ECE 6553: Homework #4

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1. The following table shows the Hamiltonian, the optimality condition for u, the costate equation, and the transversality condition for the two formulations. We can see that the are equivalent problems.

$L = 1,\ \Psi = 0$	$L = 0,\ \Psi = T$
$H = 1 + \lambda^{\mathrm{T}} f$	$H = \lambda^{\mathrm{T}} f$
$\dfrac{\partial H}{\partial u} = \lambda^{\mathrm{T}} \dfrac{\partial f}{\partial u} = 0$	$\dfrac{\partial H}{\partial u} = \lambda^{\mathrm{T}} \dfrac{\partial f}{\partial u} = 0$
$\dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x} = -\dfrac{\partial f^{\mathrm{T}}}{\partial x}\lambda$	$\dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x} = -\dfrac{\partial f^{\mathrm{T}}}{\partial x}\lambda$
$H + \left.\dfrac{\partial \Psi}{\partial T}\right\rvert_{t=T} = 1 + \lambda^{\mathrm{T}} f + 0\Big\rvert_{t=T}$ $= 1 + \lambda^{\mathrm{T}} f\Big\rvert_{t=T}$	$H + \left.\dfrac{\partial \Psi}{\partial T}\right\rvert_{t=T} = \lambda^{\mathrm{T}} f + \left.\dfrac{\partial T}{\partial T}\right\rvert_{t=T}$ $= 1 + \lambda^{\mathrm{T}} f\Big\rvert_{t=T}$

2. We add an augmented state to represent the energy constraint:

$$
\begin{aligned}
&\min_{u,T} \int_0^T \mathrm{d}t \\
&\text{s.t.}\ \ \dot{x} = u, \quad x(0) = 0, \quad x(T) = x_T \\
&\qquad \dot{\hat{x}} = u^2, \quad \hat{x}(0) = 0, \quad \hat{x}(T) = E \\
&\quad H = 1 + \lambda u + \widehat{\lambda} u^2
\end{aligned}
$$

The costate equations are

$$
\begin{aligned}
\dot{\lambda} &= -\frac{\partial H}{\partial x} = 0 \Rightarrow \lambda = c_1 \\
\dot{\widehat{\lambda}} &= -\frac{\partial H}{\partial \hat{x}} = 0 \Rightarrow \widehat{\lambda} = c_2
\end{aligned}
$$

The optimal control is given by

$$
\begin{aligned}
\frac{\partial H}{\partial u} &= \lambda + 2\widehat{\lambda} u = c_1 + 2c_2 u = 0 \\
u &= -\frac{c_1}{2c_2} = k,
\end{aligned}
$$

where k is a constant.

Applying the boundary conditions,

$$
\begin{aligned}
x(T) &= x(0) + \int_0^T \dot{x}\,\mathrm{d}t = 0 + \int_0^T u\,\mathrm{d}t = kT = x_T \\
\hat{x}(T) &= \hat{x}(0) + \int_0^T \dot{\hat{x}}\,\mathrm{d}t = 0 + \int_0^T u^2\,\mathrm{d}t = k^2 T = E
\end{aligned}
$$

The quotient of the second equation and the first is

$$\frac{k^2T}{kT} = k = \frac{E}{x_T}$$
$$\Rightarrow \boxed{u = \frac{E}{x_T}}$$
$$T = \frac{x_T}{k} = \frac{x_T}{E/x_T}$$
$$\Rightarrow \boxed{T = \frac{x_T^2}{E}}$$

3. (a) The problem is

$$\begin{aligned}
&\min_{u,T} \int_0^T \mathrm{d}t \\
&\text{s.t. } \dot{x}_1 = x_2 \\
&\qquad \dot{x}_2 = -x_1 + u \\
&\qquad u(t) \in [-1, 1] \quad \forall t \in [0, T]
\end{aligned}$$

The Hamiltonian is $H = 1 + \lambda^{\mathrm{T}} f = 1 + \lambda_1 x_2 + \lambda_2(-x_1 + u)$. Then, the costate equations are

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = \lambda_2 \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1
\end{aligned}$$

