{\partial L}{\partial x}(x)\eta + \lambda^{\mathrm{T}}\left(\frac{\partial f}{\partial x}\eta - \dot{\eta} \right) \right]\mathrm{d}t
\end{aligned}$$

A powerful idea: we want $\delta\tilde{J}(x_0;\nu) = 0\ \forall\nu$. Somehow get this in the form

$$\int_0^T \Big(\mathrm{stuff}(t)\Big)\eta(t)\,\mathrm{d}t = 0$$

We can pick $\mathrm{stuff}(t) = 0\ \forall t \in [0,T]$.

In $\delta\tilde{J}(x_0;\nu)$ we have $\dot{\eta}$ (problem!). We can solve this using *integration by parts*.

$$\int_0^T \lambda^{\mathrm{T}}\dot{\eta}\,\mathrm{d}t = \lambda^{\mathrm{T}}(T)\eta(T) - \lambda^{\mathrm{T}}(0)\eta(0) - \int_0^T \dot{\lambda}^{\mathrm{T}}\eta\,\mathrm{d}t$$

Hence,

$$\delta\tilde{J}(x_0;\nu) = \int_0^T \underbrace{\left(\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)}_{\text{pick}=0}\eta\,\mathrm{d}t - \underbrace{\lambda^{\mathrm{T}}(T)}_{\text{pick}=0}\eta(T) + \lambda^{\mathrm{T}}(0)\underbrace{\eta(0)}_{\nu}$$

We are free to pick λ freely if it gives $\delta\tilde{J} = 0$.

$$\text{Pick: } \begin{cases} \dot{\lambda}(t) = -\frac{\partial L^{\mathrm{T}}}{\partial x}(x(t)) - \frac{\partial f^{\mathrm{T}}}{\partial x}(x(t))\lambda(t) \\ \lambda(T) = 0 \qquad\qquad \text{backwards diff. eq} \end{cases}$$

Under this choice of λ we get

$$\delta \tilde{J}(x_0; \nu) = \lambda^{\mathrm{T}}(0)\nu$$

This is linear in ν so the FONC is $\lambda(0) = 0$.

Moreover, we really have a "normal" optimization problem

$$\min_{x_0 \in \mathbb{R}^m} \tilde{J}(x_0)$$

$$\delta \tilde{J}(x_0; \nu) = \frac{\partial \tilde{J}}{\partial x_0}(x_0)\nu$$

which means that

$$\frac{\partial \tilde{J}}{\partial x_0} = \lambda^{\mathrm{T}}(0)$$

If x_0^* minimizes

$$\int_0^T L(x(t))\,\mathrm{d}t$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0^* \end{cases}$$

then

$$\lambda(0) = \mathbf{0}$$

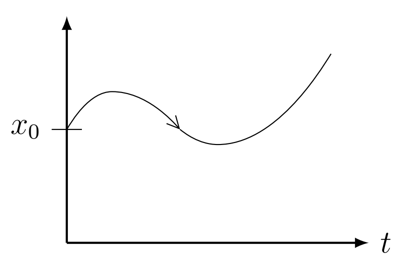
where $\lambda(t)$ satisfies

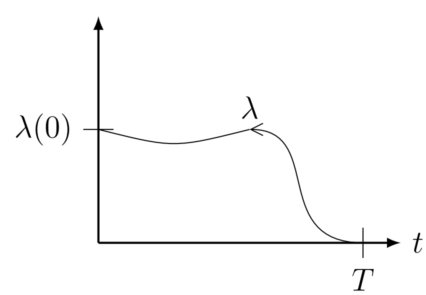
$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial L^{\mathrm{T}}}{\partial x}(x(t)) - \frac{\partial f^{\mathrm{T}}}{\partial x}(x(t))\lambda(t) \\ \lambda(T) = 0 \end{cases}$$

So what? We actually have a two-point boundary value problem.

$$\dot{x} = f(x) \qquad\qquad \dot{\lambda} = -\frac{\partial L^{\mathrm{T}}}{\partial x} - \frac{\partial f^{\mathrm{T}}}{\partial x}\lambda$$

$$x(0) = x_0 \qquad\qquad \lambda(T) = 0$$





We want to find x_0 that gives $f(x)$ such that after solving backwards for $\lambda(t)$, we find that

$$\lambda(0) = \frac{\partial \tilde{J}^{\mathrm{T}}}{\partial x_0} = 0.$$

This leads to the following:

Pick $x_{0,0}$
$k = 1$
repeat
  Simulate $x(t)$ from $x_{0,k}$ over $[0, T]$
  Simulate $\lambda(t)$ from $\lambda(T) = 0$ backwards using $x(t)$
  Update $x_{0,k}$ as $x_{0,k+1} = x_{0,k} - \gamma\lambda(0)$ ▷ $\lambda(0)$ is the gradient
  $k := k + 1$
until $\lambda(0) = 0$

An algorithm

Example: `optinit.m`

$$\dot{x} = Ax, \quad L = x^{\mathrm{T}}Qx - q, \quad Q = Q^{\mathrm{T}} \succ 0$$

$$\dot{\lambda} = -2Qx - A^{\mathrm{T}}\lambda$$

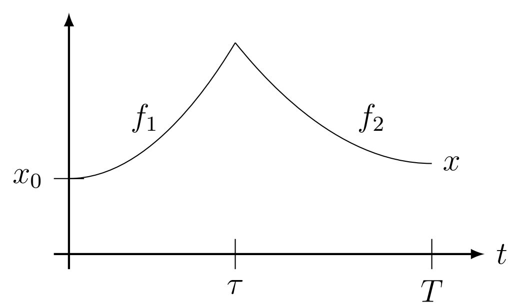
$$\lambda(0) = 0$$

2.2.2 Optimal Timing Control

When to switch between modes?

$$\dot{x} = \begin{cases} f_1(x) & \text{if } t \in [0, \tau) \\ f_2(x) & \text{if } t \in [\tau, T] \end{cases} \tag{2.1}$$

$$x(0) = x_0$$



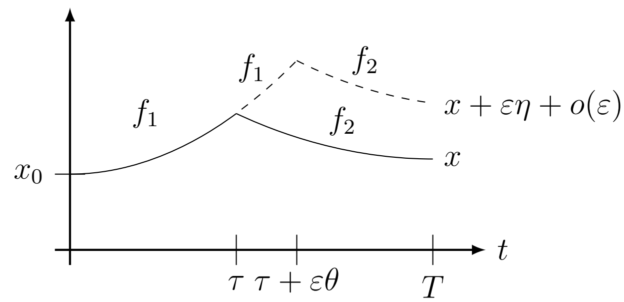
$$\min_{\tau} \int_0^T L(x(t))\,\mathrm{d}t = J(\tau)$$

s.t. (2.1) holds

Step 1: Augment cost with constraint

$$\tilde{J} = \int_0^{\tau} \Big[L(x) + \lambda^{\mathrm{T}}(f_1(x) - \dot{x})\Big]\,\mathrm{d}t + \int_{\tau}^{T} \Big[L(x) + \lambda^{\mathrm{T}}(f_2(x) - \dot{x})\Big]\,\mathrm{d}t$$

Step 2: Variation $\tau \longmapsto \tau + \varepsilon\theta$



Step 3: Compute $\delta\tilde{J}(\tau;\theta)$

$$\begin{aligned}\tilde{J}(\tau+\varepsilon\theta) = &\int_0^{\tau+\varepsilon\theta} \Big\{ L(x+\varepsilon\eta) + \lambda^{\mathrm{T}}[f_1(x+\varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \Big\}\,\mathrm{d}t \\ &+ \int_{\tau+\varepsilon\theta}^{T} \Big\{ L(x+\varepsilon\eta) + \lambda^{\mathrm{T}}[f_2(x+\varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \Big\}\,\mathrm{d}t + o(\varepsilon)\end{aligned}$$

Note that $\eta = \dot{\eta} = 0$ on $[0, \tau)$.

$$\begin{aligned}\tilde{J}(\tau+\varepsilon\theta) = &\int_0^{\tau} \Big\{ L(x) + \lambda^{\mathrm{T}}[f_1(x) - \dot{x}] \Big\}\,\mathrm{d}t \\ &+ \int_{\tau}^{\tau+\varepsilon\theta} \Big\{ \underbrace{L(x+\varepsilon\eta)}_{L(x)+\varepsilon\frac{\partial L}{\partial x}\eta} + \lambda^{\mathrm{T}}[\underbrace{f_1(x+\varepsilon\eta)}_{f_1(x)+\varepsilon\frac{\partial f_1}{\partial x}\eta} - \dot{x} - \varepsilon\dot{\eta}] \Big\}\,\mathrm{d}t \\ &+ \int_{\tau+\varepsilon\theta}^{T} \Big\{ \underbrace{L(x+\varepsilon\eta)}_{L(x)+\varepsilon\frac{\partial L}{\partial x}\eta} + \lambda^{\mathrm{T}}[\underbrace{f_2(x+\varepsilon\eta)}_{f_2(x)+\varepsilon\frac{\partial f_2}{\partial x}\eta} - \dot{x} - \varepsilon\dot{\eta}] \Big\}\,\mathrm{d}t + o(\varepsilon)\end{aligned}$$

$$
\delta\tilde{J}(\tau;\theta) = \lim_{\varepsilon\to 0} \frac{\tilde{J}(\tau+\varepsilon\theta) - \tilde{J}(\tau)}{\varepsilon}
$$

$$
\tilde{J}(\tau+\varepsilon\theta) - \tilde{J}(\tau) = \int_0^\tau 0\cdot \mathrm{d}t + \underbrace{\int_\tau^{\tau+\varepsilon\theta} \left[\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\Big(f_1(x) + \varepsilon\frac{\partial f_1}{\partial x}\eta - f_2(x) - \varepsilon\dot{\eta}\Big)\right]\mathrm{d}t}_{I_1}
$$

$$
+ \underbrace{\int_{\tau+\varepsilon\theta}^T \left[\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\Big(\varepsilon\frac{\partial f_2}{\partial x}\eta - \varepsilon\dot{\eta}\Big)\right]\mathrm{d}t}_{I_2} + o(\varepsilon)
$$

Theorem (Mean-value theorem)**.**

$$
\int_{t_1}^{t_2} h(t)\,\mathrm{d}t = (t_2 - t_1)h(\xi) \quad \textit{for some } \xi \in [t_1, t_2]
$$

The first integral is

$$
\begin{aligned}
I_1 &= \int_\tau^{\tau+\varepsilon\theta} \left\{\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\left[f_1(x) + \varepsilon\frac{\partial f_1}{\partial x}\eta - \varepsilon\dot{\eta} - f_2(x)\right]\right\}\mathrm{d}t \\
&= \varepsilon\theta\Big\{\lambda^{\mathrm{T}}(\xi)\big[f_1(x(\xi)) - f_2(x(\xi))\big]\Big\} + o(\varepsilon)
\end{aligned}
$$

Note that as $\varepsilon \to 0$, $\xi \to \tau$. Using integration by parts, the second integral is

$$
\int_\tau^T \lambda^{\mathrm{T}}\dot{\eta}\,\mathrm{d}t = \lambda^{\mathrm{T}}(T)\eta(T) - \lambda^{\mathrm{T}}(\tau)\underbrace{\eta(\tau)}_{=0} - \int_\tau^T \dot{\lambda}^{\mathrm{T}}\eta\,\mathrm{d}t
$$

$$
\begin{aligned}
I_2 &= \int_\tau^T \left[\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\Big(\varepsilon\frac{\partial f_2}{\partial x}\eta - \varepsilon\dot{\eta}\Big)\right]\mathrm{d}t - \underbrace{\int_\tau^{\tau+\varepsilon\theta} \left[\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\Big(\varepsilon\frac{\partial f_2}{\partial x}\eta - \varepsilon\dot{\eta}\Big)\right]\mathrm{d}t}_{o(\varepsilon)} \\
&= \varepsilon\int_\tau^T \left[\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f_2}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right]\eta\,\mathrm{d}t - \varepsilon\lambda^{\mathrm{T}}(T)\eta(T) + o(\varepsilon)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\delta\tilde{J}(\tau;\theta) &= \lim_{\varepsilon\to 0} \frac{\tilde{J}(\tau+\varepsilon\theta) - \tilde{J}(\tau)}{\varepsilon} \\
&= \theta\lambda^{\mathrm{T}}(\tau)\Big[f_1(x(\tau)) - f_2(x(\tau))\Big] + \int_\tau^T \left[\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f_2}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right]\eta\,\mathrm{d}t - \lambda^{\mathrm{T}}(T)\eta(T)
\end{aligned}
$$

Step 4: Select the *costate* $\lambda(t)$. The key idea is to get rid of any term that has η in it, i.e.

$$
\begin{aligned}
\dot{\lambda} &= -\frac{\partial L^{\mathrm{T}}}{\partial x} - \frac{\partial f_2^{\mathrm{T}}}{\partial x}\lambda \quad \text{on } [\tau, T] \\
\lambda(T) &= 0
\end{aligned}
$$

Step 5: With this choice of $\lambda(t)$, we have

$$\delta\tilde{J}(\tau;\theta) = \theta\lambda^{\mathrm{T}}(\tau)\Big[f_1(x(\tau)) - f_2(x(\tau))\Big] = \frac{\partial\tilde{J}}{\partial\tau}\theta.$$

Therefore,

$$\frac{\partial\tilde{J}}{\partial\tau} = \lambda^{\mathrm{T}}(\tau)\Big[f_1(x(\tau)) - f_2(x(\tau))\Big] = 0 \quad \text{(for optimality)}$$

Algorithm

Pick τ_0
$k = 0$
repeat
  Simulate x forward in time from $x(0) = x_0$
  Simulate λ backwards from $\lambda(T) = 0$
  Update τ_k as $\tau_{k+1} = \tau_k - \gamma\lambda^{\mathrm{T}}(\tau_k)\big[f_1(x(\tau_k)) - f_2(x(\tau_k))\big]$
  $k := k + 1$
until $\|\lambda^{\mathrm{T}}(f_1 - f_2)\| < \varepsilon$

Where are we going? Come up with general principles for $\min_{u\in\mathcal{U}} J(u)$:

- Costate equations
- Optimality conditions
- Algorithms
- Applications

Chapter 3

The Maximum Principle

3.1 The Bolza Problem

Up until now, we have optimized with respect to finite-dimensional parameters. Today, we will minimize with respect to $u \in \mathcal{U}$.

$$\begin{aligned}
\min_{u \in \mathcal{U}} J(u) &= \int_0^T L(x(t), u(t), t)\,\mathrm{d}t + \underbrace{\Psi(x(T))}_{\substack{\text{terminal cost} \\ \text{(parking cost)}}} \\
\text{s.t.} \quad \dot{x}(t) &= f(x(t), u(t), t) \\
x(0) &= x_0
\end{aligned}$$

Assume that f and L are C^1 in x, u and piecewise continuous in t. Then, a small change in u causes small changes in f and L. The variation: $u \longmapsto u + \varepsilon v$, $\varepsilon \in \mathbb{R}$, $v \in \mathcal{U}$. See Figure 2.1.

$$\begin{aligned}
\tilde{J}(u) &= \int_0^T \big[L(x, u, t) + \lambda^{\mathrm{T}}(f(x, u, t) - \dot{x})\big]\,\mathrm{d}t + \Psi(x(T)) \\
\tilde{J}(u + \varepsilon v) &= \int_0^T \big[L(x + \varepsilon\eta, u + \varepsilon v, t) + \lambda^{\mathrm{T}}(f(x + \varepsilon\eta, u + \varepsilon v, t) - \dot{x} - \varepsilon\dot{\eta})\big]\,\mathrm{d}t \\
&\quad + \Psi(x(T) + \varepsilon\eta(T)) + o(\varepsilon) \\
\tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \bigg[L(x + \varepsilon\eta, u + \varepsilon v, t) - L(x, u, t) \\
&\qquad + \lambda^{\mathrm{T}}\Big(f(x + \varepsilon\eta, u + \varepsilon v, t) - f(x, u, t) - \dot{x} - \varepsilon\dot{\eta} + \dot{x}\Big)\bigg]\,\mathrm{d}t \\
&\quad + \Psi(x(T) + \varepsilon\eta(T)) - \Psi(x(T)) + o(\varepsilon) \\
&= \int_0^T \bigg[\frac{\partial L}{\partial x}\varepsilon\eta + \frac{\partial L}{\partial u}\varepsilon v + \lambda^{\mathrm{T}}\Big(\frac{\partial f}{\partial x}\varepsilon\eta + \frac{\partial f}{\partial u}\varepsilon v - \varepsilon\dot{\eta}\Big)\bigg]\,\mathrm{d}t \\
&\quad + \frac{\partial \Psi}{\partial x}(x(T))\varepsilon\eta(T) + o(\varepsilon)
\end{aligned}$$

