

ECE 6553: Optimal Control Notes

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Chapter 1

Parameter Optimization

1.1 What is Optimal Control?

Optimal Maximize/minimize cost (subject to constraints): $\min_u g(u)$

With constraints,

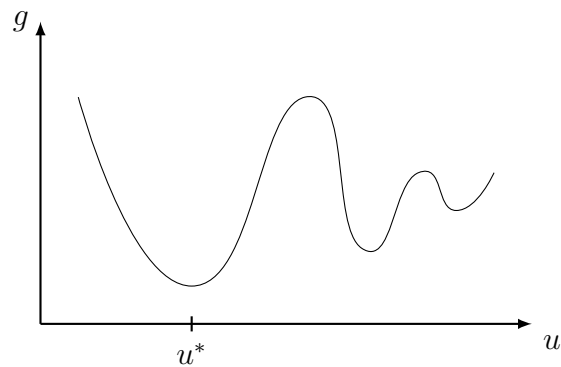
$$\begin{aligned} \min_u \quad & g(u) \\ \text{s.t.} \quad & \begin{cases} h_1(u) = 0 \\ h_2(u) \leq 0 \end{cases} \end{aligned}$$

First-order necessary condition (FONC):

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Optimality can be

- local vs global
- max vs min



Control control design: pick u such that specifications are satisfied:

$$\dot{x} = f(x, u), \quad \dot{x} = Ax + Bu,$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, and $f(\cdot)$ is the dynamics.

Actually, x and u are signals:

$$x : [0, T] \rightarrow \mathbb{R}^n, \quad u : [0, T] \rightarrow \mathbb{R}^m$$

Optimal control find the “best” u !

For “best” to mean anything, we need a cost. The big/deep question is

$$\frac{\partial \text{“cost”}}{\partial u} = 0$$

Example

Suppose we have a car with position p . Its acceleration \ddot{p} is controlled by the gas/brake input u ($\ddot{p} = u$). In order to express the dynamics of the system in the form $\dot{x} = f(x, u)$, we introduce state variables:

$$\begin{aligned} x_1 = p \\ x_2 = \dot{p} \end{aligned} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

The task is to move the car from its initial position to a stop at a distance c away.

Minimum energy problem

$$\begin{aligned} \min_u \quad & \int_0^T u^2(t) dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0, x_2(0) = 0 \\ & x_1(T) = c, x_2(T) = 0 \end{aligned}$$

Minimum time problem

$$\begin{aligned} \min_{u, T} \quad & T = \int_0^T dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0, x_2(0) = 0 \\ & x_1(T) = c, x_2(T) = 0 \\ & u(t) \in [u_{\min}, u_{\max}] \end{aligned}$$

The general optimal control problem we will solve will look like

$$\begin{aligned} \min_{u, T} \quad & \int_0^T L(x(t), u(t), t) dt + \Psi(x(T)) \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T] \\ & x(0) = x_0 \\ & x(T) \in S \\ & u(t) \in \Omega, \quad t \in [0, T] \end{aligned}$$

where $\Psi(\cdot)$ is the terminal cost and S is the terminal manifold. This is a so-called **Bolza Problem**.

What tools do we need to solve this?

1. optimality conditions $\partial \text{cost} / \partial u = 0$
2. some way of representing the optimal signal $u^*(x, t)$
3. some way of actually finding/computing the optimal controllers

1.2 Unconstrained Optimization

Let the decision variable be $u = [u_1, \dots, u_m]^T \in \mathbb{R}^m$. The cost is $g(u) \in C^1$ (C^k means k times continuously differentiable). The problem is

$$\min_u g(u), \quad g : \mathbb{R}^m \rightarrow \mathbb{R}$$

For u^* to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Definition. u^* is a (local) minimizer to g if $\exists \delta > 0$ s.t.

$$\begin{aligned} g(u^*) &\leq g(u) \quad \forall u \in B_\delta(u^*) \\ B_\delta(u^*) &= \{u \mid \|u - u^*\| \leq \delta\} \end{aligned}$$

Note:

- $\frac{\partial g}{\partial u}(u^*) \delta u \in \mathbb{R}$ and δu is $m \times 1$, so $\frac{\partial g}{\partial u}$ is a $1 \times m$ row vector. For the column vector,

$$\nabla g = \frac{\partial g^T}{\partial u} \in \mathbb{R}^m$$

- $\frac{\partial g}{\partial u} \delta u$ is an inner product

$$\langle \nabla g, \delta u \rangle = \left\langle \frac{\partial g^T}{\partial u}, \delta u \right\rangle$$

- $o(\varepsilon)$ encodes higher-order terms

$$\lim_{\varepsilon \rightarrow 0} \frac{o(\varepsilon)}{\varepsilon} = 0 \quad \text{“faster than linear”}$$

This is opposed to big-O notation:

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{O}(\varepsilon)}{\varepsilon} = c$$

- δu has direction and scale so we could write it as

$$\delta u = \varepsilon v, \quad \varepsilon \in \mathbb{R}, \quad v \in \mathbb{R}^m$$

Theorem. For u^* to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or, equivalently,

$$\frac{\partial g}{\partial u}(u^*)v = 0 \quad \forall v \in \mathbb{R}^m$$

Proof. Let u^* be a minimizer. Evaluating the cost $g(u)$ in the ball and using Taylor’s expansion,

$$g(u^* + \delta u) = g(u^*) + \frac{\partial g}{\partial u}(u^*)\delta u + o(\|\delta u\|) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$

Assume that $\frac{\partial g}{\partial u} \neq 0$. Then we could pick $v = -\frac{\partial g}{\partial u}^T(u^*)$, i.e.

$$g(u^* + \varepsilon v) = g(u^*) - \varepsilon \left\| \frac{\partial g}{\partial u}(u^*) \right\|^2 + o(\varepsilon)$$

Note that the second term is negative per our assumptions. So, for ε sufficiently small, we have

$$g\left(u^* - \varepsilon \frac{\partial g}{\partial u}(u^*)\right) < g(u^*)$$

This contradicts u^* being a minimizer. \times (crossed swords) □

Definition (Positive definite). $M = M^T \succ 0$ if

$$\begin{aligned} z^T M z &> 0 \quad \forall z \neq 0, \quad z \in \mathbb{R}^m \\ \iff M &\text{ has real and positive eigenvalues} \end{aligned}$$

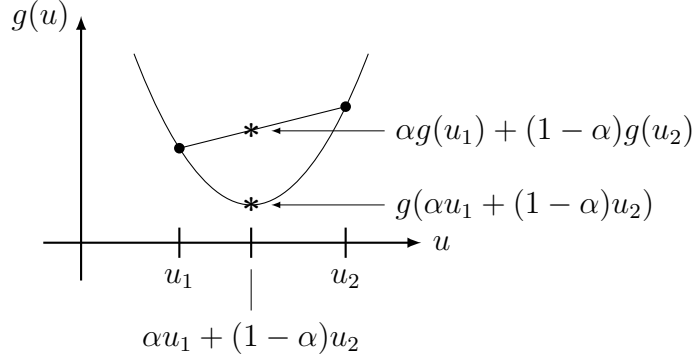
Theorem. If $g \in C^2$, then a **sufficient** condition for u^* to be a (local) minimizer is

$$1. \quad \frac{\partial g}{\partial u}(u^*) = 0$$

2. $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ (the Hessian is positive definite)

Definition. $g : \mathbb{R}^m \rightarrow \mathbb{R}$ is convex if

$$g(\alpha u_1 + (1 - \alpha)u_2) \leq \alpha g(u_1) + (1 - \alpha)g(u_2) \quad \forall \alpha \in [0, 1], \quad u_1, u_2 \in \mathbb{R}^m$$



Theorem. If $\frac{\partial^2 g}{\partial u^2}(u) \succeq 0 \quad \forall u \in \mathbb{R}^m$, then g is convex. (\iff for $g \in C^2$)

Example $\min_u u^T Q u - b^T u$ where $Q = Q^T \succ 0$ (positive definite matrix)

$$\begin{aligned} \frac{\partial g}{\partial u} &= \frac{\partial}{\partial u} (u^T Q u - b^T u) \\ &= u^T Q^T + u^T Q - b^T \\ &= 2u^T Q - b^T \\ \frac{\partial^2 g}{\partial u^2} &= 2Q \end{aligned} \quad \frac{\partial^2 g}{\partial u^2} = \begin{bmatrix} \frac{\partial^2 g}{\partial u_1^2} & \cdots & \frac{\partial^2 g}{\partial u_1 \partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 g}{\partial u_m^2} \end{bmatrix}$$

From $\frac{\partial g}{\partial u} = 2u^T Q - b^T = 0$,

$$u = \frac{1}{2} Q^{-1} b$$

To see whether this is a minimizer, consider the Hessian. Since $Q \succ 0$, it follows that $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ and $u^* = \frac{1}{2} Q^{-1} b$ is a (local) minimizer. Additionally, since $\frac{\partial^2 g}{\partial u^2} \succ 0$, g is convex and u^* is a global minimizer. In fact, since we have strict convexity ($\succ 0$ rather than $\succeq 0$), it is the unique global minimizer.

In optimal control, *local* is typically all we can ask for. In optimization, we can do better!

But wait, just because we know $\frac{\partial g}{\partial u} = 0$, it doesn't follow that we can actually find $u^* \dots$

1.3 Numerical Methods

Idea: $u_{k+1} = u_k + \text{step}_k$. What should step_k be? For small $\text{step}_k = \gamma_k v_k$,

$$g(u_k + \text{step}_k) = g(u_k) + \frac{\partial g}{\partial u}(u_k) \cdot \text{step}_k + o(\|\text{step}_k\|) = g(u_k) + \gamma_k \frac{\partial g}{\partial u}(u_k) v_k + o(\gamma_k)$$

A perfectly reasonable choice of step direction is

$$v_k = -\frac{\partial g^T}{\partial u}(u_k),$$

known as the *steepest descend* direction. This produces

$$g\left(u_k - \gamma_k \frac{\partial g}{\partial u}(u_k)\right) = g(u_k) - \gamma_k \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 + o(\gamma_k)$$

Steepest descent

$$u_{k+1} = u_k - \gamma_k \frac{\partial g^T}{\partial u}(u_k)$$

Note:

- What should γ_k be?
- This method “pretends” that $g(u)$ is linear. If we pretend $g(u)$ is quadratic, we get

$$u_{k+1} = u_k - \left(\frac{\partial^2 g}{\partial u^2}(u_k) \right)^{-1} \frac{\partial g^T}{\partial u}(u_k),$$

i.e. Newton’s Method

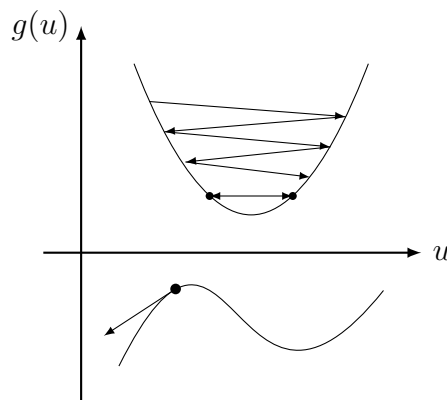
This course: steepest descent

Step-size selection?

- Choice 1: $\gamma_k = \gamma$ “small” $\forall k$; will get close to a minimizer if u_0 is close enough and γ small enough

Problems:

- You may not converge! (but you’ll get close)
- You may go off to infinity (diverge)



- Choice 2: Reduce γ_k as a function of k ; will get close to a minimizer if u_0 is close enough

Problem: slow

Theorem. If u_0 is close enough to u^* and γ_k satisfies

$$\begin{aligned} - \sum_{k=0}^{\infty} \gamma_k &= \infty \\ - \sum_{k=0}^{\infty} \gamma_k^2 &< \infty \end{aligned}$$

e.g. $\gamma_k = c/k$, then $u_k \rightarrow u^*$ as $k \rightarrow \infty$.

- Choice 3: **Armijo step-size:** Take as big a step as possible, but no larger
Pick $\alpha \in (0, 1)$, $\beta \in (0, 1)$. Let i be the smallest non-negative integer such that

$$\begin{aligned} g\left(u_k - \beta^i \frac{\partial g^T}{\partial u}(u_k)\right) - g(u_k) &< -\alpha \beta^i \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 \\ u_{k+1} &= u_k - \beta^i \frac{\partial g^T}{\partial u}(u_k) \end{aligned}$$

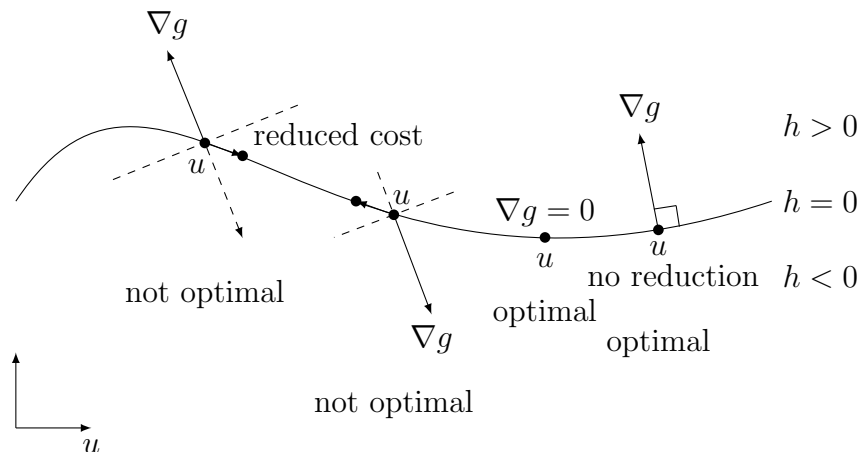
This will get to a minimizer blazingly fast if u_0 is close enough.

1.4 Constrained Optimization

Equality constraints:

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = 0 \end{aligned}$$

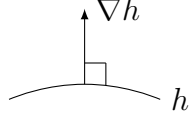
Consider $u \in \mathbb{R}^2$, $h : \mathbb{R}^2 \rightarrow \mathbb{R}$



So u is (locally) optimal if $\nabla g \parallel$ (is parallel to) the normal vector to tangent plane to h .

Fact: (HW# 1)

$$\nabla h \perp Th \quad (\text{tangent plane to } h)$$



We need $\nabla g \parallel \nabla h$ at u^* for optimality, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \alpha \frac{\partial h}{\partial u}(u^*), \quad \text{for some } \alpha \in \mathbb{R}$$

or ($\lambda = -\alpha$),

$$\frac{\partial g}{\partial u}(u^*) + \lambda \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}(g(u^*) + \lambda h(u^*)) = 0, \quad \text{for some } \lambda \in \mathbb{R}$$

More generally,

$$\begin{aligned} \min_{u \in \mathbb{R}^m} g(u) \\ \text{s.t. } h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

Note that $h(u) = [h_1(u), \dots, h_k(u)]^T$.

We need $\frac{\partial g}{\partial u}(u^*)$ to be a linear combination of $\frac{\partial h_i}{\partial u}(u^*)$, $i = 1, \dots, k$, for exactly the same reasons, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \sum_{i=1}^k \alpha_i \frac{\partial h_i}{\partial u}(u^*)$$

or ($\lambda = -[\alpha_1, \dots, \alpha_k]^T$)

$$\frac{\partial g}{\partial u}(u^*) + \lambda^T \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}(g(u^*) + \lambda^T h(u^*)) = 0, \quad \text{for some } \lambda \in \mathbb{R}^k$$

Theorem. If u^* is a minimizer to

$$\begin{aligned} \min_{u \in \mathbb{R}^m} g(u) \\ \text{s.t. } h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

then $\exists \lambda \in \mathbb{R}^k$ s.t.

$$\begin{cases} \frac{\partial L}{\partial u}(u^*, \lambda) = 0 \\ \frac{\partial L}{\partial \lambda}(u^*, \lambda) = 0 \end{cases}$$

where the Lagrangian L is given by

$$L(u, \lambda) = g(u) + \lambda^T h(u)$$

Note:

- λ are the Lagrange multipliers
- $\frac{\partial L}{\partial \lambda} = 0$ is fancy speak for $h(u^*) = 0$

Example

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & \frac{1}{2} \|u\|^2 \\ \text{s.t.} \quad & Au = b \end{aligned}$$

where A is $k \times m$, $k \leq m$. Assume $(AA^T)^{-1}$ exists (constraints are linearly independent, none of the constraints are “duplicates”, all the constraints are essential).

$$\begin{aligned} L &= \frac{1}{2} u^T u + \lambda^T (Au - b) \\ \frac{\partial L}{\partial u} &= u^T + \lambda^T A = 0 \\ u^* &= -A^T \lambda \end{aligned}$$

Using the equality constraint,

$$\begin{aligned} Au^* &= b \\ -AA^T \lambda &= b \\ \lambda &= -(AA^T)^{-1} b \\ u^* &= A^T (AA^T)^{-1} b \end{aligned}$$

Example

$$\begin{aligned} \min \quad & u_1 u_2 + u_2 u_3 + u_1 u_3 \\ \text{s.t.} \quad & u_1 + u_2 + u_3 = 3 \end{aligned}$$

$$L = u_1 u_2 + u_2 u_3 + u_1 u_3 + \lambda(u_1 + u_2 + u_3 - 3)$$

$$\begin{cases} \frac{\partial L}{\partial u_1} = u_2 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_2} = u_1 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_3} = u_2 + u_1 + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = u_1 + u_2 + u_3 = 3 \end{cases} \implies \begin{cases} u_1^* = 1 \\ u_2^* = 1 \\ u_3^* = 1 \\ \lambda = -2 \end{cases} \quad \text{optimal solution}$$

Note: This was actually the worst we can do—maximize! Even weirder: no local minimizer!

