

# ECE 6553: Optimal Control Notes

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# Chapter 1

## Parameter Optimization

### 1.1 What is Optimal Control?

**Optimal** Maximize/minimize cost (subject to constraints):  $\min_u g(u)$

With constraints,

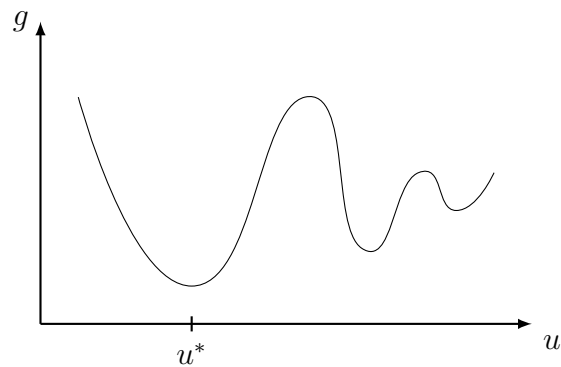
$$\begin{aligned} \min_u \quad & g(u) \\ \text{s.t.} \quad & \begin{cases} h_1(u) = 0 \\ h_2(u) \leq 0 \end{cases} \end{aligned}$$

First-order necessary condition (FONC):

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Optimality can be

- local vs global
- max vs min



**Control** control design: pick  $u$  such that specifications are satisfied:

$$\dot{x} = f(x, u), \quad \dot{x} = Ax + Bu,$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}^m$  is the control, and  $f(\cdot)$  is the dynamics.

Actually,  $x$  and  $u$  are signals:

$$x : [0, T] \rightarrow \mathbb{R}^n, \quad u : [0, T] \rightarrow \mathbb{R}^m$$

**Optimal control** find the “best”  $u$ !

For “best” to mean anything, we need a cost. The big/deep question is

$$\frac{\partial \text{“cost”}}{\partial u} = 0$$

**Example**

Suppose we have a car with position  $p$ . Its acceleration  $\ddot{p}$  is controlled by the gas/brake input  $u$  ( $\ddot{p} = u$ ). In order to express the dynamics of the system in the form  $\dot{x} = f(x, u)$ , we introduce state variables:

$$\begin{aligned} x_1 = p \\ x_2 = \dot{p} \end{aligned} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

The task is to move the car from its initial position to a stop at a distance  $c$  away.

**Minimum energy problem**

$$\begin{aligned} \min_u \quad & \int_0^T u^2(t) dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0, x_2(0) = 0 \\ & x_1(T) = c, x_2(T) = 0 \end{aligned}$$

**Minimum time problem**

$$\begin{aligned} \min_{u, T} \quad & T = \int_0^T dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0, x_2(0) = 0 \\ & x_1(T) = c, x_2(T) = 0 \\ & u(t) \in [u_{\min}, u_{\max}] \end{aligned}$$

The general optimal control problem we will solve will look like

$$\begin{aligned} \min_{u, T} \quad & \int_0^T L(x(t), u(t), t) dt + \Psi(x(T)) \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T] \\ & x(0) = x_0 \\ & x(T) \in S \\ & u(t) \in \Omega, \quad t \in [0, T] \end{aligned}$$

where  $\Psi(\cdot)$  is the terminal cost and  $S$  is the terminal manifold. This is a so-called **Bolza Problem**.

**What tools do we need to solve this?**

1. optimality conditions  $\partial \text{cost} / \partial u = 0$
2. some way of representing the optimal signal  $u^*(x, t)$
3. some way of actually finding/computing the optimal controllers

## 1.2 Unconstrained Optimization

Let the decision variable be  $u = [u_1, \dots, u_m]^T \in \mathbb{R}^m$ . The cost is  $g(u) \in C^1$  ( $C^k$  means  $k$  times continuously differentiable). The problem is

$$\min_u g(u), \quad g : \mathbb{R}^m \rightarrow \mathbb{R}$$

For  $u^*$  to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

**Definition.**  $u^*$  is a (local) minimizer to  $g$  if  $\exists \delta > 0$  s.t.

$$\begin{aligned} g(u^*) &\leq g(u) \quad \forall u \in B_\delta(u^*) \\ B_\delta(u^*) &= \{u \mid \|u - u^*\| \leq \delta\} \end{aligned}$$

**Note:**

- $\frac{\partial g}{\partial u}(u^*) \delta u \in \mathbb{R}$  and  $\delta u$  is  $m \times 1$ , so  $\frac{\partial g}{\partial u}$  is a  $1 \times m$  row vector. For the column vector,

$$\nabla g = \frac{\partial g^T}{\partial u} \in \mathbb{R}^m$$

- $\frac{\partial g}{\partial u} \delta u$  is an inner product

$$\langle \nabla g, \delta u \rangle = \left\langle \frac{\partial g^T}{\partial u}, \delta u \right\rangle$$

- $o(\varepsilon)$  encodes higher-order terms

$$\lim_{\varepsilon \rightarrow 0} \frac{o(\varepsilon)}{\varepsilon} = 0 \quad \text{“faster than linear”}$$

This is opposed to big-O notation:

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{O}(\varepsilon)}{\varepsilon} = c$$

- $\delta u$  has direction and scale so we could write it as

$$\delta u = \varepsilon v, \quad \varepsilon \in \mathbb{R}, \quad v \in \mathbb{R}^m$$

**Theorem.** For  $u^*$  to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or, equivalently,

$$\frac{\partial g}{\partial u}(u^*)v = 0 \quad \forall v \in \mathbb{R}^m$$

*Proof.* Let  $u^*$  be a minimizer. Evaluating the cost  $g(u)$  in the ball and using Taylor’s expansion,

$$g(u^* + \delta u) = g(u^*) + \frac{\partial g}{\partial u}(u^*)\delta u + o(\|\delta u\|) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$

Assume that  $\frac{\partial g}{\partial u} \neq 0$ . Then we could pick  $v = -\frac{\partial g^T}{\partial u}(u^*)$ , i.e.

$$g(u^* + \varepsilon v) = g(u^*) - \varepsilon \left\| \frac{\partial g^T}{\partial u}(u^*) \right\|^2 + o(\varepsilon)$$

Note that the second term is negative per our assumptions. So, for  $\varepsilon$  sufficiently small, we have

$$g\left(u^* - \varepsilon \frac{\partial g^T}{\partial u}(u^*)\right) < g(u^*)$$

This contradicts  $u^*$  being a minimizer.  $\times$  (crossed swords) □

**Definition** (Positive definite).  $M = M^T \succ 0$  if

$$\begin{aligned} z^T M z &> 0 \quad \forall z \neq 0, \quad z \in \mathbb{R}^m \\ \iff M &\text{ has real and positive eigenvalues} \end{aligned}$$

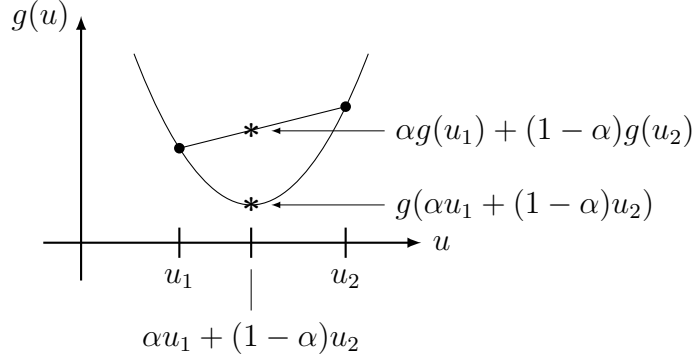
**Theorem.** If  $g \in C^2$ , then a **sufficient** condition for  $u^*$  to be a (local) minimizer is

$$1. \quad \frac{\partial g}{\partial u}(u^*) = 0$$

2.  $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$  (the Hessian is positive definite)

**Definition.**  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  is convex if

$$g(\alpha u_1 + (1 - \alpha)u_2) \leq \alpha g(u_1) + (1 - \alpha)g(u_2) \quad \forall \alpha \in [0, 1], \quad u_1, u_2 \in \mathbb{R}^m$$



**Theorem.** If  $\frac{\partial^2 g}{\partial u^2}(u) \succeq 0 \quad \forall u \in \mathbb{R}^m$ , then  $g$  is convex. ( $\iff$  for  $g \in C^2$ )

**Example**  $\min_u u^T Q u - b^T u$  where  $Q = Q^T \succ 0$  (positive definite matrix)

$$\begin{aligned} \frac{\partial g}{\partial u} &= \frac{\partial}{\partial u} (u^T Q u - b^T u) \\ &= u^T Q^T + u^T Q - b^T \\ &= 2u^T Q - b^T \\ \frac{\partial^2 g}{\partial u^2} &= 2Q \end{aligned} \quad \frac{\partial^2 g}{\partial u^2} = \begin{bmatrix} \frac{\partial^2 g}{\partial u_1^2} & \cdots & \frac{\partial^2 g}{\partial u_1 \partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 g}{\partial u_m^2} \end{bmatrix}$$

From  $\frac{\partial g}{\partial u} = 2u^T Q - b^T = 0$ ,

$$u = \frac{1}{2} Q^{-1} b$$

To see whether this is a minimizer, consider the Hessian. Since  $Q \succ 0$ , it follows that  $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$  and  $u^* = \frac{1}{2} Q^{-1} b$  is a (local) minimizer. Additionally, since  $\frac{\partial^2 g}{\partial u^2} \succ 0$ ,  $g$  is convex and  $u^*$  is a global minimizer. In fact, since we have strict convexity ( $\succ 0$  rather than  $\succeq 0$ ), it is the unique global minimizer.

In optimal control, *local* is typically all we can ask for. In optimization, we can do better!

But wait, just because we know  $\frac{\partial g}{\partial u} = 0$ , it doesn't follow that we can actually find  $u^* \dots$

## 1.3 Numerical Methods

Idea:  $u_{k+1} = u_k + \text{step}_k$ . What should  $\text{step}_k$  be? For small  $\text{step}_k = \gamma_k v_k$ ,

$$g(u_k + \text{step}_k) = g(u_k) + \frac{\partial g}{\partial u}(u_k) \cdot \text{step}_k + o(\|\text{step}_k\|) = g(u_k) + \gamma_k \frac{\partial g}{\partial u}(u_k) v_k + o(\gamma_k)$$

A perfectly reasonable choice of step direction is

$$v_k = -\frac{\partial g^T}{\partial u}(u_k),$$

known as the *steepest descend* direction. This produces

$$g\left(u_k - \gamma_k \frac{\partial g}{\partial u}(u_k)\right) = g(u_k) - \gamma_k \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 + o(\gamma_k)$$

### Steepest descent

$$u_{k+1} = u_k - \gamma_k \frac{\partial g^T}{\partial u}(u_k)$$

**Note:**

- What should  $\gamma_k$  be?
- This method “pretends” that  $g(u)$  is linear. If we pretend  $g(u)$  is quadratic, we get

$$u_{k+1} = u_k - \left( \frac{\partial^2 g}{\partial u^2}(u_k) \right)^{-1} \frac{\partial g^T}{\partial u}(u_k),$$

i.e. Newton’s Method

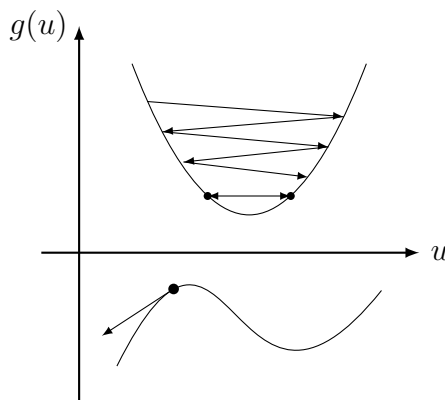
**This course:** steepest descent

### Step-size selection?

- Choice 1:  $\gamma_k = \gamma$  “small”  $\forall k$ ; will get close to a minimizer if  $u_0$  is close enough and  $\gamma$  small enough

Problems:

- You may not converge! (but you’ll get close)
- You may go off to infinity (diverge)





- Choice 2: Reduce  $\gamma_k$  as a function of  $k$ ; will get close to a minimizer if  $u_0$  is close enough

Problem: slow

**Theorem.** If  $u_0$  is close enough to  $u^*$  and  $\gamma_k$  satisfies

$$\begin{aligned} - \sum_{k=0}^{\infty} \gamma_k &= \infty \\ - \sum_{k=0}^{\infty} \gamma_k^2 &< \infty \end{aligned}$$

e.g.  $\gamma_k = c/k$ , then  $u_k \rightarrow u^*$  as  $k \rightarrow \infty$ .

- Choice 3: **Armijo step-size:** Take as big a step as possible, but no larger  
Pick  $\alpha \in (0, 1)$ ,  $\beta \in (0, 1)$ . Let  $i$  be the smallest non-negative integer such that

$$\begin{aligned} g\left(u_k - \beta^i \frac{\partial g^T}{\partial u}(u_k)\right) - g(u_k) &< -\alpha \beta^i \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 \\ u_{k+1} &= u_k - \beta^i \frac{\partial g^T}{\partial u}(u_k) \end{aligned}$$

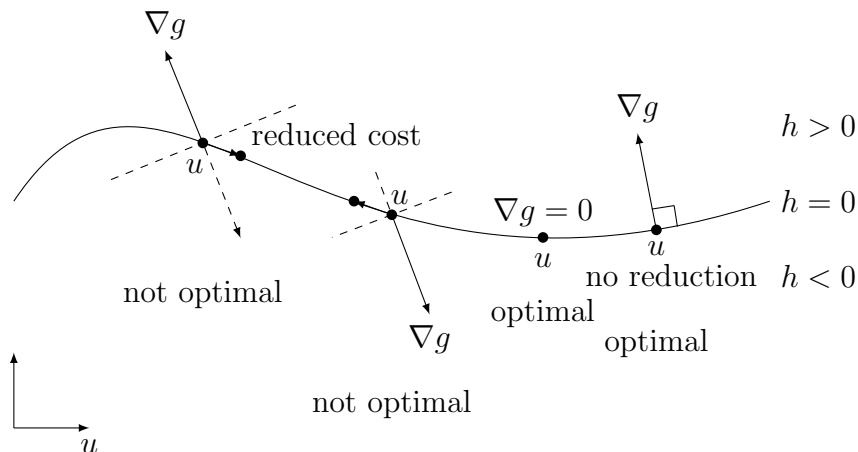
This will get to a minimizer blazingly fast if  $u_0$  is close enough.

## 1.4 Constrained Optimization

Equality constraints:

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = 0 \end{aligned}$$

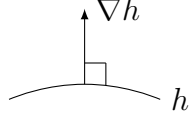
Consider  $u \in \mathbb{R}^2$ ,  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$



So  $u$  is (locally) optimal if  $\nabla g \parallel$  (is parallel to) the normal vector to tangent plane to  $h$ .

Fact: (HW# 1)

$$\nabla h \perp Th \quad (\text{tangent plane to } h)$$



We need  $\nabla g \parallel \nabla h$  at  $u^*$  for optimality, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \alpha \frac{\partial h}{\partial u}(u^*), \quad \text{for some } \alpha \in \mathbb{R}$$

or ( $\lambda = -\alpha$ ),

$$\frac{\partial g}{\partial u}(u^*) + \lambda \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}(g(u^*) + \lambda h(u^*)) = 0, \quad \text{for some } \lambda \in \mathbb{R}$$

More generally,

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

Note that  $h(u) = [h_1(u), \dots, h_k(u)]^T$ .

We need  $\frac{\partial g}{\partial u}(u^*)$  to be a linear combination of  $\frac{\partial h_i}{\partial u}(u^*)$ ,  $i = 1, \dots, k$ , for exactly the same reasons, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \sum_{i=1}^k \alpha_i \frac{\partial h_i}{\partial u}(u^*)$$

or ( $\lambda = -[\alpha_1, \dots, \alpha_k]^T$ )

$$\frac{\partial g}{\partial u}(u^*) + \lambda^T \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}(g(u^*) + \lambda^T h(u^*)) = 0, \quad \text{for some } \lambda \in \mathbb{R}^k$$

**Theorem.** If  $u^*$  is a minimizer to

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

then  $\exists \lambda \in \mathbb{R}^k$  s.t.

$$\begin{cases} \frac{\partial L}{\partial u}(u^*, \lambda) = 0 \\ \frac{\partial L}{\partial \lambda}(u^*, \lambda) = 0 \end{cases}$$

where the Lagrangian  $L$  is given by

$$L(u, \lambda) = g(u) + \lambda^T h(u)$$

**Note:**

- $\lambda$  are the Lagrange multipliers
- $\frac{\partial L}{\partial \lambda} = 0$  is fancy speak for  $h(u^*) = 0$

**Example**

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & \frac{1}{2} \|u\|^2 \\ \text{s.t.} \quad & Au = b \end{aligned}$$

where  $A$  is  $k \times m$ ,  $k \leq m$ . Assume  $(AA^T)^{-1}$  exists (constraints are linearly independent, none of the constraints are “duplicates”, all the constraints are essential).

$$\begin{aligned} L &= \frac{1}{2} u^T u + \lambda^T (Au - b) \\ \frac{\partial L}{\partial u} &= u^T + \lambda^T A = 0 \\ u^* &= -A^T \lambda \end{aligned}$$

Using the equality constraint,

$$\begin{aligned} Au^* &= b \\ -AA^T \lambda &= b \\ \lambda &= -(AA^T)^{-1} b \\ u^* &= A^T (AA^T)^{-1} b \end{aligned}$$

**Example**

$$\begin{aligned} \min \quad & u_1 u_2 + u_2 u_3 + u_1 u_3 \\ \text{s.t.} \quad & u_1 + u_2 + u_3 = 3 \end{aligned}$$

$$L = u_1 u_2 + u_2 u_3 + u_1 u_3 + \lambda(u_1 + u_2 + u_3 - 3)$$

$$\begin{cases} \frac{\partial L}{\partial u_1} = u_2 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_2} = u_1 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_3} = u_2 + u_1 + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = u_1 + u_2 + u_3 = 3 \end{cases} \implies \begin{cases} u_1^* = 1 \\ u_2^* = 1 \\ u_3^* = 1 \\ \lambda = -2 \end{cases} \quad \text{optimal solution}$$

Note: This was actually the worst we can do—maximize! Even weirder: no local minimizer!