We check the proposed λ to see whether they fit these equations:

$$\begin{aligned}
\dot{\lambda}_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big[\alpha \cos(T-t) + \beta \sin(T-t)\Big] \\
&= -\alpha \sin(T-t)\frac{\mathrm{d}}{\mathrm{d}t}(T-t) + \beta \cos(T-t)\frac{\mathrm{d}}{\mathrm{d}t}(T-t) \\
&= \alpha \sin(T-t) - \beta \cos(T-t) \\
&= \lambda_2(t) \\
\dot{\lambda}_2 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big[\alpha \sin(T-t) - \beta \cos(T-t)\Big] \\
&= \alpha \cos(T-t)\frac{\mathrm{d}}{\mathrm{d}t}(T-t) + \beta \sin(T-t)\frac{\mathrm{d}}{\mathrm{d}t}(T-t) \\
&= -\Big[\alpha \cos(T-t) + \beta \sin(T-t)\Big] \\
&= -\lambda_1(t)
\end{aligned}$$

They indeed fit the costate equations.

(b) Applying the transversality condition gives

$$\begin{aligned}
0 = H + \frac{\partial \Psi}{\partial T}\bigg|_{t=T} &= 1 + \lambda_1 x_2 + \lambda_2(-x_1 + u)\Big|_{t=T} \\
&= 1 + \lambda_2(T)u(T) \quad (\text{since } x(T) = 0)
\end{aligned}$$

Note that PMP gives the optimal control as $u = -\operatorname{sign}(\lambda_2)$, so

$$\begin{aligned} 0 &= 1 + \lambda_2(T) \cdot -\operatorname{sign}(\lambda_2(T)) \\ &= 1 - \big|\lambda_2(T)\big| \\ &= 1 - \big|\alpha \sin 0 - \beta \cos 0\big| \\ |\beta| &= 1 \\ \beta &= \pm 1 \end{aligned}$$

Additionally, this is a conservative system so H is constant. Then,

$$\begin{aligned} 0 = H + \frac{\partial \Psi}{\partial T}\bigg|_{t=0} &= 1 + \lambda_1 x_2 + \lambda_2(-x_1 + u)\Big|_{t=0} \\ &= 1 + \lambda_1(0) x_2(0) - \lambda_2(0) x_1(0) - \big|\lambda_2(0)\big| \\ &= 1 + \big[\alpha \cos T + \beta \sin T\big] x_2(0) - \big[\alpha \sin T - \beta \cos T\big] x_1(0) - \big|\alpha \sin T - \beta \cos T\big| \\ &= 1 + \big[\alpha \cos T + \beta \sin T\big] x_2(0) - \big[\alpha \sin T - \beta \cos T\big] x_1(0) - \sqrt{\alpha^2 + \beta^2} \\ &= 1 + \big[\alpha \cos T \pm \sin T\big] x_2(0) - \big[\alpha \sin T \mp \cos T\big] x_1(0) - \sqrt{\alpha^2 + 1} \\ \alpha^2 + 1 &= \Big(1 + \big[\alpha \cos T \pm \sin T\big] x_2(0) - \big[\alpha \sin T \mp \cos T\big] x_1(0)\Big)^2 \end{aligned}$$

This is quadratic in α, so α can take on four values dependent on $x(0)$ and T (and the sign of β), not necessary unique or real. For a given $x(0)$, the optimal values for α and β are the real values that result in minimum T.

4. On a time interval of length π, sine and cosine switch sign at most once. (Zero sign switches happen if they are zero at $k\pi$). Additionally, using the angle sum identity

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B),$$

we can rewrite the costate equation as a single sinusoid:

$$\lambda_2(t) = \sqrt{\alpha^2 + \beta^2} \sin(T - t + \phi), \quad \phi = \arctan\left(\frac{-\beta}{\alpha}\right).$$

Therefore, $\lambda_2(t)$ switches sign at most once on that interval and the optimal control