1.4.1 Equality Constraints

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

Theorem. If u^* is a minimizer/maximizer then $\exists \lambda \in \mathbb{R}^k$ s.t.

$$\begin{aligned} \frac{\partial L}{\partial u}(u^*, \lambda) &= 0 \\ \frac{\partial L}{\partial \lambda}(u^*, \lambda) &= 0 \quad (\iff h(u^*) = 0) \end{aligned}$$

where $L(u, \lambda) = g(u) + \lambda^T h(u)$.

Example [Entropy Maximization]

Given $S = \{x_1, \dots, x_n\}$ and a distribution over S such that it takes the value x_j with probability p_j . The entropy is

$$E(p) = \sum_{j=1}^n (-p_j \ln p_j).$$

The mean is

$$m = \sum_{j=1}^n p_j x_j.$$

Problem: Given m , find p such that E is maximized.

$$\begin{aligned} \min_p \quad & - \sum_{j=1}^n p_j \ln p_j \\ \text{s.t.} \quad & \sum_{j=1}^n p_j x_j = m \\ & \sum_{j=1}^n p_j = 1 \\ & p_j \geq 0, \quad j = 1, \dots, n \quad (\text{ignore this...}) \end{aligned}$$

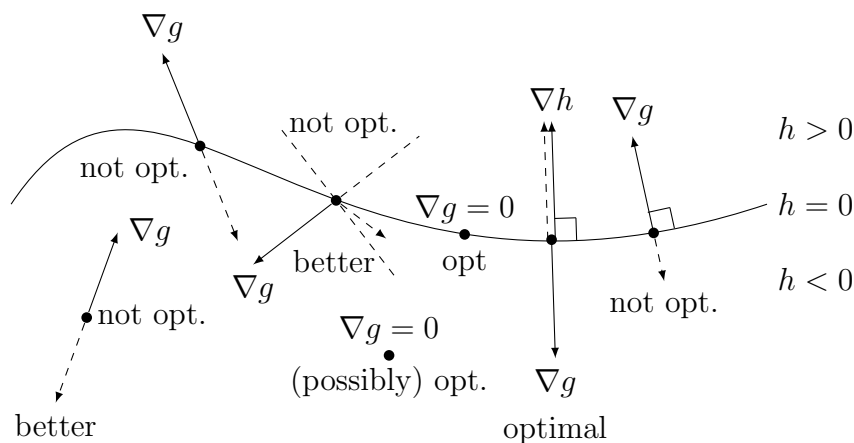
$$\begin{aligned} L &= - \sum p_j \ln p_j + \lambda_1 \left[\sum p_j x_j - m \right] + \lambda_2 \left[\sum p_j - 1 \right] \\ \frac{\partial L}{\partial p_j} &= - \ln p_j - 1 + \lambda_1 x_j + \lambda_2 = 0 \\ p_j &= e^{\lambda_2 - 1 + \lambda_1 x_j}, \quad j = 1, \dots, n \quad (p_j \geq 0 \text{ so we're ok with ignoring that}) \end{aligned}$$

$$\begin{aligned} \sum e^{\lambda_2 - 1 + \lambda_1 x_j} x_j &= m & n + 2 \text{ equations and} \\ \sum e^{\lambda_2 - 1 + \lambda_1 x_j} &= 1 & n + 2 \text{ unknowns...} \end{aligned}$$

No analytical solution, but numerically “solvable”

1.4.2 Inequality Constraints

$$\begin{array}{ll} \min_{u \in \mathcal{R}^m} & g(u) \\ \text{s.t.} & h(u) \leq \mathbf{0}, \quad h: \mathbb{R}^m \rightarrow \mathbb{R}^k \end{array}$$



We need:

- if $h(u^*) < 0$ then $\frac{\partial g}{\partial u}(u^*) = 0$
- if $h(u^*) = 0$ then we need either

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or

$$\frac{\partial g}{\partial u}(u^*) = -\lambda \frac{\partial h}{\partial u}(u^*) \quad \text{for } \lambda > 0$$

Or, even better,

$$\frac{\partial}{\partial u}(g(u^*) + \lambda h(u^*)) = 0 \quad \text{for } \lambda \geq 0,$$

where $\lambda h(u^*) = 0$. ($h < 0 \rightarrow \lambda = 0$, $h = 0 \rightarrow \lambda \geq 0$)

In general, if $u \in \mathbb{R}^m$ and $h : \mathbb{R}^m \rightarrow \mathbb{R}^k$, we have that u^* , if optimal, has to satisfy

$$\begin{aligned} \frac{\partial}{\partial u} L(u^*, \lambda) &= 0 \\ h(u^*) &\leq \mathbf{0} \\ \lambda^\text{T} h(u^*) &= 0 \\ \lambda &\geq \mathbf{0} \end{aligned}$$

where the Lagrangian is $L(u, \lambda) = g(u) + \lambda^T h(u)$. Note that if we're maximizing, the same holds except we need $\lambda \leq 0$.

Example

$$\begin{aligned} \min \quad & 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 \\ \text{s.t.} \quad & \begin{cases} u_1^2 + u_2^2 \leq 5 \\ 3u_1 + u_2 \leq 6 \end{cases} \end{aligned}$$

$$L = 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 + \lambda_1(u_1^2 + u_2^2 - 5) + \lambda_2(3u_1 + u_2 - 6)$$

FONC:

- i) $\partial L / \partial u_1 = 4u_1 + 2u_2 - 10 + 2\lambda_1u_1 + 3\lambda_2$
- ii) $\partial L / \partial u_2 = 2u_1 + 2u_2 - 10 + 2\lambda_1u_2 + \lambda_2$
- iii) $u_1^2 + u_2^2 \leq 5$
- iv) $3u_1 + u_2 \leq 6$
- v) $\lambda_1(u_1^2 + u_2^2 - 5) = 0$
- vi) $\lambda_2(3u_1 + u_2 - 6) = 0$
- vii) $\lambda_1 \geq 0$
- viii) $\lambda_2 \geq 0$

To solve, assume different constraints are active/inactive:

1. Both constraints are inactive ($u_1^2 + u_2^2 < 5$, $3u_1 + u_2 < 6$) $\implies \lambda_1 = \lambda_2 = 0$

$$\begin{cases} 4u_1 + 2u_2 - 10 = 0 \\ 2u_1 + 2u_2 - 10 = 0 \end{cases} \implies \begin{cases} u_1 = 0 \\ u_2 = 5 \end{cases}$$

Note: iii) $0^2 + 5^2 \not\leq 5$

Not feasible

2. Assume constraint 1 is active and constraint 2 is inactive ($u_1^2 + u_2^2 = 5$, $\lambda_2 = 0$)

$$\begin{cases} 4u_1 + 2u_2 - 10 + 2\lambda_1u_1 = 0 \\ 2u_1 + 2u_2 - 10 + 2\lambda_1u_2 = 0 \\ u_1^2 + u_2^2 = 5 \end{cases} \implies \begin{cases} u_1 = 1 \\ u_2 = 2 \\ \lambda_1 = 1 \end{cases}$$

$$\checkmark \lambda_1 \geq 0$$

$$\checkmark 3 \cdot 1 + 2 \leq 6$$

This is a local minimizer

3. Assume constraint 2 is active and constraint 1 is inactive
4. Assume both constraints are active

Kuhn-Tucker Conditions (KKT conditions, Karush-Kuhn-Tucker)

Problem:

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & \begin{cases} h_1(u) = 0, & h_1 : \mathbb{R}^m \rightarrow \mathbb{R}^p \\ h_2(u) \leq 0, & h_2 : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{cases} \end{aligned} \tag{1.1}$$

Theorem. Let u^* be feasible ($h_1 = 0, h_2 \leq 0$). If u^* is a minimizer to (1.1) then there exists vectors $\lambda \in \mathbb{R}^p, \mu \in \mathbb{R}^k$ with $\mu \geq \mathbf{0}$ such that

$$\begin{cases} \frac{\partial g}{\partial u}(u^*) + \lambda^T \frac{\partial h_1}{\partial u}(u^*) + \mu^T \frac{\partial h_2}{\partial u}(u^*) = 0 \\ \mu^T h_2(u^*) = 0 \end{cases}$$

Looking ahead: $\min \text{cost}(u(\cdot))$ s.t. $\dot{x} = f(x, u)$ (dynamics), where u is a function. Note the equality constraint.

Question: How do we go from $u \in \mathbb{R}^m$ to $u \in \mathcal{U}$ (function space)?

Note: Function space is a set of functions of a given kind from a set X to a set Y

1. linear function
2. square-integrable functions: $L_2[0, T] : \int_0^T \|u(t)\|^2 dt < \infty$
3. $C^\infty(\mathbb{R})$

What would ∂ “cost”/ ∂u mean?

Chapter 2

Calculus of Variations

2.1 Directional Derivatives

Recall: To minimize $g(u)$, let u^* be a candidate minimizer and pitch a perturbation on u^* of εv , where ε is the scale and v is the direction. Taking Taylor's expansion at the perturbation produces



$$g(u^* + \varepsilon v) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$

$$\text{FONC: } \frac{\partial g}{\partial u}(u^*) = 0$$

Note: $\frac{\partial g}{\partial u}(u^*)v$ tells us how much $g(u)$ increases/decreases in the direction of v .

Definition. The directional (Gateaux) derivative is given by

$$\delta g(u; v) = \lim_{\varepsilon \rightarrow 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon}$$

Example

$$g(u) = \frac{1}{2}u_1^2 - u_1 + 2u_2, \quad g : \mathbb{R}^2 \rightarrow \mathbb{R}$$

Let's consider $e_1 = [1 \ 0]^T$, $e_2 = [0 \ 1]^T$. What is $\delta g(u; e_i)$, $i = 1, 2$?

$$\begin{aligned} \delta g(u; v) &= \lim_{\varepsilon \rightarrow 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{g(u) + \varepsilon \frac{\partial g}{\partial u}(u)v + o(\varepsilon) - g(u)}{\varepsilon} \\ &= \frac{\partial g}{\partial u}(u)v \end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial u}(u) &= [u_1 - 1 \ 2] \\ \delta g(u; e_1) &= [u_1 - 1 \ 2]e_1 = u_1 - 1 \\ \delta g(u; e_2) &= [u_1 - 1 \ 2]e_2 = 2\end{aligned}$$

But the beauty of directional derivatives is that they generalize beyond vectors, $u \in \mathbb{R}^m$, to function spaces (\mathcal{U}) or other “objects” like matrices.

Example $M \in \mathbb{R}^{n \times n}$, $F(M) = M^2$
What is $\frac{\partial F}{\partial M}$? (ponder at home...)
We can easily compute $\delta F(M; N)$!

$$\begin{aligned}F(M + \varepsilon N) &= (M + \varepsilon N)(M + \varepsilon N) = M^2 + \varepsilon MN + \varepsilon NM + \varepsilon^2 N^2 \\ \delta F(M; N) &= \lim_{\varepsilon \rightarrow 0} \frac{F(M + \varepsilon N) - F(M)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon MN + \varepsilon NM + \varepsilon^2 N^2}{\varepsilon} = MN + NM\end{aligned}$$

Infinite Dimensional Optimization Let $u \in \mathcal{U}$ (function space) and let $J(u)$ be the cost:

$$\min_{u \in \mathcal{U}} J(u)$$

Theorem. If $u^* \in \mathcal{U}$ is a (local) minimizer then

$$\delta J(u^*; v) = 0, \quad \forall v \in \mathcal{U}$$

Example Find minimizer u^* to

$$J(u) = \int_0^T L(u(t)) \, dt$$

$$\begin{aligned}J(u + \varepsilon v) - J(u) &= \int_0^T L(u(t) + \varepsilon v(t)) \, dt - \int_0^T L(u(t)) \, dt, \quad u, v \in \mathcal{U} \\ &= \int_0^T \left[L(u(t)) + \varepsilon \frac{\partial L}{\partial u}(u(t))v(t) + o(\varepsilon) - L(u(t)) \right] \, dt \\ \delta J(u^*; v) &= \lim_{\varepsilon \rightarrow 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\int_0^T \varepsilon \frac{\partial L}{\partial u}(u(t))v(t) \, dt + o(\varepsilon)}{\varepsilon} \\ &= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t) \, dt\end{aligned}$$

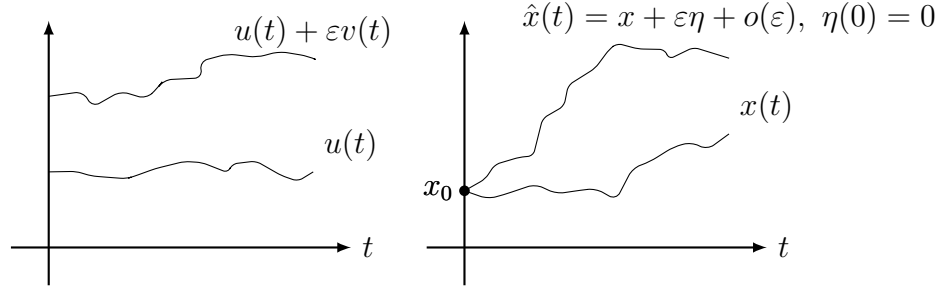


Figure 2.1: Variation in u causes a variation in x .

u^* optimizer:

$$\begin{aligned} \delta J(u^*; v) &= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t) dt = 0 \quad \forall v \in \mathcal{U} \\ &\Downarrow \\ \frac{\partial L}{\partial u}(u(t)) &= 0 \quad \forall t \in [0, T] \end{aligned}$$

But, we want *optimal control*! We want our cost to look like

$$\begin{aligned} \int_0^T L(x(t), u(t)) dt \\ \dot{x} = f(x, u) \end{aligned}$$

2.2 Calculus of Variations

What happens to $x(t)$ when $u(t)$ changes to $u(t) + \varepsilon v(t)$? Let the system be given by

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

After perturbation of u , the new system is

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}, u + \varepsilon v) \\ x(0) = x_0 \end{cases}$$

Consider

$$\tilde{x} = x + \varepsilon \eta,$$

where

$$\begin{aligned} \dot{x} &= f(x, u), & x(0) &= x_0 \\ \dot{\eta} &= \frac{\partial f}{\partial x}(x, u)\eta + \frac{\partial f}{\partial u}(x, u)v, & \eta(0) &= 0 \end{aligned}$$

Theorem. *If f is continuously differentiable in x and u then*

$$\hat{x}(t) = \tilde{x}(t) + o(\varepsilon)$$

Proof.

i) Initial conditions:

$$\begin{aligned}\hat{x}(0) &= x_0 \\ \tilde{x}(0) &= x(0) + \varepsilon\eta(0) = x_0\end{aligned}$$

ii) Dynamics:

$$\begin{aligned}\dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v) \\ \dot{\tilde{x}} &= \dot{x} + \varepsilon\dot{\eta} = f(x, u) + \varepsilon \frac{\partial f}{\partial x}(x, u)\eta + \varepsilon \frac{\partial f}{\partial u}(x, u)v \\ &= f(x + \varepsilon\eta, u + \varepsilon v) + o(\varepsilon) \\ &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon)\end{aligned}$$

We can see that the dynamics of $\hat{x}(t)$ are equal to those of $\tilde{x}(t)$ plus higher order terms:

$$\begin{aligned}\dot{\tilde{x}} &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon) \\ \dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v)\end{aligned}$$

Therefore, if our perturbation is small enough, we can model $\hat{x}(t)$ as $\tilde{x}(t)$.

□

Note: Taylor expansion with two elements is

$$\begin{aligned}h(w + \varepsilon v, z + \varepsilon y) &= h(w, z + \varepsilon y) + \frac{\partial h}{\partial w}(w, z + \varepsilon y)\varepsilon v + o(\varepsilon) \\ &= \left\{ h(w, z) + \frac{\partial h}{\partial z}(w, z)\varepsilon y + o(\varepsilon) \right\} \\ &\quad + \left\{ \frac{\partial h}{\partial w}(w, z)\varepsilon v + \underbrace{\frac{\partial^2 h}{\partial z \partial w}\varepsilon v \odot \varepsilon y}_{o(\varepsilon)} + o(\varepsilon) \right\} \\ &= h(w, z) + \frac{\partial h}{\partial z}\varepsilon y + \frac{\partial h}{\partial w}\varepsilon v + o(\varepsilon)\end{aligned}$$

Last class:

1. $u \in \mathcal{U}$ (space of functions), $J : \mathcal{U} \rightarrow \mathbb{R}$ (cost).

FONC: If u^* is optimal, then

$$\delta J(u; \nu) = 0 \quad \forall \nu \in \mathcal{U},$$

where the directional derivative is given by

$$\delta J(u; \nu) = \lim_{\varepsilon \rightarrow 0} \frac{J(u + \varepsilon \nu) - J(u)}{\varepsilon}.$$

2. If

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

then a variation in u :

$$u \mapsto u + \varepsilon \nu$$

results in a variation in x :

$$x \mapsto x + \varepsilon \eta + o(\varepsilon)$$

See Figure 2.1. Note $\eta(0) = 0$.

2.2.1 An (Almost) Optimal Control Problem

Let $\dot{x} = f(x)$, $x(0) = x_0$. Note we get to pick the initial condition!