### 1.4.1 Equality Constraints

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

**Theorem.** If  $u^*$  is a minimizer/maximizer then  $\exists \lambda \in \mathbb{R}^k$  s.t.

$$\begin{aligned} \frac{\partial L}{\partial u}(u^*, \lambda) &= 0 \\ \frac{\partial L}{\partial \lambda}(u^*, \lambda) &= 0 \quad (\iff h(u^*) = 0) \end{aligned}$$

where  $L(u, \lambda) = g(u) + \lambda^T h(u)$ .

**Example** [Entropy Maximization]

Given  $S = \{x_1, \dots, x_n\}$  and a distribution over  $S$  such that it takes the value  $x_j$  with probability  $p_j$ . The entropy is

$$E(p) = \sum_{j=1}^n (-p_j \ln p_j).$$

The mean is

$$m = \sum_{j=1}^n p_j x_j.$$

Problem: Given  $m$ , find  $p$  such that  $E$  is maximized.

$$\begin{aligned} \min_p \quad & - \sum_{j=1}^n p_j \ln p_j \\ \text{s.t.} \quad & \sum_{j=1}^n p_j x_j = m \\ & \sum_{j=1}^n p_j = 1 \\ & p_j \geq 0, \quad j = 1, \dots, n \quad (\text{ignore this...}) \end{aligned}$$

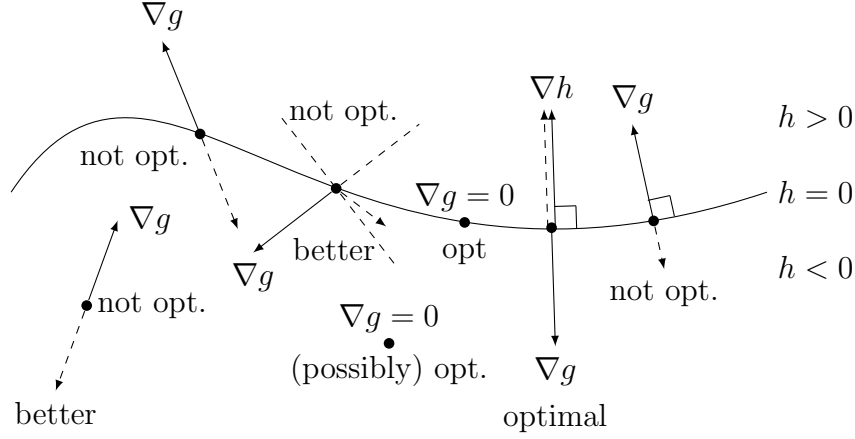
$$\begin{aligned} L &= - \sum p_j \ln p_j + \lambda_1 \left[ \sum p_j x_j - m \right] + \lambda_2 \left[ \sum p_j - 1 \right] \\ \frac{\partial L}{\partial p_j} &= - \ln p_j - 1 + \lambda_1 x_j + \lambda_2 = 0 \\ p_j &= e^{\lambda_2 - 1 + \lambda_1 x_j}, \quad j = 1, \dots, n \quad (p_j \geq 0 \text{ so we're ok with ignoring that}) \end{aligned}$$

$$\begin{aligned} \sum e^{\lambda_2 - 1 + \lambda_1 x_j} x_j &= m & n + 2 \text{ equations and} \\ \sum e^{\lambda_2 - 1 + \lambda_1 x_j} &= 1 & n + 2 \text{ unknowns...} \end{aligned}$$

No analytical solution, but numerically “solvable”

## 1.4.2 Inequality Constraints

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) \leq \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$



We need:

- if  $h(u^*) < 0$  then  $\frac{\partial g}{\partial u}(u^*) = 0$
- if  $h(u^*) = 0$  then we need either

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or

$$\frac{\partial g}{\partial u}(u^*) = -\lambda \frac{\partial h}{\partial u}(u^*) \quad \text{for } \lambda > 0$$

Or, even better,

$$\frac{\partial}{\partial u} (g(u^*) + \lambda h(u^*)) = 0 \quad \text{for } \lambda \geq 0,$$

where  $\lambda h(u^*) = 0$ . ( $h < 0 \rightarrow \lambda = 0$ ,  $h = 0 \rightarrow \lambda \geq 0$ )

In general, if  $u \in \mathbb{R}^m$  and  $h : \mathbb{R}^m \rightarrow \mathbb{R}^k$ , we have that  $u^*$ , if optimal, has to satisfy

$$\begin{aligned} \frac{\partial}{\partial u} L(u^*, \lambda) &= 0 \\ h(u^*) &\leq \mathbf{0} \\ \lambda^T h(u^*) &= 0 \\ \lambda &\geq \mathbf{0} \end{aligned}$$

where the Lagrangian is  $L(u, \lambda) = g(u) + \lambda^T h(u)$ . Note that if we're maximizing, the same holds except we need  $\lambda \leq 0$ .

### Example

$$\begin{aligned} \min \quad & 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 \\ \text{s.t.} \quad & \begin{cases} u_1^2 + u_2^2 \leq 5 \\ 3u_1 + u_2 \leq 6 \end{cases} \end{aligned}$$

$$L = 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 + \lambda_1(u_1^2 + u_2^2 - 5) + \lambda_2(3u_1 + u_2 - 6)$$

FONC:

- i)  $\partial L / \partial u_1 = 4u_1 + 2u_2 - 10 + 2\lambda_1u_1 + 3\lambda_2$
- ii)  $\partial L / \partial u_2 = 2u_1 + 2u_2 - 10 + 2\lambda_1u_2 + \lambda_2$
- iii)  $u_1^2 + u_2^2 \leq 5$
- iv)  $3u_1 + u_2 \leq 6$
- v)  $\lambda_1(u_1^2 + u_2^2 - 5) = 0$
- vi)  $\lambda_2(3u_1 + u_2 - 6) = 0$
- vii)  $\lambda_1 \geq 0$
- viii)  $\lambda_2 \geq 0$

To solve, assume different constraints are active/inactive:

1. Both constraints are inactive ( $u_1^2 + u_2^2 < 5$ ,  $3u_1 + u_2 < 6$ )  $\implies \lambda_1 = \lambda_2 = 0$

$$\begin{cases} 4u_1 + 2u_2 - 10 = 0 \\ 2u_1 + 2u_2 - 10 = 0 \end{cases} \implies \begin{cases} u_1 = 0 \\ u_2 = 5 \end{cases}$$

Note: iii)  $0^2 + 5^2 \not\leq 5$

Not feasible

2. Assume constraint 1 is active and constraint 2 is inactive ( $u_1^2 + u_2^2 = 5$ ,  $\lambda_2 = 0$ )

$$\begin{cases} 4u_1 + 2u_2 - 10 + 2\lambda_1u_1 = 0 \\ 2u_1 + 2u_2 - 10 + 2\lambda_1u_2 = 0 \\ u_1^2 + u_2^2 = 5 \end{cases} \implies \begin{cases} u_1 = 1 \\ u_2 = 2 \\ \lambda_1 = 1 \end{cases}$$

$$\checkmark \lambda_1 \geq 0$$

$$\checkmark 3 \cdot 1 + 2 \leq 6$$

This is a local minimizer

3. Assume constraint 2 is active and constraint 1 is inactive
4. Assume both constraints are active

## Kuhn-Tucker Conditions (KKT conditions, Karush-Kuhn-Tucker)

**Problem:**

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & \begin{cases} h_1(u) = 0, & h_1 : \mathbb{R}^m \rightarrow \mathbb{R}^p \\ h_2(u) \leq 0, & h_2 : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{cases} \end{aligned} \tag{1.1}$$

**Theorem.** Let  $u^*$  be feasible ( $h_1 = 0, h_2 \leq 0$ ). If  $u^*$  is a minimizer to (1.1) then there exists vectors  $\lambda \in \mathbb{R}^p, \mu \in \mathbb{R}^k$  with  $\mu \geq \mathbf{0}$  such that

$$\begin{cases} \frac{\partial g}{\partial u}(u^*) + \lambda^T \frac{\partial h_1}{\partial u}(u^*) + \mu^T \frac{\partial h_2}{\partial u}(u^*) = 0 \\ \mu^T h_2(u^*) = 0 \end{cases}$$

Looking ahead:  $\min \text{cost}(u(\cdot))$  s.t.  $\dot{x} = f(x, u)$  (dynamics), where  $u$  is a function. Note the equality constraint.

**Question:** How do we go from  $u \in \mathbb{R}^m$  to  $u \in \mathcal{U}$  (function space)?

**Note:** Function space is a set of functions of a given kind from a set  $X$  to a set  $Y$

1. linear function
2. square-integrable functions:  $L_2[0, T] : \int_0^T \|u(t)\|^2 dt < \infty$
3.  $C^\infty(\mathbb{R})$

What would  $\partial$ “cost”/ $\partial u$  mean?

# Chapter 2

## Calculus of Variations

### 2.1 Directional Derivatives

**Recall:** To minimize  $g(u)$ , let  $u^*$  be a candidate minimizer and pitch a perturbation on  $u^*$  of  $\varepsilon v$ , where  $\varepsilon$  is the scale and  $v$  is the direction. Taking Taylor's expansion at the perturbation produces



$$g(u^* + \varepsilon v) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$

$$\text{FONC: } \frac{\partial g}{\partial u}(u^*) = 0$$

**Note:**  $\frac{\partial g}{\partial u}(u^*)v$  tells us how much  $g(u)$  increases/decreases in the direction of  $v$ .

**Definition.** The directional (Gateaux) derivative is given by

$$\delta g(u; v) = \lim_{\varepsilon \rightarrow 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon}$$

**Example**

$$g(u) = \frac{1}{2}u_1^2 - u_1 + 2u_2, \quad g : \mathbb{R}^2 \rightarrow \mathbb{R}$$

Let's consider  $e_1 = [1 \ 0]^T$ ,  $e_2 = [0 \ 1]^T$ . What is  $\delta g(u; e_i)$ ,  $i = 1, 2$ ?

$$\begin{aligned} \delta g(u; v) &= \lim_{\varepsilon \rightarrow 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{g(u) + \varepsilon \frac{\partial g}{\partial u}(u)v + o(\varepsilon) - g(u)}{\varepsilon} \\ &= \frac{\partial g}{\partial u}(u)v \end{aligned}$$



$$\begin{aligned}\frac{\partial g}{\partial u}(u) &= [u_1 - 1 \ 2] \\ \delta g(u; e_1) &= [u_1 - 1 \ 2]e_1 = u_1 - 1 \\ \delta g(u; e_2) &= [u_1 - 1 \ 2]e_2 = 2\end{aligned}$$

But the beauty of directional derivatives is that they generalize beyond vectors,  $u \in \mathbb{R}^m$ , to function spaces ( $\mathcal{U}$ ) or other “objects” like matrices.

**Example**  $M \in \mathbb{R}^{n \times n}$ ,  $F(M) = M^2$

What is  $\frac{\partial F}{\partial M}$ ? (ponder at home...)

We can easily compute  $\delta F(M; N)$ !

$$\begin{aligned}F(M + \varepsilon N) &= (M + \varepsilon N)(M + \varepsilon N) = M^2 + \varepsilon MN + \varepsilon NM + \varepsilon^2 N^2 \\ \delta F(M; N) &= \lim_{\varepsilon \rightarrow 0} \frac{F(M + \varepsilon N) - F(M)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon MN + \varepsilon NM + \varepsilon^2 N^2}{\varepsilon} = MN + NM\end{aligned}$$

**Infinite Dimensional Optimization** Let  $u \in \mathcal{U}$  (function space) and let  $J(u)$  be the cost:

$$\min_{u \in \mathcal{U}} J(u)$$

**Theorem.** If  $u^* \in \mathcal{U}$  is a (local) minimizer then

$$\delta J(u^*; v) = 0, \quad \forall v \in \mathcal{U}$$

**Example** Find minimizer  $u^*$  to

$$J(u) = \int_0^T L(u(t)) \, dt$$

$$\begin{aligned}J(u + \varepsilon v) - J(u) &= \int_0^T L(u(t) + \varepsilon v(t)) \, dt - \int_0^T L(u(t)) \, dt, \quad u, v \in \mathcal{U} \\ &= \int_0^T \left[ L(u(t)) + \varepsilon \frac{\partial L}{\partial u}(u(t))v(t) + o(\varepsilon) - L(u(t)) \right] \, dt \\ \delta J(u^*; v) &= \lim_{\varepsilon \rightarrow 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\int_0^T \varepsilon \frac{\partial L}{\partial u}(u(t))v(t) \, dt + o(\varepsilon)}{\varepsilon} \\ &= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t) \, dt\end{aligned}$$

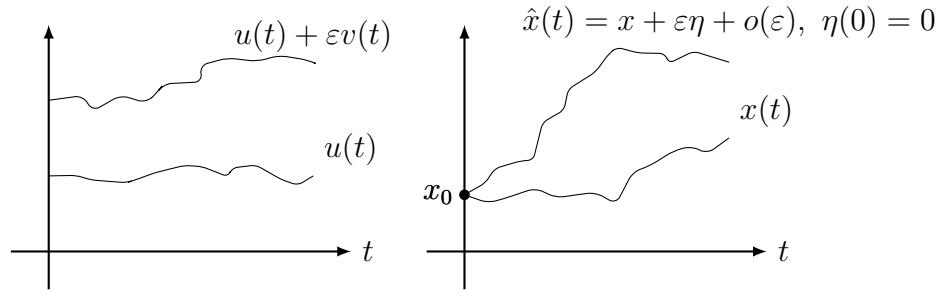


Figure 2.1: Variation in  $u$  causes a variation in  $x$ .

$u^*$  optimizer:

$$\begin{aligned} \delta J(u^*; v) &= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t) dt = 0 \quad \forall v \in \mathcal{U} \\ &\Downarrow \\ \frac{\partial L}{\partial u}(u(t)) &= 0 \quad \forall t \in [0, T] \end{aligned}$$

But, we want *optimal control*! We want our cost to look like

$$\begin{aligned} &\int_0^T L(x(t), u(t)) dt \\ &\dot{x} = f(x, u) \end{aligned}$$

## 2.2 Calculus of Variations

What happens to  $x(t)$  when  $u(t)$  changes to  $u(t) + \varepsilon v(t)$ ? Let the system be given by

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

After perturbation of  $u$ , the new system is

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}, u + \varepsilon v) \\ \hat{x}(0) = x_0 \end{cases}$$

Consider

$$\tilde{x} = x + \varepsilon \eta,$$

where

$$\begin{aligned} \dot{x} &= f(x, u), & x(0) &= x_0 \\ \dot{\eta} &= \frac{\partial f}{\partial x}(x, u)\eta + \frac{\partial f}{\partial u}(x, u)v, & \eta(0) &= 0 \end{aligned}$$

**Theorem.** *If  $f$  is continuously differentiable in  $x$  and  $u$  then*

$$\hat{x}(t) = \tilde{x}(t) + o(\varepsilon)$$

*Proof.*

i) Initial conditions:

$$\begin{aligned}\hat{x}(0) &= x_0 \\ \tilde{x}(0) &= x(0) + \varepsilon\eta(0) = x_0\end{aligned}$$

ii) Dynamics:

$$\begin{aligned}\dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v) \\ \dot{\tilde{x}} &= \dot{x} + \varepsilon\dot{\eta} = f(x, u) + \varepsilon \frac{\partial f}{\partial x}(x, u)\eta + \varepsilon \frac{\partial f}{\partial u}(x, u)v \\ &= f(x + \varepsilon\eta, u + \varepsilon v) + o(\varepsilon) \\ &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon)\end{aligned}$$

We can see that the dynamics of  $\hat{x}(t)$  are equal to those of  $\tilde{x}(t)$  plus higher order terms:

$$\begin{aligned}\dot{\tilde{x}} &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon) \\ \dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v)\end{aligned}$$

Therefore, if our perturbation is small enough, we can model  $\hat{x}(t)$  as  $\tilde{x}(t)$ .