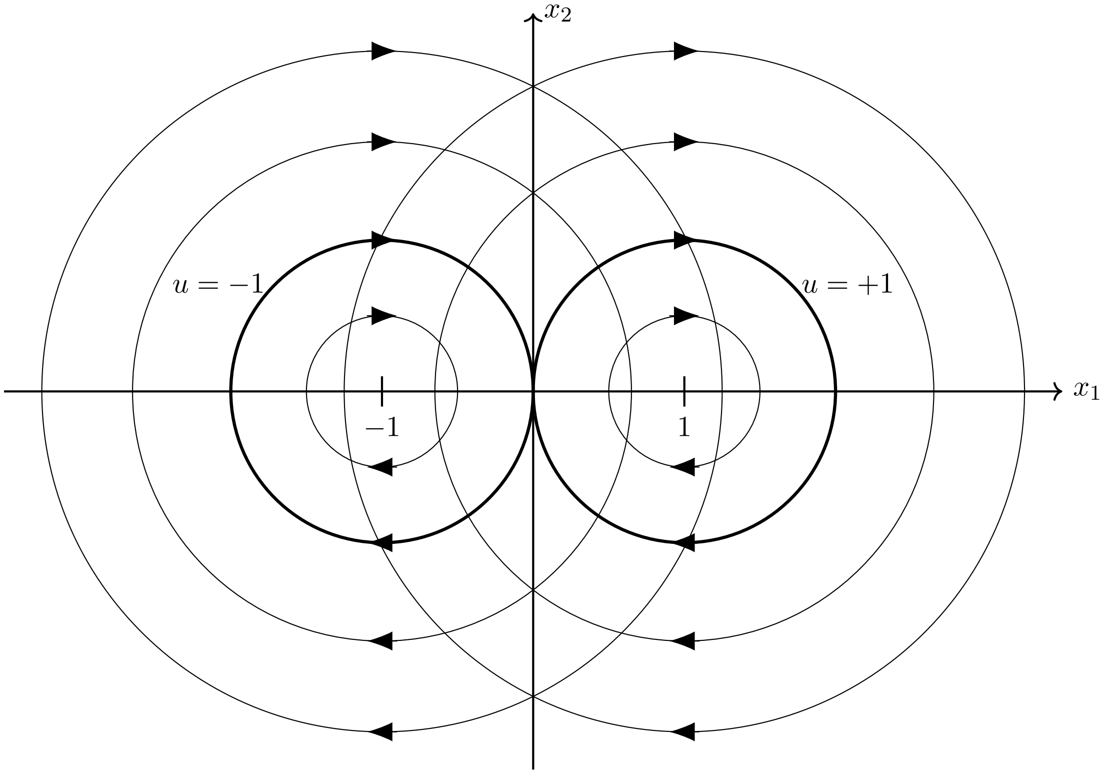
$$u = -\operatorname{sign}(\lambda_2)$$

also switches at most once (from ± 1 to ∓ 1).

5. The dynamics of the system are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + u$$

With $u = 0$, this describes clockwise movement along a circle centered at the origin. Thus, with the control $u = \pm 1$, the circles are centered at $(\pm 1, 0)$.



Note that each orbit takes the same amount of time to move around. The controller switches sign every π seconds to get to an orbit whose radius is closer to 1 (closer to the orbits that reach the origin).

6. Since the constraint is $y = Cx = 0\ \forall t$, then all derivatives of y are also zero, i.e.

$$\begin{aligned}
0 &= y = Cx \\
0 &= \dot{y} = C\dot{x} = C(Ax + Bu) = CAx + CBu \\
0 &= \ddot{y} = CA\dot{x} + CB\dot{u} = CA^2x + CABu + CB\dot{u} \\
0 &= y^{(3)} = CA^2\dot{x} + CAB\dot{u} + CB\ddot{u} = CA^3x + CA^2Bu + CAB\dot{u} + CB\ddot{u} \\
&\vdots \\
0 &= y^{(d)} = CA^dx + CA^{d-1}Bu + \sum_{k=0}^{d-2} CA^kBu^{(d-1-k)}
\end{aligned}$$

Since this system has relative degree d, $CA^kB = 0$ for $k < d - 1$, so

$$0 = CA^dx + CA^{d-1}Bu$$
$$u = -(CA^{d-1}B)^{-1}CA^dx$$

Note that the optimal control only depends on the state.

The optimality conditions of the system are

$$\boxed{\begin{aligned} H &= L + \lambda^{\mathrm{T}} f = x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u + \lambda^{\mathrm{T}}(Ax + Bu) \\ u &= -(CA^{d-1}B)^{-1} CA^{d} x \\ \dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} = -2Qx - A^{\mathrm{T}}\lambda \\ Cx_0 &= 0 \end{aligned}}$$

The optimal control and the costate are related by

$$\frac{\partial H}{\partial u} = 2u^{\mathrm{T}} R + \lambda^{\mathrm{T}} B = 0$$
$$u = \frac{1}{2} R^{-1} B^{\mathrm{T}} \lambda$$