Problem

$$\begin{aligned} \min_{x_0 \in \mathbb{R}^m} J(x_0) &= \int_0^T L(x(t)) \, dt \\ \text{s.t. } \begin{cases} \dot{x}(t) = f(x(t)) & \text{the constraint! (equality)} \\ x(0) = x_0 \end{cases} \end{aligned}$$

Note every constraint needs a Lagrange multiplier. We have infinitely many constraints:

$$\dot{x}(t) = f(x(t)) \quad \forall t \in [0, T]$$

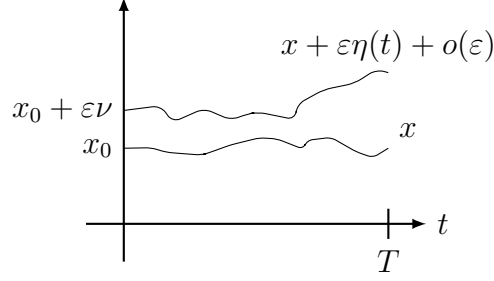
We need $\lambda(t)$ as a function of t . Also, the sum in the Lagrangian has to become an integral. The continuous-time Lagrangian thus becomes

$$\tilde{J}(x_0, \lambda) = \int_0^T \left[L(x(t)) + \lambda^T(t)(f(x(t)) - \dot{x}(t)) \right] dt$$

The task is to perturb x_0 as $x_0 \mapsto x_0 + \varepsilon \nu$, $\nu \in \mathbb{R}^m$ and compute

$$\delta \tilde{J}(x_0; \nu) = \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(x_0 + \varepsilon \nu) - \tilde{J}(x_0)}{\varepsilon}$$

and make this equal to 0 $\forall \nu \in \mathbb{R}^m$. The variation in x is



Note:

- x_0 decision variable
- ν variation in x_0
- $x(t)$ trajectory starting at x_0
- $\eta(t)$ change in trajectory resulting from ν -variation in x_0
- $\lambda(t)$ time-varying Lagrange multiplier

$$\begin{aligned}
\tilde{J}(x_0 + \varepsilon \nu) &= \int_0^T \left\{ L(x(t)) + \lambda^T(t) [f(x(t) + \varepsilon \eta(t)) - \dot{x}(t) - \varepsilon \dot{\eta}(t)] \right\} dt + o(\varepsilon) \\
&= \int_0^T \left[L(x) + \varepsilon \frac{\partial L}{\partial x}(x) \eta + \lambda^T \left(f(x) + \varepsilon \frac{\partial f}{\partial x}(x) \eta - \dot{x} - \varepsilon \dot{\eta} \right) \right] dt + o(\varepsilon) \\
\tilde{J}(x_0 + \varepsilon \nu) - \tilde{J}(x_0) &= \int_0^T \left[\varepsilon \frac{\partial L}{\partial x}(x) \eta + \lambda^T \left(\varepsilon \frac{\partial f}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt + o(\varepsilon) \\
\delta \tilde{J}(x_0; \nu) &= \int_0^T \left[\frac{\partial L}{\partial x}(x) \eta + \lambda^T \left(\frac{\partial f}{\partial x} \eta - \dot{\eta} \right) \right] dt
\end{aligned}$$

A powerful idea: we want $\delta \tilde{J}(x_0; \nu) = 0 \forall \nu$. Somehow get this in the form

$$\int_0^T (\text{stuff}(t)) \eta(t) dt = 0$$

We can pick $\text{stuff}(t) = 0 \forall t \in [0, T]$.

In $\delta \tilde{J}(x_0; \nu)$ we have $\dot{\eta}$ (problem!). We can solve this using *integration by parts*.

$$\int_0^T \lambda^T \dot{\eta} dt = \lambda^T(T) \eta(T) - \lambda^T(0) \eta(0) - \int_0^T \dot{\lambda}^T \eta dt$$

Hence,

$$\delta \tilde{J}(x_0; \nu) = \int_0^T \underbrace{\left(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T \right)}_{\text{pick}=0} \eta dt - \underbrace{\lambda^T(T) \eta(T)}_{\text{pick}=0} + \lambda^T(0) \underbrace{\eta(0)}_{\nu}$$

We are free to pick λ freely if it gives $\delta \tilde{J} = 0$.

$$\text{Pick: } \begin{cases} \dot{\lambda}(t) = -\frac{\partial L^T}{\partial x}(x(t)) - \frac{\partial f^T}{\partial x}(x(t)) \lambda(t) \\ \lambda(T) = 0 \end{cases} \quad \text{backwards diff. eq}$$

Under this choice of λ we get

$$\delta \tilde{J}(x_0; \nu) = \lambda^T(0) \nu$$

This is linear in ν so the FONC is $\lambda(0) = 0$.

Moreover, we really have a “normal” optimization problem

$$\begin{aligned} \min_{x_0 \in \mathbb{R}^m} \quad & \tilde{J}(x_0) \\ \delta \tilde{J}(x_0; \nu) = \frac{\partial \tilde{J}}{\partial x_0}(x_0) \nu \end{aligned}$$

which means that

$$\frac{\partial \tilde{J}}{\partial x_0} = \lambda^T(0)$$

If x_0^* minimizes

$$\begin{aligned} & \int_0^T L(x(t)) \, dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0^* \end{cases} \end{aligned}$$

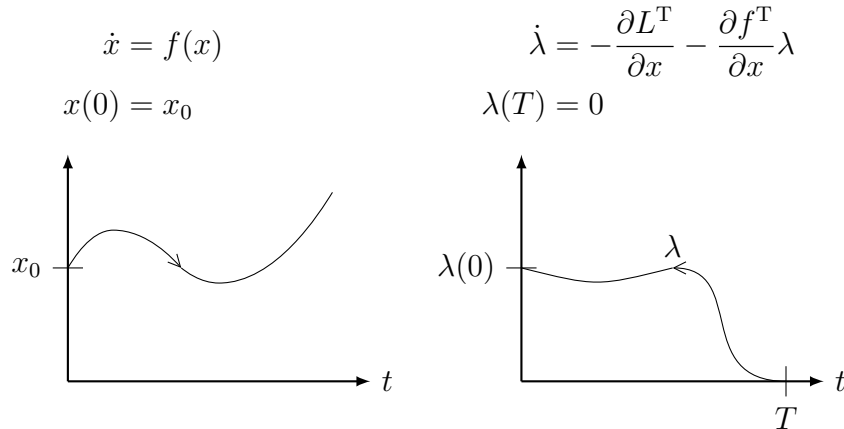
then

$$\lambda(0) = \mathbf{0}$$

where $\lambda(t)$ satisfies

$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial L^T}{\partial x}(x(t)) - \frac{\partial f^T}{\partial x}(x(t)) \lambda(t) \\ \lambda(T) = 0 \end{cases}$$

So what? We actually have a two-point boundary value problem.



We want to find x_0 that gives $f(x)$ such that after solving backwards for $\lambda(t)$, we find that

$$\lambda(0) = \frac{\partial \tilde{J}^T}{\partial x_0} = 0.$$

This leads to the following:

Pick $x_{0,0}$

$k = 1$

repeat

 Simulate $x(t)$ from $x_{0,k}$ over $[0, T]$

 Simulate $\lambda(t)$ from $\lambda(T) = 0$ backwards using $x(t)$

 Update $x_{0,k}$ as $x_{0,k+1} = x_{0,k} - \gamma\lambda(0)$

▷ $\lambda(0)$ is the gradient

$k := k + 1$

until $\lambda(0) = 0$

An algorithm

Example: `optinit.m`

$$\dot{x} = Ax, \quad L = x^T Q x - q, \quad Q = Q^T \succ 0$$

$$\dot{\lambda} = -2Qx - A^T \lambda$$

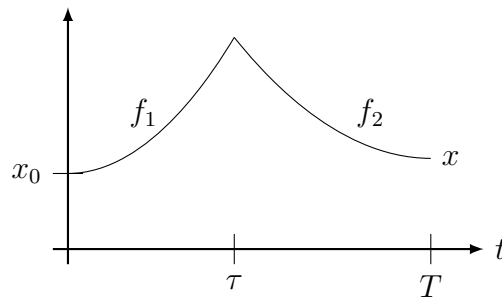
$$\lambda(0) = 0$$

2.2.2 Optimal Timing Control

When to switch between modes?

$$\dot{x} = \begin{cases} f_1(x) & \text{if } t \in [0, \tau) \\ f_2(x) & \text{if } t \in [\tau, T] \end{cases} \quad (2.1)$$

$$x(0) = x_0$$

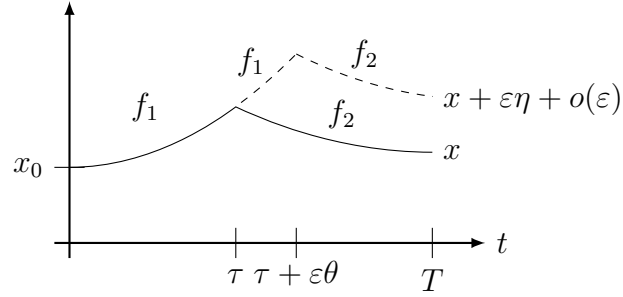


$$\begin{aligned} \min_{\tau} \int_0^T L(x(t)) \, dt &= J(\tau) \\ \text{s.t. (2.1) holds} \end{aligned}$$

Step 1: Augment cost with constraint

$$\tilde{J} = \int_0^{\tau} \left[L(x) + \lambda^T (f_1(x) - \dot{x}) \right] dt + \int_{\tau}^T \left[L(x) + \lambda^T (f_2(x) - \dot{x}) \right] dt$$

Step 2: Variation $\tau \mapsto \tau + \varepsilon\theta$



Step 3: Compute $\delta\tilde{J}(\tau; \theta)$

$$\begin{aligned}\tilde{J}(\tau + \varepsilon\theta) &= \int_0^{\tau + \varepsilon\theta} \left\{ L(x + \varepsilon\eta) + \lambda^T[f_1(x + \varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \right\} dt \\ &\quad + \int_{\tau + \varepsilon\theta}^T \left\{ L(x + \varepsilon\eta) + \lambda^T[f_2(x + \varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \right\} dt + o(\varepsilon)\end{aligned}$$

Note that $\eta = \dot{\eta} = 0$ on $[0, \tau)$.

$$\begin{aligned}\tilde{J}(\tau + \varepsilon\theta) &= \int_0^{\tau} \left\{ L(x) + \lambda^T[f_1(x) - \dot{x}] \right\} dt \\ &\quad + \int_{\tau}^{\tau + \varepsilon\theta} \left\{ \underbrace{L(x + \varepsilon\eta)}_{L(x) + \varepsilon \frac{\partial L}{\partial x} \eta} + \lambda^T \left[\underbrace{f_1(x + \varepsilon\eta)}_{f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta} - \dot{x} - \varepsilon\dot{\eta} \right] \right\} dt \\ &\quad + \int_{\tau + \varepsilon\theta}^T \left\{ \underbrace{L(x + \varepsilon\eta)}_{L(x) + \varepsilon \frac{\partial L}{\partial x} \eta} + \lambda^T \left[\underbrace{f_2(x + \varepsilon\eta)}_{f_2(x) + \varepsilon \frac{\partial f_2}{\partial x} \eta} - \dot{x} - \varepsilon\dot{\eta} \right] \right\} dt + o(\varepsilon)\end{aligned}$$

$$\begin{aligned}
\delta \tilde{J}(\tau; \theta) &= \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau)}{\varepsilon} \\
\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau) &= \int_0^\tau 0 \cdot dt + \underbrace{\int_\tau^{\tau+\varepsilon\theta} \left[\varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left(f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta - f_2(x) - \varepsilon \dot{\eta} \right) \right] dt}_{I_1} \\
&\quad + \underbrace{\int_{\tau+\varepsilon\theta}^T \left[\varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left(\varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt}_{I_2} + o(\varepsilon)
\end{aligned}$$

Theorem (Mean-value theorem).

$$\int_{t_1}^{t_2} h(t) dt = (t_2 - t_1)h(\xi) \quad \text{for some } \xi \in [t_1, t_2]$$

The first integral is

$$\begin{aligned}
I_1 &= \int_\tau^{\tau+\varepsilon\theta} \left\{ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left[f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta - \varepsilon \dot{\eta} - f_2(x) \right] \right\} dt \\
&= \varepsilon \theta \left\{ \lambda^\top(\xi) [f_1(x(\xi)) - f_2(x(\xi))] \right\} + o(\varepsilon)
\end{aligned}$$

Note that as $\varepsilon \rightarrow 0$, $\xi \rightarrow \tau$. Using integration by parts, the second integral is

$$\begin{aligned}
\int_\tau^T \lambda^\top \dot{\eta} dt &= \lambda^\top(T) \eta(T) - \lambda^\top(\tau) \underbrace{\eta(\tau)}_{=0} - \int_\tau^T \dot{\lambda}^\top \eta dt \\
I_2 &= \int_\tau^T \left[\varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left(\varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt - \underbrace{\int_\tau^{\tau+\varepsilon\theta} \left[\varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left(\varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt}_{o(\varepsilon)} \\
&= \varepsilon \int_\tau^T \left[\frac{\partial L}{\partial x} + \lambda^\top \frac{\partial f_2}{\partial x} + \dot{\lambda}^\top \right] \eta dt - \varepsilon \lambda^\top(T) \eta(T) + o(\varepsilon)
\end{aligned}$$

Hence,

$$\begin{aligned}
\delta \tilde{J}(\tau; \theta) &= \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau)}{\varepsilon} \\
&= \theta \lambda^\top(\tau) [f_1(x(\tau)) - f_2(x(\tau))] + \int_\tau^T \left[\frac{\partial L}{\partial x} + \lambda^\top \frac{\partial f_2}{\partial x} + \dot{\lambda}^\top \right] \eta dt - \lambda^\top(T) \eta(T)
\end{aligned}$$

Step 4: Select the *costate* $\lambda(t)$. The key idea is to get rid of any term that has η in it, i.e.

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial L}{\partial x} - \frac{\partial f_2^\top}{\partial x} \lambda \quad \text{on } [\tau, T] \\
\lambda(T) &= 0
\end{aligned}$$

Step 5: With this choice of $\lambda(t)$, we have

$$\delta \tilde{J}(\tau; \theta) = \theta \lambda^T(\tau) [f_1(x(\tau)) - f_2(x(\tau))] = \frac{\partial \tilde{J}}{\partial \tau} \theta.$$

Therefore,

$$\frac{\partial \tilde{J}}{\partial \tau} = \lambda^T(\tau) [f_1(x(\tau)) - f_2(x(\tau))] = 0 \quad (\text{for optimality})$$

Algorithm

```

Pick  $\tau_0$ 
 $k = 0$ 
repeat
    Simulate  $x$  forward in time from  $x(0) = x_0$ 
    Simulate  $\lambda$  backwards from  $\lambda(T) = 0$ 
    Update  $\tau_k$  as  $\tau_{k+1} = \tau_k - \gamma \lambda^T(\tau_k) [f_1(x(\tau_k)) - f_2(x(\tau_k))]$ 
     $k := k + 1$ 
until  $\|\lambda^T(f_1 - f_2)\| < \varepsilon$ 

```

Where are we going? Come up with general principles for $\min_{u \in \mathcal{U}} J(u)$:

- Costate equations
- Optimality conditions
- Algorithms
- Applications

Chapter 3

The Maximum Principle

3.1 The Bolza Problem

Up until now, we have optimized with respect to finite-dimensional parameters. Today, we will minimize with respect to $u \in \mathcal{U}$.

$$\begin{aligned} \min_{u \in \mathcal{U}} J(u) &= \int_0^T L(x(t), u(t), t) dt + \underbrace{\Psi(x(T))}_{\substack{\text{terminal cost} \\ \text{(parking cost)}}} \\ \text{s.t. } \dot{x}(t) &= f(x(t), u(t), t) \\ x(0) &= x_0 \end{aligned}$$

Assume that f and L are C^1 in x, u and piecewise continuous in t . Then, a small change in u causes small changes in f and L . The variation: $u \mapsto u + \varepsilon v$, $\varepsilon \in \mathbb{R}$, $v \in \mathcal{U}$. See Figure 2.1.

$$\begin{aligned} \tilde{J}(u) &= \int_0^T [L(x, u, t) + \lambda^T(f(x, u, t) - \dot{x})] dt + \Psi(x(T)) \\ \tilde{J}(u + \varepsilon v) &= \int_0^T [L(x + \varepsilon \eta, u + \varepsilon v, t) + \lambda^T(f(x + \varepsilon \eta, u + \varepsilon v, t) - \dot{x} - \varepsilon \dot{\eta})] dt \\ &\quad + \Psi(x(T) + \varepsilon \eta(T)) + o(\varepsilon) \\ \tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \left[L(x + \varepsilon \eta, u + \varepsilon v, t) - L(x, u, t) \right. \\ &\quad \left. + \lambda^T(f(x + \varepsilon \eta, u + \varepsilon v, t) - f(x, u, t) - \dot{x} - \varepsilon \dot{\eta} + \dot{x}) \right] dt \\ &\quad + \Psi(x(T) + \varepsilon \eta(T)) - \Psi(x(T)) + o(\varepsilon) \\ &= \int_0^T \left[\frac{\partial L}{\partial x} \varepsilon \eta + \frac{\partial L}{\partial u} \varepsilon v + \lambda^T \left(\frac{\partial f}{\partial x} \varepsilon \eta + \frac{\partial f}{\partial u} \varepsilon v - \varepsilon \dot{\eta} \right) \right] dt \\ &\quad + \frac{\partial \Psi}{\partial x}(x(T)) \varepsilon \eta(T) + o(\varepsilon) \end{aligned}$$

(See Taylor expansion with respect to two variables.)

$$\begin{aligned}\delta\tilde{J}(u;v) &= \int_0^T \left(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) v \, dt + \int_0^T \left[\left(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} \right) \eta - \dot{\lambda}^T \eta \right] \, dt \\ &\quad + \frac{\partial \Psi}{\partial x}(x(T))\eta(T)\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_0^T \dot{\lambda}^T \eta \, dt &= \lambda^T(T)\eta(T) - \lambda^T(0)\eta(0) - \int_0^T \dot{\lambda}^T \eta \, dt \\ &= \lambda^T(T)\eta(T) - \int_0^T \dot{\lambda}^T \eta \, dt \\ \delta\tilde{J}(u;v) &= \int_0^T \left(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) v \, dt + \int_0^T \left(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T \right) \eta \, dt \\ &\quad + \left(\frac{\partial \Psi}{\partial x}(x(T)) - \lambda^T(T) \right) \eta(T)\end{aligned}$$