(See Taylor expansion with respect to two variables.)

$$\delta \tilde{J}(u;v) = \int_0^T \left(\frac{\partial L}{\partial u} + \lambda^{\mathrm{T}} \frac{\partial f}{\partial u}\right) v \,\mathrm{d}t + \int_0^T \left[\left(\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}} \frac{\partial f}{\partial x}\right) \eta - \lambda^{\mathrm{T}} \dot{\eta}\right] \mathrm{d}t \\ + \frac{\partial \Psi}{\partial x}(x(T))\eta(T)$$

Integrating by parts,

$$\begin{aligned} \int_0^T \lambda^{\mathrm{T}} \dot{\eta} \,\mathrm{d}t &= \lambda^{\mathrm{T}}(T)\eta(T) - \lambda^{\mathrm{T}}(0)\eta(0) - \int_0^T \dot{\lambda}^{\mathrm{T}} \eta \,\mathrm{d}t \\ &= \lambda^{\mathrm{T}}(T)\eta(T) - \int_0^T \dot{\lambda}^{\mathrm{T}} \eta \,\mathrm{d}t \end{aligned}$$

$$\delta \tilde{J}(u;v) = \int_0^T \left(\frac{\partial L}{\partial u} + \lambda^{\mathrm{T}} \frac{\partial f}{\partial u}\right) v \,\mathrm{d}t + \int_0^T \left(\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}} \frac{\partial f}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right) \eta \,\mathrm{d}t \\ + \left(\frac{\partial \Psi}{\partial x}(x(T)) - \lambda^{\mathrm{T}}(T)\right) \eta(T)$$

For optimality, we need the directional derivative to be zero for every $v \in \mathcal{U}$, where v represents the direction of the derivative. Therefore, the term $(\frac{\partial L}{\partial u} + \lambda^{\mathrm{T}} \frac{\partial f}{\partial u})$ in the first integral has to be identically zero. Thus, we need

$$\begin{cases} \dfrac{\partial L}{\partial u} + \lambda^{\mathrm{T}} \dfrac{\partial f}{\partial u} = 0, & \forall t \in [0,T] \\ \dfrac{\partial L}{\partial x} + \lambda^{\mathrm{T}} \dfrac{\partial f}{\partial x} + \dot{\lambda}^{\mathrm{T}} = 0, & \forall t \in [0,T] \\ \dfrac{\partial \Psi}{\partial x}(x(T)) - \lambda^{\mathrm{T}}(T) = 0 \end{cases}$$

Definition. Let the *Hamiltonian* $H(x,u,t,\lambda)$ be given by

$$H(x,u,t,\lambda) = L(x,u,t) + \lambda^{\mathrm{T}} f(x,u,t)$$

Theorem. *For u to solve the Bolza problem, it has to satisfy*

$$\frac{\partial H}{\partial u}(x,u,t,\lambda) = 0,$$

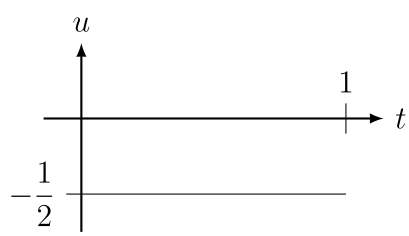
where the costate *satisfies*

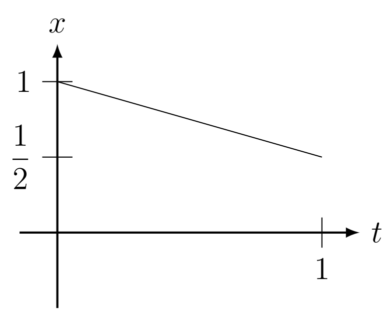
$$\begin{cases} \dot{\lambda} = -\dfrac{\partial H}{\partial x}^T(x,u,t,\lambda) \\ \lambda(T) = \dfrac{\partial \Psi}{\partial x}^T(x(T)) \end{cases}$$

Example

$$\min_u \int_0^1 \frac{1}{2}u^2(t)\,\mathrm{d}t + \frac{1}{2}x^2(1)$$
$$\text{s.t.} \begin{cases} \dot{x} = u, & x, u \in \mathbb{R} \\ x(0) = 1 \end{cases}$$

$$\begin{aligned}
H &= \frac{1}{2}u^2 + \lambda u \\
\frac{\partial H}{\partial u} &= u + \lambda = 0 \Longrightarrow u = -\lambda \\
\dot{\lambda} &= -\frac{\partial H}{\partial x} = 0 \Longrightarrow \lambda(t) = c \\
\lambda(T) &= c = \frac{\partial \Psi}{\partial x}(x(1)) = x(1) \\
\dot{x} &= u = -c \Longrightarrow x(t) = -ct + x(0) = -ct + 1 \\
x(1) &= -c + 1 \\
\lambda(1) &= c = x(1) = -c + 1 \Longrightarrow c = \frac{1}{2} \\
&\boxed{u^* = -\frac{1}{2}}
\end{aligned}$$





We really used five different equations to solve this!

i) $\frac{\partial H}{\partial u} = 0$

ii) $\dot{\lambda} = -\frac{\partial H^{\mathrm{T}}}{\partial x}$

iii) $\lambda(T) = \dfrac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T))$

iv) $\dot{x} = f(x, u, t)$

v) $x(0) = x_0$

There is a sixth condition that is pretty useful if L and f do not depend on t ($L(x, u)$, $f(x, u)$). This is called a *conservative system*. Then, along optimal trajectories (equations i-v are satisfied), the total time derivative of the Hamiltonian is

$$\frac{\mathrm{d}}{\mathrm{d}t} H = \underbrace{\frac{\partial H}{\partial t}}_{\substack{0 \\ H(x,u,\lambda)}} + \underbrace{\frac{\partial H}{\partial x}}_{-\dot{\lambda}^{\mathrm{T}}} \dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{\substack{0 \\ u \text{ is optimal}}} \dot{u} + \underbrace{\frac{\partial H}{\partial \lambda}}_{f^{\mathrm{T}} = \dot{x}^{\mathrm{T}}} \dot{\lambda} = -\dot{\lambda}^{\mathrm{T}} \dot{x} + \dot{x}^{\mathrm{T}} \dot{\lambda} = 0$$

Therefore, for conservative systems,

vi) H is constant along optimal trajectories. (Hamilton's Principle in analytical mechanics)

Back to the example,

$$H = \frac{1}{2} u^2 + \lambda u = \frac{1}{2} c^2 - c^2 = -\frac{1}{2} c^2 = -\frac{1}{8}$$

The Hamiltonian

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^{\mathrm{T}} f(x, u, t)$$

lets us write the Lagrangian as

$$\tilde{J}(u) = \int_0^T \left[L + \lambda^{\mathrm{T}}(f - \dot{x}) \right] \mathrm{d}t + \Psi = \int_0^T \left(H - \lambda^{\mathrm{T}} \dot{x} \right) \mathrm{d}t + \Psi$$

The optimality conditions are

$$\frac{\partial H}{\partial u} = 0, \tag{3.1}$$

where

$$\begin{cases} \dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x} \\ \lambda(T) = \dfrac{\partial \Psi}{\partial x}(x(T)) \end{cases} \tag{3.2}$$

Example Hamilton's Principle

Let q be the generalized coordinates (positions and angles). Then, $\dot{q} = u$ are generalized velocities, which we assume we can control. Let $T(q,u) = u^{\mathrm{T}} M(q) u$, $M \succ 0$, be the kinetic energy and $V(q)$ be the potential energy.

For conservative systems, the following quantity is minimized:

$$\int_0^T \underbrace{\big[T(q,u) - V(q)\big]}_{\substack{L(q,u) = \\ \text{Lagrange's "action function"}}} \mathrm{d}t$$

The Hamiltonian is

$$H(q,u,\lambda) = L(q,u) + \lambda^{\mathrm{T}} f(q,u) = L(q,u) + \lambda^{\mathrm{T}} u$$

In mechanics, λ is called a generalized momentum, satisfying

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H}{\partial q}^{\mathrm{T}} = -\frac{\partial L}{\partial q}^{\mathrm{T}} + 0 \\ 0 &= \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^{\mathrm{T}} \implies \lambda = -\frac{\partial L}{\partial u}^{\mathrm{T}} \\ \dot{\lambda} &= -\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial u}^{\mathrm{T}} = -\frac{\partial L}{\partial q}^{\mathrm{T}} \end{aligned}$$

This produces the Euler-Lagrange Equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0}$$

Recall, along optimal trajectories

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \underbrace{\frac{\partial H}{\partial t}}_{\substack{=0 \text{ if } L \text{ and } f \text{ do not} \\ \text{depend explicitly on } t}} + \overbrace{\frac{\partial H}{\partial x}}^{-\dot{\lambda}^{\mathrm{T}} \dot{x}} \dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{=0} \dot{u} \underbrace{\frac{\partial H}{\partial \lambda}}_{f^{\mathrm{T}} = \dot{x}^{\mathrm{T}}} \dot{\lambda} = -\dot{\lambda}^{\mathrm{T}} \dot{x} + \dot{x}^{\mathrm{T}} \dot{\lambda} = 0$$

Therefore, along optimal trajectories, the Hamiltonian is constant!

We had

$$\begin{aligned} H &= L + \lambda^{\mathrm{T}} u \\ \frac{\partial H}{\partial u} &= \lambda^{\mathrm{T}} + \frac{\partial L}{\partial u} = 0 \end{aligned}$$

Along optimal trajectories,

$$H = L - \frac{\partial L}{\partial u} u$$

Recall, $L(q,u) = T(q,u) - V(q)$.

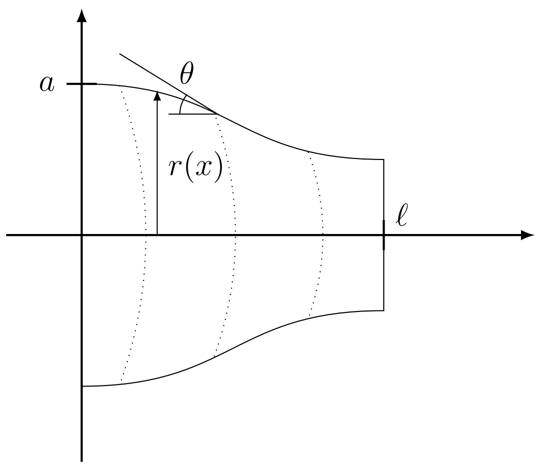
$$\frac{\partial L}{\partial u} = \frac{\partial T}{\partial u} - 0$$
$$T(q,u) = u^{\mathrm{T}} M(q) u$$
$$\frac{\partial T}{\partial u} = 2u^{\mathrm{T}} M$$

So,

$$H = \underbrace{T}_{u^{\mathrm{T}}Mu} - V - 2u^{\mathrm{T}} M u = -(V + u^{\mathrm{T}} M u) = -(V + T)$$

Therefore, the total energy (kinetic plus potential energy) remains constant for conservative systems.

Example minimum drag nose shape (Newton 1686)



The drag is

$$D = -2\pi q \int_{x=0}^{\ell} C_p(\theta) r \, \mathrm{d}r,$$

where q is a pressure constant and $C_p(\theta) = 2\sin^2\theta$ is Newton's pressure formula.

Geometry tells us

$$\frac{\mathrm{d}r}{\mathrm{d}x} = -\tan\theta = -u$$

Choose the control as $\tan\theta$. Manipulating the drag,

$$\frac{D}{4\pi q} = \int_0^{\ell} \frac{ru^3}{1+u^2} \, \mathrm{d}x + \frac{1}{2} r(\ell)^2$$

The optimal control problem is

$$\min_u \int_0^{\ell} \frac{ru^3}{1+u^2} \, \mathrm{d}x + \frac{1}{2} r(\ell)^2$$
$$\text{s.t. } \frac{\mathrm{d}r}{\mathrm{d}x} = -u$$

This is in the standard form with the following changes of variables:

$$
\begin{aligned}
\ell &\longleftarrow T \\
x &\longleftarrow t \\
r &\longleftarrow x
\end{aligned}
$$

Refer to (3.1) and (3.2) for the following steps.

$$
\begin{aligned}
H &= \frac{ru^3}{1+u^2} - \lambda u \\
\frac{\partial H}{\partial u} &= \frac{3ru^2(1+u^2) - ru^3 \cdot 2u}{(1+u^2)^2} - \lambda \\
&= \frac{ru^4 + 3ru^2}{(1+u^2)^2} - \lambda = 0 \\
\lambda &= \frac{ru^2(u^2+3)}{(1+u^2)^2} \qquad (3.3) \\
\frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2} \\
\lambda(\ell) &= r(\ell)
\end{aligned}
$$

Right now, we know

$$
\begin{cases}
\dfrac{\mathrm{d}r}{\mathrm{d}x} = -u \\
r(0) = a \\
\dfrac{\mathrm{d}\lambda}{\mathrm{d}x} = -\dfrac{u^3}{1+u^2} \\
\lambda(\ell) = r(\ell)
\end{cases}
$$

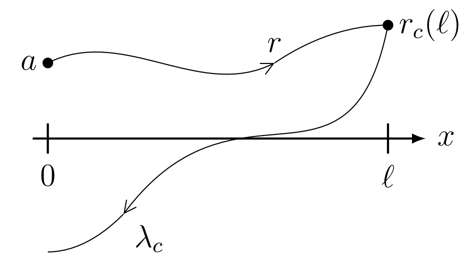
We need to remove u and get a function of r and λ instead. However, it is difficult to solve (3.3). Maybe $H = \text{const.}$ gives us something nicer?

$$
\begin{aligned}
H &= \frac{ru^3}{1+u^2} - \lambda u \\
&= \frac{ru^3}{1+u^2} - \frac{ru^2(u^2+3)}{(1+u^2)^2} u \\
&= -\frac{2ru^3}{(1+u^2)^2} = c
\end{aligned}
$$

Assume we can find $u = G(r, c)$, either numerically or some other way. So, now we have

$$
\begin{cases}
\dfrac{\mathrm{d}r}{\mathrm{d}x} = -G(r,c) \\
r(0) = a \\
\dfrac{\mathrm{d}\lambda}{\mathrm{d}x} = -\dfrac{G^3(r,c)}{1+G^2(r,c)} \\
\lambda(\ell) = r(\ell)
\end{cases}
$$

We do not know c, but we can guess c and simulate r forward in "time" (x) from $r(0) = a$. Then, we simulate λ backwards from $r(\ell)$.

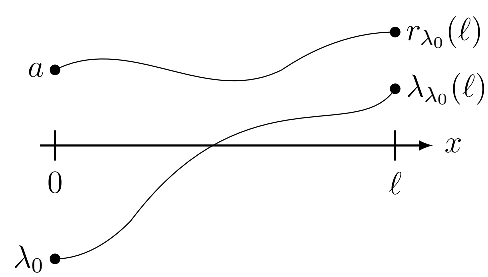


Problem: we can do this for any c. Which c is it? *Last 15 minutes was a dead end!*

Back to $u = F(r, \lambda)$. Assume we have F (numerically).

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}x} &= -F(r, \lambda) \\
r(0) &= a \\
\frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{F^3(r, \lambda)}{1 + F^2(r, \lambda)} \\
\lambda(\ell) &= r(\ell)
\end{aligned}
$$

The mistake before was that the simulation forward from a depends on λ.



Therefore, we should guess λ_0 and simulate both r and λ to get $r_{\lambda_0}(\ell)$ and $\lambda_{\lambda_0}(\ell)$. We need

$$r_{\lambda_0}(\ell) = \lambda_{\lambda_0}$$

for optimality. To do this, we need numerics.