□

Note: Taylor expansion with two elements is

$$\begin{aligned}h(w + \varepsilon v, z + \varepsilon y) &= h(w, z + \varepsilon y) + \frac{\partial h}{\partial w}(w, z + \varepsilon y)\varepsilon v + o(\varepsilon) \\ &= \left\{ h(w, z) + \frac{\partial h}{\partial z}(w, z)\varepsilon y + o(\varepsilon) \right\} \\ &\quad + \left\{ \frac{\partial h}{\partial w}(w, z)\varepsilon v + \underbrace{\frac{\partial^2 h}{\partial z \partial w}\varepsilon v \odot \varepsilon y}_{o(\varepsilon)} + o(\varepsilon) \right\} \\ &= h(w, z) + \frac{\partial h}{\partial z}\varepsilon y + \frac{\partial h}{\partial w}\varepsilon v + o(\varepsilon)\end{aligned}$$

**Last class:**

1.  $u \in \mathcal{U}$  (space of functions),  $J : \mathcal{U} \rightarrow \mathbb{R}$  (cost).

FONC: If  $u^*$  is optimal, then

$$\delta J(u; \nu) = 0 \quad \forall \nu \in \mathcal{U},$$

where the directional derivative is given by

$$\delta J(u; \nu) = \lim_{\varepsilon \rightarrow 0} \frac{J(u + \varepsilon \nu) - J(u)}{\varepsilon}.$$

2. If

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

then a variation in  $u$ :

$$u \mapsto u + \varepsilon \nu$$

results in a variation in  $x$ :

$$x \mapsto x + \varepsilon \eta + o(\varepsilon)$$

See Figure 2.1. Note  $\eta(0) = 0$ .

### 2.2.1 An (Almost) Optimal Control Problem

Let  $\dot{x} = f(x)$ ,  $x(0) = x_0$ . Note we get to pick the initial condition!

**Problem**

$$\begin{aligned} \min_{x_0 \in \mathbb{R}^m} J(x_0) &= \int_0^T L(x(t)) \, dt \\ \text{s.t. } \begin{cases} \dot{x}(t) = f(x(t)) & \text{the constraint! (equality)} \\ x(0) = x_0 \end{cases} \end{aligned}$$

Note every constraint needs a Lagrange multiplier. We have infinitely many constraints:

$$\dot{x}(t) = f(x(t)) \quad \forall t \in [0, T]$$

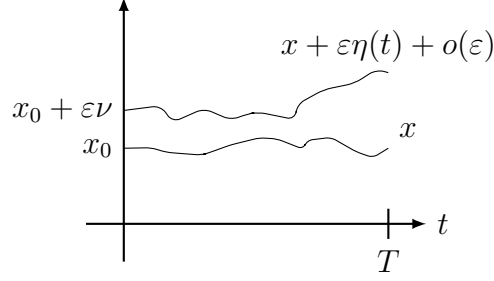
We need  $\lambda(t)$  as a function of  $t$ . Also, the sum in the Lagrangian has to become an integral. The continuous-time Lagrangian thus becomes

$$\tilde{J}(x_0, \lambda) = \int_0^T \left[ L(x(t)) + \lambda^T(t)(f(x(t)) - \dot{x}(t)) \right] dt$$

The task is to perturb  $x_0$  as  $x_0 \mapsto x_0 + \varepsilon \nu$ ,  $\nu \in \mathbb{R}^m$  and compute

$$\delta \tilde{J}(x_0; \nu) = \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(x_0 + \varepsilon \nu) - \tilde{J}(x_0)}{\varepsilon}$$

and make this equal to 0  $\forall \nu \in \mathbb{R}^m$ . The variation in  $x$  is



Note:

- $x_0$  decision variable
- $\nu$  variation in  $x_0$
- $x(t)$  trajectory starting at  $x_0$
- $\eta(t)$  change in trajectory resulting from  $\nu$ -variation in  $x_0$
- $\lambda(t)$  time-varying Lagrange multiplier

$$\begin{aligned}
 \tilde{J}(x_0 + \varepsilon \nu) &= \int_0^T \left\{ L(x(t)) + \lambda^T(t) [f(x(t) + \varepsilon \eta(t)) - \dot{x}(t) - \varepsilon \dot{\eta}(t)] \right\} dt + o(\varepsilon) \\
 &= \int_0^T \left[ L(x) + \varepsilon \frac{\partial L}{\partial x}(x) \eta + \lambda^T \left( f(x) + \varepsilon \frac{\partial f}{\partial x}(x) \eta - \dot{x} - \varepsilon \dot{\eta} \right) \right] dt + o(\varepsilon) \\
 \tilde{J}(x_0 + \varepsilon \nu) - \tilde{J}(x_0) &= \int_0^T \left[ \varepsilon \frac{\partial L}{\partial x}(x) \eta + \lambda^T \left( \varepsilon \frac{\partial f}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt + o(\varepsilon) \\
 \delta \tilde{J}(x_0; \nu) &= \int_0^T \left[ \frac{\partial L}{\partial x}(x) \eta + \lambda^T \left( \frac{\partial f}{\partial x} \eta - \dot{\eta} \right) \right] dt
 \end{aligned}$$

A powerful idea: we want  $\delta \tilde{J}(x_0; \nu) = 0 \forall \nu$ . Somehow get this in the form

$$\int_0^T \left( \text{stuff}(t) \right) \eta(t) dt = 0$$

We can pick  $\text{stuff}(t) = 0 \forall t \in [0, T]$ .

In  $\delta \tilde{J}(x_0; \nu)$  we have  $\dot{\eta}$  (problem!). We can solve this using *integration by parts*.

$$\int_0^T \lambda^T \dot{\eta} dt = \lambda^T(T) \eta(T) - \lambda^T(0) \eta(0) - \int_0^T \dot{\lambda}^T \eta dt$$

Hence,

$$\delta \tilde{J}(x_0; \nu) = \int_0^T \underbrace{\left( \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T \right)}_{\text{pick}=0} \eta dt - \underbrace{\lambda^T(T) \eta(T)}_{\text{pick}=0} + \lambda^T(0) \underbrace{\eta(0)}_{\nu}$$

We are free to pick  $\lambda$  freely if it gives  $\delta \tilde{J} = 0$ .

$$\text{Pick: } \begin{cases} \dot{\lambda}(t) = -\frac{\partial L^T}{\partial x}(x(t)) - \frac{\partial f^T}{\partial x}(x(t)) \lambda(t) \\ \lambda(T) = 0 \end{cases} \quad \text{backwards diff. eq}$$

Under this choice of  $\lambda$  we get

$$\delta \tilde{J}(x_0; \nu) = \lambda^T(0) \nu$$

This is linear in  $\nu$  so the FONC is  $\lambda(0) = 0$ .

Moreover, we really have a “normal” optimization problem

$$\begin{aligned} \min_{x_0 \in \mathbb{R}^m} \quad & \tilde{J}(x_0) \\ \delta \tilde{J}(x_0; \nu) = \frac{\partial \tilde{J}}{\partial x_0}(x_0) \nu \end{aligned}$$

which means that

$$\frac{\partial \tilde{J}}{\partial x_0} = \lambda^T(0)$$

If  $x_0^*$  minimizes

$$\begin{aligned} & \int_0^T L(x(t)) \, dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0^* \end{cases} \end{aligned}$$

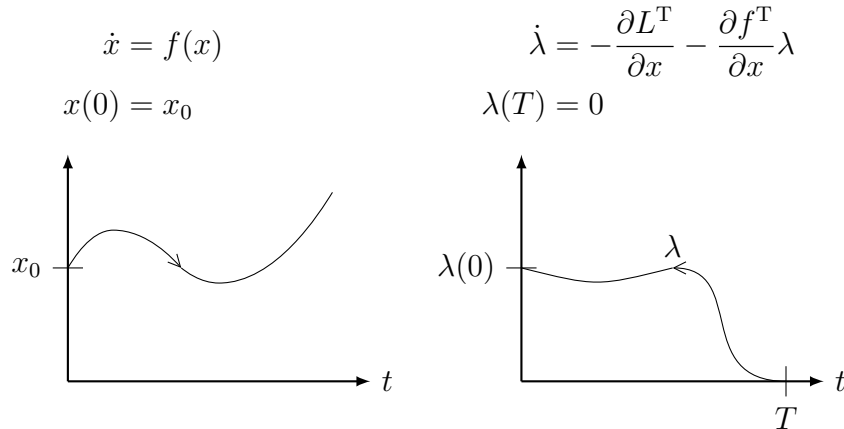
then

$$\lambda(0) = \mathbf{0}$$

where  $\lambda(t)$  satisfies

$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial L^T}{\partial x}(x(t)) - \frac{\partial f^T}{\partial x}(x(t)) \lambda(t) \\ \lambda(T) = 0 \end{cases}$$

**So what?** We actually have a two-point boundary value problem.



We want to find  $x_0$  that gives  $f(x)$  such that after solving backwards for  $\lambda(t)$ , we find that

$$\lambda(0) = \frac{\partial \tilde{J}^T}{\partial x_0} = 0.$$

This leads to the following:

---

Pick  $x_{0,0}$

$k = 1$

**repeat**

    Simulate  $x(t)$  from  $x_{0,k}$  over  $[0, T]$

    Simulate  $\lambda(t)$  from  $\lambda(T) = 0$  backwards using  $x(t)$

    Update  $x_{0,k}$  as  $x_{0,k+1} = x_{0,k} - \gamma\lambda(0)$

▷  $\lambda(0)$  is the gradient

$k := k + 1$

**until**  $\lambda(0) = 0$

---

## An algorithm

**Example:** `optinit.m`

$$\dot{x} = Ax, \quad L = x^T Q x - q, \quad Q = Q^T \succ 0$$

$$\dot{\lambda} = -2Qx - A^T \lambda$$

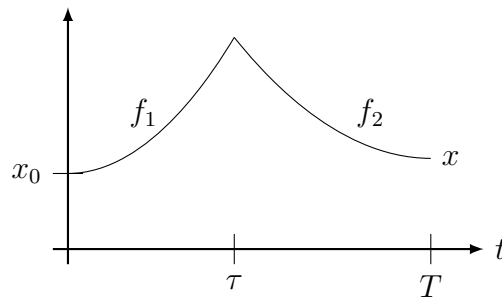
$$\lambda(0) = 0$$

### 2.2.2 Optimal Timing Control

When to switch between modes?

$$\dot{x} = \begin{cases} f_1(x) & \text{if } t \in [0, \tau) \\ f_2(x) & \text{if } t \in [\tau, T] \end{cases} \quad (2.1)$$

$$x(0) = x_0$$



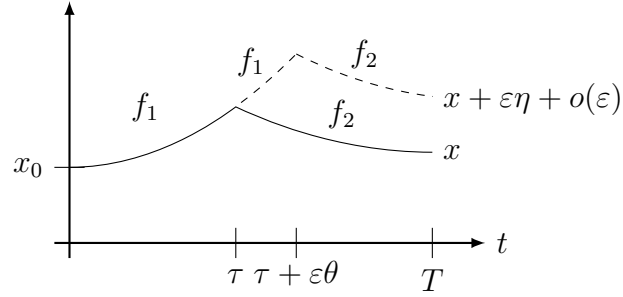
$$\min_{\tau} \int_0^T L(x(t)) \, dt = J(\tau)$$

s.t. (2.1) holds

Step 1: Augment cost with constraint

$$\tilde{J} = \int_0^{\tau} \left[ L(x) + \lambda^T (f_1(x) - \dot{x}) \right] dt + \int_{\tau}^T \left[ L(x) + \lambda^T (f_2(x) - \dot{x}) \right] dt$$

Step 2: Variation  $\tau \mapsto \tau + \varepsilon\theta$



Step 3: Compute  $\delta\tilde{J}(\tau; \theta)$

$$\begin{aligned}\tilde{J}(\tau + \varepsilon\theta) &= \int_0^{\tau + \varepsilon\theta} \left\{ L(x + \varepsilon\eta) + \lambda^T [f_1(x + \varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \right\} dt \\ &\quad + \int_{\tau + \varepsilon\theta}^T \left\{ L(x + \varepsilon\eta) + \lambda^T [f_2(x + \varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \right\} dt + o(\varepsilon)\end{aligned}$$

Note that  $\eta = \dot{\eta} = 0$  on  $[0, \tau)$ .

$$\begin{aligned}\tilde{J}(\tau + \varepsilon\theta) &= \int_0^{\tau} \left\{ L(x) + \lambda^T [f_1(x) - \dot{x}] \right\} dt \\ &\quad + \int_{\tau}^{\tau + \varepsilon\theta} \left\{ \underbrace{L(x + \varepsilon\eta)}_{L(x) + \varepsilon \frac{\partial L}{\partial x} \eta} + \lambda^T \left[ \underbrace{f_1(x + \varepsilon\eta)}_{f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta} - \dot{x} - \varepsilon\dot{\eta} \right] \right\} dt \\ &\quad + \int_{\tau + \varepsilon\theta}^T \left\{ \underbrace{L(x + \varepsilon\eta)}_{L(x) + \varepsilon \frac{\partial L}{\partial x} \eta} + \lambda^T \left[ \underbrace{f_2(x + \varepsilon\eta)}_{f_2(x) + \varepsilon \frac{\partial f_2}{\partial x} \eta} - \dot{x} - \varepsilon\dot{\eta} \right] \right\} dt + o(\varepsilon)\end{aligned}$$



$$\begin{aligned}
\delta \tilde{J}(\tau; \theta) &= \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau)}{\varepsilon} \\
\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau) &= \int_0^\tau 0 \cdot dt + \underbrace{\int_\tau^{\tau+\varepsilon\theta} \left[ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left( f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta - f_2(x) - \varepsilon \dot{\eta} \right) \right] dt}_{I_1} \\
&\quad + \underbrace{\int_{\tau+\varepsilon\theta}^T \left[ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left( \varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt}_{I_2} + o(\varepsilon)
\end{aligned}$$

**Theorem** (Mean-value theorem).

$$\int_{t_1}^{t_2} h(t) dt = (t_2 - t_1)h(\xi) \quad \text{for some } \xi \in [t_1, t_2]$$

The first integral is

$$\begin{aligned}
I_1 &= \int_\tau^{\tau+\varepsilon\theta} \left\{ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left[ f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta - \varepsilon \dot{\eta} - f_2(x) \right] \right\} dt \\
&= \varepsilon \theta \left\{ \lambda^\top(\xi) [f_1(x(\xi)) - f_2(x(\xi))] \right\} + o(\varepsilon)
\end{aligned}$$

Note that as  $\varepsilon \rightarrow 0$ ,  $\xi \rightarrow \tau$ . Using integration by parts, the second integral is

$$\begin{aligned}
\int_\tau^T \lambda^\top \dot{\eta} dt &= \lambda^\top(T) \eta(T) - \lambda^\top(\tau) \underbrace{\eta(\tau)}_{=0} - \int_\tau^T \dot{\lambda}^\top \eta dt \\
I_2 &= \int_\tau^T \left[ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left( \varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt - \underbrace{\int_\tau^{\tau+\varepsilon\theta} \left[ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left( \varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt}_{o(\varepsilon)} \\
&= \varepsilon \int_\tau^T \left[ \frac{\partial L}{\partial x} + \lambda^\top \frac{\partial f_2}{\partial x} + \dot{\lambda}^\top \right] \eta dt - \varepsilon \lambda^\top(T) \eta(T) + o(\varepsilon)
\end{aligned}$$

Hence,

$$\begin{aligned}
\delta \tilde{J}(\tau; \theta) &= \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau)}{\varepsilon} \\
&= \theta \lambda^\top(\tau) [f_1(x(\tau)) - f_2(x(\tau))] + \int_\tau^T \left[ \frac{\partial L}{\partial x} + \lambda^\top \frac{\partial f_2}{\partial x} + \dot{\lambda}^\top \right] \eta dt - \lambda^\top(T) \eta(T)
\end{aligned}$$

Step 4: Select the *costate*  $\lambda(t)$ . The key idea is to get rid of any term that has  $\eta$  in it, i.e.