For optimality, we need the directional derivative to be zero for every $v \in \mathcal{U}$, where v represents the direction of the derivative. Therefore, the term $(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u})$ in the first integral has to be identically zero. Thus, we need

$$\begin{cases} \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0, & \forall t \in [0, T] \\ \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T = 0, & \forall t \in [0, T] \\ \frac{\partial \Psi}{\partial x}(x(T)) - \lambda^T(T) = 0 \end{cases}$$

Definition. Let the *Hamiltonian* $H(x, u, t, \lambda)$ be given by

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^T f(x, u, t)$$

Theorem. For u to solve the Bolza problem, it has to satisfy

$$\frac{\partial H}{\partial u}(x, u, t, \lambda) = 0,$$

where the costate satisfies

$$\begin{cases} \dot{\lambda} = -\frac{\partial H^T}{\partial x}(x, u, t, \lambda) \\ \lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T)) \end{cases}$$

Example

$$\begin{aligned} \min_u \quad & \int_0^1 \frac{1}{2} u^2(t) \, dt + \frac{1}{2} x^2(1) \\ \text{s.t.} \quad & \begin{cases} \dot{x} = u, & x, u \in \mathbb{R} \\ x(0) = 1 \end{cases} \end{aligned}$$

$$H = \frac{1}{2} u^2 + \lambda u$$

$$\frac{\partial H}{\partial u} = u + \lambda = 0 \implies u = -\lambda$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0 \implies \lambda(t) = c$$

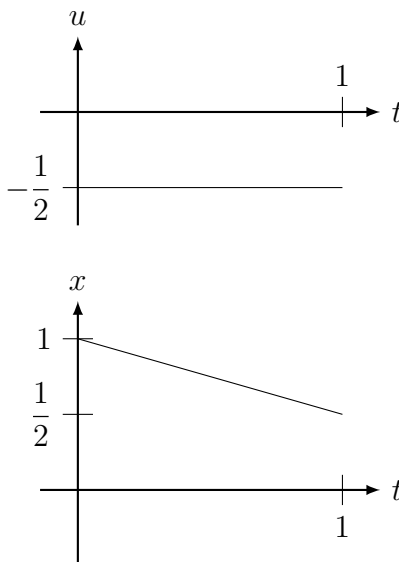
$$\lambda(T) = c = \frac{\partial \Psi}{\partial x}(x(1)) = x(1)$$

$$\dot{x} = u = -c \implies x(t) = -ct + x(0) = -ct + 1$$

$$x(1) = -c + 1$$

$$\lambda(1) = c = x(1) = -c + 1 \implies c = \frac{1}{2}$$

$$\boxed{u^* = -\frac{1}{2}}$$



We really used five different equations to solve this!

i) $\frac{\partial H}{\partial u} = 0$

ii) $\dot{\lambda} = -\frac{\partial H^T}{\partial x}$

$$\text{iii) } \lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T))$$

$$\text{iv) } \dot{x} = f(x, u, t)$$

$$\text{v) } x(0) = x_0$$

There is a sixth condition that is pretty useful if L and f do not depend on t ($L(x, u)$, $f(x, u)$). This is called a *conservative system*. Then, along optimal trajectories (equations i-v are satisfied), the total time derivative of the Hamiltonian is

$$\frac{d}{dt}H = \underbrace{\frac{\partial H}{\partial t}}_{0}_{H(x,u,\lambda)} + \underbrace{\frac{\partial H}{\partial x}}_{-\dot{\lambda}^T} \dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{0}_{u \text{ is optimal}} \dot{u} + \underbrace{\frac{\partial H}{\partial \lambda}}_{f^T = \dot{x}^T} \dot{\lambda} = -\dot{\lambda}^T \dot{x} + \dot{x}^T \dot{\lambda} = 0$$

Therefore, for conservative systems,

vi) H is constant along optimal trajectories. (Hamilton's Principle in analytical mechanics)

Back to the example,

$$H = \frac{1}{2}u^2 + \lambda u = \frac{1}{2}c^2 - c^2 = -\frac{1}{2}c^2 = -\frac{1}{8}$$

The Hamiltonian

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^T f(x, u, t)$$

lets us write the Lagrangian as

$$\tilde{J}(u) = \int_0^T [L + \lambda^T(f - \dot{x})] dt + \Psi = \int_0^T (H - \lambda^T \dot{x}) dt + \Psi$$

The optimality conditions are

$$\frac{\partial H}{\partial u} = 0, \tag{3.1}$$

where

$$\begin{cases} \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = \frac{\partial \Psi}{\partial x}(x(T)) \end{cases} \tag{3.2}$$

Example Hamilton's Principle

Let q be the generalized coordinates (positions and angles). Then, $\dot{q} = u$ are generalized velocities, which we assume we can control. Let $T(q, u) = u^T M(q)u$, $M \succ 0$, be the kinetic energy and $V(q)$ be the potential energy.

For conservative systems, the following quantity is minimized:

$$\int_0^T \underbrace{[T(q, u) - V(q)]}_{L(q, u) = \text{Lagrange's "action function"}} dt$$

The Hamiltonian is

$$H(q, u, \lambda) = L(q, u) + \lambda^T f(q, u) = L(q, u) + \lambda^T u$$

In mechanics, λ is called a generalized momentum, satisfying

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H^T}{\partial q} = -\frac{\partial L^T}{\partial q} + 0 \\ 0 &= \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \implies \lambda = -\frac{\partial L^T}{\partial u} \\ \dot{\lambda} &= -\frac{d}{dt} \frac{\partial L^T}{\partial u} = -\frac{\partial L^T}{\partial q} \end{aligned}$$

This produces the Euler-Lagrange Equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

Recall, along optimal trajectories

$$\begin{aligned} \frac{dH}{dt} &= \underbrace{\frac{\partial H}{\partial t}}_{=0 \text{ if } L \text{ and } f \text{ do not depend explicitly on } t} + \underbrace{\frac{\partial H}{\partial x}}_{-\dot{\lambda}^T \dot{x}} \dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{=0} \dot{u} + \underbrace{\frac{\partial H}{\partial \lambda}}_{f^T = \dot{x}^T} \dot{\lambda} = -\dot{\lambda}^T \dot{x} + \dot{x}^T \dot{\lambda} = 0 \end{aligned}$$

Therefore, along optimal trajectories, the Hamiltonian is constant!

We had

$$\begin{aligned} H &= L + \lambda^T u \\ \frac{\partial H}{\partial u} &= \lambda^T + \frac{\partial L}{\partial u} = 0 \end{aligned}$$

Along optimal trajectories,

$$H = L - \frac{\partial L}{\partial u} u$$

Recall, $L(q, u) = T(q, u) - V(q)$.

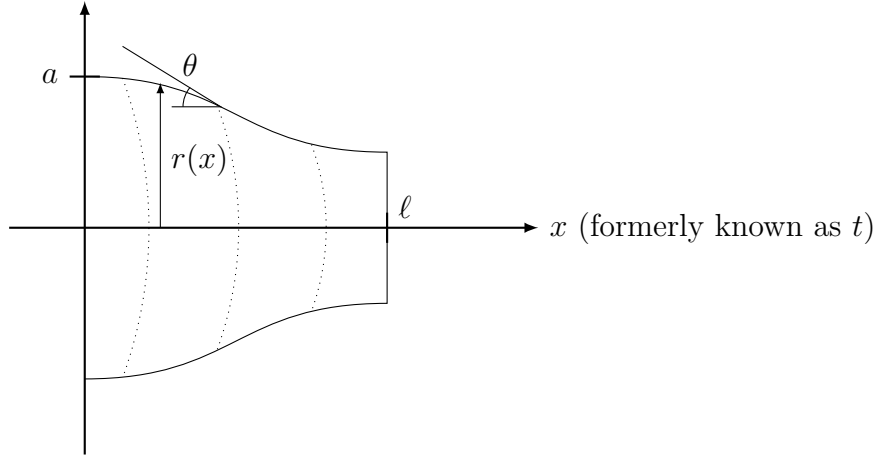
$$\begin{aligned} \frac{\partial L}{\partial u} &= \frac{\partial T}{\partial u} - 0 \\ T(q, u) &= u^T M(q) u \\ \frac{\partial T}{\partial u} &= 2u^T M \end{aligned}$$

So,

$$H = \underbrace{T}_{u^T M u} - V - 2u^T M u = -(V + u^T M u) = -(V + T)$$

Therefore, the total energy (kinetic plus potential energy) remains constant for conservative systems.

Example minimum drag nose shape (Newton 1686)



The drag is

$$D = -2\pi q \int_{x=0}^{\ell} C_p(\theta) r \, dr,$$

where q is a pressure constant and $C_p(\theta) = 2 \sin^2 \theta$ is Newton's pressure formula.

Geometry tells us

$$\frac{dr}{dx} = -\tan \theta = -u$$

Choose the control as $\tan \theta$. Manipulating the drag,

$$\frac{D}{4\pi q} = \int_0^{\ell} \frac{ru^3}{1+u^2} dx + \frac{1}{2}r(\ell)^2$$

The optimal control problem is

$$\begin{aligned} \min_u \quad & \int_0^{\ell} \frac{ru^3}{1+u^2} dx + \frac{1}{2}r(\ell)^2 \\ \text{s.t.} \quad & \frac{dr}{dx} = -u \end{aligned}$$

This is in the standard form with the following changes of variables:

$$\begin{aligned} \ell &\longleftarrow T \\ x &\longleftarrow t \\ r &\longleftarrow x \end{aligned}$$

Refer to (3.1) and (3.2) for the following steps.

$$\begin{aligned}
H &= \frac{ru^3}{1+u^2} - \lambda u \\
\frac{\partial H}{\partial u} &= \frac{3ru^2(1+u^2) - ru^3 \cdot 2u}{(1+u^2)^2} - \lambda \\
&= \frac{ru^4 + 3ru^2}{(1+u^2)^2} - \lambda = 0 \\
\lambda &= \frac{ru^2(u^2 + 3)}{(1+u^2)^2} \\
\frac{d\lambda}{dx} &= -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2} \\
\lambda(\ell) &= r(\ell)
\end{aligned} \tag{3.3}$$

Right now, we know

$$\begin{cases} \frac{dr}{dx} = -u \\ r(0) = a \\ \frac{d\lambda}{dx} = -\frac{u^3}{1+u^2} \\ \lambda(\ell) = r(\ell) \end{cases}$$

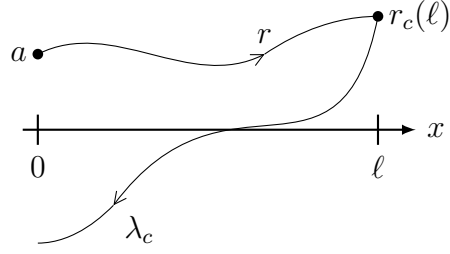
We need to remove u and get a function of r and λ instead. However, it is difficult to solve (3.3). Maybe $H = \text{const.}$ gives us something nicer?

$$\begin{aligned}
H &= \frac{ru^3}{1+u^2} - \lambda u \\
&= \frac{ru^3}{1+u^2} - \frac{ru^2(u^2 + 3)}{(1+u^2)^2} u \\
&= -\frac{2ru^3}{(1+u^2)^2} = c
\end{aligned}$$

Assume we can find $u = G(r, c)$, either numerically or some other way. So, now we have

$$\begin{cases} \frac{dr}{dx} = -G(r, c) \\ r(0) = a \\ \frac{d\lambda}{dx} = -\frac{G^3(r, c)}{1+G^2(r, c)} \\ \lambda(\ell) = r(\ell) \end{cases}$$

We do not know c , but we can guess c and simulate r forward in “time” (x) from $r(0) = a$. Then, we simulate λ backwards from $r(\ell)$.

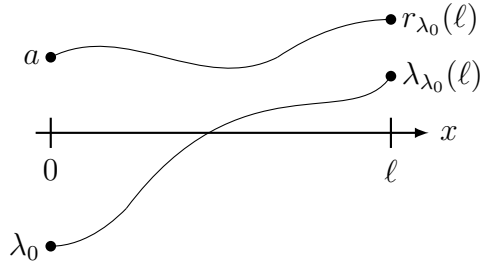


Problem: we can do this for any c . Which c is it? *Last 15 minutes was a dead end!*

Back to $u = F(r, \lambda)$. Assume we have F (numerically).

$$\begin{aligned}\frac{dr}{dx} &= -F(r, \lambda) \\ r(0) &= a \\ \frac{d\lambda}{dx} &= -\frac{F^3(r, \lambda)}{1 + F^2(r, \lambda)} \\ \lambda(\ell) &= r(\ell)\end{aligned}$$

The mistake before was that the simulation forward from a depends on λ .



Therefore, we should guess λ_0 and simulate both r and λ to get $r_{\lambda_0}(\ell)$ and $\lambda_{\lambda_0}(\ell)$. We need

$$r_{\lambda_0}(\ell) = \lambda_{\lambda_0}$$

for optimality. To do this, we need numerics.

Terminal Constraints

Let $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ and solve

$$\begin{aligned}\min_{u \in \mathcal{U}} \quad & \int_0^T L(x, u, t) dt + \Psi(x(T)) \\ \text{s.t.} \quad & \dot{x} = f(x, u, t) \\ & x(0) = x_0 \\ & x_i(T) = x_{iT} \quad \text{given for } i \in \mathcal{T} \subset \{1, \dots, n\}\end{aligned}$$

First, we augment the cost:

$$\begin{aligned}
\tilde{J}(u) &= \int_0^T [L + \lambda^T(f - \dot{x})] dt + \Psi \\
&= \int_0^T (H - \lambda^T \dot{x}) dt + \Psi \\
\tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \left(\varepsilon \frac{\partial H}{\partial u} v + \varepsilon \frac{\partial H}{\partial x} \eta - \varepsilon \lambda^T \dot{\eta} \right) dt + \varepsilon \frac{\partial \Psi}{\partial x}(x(T)) \eta(T) + o(\varepsilon) \\
\delta \tilde{J}(u; v) &= \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \int_0^T \frac{\partial H}{\partial u} v dt \\
&\quad + \lambda^T(0) \eta(0) - \lambda^T(T) \eta(T) + \frac{\partial \Psi}{\partial x}(x(T)) \eta(T)
\end{aligned}$$

As always,

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial H^T}{\partial x} \\
\frac{\partial H}{\partial u} &= 0 \quad (\text{FONC})
\end{aligned}$$

Additionally,

$$\begin{aligned}
\eta(0) &= 0 \\
\eta_i(T) &= 0 \quad \text{for } i \in \mathcal{T}
\end{aligned}$$

Note that if $x(T) = x_T$ is given, then $x(T) = x(T) + \varepsilon \eta(T) + o(\varepsilon)$, so $\eta(T) = 0$. Here, we have $x_i(T) = x_{iT}$ fixed for $i \in \mathcal{T}$ so $\eta_i(T) = 0$ for $i \in \mathcal{T}$.

For optimality, we want

$$\begin{aligned}
\left[-\lambda^T(T) + \frac{\partial \Psi}{\partial x}(x(T)) \right] \eta(T) &= 0 \quad \text{for all admissible variations} \\
\left[\frac{\partial \Psi}{\partial x_1} - \lambda_1, \quad \dots, \quad \frac{\partial \Psi}{\partial x_n} - \lambda_n \right] \begin{bmatrix} \eta_1(T) \\ \vdots \\ \eta_n(T) \end{bmatrix} &= 0
\end{aligned}$$

Hence, we need

$$\begin{aligned}
\lambda_j(T) &= \frac{\partial \Psi}{\partial x_j}(x(T)) \quad \text{if } j \notin \mathcal{T} \\
\lambda_i(T) &= \text{free} \quad \text{if } i \in \mathcal{T}
\end{aligned}$$

So we have

$$\begin{cases} \dot{x} = f \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x}, \end{cases}$$

an ODE with $2n$ variables. We need $2n$ boundary conditions for this ODE to be well-posed.

At $t = 0$		At $t = T$	
$x(0) = x_0$	$[n]$	$x_i(T) = x_{iT}, i \in \mathcal{T}$	$[q]$
		$ \mathcal{T} = q$	
		$x_j(T)$ free, $j \notin \mathcal{T}$	$[0]$
$\lambda(0)$ free	$[0]$	$\lambda_i(T)$ free, $i \in \mathcal{T}$	$[0]$
		$\lambda_j(T) = \frac{\partial \Psi}{\partial x_j}(x(T)), j \notin \mathcal{T}$	$[n - q]$

So we have $n + q + (n - q) = 2n$ boundary conditions.

We could even fix some but not all of $x(0)$, i.e.

$$\begin{aligned} x_i(0) &= x_{i0} & \text{if } i \in \mathcal{I} \\ x_j(0) &= \text{free} & \text{if } j \notin \mathcal{I} \end{aligned}$$

Recall,

$$\delta \tilde{J}(u; v) = \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta \, dt + \int_0^T \frac{\partial H}{\partial u} v \, dt + \lambda^T(0) \eta(0) + \left[\lambda^T(T) - \frac{\partial \Psi}{\partial x}(x(T)) \right] \eta(T)$$

For $x_i(0) = x_{i0}$ fixed, we have $\eta_i(0) = 0$ and $\lambda_i(0)$ free. For $x_j(0)$ free, we have $\eta_j(0)$ free and $\lambda_j(0) = 0$.

To ponder, what if $J = \int L \, dt + \Psi(x(T)) + \Theta(x(0))$?