Terminal Constraints

Let $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ and solve

$$
\begin{aligned}
\min_{u \in \mathcal{U}} \int_0^T & L(x, u, t)\, \mathrm{d}t + \Psi(x(T)) \\
\text{s.t.} \qquad \dot{x} &= f(x, u, t) \\
x(0) &= x_0 \\
x_i(T) &= x_{iT} \quad \text{given for } i \in \mathcal{T} \subset \{1, \ldots, n\}
\end{aligned}
$$

First, we augment the cost:

$$
\begin{aligned}
\tilde{J}(u) &= \int_0^T \left[L + \lambda^{\mathrm{T}}(f - \dot{x})\right] \mathrm{d}t + \Psi \\
&= \int_0^T (H - \lambda^{\mathrm{T}}\dot{x})\,\mathrm{d}t + \Psi \\
\tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \left(\varepsilon \frac{\partial H}{\partial u} v + \varepsilon \frac{\partial H}{\partial x}\eta - \varepsilon \lambda^{\mathrm{T}}\dot{\eta}\right) \mathrm{d}t + \varepsilon \frac{\partial \Psi}{\partial x}(x(T))\eta(T) + o(\varepsilon) \\
\delta \tilde{J}(u; v) &= \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right) \eta \,\mathrm{d}t + \int_0^T \frac{\partial H}{\partial u} v \,\mathrm{d}t \\
&\quad + \lambda^{\mathrm{T}}(0)\eta(0) - \lambda^{\mathrm{T}}(T)\eta(T) + \frac{\partial \Psi}{\partial x}(x(T))\eta(T)
\end{aligned}
$$

As always,

$$
\begin{aligned}
\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} \\
\frac{\partial H}{\partial u} &= 0 \quad \text{(FONC)}
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
\eta(0) &= 0 \\
\eta_i(T) &= 0 \quad \text{for } i \in \mathcal{T}
\end{aligned}
$$

Note that if $x(T) = x_T$ is given, then $x(T) = x(T) + \varepsilon\eta(T) + o(\varepsilon)$, so $\eta(T) = 0$. Here, we have $x_i(T) = x_{iT}$ fixed for $i \in \mathcal{T}$ so $\eta_i(T) = 0$ for $i \in \mathcal{T}$.

For optimality, we want

$$
\left[-\lambda^{\mathrm{T}}(T) + \frac{\partial \Psi}{\partial x}(x(T))\right] \eta(T) = 0 \quad \text{for all } \textit{admissible} \text{ variations}
$$

$$
\begin{bmatrix} \dfrac{\partial \Psi}{\partial x_1} - \lambda_1, & \cdots, & \dfrac{\partial \Psi}{\partial x_n} - \lambda_n \end{bmatrix} \begin{bmatrix} \eta_1(T) \\ \vdots \\ \eta_n(T) \end{bmatrix} = 0
$$

Hence, we need

$$
\begin{aligned}
\lambda_j(T) &= \frac{\partial \Psi}{\partial x_j}(x(T)) \quad && \text{if } j \notin \mathcal{T} \\
\lambda_i(T) &= \text{free} && \text{if } i \in \mathcal{T}
\end{aligned}
$$

So we have

$$
\begin{cases}
\dot{x} = f \\
\dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x},
\end{cases}
$$

an ODE with $2n$ variables. We need $2n$ boundary conditions for this ODE to be well-posed.

At $t=0$		At $t=T$	
$x(0)=x_0$	$[n]$	$x_i(T)=x_{iT},\ i\in\mathcal{T}$ $\lvert\mathcal{T}\rvert=q$	$[q]$
		$x_j(T)$ free, $j\notin\mathcal{T}$	$[0]$
$\lambda(0)$ free	$[0]$	$\lambda_i(T)$ free, $i\in\mathcal{T}$	$[0]$
		$\lambda_j(T)=\dfrac{\partial\Psi}{\partial x_j}(x(T)),\ j\notin\mathcal{T}$	$[n-q]$

So we have $n+q+(n-q)=2n$ boundary conditions.

We could even fix some but not all of $x(0)$, i.e.

$$\begin{aligned} x_i(0) &= x_{i0} \quad \text{if } i\in\mathcal{I} \\ x_j(0) &= \text{free} \quad \text{if } j\notin\mathcal{I} \end{aligned}$$

Recall,

$$\delta\tilde{J}(u;v)=\int_0^T\left(\frac{\partial H}{\partial x}+\dot{\lambda}^{\mathrm{T}}\right)\eta\,\mathrm{d}t+\int_0^T\frac{\partial H}{\partial u}v\,\mathrm{d}t+\lambda^{\mathrm{T}}(0)\eta(0)+\left[\lambda^{\mathrm{T}}(T)-\frac{\partial\Psi}{\partial x}(x(T))\right]\eta(T)$$

For $x_i(0)=x_{i0}$ fixed, we have $\eta_i(0)=0$ and $\lambda_i(0)$ free. For $x_j(0)$ free, we have $\eta_j(0)$ free and $\lambda_j(0)=0$.

To ponder, what if $J=\int L\,\mathrm{d}t+\Psi(x(T))+\Theta(x(0))$?

To summarize, the minimizer to

$$\begin{aligned} &\min_{u\in\mathcal{U}}\int_0^T L(x,u,t)\,\mathrm{d}t+\Psi(x(T)) \\ &\text{s.t.} \qquad \dot{x}=f(x,u,t) \\ &\qquad x_i(0)=x_{i0},\quad i\in\mathcal{I} \\ &\qquad x_j(T)=x_{jT}\quad j\in\mathcal{T} \end{aligned}$$

has to satisfy

$$\begin{aligned} &\frac{\partial H}{\partial u}=0 \\ &\dot{\lambda}=-\frac{\partial H^{\mathrm{T}}}{\partial x} \\ &\lambda_i(0)=0,\quad i\notin\mathcal{I} \\ &\lambda_j(T)=\frac{\partial\Psi}{\partial x_j}(x(T)),\quad j\notin\mathcal{T} \end{aligned}$$

Example

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = f(x_1, x_2, x_3, x_4)
$$

$$x_1(0) = 1,\ x_3(0) = 7,\ x_4(0) = 0,\ x_1(1) = 2$$

$$\mathcal{I} = \{1, 3, 4\},\ \mathcal{T} = \{1\}$$

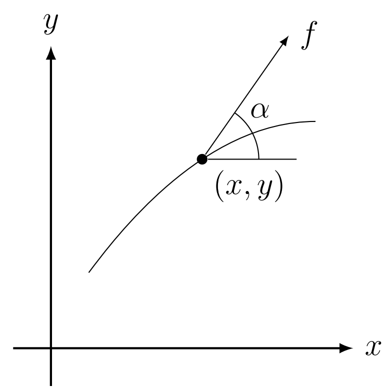
$$\min \int_0^1 L(x, u)\,\mathrm{d}t + \left(x_2^2(1) - x_3^2(1) + 7x_1(1) + 14\right)$$

Note there are 4 boundary conditions on x so there must be 4 boundary conditions on λ:

$\lambda_1(0)$ free/unspecified	$\lambda_1(1)$ free
$\lambda_2(0) = 0$	$\lambda_2(1) = 2x_2(1)$
$\lambda_3(0)$ free	$\lambda_3(1) = -2x_3(1)$
$\lambda_4(0)$ free	$\lambda_4(1) = 0$

Example

A force f acts on a particle at position (x, y) (mass = 1).



$$
\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= |f| \cos\alpha \\
\dot{v}_y &= |f| \sin\alpha \\
\alpha &= \text{control variable}
\end{aligned}
$$

Assume we only care about where the particle ends up (to be specified later), i.e. $L = 0$.

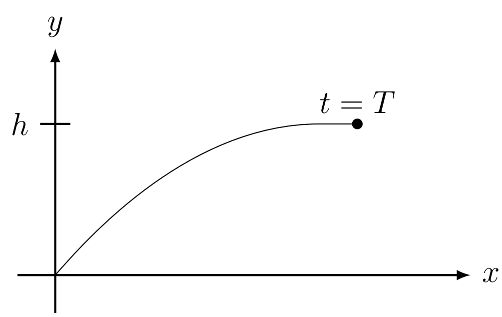
$$
H = \begin{bmatrix} \lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f| \cos\alpha \\ |f| \sin\alpha \end{bmatrix}
$$

$$
\begin{aligned}
\dot{\lambda}_x &= -\frac{\partial H}{\partial x} = 0 &&\implies \lambda_x(t) = c_1\\
\dot{\lambda}_y &= -\frac{\partial H}{\partial y} = 0 &&\implies \lambda_y(t) = c_2\\
\dot{\lambda}_{v_x} &= -\frac{\partial H}{\partial v_x} = -\lambda_x &&\implies \lambda_{v_x}(t) = -c_1 t + c_3\\
\dot{\lambda}_{v_y} &= -\frac{\partial H}{\partial v_y} = -\lambda_y &&\implies \lambda_{v_y}(t) = -c_2 t + c_4
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{\partial H}{\partial \alpha} &= -\lambda_{v_x}|f|\sin\alpha + \lambda_{v_y}|f|\cos\alpha = 0\\
\tan\alpha &= \frac{\lambda_{v_y}}{\lambda_{v_x}} = \frac{-c_2 t + c_4}{-c_1 t + c_3}
\end{aligned}
$$

We want to drive the particle from $[0,0,0,0]^{\mathrm{T}}$ to a path parallel to the x-axis with $y(T) = h$.



Choose $\Psi = -v_x$,

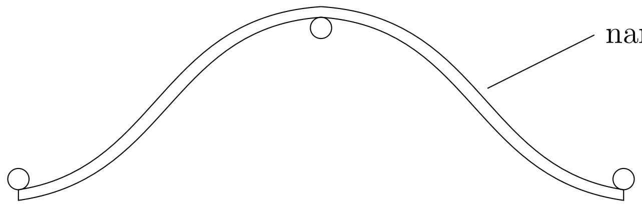
$$
\begin{aligned}
y(T) &= h \quad & v_y(T) &= 0\\
x(T) &\text{ free} \quad & v_x(T) &\text{ free, but costs}\\
\lambda_i(0) &\text{ free} & &\\
\lambda_y(T) &\text{ free} \quad & \lambda_{v_y}(T) &\text{ free}\\
\lambda_x(T) &= 0 \quad & \lambda_{v_x}(T) &= -1
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \lambda_x(t) = 0\\
\implies \lambda_{v_x} &= -c_1 t + c_3 = c_3 = -1\\
\implies \tan\alpha &= -\frac{-c_2 t + c_4}{-1} = c_2 t + c_4
\end{aligned}
$$

How do we find c_2 and c_4? Plug into $\dot{x}$ and $\dot{\lambda}$ and try to satisfy the remaining boundary conditions. (This is hard=numerics.)

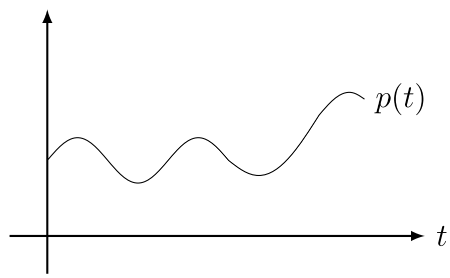
3.2 Splines

From ship building. Splines are used a lot in path-planning, e.g. cubic splines.



But, they are solutions to optimal control problems.

Let $p(t)$ be a curve we'd like to shape.



We want to minimize the "energy" put into the curve, a.k.a acceleration. Let $x_1 = p$ and $x_2 = \dot{p}$, so

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

3.2.1 Minimum-Energy

$$\begin{aligned}
&\min_{u \in \mathcal{U}} \frac{1}{2} \int_0^T u^2(t)\,\mathrm{d}t \quad + \text{Boundary conditions on } x \\
H &= L + \lambda^{\mathrm{T}} f = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u \\
\frac{\partial H}{\partial u} &= u + \lambda_2 = 0 \implies u = -\lambda_2 \\
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = c_1 \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -c_1 t + c_2 \\
u &= -\lambda_2 = c_1 t - c_2 \\
\dot{x}_2 &= u = c_1 t - c_2 \implies x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3 \\
\dot{x}_1 &= x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3 \\
&\implies x_1 = \frac{c_1}{6} t^3 - \frac{c_2}{2} t^2 + c_3 t + c_4
\end{aligned}$$

$p(t)$ is a cubic polynomial!

What about boundary conditions?

Let $T = 1$, $p(0)$ given, $p(1)$ given, $\dot{p}(0) = 0$, $\dot{p}(1) = 0$, e.g. $p(0) = 0$, $p(1) = 1$. Since the boundary conditions for x are all specified, those for the costate are free.

$$\left.\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= 0 \\ x_1(1) &= 1 \\ x_2(1) &= 0 \end{aligned}\right\} \implies \left\{\begin{aligned} &\lambda_1(0) \\ &\lambda_2(0) \\ &\lambda_1(1) \\ &\lambda_2(1) \end{aligned}\right. \quad \text{free/unspecified}$$

$$\begin{aligned} x_2(0) &= c_3 = 0 & x_1(1) &= \frac{2c_2}{6} - \frac{c_2}{2} = 1 \\ x_1(0) &= c_4 = 0 & c_2 &= -6 \\ x_2(1) &= \frac{c_1}{2} - c_2 + \underbrace{c_3}_{0} = 0 & c_1 &= -12 \\ c_1 &= 2c_2 & & \end{aligned}$$

$$\begin{aligned} \implies p(t) &= -2t^3 + 3t^2 \\ u(t) &= -12t + 6 \end{aligned}$$

Or, what if $\dot{p}(0)$, $\dot{p}(1)$ are not specified?

$$\left.\begin{aligned} &x_1(0) = 0 \\ &x_2(0) \text{ unspec.} \\ &x_1(1) = 1 \\ &x_2(1) \text{ unspec.} \end{aligned}\right\} \implies \left\{\begin{aligned} &\lambda_1(0) \text{ unspec.} \\ &\lambda_2(0) = 0 \\ &\lambda_1(1) \text{ unspec.} \\ &\lambda_2(1) = 0 \end{aligned}\right.$$

$$\left.\begin{aligned} \lambda_2(0) &= c_2 = 0 \\ \lambda_2(1) &= -c_1 + c_2 = 0 \end{aligned}\right\} \implies u = c_1 t - c_2 = 0$$

$$\left.\begin{aligned} x_1(0) &= c_4 = 0 \\ x_1(1) &= c_3 = 1 \end{aligned}\right\} \implies p(t) = t$$

What did we do?

Case 1:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1/2 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_2(0) \\ x_2(1) \end{bmatrix}$$

Case 2:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ \lambda_2(0) \\ \lambda_2(1) \end{bmatrix}$$

3.2.2 Generalized Splines

We had $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This A is nilpotent ($A^k = 0$ for some $k \in \mathbb{Z}^+$). This means e^{At} is a polynomial in t. (This e^{At} is cubic.)

In general, e^{At} is a mix of polynomials, exponentials, and trignometric terms. The eigenvalues of A determine the form of $x(t)$.

$$\begin{aligned} \dot{x} &= Ax &&\Longrightarrow x(t) = e^{At}x(0) \\ \dot{x} &= Ax + Bu &&\Longrightarrow x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau \end{aligned}$$

The general problem to solve is

$$\begin{aligned} &\min_{u \in \mathcal{U}} \int_0^T \frac{1}{2}\|u\|^2\,\mathrm{d}t \\ &\text{s.t. } \dot{x} = Ax + Bu \\ &\quad + \text{Boundary conditions} \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2}\|u\|^2 + \lambda^{\mathrm{T}}(Ax + Bu) \\ \frac{\partial H}{\partial u} &= u^{\mathrm{T}} + \lambda^{\mathrm{T}}B = 0 \\ &\Rightarrow u = -B^{\mathrm{T}}\lambda \\ \dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} = -A^{\mathrm{T}}\lambda \end{aligned}$$

We have the Hamiltonian Dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BB^{\mathrm{T}} \\ 0 & -A^{\mathrm{T}} \end{bmatrix}}_{M} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

Where we used $\dot{x} = Ax + Bu = Ax - BB^{\mathrm{T}}\lambda$. Then,

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

Suppose we want to drive from $x(0) = x_0$ to $x(T) = x_T$.

$$\begin{bmatrix} x_T \\ \lambda(T) \end{bmatrix} = e^{MT} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix} = \begin{bmatrix} N_{xx} & N_{x\lambda} \\ N_{\lambda x} & N_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix}$$
$$x_T = N_{xx}x_0 + N_{x\lambda}\lambda(0)$$

$N_{x\lambda}$ is invertible if (A, B) is completely controllable. Assume it is.

$$\begin{gathered}
\lambda(0) = N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \\
\implies \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} x_0 \\ N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \end{bmatrix} \\
\implies u(t) = -B^{\mathrm{T}}\lambda(t)
\end{gathered}$$

This is the optimal trajectory, but there is no feedback. We will consider closed-loop systems after the midterm.

As a preview, we need to find λ as a function of x. For example, $u = -R^{-1}B^{\mathrm{T}}Px$ minimizes $u^{\mathrm{T}}Ru$, so $\lambda = Px$ where P is the solution to the Riccati equation.

