Homework 1

ECE6553 Optimal Control

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Due: January 26, 2017 (and Feb. 2 for DL students – section Q)

1

a

In class it was claimed that for a given function $g : \mathbb{R}^m \to \mathbb{R}$, its gradient

$$\nabla g(u) = \frac{\partial g(u)^T}{\partial u}$$

points in the direction in which g grows the most, at the point $u \in \mathbb{R}^m$. Show that this is indeed the case.

b

Similarly, it was claimed that, given the constraint

$$h(u) = 0, \quad h : \mathbb{R}^m \to \mathbb{R},$$

the gradient $\nabla h(u)$ is orthogonal to the tangent plane to the constraint set at the point u. Show that this is indeed the case.

2

$$\text{Minimize } (u_1 - 2)^2 + 2(u_2 - 1)^2$$

$$\begin{array}{ll} \text{subject to} & u_1 + 4u_2 \leq 3 \\ & u_1 \geq u_2. \end{array}$$

3

a

Assume that you are a manufacturer who have committed to shipping q units of a particular product to a store. (The units can be real numbers - don't worry about making sure they are integers). There are two ways to ship the product and you can combine these two options. You can use shipping option 1, which costs αu^2 to ship u units (for some $\alpha > 0$), or shipping option 2, which costs βu^2 to ship u units ($\beta > 0$). Come up with the best combination of the two shipping options.

b

How do you know this is the best combination and not the worst combination?

4

Let $g \in C^1$ be convex, which means that

$$g(\alpha u_1 + (1-\alpha)u_2) \leq \alpha g(u_1) + (1-\alpha)g(u_2), \ \forall u_1, u_2 \in \mathbb{R}^m, \ \alpha \in [0,1].$$

Assume that

$$\frac{\partial g}{\partial u}(u^\star) = 0.$$

Show that $u^\star$ is a *global* minimum to g.

5

An alternative to the Armijo rule for picking the step-size is to do a line minimization, i.e., to pick the step-size that minimizes the cost in the descent direction, i.e.,

$$u_{k+1} = u_k + \gamma_k^{\min} d_k,$$

where d_k is the descent direction, and

$$\gamma_k^{\min} = \text{argmin}_\gamma \ g(u_k + \gamma d_k),$$

given the cost function $g(u)$.

If the direction is the negative gradient to g, i.e., $d_k = -\nabla g(u_k)$, and the cost is given by

$$g(u) = \frac{1}{2}u^T Q u + b^T u,$$

where $Q = Q^T \succ 0$, what is the step-size $\gamma_k^{\min}$?

6

a

For the matrices in Question 5, let

$$Q = \begin{bmatrix} 5 & 0 & 8 & -1 & -3 \\ 0 & 10 & 9 & 7 & 11 \\ 8 & 9 & 25 & 0 & 6 \\ -1 & 7 & 0 & 19 & 5 \\ -3 & 11 & 6 & 5 & 18 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Write a matlab program that compares the Line Search step-size developed in Question 5 to the Armijo step-size, with the choice of $\alpha = \beta = 0.5$ in the Armijo algorithm. Starting from $u_0 = [1,1,1,1,1]^T$, plot the costs in the same figure (iteration number on the x-axis and cost on the y-axis), when using Armijo and Line Search. Include the plot and the m-file in your homework.

b

Based on the plot, answer the following questions:

- Which method is faster?
- Is this always the case? (No need for a formal proof. An informed discussion is sufficient.)
- If one method is always better, why not always use that one?