To summarize, the minimizer to

$$\begin{aligned} \min_{u \in \mathcal{U}} & \int_0^T L(x, u, t) \, dt + \Psi(x(T)) \\ \text{s.t.} & \quad \dot{x} = f(x, u, t) \\ & \quad x_i(0) = x_{i0}, \quad i \in \mathcal{I} \\ & \quad x_j(T) = x_{jT} \quad j \in \mathcal{T} \end{aligned}$$

has to satisfy

$$\begin{aligned} \frac{\partial H}{\partial u} &= 0 \\ \dot{\lambda} &= -\frac{\partial H^T}{\partial x} \\ \lambda_i(0) &= 0, \quad i \notin \mathcal{I} \\ \lambda_j(T) &= \frac{\partial \Psi}{\partial x_j}(x(T)), \quad j \notin \mathcal{T} \end{aligned}$$

Example

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = f(x_1, x_2, x_3, x_4)$$

$$x_1(0) = 1, x_3(0) = 7, x_4(0) = 0, x_1(1) = 2$$

$$\mathcal{I} = \{1, 3, 4\}, \mathcal{T} = \{1\}$$

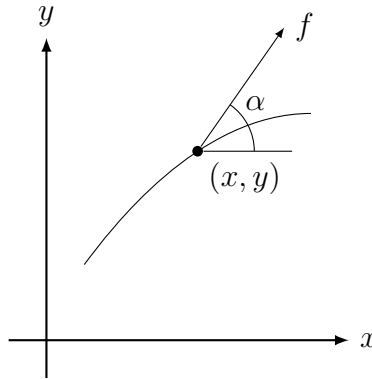
$$\min \int_0^1 L(x, u) dt + (x_2^2(1) - x_3^2(1) + 7x_1(1) + 14)$$

Note there are 4 boundary conditions on x so there must be 4 boundary conditions on λ :

$\lambda_1(0)$ free/unspecified	$\lambda_1(1)$ free
$\lambda_2(0) = 0$	$\lambda_2(1) = 2x_2(1)$
$\lambda_3(0)$ free	$\lambda_3(1) = -2x_3(1)$
$\lambda_4(0)$ free	$\lambda_4(1) = 0$

Example

A force f acts on a particle at position (x, y) (mass = 1).



$$\dot{x} = v_x$$

$$\dot{y} = v_y$$

$$\dot{v}_x = |f| \cos \alpha$$

$$\dot{v}_y = |f| \sin \alpha$$

$$\alpha = \text{control variable}$$

Assume we only care about where the particle ends up (to be specified later), i.e. $L = 0$.

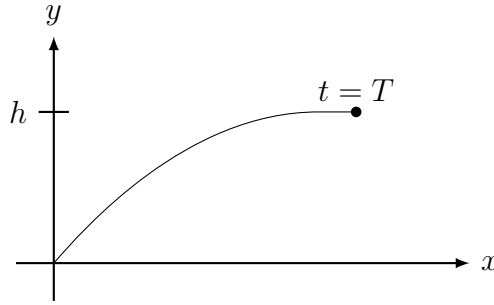
$$H = \begin{bmatrix} \lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f| \cos \alpha \\ |f| \sin \alpha \end{bmatrix}$$

$$\begin{aligned}
\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 &\implies \lambda_x(t) = c_1 \\
\dot{\lambda}_y = -\frac{\partial H}{\partial y} = 0 &\implies \lambda_y(t) = c_2 \\
\dot{\lambda}_{v_x} = -\frac{\partial H}{\partial v_x} = -\lambda_x &\implies \lambda_{v_x}(t) = -c_1 t + c_3 \\
\dot{\lambda}_{v_y} = -\frac{\partial H}{\partial v_y} = -\lambda_y &\implies \lambda_{v_y}(t) = -c_2 t + c_4
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{\partial H}{\partial \alpha} &= -\lambda_{v_x}|f|\sin \alpha + \lambda_{v_y}|f|\cos \alpha = 0 \\
\tan \alpha &= \frac{\lambda_{v_y}}{\lambda_{v_x}} = \frac{-c_2 t + c_4}{-c_1 t + c_3}
\end{aligned}$$

We want to drive the particle from $[0, 0, 0, 0]^T$ to a path parallel to the x-axis with $y(T) = h$.



Choose $\Psi = -v_x$,

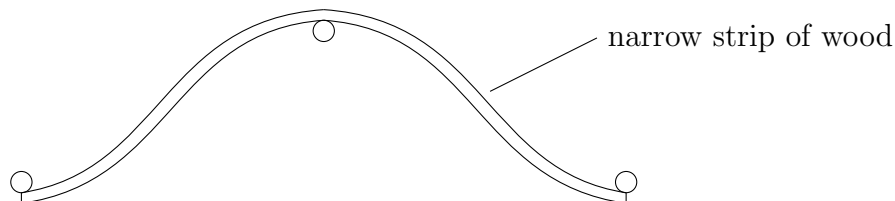
$$\begin{aligned}
y(T) &= h & v_y(T) &= 0 \\
x(T) &\text{ free} & v_x(T) &\text{ free, but costs} \\
\lambda_i(0) &\text{ free} \\
\lambda_y(T) &\text{ free} & \lambda_{v_y}(T) &\text{ free} \\
\lambda_x(T) &= 0 & \lambda_{v_x}(T) &= -1
\end{aligned}$$

$$\begin{aligned}
c_1 &= \lambda_x(t) = 0 \\
\implies \lambda_{v_x} &= -c_1 t + c_3 = c_3 = -1 \\
\implies \tan \alpha &= -\frac{-c_2 t + c_4}{-1} = c_2 t + c_4
\end{aligned}$$

How do we find c_2 and c_4 ? Plug into \dot{x} and $\dot{\lambda}$ and try to satisfy the remaining boundary conditions. (This is hard=numerics.)

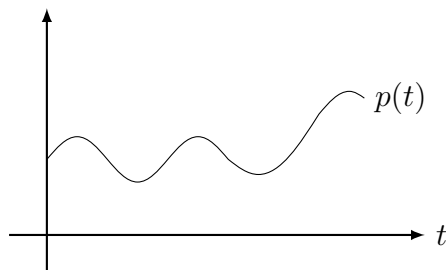
3.2 Splines

From ship building. Splines are used a lot in path-planning, e.g. cubic splines.



But, they are solutions to optimal control problems.

Let $p(t)$ be a curve we'd like to shape.



We want to minimize the “energy” put into the curve, a.k.a acceleration. Let $x_1 = p$ and $x_2 = \dot{p}$, so

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

3.2.1 Minimum-Energy

$$\begin{aligned} & \min_{u \in \mathcal{U}} \frac{1}{2} \int_0^T u^2(t) dt \quad + \text{Boundary conditions on } x \\ H &= L + \lambda^T f = \frac{1}{2} u^2 + \lambda_1 x_2 + \lambda_2 u \\ \frac{\partial H}{\partial u} &= u + \lambda_2 = 0 \implies u = -\lambda_2 \\ \dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = c_1 \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -c_1 t + c_2 \\ u &= -\lambda_2 = c_1 t - c_2 \\ \dot{x}_2 &= u = c_1 t - c_2 \implies x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3 \\ \dot{x}_1 &= x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3 \\ \implies x_1 &= \frac{c_1}{6} t^3 - \frac{c_2}{2} t^2 + c_3 t + c_4 \end{aligned}$$

$p(t)$ is a cubic polynomial!

What about boundary conditions?

Let $T = 1$, $p(0)$ given, $p(1)$ given, $\dot{p}(0) = 0$, $\dot{p}(1) = 0$, e.g. $p(0) = 0$, $p(1) = 1$. Since the boundary conditions for x are all specified, those for the costate are free.

$$\left. \begin{array}{l} x_1(0) = 0 \\ x_2(0) = 0 \\ x_1(1) = 1 \\ x_2(1) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lambda_1(0) \\ \lambda_2(0) \\ \lambda_1(1) \\ \lambda_2(1) \end{array} \right. \text{ free/unspecified}$$

$$\begin{array}{ll} x_2(0) = c_3 = 0 & x_1(1) = \frac{2c_2}{6} - \frac{c_2}{2} = 1 \\ x_1(0) = c_4 = 0 & c_2 = -6 \\ x_2(1) = \frac{c_1}{2} - c_2 + \underbrace{c_3}_0 = 0 & c_1 = -12 \\ c_1 = 2c_2 & \end{array}$$

$$\begin{aligned} \Rightarrow p(t) &= -2t^3 + 3t^2 \\ u(t) &= -12t + 6 \end{aligned}$$

Or, what if $\dot{p}(0)$, $\dot{p}(1)$ are not specified?

$$\left. \begin{array}{l} x_1(0) = 0 \\ x_2(0) \text{ unspec.} \\ x_1(1) = 1 \\ x_2(1) \text{ unspec.} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lambda_1(0) \text{ unspec.} \\ \lambda_2(0) = 0 \\ \lambda_1(1) \text{ unspec.} \\ \lambda_2(1) = 0 \end{array} \right.$$

$$\left. \begin{array}{l} \lambda_2(0) = c_2 = 0 \\ \lambda_2(1) = -c_1 + c_2 = 0 \end{array} \right\} \Rightarrow u = c_1 t - c_2 = 0$$

$$\left. \begin{array}{l} x_1(0) = c_4 = 0 \\ x_1(1) = c_3 = 1 \end{array} \right\} \Rightarrow p(t) = t$$

What did we do?

Case 1:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1/2 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_2(0) \\ x_2(1) \end{bmatrix}$$

Case 2:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ \lambda_2(0) \\ \lambda_2(1) \end{bmatrix}$$

3.2.2 Generalized Splines

We had $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This A is nilpotent ($A^k = 0$ for some $k \in \mathbb{Z}^+$). This means e^{At} is a polynomial in t . (This e^{At} is cubic.)

In general, e^{At} is a mix of polynomials, exponentials, and trigonometric terms. The eigenvalues of A determine the form of $x(t)$.

$$\begin{aligned} \dot{x} &= Ax &\implies x(t) &= e^{At}x(0) \\ \dot{x} &= Ax + Bu &\implies x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \end{aligned}$$

The general problem to solve is

$$\begin{aligned} \min_{u \in \mathcal{U}} & \int_0^T \frac{1}{2} \|u\|^2 dt \\ \text{s.t. } & \dot{x} = Ax + Bu \\ & + \text{Boundary conditions} \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2} \|u\|^2 + \lambda^T (Ax + Bu) \\ \frac{\partial H}{\partial u} &= u^T + \lambda^T B = 0 \\ &\implies u = -B^T \lambda \\ \dot{\lambda} &= -\frac{\partial H^T}{\partial x} = -A^T \lambda \end{aligned}$$

We have the Hamiltonian Dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BB^T \\ 0 & -A^T \end{bmatrix}}_M \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

Where we used $\dot{x} = Ax + Bu = Ax - BB^T \lambda$. Then,

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

Suppose we want to drive from $x(0) = x_0$ to $x(T) = x_T$.

$$\begin{aligned} \begin{bmatrix} x_T \\ \lambda(T) \end{bmatrix} &= e^{MT} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix} = \begin{bmatrix} N_{xx} & N_{x\lambda} \\ N_{\lambda x} & N_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix} \\ x_T &= N_{xx}x_0 + N_{x\lambda}\lambda(0) \end{aligned}$$

$N_{x\lambda}$ is invertible if (A, B) is completely controllable. Assume it is.

$$\begin{aligned}\lambda(0) &= N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \\ \implies \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} &= e^{Mt} \begin{bmatrix} x_0 \\ N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \end{bmatrix} \\ \implies u(t) &= -B^T \lambda(t)\end{aligned}$$

This is the optimal trajectory, but there is no feedback. We will consider closed-loop systems after the midterm.

As a preview, we need to find λ as a function of x . For example, $u = -R^{-1}B^T Px$ minimizes $u^T Ru$, so $\lambda = Px$ where P is the solution to the Riccati equation.

3.3 Numerical Methods

Optimal control boils down to solving two sets of differential equations:

$$\begin{aligned}\dot{x} &= f(x, u) & \frac{\partial H}{\partial u}(x, u, \lambda) &= 0 \\ \dot{\lambda} &= -\frac{\partial H^T}{\partial x}(x, u, \lambda) & u &= F(x, \lambda) \\ \implies & \begin{cases} \dot{x} = f(x, F(x, \lambda)) \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x}(x, F(x, \lambda), \lambda) \end{cases}\end{aligned}$$

The equations are functions of x and λ . They are completely determined by the boundary conditions on $x(0)$, $x(T)$, $\lambda(0)$, $\lambda(T)$. This is known as the *Boundary Value Problem*. This is solved using *test shooting*:

1. Guess initial conditions
2. Simulate forward in time
3. Update the guess (cleverly...)

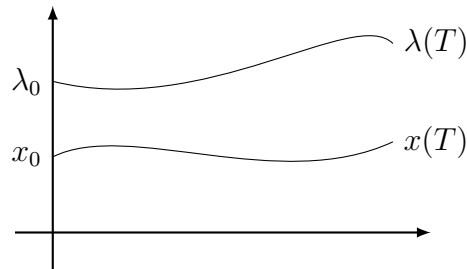
Exmample: Bolza problem

$$\begin{aligned}\min_{u \in \mathcal{U}} & \int_0^T L(x, u) dt + \Psi(x(T)) \\ \text{s.t.} & \begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases} \\ & H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u) \\ & u^*(x, \lambda) \text{ satisfies } \frac{\partial H}{\partial u} = 0\end{aligned}$$

The optimal control satisfies

$$\begin{cases} x = f(x, u^*(x, \lambda)) \\ x(0) = x_0 \\ \lambda = -\frac{\partial H^T}{\partial x}(x, u^*(x, \lambda), \lambda) \\ \lambda(T) = \frac{\partial \Psi}{\partial x}(x(T)) \end{cases}$$

Algorithm Guess λ_0 and solve for $x(t)$, $\lambda(t)$.



Let's define a cost:

$$\left\| \lambda(T) - \frac{\partial \Psi^T}{\partial x}(x(T)) \right\|^2 = g(\lambda_0)$$

Update λ_0 through

$$\lambda_0 := \lambda_0 - \gamma \frac{\partial g^T}{\partial \lambda_0}(\lambda_0)$$

↑
any choice of step size works

Repeat

Problem: What is $\partial g / \partial \lambda_0$? We estimate $\partial g / \partial \lambda_0$ numerically. This is where “test shooting” comes into play.

Let e_i be the i th unit vector, $i = 1, \dots, n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\frac{\partial g}{\partial \lambda_0} = \left(\frac{\partial g}{\partial \lambda_{0,1}}, \frac{\partial g}{\partial \lambda_{0,2}}, \dots, \frac{\partial g}{\partial \lambda_{0,n}} \right)$$

The i th component of $\partial g / \partial \lambda_0$ is given by the directional derivative

$$\frac{\partial g}{\partial \lambda_{0,i}} = \frac{\partial g}{\partial \lambda_0} \cdot e_i = \delta g(\lambda_0; e_i) = \lim_{\varepsilon \rightarrow 0} \frac{g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)}{\varepsilon}$$

So, if $x \in \mathbb{R}^n$ (and thus so is λ_0), we have to do this n times (with a small ε) and get the full derivative $\partial g / \partial \lambda_0$.

```

Given  $\lambda_0, g(\lambda_0)$ 
for  $i = 1$  to  $n$  do
    Compute  $g(\lambda_0 + \varepsilon e_i)$ 
     $dg_i = \frac{1}{\varepsilon}[g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)]$ 
end for
 $\frac{\partial g}{\partial \lambda_0} = [dg_1, \dots, dg_n]$ 

```

Algorithm

Example LQ

$$\begin{aligned}
 & \min_u \frac{1}{2} \int_0^1 (x^T Q x + u^T R u) dt + \frac{1}{2} x^T(1) S x(1) \\
 & \text{s.t.} \quad \begin{cases} \dot{x} = A x + B u \\ x(0) = x_0 \end{cases} \\
 & \quad Q, R, S \succ 0
 \end{aligned}$$

$$\begin{aligned}
 H &= \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \lambda^T (A x + B u) \\
 \frac{\partial H}{\partial u} &= u^T R + \lambda^T B = 0 \\
 u^* &= -R^{-1} B^T \lambda \\
 \dot{\lambda} &= -\frac{\partial H^T}{\partial x} = -Q x - A^T \lambda \\
 \lambda(1) &= \frac{\partial \Psi^T}{\partial x}(x(1)) = S x(1)
 \end{aligned}$$

So putting it all together,

$$\begin{aligned}
 \dot{x} &= A x - B R^{-1} B^T \lambda & x(0) &= x_0 \\
 \dot{\lambda} &= -Q x - A^T \lambda & \lambda(1) &= S x(1)
 \end{aligned}$$

Example Newton's nose shape problem (revisited, see previous)

$$\begin{aligned}
 & \min_u \int_0^\ell \frac{r u^3}{1 + u^2} dx + \frac{1}{2} r(\ell)^2 \\
 & \text{s.t.} \quad \frac{dr}{dx} = -u \quad r(0) = a
 \end{aligned}$$

$$H = \frac{ru^3}{1+u^2} + \lambda(-u)$$

$$\frac{\partial H}{\partial u} = \frac{ru^2(3+u^2)}{(1+u^2)^2} - \lambda = 0$$

We solve the above numerically to get $u^*(r, \lambda)$.

$$\frac{\partial \lambda}{\partial x} = -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2}$$

$$\lambda(\ell) = r(\ell)$$

So, we have

$$\frac{dr}{dx} = -u \quad r(0) = a \quad u = F(x, \lambda)$$

$$\frac{d\lambda}{dx} = -\frac{u^3}{1+u^2} \quad \lambda(\ell) = r(\ell)$$

Example Fixed terminal constraints (revisited, see previous)

$$\min_{\alpha} -v_x(T) \quad \alpha = \text{control}$$

$$\text{s.t.} \quad \begin{aligned} \dot{x} &= v_x & x(0) &= 0 \\ \dot{y} &= v_y & y(0) &= 0 \\ \dot{v}_x &= |f| \cos \alpha & v_x(0) &= 0 \\ \dot{v}_y &= |f| \sin \alpha & v_y(0) &= 0 \\ y(T) &= h \\ v_y(T) &= 0 \end{aligned}$$

$$H = -v_x(T) + \begin{bmatrix} \lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f| \cos \alpha \\ |f| \sin \alpha \end{bmatrix}$$

$$\frac{dH}{d\alpha} = 0 \Rightarrow \tan \alpha = \frac{\lambda_{v_y}}{\lambda_{v_x}}$$

$$\dot{\lambda}_x = 0$$

$$\dot{\lambda}_y = 0$$

$$\dot{\lambda}_{v_x} = -\lambda_x$$

$$\dot{\lambda}_{v_y} = -\lambda_y$$

$$\boldsymbol{\lambda}(0) \text{ unspecified}$$

$$\lambda_x(T) = \frac{\partial \Psi^T}{\partial x}(x(T)) = 0$$

$$\lambda_y(T) \text{ unspecified}$$

$$\lambda_{v_x}(T) = \frac{\partial \Psi^T}{\partial v_x}(v_x(T)) = -1$$

$$\lambda_{v_y}(T) \text{ unspecified}$$

Again, we guess λ_0 and solve forward in time. But, we have terminal constraints on y and v_y as well.

$$g(\lambda_0) = \frac{1}{2} \left[(y(T) - h)^2 + (v_y(T))^2 + (\lambda_x(T))^2 + (\lambda_{v_x} + 1)^2 \right]$$

3.4 Terminal Manifolds

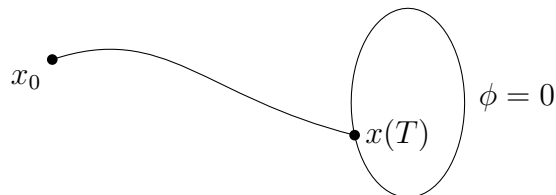
We can solve

$$\begin{aligned} \min_{u \in \mathcal{U}} \int_0^T L(x, u, t) dt + \Psi(x(T)) \\ \text{s.t. } \dot{x} = f(x, u, t) \end{aligned}$$

with all sorts of boundary conditions on x :

- $x(0) = x_0$, $x(T)$ free (typical)
- $x_i(0) = x_{i0}$, $i \in \mathcal{I}$ and $x_j(T) = x_{jT}$, $j \in \mathcal{T}$

But what if we want $x(T)$ to belong to a set?