3.3 Numerical Methods

Optimal control boils down to solving two sets of differential equations:

$$\begin{aligned}
\dot{x} &= f(x, u) & \frac{\partial H}{\partial u}(x, u, \lambda) &= 0 \\
\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x}(x, u, \lambda) & u &= F(x, \lambda)
\end{aligned}$$
$$\implies \begin{cases} \dot{x} = f(x, F(x, \lambda)) \\ \dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x}(x, F(x, \lambda), \lambda) \end{cases}$$

The equations are functions of x and λ. They are completely determined by the boundary conditions on $x(0)$, $x(T)$, $\lambda(0)$, $\lambda(T)$. This is known as the *Boundary Value Problem.* This is solved using *test shooting*:

1. Guess initial conditions
2. Simulate forward in time
3. Update the guess (cleverly...)

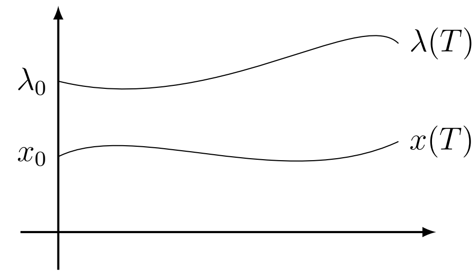
Exmaple: Bolza problem

$$\begin{aligned}
&\min_{u \in \mathcal{U}} \int_0^T L(x, u)\,\mathrm{d}t + \Psi(x(T)) \\
&\text{s.t. } \begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases} \\
&H(x, u, \lambda) = L(x, u) + \lambda^{\mathrm{T}} f(x, u) \\
&u^*(x, \lambda) \text{ satisfies } \frac{\partial H}{\partial u} = 0
\end{aligned}$$

The optimal control satisfies

$$\begin{cases} x = f(x, u^*(x,\lambda)) \\ x(0) = x_0 \\ \lambda = -\dfrac{\partial H^{\mathrm{T}}}{\partial x}(x, u^*(x,\lambda), \lambda) \\ \lambda(T) = \dfrac{\partial \Psi}{\partial x}(x(T)) \end{cases}$$

Algorithm Guess λ_0 and solve for $x(t)$, $\lambda(t)$.



Let's define a cost:

$$\left\| \lambda(T) - \frac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) \right\|^2 = g(\lambda_0)$$

Update λ_0 through

$$\lambda_0 \coloneqq \lambda_0 - \underset{\substack{\uparrow \\ \text{any choice of step size works}}}{\gamma} \frac{\partial g^{\mathrm{T}}}{\partial \lambda_0}(\lambda_0)$$

Repeat

Problem: What is $\partial g/\partial \lambda_0$? We estimate $\partial g/\partial \lambda_0$ numerically. This is where "test shooting" comes into play.

Let e_i be the ith unit vector, $i = 1, \ldots, n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \; \ldots, \; e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\frac{\partial g}{\partial \lambda_0} = \left(\frac{\partial g}{\partial \lambda_{0,1}}, \frac{\partial g}{\partial \lambda_{0,2}}, \ldots, \frac{\partial g}{\partial \lambda_{o,n}} \right)$$

The ith component of $\partial g/\partial \lambda_0$ is given by the directional derivative

$$\frac{\partial g}{\partial \lambda_{0,i}} = \frac{\partial g}{\partial \lambda_0} \cdot e_i = \delta g(\lambda_0; e_i) = \lim_{\varepsilon \to 0} \frac{g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)}{\varepsilon}$$

So, if $x \in \mathbb{R}^n$ (and thus so is λ_0), we have to do this n times (with a small ε) and get the full derivative $\partial g/\partial \lambda_0$.

Given λ_0, $g(\lambda_0)$
for $i = 1$ to n **do**
  Compute $g(\lambda_0 + \varepsilon e_i)$
  $dg_i = \frac{1}{\varepsilon}[g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)]$
end for
$\frac{\partial g}{\partial \lambda_0} = [dg_1, \ldots, dg_n]$

Algorithm

Example LQ

$$
\begin{aligned}
&\min_u \frac{1}{2}\int_0^1 (x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)\,\mathrm{d}t + \frac{1}{2}x^{\mathrm{T}}(1)Sx(1) \\
&\text{s.t.} \begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases} \\
&\quad Q, R, S \succ 0
\end{aligned}
$$

$$
\begin{aligned}
H &= \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + \lambda^{\mathrm{T}}(Ax + Bu) \\
\frac{\partial H}{\partial u} &= u^{\mathrm{T}}R + \lambda^{\mathrm{T}}B = 0 \\
u^* &= -R^{-1}B^{\mathrm{T}}\lambda \\
\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} = -Qx - A^{\mathrm{T}}\lambda \\
\lambda(1) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(1)) = Sx(1)
\end{aligned}
$$

So putting it all together,

$$
\begin{aligned}
\dot{x} &= Ax - BR^{-1}B^{\mathrm{T}}\lambda \qquad && x(0) = x_0 \\
\dot{\lambda} &= -Qx - A^{\mathrm{T}}\lambda && \lambda(1) = Sx(1)
\end{aligned}
$$

Example Newton's nose shape problem (revisited, see previous)

$$
\begin{aligned}
&\min_u \int_0^\ell \frac{ru^3}{1+u^2}\,\mathrm{d}x + \frac{1}{2}r(\ell)^2 \\
&\text{s.t.} \ \frac{\mathrm{d}r}{\mathrm{d}x} = -u \qquad r(0) = a
\end{aligned}
$$

$$H = \frac{ru^3}{1+u^2} + \lambda(-u)$$
$$\frac{\partial H}{\partial u} = \frac{ru^2(3+u^2)}{(1+u^2)^2} - \lambda = 0$$

We solve the above numerically to get $u^*(r, \lambda)$.

$$\frac{\partial \lambda}{\partial x} = -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2}$$
$$\lambda(\ell) = r(\ell)$$

So, we have

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}x} &= -u & r(0) &= a & u &= F(x, \lambda) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{u^3}{1+u^2} & \lambda(\ell) &= r(\ell) \end{aligned}$$

Example Fixed terminal constraints (revisited, see previous)

$$\begin{aligned} &\min_{\alpha} -v_x(T) \qquad \alpha = \text{control} \\ &\text{s.t.} \qquad \dot{x} = v_x \qquad x(0) = 0 \\ &\qquad \dot{y} = v_y \qquad y(0) = 0 \\ &\qquad \dot{v}_x = |f| \cos\alpha \quad v_x(0) = 0 \\ &\qquad \dot{v}_y = |f| \sin\alpha \quad v_y(0) = 0 \\ &\qquad y(T) = h \\ &\qquad v_y(T) = 0 \end{aligned}$$

$$
\begin{aligned}
H &= -v_x(T) + \begin{bmatrix}\lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y}\end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f|\cos\alpha \\ |f|\sin\alpha \end{bmatrix} \\
\frac{\mathrm{d}H}{\mathrm{d}\alpha} &= 0 \Rightarrow \tan\alpha = \frac{\lambda_{v_y}}{\lambda_{v_x}} \\
\dot{\lambda}_x &= 0 \\
\dot{\lambda}_y &= 0 \\
\dot{\lambda}_{v_x} &= -\lambda_x \\
\dot{\lambda}_{v_y} &= -\lambda_y \\
\boldsymbol{\lambda}(0) &\text{ unspecified} \\
\lambda_x(T) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) = 0 \\
\lambda_y(T) &\text{ unspecified} \\
\lambda_{v_x}(T) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial v_x}(v_x(T)) = -1 \\
\lambda_{v_y}(T) &\text{ unspecified}
\end{aligned}
$$

Again, we guess λ_0 and solve forward in time. But, we have terminal constraints on y and v_y as well.

$$
g(\lambda_0) = \frac{1}{2}\Big[(y(T) - h)^2 + (v_y(T))^2 + (\lambda_x(T))^2 + (\lambda_{v_x} + 1)^2\Big]
$$

3.4 Terminal Manifolds

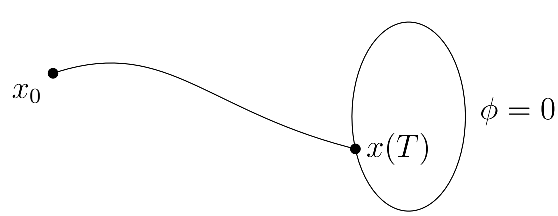
We can solve

$$
\begin{aligned}
&\min_{u\in\mathcal{U}} \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T)) \\
&\text{s.t. } \dot{x} = f(x,u,t)
\end{aligned}
$$

with all sorts of boundary conditions on x:

- $x(0) = x_0$, $x(T)$ free (typical)
- $x_i(0) = x_{i0}$, $i \in \mathcal{I}$ and $x_j(T) = x_{jT}$, $j \in \mathcal{T}$

But what if we want $x(T)$ to belong to a set?



Problem

$$\begin{aligned}
&\min_{u\in\mathcal{U}} \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T))\\
&\text{s.t. } \dot{x} = f(x,u,t), \quad x\in\mathbb{R}^n\\
&\qquad x(0) = x_0\\
&\qquad \phi(x(T)) = 0, \quad \phi:\mathbb{R}^n\to\mathbb{R}^q,\ q\le n
\end{aligned}$$

The augmented cost is

$$\tilde{J} = \int_0^T [H(x,u,t,\lambda) - \lambda^{\mathrm{T}}\dot{x}]\,\mathrm{d}t + \Psi(x(T)) + \underset{\substack{\uparrow\\ \text{q-dimensional}\\ \text{Lagrange multiplier}}}{\nu}^{\mathrm{T}}\phi(x(T))$$

Let $\Phi(x(T),\nu) = \Psi(x(T)) + \nu^{\mathrm{T}}\phi(x(T))$. Then,

$$\tilde{J} = \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta\,\mathrm{d}t + \Phi(x(T),\nu)$$

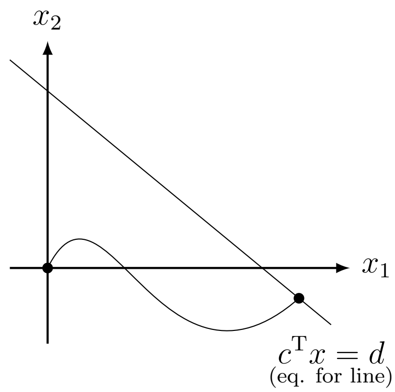
We know how to solve this! With $u\mapsto u+\varepsilon v$, $x\mapsto x+\varepsilon\eta + o(\varepsilon)$,

$$\begin{aligned}
\delta\tilde{J} = &\int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta\,\mathrm{d}t + \int_0^T \frac{\partial H}{\partial u} v\,\mathrm{d}t + \frac{\partial \Phi}{\partial x}(x(T),\nu)\eta(T)\\
&- \lambda^{\mathrm{T}}(T)\eta(T) + \lambda^{\mathrm{T}}(0)\underbrace{\eta(0)}_{=0}
\end{aligned}$$

$$\begin{cases}
\dfrac{\partial H}{\partial u} = 0\\
\dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x}\\
\lambda(T) = \dfrac{\partial \Phi^{\mathrm{T}}}{\partial x}(x(T), \overset{q\text{ new variables}}{\overset{\downarrow}{\nu}})\\
\phi(x(T)) = 0 \ \leftarrow q \text{ new equations}
\end{cases}$$
$$\Longrightarrow u^*$$

Spling to line

$$\begin{aligned}
&\min_u \frac{1}{2}\int_0^1 u^2(t)\,\mathrm{d}t\\
&\text{s.t. } \begin{cases}
\dot{x}_1 = x_2\\
\dot{x}_2 = u\\
x_1(0) = 0,\ x_2(0) = 0\\
c_1x_1(1) + c_2x_2(1) = d
\end{cases}
\end{aligned}$$



$$
\begin{aligned}
H &= \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u \\
\frac{\partial H}{\partial u} &= u + \lambda_2 \Longrightarrow u = -\lambda_2 \\
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \Longrightarrow \lambda_1 = k_1 \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \Longrightarrow \lambda_2 = -k_1 t + k_2 \\
\phi(x(1)) &= c_1 x_1(1) + c_2 x_2(1) - d \\
\Psi &= 0 \Longrightarrow \Phi = \nu(c_1 x_1(1) + c_2 x_2(1) - d) \\
\lambda_1(1) &= \frac{\partial \Phi}{\partial x_1} = \nu c_1 \\
\lambda_2(1) &= \frac{\partial \Phi}{\partial x_2} = \nu c_2
\end{aligned}
$$

So,

$$
\begin{aligned}
\lambda_1(1) &= \nu c_1 = k_1 \\
\lambda_2(1) &= \nu c_2 = -k_1 + k_2 \\
k_2 &= \nu(c_1 + c_2) \\
\dot{x}_2 &= u = -\lambda_2 = k_1 t - k_2 \\
x_2 &= \frac{k_1}{2}t^2 - k_2 t + 0 \\
\dot{x}_1 &= x_2 \\
x_1 &= \frac{k_1}{6}t^3 - \frac{k_2}{2}t^2 + 0
\end{aligned}
$$

Substituting k_1 and k_2 into $c_1 x_1(1) + c_2 x_2(1) = d$,

$$
\begin{aligned}
\nu\left(-\frac{c_1^2}{3} - c_1 c_2 - c_2^2\right) &= d \\
\nu &= -\frac{d}{c_1^2/3 + c_1 c_2 + c_2^2}
\end{aligned}
$$

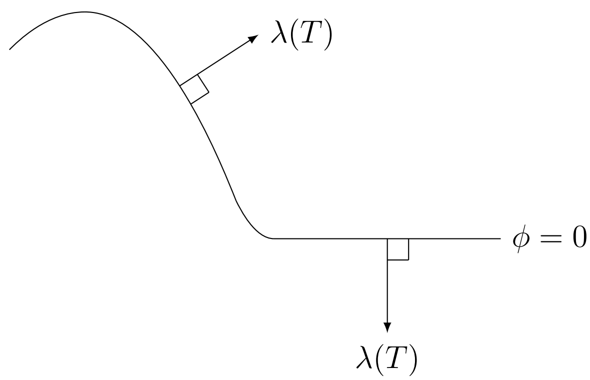
And finally

$$\boxed{u = k_1 t - k_2 = \frac{d}{c_1^2/3 + c_1 c_2 + c_2^2}(c_1 + c_2 - c_1 t)}$$

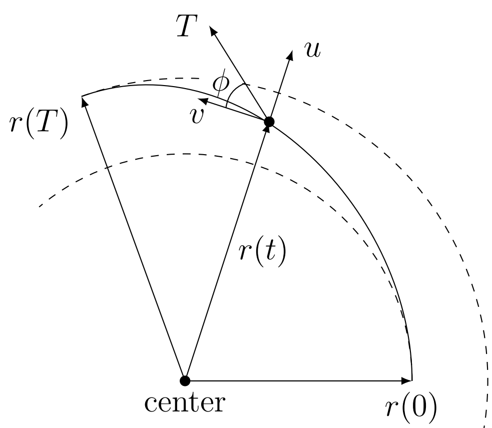
As a final observation if $\Psi = 0$ then

$$\lambda(T) = \nu^{\mathrm{T}} \frac{\partial \phi}{\partial x}(x(T)),$$

which means $\lambda(T)$ is orthogonal to the tangent plane to $\phi(x(T))$.