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial L}{\partial x} - \frac{\partial f_2^\top}{\partial x} \lambda \quad \text{on } [\tau, T] \\
\lambda(T) &= 0
\end{aligned}$$

Step 5: With this choice of  $\lambda(t)$ , we have

$$\delta \tilde{J}(\tau; \theta) = \theta \lambda^T(\tau) [f_1(x(\tau)) - f_2(x(\tau))] = \frac{\partial \tilde{J}}{\partial \tau} \theta.$$

Therefore,

$$\frac{\partial \tilde{J}}{\partial \tau} = \lambda^T(\tau) [f_1(x(\tau)) - f_2(x(\tau))] = 0 \quad (\text{for optimality})$$

## Algorithm

---

Pick  $\tau_0$

$k = 0$

**repeat**

    Simulate  $x$  forward in time from  $x(0) = x_0$

    Simulate  $\lambda$  backwards from  $\lambda(T) = 0$

    Update  $\tau_k$  as  $\tau_{k+1} = \tau_k - \gamma \lambda^T(\tau_k) [f_1(x(\tau_k)) - f_2(x(\tau_k))]$

$k := k + 1$

**until**  $\|\lambda^T(f_1 - f_2)\| < \varepsilon$

---

Where are we going? Come up with general principles for  $\min_{u \in \mathcal{U}} J(u)$ :

- Costate equations
- Optimality conditions
- Algorithms
- Applications

# Chapter 3

## The Maximum Principle

### 3.1 The Bolza Problem

Up until now, we have optimized with respect to finite-dimensional parameters. Today, we will minimize with respect to  $u \in \mathcal{U}$ .

$$\begin{aligned} \min_{u \in \mathcal{U}} J(u) &= \int_0^T L(x(t), u(t), t) dt + \underbrace{\Psi(x(T))}_{\substack{\text{terminal cost} \\ \text{(parking cost)}}} \\ \text{s.t. } \dot{x}(t) &= f(x(t), u(t), t) \\ x(0) &= x_0 \end{aligned}$$

Assume that  $f$  and  $L$  are  $C^1$  in  $x, u$  and piecewise continuous in  $t$ . Then, a small change in  $u$  causes small changes in  $f$  and  $L$ . The variation:  $u \mapsto u + \varepsilon v$ ,  $\varepsilon \in \mathbb{R}$ ,  $v \in \mathcal{U}$ . See Figure 2.1.

$$\begin{aligned} \tilde{J}(u) &= \int_0^T [L(x, u, t) + \lambda^T(f(x, u, t) - \dot{x})] dt + \Psi(x(T)) \\ \tilde{J}(u + \varepsilon v) &= \int_0^T [L(x + \varepsilon \eta, u + \varepsilon v, t) + \lambda^T(f(x + \varepsilon \eta, u + \varepsilon v, t) - \dot{x} - \varepsilon \dot{\eta})] dt \\ &\quad + \Psi(x(T) + \varepsilon \eta(T)) + o(\varepsilon) \\ \tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \left[ L(x + \varepsilon \eta, u + \varepsilon v, t) - L(x, u, t) \right. \\ &\quad \left. + \lambda^T(f(x + \varepsilon \eta, u + \varepsilon v, t) - f(x, u, t) - \dot{x} - \varepsilon \dot{\eta} + \dot{x}) \right] dt \\ &\quad + \Psi(x(T) + \varepsilon \eta(T)) - \Psi(x(T)) + o(\varepsilon) \\ &= \int_0^T \left[ \frac{\partial L}{\partial x} \varepsilon \eta + \frac{\partial L}{\partial u} \varepsilon v + \lambda^T \left( \frac{\partial f}{\partial x} \varepsilon \eta + \frac{\partial f}{\partial u} \varepsilon v - \varepsilon \dot{\eta} \right) \right] dt \\ &\quad + \frac{\partial \Psi}{\partial x}(x(T)) \varepsilon \eta(T) + o(\varepsilon) \end{aligned}$$

(See Taylor expansion with respect to two variables.)

$$\begin{aligned}\delta\tilde{J}(u;v) &= \int_0^T \left( \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) v \, dt + \int_0^T \left[ \left( \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} \right) \eta - \lambda^T \dot{\eta} \right] \, dt \\ &\quad + \frac{\partial \Psi}{\partial x}(x(T))\eta(T)\end{aligned}$$

Integrating by parts,

$$\begin{aligned}\int_0^T \lambda^T \dot{\eta} \, dt &= \lambda^T(T)\eta(T) - \lambda^T(0)\eta(0) - \int_0^T \dot{\lambda}^T \eta \, dt \\ &= \lambda^T(T)\eta(T) - \int_0^T \dot{\lambda}^T \eta \, dt \\ \delta\tilde{J}(u;v) &= \int_0^T \left( \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) v \, dt + \int_0^T \left( \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T \right) \eta \, dt \\ &\quad + \left( \frac{\partial \Psi}{\partial x}(x(T)) - \lambda^T(T) \right) \eta(T)\end{aligned}$$

For optimality, we need the directional derivative to be zero for every  $v \in \mathcal{U}$ , where  $v$  represents the direction of the derivative. Therefore, the term  $(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u})$  in the first integral has to be identically zero. Thus, we need

$$\begin{cases} \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0, & \forall t \in [0, T] \\ \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T = 0, & \forall t \in [0, T] \\ \frac{\partial \Psi}{\partial x}(x(T)) - \lambda^T(T) = 0 \end{cases}$$

**Definition.** Let the *Hamiltonian*  $H(x, u, t, \lambda)$  be given by

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^T f(x, u, t)$$

**Theorem.** For  $u$  to solve the Bolza problem, it has to satisfy

$$\frac{\partial H}{\partial u}(x, u, t, \lambda) = 0,$$

where the costate satisfies

$$\begin{cases} \dot{\lambda} = -\frac{\partial H^T}{\partial x}(x, u, t, \lambda) \\ \lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T)) \end{cases}$$

## Example

$$\begin{aligned} \min_u \quad & \int_0^1 \frac{1}{2} u^2(t) \, dt + \frac{1}{2} x^2(1) \\ \text{s.t.} \quad & \begin{cases} \dot{x} = u, & x, u \in \mathbb{R} \\ x(0) = 1 \end{cases} \end{aligned}$$

$$H = \frac{1}{2} u^2 + \lambda u$$

$$\frac{\partial H}{\partial u} = u + \lambda = 0 \implies u = -\lambda$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0 \implies \lambda(t) = c$$

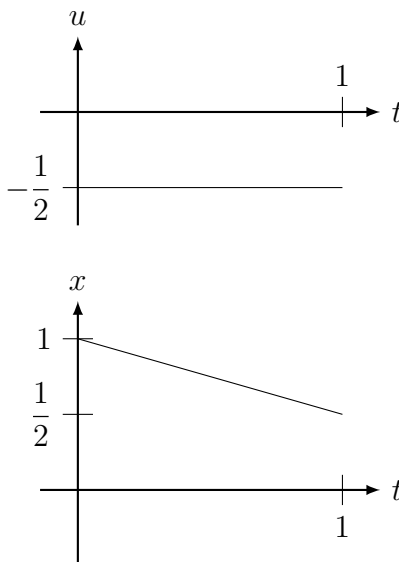
$$\lambda(T) = c = \frac{\partial \Psi}{\partial x}(x(1)) = x(1)$$

$$\dot{x} = u = -c \implies x(t) = -ct + x(0) = -ct + 1$$

$$x(1) = -c + 1$$

$$\lambda(1) = c = x(1) = -c + 1 \implies c = \frac{1}{2}$$

$$\boxed{u^* = -\frac{1}{2}}$$



We really used five different equations to solve this!

i)  $\frac{\partial H}{\partial u} = 0$

ii)  $\dot{\lambda} = -\frac{\partial H^T}{\partial x}$

$$\text{iii) } \lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T))$$

$$\text{iv) } \dot{x} = f(x, u, t)$$

$$\text{v) } x(0) = x_0$$

There is a sixth condition that is pretty useful if  $L$  and  $f$  do not depend on  $t$  ( $L(x, u)$ ,  $f(x, u)$ ). This is called a *conservative system*. Then, along optimal trajectories (equations i-v are satisfied), the total time derivative of the Hamiltonian is

$$\frac{d}{dt}H = \underbrace{\frac{\partial H}{\partial t}}_{0}_{H(x,u,\lambda)} + \underbrace{\frac{\partial H}{\partial x}}_{-\dot{\lambda}^T} \dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{0}_{u \text{ is optimal}} \dot{u} + \underbrace{\frac{\partial H}{\partial \lambda}}_{f^T = \dot{x}^T} \dot{\lambda} = -\dot{\lambda}^T \dot{x} + \dot{x}^T \dot{\lambda} = 0$$

Therefore, for conservative systems,

vi)  $H$  is constant along optimal trajectories. (Hamilton's Principle in analytical mechanics)

Back to the example,

$$H = \frac{1}{2}u^2 + \lambda u = \frac{1}{2}c^2 - c^2 = -\frac{1}{2}c^2 = -\frac{1}{8}$$

The Hamiltonian

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^T f(x, u, t)$$

lets us write the Lagrangian as

$$\tilde{J}(u) = \int_0^T [L + \lambda^T(f - \dot{x})] dt + \Psi = \int_0^T (H - \lambda^T \dot{x}) dt + \Psi$$

The optimality conditions are

$$\frac{\partial H}{\partial u} = 0, \tag{3.1}$$

where

$$\begin{cases} \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = \frac{\partial \Psi}{\partial x}(x(T)) \end{cases} \tag{3.2}$$

**Example** Hamilton's Principle

Let  $q$  be the generalized coordinates (positions and angles). Then,  $\dot{q} = u$  are generalized velocities, which we assume we can control. Let  $T(q, u) = u^T M(q)u$ ,  $M \succ 0$ , be the kinetic energy and  $V(q)$  be the potential energy.

For conservative systems, the following quantity is minimized:

$$\int_0^T \underbrace{[T(q, u) - V(q)]}_{L(q, u) = \text{Lagrange's "action function"}} dt$$

The Hamiltonian is

$$H(q, u, \lambda) = L(q, u) + \lambda^T f(q, u) = L(q, u) + \lambda^T u$$

In mechanics,  $\lambda$  is called a generalized momentum, satisfying

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H^T}{\partial q} = -\frac{\partial L^T}{\partial q} + 0 \\ 0 &= \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \implies \lambda = -\frac{\partial L^T}{\partial u} \\ \dot{\lambda} &= -\frac{d}{dt} \frac{\partial L^T}{\partial u} = -\frac{\partial L^T}{\partial q} \end{aligned}$$

This produces the Euler-Lagrange Equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

Recall, along optimal trajectories

$$\begin{aligned} \frac{dH}{dt} &= \underbrace{\frac{\partial H}{\partial t}}_{=0 \text{ if } L \text{ and } f \text{ do not depend explicitly on } t} + \underbrace{\frac{\partial H}{\partial x}}_{-\dot{\lambda}^T \dot{x}} \dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{=0} \dot{u} + \underbrace{\frac{\partial H}{\partial \lambda}}_{f^T = \dot{x}^T} \dot{\lambda} = -\dot{\lambda}^T \dot{x} + \dot{x}^T \dot{\lambda} = 0 \end{aligned}$$

Therefore, along optimal trajectories, the Hamiltonian is constant!

We had

$$\begin{aligned} H &= L + \lambda^T u \\ \frac{\partial H}{\partial u} &= \lambda^T + \frac{\partial L}{\partial u} = 0 \end{aligned}$$

Along optimal trajectories,

$$H = L - \frac{\partial L}{\partial u} u$$

Recall,  $L(q, u) = T(q, u) - V(q)$ .

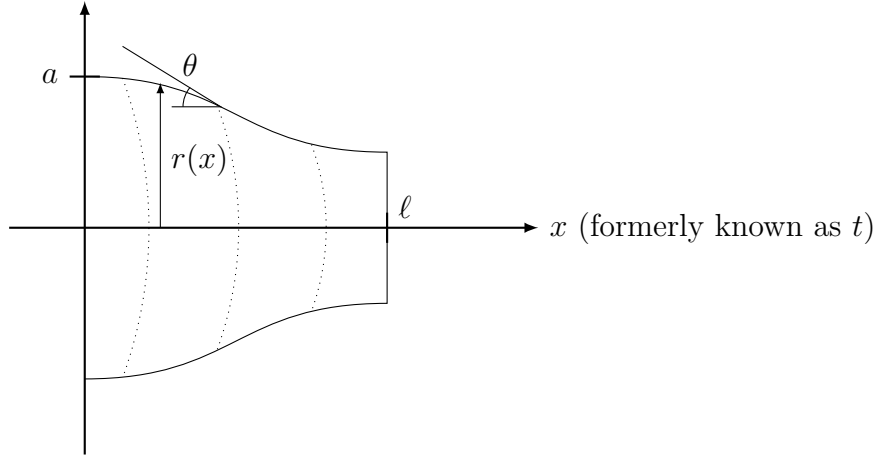
$$\begin{aligned} \frac{\partial L}{\partial u} &= \frac{\partial T}{\partial u} - 0 \\ T(q, u) &= u^T M(q) u \\ \frac{\partial T}{\partial u} &= 2u^T M \end{aligned}$$

So,

$$H = \underbrace{T}_{u^T M u} - V - 2u^T M u = -(V + u^T M u) = -(V + T)$$

Therefore, the total energy (kinetic plus potential energy) remains constant for conservative systems.

**Example** minimum drag nose shape (Newton 1686)



The drag is

$$D = -2\pi q \int_{x=0}^{\ell} C_p(\theta) r \, dr,$$

where  $q$  is a pressure constant and  $C_p(\theta) = 2 \sin^2 \theta$  is Newton's pressure formula.

Geometry tells us

$$\frac{dr}{dx} = -\tan \theta = -u$$

Choose the control as  $\tan \theta$ . Manipulating the drag,

$$\frac{D}{4\pi q} = \int_0^{\ell} \frac{ru^3}{1+u^2} \, dx + \frac{1}{2}r(\ell)^2$$

The optimal control problem is

$$\begin{aligned} \min_u \quad & \int_0^{\ell} \frac{ru^3}{1+u^2} \, dx + \frac{1}{2}r(\ell)^2 \\ \text{s.t.} \quad & \frac{dr}{dx} = -u \end{aligned}$$

This is in the standard form with the following changes of variables:

$$\begin{aligned} \ell &\longleftarrow T \\ x &\longleftarrow t \\ r &\longleftarrow x \end{aligned}$$



Refer to (3.1) and (3.2) for the following steps.

$$\begin{aligned}
H &= \frac{ru^3}{1+u^2} - \lambda u \\
\frac{\partial H}{\partial u} &= \frac{3ru^2(1+u^2) - ru^3 \cdot 2u}{(1+u^2)^2} - \lambda \\
&= \frac{ru^4 + 3ru^2}{(1+u^2)^2} - \lambda = 0 \\
\lambda &= \frac{ru^2(u^2 + 3)}{(1+u^2)^2} \\
\frac{d\lambda}{dx} &= -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2} \\
\lambda(\ell) &= r(\ell)
\end{aligned} \tag{3.3}$$

Right now, we know

$$\begin{cases} \frac{dr}{dx} = -u \\ r(0) = a \\ \frac{d\lambda}{dx} = -\frac{u^3}{1+u^2} \\ \lambda(\ell) = r(\ell) \end{cases}$$

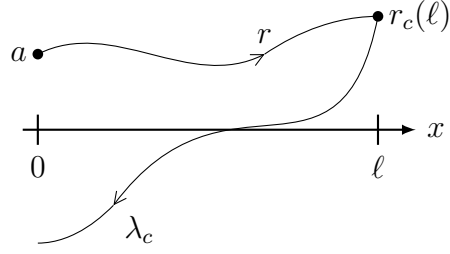
We need to remove  $u$  and get a function of  $r$  and  $\lambda$  instead. However, it is difficult to solve (3.3). Maybe  $H = \text{const.}$  gives us something nicer?

$$\begin{aligned}
H &= \frac{ru^3}{1+u^2} - \lambda u \\
&= \frac{ru^3}{1+u^2} - \frac{ru^2(u^2 + 3)}{(1+u^2)^2} u \\
&= -\frac{2ru^3}{(1+u^2)^2} = c
\end{aligned}$$

Assume we can find  $u = G(r, c)$ , either numerically or some other way. So, now we have

$$\begin{cases} \frac{dr}{dx} = -G(r, c) \\ r(0) = a \\ \frac{d\lambda}{dx} = -\frac{G^3(r, c)}{1+G^2(r, c)} \\ \lambda(\ell) = r(\ell) \end{cases}$$

We do not know  $c$ , but we can guess  $c$  and simulate  $r$  forward in “time” ( $x$ ) from  $r(0) = a$ . Then, we simulate  $\lambda$  backwards from  $r(\ell)$ .

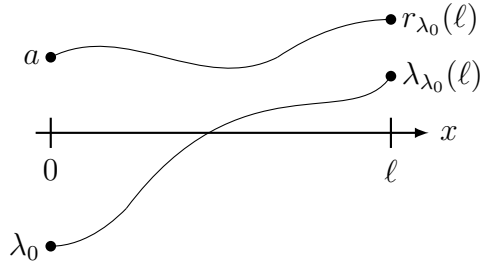


Problem: we can do this for any  $c$ . Which  $c$  is it? *Last 15 minutes was a dead end!*

Back to  $u = F(r, \lambda)$ . Assume we have  $F$  (numerically).

$$\begin{aligned}\frac{dr}{dx} &= -F(r, \lambda) \\ r(0) &= a \\ \frac{d\lambda}{dx} &= -\frac{F^3(r, \lambda)}{1 + F^2(r, \lambda)} \\ \lambda(\ell) &= r(\ell)\end{aligned}$$

The mistake before was that the simulation forward from  $a$  depends on  $\lambda$ .



Therefore, we should guess  $\lambda_0$  and simulate both  $r$  and  $\lambda$  to get  $r_{\lambda_0}(\ell)$  and  $\lambda_{\lambda_0}(\ell)$ . We need

$$r_{\lambda_0}(\ell) = \lambda_{\lambda_0}$$

for optimality. To do this, we need numerics.