Problem

$$\begin{aligned}
& \min_{u \in \mathcal{U}} \int_0^T L(x, u, t) dt + \Psi(x(T)) \\
& \text{s.t. } \dot{x} = f(x, u, t), \quad x \in \mathbb{R}^n \\
& \quad x(0) = x_0 \\
& \quad \phi(x(T)) = 0, \quad \phi : \mathbb{R}^n \rightarrow \mathbb{R}^q, \quad q \leq n
\end{aligned}$$

The augmented cost is

$$\tilde{J} = \int_0^T [H(x, u, t, \lambda) - \lambda^T \dot{x}] dt + \Psi(x(T)) + \underbrace{\nu^T \phi(x(T))}_{\substack{\text{q-dimensional} \\ \text{Lagrange multiplier}}}$$

Let $\Phi(x(T), \nu) = \Psi(x(T)) + \nu^T \phi(x(T))$. Then,

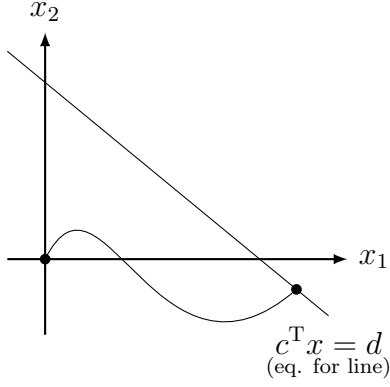
$$\tilde{J} = \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \Phi(x(T), \nu)$$

We know how to solve this! With $u \mapsto u + \varepsilon v$, $x \mapsto x + \varepsilon \eta + o(\varepsilon)$,

$$\begin{aligned}
\delta \tilde{J} &= \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \int_0^T \frac{\partial H}{\partial u} v dt + \underbrace{\frac{\partial \Phi}{\partial x}(x(T), \nu) \eta(T)}_{=0} \\
&\quad - \lambda^T(T) \eta(T) + \lambda^T(0) \underbrace{\eta(0)}_{=0} \\
&\left\{ \begin{array}{l} \frac{\partial H}{\partial u} = 0 \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = \frac{\partial \Phi^T}{\partial x}(x(T), \nu) \\ \phi(x(T)) = 0 \end{array} \right. \quad \begin{array}{l} \text{q new variables} \\ \downarrow \\ \leftarrow \text{q new equations} \end{array} \\
&\implies u^*
\end{aligned}$$

Spring to line

$$\begin{aligned}
& \min_u \frac{1}{2} \int_0^1 u^2(t) dt \\
& \text{s.t. } \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ x_1(0) = 0, \quad x_2(0) = 0 \\ c_1 x_1(1) + c_2 x_2(1) = d \end{cases}
\end{aligned}$$



$$H = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u$$

$$\frac{\partial H}{\partial u} = u + \lambda_2 \implies u = -\lambda_2$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = k_1$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -k_1 t + k_2$$

$$\phi(x(1)) = c_1 x_1(1) + c_2 x_2(1) - d$$

$$\Psi = 0 \implies \Phi = \nu(c_1 x_1(1) + c_2 x_2(1) - d)$$

$$\lambda_1(1) = \frac{\partial \Phi}{\partial x_1} = \nu c_1$$

$$\lambda_2(1) = \frac{\partial \Phi}{\partial x_2} = \nu c_2$$

So,

$$\lambda_1(1) = \nu c_1 = k_1$$

$$\lambda_2(1) = \nu c_2 = -k_1 + k_2$$

$$k_2 = \nu(c_1 + c_2)$$

$$\dot{x}_2 = u = -\lambda_2 = k_1 t - k_2$$

$$x_2 = \frac{k_1}{2}t^2 - k_2 t + 0$$

$$\dot{x}_1 = x_2$$

$$x_1 = \frac{k_1}{6}t^3 - \frac{k_2}{2}t^2 + 0$$

Substituting k_1 and k_2 into $c_1 x_1(1) + c_2 x_2(1) = d$,

$$\nu \left(-\frac{c_1^2}{3} - c_1 c_2 - c_2^2 \right) = d$$

$$\nu = -\frac{d}{c_1^2/3 + c_1 c_2 + c_2^2}$$

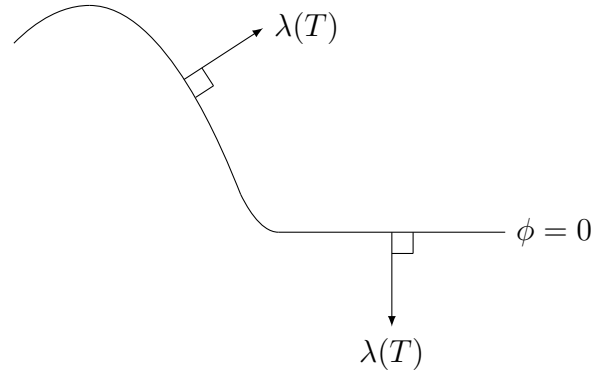
And finally

$$u = k_1 t - k_2 = \frac{d}{c_1^2/3 + c_1 c_2 + c_2^2} (c_1 + c_2 - c_1 t)$$

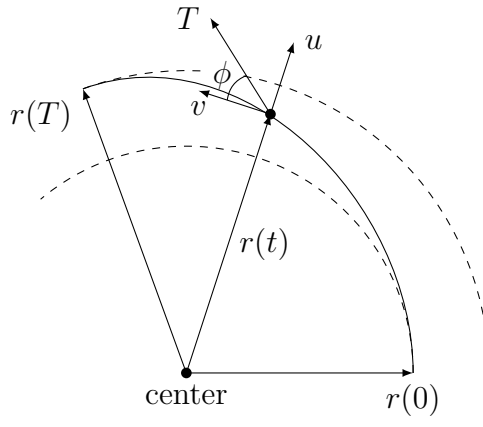
As a final observation if $\Psi = 0$ then

$$\lambda(T) = \nu^T \frac{\partial \phi}{\partial x}(x(T)),$$

which means $\lambda(T)$ is orthogonal to the tangent plane to $\phi(x(T))$.



Example Maximum orbit transform (e.g. Hidden Figures)



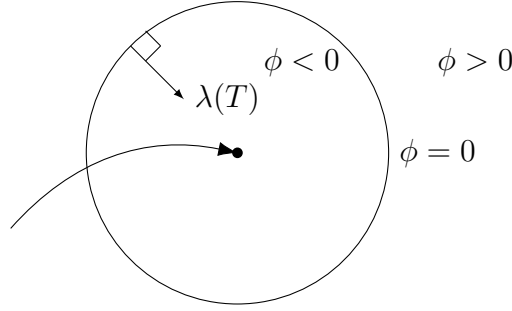
- r = radial distance from spacecraft to center
- u = radial velocity
- v = tangential velocity
- m = mass of spacecraft
- \dot{m} = $-$ fuel consumption rate
- ϕ = thrust angle (control input)
- T = thrust

$$\begin{aligned}
& \max_{\phi} r(T) \iff \min_{\phi} -r(T) \\
& \text{s.t.} \quad \begin{cases} \dot{r} = u \\ \dot{u} = \frac{v^2}{r} - \frac{g}{r^2} + \frac{T \sin \phi}{m_0 - |\dot{m}|t} \\ \dot{v} = -\frac{uv}{r} + \frac{T \cos \phi}{m_0 - |\dot{m}|t} \\ r(0) = r_0 \\ u(0) = 0 \\ v(0) = \sqrt{\frac{g}{r_0}} \\ u(T) = 0 = \phi_1 \\ v(T) = \sqrt{\frac{g}{r(T)}} = \phi_2 \end{cases} \\
\\
& H = \lambda_r u + \lambda_u \left(\frac{v^2}{r} - \frac{g}{r^2} + \frac{T \sin \phi}{m_0 - |\dot{m}|t} \right) + \lambda_v \left(-\frac{uv}{r} + \frac{T \cos \phi}{m_0 - |\dot{m}|t} \right) \\
& \Phi = \underbrace{\nu_1 u(T) + \nu_2 \left(v(T) - \sqrt{\frac{g}{r(T)}} \right)}_{\nu^T \phi} \underbrace{-r(T)}_{\Psi} \\
\\
& \frac{\partial H}{\partial \phi} = \frac{\lambda_u T \cos \phi - \lambda_v T \sin \phi}{m_0 - |\dot{m}|t} = 0 \\
& \Rightarrow \tan \phi = \frac{\lambda_u}{\lambda_v} \\
\\
& \dot{\lambda}_r = -\frac{\partial H}{\partial r} = -\lambda_u \left(-\frac{v^2}{r^2} + \frac{2g}{r^3} \right) - \lambda_v \cdot \frac{uv}{r^2} \\
& \dot{\lambda}_u = -\frac{\partial H}{\partial u} = -\lambda_r + \lambda_v \cdot \frac{v}{r} \\
& \dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_u \cdot \frac{2v}{r} + \lambda_v \cdot \frac{u}{r} \\
\\
& \begin{cases} \lambda_r(T) = \frac{\partial \Phi}{\partial r} = -1 + \frac{\nu_2 \sqrt{g}}{2(r(T))^{3/2}} \\ \lambda_u(T) = \frac{\partial \Phi}{\partial u} = \nu_1 \\ \lambda_v(T) = \frac{\partial \Phi}{\partial v} = \nu_2 \\ u(T) = 0 \\ v(T) = \sqrt{\frac{g}{r(T)}} \end{cases}
\end{aligned}$$

This needs numerics to solve.

3.4.1 Terminal manifold with inequality constraints

$$\begin{aligned} \min_u \int_0^T L \, dt + \Psi \\ \dot{x} &= f(x, u) \\ \phi(x(T)) &\leq 0 \end{aligned}$$



Repeat process: $\tilde{J} = \int (H - \lambda^T \dot{x}) \, dt + \Psi + \nu^T \phi$. The optimality conditions are

$$\begin{cases} \frac{\partial H}{\partial u} = 0 \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T)) + \nu^T \frac{\partial \phi^T}{\partial x}(x(T)) \\ \nu \geq 0 \\ \phi(x(T)) \leq 0 \\ \nu^T \phi(x(T)) = 0 \quad (\text{KKT}) \end{cases}$$

3.4.2 Initial manifold

$$\begin{aligned} \min_{x_0, u} \int L + \Psi(x(T)) + \Theta(x(0)) \\ \text{s.t. } \dot{x} &= f(x, u) \\ \phi(x(T)) &= 0 \\ \xi(x(0)) &= 0 \end{aligned}$$

$$\begin{aligned} \tilde{J} &= \int (H - \lambda^T \dot{x}) \, dt + \Psi(x(T)) + \Theta(x(0)) + \nu_\phi^T \phi(x(T)) + \nu_\xi^T \xi(x(0)) \\ \delta \tilde{J} &= \int \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta + \frac{\partial H}{\partial u} v \right] \, dt + \left[\frac{\partial \Psi}{\partial x}(x(T)) + \nu_\phi^T \frac{\partial \phi}{\partial x}(x(T)) - \lambda^T(T) \right] \eta(T) \\ &\quad + \left[\frac{\partial \Theta}{\partial x}(x(0)) + \nu_\xi^T \frac{\partial \xi}{\partial x}(x(0)) + \lambda^T(0) \right] \eta(0) \end{aligned}$$

The optimality conditions are

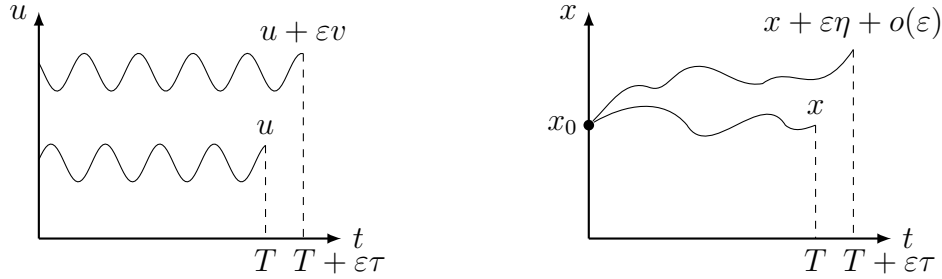
$$\begin{cases} \frac{\partial H}{\partial u} = 0 \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = -\frac{\partial \Psi^T}{\partial x}(x(T)) - \nu_\phi^T \frac{\partial \phi^T}{\partial x}(x(T)) \\ \lambda(0) = -\frac{\partial \Theta^T}{\partial x}(x(0)) - \nu_\xi^T \frac{\partial \xi^T}{\partial x}(x(0)) \end{cases}$$

3.4.3 Unspecified Terminal Times

For example, instead of driving to the moon using minimum fuel, we want to get there as soon as possible:

$$\min_{u, T} \int_0^T L(x, u, t) dt + \Psi(x(T), T).$$

The variations are $u \mapsto u + \varepsilon v$ and $T \mapsto T + \varepsilon \tau$



$$\begin{aligned} \tilde{J}(u, T) &= \int_0^T [L(x, u, t) + \lambda^T(f(x) - \dot{x})] dt + \Psi(x(T), T) \\ &= \int_0^T [H - \lambda^T \dot{x}] dt + \Psi \\ \tilde{J}(u + \varepsilon v, T + \varepsilon \tau) &= \int_0^T [H(x + \varepsilon \eta, u + \varepsilon v, t, \lambda) - \lambda^T(\dot{x} + \varepsilon \dot{\eta})] dt \\ &\quad + \int_T^{T + \varepsilon \tau} [H(x + \varepsilon \eta, u + \varepsilon v, t, \lambda) - \lambda^T(\dot{x} + \varepsilon \dot{\eta})] dt \\ &\quad + \Psi(x(T + \varepsilon \tau) + \varepsilon \eta(T + \varepsilon \tau), T + \varepsilon \tau) \end{aligned}$$

$$\begin{aligned}
\tilde{J}(u + \varepsilon v, T + \varepsilon \tau) - \tilde{J}(u, T) &= \varepsilon \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta \, dt + \varepsilon \int_0^T \frac{\partial H}{\partial u} v \, dt \\
&\quad - \varepsilon \lambda^T(T) \eta(T) + \varepsilon \lambda^T(0) \eta(0) + o(\varepsilon) \\
&\quad + \underbrace{\int_T^{T+\varepsilon \tau} [H(x + \varepsilon \eta, u + \varepsilon v, t, \lambda) - \lambda^T(\dot{x} + \varepsilon \dot{\eta})] \, dt}_{\text{(I)}} \\
&\quad + \underbrace{\Psi(x(T + \varepsilon \tau) + \varepsilon \eta(T + \varepsilon \tau), T + \varepsilon \tau) - \Psi(x(T), T)}_{\text{(II)}}
\end{aligned} \tag{3.4}$$

For term I, use the mean value theorem to get rid of terms inside the integral that have a ε before them:

$$\begin{aligned}
&\int_T^{T+\varepsilon \tau} [L + \lambda^T(f - \dot{x} - \varepsilon \dot{\eta})] \, dt \\
&= \int_T^{T+\varepsilon \tau} \left[L(x, u, t) + \varepsilon \frac{\partial L}{\partial x} \eta + \varepsilon \frac{\partial L}{\partial u} v + \lambda^T \left(f + \varepsilon \frac{\partial f}{\partial x} \eta + \varepsilon \frac{\partial f}{\partial u} v - \dot{x} - \varepsilon \dot{\eta} \right) \right] \, dt + o(\varepsilon) \\
&= \varepsilon \tau [L + \lambda^T(f - \dot{x})] \Big|_{t=\xi} + o(\varepsilon) = \varepsilon \tau L \Big|_{t=\xi} + o(\varepsilon) \\
&= \varepsilon \tau L(x(\xi), u(\xi), \xi) + o(\varepsilon), \quad \xi \in [T, T + \varepsilon \xi]
\end{aligned} \tag{3.5}$$

Note that as $\varepsilon \rightarrow 0$, $\xi \rightarrow T$.