Example Maximum orbit transform (e.g. Hidden Figures)



$$\begin{aligned}
r &= \text{radial distance from spacecraft to} \\
&\quad\ \text{center} \\
u &= \text{radial velocity} \\
v &= \text{tangential velocity} \\
m &= \text{mass of spacecraft} \\
\dot{m} &= -\text{fuel consumption rate} \\
\phi &= \text{thrust angle (control input)} \\
T &= \text{thrust}
\end{aligned}$$

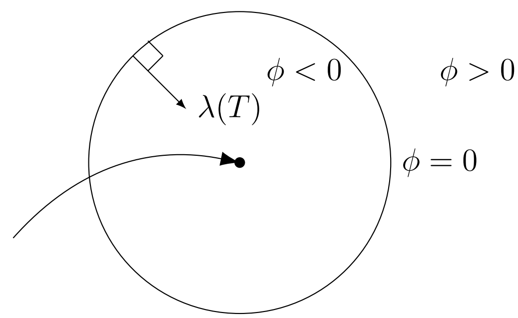
$$
\begin{aligned}
&\max_{\phi} r(T) \Longleftrightarrow \min_{\phi} -r(T) \\
&\text{s.t.} \quad \begin{cases}
\dot{r} = u \\
\dot{u} = \dfrac{v^2}{r} - \dfrac{g}{r^2} + \dfrac{T\sin\phi}{m_0 - |\dot{m}|t} \\
\dot{v} = -\dfrac{uv}{r} + \dfrac{T\cos\phi}{m_0 - |\dot{m}|t} \\
r(0) = r_0 \\
u(0) = 0 \\
v(0) = \sqrt{\dfrac{g}{r_0}} \\
u(T) = 0 = \phi_1 \\
v(T) = \sqrt{\dfrac{g}{r(T)}} = \phi_2
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
H &= \lambda_r u + \lambda_u \left(\frac{v^2}{r} - \frac{g}{r^2} + \frac{T\sin\phi}{m_0 - |\dot{m}|t}\right) + \lambda_v \left(-\frac{uv}{r} + \frac{T\cos\phi}{m_0 - |\dot{m}|t}\right) \\
\Phi &= \underbrace{\nu_1 u(T) + \nu_2 \left(v(T) - \sqrt{\frac{g}{r(T)}}\right)}_{\nu^{\mathrm{T}}\phi} \underbrace{-r(T)}_{\Psi} \\
\frac{\partial H}{\partial \phi} &= \frac{\lambda_u T\cos\phi - \lambda_v T\sin\phi}{m_0 - |\dot{m}|t} = 0 \\
&\Rightarrow \tan\phi = \frac{\lambda_u}{\lambda_v} \\
\dot{\lambda}_r &= -\frac{\partial H}{\partial r} = -\lambda_u \left(-\frac{v^2}{r^2} + \frac{2g}{r^3}\right) - \lambda_v \cdot \frac{uv}{r^2} \\
\dot{\lambda}_u &= -\frac{\partial H}{\partial u} = -\lambda_r + \lambda_v \cdot \frac{v}{r} \\
\dot{\lambda}_v &= -\frac{\partial H}{\partial v} = -\lambda_u \cdot \frac{2v}{r} + \lambda_v \cdot \frac{u}{r} \\
&\begin{cases}
\lambda_r(T) = \dfrac{\partial \Phi}{\partial r} = -1 + \dfrac{\nu_2 \sqrt{g}}{2(r(T))^{3/2}} \\
\lambda_u(T) = \dfrac{\partial \Phi}{\partial u} = \nu_1 \\
\lambda_v(T) = \dfrac{\partial \Phi}{\partial v} = \nu_2 \\
u(T) = 0 \\
v(T) = \sqrt{\dfrac{g}{r(T)}}
\end{cases}
\end{aligned}
$$

This needs numerics to solve.

3.4.1 Terminal manifold with inequality constraints

$$
\begin{aligned}
&\min_u \int_0^T L\,\mathrm{d}t + \Psi \\
&\dot{x} = f(x,u) \\
&\phi(x(T)) \le 0
\end{aligned}
$$



Repeat process: $\tilde{J} = \int (H - \lambda^{\mathrm{T}}\dot{x})\,\mathrm{d}t + \Psi + \nu^{\mathrm{T}}\phi$. The optimality conditions are

$$
\begin{cases}
\dfrac{\partial H}{\partial u} = 0 \\
\dot{\lambda} = -\dfrac{\partial H}{\partial x}^{\mathrm{T}} \\
\lambda(T) = \dfrac{\partial \Psi}{\partial x}^{\mathrm{T}}(x(T)) + \nu^{\mathrm{T}}\dfrac{\partial \phi}{\partial x}^{\mathrm{T}}(x(T)) \\
\nu \ge 0 \\
\phi(x(T)) \le 0 \\
\nu^{\mathrm{T}}\phi(x(T)) = 0 \quad \text{(KKT)}
\end{cases}
$$

3.4.2 Initial manifold

$$
\begin{aligned}
\min_{x_0,u} &\int L + \Psi(x(T)) + \Theta(x(0)) \\
\text{s.t.}\ \ &\dot{x} = f(x,u) \\
&\phi(x(T)) = 0 \\
&\xi(x(0)) = 0
\end{aligned}
$$

$$
\begin{aligned}
\tilde{J} &= \int (H - \lambda^{\mathrm{T}}\dot{x})\,\mathrm{d}t + \Psi(x(T)) + \Theta(x(0)) + \nu_\phi^{\mathrm{T}}\phi(x(T)) + \nu_\xi^{\mathrm{T}}\xi(x(0)) \\
\delta\tilde{J} &= \int \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta + \frac{\partial H}{\partial u}v\right]\mathrm{d}t + \left[\frac{\partial \Psi}{\partial x}(x(T)) + \nu_\phi^{\mathrm{T}}\frac{\partial \phi}{\partial x}(x(T)) - \lambda^{\mathrm{T}}(T)\right]\eta(T) \\
&\quad + \left[\frac{\partial \Theta}{\partial x}(x(0)) + \nu_\xi^{\mathrm{T}}\frac{\partial \xi}{\partial x}(x(0)) + \lambda^{\mathrm{T}}(0)\right]\eta(0)
\end{aligned}
$$

The optimality conditions are

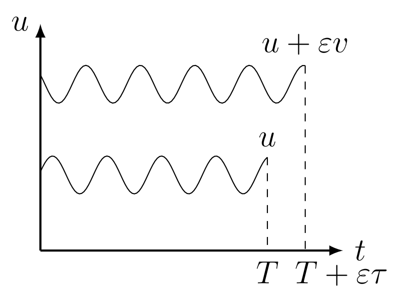
$$\begin{cases} \dfrac{\partial H}{\partial u} = 0 \\ \dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x} \\ \lambda(T) = -\dfrac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) - \nu_\phi^{\mathrm{T}} \dfrac{\partial \phi^{\mathrm{T}}}{\partial x}(x(T)) \\ \lambda(0) = -\dfrac{\partial \Theta^{\mathrm{T}}}{\partial x}(x(0)) - \nu_\xi^{\mathrm{T}} \dfrac{\partial \xi^{\mathrm{T}}}{\partial x}(x(0)) \end{cases}$$

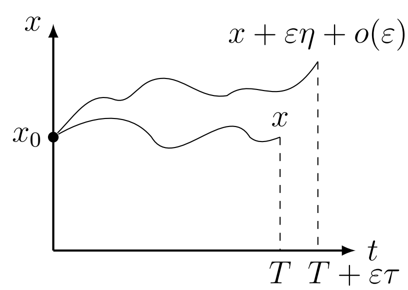
3.4.3 Unspecified Terminal Times

For example, instead of driving to the moon using minimum fuel, we want to get there as soon as possible:

$$\min_{u,T} \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T),T).$$

The variations are $u \mapsto u + \varepsilon v$ and $T \mapsto T + \varepsilon\tau$





$$\begin{aligned} \tilde{J}(u,T) &= \int_0^T [L(x,u,t) + \lambda^{\mathrm{T}}(f(x) - \dot{x})]\,\mathrm{d}t + \Psi(x(T),T) \\ &= \int_0^T [H - \lambda^{\mathrm{T}}\dot{x}]\,\mathrm{d}t + \Psi \\ \tilde{J}(u+\varepsilon v, T+\varepsilon\tau) &= \int_0^T [H(x+\varepsilon\eta, u+\varepsilon v, t, \lambda) - \lambda^{\mathrm{T}}(\dot{x}+\varepsilon\dot{\eta})]\,\mathrm{d}t \\ &\quad + \int_T^{T+\varepsilon\tau} [H(x+\varepsilon\eta, u+\varepsilon v, t, \lambda) - \lambda^{\mathrm{T}}(\dot{x}+\varepsilon\dot{\eta})]\,\mathrm{d}t \\ &\quad + \Psi(x(T+\varepsilon\tau) + \varepsilon\eta(T+\varepsilon\tau), T+\varepsilon\tau) \end{aligned}$$

$$
\begin{aligned}
\tilde{J}(u+\varepsilon v, T+\varepsilon\tau) - \tilde{J}(u,T) = \varepsilon \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta \,\mathrm{d}t + \varepsilon \int_0^T \frac{\partial H}{\partial u} v \,\mathrm{d}t& \\
- \varepsilon \lambda^{\mathrm{T}}(T)\eta(T) + \varepsilon\lambda^{\mathrm{T}}(0)\eta(0) + o(\varepsilon)& \\
+ \underbrace{\int_T^{T+\varepsilon\tau} [H(x+\varepsilon\eta, u+\varepsilon v, t, \lambda) - \lambda^{\mathrm{T}}(\dot{x}+\varepsilon\dot{\eta})] \,\mathrm{d}t}_{\text{(I)}}& \\
+ \underbrace{\Psi(x(T+\varepsilon\tau) + \varepsilon\eta(T+\varepsilon\tau), T+\varepsilon\tau) - \Psi(x(T),T)}_{\text{(II)}}&
\end{aligned} \tag{3.4}
$$

For term I, use the mean value theorem to get rid of terms inside the integral that have a ε before them:

$$
\begin{aligned}
&\int_T^{T+\varepsilon\tau} [L + \lambda^{\mathrm{T}}(f - \dot{x} - \varepsilon\dot{\eta})] \,\mathrm{d}t \\
&\quad = \int_T^{T+\varepsilon\tau} \left[L(x,u,t) + \varepsilon\frac{\partial L}{\partial x}\eta + \varepsilon\frac{\partial L}{\partial u} v + \lambda^{\mathrm{T}}\left(f + \varepsilon\frac{\partial f}{\partial x}\eta + \varepsilon\frac{\partial f}{\partial u} v - \dot{x} - \varepsilon\dot{\eta}\right)\right] \mathrm{d}t + o(\varepsilon) \\
&\quad = \varepsilon\tau\big[L + \lambda^{\mathrm{T}}(f - \dot{x})\big]\Big|_{t=\xi} + o(\varepsilon) = \varepsilon\tau L\Big|_{t=\xi} + o(\varepsilon) \\
&\quad = \varepsilon\tau L(x(\xi), u(\xi), \xi) + o(\varepsilon), \quad \xi \in [T, T+\varepsilon\xi]
\end{aligned} \tag{3.5}
$$

Note that as $\varepsilon \to 0$, $\xi \to T$.

For term II, we further split it into two parts:

$$
\Psi(x+\varepsilon\eta, T+\varepsilon\tau) - \Psi(x,T) = \underbrace{\Psi(x, T+\varepsilon\tau)}_{\text{(II.a)}} + \underbrace{\varepsilon\frac{\partial \Psi}{\partial x}(x, T+\varepsilon\tau)\eta(T+\varepsilon\tau)}_{\text{(II.b)}} - \Psi(x,T)
$$

$$
\begin{aligned}
\text{(II.a)} \Longrightarrow \Psi(x, T+\varepsilon\tau) &= \Psi(x(T), T+\varepsilon\tau) + \varepsilon\frac{\partial\Psi}{\partial x}(x(T), T+\varepsilon\tau)\dot{x}(T)\tau + o(\varepsilon) \\
&= \Psi(x(T),T) + \varepsilon\frac{\partial\Psi}{\partial x}(x(T),T)\dot{x}(T)\tau + \varepsilon\frac{\partial\Psi}{\partial T}(x(T),T)\tau + o(\varepsilon) \\
\text{(II.b)} \Longrightarrow \varepsilon\frac{\partial\Psi}{\partial x}&(x, T+\varepsilon\tau)\eta(T+\varepsilon\tau) \\
&= \varepsilon\left[\frac{\partial\Psi}{\partial x}(x(T),T) + \varepsilon\frac{\partial^2\Psi}{\partial x^2}\dot{x}\tau + \varepsilon\frac{\partial^2\Psi}{\partial T\partial x}\tau + o(\varepsilon)\right] \\
&\quad \times \left[\eta(T) + \varepsilon\dot{\eta}(T)\tau + o(\varepsilon)\right] \\
&= \varepsilon\frac{\partial\Psi}{\partial x}(x(T),T)\eta(T) + o(\varepsilon) \\
\text{(II)} \Longrightarrow \Psi(x+\varepsilon\eta, &T+\varepsilon\tau) - \Psi(x,T) \\
&= \varepsilon\frac{\partial\Psi}{\partial x}(x(T),T)[\dot{x}(T)\tau + \eta(T)] + \varepsilon\frac{\partial\Psi}{\partial T}(x(T),T)\tau + o(\varepsilon)
\end{aligned} \tag{3.6}
$$

Substituting (3.5) and (3.6) into (3.4) and taking the directional derivative,

$$\delta\tilde{J} = \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta \, \mathrm{d}t + \int_0^T \frac{\partial H}{\partial u} v \, \mathrm{d}t + \lambda^{\mathrm{T}}(0)\eta(0)$$
$$+ \left[L + \frac{\partial \Psi}{\partial T} + \frac{\partial \Psi}{\partial x} f\right]\tau\Big|_{t=T} + \left(\frac{\partial \Psi}{\partial x} - \lambda^{\mathrm{T}}\right)\eta\Big|_{t=T}$$

So we have a mix of old and new:

$$\begin{aligned} \text{old: } \frac{\partial H}{\partial u} &= 0 \\ \dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} \\ \lambda(T) &= \frac{\partial \Psi}{\partial x}\Big|_T \\ \text{new: } L + \frac{\partial \Psi}{\partial T} + \lambda^{\mathrm{T}} f\Big|_T &= 0 \end{aligned}$$

This last condition is known as the *Transversality condition.*

Example Pure minimum time question

$$\min_{u,T} \int_0^T \mathrm{d}t$$
$$\dot{x} = f(x,u)$$
$$x(0) = x_0$$
$$x(T) = x_T$$
$$H = L + \lambda^{\mathrm{T}} f = 1 + \lambda^{\mathrm{T}} f$$

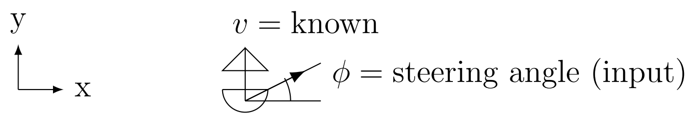
The transversality condition is

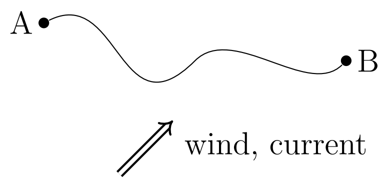
$$L + \frac{\partial \Psi}{\partial T} + \lambda^{\mathrm{T}} f\Big|_T = 0$$
$$\lambda^{\mathrm{T}} f\big|_T = -1$$
$$H(T) = 1 + \lambda^{\mathrm{T}} f\big|_T = 1 - 1 = 0$$

But this is a conservative system, so H is a constant. Therefore,

$$H(t) = 0 \quad \forall t \in [0,T]$$

Example Zermelo's problem: sail from A to B as quickly as possible in the presence of known winds and currents.





The dynamics are

$$\begin{aligned}\dot{x} &= v\cos\phi + c_1(x,y) \\ \dot{y} &= v\sin\phi + c_2(x,y)\end{aligned} \qquad \lambda = \begin{bmatrix}\lambda_x \\ \lambda_y\end{bmatrix}$$

For minimum time, $L = 1$.

$$\begin{aligned}H &= 1 + \lambda_x(v\cos\phi + c_1) + \lambda_y(v\sin\phi + c_2) \\ 0 &= \frac{\partial H}{\partial \phi} = -v\lambda_x \sin\phi + v\lambda_y \cos\phi \\ \phi &= \tan^{-1}\left(\frac{\lambda_y}{\lambda_x}\right)\end{aligned}$$

Since this is a conservative system and $\partial\Psi/\partial T = 0$, then $H(t) = H(T) = 0$.

$$\begin{aligned}-1 &= \lambda_x(v\cos\phi + c_1) + \lambda_y(v\sin\phi + c_2) \\ \lambda_x &= -\frac{\cos\phi}{v + c_1\cos\phi + c_2\sin\phi} \\ \lambda_y &= -\frac{\sin\phi}{v + c_1\cos\phi + c_2\sin\phi} \\ \dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} \\ \dot{\lambda}_x &= -\lambda_x \frac{\partial c_1}{\partial x} - \lambda_y \frac{\partial c_2}{\partial x} \\ \dot{\lambda}_y &= -\lambda_x \frac{\partial c_1}{\partial y} - \lambda_y \frac{\partial c_2}{\partial y} \\ \dot{\phi} &= \sin^2\phi \frac{\partial c_2}{\partial x} + \sin\phi\cos\phi\left(\frac{\partial c_1}{\partial x} - \frac{\partial c_2}{\partial y}\right) - \cos^2\phi \frac{\partial c_1}{\partial y}\end{aligned}$$

This is an ODE that completely determines ϕ if we just had ϕ_0.

Example We want to drive a car and stop at a stop sign as quickly as possible. Assume that the stop sign is at the origin, and our control is the acceleration ($\ddot{x} = u$).

$$\begin{aligned} &\min_{u,T} \int_0^T \mathrm{d}t \\ &\text{s.t.} \quad \begin{cases} \dot{x}_1 = x_2, & x(0) = x_0 \\ \dot{x}_2 = u, & x(T) = 0 \end{cases} \end{aligned}$$

Recall the transversality condition:

$$H + \left.\frac{\partial \Psi}{\partial T}\right|_{t=T} = 0.$$

For minimum-time problems, $L = 1$ and $\Psi = 0$, so $\lambda^{\mathrm{T}} f|_{t=T} = -1$.

$$\begin{gathered} H = 1 + \lambda_1 x_2 + \lambda_2 u \\ \lambda_1(T) \underbrace{x_2(T)}_{=0 \text{ (rest)}} + \lambda_2(T) u(T) = -1 \\ \boxed{\lambda_2(T) u(T) = -1} \\ \frac{\partial H}{\partial u} = \boxed{\lambda_2 = 0}, \end{gathered}$$

i.e. $0 \cdot u(T) = -1$!? This problem is ill-posed; we need to go infinitely fast…

Idea 1: Constrain u. We don't know how to do this.