### Terminal Constraints

Let  $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$  and solve

$$\begin{aligned}\min_{u \in \mathcal{U}} \quad & \int_0^T L(x, u, t) dt + \Psi(x(T)) \\ \text{s.t.} \quad & \dot{x} = f(x, u, t) \\ & x(0) = x_0 \\ & x_i(T) = x_{iT} \quad \text{given for } i \in \mathcal{T} \subset \{1, \dots, n\}\end{aligned}$$

First, we augment the cost:

$$\begin{aligned}
\tilde{J}(u) &= \int_0^T [L + \lambda^T(f - \dot{x})] dt + \Psi \\
&= \int_0^T (H - \lambda^T \dot{x}) dt + \Psi \\
\tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \left( \varepsilon \frac{\partial H}{\partial u} v + \varepsilon \frac{\partial H}{\partial x} \eta - \varepsilon \lambda^T \dot{\eta} \right) dt + \varepsilon \frac{\partial \Psi}{\partial x}(x(T)) \eta(T) + o(\varepsilon) \\
\delta \tilde{J}(u; v) &= \int_0^T \left( \frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \int_0^T \frac{\partial H}{\partial u} v dt \\
&\quad + \lambda^T(0) \eta(0) - \lambda^T(T) \eta(T) + \frac{\partial \Psi}{\partial x}(x(T)) \eta(T)
\end{aligned}$$

As always,

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial H^T}{\partial x} \\
\frac{\partial H}{\partial u} &= 0 \quad (\text{FONC})
\end{aligned}$$

Additionally,

$$\begin{aligned}
\eta(0) &= 0 \\
\eta_i(T) &= 0 \quad \text{for } i \in \mathcal{T}
\end{aligned}$$

Note that if  $x(T) = x_T$  is given, then  $x(T) = x(T) + \varepsilon \eta(T) + o(\varepsilon)$ , so  $\eta(T) = 0$ . Here, we have  $x_i(T) = x_{iT}$  fixed for  $i \in \mathcal{T}$  so  $\eta_i(T) = 0$  for  $i \in \mathcal{T}$ .

For optimality, we want

$$\begin{aligned}
\left[ -\lambda^T(T) + \frac{\partial \Psi}{\partial x}(x(T)) \right] \eta(T) &= 0 \quad \text{for all admissible variations} \\
\left[ \frac{\partial \Psi}{\partial x_1} - \lambda_1, \quad \dots, \quad \frac{\partial \Psi}{\partial x_n} - \lambda_n \right] \begin{bmatrix} \eta_1(T) \\ \vdots \\ \eta_n(T) \end{bmatrix} &= 0
\end{aligned}$$

Hence, we need

$$\begin{aligned}
\lambda_j(T) &= \frac{\partial \Psi}{\partial x_j}(x(T)) \quad \text{if } j \notin \mathcal{T} \\
\lambda_i(T) &= \text{free} \quad \text{if } i \in \mathcal{T}
\end{aligned}$$

So we have

$$\begin{cases} \dot{x} = f \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x}, \end{cases}$$

an ODE with  $2n$  variables. We need  $2n$  boundary conditions for this ODE to be well-posed.

At $t = 0$		At $t = T$	
$x(0) = x_0$	$[n]$	$x_i(T) = x_{iT}, i \in \mathcal{T}$	$[q]$
		$ \mathcal{T}  = q$	
		$x_j(T)$ free, $j \notin \mathcal{T}$	$[0]$
$\lambda(0)$ free	$[0]$	$\lambda_i(T)$ free, $i \in \mathcal{T}$	$[0]$
		$\lambda_j(T) = \frac{\partial \Psi}{\partial x_j}(x(T)), j \notin \mathcal{T}$	$[n - q]$

So we have  $n + q + (n - q) = 2n$  boundary conditions.

We could even fix some but not all of  $x(0)$ , i.e.

$$\begin{aligned} x_i(0) &= x_{i0} & \text{if } i \in \mathcal{I} \\ x_j(0) &= \text{free} & \text{if } j \notin \mathcal{I} \end{aligned}$$

Recall,

$$\delta \tilde{J}(u; v) = \int_0^T \left( \frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta \, dt + \int_0^T \frac{\partial H}{\partial u} v \, dt + \lambda^T(0) \eta(0) + \left[ \lambda^T(T) - \frac{\partial \Psi}{\partial x}(x(T)) \right] \eta(T)$$

For  $x_i(0) = x_{i0}$  fixed, we have  $\eta_i(0) = 0$  and  $\lambda_i(0)$  free. For  $x_j(0)$  free, we have  $\eta_j(0)$  free and  $\lambda_j(0) = 0$ .

To ponder, what if  $J = \int L \, dt + \Psi(x(T)) + \Theta(x(0))$ ?

To summarize, the minimizer to

$$\begin{aligned} \min_{u \in \mathcal{U}} & \int_0^T L(x, u, t) \, dt + \Psi(x(T)) \\ \text{s.t.} & \quad \dot{x} = f(x, u, t) \\ & \quad x_i(0) = x_{i0}, \quad i \in \mathcal{I} \\ & \quad x_j(T) = x_{jT} \quad j \in \mathcal{T} \end{aligned}$$

has to satisfy

$$\begin{aligned} \frac{\partial H}{\partial u} &= 0 \\ \dot{\lambda} &= -\frac{\partial H^T}{\partial x} \\ \lambda_i(0) &= 0, \quad i \notin \mathcal{I} \\ \lambda_j(T) &= \frac{\partial \Psi}{\partial x_j}(x(T)), \quad j \notin \mathcal{T} \end{aligned}$$

### Example

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = f(x_1, x_2, x_3, x_4)$$

$$x_1(0) = 1, x_3(0) = 7, x_4(0) = 0, x_1(1) = 2$$

$$\mathcal{I} = \{1, 3, 4\}, \mathcal{T} = \{1\}$$

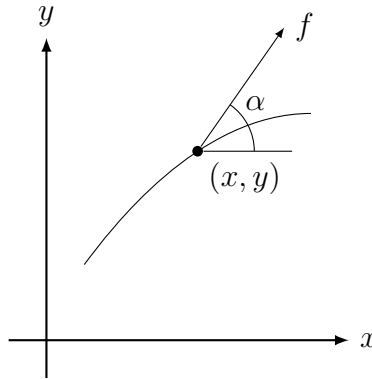
$$\min \int_0^1 L(x, u) dt + (x_2^2(1) - x_3^2(1) + 7x_1(1) + 14)$$

Note there are 4 boundary conditions on  $x$  so there must be 4 boundary conditions on  $\lambda$ :

$\lambda_1(0)$ free/unspecified	$\lambda_1(1)$ free
$\lambda_2(0) = 0$	$\lambda_2(1) = 2x_2(1)$
$\lambda_3(0)$ free	$\lambda_3(1) = -2x_3(1)$
$\lambda_4(0)$ free	$\lambda_4(1) = 0$

### Example

A force  $f$  acts on a particle at position  $(x, y)$  (mass = 1).



$$\dot{x} = v_x$$

$$\dot{y} = v_y$$

$$\dot{v}_x = |f| \cos \alpha$$

$$\dot{v}_y = |f| \sin \alpha$$

$$\alpha = \text{control variable}$$

Assume we only care about where the particle ends up (to be specified later), i.e.  $L = 0$ .

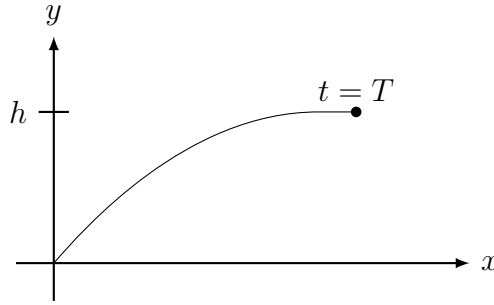
$$H = \begin{bmatrix} \lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f| \cos \alpha \\ |f| \sin \alpha \end{bmatrix}$$

$$\begin{aligned}
\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 &\implies \lambda_x(t) = c_1 \\
\dot{\lambda}_y = -\frac{\partial H}{\partial y} = 0 &\implies \lambda_y(t) = c_2 \\
\dot{\lambda}_{v_x} = -\frac{\partial H}{\partial v_x} = -\lambda_x &\implies \lambda_{v_x}(t) = -c_1 t + c_3 \\
\dot{\lambda}_{v_y} = -\frac{\partial H}{\partial v_y} = -\lambda_y &\implies \lambda_{v_y}(t) = -c_2 t + c_4
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{\partial H}{\partial \alpha} &= -\lambda_{v_x}|f|\sin \alpha + \lambda_{v_y}|f|\cos \alpha = 0 \\
\tan \alpha &= \frac{\lambda_{v_y}}{\lambda_{v_x}} = \frac{-c_2 t + c_4}{-c_1 t + c_3}
\end{aligned}$$

We want to drive the particle from  $[0, 0, 0, 0]^T$  to a path parallel to the x-axis with  $y(T) = h$ .



Choose  $\Psi = -v_x$ ,

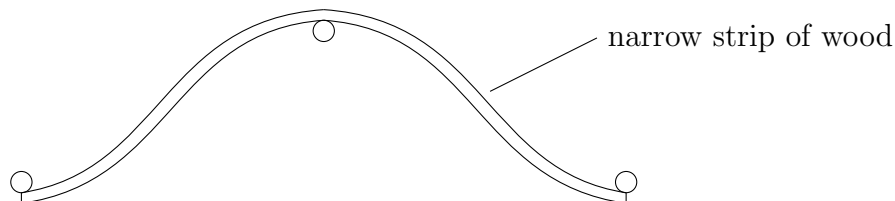
$$\begin{aligned}
y(T) &= h & v_y(T) &= 0 \\
x(T) &\text{ free} & v_x(T) &\text{ free, but costs} \\
\lambda_i(0) &\text{ free} \\
\lambda_y(T) &\text{ free} & \lambda_{v_y}(T) &\text{ free} \\
\lambda_x(T) &= 0 & \lambda_{v_x}(T) &= -1
\end{aligned}$$

$$\begin{aligned}
c_1 &= \lambda_x(t) = 0 \\
\implies \lambda_{v_x} &= -c_1 t + c_3 = c_3 = -1 \\
\implies \tan \alpha &= -\frac{-c_2 t + c_4}{-1} = c_2 t + c_4
\end{aligned}$$

How do we find  $c_2$  and  $c_4$ ? Plug into  $\dot{x}$  and  $\dot{\lambda}$  and try to satisfy the remaining boundary conditions. (This is hard=numerics.)

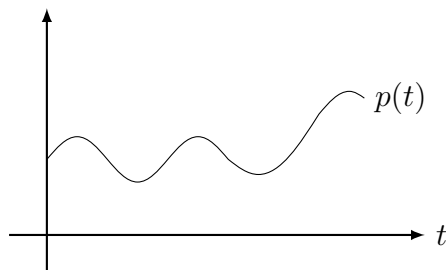
## 3.2 Splines

From ship building. Splines are used a lot in path-planning, e.g. cubic splines.



But, they are solutions to optimal control problems.

Let  $p(t)$  be a curve we'd like to shape.



We want to minimize the “energy” put into the curve, a.k.a acceleration. Let  $x_1 = p$  and  $x_2 = \dot{p}$ , so

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

### 3.2.1 Minimum-Energy

$$\begin{aligned} & \min_{u \in \mathcal{U}} \frac{1}{2} \int_0^T u^2(t) dt \quad + \text{Boundary conditions on } x \\ H &= L + \lambda^T f = \frac{1}{2} u^2 + \lambda_1 x_2 + \lambda_2 u \\ \frac{\partial H}{\partial u} &= u + \lambda_2 = 0 \implies u = -\lambda_2 \\ \dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = c_1 \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -c_1 t + c_2 \\ u &= -\lambda_2 = c_1 t - c_2 \\ \dot{x}_2 &= u = c_1 t - c_2 \implies x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3 \\ \dot{x}_1 &= x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3 \\ \implies x_1 &= \frac{c_1}{6} t^3 - \frac{c_2}{2} t^2 + c_3 t + c_4 \end{aligned}$$

$p(t)$  is a cubic polynomial!

What about boundary conditions?

Let  $T = 1$ ,  $p(0)$  given,  $p(1)$  given,  $\dot{p}(0) = 0$ ,  $\dot{p}(1) = 0$ , e.g.  $p(0) = 0$ ,  $p(1) = 1$ . Since the boundary conditions for  $x$  are all specified, those for the costate are free.

$$\left. \begin{array}{l} x_1(0) = 0 \\ x_2(0) = 0 \\ x_1(1) = 1 \\ x_2(1) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lambda_1(0) \\ \lambda_2(0) \\ \lambda_1(1) \\ \lambda_2(1) \end{array} \right. \text{ free/unspecified}$$

$$\begin{array}{ll} x_2(0) = c_3 = 0 & x_1(1) = \frac{2c_2}{6} - \frac{c_2}{2} = 1 \\ x_1(0) = c_4 = 0 & c_2 = -6 \\ x_2(1) = \frac{c_1}{2} - c_2 + \underbrace{c_3}_0 = 0 & c_1 = -12 \\ c_1 = 2c_2 & \end{array}$$

$$\begin{aligned} \Rightarrow p(t) &= -2t^3 + 3t^2 \\ u(t) &= -12t + 6 \end{aligned}$$

Or, what if  $\dot{p}(0)$ ,  $\dot{p}(1)$  are not specified?

$$\left. \begin{array}{l} x_1(0) = 0 \\ x_2(0) \text{ unspec.} \\ x_1(1) = 1 \\ x_2(1) \text{ unspec.} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \lambda_1(0) \text{ unspec.} \\ \lambda_2(0) = 0 \\ \lambda_1(1) \text{ unspec.} \\ \lambda_2(1) = 0 \end{array} \right.$$

$$\left. \begin{array}{l} \lambda_2(0) = c_2 = 0 \\ \lambda_2(1) = -c_1 + c_2 = 0 \end{array} \right\} \Rightarrow u = c_1 t - c_2 = 0$$

$$\left. \begin{array}{l} x_1(0) = c_4 = 0 \\ x_1(1) = c_3 = 1 \end{array} \right\} \Rightarrow p(t) = t$$

What did we do?

Case 1:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1/2 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_2(0) \\ x_2(1) \end{bmatrix}$$

Case 2:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ \lambda_2(0) \\ \lambda_2(1) \end{bmatrix}$$



### 3.2.2 Generalized Splines

We had  $\dot{x} = Ax + Bu$  with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This  $A$  is nilpotent ( $A^k = 0$  for some  $k \in \mathbb{Z}^+$ ). This means  $e^{At}$  is a polynomial in  $t$ . (This  $e^{At}$  is cubic.)

In general,  $e^{At}$  is a mix of polynomials, exponentials, and trigonometric terms. The eigenvalues of  $A$  determine the form of  $x(t)$ .

$$\begin{aligned} \dot{x} &= Ax &\implies x(t) &= e^{At}x(0) \\ \dot{x} &= Ax + Bu &\implies x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \end{aligned}$$

The general problem to solve is

$$\begin{aligned} \min_{u \in \mathcal{U}} \int_0^T \frac{1}{2} \|u\|^2 dt \\ \text{s.t. } \dot{x} &= Ax + Bu \\ &+ \text{Boundary conditions} \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2} \|u\|^2 + \lambda^T (Ax + Bu) \\ \frac{\partial H}{\partial u} &= u^T + \lambda^T B = 0 \\ &\Rightarrow u = -B^T \lambda \\ \dot{\lambda} &= -\frac{\partial H^T}{\partial x} = -A^T \lambda \end{aligned}$$

We have the Hamiltonian Dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BB^T \\ 0 & -A^T \end{bmatrix}}_M \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

Where we used  $\dot{x} = Ax + Bu = Ax - BB^T \lambda$ . Then,

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

Suppose we want to drive from  $x(0) = x_0$  to  $x(T) = x_T$ .

$$\begin{aligned} \begin{bmatrix} x_T \\ \lambda(T) \end{bmatrix} &= e^{MT} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix} = \begin{bmatrix} N_{xx} & N_{x\lambda} \\ N_{\lambda x} & N_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix} \\ x_T &= N_{xx}x_0 + N_{x\lambda}\lambda(0) \end{aligned}$$

$N_{x\lambda}$  is invertible if  $(A, B)$  is completely controllable. Assume it is.

$$\begin{aligned}\lambda(0) &= N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \\ \Rightarrow \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} &= e^{Mt} \begin{bmatrix} x_0 \\ N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \end{bmatrix} \\ \Rightarrow u(t) &= -B^T \lambda(t)\end{aligned}$$

This is the optimal trajectory, but there is no feedback. We will consider closed-loop systems after the midterm.

As a preview, we need to find  $\lambda$  as a function of  $x$ . For example,  $u = -R^{-1}B^T Px$  minimizes  $u^T Ru$ , so  $\lambda = Px$  where  $P$  is the solution to the Riccati equation.