For term II, we further split it into two parts:

$$\Psi(x + \varepsilon \eta, T + \varepsilon \tau) - \Psi(x, T) = \underbrace{\Psi(x, T + \varepsilon \tau)}_{\text{(II.a)}} + \underbrace{\varepsilon \frac{\partial \Psi}{\partial x}(x, T + \varepsilon \tau) \eta(T + \varepsilon \tau) - \Psi(x, T)}_{\text{(II.b)}}$$

$$\begin{aligned}
\text{(II.a)} \implies \Psi(x, T + \varepsilon \tau) &= \Psi(x(T), T + \varepsilon \tau) + \varepsilon \frac{\partial \Psi}{\partial x}(x(T), T + \varepsilon \tau) \dot{x}(T) \tau + o(\varepsilon) \\
&= \Psi(x(T), T) + \varepsilon \frac{\partial \Psi}{\partial x}(x(T), T) \dot{x}(T) \tau + \varepsilon \frac{\partial \Psi}{\partial T}(x(T), T) \tau + o(\varepsilon)
\end{aligned}$$

$$\begin{aligned}
\text{(II.b)} \implies \varepsilon \frac{\partial \Psi}{\partial x}(x, T + \varepsilon \tau) \eta(T + \varepsilon \tau) &= \varepsilon \left[\frac{\partial \Psi}{\partial x}(x(T), T) + \varepsilon \frac{\partial^2 \Psi}{\partial x^2} \dot{x} \tau + \varepsilon \frac{\partial^2 \Psi}{\partial T \partial x} \tau + o(\varepsilon) \right] \\
&\quad \times [\eta(T) + \varepsilon \dot{\eta}(T) \tau + o(\varepsilon)] \\
&= \varepsilon \frac{\partial \Psi}{\partial x}(x(T), T) \eta(T) + o(\varepsilon) \\
\text{(II)} \implies \Psi(x + \varepsilon \eta, T + \varepsilon \tau) - \Psi(x, T) &= \varepsilon \frac{\partial \Psi}{\partial x}(x(T), T) [\dot{x}(T) \tau + \eta(T)] + \varepsilon \frac{\partial \Psi}{\partial T}(x(T), T) \tau + o(\varepsilon) \tag{3.6}
\end{aligned}$$

Substituting (3.5) and (3.6) into (3.4) and taking the directional derivative,

$$\begin{aligned}\delta\tilde{J} = & \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta \, dt + \int_0^T \frac{\partial H}{\partial u} v \, dt + \lambda^T(0)\eta(0) \\ & + \left[L + \frac{\partial \Psi}{\partial T} + \frac{\partial \Psi}{\partial x} f \right] \tau \Big|_{t=T} + \left(\frac{\partial \Psi}{\partial x} - \lambda^T \right) \eta \Big|_{t=T}\end{aligned}$$

So we have a mix of old and new:

$$\begin{aligned}\text{old: } & \frac{\partial H}{\partial u} = 0 \\ & \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ & \lambda(T) = \frac{\partial \Psi}{\partial x} \Big|_T \\ \text{new: } & L + \frac{\partial \Psi}{\partial T} + \lambda^T f \Big|_T = 0\end{aligned}$$

This last condition is known as the *Transversality condition*.

Example Pure minimum time question

$$\begin{aligned}\min_{u,T} & \int_0^T dt \\ & \dot{x} = f(x, u) \\ & x(0) = x_0 \\ & x(T) = x_T \\ & H = L + \lambda^T f = 1 + \lambda^T f\end{aligned}$$

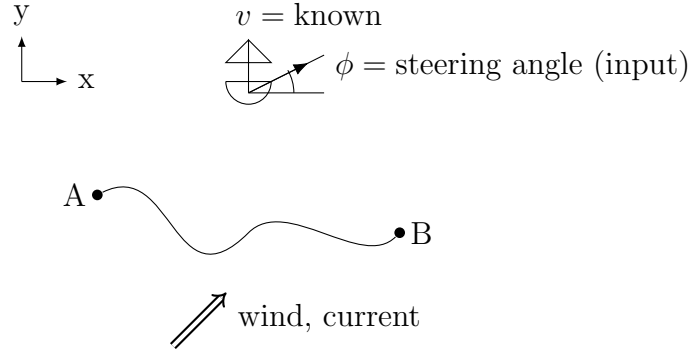
The transversality condition is

$$\begin{aligned}L + \frac{\partial \Psi}{\partial T} + \lambda^T f \Big|_T &= 0 \\ \lambda^T f \Big|_T &= -1 \\ H(T) = 1 + \lambda^T f \Big|_T &= 1 - 1 = 0\end{aligned}$$

But this is a conservative system, so H is a constant. Therefore,

$$H(t) = 0 \quad \forall t \in [0, T]$$

Example Zermelo's problem: sail from A to B as quickly as possible in the presence of known winds and currents.



The dynamics are

$$\begin{aligned}\dot{x} &= v \cos \phi + c_1(x, y) \\ \dot{y} &= v \sin \phi + c_2(x, y)\end{aligned} \quad \lambda = \begin{bmatrix} \lambda_x \\ \lambda_y \end{bmatrix}$$

For minimum time, $L = 1$.

$$\begin{aligned}H &= 1 + \lambda_x(v \cos \phi + c_1) + \lambda_y(v \sin \phi + c_2) \\ 0 &= \frac{\partial H}{\partial \phi} = -v \lambda_x \sin \phi + v \lambda_y \cos \phi \\ \phi &= \tan^{-1} \left(\frac{\lambda_y}{\lambda_x} \right)\end{aligned}$$

Since this is a conservative system and $\partial \Psi / \partial T = 0$, then $H(t) = H(T) = 0$.

$$\begin{aligned}-1 &= \lambda_x(v \cos \phi + c_1) + \lambda_y(v \sin \phi + c_2) \\ \lambda_x &= -\frac{\cos \phi}{v + c_1 \cos \phi + c_2 \sin \phi} \\ \lambda_y &= -\frac{\sin \phi}{v + c_1 \cos \phi + c_2 \sin \phi} \\ \dot{\lambda} &= -\frac{\partial H^T}{\partial x} \\ \dot{\lambda}_x &= -\lambda_x \frac{\partial c_1}{\partial x} - \lambda_y \frac{\partial c_2}{\partial x} \\ \dot{\lambda}_y &= -\lambda_x \frac{\partial c_1}{\partial y} - \lambda_y \frac{\partial c_2}{\partial y} \\ \dot{\phi} &= \sin^2 \phi \frac{\partial c_2}{\partial x} + \sin \phi \cos \phi \left(\frac{\partial c_1}{\partial x} - \frac{\partial c_2}{\partial y} \right) - \cos^2 \phi \frac{\partial c_1}{\partial y}\end{aligned}$$

This is an ODE that completely determines ϕ if we just had ϕ_0 .

Example We want to drive a car and stop at a stop sign as quickly as possible. Assume that the stop sign is at the origin, and our control is the acceleration ($\ddot{x} = u$).

$$\begin{aligned} \min_{u,T} \quad & \int_0^T dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2, & x(0) = x_0 \\ \dot{x}_2 = u, & x(T) = 0 \end{cases} \end{aligned}$$

Recall the transversality condition:

$$H + \frac{\partial \Psi}{\partial T} \Big|_{t=T} = 0.$$

For minimum-time problems, $L = 1$ and $\Psi = 0$, so $\lambda^T f|_{t=T} = -1$.

$$\begin{aligned} H &= 1 + \lambda_1 x_2 + \lambda_2 u \\ \lambda_1(T) \underbrace{x_2(T)}_{=0 \text{ (rest)}} + \lambda_2(T) u(T) &= -1 \\ \boxed{\lambda_2(T) u(T) = -1} \\ \frac{\partial H}{\partial u} &= \boxed{\lambda_2 = 0}, \end{aligned}$$

i.e. $0 \cdot u(T) = -1$? This problem is ill-posed; we need to go infinitely fast...

Idea 1: Constrain u . We don't know how to do this.

Idea 2: Pay for gas. This is a design choice.

For the second idea,

$$\begin{aligned} \min_{u,T} \quad & \int_0^T \frac{1}{2} u^2(t) dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2, & x(0) = x_0 \\ \dot{x}_2 = u, & x(T) = 0 \end{cases} \\ H &= \frac{1}{2} u^2 + \lambda_1 x_2 + \lambda_2 u \\ \frac{1}{2} u^2(T) + \lambda_1(T) x_2(T) + \lambda_2(T) u(T) &= 0 \\ \boxed{\frac{1}{2} u^2(T) + \lambda_2(T) u(T) = 0} \\ \frac{\partial H}{\partial u} = u + \lambda_2 = 0 &\implies u = -\lambda_2 \\ \frac{1}{2} \lambda_2^2(T) - \lambda_2^2(T) &= 0 \\ \boxed{\lambda_2(T) = 0} \end{aligned}$$

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = c \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -ct + d \\
\lambda_2(T) &= -cT + d = 0 \implies T = \frac{d}{c} \\
\dot{x}_2 &= u = -\lambda_2 = ct - d \\
x_2 &= c\frac{t^2}{2} - dt + x_{2,0} \\
\dot{x}_1 = x_2 &\implies x_1 = c\frac{t^3}{6} - d\frac{t^2}{2} + x_{2,0}t + x_{1,0} \\
\begin{cases} x_1(T) = c\frac{T^3}{6} - d\frac{T^2}{2} + x_{2,0}T + x_{1,0} = 0 \\ x_2(T) = c\frac{T^2}{2} - dT + x_{2,0} = 0 \\ T = \frac{d}{c} \end{cases} \\
\implies \begin{cases} c = -\frac{2x_{2,0}^2}{3x_{1,0}} \\ d = \sqrt{-\frac{4x_{2,0}^3}{3x_{1,0}}} \\ T = \frac{d}{c} \\ u = ct - d \end{cases}
\end{aligned}$$

Fine, but we really want to get there as quickly as possible! We have to constrain u , e.g. $u(t) \in [-1, 1]$, $\forall t \in [0, T]$. How do we deal with the constraints on u ?

3.5 Hamilton's Minor "Mistake"

$$\begin{aligned}
&\min_{u \in \mathcal{U}_{\text{constr.}}} \int_0^T L(x, u, t) dt + \Psi(x(T)) \\
&\text{s.t. } \dot{x} = f(x, u, t) \\
&\quad x(0) = x_0 \\
&\quad (u(t) \in U)
\end{aligned}$$

Augment the cost:

$$\tilde{J}(u) = \int_0^T \left(H(x, u, t, \lambda) - \lambda^\top \dot{x} \right) dt + \Psi(x(T))$$

Vary $u \mapsto u + \varepsilon v$ s.t. $u + \varepsilon v \in \mathcal{U}_{\text{constr.}} \Rightarrow x \mapsto x + \varepsilon \eta + o(\varepsilon)$:

$$\tilde{J}(u + \varepsilon v) = \int_0^T \left(H(x + \varepsilon \eta, u + \varepsilon v, t, \lambda) - \lambda^T \dot{x} - \lambda^T \varepsilon \dot{\eta} \right) dt + \Psi(x(T) + \varepsilon \eta(T)) + o(\varepsilon)$$

Instead of computing $\delta \tilde{J}(u; v)$, let's check $\Delta \tilde{J} = \tilde{J}(u + \varepsilon v) - \tilde{J}(u)$. If $\Delta \tilde{J} \geq 0 \forall v$ s.t. $u + \varepsilon v \in \mathcal{U}_{\text{constr.}}$ for ε small enough, then u is a local minimum!

$$\begin{aligned} \Delta \tilde{J} &= \int_0^T \left[H(x + \varepsilon \eta, u + \varepsilon v, t, \lambda) - H(x, u, t, \lambda) - \lambda^T (\dot{x} + \varepsilon \dot{\eta} - \dot{x}) \right] dt \\ &\quad + \Psi(x(T) + \varepsilon \eta(T)) - \Psi(x(T)) + o(\varepsilon) \end{aligned}$$

Only Taylor expanding w.r.t. x :

$$\begin{aligned} \Delta \tilde{J} &= \int_0^T \left[\varepsilon \frac{\partial H}{\partial x}(x, u, t, \lambda) \eta - \varepsilon \lambda^T \dot{\eta} \right] dt + \int_0^T [H(x, u + \varepsilon v, t, \lambda) - H(x, u, t, \lambda)] dt \\ &\quad + \varepsilon \frac{\partial \Psi}{\partial x}(x(T)) \eta(T) + o(\varepsilon) \\ &= \varepsilon \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \varepsilon \lambda^T(0) \eta(0) - \varepsilon \lambda^T(T) \eta(T) + \varepsilon \frac{\partial \Psi}{\partial x}(x(T)) \eta(T) \\ &\quad + \int_0^T [H(x, u + \varepsilon v, t, \lambda) - H(x, u, t, \lambda)] dt + o(\varepsilon) \end{aligned}$$

With $\dot{\lambda} = -\partial H^T / \partial x$ and $\lambda(T) = \partial \Psi(x(T)) / \partial x$,

$$\Delta \tilde{J} = \int_0^T [H(x, u + \varepsilon v, t, \lambda) - H(x, u, t, \lambda)] dt + o(\varepsilon)$$

Here, Hamilton did Taylor's expansion and set $\partial H / \partial u = 0$. Instead, Pontryagin desired $\Delta \tilde{J} \geq 0 \forall v | u + \varepsilon v \in \mathcal{U}_{\text{constr.}}, \varepsilon$ small enough, i.e. we need

$$H(x, u^* + \varepsilon v, t, \lambda) \geq H(x, u^*, t, \lambda)$$

$\forall t \in [0, t], \forall v | u + \varepsilon v \in \mathcal{U}_{\text{constr.}}, \varepsilon$ small enough. That is, we need

$$u^* = \arg \min_u H(x, u, t, \lambda)$$

In summary,

$$\text{Hamilton: } \frac{\partial H}{\partial u} = 0$$

$$\text{Pontryagin: } \min_u H$$

Theorem (Pontryagin's Maximum Principle (PMP)). *Consider the problem:*

$$\begin{aligned} \min_{u, T} \quad & \int_0^T L(x, u, t) dt + \Psi(x(T), T) \\ \text{s.t.} \quad & \dot{x} = f(x, u, t) \\ & u(t) \in U(x, t), \quad \forall t \in [0, T] \\ & x_i(0) = x_{i0}, \quad i \in \mathcal{I} \\ & x_j(T) = x_{jT}, \quad j \in \mathcal{T} \end{aligned}$$

The necessary condition for optimality is

$$\begin{aligned}
H &= L + \lambda^T f \\
\dot{\lambda} &= -\frac{\partial H^T}{\partial x} \\
\lambda_j(0) &= 0, \quad j \notin \mathcal{I} \\
\lambda_i(T) &= \frac{\partial \Psi}{\partial x_i}(x(T)), \quad i \notin \mathcal{T} \\
H + \frac{\partial \Psi}{\partial T} \Big|_{t=T} &= 0 \\
u^*(x, t, \lambda) &= \arg \min_{u \in U(x, t)} H(x, u, t, \lambda)
\end{aligned}$$

We have two paths to solve optimality problems: we always start with the Hamiltonian, find the costate dynamics and boundary conditions, and apply the transversality condition; then, we can either apply calculus of variations (COV) or Pontryagin's Maximum Principle (PMP). COV only works for unconstrained problems, while with PMP we can deal with constraints.

3.6 Bang-Bang Control

Return to the car problem:

$$\begin{aligned}
&\min_{u, T} \int_0^T dt \\
&\text{s.t.} \quad \begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{1,0}, & x_1(T) = 0 \\ \dot{x}_2 = u, & x_2(0) = x_{2,0}, & x_2(T) = 0 \\ u(t) \in [-1, 1] & \forall t \in [0, T] \end{cases}
\end{aligned}$$

How do we minimize H w.r.t. u ?

$$H = 1 + \lambda_1 x_2 + \lambda_2 u$$

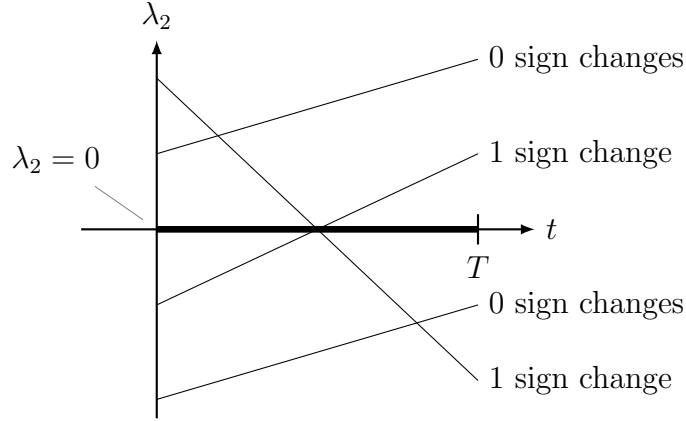
Clearly, we minimize H by letting

$$u = \begin{cases} -1, & \lambda_2 > 0 \\ +1, & \lambda_2 < 0 = -\text{sign}(\lambda_2) \\ ??, & \lambda_2 = 0 \end{cases}$$

Therefore, the optimal u switches between -1 and $+1$ (bang-bang control).

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = c \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -ct + d
\end{aligned}$$

Notice that $\lambda_2(t)$ is a line, so it has at most one sign change. Thus, u also changes sign (from ± 1 to ∓ 1) at most one time.