Idea 2: Pay for gas. This is a design choice.

For the second idea,

$$\begin{aligned} &\min_{u,T} \int_0^T \frac{1}{2} u^2(t) \, \mathrm{d}t \\ &\text{s.t.} \quad \begin{cases} \dot{x}_1 = x_2, & x(0) = x_0 \\ \dot{x}_2 = u, & x(T) = 0 \end{cases} \end{aligned}$$

$$\begin{gathered} H = \frac{1}{2} u^2 + \lambda_1 x_2 + \lambda_2 u \\ \frac{1}{2} u^2(T) + \lambda_1(T) x_2(T) + \lambda_2(T) u(T) = 0 \\ \boxed{\frac{1}{2} u^2(T) + \lambda_2(T) u(T) = 0} \\ \frac{\partial H}{\partial u} = u + \lambda_2 = 0 \Longrightarrow u = -\lambda_2 \\ \frac{1}{2} \lambda_2^2(T) - \lambda_2^2(T) = 0 \\ \boxed{\lambda_2(T) = 0} \end{gathered}$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \Longrightarrow \lambda_1 = c$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \Longrightarrow \lambda_2 = -ct + d$$

$$\lambda_2(T) = -cT + d = 0 \Longrightarrow T = \frac{d}{c}$$

$$\dot{x}_2 = u = -\lambda_2 = ct - d$$

$$x_2 = c\frac{t^2}{2} - dt + x_{2,0}$$

$$\dot{x}_1 = x_2 \Longrightarrow x_1 = c\frac{t^3}{6} - d\frac{t^2}{2} + x_{2,0}t + x_{1,0}$$

$$\begin{cases} x_1(T) = c\frac{T^3}{6} - d\frac{T^2}{2} + x_{2,0}T + x_{1,0} = 0 \\ x_2(T) = c\frac{T^2}{2} - dT + x_{2,0} = 0 \\ T = \frac{d}{c} \end{cases}$$

$$\Longrightarrow \begin{cases} c = -\frac{2}{3}\frac{x_{2,0}^2}{x_{1,0}} \\ d = \sqrt{-\frac{4}{3}\frac{x_{2,0}^3}{x_{1,0}}} \\ T = \frac{d}{c} \\ u = ct - d \end{cases}$$

Fine, but we really want to get there as quickly as possible! We have to constrain u, e.g. $u(t) \in [-1, 1]$, $\forall t \in [0, T]$. How do we deal with the constraints on u?

3.5 Hamilton's Minor "Mistake"

$$\begin{aligned} \min_{u \in \mathcal{U}_{\text{constr.}}} \ & \int_0^T L(x, u, t)\,\mathrm{d}t + \Psi(x(T)) \\ \text{s.t. } & \dot{x} = f(x, u, t) \\ & x(0) = x_0 \\ & (u(t) \in U) \end{aligned}$$

Augment the cost:

$$\tilde{J}(u) = \int_0^T \Big(H(x, u, t, \lambda) - \lambda^{\mathrm{T}}\dot{x}\Big)\,\mathrm{d}t + \Psi(x(T))$$

Vary $u \mapsto u + \varepsilon v$ s.t. $u + \varepsilon v \in \mathcal{U}_{\text{constr.}} \Rightarrow x \mapsto x + \varepsilon\eta + o(\varepsilon)$:

$$\tilde{J}(u+\varepsilon v) = \int_0^T \Big(H(x+\varepsilon\eta, u+\varepsilon v, t, \lambda) - \lambda^{\mathrm{T}}\dot{x} - \lambda^{\mathrm{T}}\varepsilon\dot{\eta} \Big)\,\mathrm{d}t + \Psi(x(T)+\varepsilon\eta(T)) + o(\varepsilon)$$

Instead of computing $\delta\tilde{J}(u;v)$, let's check $\Delta\tilde{J} = \tilde{J}(u+\varepsilon v) - \tilde{J}(u)$. If $\Delta\tilde{J} \geq 0\ \forall v$ s.t. $u + \varepsilon v \in \mathcal{U}_{\text{constr.}}$ for ε small enough, then u is a local minimum!

$$\begin{aligned}\Delta\tilde{J} = \int_0^T &\Big[H(x+\varepsilon\eta, u+\varepsilon v, t, \lambda) - H(x,u,t,\lambda) - \lambda^{\mathrm{T}}(\dot{x} + \varepsilon\dot{\eta} - \dot{x}) \Big]\,\mathrm{d}t \\ &+ \Psi(x(T)+\varepsilon\eta(T)) - \Psi(x(T)) + o(\varepsilon)\end{aligned}$$

Only Taylor expanding w.r.t. x:

$$\begin{aligned}\Delta\tilde{J} &= \int_0^T \Big[\varepsilon\frac{\partial H}{\partial x}(x,u,t,\lambda)\eta - \varepsilon\lambda^{\mathrm{T}}\dot{\eta} \Big]\,\mathrm{d}t + \int_0^T \big[H(x,u+\varepsilon v,t,\lambda) - H(x,u,t,\lambda) \big]\,\mathrm{d}t \\ &\quad + \varepsilon\frac{\partial\Psi}{\partial x}(x(T))\eta(T) + o(\varepsilon) \\ &= \varepsilon\int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}} \right)\eta\,\mathrm{d}t + \varepsilon\lambda^{\mathrm{T}}(0)\eta(0) - \varepsilon\lambda^{\mathrm{T}}(T)\eta(T) + \varepsilon\frac{\partial\Psi}{\partial x}(x(T))\eta(T) \\ &\quad + \int_0^T \big[H(x,u+\varepsilon v,t,\lambda) - H(x,u,t,\lambda) \big]\,\mathrm{d}t + o(\varepsilon)\end{aligned}$$

With $\dot{\lambda} = -\partial H^{\mathrm{T}}/\partial x$ and $\lambda(T) = \partial\Psi(x(T))/\partial x$,

$$\Delta\tilde{J} = \int_0^T \big[H(x,u+\varepsilon v,t,\lambda) - H(x,u,t,\lambda) \big]\,\mathrm{d}t + o(\varepsilon)$$

Here, Hamilton did Taylor's expansion and set $\partial H/\partial u = 0$. Instead, Pontryagin desired $\Delta\tilde{J} \geq 0\ \forall v | u + \varepsilon v \in \mathcal{U}_{\text{constr.}}$, ε small enough, i.e. we need

$$H(x, u^* + \varepsilon v, t, \lambda) \geq H(x, u^*, t, \lambda)$$

$\forall t \in [0,t]$, $\forall v | u + \varepsilon v \in \mathcal{U}_{\text{constr.}}$, ε small enough. That is, we need

$$u^* = \arg\min_u H(x,u,t,\lambda)$$

In summary,

$$\text{Hamilton: } \frac{\partial H}{\partial u} = 0$$
$$\text{Pontryagin: } \min_u H$$

Theorem (Pontryagin's Maximum Principle (PMP))**.** *Consider the problem:*

$$\begin{aligned}\min_{u,T}\ & \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T),T) \\ s.t.\ & \dot{x} = f(x,u,t) \\ & u(t) \in U(x,t), \quad \forall t \in [0,T] \\ & x_i(0) = x_{i0}, \qquad i \in \mathcal{I} \\ & x_j(T) = x_{jT}, \qquad j \in \mathcal{T}\end{aligned}$$

The necessary condition for optimality is

$$
\begin{aligned}
&H = L + \lambda^T f \\
&\dot{\lambda} = -\frac{\partial H}{\partial x}^T \\
&\lambda_j(0) = 0, \quad j \notin \mathcal{I} \\
&\lambda_i(T) = \frac{\partial \Psi}{\partial x_i}(x(T)), \quad i \notin \mathcal{T} \\
&H + \left.\frac{\partial \Psi}{\partial T}\right|_{t=T} = 0 \\
&u^*(x,t,\lambda) = \arg\min_{u \in U(x,t)} H(x,u,t,\lambda)
\end{aligned}
$$

We have two paths to solve optimality problems: we always start with the Hamiltonian, find the costate dynamics and boundary conditions, and apply the transversality condition; then, we can either apply calculus of variations (COV) or Pontryagin's Maximum Principle (PMP). COV only works for unconstrained problems, while with PMP we can deal with constraints.

3.6 Bang-Bang Control

Return to the car problem:

$$
\begin{aligned}
&\min_{u,T} \int_0^T \mathrm{d}t \\
&\text{s.t.} \begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{1,0}, \quad x_1(T) = 0 \\ \dot{x}_2 = u, & x_2(0) = x_{2,0}, \quad x_2(T) = 0 \\ u(t) \in [-1,1] & \forall t \in [0,T] \end{cases}
\end{aligned}
$$

How do we minimize H w.r.t. u?

$$H = 1 + \lambda_1 x_2 + \lambda_2 u$$

Clearly, we minimize H by letting

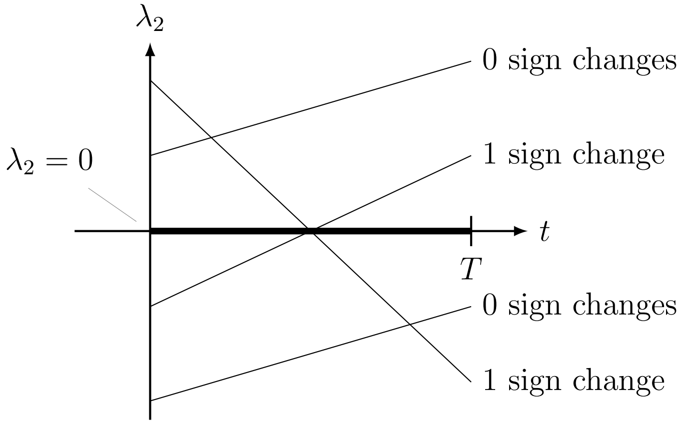
$$u = \begin{cases} -1, & \lambda_2 > 0 \\ +1, & \lambda_2 < 0 = -\operatorname{sign}(\lambda_2) \\ ??, & \lambda_2 = 0 \end{cases}$$

Therefore, the optimal u switches between -1 and $+1$ (bang-bang control).

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \Longrightarrow \lambda_1 = c$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \Longrightarrow \lambda_2 = -ct + d$$

Notice that $\lambda_2(t)$ is a line, so it has at most one sign change. Thus, u also changes sign (from ± 1 to ∓ 1) at most one time.



Let's solve this for all x_0!

i) Assume $\lambda_2 > 0$ $\forall t \in [0, T]$, $\therefore u = -1 \quad \forall t \in [0, T]$

$$\begin{aligned}
\dot{x}_2 &= -1 \implies x_2 = -t + k_1 \\
x_2(T) &= 0 = -T + k_1 \implies k_1 = T \\
x_2(t) &= T - t \implies x_2 > 0,\ t \in [0, T) \\
\dot{x}_1 &= x_2 = T - t \implies x_1 = -\frac{t^2}{2} + Tt + k_2 \\
x_1(T) &= 0 = -\frac{T^2}{2} + T^2 + k_2 \implies k_2 = -\frac{T^2}{2} \\
x_1(t) &= -\frac{t^2}{2} + Tt - \frac{T^2}{2} = -\frac{(T-t)^2}{2} \quad (< 0,\ t \in [0, T)) \\
&= -\frac{x_2^2(t)}{2}
\end{aligned}$$

Let's consider the curve

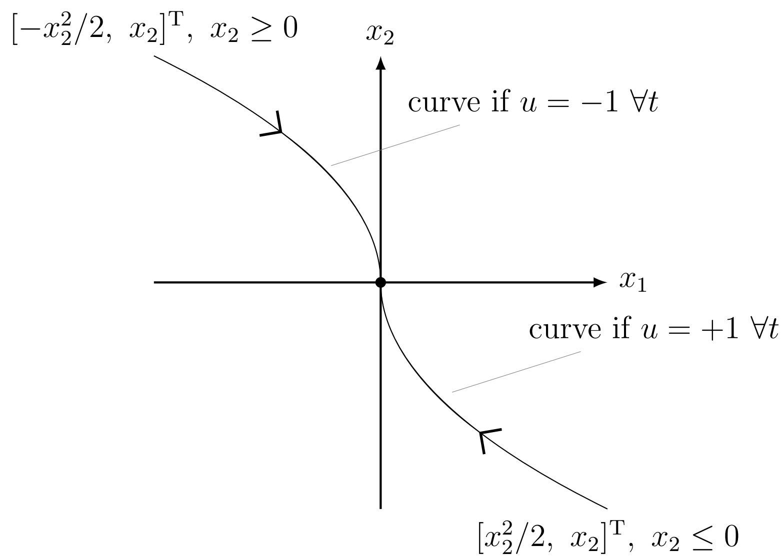
$$\begin{bmatrix} -x_2^2/2 \\ x_2 \end{bmatrix}$$

for $x_2 \geq 0$. If x_0 lies on this curve, use $u = -1$ and drive to the origin.

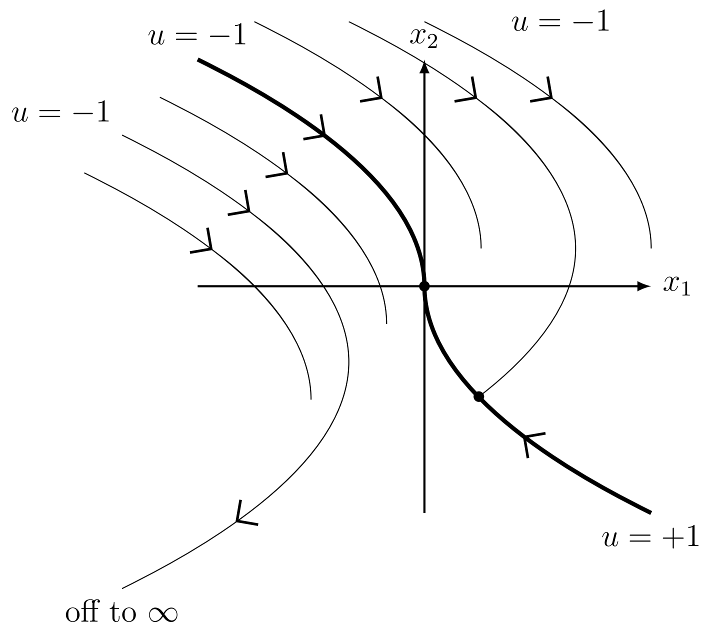
ii) Assume $u = +1 \quad \forall t \in [0, T]$

$$\begin{aligned}
x_2 &= t - T \quad (\leq 0 \text{ on } [0, T]) \\
x_1 &= \frac{x_2^2}{2} \quad (\geq 0)
\end{aligned}$$

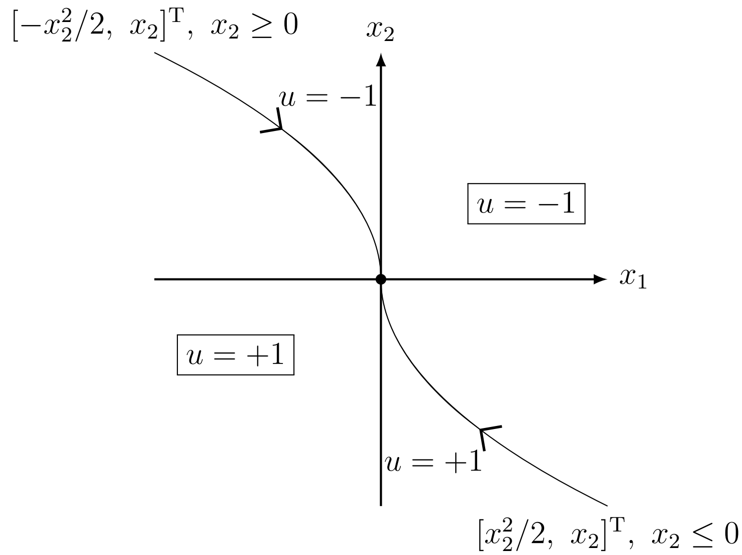
For this curve, use $u = +1$.



What happens when we do not start on the curves? We start with a certain u depending on x_0 and perform a single switch of u when we encounter one of the initial curves that travel to the origin. Note that for the case $\lambda_2 = 0 \forall t$, we start at the stop sign at rest, so the control does not matter.