### 3.3 Numerical Methods

Optimal control boils down to solving two sets of differential equations:

$$\begin{aligned}\dot{x} &= f(x, u) & \frac{\partial H}{\partial u}(x, u, \lambda) &= 0 \\ \dot{\lambda} &= -\frac{\partial H^T}{\partial x}(x, u, \lambda) & u &= F(x, \lambda) \\ \Rightarrow & \begin{cases} \dot{x} = f(x, F(x, \lambda)) \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x}(x, F(x, \lambda), \lambda) \end{cases}\end{aligned}$$

The equations are functions of  $x$  and  $\lambda$ . They are completely determined by the boundary conditions on  $x(0)$ ,  $x(T)$ ,  $\lambda(0)$ ,  $\lambda(T)$ . This is known as the *Boundary Value Problem*. This is solved using *test shooting*:

1. Guess initial conditions
2. Simulate forward in time
3. Update the guess (cleverly...)

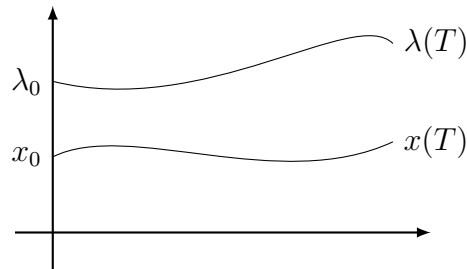
**Exmample:** Bolza problem

$$\begin{aligned}\min_{u \in \mathcal{U}} & \int_0^T L(x, u) dt + \Psi(x(T)) \\ \text{s.t.} & \begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases} \\ & H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u) \\ & u^*(x, \lambda) \text{ satisfies } \frac{\partial H}{\partial u} = 0\end{aligned}$$

The optimal control satisfies

$$\begin{cases} x = f(x, u^*(x, \lambda)) \\ x(0) = x_0 \\ \lambda = -\frac{\partial H^T}{\partial x}(x, u^*(x, \lambda), \lambda) \\ \lambda(T) = \frac{\partial \Psi}{\partial x}(x(T)) \end{cases}$$

**Algorithm** Guess  $\lambda_0$  and solve for  $x(t)$ ,  $\lambda(t)$ .



Let's define a cost:

$$\left\| \lambda(T) - \frac{\partial \Psi^T}{\partial x}(x(T)) \right\|^2 = g(\lambda_0)$$

Update  $\lambda_0$  through

$$\lambda_0 := \lambda_0 - \gamma \frac{\partial g^T}{\partial \lambda_0}(\lambda_0)$$

↑  
any choice of step size works

Repeat

Problem: What is  $\partial g / \partial \lambda_0$ ? We estimate  $\partial g / \partial \lambda_0$  numerically. This is where “test shooting” comes into play.

Let  $e_i$  be the  $i$ th unit vector,  $i = 1, \dots, n$ :

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\frac{\partial g}{\partial \lambda_0} = \left( \frac{\partial g}{\partial \lambda_{0,1}}, \frac{\partial g}{\partial \lambda_{0,2}}, \dots, \frac{\partial g}{\partial \lambda_{0,n}} \right)$$

The  $i$ th component of  $\partial g / \partial \lambda_0$  is given by the directional derivative

$$\frac{\partial g}{\partial \lambda_{0,i}} = \frac{\partial g}{\partial \lambda_0} \cdot e_i = \delta g(\lambda_0; e_i) = \lim_{\varepsilon \rightarrow 0} \frac{g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)}{\varepsilon}$$

So, if  $x \in \mathbb{R}^n$  (and thus so is  $\lambda_0$ ), we have to do this  $n$  times (with a small  $\varepsilon$ ) and get the full derivative  $\partial g / \partial \lambda_0$ .

---

```

Given  $\lambda_0, g(\lambda_0)$ 
for  $i = 1$  to  $n$  do
    Compute  $g(\lambda_0 + \varepsilon e_i)$ 
     $dg_i = \frac{1}{\varepsilon}[g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)]$ 
end for
 $\frac{\partial g}{\partial \lambda_0} = [dg_1, \dots, dg_n]$ 

```

---

## Algorithm

**Example** LQ

$$\begin{aligned}
 & \min_u \frac{1}{2} \int_0^1 (x^T Q x + u^T R u) dt + \frac{1}{2} x^T(1) S x(1) \\
 & \text{s.t.} \quad \begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases} \\
 & \quad Q, R, S \succ 0
 \end{aligned}$$

$$\begin{aligned}
 H &= \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \lambda^T (Ax + Bu) \\
 \frac{\partial H}{\partial u} &= u^T R + \lambda^T B = 0 \\
 u^* &= -R^{-1} B^T \lambda \\
 \dot{\lambda} &= -\frac{\partial H^T}{\partial x} = -Qx - A^T \lambda \\
 \lambda(1) &= \frac{\partial \Psi^T}{\partial x}(x(1)) = Sx(1)
 \end{aligned}$$

So putting it all together,

$$\begin{aligned}
 \dot{x} &= Ax - BR^{-1}B^T \lambda & x(0) &= x_0 \\
 \dot{\lambda} &= -Qx - A^T \lambda & \lambda(1) &= Sx(1)
 \end{aligned}$$

**Example** Newton's nose shape problem (revisited, see previous)

$$\begin{aligned}
 & \min_u \int_0^\ell \frac{ru^3}{1+u^2} dx + \frac{1}{2} r(\ell)^2 \\
 & \text{s.t.} \quad \frac{dr}{dx} = -u \quad r(0) = a
 \end{aligned}$$

$$H = \frac{ru^3}{1+u^2} + \lambda(-u)$$

$$\frac{\partial H}{\partial u} = \frac{ru^2(3+u^2)}{(1+u^2)^2} - \lambda = 0$$

We solve the above numerically to get  $u^*(r, \lambda)$ .

$$\frac{\partial \lambda}{\partial x} = -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2}$$

$$\lambda(\ell) = r(\ell)$$

So, we have

$$\frac{dr}{dx} = -u \quad r(0) = a \quad u = F(x, \lambda)$$

$$\frac{d\lambda}{dx} = -\frac{u^3}{1+u^2} \quad \lambda(\ell) = r(\ell)$$

**Example** Fixed terminal constraints (revisited, see previous)

$$\min_{\alpha} -v_x(T) \quad \alpha = \text{control}$$

$$\text{s.t.} \quad \begin{aligned} \dot{x} &= v_x & x(0) &= 0 \\ \dot{y} &= v_y & y(0) &= 0 \\ \dot{v}_x &= |f| \cos \alpha & v_x(0) &= 0 \\ \dot{v}_y &= |f| \sin \alpha & v_y(0) &= 0 \\ y(T) &= h \\ v_y(T) &= 0 \end{aligned}$$

$$H = -v_x(T) + \begin{bmatrix} \lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f| \cos \alpha \\ |f| \sin \alpha \end{bmatrix}$$

$$\frac{dH}{d\alpha} = 0 \Rightarrow \tan \alpha = \frac{\lambda_{v_y}}{\lambda_{v_x}}$$

$$\dot{\lambda}_x = 0$$

$$\dot{\lambda}_y = 0$$

$$\dot{\lambda}_{v_x} = -\lambda_x$$

$$\dot{\lambda}_{v_y} = -\lambda_y$$

$$\boldsymbol{\lambda}(0) \text{ unspecified}$$

$$\lambda_x(T) = \frac{\partial \Psi^T}{\partial x}(x(T)) = 0$$

$$\lambda_y(T) \text{ unspecified}$$

$$\lambda_{v_x}(T) = \frac{\partial \Psi^T}{\partial v_x}(v_x(T)) = -1$$

$$\lambda_{v_y}(T) \text{ unspecified}$$

Again, we guess  $\lambda_0$  and solve forward in time. But, we have terminal constraints on  $y$  and  $v_y$  as well.

$$g(\lambda_0) = \frac{1}{2} \left[ (y(T) - h)^2 + (v_y(T))^2 + (\lambda_x(T))^2 + (\lambda_{v_x} + 1)^2 \right]$$

### 3.4 Terminal Manifolds

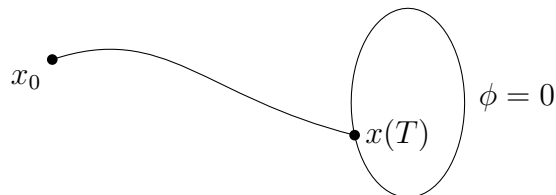
We can solve

$$\begin{aligned} \min_{u \in \mathcal{U}} \int_0^T L(x, u, t) dt + \Psi(x(T)) \\ \text{s.t. } \dot{x} = f(x, u, t) \end{aligned}$$

with all sorts of boundary conditions on  $x$ :

- $x(0) = x_0$ ,  $x(T)$  free (typical)
- $x_i(0) = x_{i0}$ ,  $i \in \mathcal{I}$  and  $x_j(T) = x_{jT}$ ,  $j \in \mathcal{T}$

But what if we want  $x(T)$  to belong to a set?



## Problem

$$\begin{aligned}
& \min_{u \in \mathcal{U}} \int_0^T L(x, u, t) dt + \Psi(x(T)) \\
& \text{s.t. } \dot{x} = f(x, u, t), \quad x \in \mathbb{R}^n \\
& \quad x(0) = x_0 \\
& \quad \phi(x(T)) = 0, \quad \phi : \mathbb{R}^n \rightarrow \mathbb{R}^q, \quad q \leq n
\end{aligned}$$

The augmented cost is

$$\tilde{J} = \int_0^T [H(x, u, t, \lambda) - \lambda^T \dot{x}] dt + \Psi(x(T)) + \underbrace{\nu^T \phi(x(T))}_{\substack{\text{q-dimensional} \\ \text{Lagrange multiplier}}}$$

Let  $\Phi(x(T), \nu) = \Psi(x(T)) + \nu^T \phi(x(T))$ . Then,

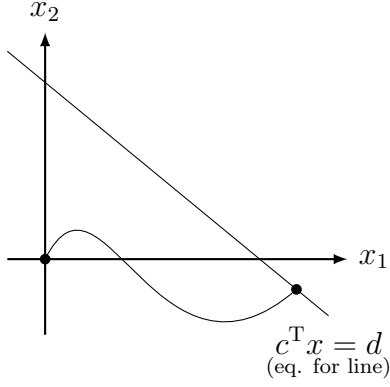
$$\tilde{J} = \int_0^T \left( \frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \Phi(x(T), \nu)$$

We know how to solve this! With  $u \mapsto u + \varepsilon v$ ,  $x \mapsto x + \varepsilon \eta + o(\varepsilon)$ ,

$$\begin{aligned}
\delta \tilde{J} &= \int_0^T \left( \frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \int_0^T \frac{\partial H}{\partial u} v dt + \underbrace{\frac{\partial \Phi}{\partial x}(x(T), \nu) \eta(T)}_{=0} \\
&\quad - \lambda^T(T) \eta(T) + \lambda^T(0) \eta(0) \\
&\left\{ \begin{array}{l} \frac{\partial H}{\partial u} = 0 \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = \frac{\partial \Phi^T}{\partial x}(x(T), \nu) \\ \phi(x(T)) = 0 \end{array} \right. \quad \begin{array}{l} q \text{ new variables} \\ \downarrow \\ \leftarrow q \text{ new equations} \end{array} \\
&\implies u^*
\end{aligned}$$

## Spring to line

$$\begin{aligned}
& \min_u \frac{1}{2} \int_0^1 u^2(t) dt \\
& \text{s.t. } \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ x_1(0) = 0, \quad x_2(0) = 0 \\ c_1 x_1(1) + c_2 x_2(1) = d \end{cases}
\end{aligned}$$



$$H = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u$$

$$\frac{\partial H}{\partial u} = u + \lambda_2 \implies u = -\lambda_2$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = k_1$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -k_1 t + k_2$$

$$\phi(x(1)) = c_1 x_1(1) + c_2 x_2(1) - d$$

$$\Psi = 0 \implies \Phi = \nu(c_1 x_1(1) + c_2 x_2(1) - d)$$

$$\lambda_1(1) = \frac{\partial \Phi}{\partial x_1} = \nu c_1$$

$$\lambda_2(1) = \frac{\partial \Phi}{\partial x_2} = \nu c_2$$

So,

$$\lambda_1(1) = \nu c_1 = k_1$$

$$\lambda_2(1) = \nu c_2 = -k_1 + k_2$$

$$k_2 = \nu(c_1 + c_2)$$

$$\dot{x}_2 = u = -\lambda_2 = k_1 t - k_2$$

$$x_2 = \frac{k_1}{2}t^2 - k_2 t + 0$$

$$\dot{x}_1 = x_2$$

$$x_1 = \frac{k_1}{6}t^3 - \frac{k_2}{2}t^2 + 0$$

Substituting  $k_1$  and  $k_2$  into  $c_1 x_1(1) + c_2 x_2(1) = d$ ,

$$\nu \left( -\frac{c_1^2}{3} - c_1 c_2 - c_2^2 \right) = d$$

$$\nu = -\frac{d}{c_1^2/3 + c_1 c_2 + c_2^2}$$



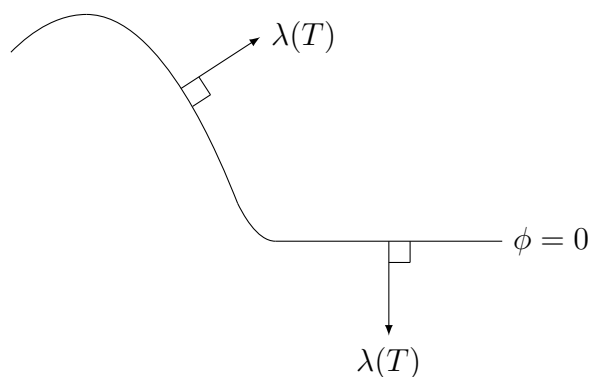
And finally

$$u = k_1 t - k_2 = \frac{d}{c_1^2/3 + c_1 c_2 + c_2^2} (c_1 + c_2 - c_1 t)$$

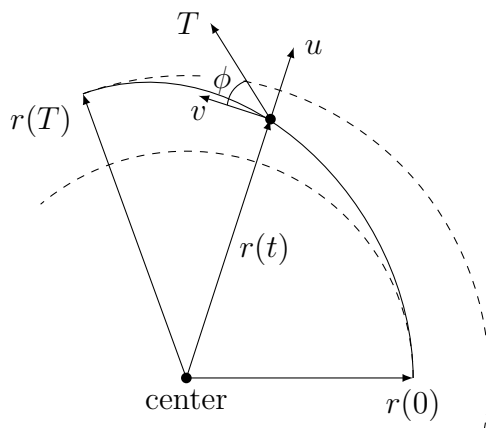
As a final observation if  $\Psi = 0$  then

$$\lambda(T) = \nu^T \frac{\partial \phi}{\partial x}(x(T)),$$

which means  $\lambda(T)$  is orthogonal to the tangent plane to  $\phi(x(T))$ .



**Example** Maximum orbit transform (e.g. Hidden Figures)



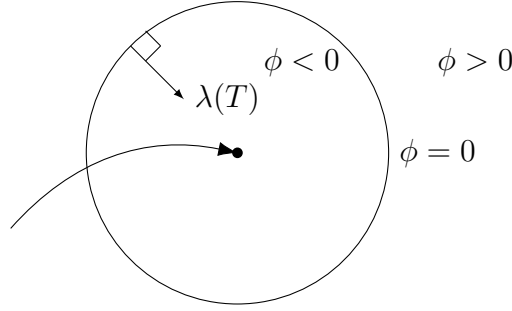
- $r$  = radial distance from spacecraft to center
- $u$  = radial velocity
- $v$  = tangential velocity
- $m$  = mass of spacecraft
- $\dot{m}$  =  $-$ fuel consumption rate
- $\phi$  = thrust angle (control input)
- $T$  = thrust

$$\begin{aligned}
& \max_{\phi} r(T) \iff \min_{\phi} -r(T) \\
& \text{s.t.} \quad \begin{cases} \dot{r} = u \\ \dot{u} = \frac{v^2}{r} - \frac{g}{r^2} + \frac{T \sin \phi}{m_0 - |\dot{m}|t} \\ \dot{v} = -\frac{uv}{r} + \frac{T \cos \phi}{m_0 - |\dot{m}|t} \\ r(0) = r_0 \\ u(0) = 0 \\ v(0) = \sqrt{\frac{g}{r_0}} \\ u(T) = 0 = \phi_1 \\ v(T) = \sqrt{\frac{g}{r(T)}} = \phi_2 \end{cases} \\
\\
& H = \lambda_r u + \lambda_u \left( \frac{v^2}{r} - \frac{g}{r^2} + \frac{T \sin \phi}{m_0 - |\dot{m}|t} \right) + \lambda_v \left( -\frac{uv}{r} + \frac{T \cos \phi}{m_0 - |\dot{m}|t} \right) \\
& \Phi = \underbrace{\nu_1 u(T) + \nu_2 \left( v(T) - \sqrt{\frac{g}{r(T)}} \right)}_{\nu^T \phi} \underbrace{-r(T)}_{\Psi} \\
\\
& \frac{\partial H}{\partial \phi} = \frac{\lambda_u T \cos \phi - \lambda_v T \sin \phi}{m_0 - |\dot{m}|t} = 0 \\
& \Rightarrow \tan \phi = \frac{\lambda_u}{\lambda_v} \\
\\
& \dot{\lambda}_r = -\frac{\partial H}{\partial r} = -\lambda_u \left( -\frac{v^2}{r^2} + \frac{2g}{r^3} \right) - \lambda_v \cdot \frac{uv}{r^2} \\
& \dot{\lambda}_u = -\frac{\partial H}{\partial u} = -\lambda_r + \lambda_v \cdot \frac{v}{r} \\
& \dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_u \cdot \frac{2v}{r} + \lambda_v \cdot \frac{u}{r} \\
\\
& \begin{cases} \lambda_r(T) = \frac{\partial \Phi}{\partial r} = -1 + \frac{\nu_2 \sqrt{g}}{2(r(T))^{3/2}} \\ \lambda_u(T) = \frac{\partial \Phi}{\partial u} = \nu_1 \\ \lambda_v(T) = \frac{\partial \Phi}{\partial v} = \nu_2 \\ u(T) = 0 \\ v(T) = \sqrt{\frac{g}{r(T)}} \end{cases}
\end{aligned}$$

This needs numerics to solve.