Let's solve this for all x_0 !

i) Assume $\lambda_2 > 0 \forall t \in [0, T]$, $\therefore u = -1 \quad \forall t \in [0, T]$

$$\begin{aligned} \dot{x}_2 &= -1 \implies x_2 = -t + k_1 \\ x_2(T) &= 0 = -T + k_1 \implies k_1 = T \\ x_2(t) &= T - t \implies x_2 > 0, \quad t \in [0, T) \\ \dot{x}_1 &= x_2 = T - t \implies x_1 = -\frac{t^2}{2} + Tt + k_2 \\ x_1(T) &= 0 = -\frac{T^2}{2} + T^2 + k_2 \implies k_2 = -\frac{T^2}{2} \\ x_1(t) &= -\frac{t^2}{2} + Tt - \frac{T^2}{2} = -\frac{(T-t)^2}{2} \quad (< 0, \quad t \in [0, T)) \\ &= -\frac{x_2^2(t)}{2} \end{aligned}$$

Let's consider the curve

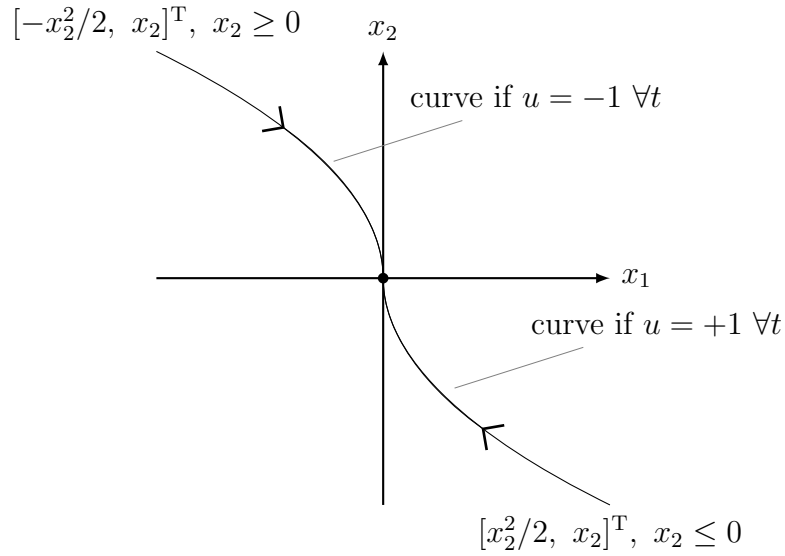
$$\begin{bmatrix} -x_2^2/2 \\ x_2 \end{bmatrix}$$

for $x_2 \geq 0$. If x_0 lies on this curve, use $u = -1$ and drive to the origin.

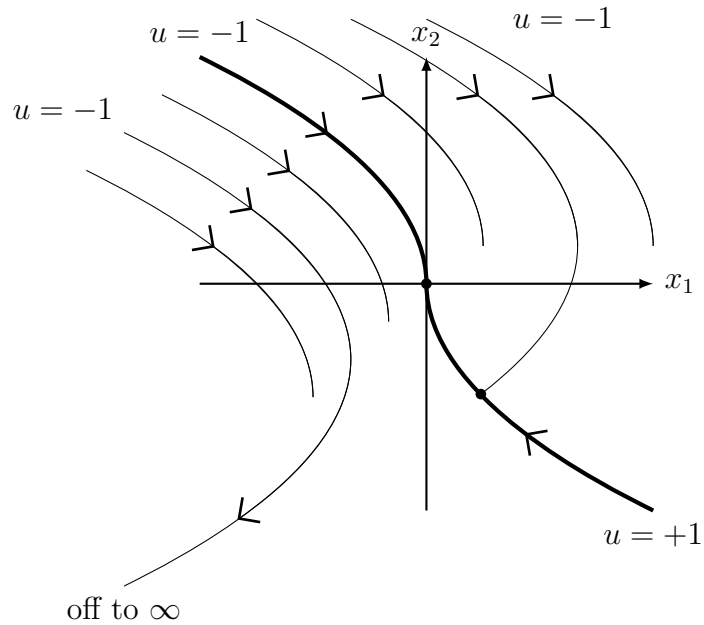
ii) Assume $u = +1 \quad \forall t \in [0, T]$

$$\begin{aligned} x_2 &= t - T \quad (\leq 0 \text{ on } [0, T]) \\ x_1 &= \frac{x_2^2}{2} \quad (\geq 0) \end{aligned}$$

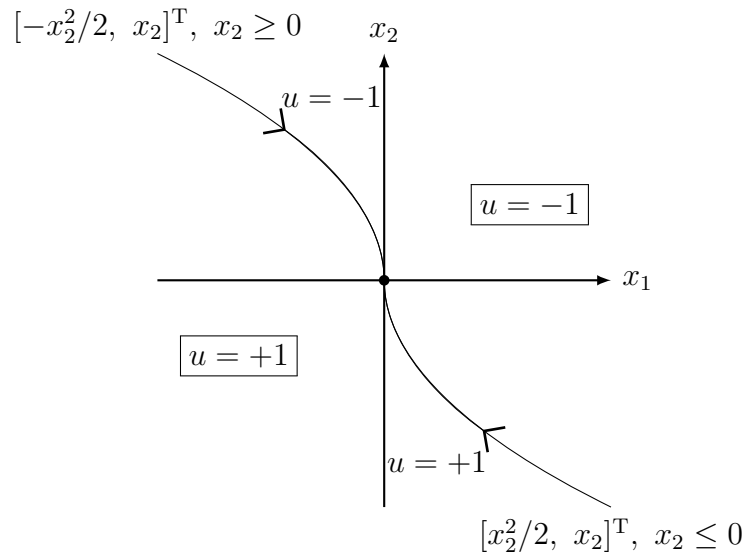
For this curve, use $u = +1$.



What happens when we do not start on the curves? We start with a certain u depending on x_0 and perform a single switch of u when we encounter one of the initial curves that travel to the origin. Note that for the case $\lambda_2 = 0 \forall t$, we start at the stop sign at rest, so the control does not matter.



The optimal solution is given by the following *switching curve*.



Note 1: Bang-bang control typically involves

- a) finding the number of switches
- b) find the switching surfaces

Note 2: This is a feedback law! (u depends on x !!)

3.6.1 Linear Systems (scalar input)

$$\begin{aligned}
 & \min_{u,T} \int_0^T dt \\
 & \text{s.t. } \dot{x} = Ax + Bu \\
 & \quad x(0) = x_0, \quad x(T) = 0 \\
 & \quad u \in [-1, 1] \\
 & \quad H = 1 + \lambda^T (Ax + Bu) \\
 & \quad u = -\text{sign}(\lambda^T B) \quad (\text{bang-bang})
 \end{aligned}$$

Aside...

$$\begin{aligned}
 \dot{x} &= f(x) + g(x)u \quad (\text{control affine}) \\
 H &= 1 + \lambda^T f + \lambda^T g u \\
 u &= -\text{sign}(\lambda^T g(x)) \quad (\text{bang-bang})
 \end{aligned}$$

Back to linear...

$$\begin{aligned}
 \dot{\lambda} &= -\frac{\partial H^T}{\partial x} = -A^T \lambda \\
 \lambda(t) &= e^{-A^T t} \lambda_0 \\
 u(t) &= -\text{sign}(\lambda_0^T e^{-A^T t} B)
 \end{aligned}$$

How do we find λ_0 ?

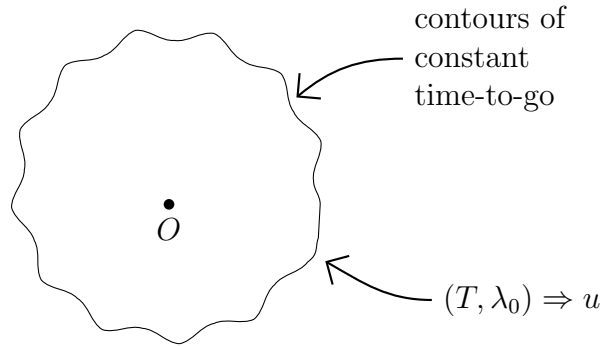
$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ x(T) &= e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau) d\tau \\ x(T) &= 0 = e^{AT}x_0 - \int_0^T e^{A(T-t)}B \operatorname{sign}(\lambda_0^T e^{-At}B) dt\end{aligned}\tag{3.7}$$

Problem 1: Given x_0 , figure out λ_0 from (3.7). Then, $u = -\operatorname{sign}(\lambda_0^T e^{-At}B)$. This has to be done numerically in general (not super simple...).

Problem 2: Find all x_0 s from which it takes the same amount of time to get to $x(T) = 0$.

$$\begin{aligned}e^{At}x_0 &= \int_0^T e^{A(T-t)}B \operatorname{sign}(\lambda_0^T e^{-At}B) dt \\ x_0 &= \int_0^T e^{-At}B \operatorname{sign}(\lambda_0^T e^{-At}B) dt\end{aligned}$$

Fix T . By varying λ_0 , we will get the x_0 s that take time T to go to $x(T) = 0$ optimally.



So by solving problem 2, we find λ_0 associated with all x_0 , i.e. we have “solved” problem 1 as well.

3.7 Integral Constraints (Isoperimetric)

Recall PMP is

$$\min_{u \in U(x,t)} H(x, u, \lambda, t)$$

We have seen $U = [-1, 1]$ in the context of bang-bang control. Now, we consider integral constraints of the form

$$C = \int_0^T N(x, u, t) dt \quad (\in \mathbb{R}^p)$$

Let $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$. Introduce p new states $\hat{x} = [x_{n+1}, \dots, x_{n+p}]^T$, where

$$\hat{x}(t) = \int_0^t N(x(\tau), u(\tau), \tau) d\tau$$

and $\dot{\hat{x}}(t) = N(x, u, t)$. Its boundary conditions are $\hat{x}(0) = 0$ and $\hat{x}(T) = C$. The Hamiltonian is

$$\begin{aligned} H(x, \hat{x}, u, t, \lambda) &= L(x, u, t) + \lambda^T f(x, u, t) + \hat{\lambda}^T N(x, u, t) \\ \dot{\lambda} &= -\frac{\partial H^T}{\partial x} = -\frac{\partial L^T}{\partial x} - \frac{\partial f^T}{\partial x} \lambda - \frac{\partial N^T}{\partial x} \hat{\lambda} \\ \dot{\hat{\lambda}} &= -\frac{\partial H^T}{\partial \hat{x}} = 0 \implies \hat{\lambda} \text{ is constant} \end{aligned}$$

Moreover, this is now an unconstrained problem, i.e.

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} + \hat{\lambda}^T \frac{\partial N}{\partial u} = 0$$

Going back to the car problem of stopping at the origin, suppose we want to use up exactly the “energy”

$$E = \int_0^T u^2(t) dt.$$

If possible, it is better to transform an inequality constraint to an equality constraint.

$$\begin{aligned} \min_{u, T} \quad & \int_0^T dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{10}, & x_1(T) = 0 \\ \dot{x}_2 = u, & x_2(0) = x_{20}, & x_2(T) = 0 \\ \dot{x}_3 = u^2, & x_3(0) = 0, & x_3(T) = E \end{cases} \end{aligned}$$

As we have seen, without the energy constraint this is an ill-posed problem.

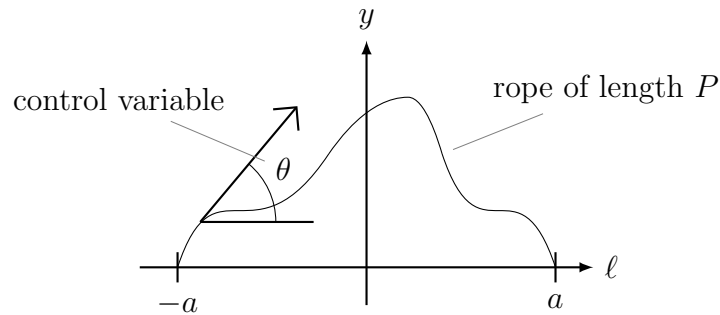
$$\begin{aligned} H &= 1 + \lambda_1 x_2 + \lambda_2 u + \lambda_3 u^2 \\ \lambda_3 &= \text{constant} \\ \dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = c \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -ct + d \\ \frac{\partial H}{\partial u} &= \lambda_2 + 2\lambda_3 u = 0 \\ \implies u &= -\frac{\lambda_2}{2\lambda_3} = \frac{c}{2\lambda_3} t - \frac{d}{2\lambda_3} \quad (\text{linear in time}) \\ \dot{x}_2 = u &\implies x_2 = \frac{c}{4\lambda_3} t^2 - \frac{d}{2\lambda_3} t + x_{20} \end{aligned}$$

$$\begin{aligned}
\dot{x}_1 = x_2 &\implies x_1 = \frac{c}{12\lambda_3}t^3 - \frac{d}{4\lambda_3}t^2 + x_{20}t + x_{10} \\
\dot{x}_3 = u^2 &\implies x_3 = \frac{c^2}{12\lambda_3^2}t^3 + \frac{d^2}{4\lambda_3^2}t - \frac{cd}{4\lambda_3^2}t^2 \\
H + \frac{\partial\Psi}{\partial T}\Big|_T &= 0 \\
1 + \lambda_1 x_2 + \lambda_2 u + \lambda_3 u^2 + 0\Big|_T &= 0 \\
1 + (d - cT) \left(\frac{c}{2\lambda_3}T - \frac{d}{2\lambda_3} \right) + \lambda_3 \left(\frac{c}{2\lambda_3}T - \frac{d}{2\lambda_3} \right) &= 0
\end{aligned}$$

The boundary conditions ($x_1(T) = 0$, $x_2(T) = 0$, $x_3(T) = E$) and the transversality condition give four equations for four unknowns.

$$\begin{cases} T = \left(\frac{3}{E}\right)^{1/3} \\ c = -\frac{2}{3}T \\ d = -\frac{T^2}{3} \\ \lambda_3 = \frac{T^4}{18} \end{cases} \implies u = \dots$$

Dido's Problem Given a strip of oxbide, enclose the most area along the Mediterranean Sea. This region has a fixed width and is bounded to the south by the ℓ axis (the sea). Historically, this became the city Carthage.



The area of this region is

$$\int_{-a}^a y \, d\ell.$$

The dynamics are

$$\frac{dy}{d\ell} = \tan \theta.$$

The constraint is

$$P = \int_{-a}^a \frac{1}{\cos \theta} d\ell.$$

The problem becomes

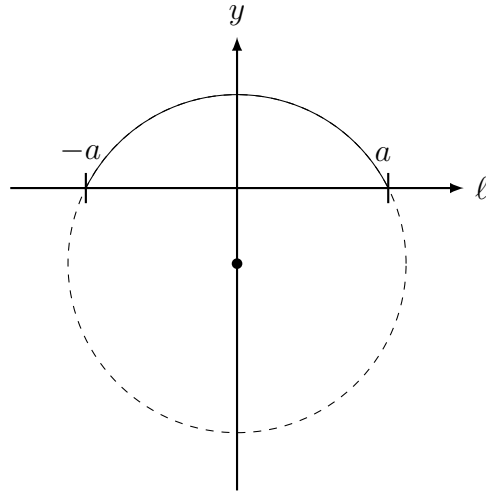
$$\begin{aligned} \min_{\theta} & - \int_{-a}^a y(\ell) d\ell \\ \text{s.t.} & \frac{dy}{d\ell} = \tan \theta, \quad y(-a) = 0, \quad y(a) = 0 \\ & \frac{d\hat{y}}{d\ell} = \frac{1}{\cos \theta}, \quad \hat{y}(-a) = 0, \quad \hat{y}(a) = P \end{aligned}$$

$$\begin{aligned} H &= -y + \lambda \tan \theta + \hat{\theta} \frac{1}{\cos \theta} \\ \hat{\lambda} &= \text{constant} \\ \frac{d\lambda}{d\ell} &= -\frac{\partial H}{\partial y} = 1 \implies \lambda(\ell) = \ell + c \\ \frac{\partial H}{\partial \theta} &= 0 = \lambda(1 + \tan^2 \theta) + \hat{\lambda} \frac{\tan \theta}{\cos \theta} \\ \sin \theta(\ell) &= -\frac{\ell + c}{\hat{\lambda}} \end{aligned}$$

Let $\sin \alpha / \alpha = 2a/P$. The optimal shape is a circular arc centered at $\ell = 0$ and

$$y = -\frac{P \cos \alpha}{2\alpha},$$

with radius $P/2\alpha$. (This produces the semi-circular city of Carthage!)



Note that this formulation cannot handle $P > \pi a$. In reality, a is also undefined and chosen so that the solution is exactly a semicircle with $P = \pi a$.

The punchline is integral constraints are no big deal. What about other constraints?

3.8 Control Constraints

Suppose the control constraint is $u(t) \in U(t)$, e.g. $h(u, t) = 0$ or $h(u, t) \leq 0$.

$$\begin{aligned} \min_u H(x, u, \lambda, t) \\ \text{s.t. } h(u, t) = 0 \end{aligned}$$

Introduce a Lagrange multiplier:

$$\begin{aligned} \tilde{H} &= H + \mu^T h \\ \left. \begin{aligned} \frac{\partial \tilde{H}}{\partial u} &= 0 \\ h &= 0 \end{aligned} \right\} &\implies u^*(x, t, \lambda) \end{aligned}$$

We still have

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H^T}{\partial x}(x, t, \lambda, u^*(x, t, \lambda)) \\ \dot{x} &= f(x, u, t) = f(x, u^*(x, t, \lambda), t) \\ &\quad + \text{Boundary cond. on } x \text{ and } \lambda \end{aligned}$$

The only change from the unconstrained control version is the method by which $u^*(x, t, \lambda)$ is found.