The optimal solution is given by the following *switching curve*.



Note 1: Bang-bang control typically involves

a) finding the number of switches
b) find the switching surfaces

Note 2: This is a feedback law! (u depends on x!!)

3.6.1 Linear Systems (scalar input)

$$
\begin{aligned}
&\min_{u,T} \int_0^T \mathrm{d}t \\
&\text{s.t. } \dot{x} = Ax + Bu \\
&\qquad x(0) = x_0, \quad x(T) = 0 \\
&\qquad u \in [-1, 1] \\
&\quad H = 1 + \lambda^{\mathrm{T}}(Ax + Bu) \\
&\quad u = -\operatorname{sign}(\lambda^{\mathrm{T}} B) \quad \text{(bang-bang)}
\end{aligned}
$$

Aside...

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u \quad \text{(control affine)} \\
H &= 1 + \lambda^{\mathrm{T}} f + \lambda^{\mathrm{T}} g u \\
u &= -\operatorname{sign}(\lambda^{\mathrm{T}} g(x)) \quad \text{(bang-bang)}
\end{aligned}
$$

Back to linear...

$$
\begin{aligned}
\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} = -A^{\mathrm{T}}\lambda \\
\lambda(t) &= e^{-A^{\mathrm{T}} t}\lambda_0 \\
u(t) &= -\operatorname{sign}\left(\lambda_0^{\mathrm{T}} e^{-At} B\right)
\end{aligned}
$$

How do we find λ_0?

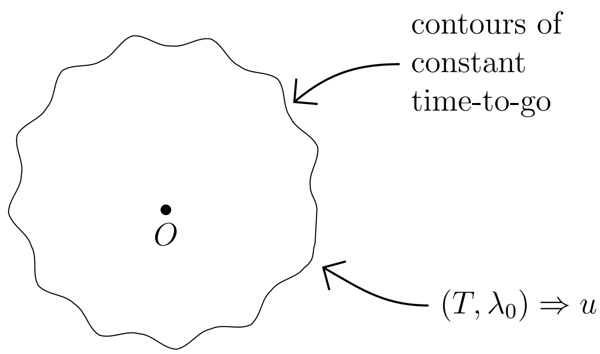
$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ x(T) &= e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)\,\mathrm{d}\tau \\ x(T) &= 0 = e^{AT}x_0 - \int_0^T e^{A(T-t)}B\,\mathrm{sign}\left(\lambda_0^{\mathrm{T}}e^{-At}B\right)\mathrm{d}t \end{aligned} \tag{3.7}$$

Problem 1: Given x_0, figure out λ_0 from (3.7). Then, $u = -\,\mathrm{sign}\left(\lambda_0^{\mathrm{T}}e^{-At}B\right)$. This has to be done numerically in general (not super simple. . .).

Problem 2: Find all x_0s from which it takes the same amount of time to get to $x(T) = 0$.

$$\begin{aligned} e^{At}x_0 &= \int_0^T e^{A(T-t)}B\,\mathrm{sign}\left(\lambda_0^{\mathrm{T}}e^{-At}B\right)\mathrm{d}t \\ x_0 &= \int_0^T e^{-At}B\,\mathrm{sign}\left(\lambda_0^{\mathrm{T}}e^{-At}B\right)\mathrm{d}t \end{aligned}$$

Fix T. By varying λ_0, we will get the x_0s that take time T to go to $x(T) = 0$ optimally.



So by solving problem 2, we find λ_0 associated with all x_0, i.e. we have "solved" problem 1 as well.

3.7 Integral Constraints (Isoperimetric)

Recall PMP is

$$\min_{u \in U(x,t)} H(x, u, \lambda, t)$$

We have see $U = [-1, 1]$ in the context of bang-bang control. Now, we consider integral constraints of the form

$$C = \int_0^T N(x, u, t)\,\mathrm{d}t \quad (\in \mathbb{R}^p)$$

Let $x = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$. Introduce p new states $\hat{x} = [x_{n+1}, \dots, x_{n+p}]^{\mathrm{T}}$, where

$$\hat{x}(t) = \int_0^t N(x(\tau), u(\tau), \tau)\,\mathrm{d}\tau$$

and $\dot{\hat{x}}(t) = N(x, u, t)$. Its boundary conditions are $\hat{x}(0) = 0$ and $\hat{x}(T) = C$. The Hamiltonian is

$$\begin{aligned} H(x, \hat{x}, u, t, \lambda) &= L(x, u, t) + \lambda^{\mathrm{T}} f(x, u, t) + \widehat{\lambda}^{\mathrm{T}} N(x, u, t) \\ \dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} = -\frac{\partial L^{\mathrm{T}}}{\partial x} - \frac{\partial f^{\mathrm{T}}}{\partial x}\lambda - \frac{\partial N^{\mathrm{T}}}{\partial x}\widehat{\lambda} \\ \dot{\widehat{\lambda}} &= -\frac{\partial H^{\mathrm{T}}}{\partial \hat{x}} = 0 \Longrightarrow \widehat{\lambda} \text{ is constant} \end{aligned}$$

Moreover, this is now an unconstrained problem, i.e.

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial u} + \widehat{\lambda}^{\mathrm{T}}\frac{\partial N}{\partial u} = 0$$

Going back to the car problem of stopping at the origin, suppose we want to use up exactly the "energy"

$$E = \int_0^T u^2(t)\,\mathrm{d}t.$$

If possible, it is better to transform an inequality constraint to an equality constraint.

$$\begin{aligned} &\min_{u,T} \int_0^T \mathrm{d}t \\ &\text{s.t.} \begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{10}, \quad x_1(T) = 0 \\ \dot{x}_2 = u, & x_2(0) = x_{20}, \quad x_2(T) = 0 \\ \dot{x}_3 = u^2, & x_3(0) = 0, \qquad x_3(T) = E \end{cases} \end{aligned}$$

As we have seen, without the energy constraint this is an ill-posed problem.

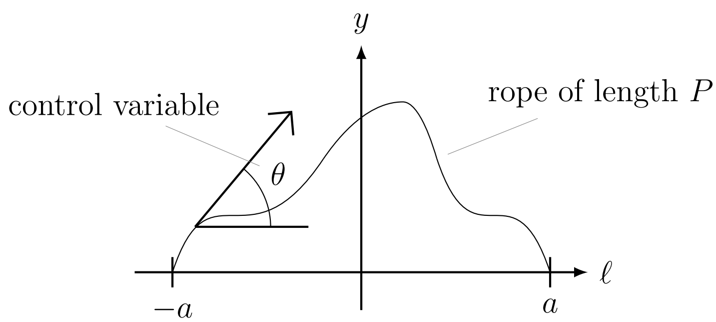
$$\begin{aligned} H &= 1 + \lambda_1 x_2 + \lambda_2 u + \lambda_3 u^2 \\ \lambda_3 &= \text{constant} \\ \dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \Longrightarrow \lambda_1 = c \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \Longrightarrow \lambda_2 = -ct + d \\ \frac{\partial H}{\partial u} &= \lambda_2 + 2\lambda_3 u = 0 \\ \Rightarrow u &= -\frac{\lambda_2}{2\lambda_3} = \frac{c}{2\lambda_3}t - \frac{d}{2\lambda_3} \quad \text{(linear in time)} \\ \dot{x}_2 = u &\Longrightarrow x_2 = \frac{c}{4\lambda_3}t^2 - \frac{d}{2\lambda_3}t + x_{20} \end{aligned}$$

$$
\begin{aligned}
\dot{x}_1 = x_2 \Longrightarrow x_1 &= \frac{c}{12\lambda_3}t^3 - \frac{d}{4\lambda_3}t^2 + x_{20}t + x_{10} \\
\dot{x}_3 = u^2 \Longrightarrow x_3 &= \frac{c^2}{12\lambda_3^2}t^3 + \frac{d^2}{4\lambda_3^2}t - \frac{cd}{4\lambda_3^2}t^2 \\
H + \left.\frac{\partial \Psi}{\partial T}\right|_T &= 0 \\
1 + \lambda_1 x_2 + \lambda_2 u + \lambda_3 u^2 + 0\Big|_T &= 0 \\
1 + (d - cT)\left(\frac{c}{2\lambda_3}T - \frac{d}{2\lambda_3}\right) + \lambda_3\left(\frac{c}{2\lambda_3}T - \frac{d}{2\lambda_3}\right) &= 0
\end{aligned}
$$

The boundary conditions ($x_1(T) = 0$, $x_2(T) = 0$, $x_3(T) = E$) and the transversality condition give four equations for four unknowns.

$$
\begin{cases}
T = \left(\frac{3}{E}\right)^{1/3} \\
c = -\frac{2}{3}T \\
d = -\frac{T^2}{3} \\
\lambda_3 = \frac{T^4}{18}
\end{cases}
\qquad \Longrightarrow u = \ldots
$$

Dido's Problem Given a strip of oxhide, enclose the most area along the Mediterranean Sea. This region has a fixed width and is bounded to the south by the ℓ axis (the sea). Historically, this became the city Carthage.



The area of this region is

$$\int_{-a}^{a} y \, \mathrm{d}\ell.$$

The dynamics are

$$\frac{\mathrm{d}y}{\mathrm{d}\ell} = \tan\theta.$$

The constraint is

$$P = \int_{-a}^{a} \frac{1}{\cos\theta}\, \mathrm{d}\ell.$$

The problem becomes

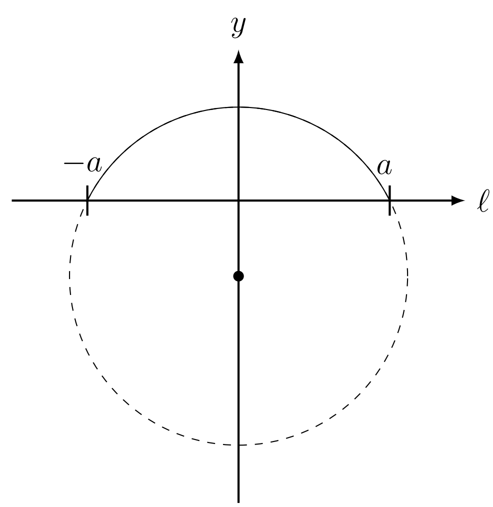
$$\begin{aligned}
&\min_{\theta} \; -\int_{-a}^{a} y(\ell)\, \mathrm{d}\ell \\
&\text{s.t. } \frac{\mathrm{d}y}{\mathrm{d}\ell} = \tan\theta, \quad y(-a) = 0, \quad y(a) = 0 \\
&\phantom{\text{s.t. }} \frac{\mathrm{d}\hat{y}}{\mathrm{d}\ell} = \frac{1}{\cos\theta}, \quad \hat{y}(-a) = 0, \quad \hat{y}(a) = P
\end{aligned}$$

$$\begin{aligned}
H &= -y + \lambda \tan\theta + \widehat{\theta}\frac{1}{\cos\theta} \\
\widehat{\lambda} &= \text{constant} \\
\frac{\mathrm{d}\lambda}{\mathrm{d}\ell} &= -\frac{\partial H}{\partial y} = 1 \Longrightarrow \lambda(\ell) = \ell + c \\
\frac{\partial H}{\partial \theta} &= 0 = \lambda(1 + \tan^2\theta) + \widehat{\lambda}\,\frac{\tan\theta}{\cos\theta} \\
\sin\theta(\ell) &= -\frac{\ell + c}{\widehat{\lambda}}
\end{aligned}$$

Let $\sin\alpha/\alpha = 2a/P$. The optimal shape is a circular arc centered at $\ell = 0$ and

$$y = -\frac{P\cos\alpha}{2\alpha},$$

with radius $P/2\alpha$. (This produces the semi-circular city of Carthage!)



Note that this formulation cannot handle $P > \pi a$. In reality, a is also undefined and chosen so that the solution is exacty a semicircle with $P = \pi a$.

The punchline is integral constraints are no big deal. What about other constraints?

3.8 Control Constraints

Suppose the control constraint is $u(t) \in U(t)$, e.g. $h(u,t) = 0$ or $h(u,t) \leq 0$.

$$\begin{aligned}
&\min_u H(x,u,\lambda,t) \\
&\text{s.t. } h(u,t) = 0
\end{aligned}$$

Introduce a Lagrange multiplier:

$$\begin{aligned}
&\tilde{H} = H + \mu^{\mathrm{T}} h \\
&\left.\begin{aligned} \frac{\partial \tilde{H}}{\partial u} = 0 \\ h = 0 \end{aligned}\right\} \Longrightarrow u^*(x,t,\lambda)
\end{aligned}$$

We still have

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x}(x,t,\lambda,u^*(x,t,\lambda)) \\
\dot{x} &= f(x,u,t) = f(x,u^*(x,t,\lambda),t) \\
&\quad + \text{Boundary cond. on } x \text{ and } \lambda
\end{aligned}$$

The only change from the unconstrained control version is the method by which $u^*(x,t,\lambda)$ is found.

Example

$$\begin{aligned}
&\min_u \frac{1}{2}\int_0^T u^2(t)\,\mathrm{d}t + \frac{1}{2}\|x(T)\|^2 \\
&\text{s.t. } \dot{x} = g(t)u, \quad g(t) \in \mathbb{R}^n \\
&\quad\;\; |u(t)| \leq 1 \; \forall t \\
&\quad\;\; \Rightarrow \begin{cases} u(t) - 1 \leq 0 \\ -u(t) - 1 \leq 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
H &= \frac{1}{2}u^2 + \lambda^{\mathrm{T}} g u \\
\widetilde{H} &= \frac{1}{2}u^2 + \lambda^{\mathrm{T}} g u + \mu_1(u-1) + \mu_2(-u-1) \\
\dot{\lambda} &= -\frac{\partial \widetilde{H}^{\mathrm{T}}}{\partial x} = 0 \Longrightarrow \lambda = \text{const} \\
\lambda(T) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial x} = x(T) \Longrightarrow \lambda(t) = x(T) \; \forall t
\end{aligned}$$

Now, let's find u by minimizing H. Assume $|u| < 1$ (no constraints active), so $\mu_1 = \mu_2 = 0$. Then,

$$\frac{\partial \widetilde{H}}{\partial u} = u + \lambda^{\mathrm{T}} g = 0 \Longrightarrow u(t) = -x^{\mathrm{T}}(T) g(t),$$

as long as $|x^{\mathrm{T}}(T)g(t)| < 1$. Assume $u = -1$, so $\mu_1 = 0$ and $\mu_2 \geq 0$. Then,

$$\begin{aligned}\frac{\partial \widetilde{H}}{\partial u} &= u + \lambda^{\mathrm{T}} g - \mu_2 = 0 \\ x^{\mathrm{T}}(T)g(t) &= \mu_2 + 1 \geq 1\end{aligned}$$

We get a similar results assuming $u = 1$. The optimal control law is

$$\begin{aligned}u(t) &= \begin{cases} -x^{\mathrm{T}}(T)g(t), & |x^{\mathrm{T}}(T)g(t)| < 1 \\ -1, & x^{\mathrm{T}}(T)g(t) \geq 1 \\ +1, & x^{\mathrm{T}}(T)g(t) \leq -1 \end{cases} \\ u(t) &= -\operatorname{Sat}(x^{\mathrm{T}}(T)g(t))\end{aligned}$$

where

$$\operatorname{Sat}(\xi) = \begin{cases} \xi, & |\xi| \leq 1 \\ \operatorname{sign}(\xi), & \text{otherwise} \end{cases}$$

Problem: we don't know $x(T)$! We have to solve this numerically through

$$\begin{aligned}x(t) &= x(0) + \int_0^t \dot{x}(\tau)\,\mathrm{d}\tau \\ x(T) &= x_0 - \int_0^T g(t)\operatorname{Sat}\left(x^{\mathrm{T}}(T)g(t)\right)\mathrm{d}t\end{aligned}$$

Example

$$\begin{aligned}&\min_u \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T)) \\ &\text{s.t. } \dot{x} = f(x,u,t), \quad x(0) = x_0 \\ &\qquad h(x,u,t) = 0 \ \forall t\end{aligned}$$

$$\begin{aligned}\widetilde{H} &= L + \lambda^{\mathrm{T}} f + \mu^{\mathrm{T}} h \\ \dot{\lambda} &= -\frac{\partial \widetilde{H}^{\mathrm{T}}}{\partial x} = -\frac{\partial L^{\mathrm{T}}}{\partial x} - \frac{\partial f^{\mathrm{T}}}{\partial x}\lambda - \frac{\partial h^{\mathrm{T}}}{\partial x}\mu \\ \lambda(T) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) \\ \frac{\partial \widetilde{H}}{\partial u} &= 0 \\ h &= 0\end{aligned}$$

Example

$$\begin{aligned}&\min_u \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T)) \\ &\text{s.t. } \dot{x} = f(x,u), \quad x(0) = x_0 \\ &\qquad h(x) = 0\end{aligned}$$

Problem: We need a constraint involving u. First, we need $h(x_0) = 0$; otherwise we have no chance. Then, if

$$\frac{\mathrm{d}}{\mathrm{d}t} h(x(t)) = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x} f(x, u) = 0,$$

we have $h(x(t)) = 0 \ \forall t$. This derivative is the Lie derivative of h along f ($L_f h = (\partial h/\partial x) f$).

$$\widetilde{H} = L + \lambda^{\mathrm{T}} f + \mu^{\mathrm{T}} \frac{\partial h}{\partial x} f$$

Problem: $(\partial h/\partial x) f$ is not guaranteed to have u in it, e.g.

$$h = x_1, \quad f = \begin{bmatrix} 17x_2 \\ \sin(x_1)u \end{bmatrix}$$
$$\frac{\partial h}{\partial x} f = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 17x_2 \\ \sin(x_1)u \end{bmatrix} = 17x_2$$

So, we keep taking derivatives until u shows up. (If u never shows up, then the control has no effect on the state.)