### 3.4.1 Terminal manifold with inequality constraints

$$\begin{aligned} \min_u \int_0^T L \, dt + \Psi \\ \dot{x} &= f(x, u) \\ \phi(x(T)) &\leq 0 \end{aligned}$$



Repeat process:  $\tilde{J} = \int (H - \lambda^T \dot{x}) \, dt + \Psi + \nu^T \phi$ . The optimality conditions are

$$\begin{cases} \frac{\partial H}{\partial u} = 0 \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T)) + \nu^T \frac{\partial \phi^T}{\partial x}(x(T)) \\ \nu \geq 0 \\ \phi(x(T)) \leq 0 \\ \nu^T \phi(x(T)) = 0 \quad (\text{KKT}) \end{cases}$$

### 3.4.2 Initial manifold

$$\begin{aligned} \min_{x_0, u} \int L + \Psi(x(T)) + \Theta(x(0)) \\ \text{s.t. } \dot{x} &= f(x, u) \\ \phi(x(T)) &= 0 \\ \xi(x(0)) &= 0 \end{aligned}$$

$$\begin{aligned} \tilde{J} &= \int (H - \lambda^T \dot{x}) \, dt + \Psi(x(T)) + \Theta(x(0)) + \nu_\phi^T \phi(x(T)) + \nu_\xi^T \xi(x(0)) \\ \delta \tilde{J} &= \int \left[ \left( \frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta + \frac{\partial H}{\partial u} v \right] \, dt + \left[ \frac{\partial \Psi}{\partial x}(x(T)) + \nu_\phi^T \frac{\partial \phi}{\partial x}(x(T)) - \lambda^T(T) \right] \eta(T) \\ &\quad + \left[ \frac{\partial \Theta}{\partial x}(x(0)) + \nu_\xi^T \frac{\partial \xi}{\partial x}(x(0)) + \lambda^T(0) \right] \eta(0) \end{aligned}$$

The optimality conditions are

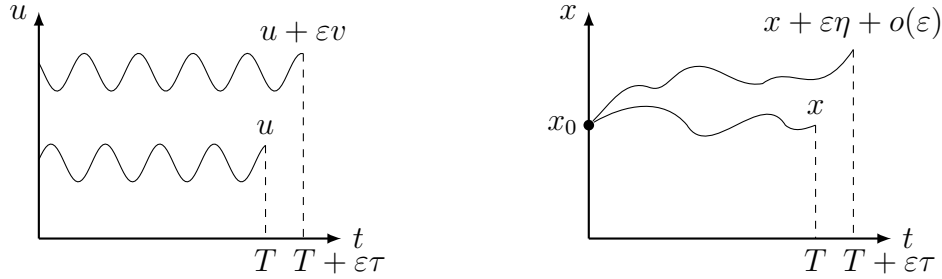
$$\begin{cases} \frac{\partial H}{\partial u} = 0 \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = -\frac{\partial \Psi^T}{\partial x}(x(T)) - \nu_\phi^T \frac{\partial \phi^T}{\partial x}(x(T)) \\ \lambda(0) = -\frac{\partial \Theta^T}{\partial x}(x(0)) - \nu_\xi^T \frac{\partial \xi^T}{\partial x}(x(0)) \end{cases}$$

### 3.4.3 Unspecified Terminal Times

For example, instead of driving to the moon using minimum fuel, we want to get there as soon as possible:

$$\min_{u, T} \int_0^T L(x, u, t) dt + \Psi(x(T), T).$$

The variations are  $u \mapsto u + \varepsilon v$  and  $T \mapsto T + \varepsilon \tau$



$$\begin{aligned} \tilde{J}(u, T) &= \int_0^T [L(x, u, t) + \lambda^T(f(x) - \dot{x})] dt + \Psi(x(T), T) \\ &= \int_0^T [H - \lambda^T \dot{x}] dt + \Psi \\ \tilde{J}(u + \varepsilon v, T + \varepsilon \tau) &= \int_0^T [H(x + \varepsilon \eta, u + \varepsilon v, t, \lambda) - \lambda^T(\dot{x} + \varepsilon \dot{\eta})] dt \\ &\quad + \int_T^{T + \varepsilon \tau} [H(x + \varepsilon \eta, u + \varepsilon v, t, \lambda) - \lambda^T(\dot{x} + \varepsilon \dot{\eta})] dt \\ &\quad + \Psi(x(T + \varepsilon \tau) + \varepsilon \eta(T + \varepsilon \tau), T + \varepsilon \tau) \end{aligned}$$

$$\begin{aligned}
\tilde{J}(u + \varepsilon v, T + \varepsilon \tau) - \tilde{J}(u, T) &= \varepsilon \int_0^T \left( \frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta \, dt + \varepsilon \int_0^T \frac{\partial H}{\partial u} v \, dt \\
&\quad - \varepsilon \lambda^T(T) \eta(T) + \varepsilon \lambda^T(0) \eta(0) + o(\varepsilon) \\
&\quad + \underbrace{\int_T^{T+\varepsilon \tau} [H(x + \varepsilon \eta, u + \varepsilon v, t, \lambda) - \lambda^T(\dot{x} + \varepsilon \dot{\eta})] \, dt}_{\text{(I)}} \\
&\quad + \underbrace{\Psi(x(T + \varepsilon \tau) + \varepsilon \eta(T + \varepsilon \tau), T + \varepsilon \tau) - \Psi(x(T), T)}_{\text{(II)}}
\end{aligned} \tag{3.4}$$

For term I, use the mean value theorem to get rid of terms inside the integral that have a  $\varepsilon$  before them:

$$\begin{aligned}
&\int_T^{T+\varepsilon \tau} [L + \lambda^T(f - \dot{x} - \varepsilon \dot{\eta})] \, dt \\
&= \int_T^{T+\varepsilon \tau} \left[ L(x, u, t) + \varepsilon \frac{\partial L}{\partial x} \eta + \varepsilon \frac{\partial L}{\partial u} v + \lambda^T \left( f + \varepsilon \frac{\partial f}{\partial x} \eta + \varepsilon \frac{\partial f}{\partial u} v - \dot{x} - \varepsilon \dot{\eta} \right) \right] \, dt + o(\varepsilon) \\
&= \varepsilon \tau [L + \lambda^T(f - \dot{x})] \Big|_{t=\xi} + o(\varepsilon) = \varepsilon \tau L \Big|_{t=\xi} + o(\varepsilon) \\
&= \varepsilon \tau L(x(\xi), u(\xi), \xi) + o(\varepsilon), \quad \xi \in [T, T + \varepsilon \xi]
\end{aligned} \tag{3.5}$$

Note that as  $\varepsilon \rightarrow 0$ ,  $\xi \rightarrow T$ .

For term II, we further split it into two parts:

$$\Psi(x + \varepsilon \eta, T + \varepsilon \tau) - \Psi(x, T) = \underbrace{\Psi(x, T + \varepsilon \tau)}_{\text{(II.a)}} + \underbrace{\varepsilon \frac{\partial \Psi}{\partial x}(x, T + \varepsilon \tau) \eta(T + \varepsilon \tau) - \Psi(x, T)}_{\text{(II.b)}}$$

$$\begin{aligned}
\text{(II.a)} \implies \Psi(x, T + \varepsilon \tau) &= \Psi(x(T), T + \varepsilon \tau) + \varepsilon \frac{\partial \Psi}{\partial x}(x(T), T + \varepsilon \tau) \dot{x}(T) \tau + o(\varepsilon) \\
&= \Psi(x(T), T) + \varepsilon \frac{\partial \Psi}{\partial x}(x(T), T) \dot{x}(T) \tau + \varepsilon \frac{\partial \Psi}{\partial T}(x(T), T) \tau + o(\varepsilon)
\end{aligned}$$

$$\begin{aligned}
\text{(II.b)} \implies \varepsilon \frac{\partial \Psi}{\partial x}(x, T + \varepsilon \tau) \eta(T + \varepsilon \tau) &= \varepsilon \left[ \frac{\partial \Psi}{\partial x}(x(T), T) + \varepsilon \frac{\partial^2 \Psi}{\partial x^2} \dot{x} \tau + \varepsilon \frac{\partial^2 \Psi}{\partial T \partial x} \tau + o(\varepsilon) \right] \\
&\quad \times [\eta(T) + \varepsilon \dot{\eta}(T) \tau + o(\varepsilon)] \\
&= \varepsilon \frac{\partial \Psi}{\partial x}(x(T), T) \eta(T) + o(\varepsilon) \\
\text{(II)} \implies \Psi(x + \varepsilon \eta, T + \varepsilon \tau) - \Psi(x, T) &= \varepsilon \frac{\partial \Psi}{\partial x}(x(T), T) [\dot{x}(T) \tau + \eta(T)] + \varepsilon \frac{\partial \Psi}{\partial T}(x(T), T) \tau + o(\varepsilon) \tag{3.6}
\end{aligned}$$

Substituting (3.5) and (3.6) into (3.4) and taking the directional derivative,

$$\begin{aligned}\delta\tilde{J} = & \int_0^T \left( \frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta \, dt + \int_0^T \frac{\partial H}{\partial u} v \, dt + \lambda^T(0)\eta(0) \\ & + \left[ L + \frac{\partial \Psi}{\partial T} + \frac{\partial \Psi}{\partial x} f \right] \tau \Big|_{t=T} + \left( \frac{\partial \Psi}{\partial x} - \lambda^T \right) \eta \Big|_{t=T}\end{aligned}$$

So we have a mix of old and new:

$$\begin{aligned}\text{old: } & \frac{\partial H}{\partial u} = 0 \\ & \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ & \lambda(T) = \frac{\partial \Psi}{\partial x} \Big|_T \\ \text{new: } & L + \frac{\partial \Psi}{\partial T} + \lambda^T f \Big|_T = 0\end{aligned}$$

This last condition is known as the *Transversality condition*.

**Example** Pure minimum time question

$$\begin{aligned}\min_{u,T} & \int_0^T dt \\ & \dot{x} = f(x, u) \\ & x(0) = x_0 \\ & x(T) = x_T \\ & H = L + \lambda^T f = 1 + \lambda^T f\end{aligned}$$

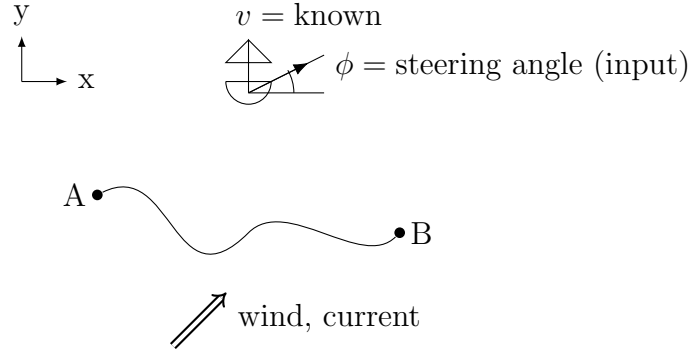
The transversality condition is

$$\begin{aligned}L + \frac{\partial \Psi}{\partial T} + \lambda^T f \Big|_T &= 0 \\ \lambda^T f \Big|_T &= -1 \\ H(T) = 1 + \lambda^T f \Big|_T &= 1 - 1 = 0\end{aligned}$$

But this is a conservative system, so  $H$  is a constant. Therefore,

$$H(t) = 0 \quad \forall t \in [0, T]$$

**Example** Zermelo's problem: sail from A to B as quickly as possible in the presence of known winds and currents.



The dynamics are

$$\begin{aligned} \dot{x} &= v \cos \phi + c_1(x, y) \\ \dot{y} &= v \sin \phi + c_2(x, y) \end{aligned} \quad \lambda = \begin{bmatrix} \lambda_x \\ \lambda_y \end{bmatrix}$$

For minimum time,  $L = 1$ .

$$H = 1 + \lambda_x(v \cos \phi + c_1) + \lambda_y(v \sin \phi + c_2)$$

$$0 = \frac{\partial H}{\partial \phi} = -v \lambda_x \sin \phi + v \lambda_y \cos \phi$$

$$\phi = \tan^{-1} \left( \frac{\lambda_y}{\lambda_x} \right)$$

Since this is a conservative system and  $\partial \Psi / \partial T = 0$ , then  $H(t) = H(T) = 0$ .

$$-1 = \lambda_x(v \cos \phi + c_1) + \lambda_y(v \sin \phi + c_2)$$

$$\lambda_x = -\frac{\cos \phi}{v + c_1 \cos \phi + c_2 \sin \phi}$$

$$\lambda_y = -\frac{\sin \phi}{v + c_1 \cos \phi + c_2 \sin \phi}$$

$$\dot{\lambda} = -\frac{\partial H^T}{\partial x}$$

$$\dot{\lambda}_x = -\lambda_x \frac{\partial c_1}{\partial x} - \lambda_y \frac{\partial c_2}{\partial x}$$

$$\dot{\lambda}_y = -\lambda_x \frac{\partial c_1}{\partial y} - \lambda_y \frac{\partial c_2}{\partial y}$$

$$\dot{\phi} = \sin^2 \phi \frac{\partial c_2}{\partial x} + \sin \phi \cos \phi \left( \frac{\partial c_1}{\partial x} - \frac{\partial c_2}{\partial y} \right) - \cos^2 \phi \frac{\partial c_1}{\partial y}$$

This is an ODE that completely determines  $\phi$  if we just had  $\phi_0$ .

**Example** We want to drive a car and stop at a stop sign as quickly as possible. Assume that the stop sign is at the origin, and our control is the acceleration ( $\ddot{x} = u$ ).

$$\begin{aligned} \min_{u,T} \quad & \int_0^T dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2, & x(0) = x_0 \\ \dot{x}_2 = u, & x(T) = 0 \end{cases} \end{aligned}$$

Recall the transversality condition:

$$H + \frac{\partial \Psi}{\partial T} \Big|_{t=T} = 0.$$

For minimum-time problems,  $L = 1$  and  $\Psi = 0$ , so  $\lambda^T f|_{t=T} = -1$ .

$$\begin{aligned} H &= 1 + \lambda_1 x_2 + \lambda_2 u \\ \lambda_1(T) \underbrace{x_2(T)}_{=0 \text{ (rest)}} + \lambda_2(T) u(T) &= -1 \\ \boxed{\lambda_2(T) u(T) = -1} \\ \frac{\partial H}{\partial u} &= \boxed{\lambda_2 = 0}, \end{aligned}$$

i.e.  $0 \cdot u(T) = -1$ ? This problem is ill-posed; we need to go infinitely fast...

**Idea 1:** Constrain  $u$ . We don't know how to do this.

**Idea 2:** Pay for gas. This is a design choice.

For the second idea,

$$\begin{aligned} \min_{u,T} \quad & \int_0^T \frac{1}{2} u^2(t) dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2, & x(0) = x_0 \\ \dot{x}_2 = u, & x(T) = 0 \end{cases} \\ H &= \frac{1}{2} u^2 + \lambda_1 x_2 + \lambda_2 u \\ \frac{1}{2} u^2(T) + \lambda_1(T) x_2(T) + \lambda_2(T) u(T) &= 0 \\ \boxed{\frac{1}{2} u^2(T) + \lambda_2(T) u(T) = 0} \\ \frac{\partial H}{\partial u} = u + \lambda_2 = 0 &\implies u = -\lambda_2 \\ \frac{1}{2} \lambda_2^2(T) - \lambda_2^2(T) &= 0 \\ \boxed{\lambda_2(T) = 0} \end{aligned}$$



$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = c \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -ct + d \\
\lambda_2(T) &= -cT + d = 0 \implies T = \frac{d}{c} \\
\dot{x}_2 &= u = -\lambda_2 = ct - d \\
x_2 &= c\frac{t^2}{2} - dt + x_{2,0} \\
\dot{x}_1 &= x_2 \implies x_1 = c\frac{t^3}{6} - d\frac{t^2}{2} + x_{2,0}t + x_{1,0} \\
\begin{cases} x_1(T) = c\frac{T^3}{6} - d\frac{T^2}{2} + x_{2,0}T + x_{1,0} = 0 \\ x_2(T) = c\frac{T^2}{2} - dT + x_{2,0} = 0 \\ T = \frac{d}{c} \end{cases} \\
\implies \begin{cases} c = -\frac{2x_{2,0}^2}{3x_{1,0}} \\ d = \sqrt{-\frac{4x_{2,0}^3}{3x_{1,0}}} \\ T = \frac{d}{c} \\ u = ct - d \end{cases}
\end{aligned}$$

Fine, but we really want to get there as quickly as possible! We have to constrain  $u$ , e.g.  $u(t) \in [-1, 1]$ ,  $\forall t \in [0, T]$ . How do we deal with the constraints on  $u$ ?