3.9 A Look Forward

So far, we found $u(t)$ over the horizon $[0, T]$. This is, in general, not robust. We need to know f exactly. We also need to know $x(0)$. What to do?

There are three paths forward:

1. If we're super lucky, we get $u(x, t)$ directly from PMP, like in the bang-bang example with switching surfaces.

2. Go from PMP to LQ (linear system, quadratic cost). This is used a lot.

3. Use Model-Predictive Control (MPC). In this, at time t_c (current time), we are at state x_c. We solve an optimal control problem:

$$\begin{aligned} &\min_u \int_{t_c}^{t_c+\Delta T} L(x, u, t)\,\mathrm{d}t + \Psi(x(t_c + \Delta T)) \\ &\text{s.t. } \dot{x} = f(x, u, t) \\ &\qquad x(t_c) = x_c \end{aligned}$$

   where ΔT is the prediction horizon. This problem can be solved using PMP, producing $u(t)$, $t \in [t_c, t_c + \Delta T]$. Instead of using $u(t)$, only use $u(t_c)$ at time t_c. This control solution depends on x_c, so we really have a feedback law $u(x_c, t_c)$. (In practice, we use $u(x_c, t_c)$ over a small interval of length δ.) Then, we resolve the optimal control problem.

   The features of MPC are

(a) Turns open-loop into closed-loop
(b) Used a lot
(c) Requires computation, but once a solution is found, it can be reused as initial conditions...
(d) Use with caution! A solution may be optimal over $[t_c, t_c + \Delta T]$ but it may still be bad (unstable) over $[t_c, \infty)$.

Chapter 4

Linear-Quadratic Control

4.1 Towards Global Optimal Control

Consider a discrete-time system

$$x_{k+1} = F(x_k, u_k),$$

where x_k is the state at time k and u_k is the input at time k.

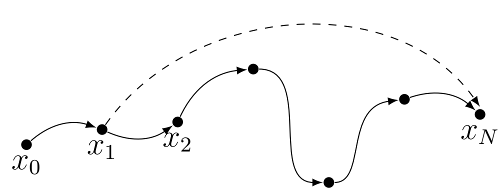
Let $c(x_k, u_k) \in \mathbb{R}$ be the cost associated with doing u_k at x_k.

Let $u = u_0, u_1, \ldots, u_{N-1}$ and assume x_0 is given. The total cost over N steps using u is

$$V_N^u(x_0) = \sum_{k=0}^{N-1} c(x_k, u_k) + \Theta(x_N),$$

where $\Theta(x_N)$ is the terminal cost.

Assume we've found the *globally* minimizing u^*. The best path over N steps is represented by the figure below.



Consider the dashed path. There is no way this path is better from x_1 to x_N using $N-1$ steps. Therefore, the solid path from x_1 to x_N is the best path over $N-1$ steps.

Definition (Bellman's Principle of Optimality)**.** Let u^* be optimal, with corresponding state sequence x^*.

$$\begin{aligned} V_N^*(x_0) &= \sum_{k=0}^{N-1} c(x_k^*, u_k^*) + \Theta(x_N^*) \\ &= c(x_0, u_0^*) + \sum_{k=1}^{N-1} c(x_k^*, u_k^*) + \Theta(x_N^*) \\ &= c(x_0, u_0^*) + V_{N-1}^*(x_1^*) \\ V_N^*(x) &= c(x, u_0^*) + V_{N-1}^*\big(F(x, u_0^*)\big) \end{aligned}$$

Equivalently,

$$V_N^*(x) = \min_u \Big\{ c(x,u) + V_{N-1}^*\big(F(x,u)\big) \Big\}$$

Theorem (Bellman's Equation)**.** *The optimal cost-to-go satisfies*

$$\begin{cases} V_k^*(x) = \min_u \Big\{ c(x,u) + V_{k-1}^*\big(F(x,u)\big) \Big\}, \quad k = 1, \ldots, N \\ V_0^*(x) = \Theta(x) \end{cases}$$

What does this have to do with optimal control? We need to reformulate the cost function J in an analogous manner. Let

$$J^*(x_t, t) = \int_t^T L(x^*(s), u^*(s))\,\mathrm{d}s + \Psi(x^*(T)),$$

where $x^*(t) = x_t$, u^* is *globally* optimal, and $\dot{x}^* = f(x^*, u^*)$. $J^*(x_t, t)$ is the optimal cost-to-go over $[t, T]$ starting at x_t. Let's discretize time with sample time Δt.

$$\begin{aligned} J^*(x_t, t) &= \int_t^{t+\Delta t} L(x^*(s), u^*(s))\,\mathrm{d}s + \int_{t+\Delta t}^T L(x^*(s), u^*(s))\,\mathrm{d}s + \Psi(x^*(T)) \\ &= \int_t^{t+\Delta t} L(x^*(s), u^*(s))\,\mathrm{d}s + J^*(x_{t+\Delta t}^*, t + \Delta t) \end{aligned}$$

Note $x_{t+\Delta t}^* = x_t + f(x_t, u^*(t))\Delta t + o(\Delta t)$. Also, assume u^* is constant over $[t, t+\Delta t]$.

$$\begin{aligned} \int_t^{t+\Delta t} L(x^*(s), u_t^*)\,\mathrm{d}s &= \Delta t L(x_t, u_t^*) + o(\Delta t) \\ \because J^*(x_t, t) &= \Delta t L(x_t, u_t^*) + J^*\big(x_t + \Delta t f(x_t, u_t^*), t + \Delta t\big) + o(\Delta t) \\ J^*(x, t) &= \min_u \Big\{ \Delta t L(x, u) + J^*\big(x + \Delta t f(x, u), t + \Delta t\big) \Big\} + o(\Delta t) \end{aligned}$$

Hence $J^*(x,t) \sim V_k^*(x)$ and $\Delta t L(x,u) \sim c(x,u)$. Also, $J^*(x,T) = \Psi(x)$, so $\Psi \sim \Theta$.

Bellman's equation produces

$$\begin{aligned} J^*(x,t) &= \min_u \Big\{ \Delta t L(x,u) + J^*\big(x + \Delta t f(x,u), t + \Delta t\big) \Big\} + o(\Delta t), \\ &\qquad\qquad\qquad\qquad\qquad\qquad t = 0, \Delta t, 2\Delta t, \ldots, T - \Delta t \\ J^*(x,T) &= \Psi(x) \end{aligned}$$

But we need this in continuous time. Taylor expansion produces

$$\begin{aligned} J^*\big(x + \Delta t f(x,u), t + \Delta t\big) &= J^*(x,t) + \frac{\partial J^*(x,t)}{\partial x} \Delta t f(x,u) + \frac{\partial J^*(x,t)}{\partial t} \Delta t + o(\Delta t) \\ J^*(x,t) &= \min_u \left\{ \Delta t L(x,u) + J^*(x,t) + \frac{\partial J^*(x,t)}{\partial x} \Delta t f(x,u) + \frac{\partial J^*(x,t)}{\partial t} \Delta t \right\} + o(\Delta t) \\ J^*(x,t) - J^*(x,t) - \frac{\partial J^*(x,t)}{\partial t} \Delta t &= \min_u \left\{ \Delta t L(x,u) + \frac{\partial J^*(x,t)}{\partial x} \Delta t f(x,u) \right\} + o(\Delta t) \end{aligned}$$

Dividing both sides by Δt and taking the limit as $\Delta t \to 0$,

$$-\frac{\partial J^*(x,t)}{\partial t} = \min_u \left\{ L(x,u) + \frac{\partial J^*(x,t)}{\partial x} f(x,u) \right\}$$

This is known as the Hamilton-Jacobi-Bellman (HJB) equation.

Theorem. u^* *is a global minimizer to*

$$\begin{aligned} &\min_u \int_0^T L(x,u,t)\, \mathrm{d}t + \Psi(x(T)) \\ &s.t.\ \dot{x} = f(x,u) \end{aligned}$$

if and only if u^* *solves the HJB equation*

$$-\frac{\partial J^*(x,t)}{\partial t} = \min_u \left\{ L(x,u) + \frac{\partial J^*(x,t)}{\partial x} f(x,u) \right\}, \quad t \in [0,T),$$

where $J^*(x,T) = \Psi(T)$,

$$J^*(x_t,t) = \int_t^T L(x^*(s), u^*(s), s)\, \mathrm{d}s + \Psi(x^*(T)),$$

$x^*(t) = x_t$, *and* $\dot{x}^* = f(x^*, u^*, t)$.

Note:

1. The HJB equation is a partial differential equation (PDE) rather than an ODE (hard to solve in general).
2. It is solvable when we have linear dynamics and quadratic costs (LQ).

4.2 Linear-Quadratic Problems

$$
\begin{aligned}
\min_u \; & \frac{1}{2}\int_0^T \Big[x^{\mathrm{T}}(t)Q(t)x(t) + u^{\mathrm{T}}(t)R(t)u(t)\Big]\,\mathrm{d}t + \frac{1}{2}x^{\mathrm{T}}(T)Sx(T), \\
& \qquad\qquad Q(t) = Q^{\mathrm{T}}(t) \succeq 0,\; S = S^{\mathrm{T}} \succeq 0,\; R(t) = R^{\mathrm{T}}(t) \succ 0 \\
\text{s.t. } & \dot{x}(t) = A(t)x(t) + B(t)u(t) \\
& x(0) = x_0
\end{aligned}
$$

HJB states

$$
\begin{aligned}
-\frac{\partial J^*}{\partial t} &= \min_u \left\{ \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + \frac{\partial J^*}{\partial x}(Ax + Bu) \right\} \\
J^*(x, T) &= \frac{1}{2}x^{\mathrm{T}}Sx
\end{aligned}
$$

Minimizing the first equation with respect to u produces

$$
\begin{aligned}
\frac{\partial\{\cdot\}}{\partial u} = u^{\mathrm{T}}R + \frac{\partial J^*}{\partial x}B &= 0 \\
Ru + B^{\mathrm{T}}\frac{\partial J^{*\mathrm{T}}}{\partial x} &= 0 \\
u &= -R^{-1}B^{\mathrm{T}}\frac{\partial J^{*\mathrm{T}}}{\partial x} \\
\frac{\partial^2\{\cdot\}}{\partial u^2} = R \succ 0 &\Rightarrow u^* \text{ is the global minimizer}
\end{aligned}
$$

Going back to HJB,

$$
\begin{aligned}
-\frac{\partial J^*}{\partial t} &= \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}\frac{\partial J^*}{\partial x}BR^{-1}RR^{-1}B^{\mathrm{T}}\frac{\partial J^{*\mathrm{T}}}{\partial x} + \frac{\partial J^*}{\partial x}Ax - \frac{\partial J^*}{\partial x}BR^{-1}B^{\mathrm{T}}\frac{\partial J^{*\mathrm{T}}}{\partial x} \\
&= \frac{1}{2}x^{\mathrm{T}}Qx + \frac{\partial J^*}{\partial x}Ax - \frac{1}{2}\frac{\partial J^*}{\partial x}BR^{-1}B^{\mathrm{T}}\frac{\partial J^{*\mathrm{T}}}{\partial x}
\end{aligned}
$$

We still have a PDE to solve. Note $J^*(x,T) = \frac{1}{2}x^{\mathrm{T}}Sx$. Maybe $J^*(x,t) = \frac{1}{2}x^{\mathrm{T}}P(t)x$ for some $P(t) = P^{\mathrm{T}}(t) \succeq 0$. Let's try:

$$
\begin{aligned}
\frac{\partial J^*}{\partial t} &= \frac{1}{2}x^{\mathrm{T}}\dot{P}x \\
\frac{\partial J^*}{\partial x} &= x^{\mathrm{T}}P \\
-\frac{1}{2}x^{\mathrm{T}}\dot{P}x &= \frac{1}{2}x^{\mathrm{T}}Qx + x^{\mathrm{T}}PAx - \frac{1}{2}x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px \\
&= \frac{1}{2}x^{\mathrm{T}}\Big(Q + 2PA - PBR^{-1}B^{\mathrm{T}}P\Big)x
\end{aligned}
$$

Note $x^{\mathrm{T}}PAx \in \mathbb{R}$ so $x^{\mathrm{T}}PAx = x^{\mathrm{T}}A^{\mathrm{T}}Px = \frac{1}{2}x^{\mathrm{T}}A^{\mathrm{T}}Px + \frac{1}{2}x^{\mathrm{T}}PAx = \frac{1}{2}x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x$.

$$
\Longrightarrow -\frac{1}{2}x^{\mathrm{T}}\dot{P}x = \frac{1}{2}x^{\mathrm{T}}\Big(Q + PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P\Big)x
$$

This has to hold for all x, i.e. P satisfies

$$\begin{cases} \dot{P} = -Q - PA - A^{\mathrm{T}}P + PBR^{-1}B^{\mathrm{T}}P \\ P(T) = S \end{cases}$$

This is known as the differential Riccati equation (RE/DRE). Luckily for us, we can actually solve RE "analytically" (almost if A, B, R, Q depend on t, and completely if they do not).

Theorem. *The optimal control is $u^* = -R^{-1}B^T P(t)x$, where $P(t) = P^T(t) \succeq 0$ solves the RE.*

Example Scalar example posted on T-square:

$$\begin{aligned} &\min \int_0^1 (qx^2 + ru^2)\,\mathrm{d}t + sx^2(1), \quad q, s \geq 0,\ r > 0 \\ &\text{s.t. } \dot{x} = ax + bu, \quad x, u \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} u &= -R^{-1}B^{\mathrm{T}}Px = -\frac{bp}{r}x \\ \dot{p} &= -q - 2ap + \frac{b^2}{r}p^2 \\ p(1) &= s \end{aligned}$$

How do we solve the RE?

$$\begin{aligned} \dot{x} &= Ax + Bu = \underbrace{(A - BR^{-1}B^{\mathrm{T}}P)}_{N(t)}x \\ x(t) &= \Phi(t,0)x(0) = \Phi(t,T)x(T), \end{aligned}$$

where Φ is the state transition matrix. Note this is also the zero-input response.

Let $X(t) = \Phi(t,T) \in \mathbb{R}^{n \times n}$. We know from ECE 6550 that

$$\begin{aligned} \dot{X} &= (A - BR^{-1}B^{\mathrm{T}}P)X \\ X(T) &= \mathrm{I} \end{aligned}$$

Let $Y = PX$. Then,

$$\begin{aligned} \dot{Y} &= \dot{P}X + P\dot{X} \\ &= \Big(-Q - A^{\mathrm{T}}P - PA + PBR^{-1}B^{\mathrm{T}}P\Big)X + P\Big(A - BR^{-1}B^{\mathrm{T}}P\Big)X \\ &= -QX - A^{\mathrm{T}}Y \\ Y(T) &= S \\ &\Longrightarrow \begin{cases} \dot{X} = AX - BR^{-1}B^{\mathrm{T}}Y \\ \dot{Y} = -QX - A^{\mathrm{T}}Y \\ X(T) = \mathrm{I} \\ Y(T) = S \end{cases} \end{aligned}$$

Note that $P = YX^{-1}$, where X is always invertible since it is a state transition matrix.

Assume that A, B, Q, R do not depend on time. Then,

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix}}_{M \in \mathbb{R}^{2n \times 2n}} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{M(t-T)} \begin{bmatrix} \mathrm{I} \\ S \end{bmatrix}$$

We've traded a quadratic $n \times n$ ODE for a linear $2n \times 2n$ ODE!