### 3.5 Hamilton's Minor "Mistake"

$$\begin{aligned}
&\min_{u \in \mathcal{U}_{\text{constr.}}} \int_0^T L(x, u, t) dt + \Psi(x(T)) \\
&\text{s.t. } \dot{x} = f(x, u, t) \\
&\quad x(0) = x_0 \\
&\quad (u(t) \in U)
\end{aligned}$$

Augment the cost:

$$\tilde{J}(u) = \int_0^T \left( H(x, u, t, \lambda) - \lambda^\top \dot{x} \right) dt + \Psi(x(T))$$

Vary  $u \mapsto u + \varepsilon v$  s.t.  $u + \varepsilon v \in \mathcal{U}_{\text{constr.}} \Rightarrow x \mapsto x + \varepsilon \eta + o(\varepsilon)$ :

$$\tilde{J}(u + \varepsilon v) = \int_0^T \left( H(x + \varepsilon \eta, u + \varepsilon v, t, \lambda) - \lambda^T \dot{x} - \lambda^T \varepsilon \dot{\eta} \right) dt + \Psi(x(T) + \varepsilon \eta(T)) + o(\varepsilon)$$

Instead of computing  $\delta \tilde{J}(u; v)$ , let's check  $\Delta \tilde{J} = \tilde{J}(u + \varepsilon v) - \tilde{J}(u)$ . If  $\Delta \tilde{J} \geq 0 \forall v$  s.t.  $u + \varepsilon v \in \mathcal{U}_{\text{constr.}}$  for  $\varepsilon$  small enough, then  $u$  is a local minimum!

$$\begin{aligned} \Delta \tilde{J} &= \int_0^T \left[ H(x + \varepsilon \eta, u + \varepsilon v, t, \lambda) - H(x, u, t, \lambda) - \lambda^T (\dot{x} + \varepsilon \dot{\eta} - \dot{x}) \right] dt \\ &\quad + \Psi(x(T) + \varepsilon \eta(T)) - \Psi(x(T)) + o(\varepsilon) \end{aligned}$$

Only Taylor expanding w.r.t.  $x$ :

$$\begin{aligned} \Delta \tilde{J} &= \int_0^T \left[ \varepsilon \frac{\partial H}{\partial x}(x, u, t, \lambda) \eta - \varepsilon \lambda^T \dot{\eta} \right] dt + \int_0^T [H(x, u + \varepsilon v, t, \lambda) - H(x, u, t, \lambda)] dt \\ &\quad + \varepsilon \frac{\partial \Psi}{\partial x}(x(T)) \eta(T) + o(\varepsilon) \\ &= \varepsilon \int_0^T \left( \frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \varepsilon \lambda^T(0) \eta(0) - \varepsilon \lambda^T(T) \eta(T) + \varepsilon \frac{\partial \Psi}{\partial x}(x(T)) \eta(T) \\ &\quad + \int_0^T [H(x, u + \varepsilon v, t, \lambda) - H(x, u, t, \lambda)] dt + o(\varepsilon) \end{aligned}$$

With  $\dot{\lambda} = -\partial H^T / \partial x$  and  $\lambda(T) = \partial \Psi(x(T)) / \partial x$ ,

$$\Delta \tilde{J} = \int_0^T [H(x, u + \varepsilon v, t, \lambda) - H(x, u, t, \lambda)] dt + o(\varepsilon)$$

Here, Hamilton did Taylor's expansion and set  $\partial H / \partial u = 0$ . Instead, Pontryagin desired  $\Delta \tilde{J} \geq 0 \forall v | u + \varepsilon v \in \mathcal{U}_{\text{constr.}}, \varepsilon$  small enough, i.e. we need

$$H(x, u^* + \varepsilon v, t, \lambda) \geq H(x, u^*, t, \lambda)$$

$\forall t \in [0, t], \forall v | u + \varepsilon v \in \mathcal{U}_{\text{constr.}}, \varepsilon$  small enough. That is, we need

$$u^* = \arg \min_u H(x, u, t, \lambda)$$

In summary,

$$\text{Hamilton: } \frac{\partial H}{\partial u} = 0$$

$$\text{Pontryagin: } \min_u H$$

**Theorem** (Pontryagin's Maximum Principle (PMP)). *Consider the problem:*

$$\begin{aligned} \min_{u, T} \quad & \int_0^T L(x, u, t) dt + \Psi(x(T), T) \\ \text{s.t.} \quad & \dot{x} = f(x, u, t) \\ & u(t) \in U(x, t), \quad \forall t \in [0, T] \\ & x_i(0) = x_{i0}, \quad i \in \mathcal{I} \\ & x_j(T) = x_{jT}, \quad j \in \mathcal{T} \end{aligned}$$

*The necessary condition for optimality is*

$$\begin{aligned}
H &= L + \lambda^T f \\
\dot{\lambda} &= -\frac{\partial H^T}{\partial x} \\
\lambda_j(0) &= 0, \quad j \notin \mathcal{I} \\
\lambda_i(T) &= \frac{\partial \Psi}{\partial x_i}(x(T)), \quad i \notin \mathcal{T} \\
H + \frac{\partial \Psi}{\partial T} \Big|_{t=T} &= 0 \\
u^*(x, t, \lambda) &= \arg \min_{u \in U(x, t)} H(x, u, t, \lambda)
\end{aligned}$$

We have two paths to solve optimality problems: we always start with the Hamiltonian, find the costate dynamics and boundary conditions, and apply the transversality condition; then, we can either apply calculus of variations (COV) or Pontryagin's Maximum Principle (PMP). COV only works for unconstrained problems, while with PMP we can deal with constraints.

## 3.6 Bang-Bang Control

Return to the car problem:

$$\begin{aligned}
&\min_{u, T} \int_0^T dt \\
&\text{s.t.} \quad \begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{1,0}, & x_1(T) = 0 \\ \dot{x}_2 = u, & x_2(0) = x_{2,0}, & x_2(T) = 0 \\ u(t) \in [-1, 1] & \forall t \in [0, T] \end{cases}
\end{aligned}$$

How do we minimize  $H$  w.r.t.  $u$ ?

$$H = 1 + \lambda_1 x_2 + \lambda_2 u$$

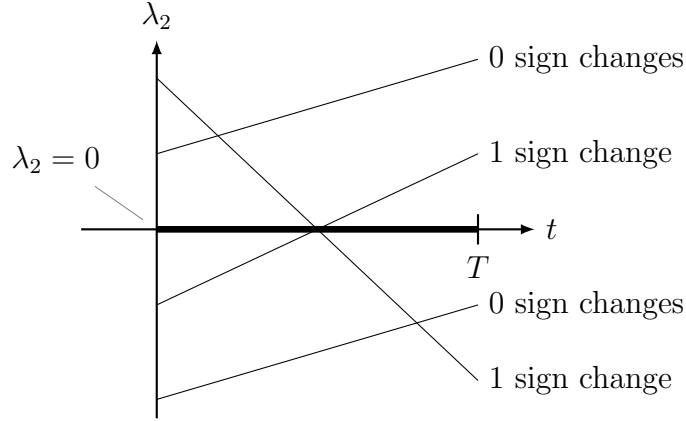
Clearly, we minimize  $H$  by letting

$$u = \begin{cases} -1, & \lambda_2 > 0 \\ +1, & \lambda_2 < 0 = -\text{sign}(\lambda_2) \\ ??, & \lambda_2 = 0 \end{cases}$$

Therefore, the optimal  $u$  switches between  $-1$  and  $+1$  (bang-bang control).

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = c \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -ct + d
\end{aligned}$$

Notice that  $\lambda_2(t)$  is a line, so it has at most one sign change. Thus,  $u$  also changes sign (from  $\pm 1$  to  $\mp 1$ ) at most one time.



Let's solve this for all  $x_0$ !

i) Assume  $\lambda_2 > 0 \forall t \in [0, T]$ ,  $\therefore u = -1 \quad \forall t \in [0, T]$

$$\begin{aligned}
 \dot{x}_2 &= -1 \implies x_2 = -t + k_1 \\
 x_2(T) &= 0 = -T + k_1 \implies k_1 = T \\
 x_2(t) &= T - t \implies x_2 > 0, \quad t \in [0, T) \\
 \dot{x}_1 &= x_2 = T - t \implies x_1 = -\frac{t^2}{2} + Tt + k_2 \\
 x_1(T) &= 0 = -\frac{T^2}{2} + T^2 + k_2 \implies k_2 = -\frac{T^2}{2} \\
 x_1(t) &= -\frac{t^2}{2} + Tt - \frac{T^2}{2} = -\frac{(T-t)^2}{2} \quad (< 0, \quad t \in [0, T)) \\
 &= -\frac{x_2^2(t)}{2}
 \end{aligned}$$

Let's consider the curve

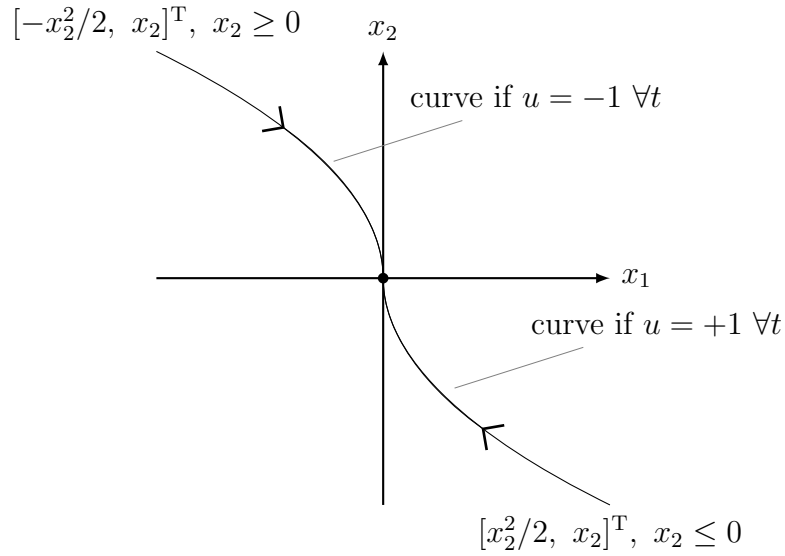
$$\begin{bmatrix} -x_2^2/2 \\ x_2 \end{bmatrix}$$

for  $x_2 \geq 0$ . If  $x_0$  lies on this curve, use  $u = -1$  and drive to the origin.

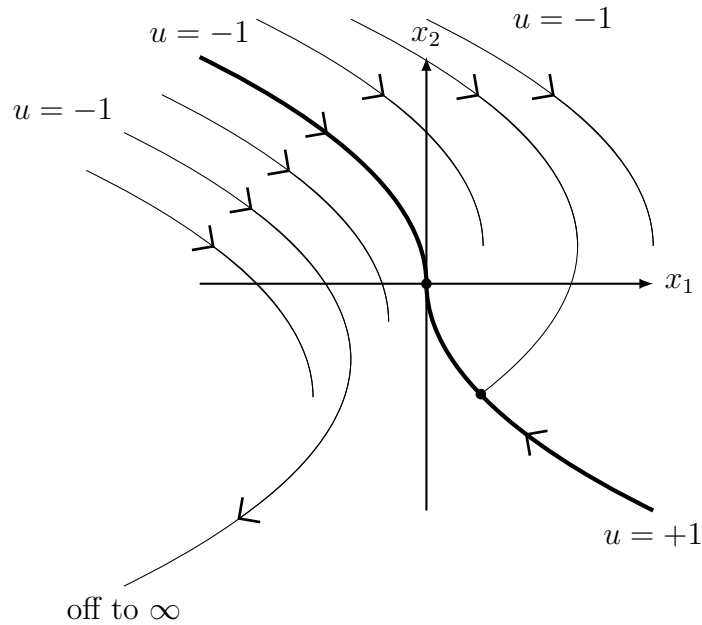
ii) Assume  $u = +1 \quad \forall t \in [0, T]$

$$\begin{aligned}
 x_2 &= t - T \quad (\leq 0 \text{ on } [0, T]) \\
 x_1 &= \frac{x_2^2}{2} \quad (\geq 0)
 \end{aligned}$$

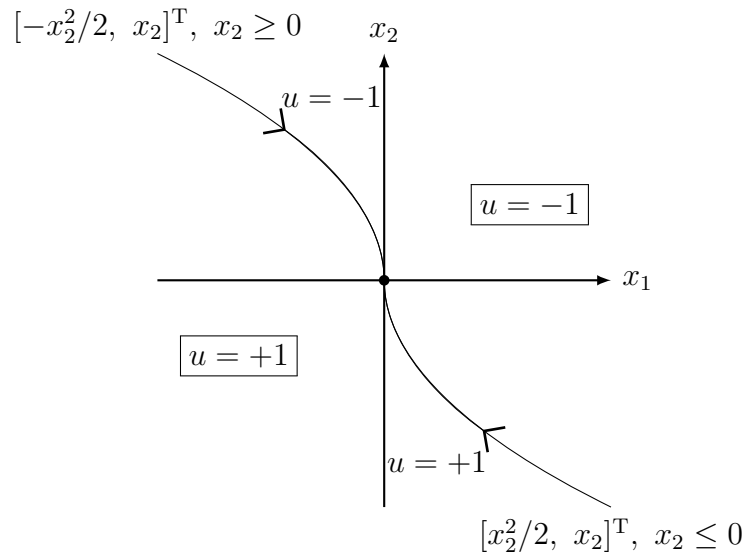
For this curve, use  $u = +1$ .



What happens when we do not start on the curves? We start with a certain  $u$  depending on  $x_0$  and perform a single switch of  $u$  when we encounter one of the initial curves that travel to the origin. Note that for the case  $\lambda_2 = 0 \forall t$ , we start at the stop sign at rest, so the control does not matter.



The optimal solution is given by the following *switching curve*.



**Note 1:** Bang-bang control typically involves

- a) finding the number of switches
- b) find the switching surfaces

**Note 2:** This is a feedback law! ( $u$  depends on  $x$ !!)

### 3.6.1 Linear Systems (scalar input)

$$\begin{aligned}
 & \min_{u,T} \int_0^T dt \\
 & \text{s.t. } \dot{x} = Ax + Bu \\
 & \quad x(0) = x_0, \quad x(T) = 0 \\
 & \quad u \in [-1, 1] \\
 & \quad H = 1 + \lambda^T (Ax + Bu) \\
 & \quad u = -\text{sign}(\lambda^T B) \quad (\text{bang-bang})
 \end{aligned}$$

Aside...

$$\begin{aligned}
 \dot{x} &= f(x) + g(x)u \quad (\text{control affine}) \\
 H &= 1 + \lambda^T f + \lambda^T g u \\
 u &= -\text{sign}(\lambda^T g(x)) \quad (\text{bang-bang})
 \end{aligned}$$

Back to linear...

$$\begin{aligned}
 \dot{\lambda} &= -\frac{\partial H^T}{\partial x} = -A^T \lambda \\
 \lambda(t) &= e^{-A^T t} \lambda_0 \\
 u(t) &= -\text{sign}(\lambda_0^T e^{-A^T t} B)
 \end{aligned}$$

How do we find  $\lambda_0$ ?

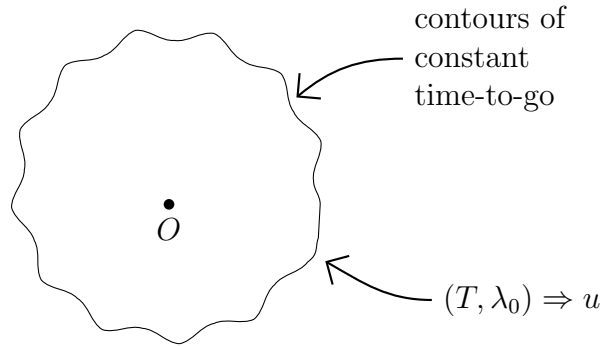
$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ x(T) &= e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau) d\tau \\ x(T) = 0 &= e^{AT}x_0 - \int_0^T e^{A(T-t)}B \operatorname{sign}(\lambda_0^T e^{-At}B) dt\end{aligned}\tag{3.7}$$

*Problem 1:* Given  $x_0$ , figure out  $\lambda_0$  from (3.7). Then,  $u = -\operatorname{sign}(\lambda_0^T e^{-At}B)$ . This has to be done numerically in general (not super simple...).

*Problem 2:* Find all  $x_0$ s from which it takes the same amount of time to get to  $x(T) = 0$ .

$$\begin{aligned}e^{At}x_0 &= \int_0^T e^{A(T-t)}B \operatorname{sign}(\lambda_0^T e^{-At}B) dt \\ x_0 &= \int_0^T e^{-At}B \operatorname{sign}(\lambda_0^T e^{-At}B) dt\end{aligned}$$

Fix  $T$ . By varying  $\lambda_0$ , we will get the  $x_0$ s that take time  $T$  to go to  $x(T) = 0$  optimally.



So by solving problem 2, we find  $\lambda_0$  associated with all  $x_0$ , i.e. we have “solved” problem 1 as well.