ECE 6553: Optimal Control Notes

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Contents

Chapter 1

Parameter Optimization

1.1 What is Optimal Control?

Optimal Maximize/minimize cost (subject to constraints): $\min_u g(u)$
With constraints,

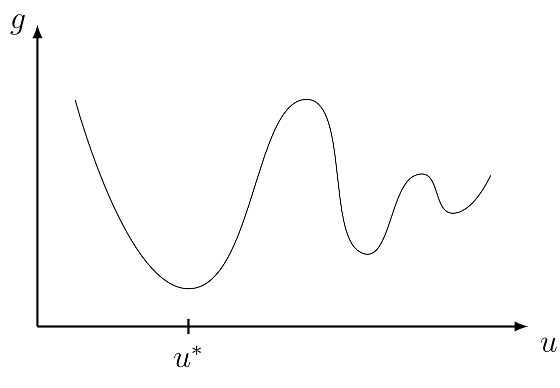
$$\min_u \; g(u)$$
$$\text{s.t.} \; \begin{cases} h_1(u) = 0 \\ h_2(u) \leq 0 \end{cases}$$

First-order necessary condition (FONC):

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Optimality can be

- local vs global
- max vs min



Control control design: pick u such that specifications are satisfied:

$$\dot{x} = f(x, u), \qquad \dot{x} = Ax + Bu,$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, and $f(\cdot)$ is the dynamics.

Actually, x and u are signals:

$$x : [0, T] \to \mathbb{R}^n, \qquad u : [0, T] \to \mathbb{R}^m$$

Optimal control find the "best" u!

For "best" to mean anything, we need a cost. The big/deep question is

$$\frac{\partial \text{"cost"}}{\partial u} = 0$$

Example

Suppose we have a car with position p. Its acceleration $\ddot{p}$ is controlled by the gas/brake input u ($\ddot{p} = u$). In order to express the dynamics of the system in the form $\dot{x} = f(x, u)$, we introduce state variables:

$$\begin{aligned} x_1 &= p \\ x_2 &= \dot{p} \end{aligned} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

The task is to move the car from its initial position to a stop at a distance c away.

Minimum energy problem

$$\begin{aligned} \min_u \quad & \int_0^T u^2(t)\,\mathrm{d}t \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0,\ x_2(0) = 0 \\ & x_1(T) = c,\ x_2(T) = 0 \end{aligned}$$

Minimum time problem

$$\begin{aligned} \min_{u,T} \quad & T = \int_0^T \mathrm{d}t \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0,\ x_2(0) = 0 \\ & x_1(T) = c,\ x_2(T) = 0 \\ & u(t) \in [u_{\min}, u_{\max}] \end{aligned}$$

The general optimal control problem we will solve will look like

$$
\begin{aligned}
\min_{u,T} \quad & \int_0^T L(x(t), u(t), t)\,\mathrm{d}t + \Psi(x(T)) \\
\text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t), t),\ \ t \in [0, T] \\
& x(0) = x_0 \\
& x(T) \in S \\
& u(t) \in \Omega,\ \ t \in [0, T]
\end{aligned}
$$

where $\Psi(\cdot)$ is the terminal cost and S is the terminal manifold. This is a so-called **Bolza Problem**.

What tools do we need to solve this?

1. optimality conditions $\partial \text{cost}/\partial u = 0$
2. some way of representing the optimal signal $u^*(x, t)$
3. some way of actually finding/computing the optimal controllers

1.2 Unconstrained Optimization

Let the decision variable be $u = [u_1, \ldots, u_m]^{\mathrm{T}} \in \mathbb{R}^m$. The cost is $g(u) \in C^1$ (C^k means k times continuously differentiable). The problem is

$$
\min_u g(u), \quad g : \mathbb{R}^m \to \mathbb{R}
$$

For u^* to be a minimizer, we need

$$
\frac{\partial g}{\partial u}(u^*) = 0
$$

Definition. u^* is a (local) minimizer to g if $\exists \delta > 0$ s.t.

$$
\begin{gathered}
g(u^*) \le g(u) \quad \forall u \in B_\delta(u^*) \\
B_\delta(u^*) = \{u \mid \|u - u^*\| \le \delta\}
\end{gathered}
$$

Note:

- $\dfrac{\partial g}{\partial u}(u^*)\delta u \in \mathbb{R}$ and δu is $m \times 1$, so $\dfrac{\partial g}{\partial u}$ is a $1 \times m$ row vector. For the column vector,

$$
\nabla g = \frac{\partial g}{\partial u}^{\mathrm{T}} \in \mathbb{R}^m
$$

- $\dfrac{\partial g}{\partial u}\,\delta u$ is an inner product

$$
\langle \nabla g, \delta u \rangle = \left\langle \frac{\partial g}{\partial u}^{\mathrm{T}}, \delta u \right\rangle
$$

- $o(\varepsilon)$ encodes higher-order terms

$$\lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0 \qquad \text{“faster than linear”}$$

  This is opposed to big-O notation:

$$\lim_{\varepsilon \to 0} \frac{\mathcal{O}(\varepsilon)}{\varepsilon} = c$$

- δu has direction and scale so we could write it as

$$\delta u = \varepsilon v, \quad \varepsilon \in \mathbb{R},\ v \in \mathbb{R}^m$$

Theorem. *For u^* to be a minimizer, we need*

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or, equivalently,

$$\frac{\partial g}{\partial u}(u^*)v = 0 \quad \forall v \in \mathbb{R}^m$$

Proof. Let u^* be a minimizer. Evaluating the cost $g(u)$ in the ball and using Taylor's expansion,

$$g(u^* + \delta u) = g(u^*) + \frac{\partial g}{\partial u}(u^*)\delta u + o(\|\delta u\|) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$

Assume that $\frac{\partial g}{\partial u} \neq 0$. Then we could pick $v = -\frac{\partial g^{\mathrm{T}}}{\partial u}(u^*)$, i.e.

$$g(u^* + \varepsilon v) = g(u^*) - \varepsilon \left\| \frac{\partial g^{\mathrm{T}}}{\partial u}(u^*) \right\|^2 + o(\varepsilon)$$

Note that the second term is negative per our assumptions. So, for ε sufficiently small, we have

$$g\Big(u^* - \varepsilon \frac{\partial g^{\mathrm{T}}}{\partial u}(u^*)\Big) < g(u^*)$$

This contradicts u^* being a minimizer. ⚔ (crossed swords) □

Definition (Positive definite)**.** $M = M^{\mathrm{T}} \succ 0$ if

$$z^{\mathrm{T}} M z > 0 \quad \forall z \neq 0,\ z \in \mathbb{R}^m$$
$$\iff M \text{ has real and positive eigenvalues}$$

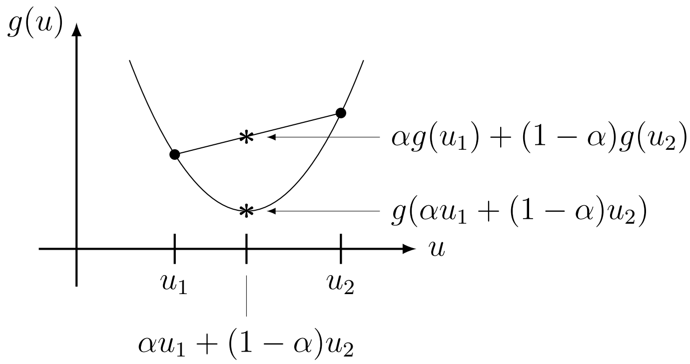
Theorem. *If $g \in C^2$, then a **sufficient** condition for u^* to be a (local) minimizer is*

1. $\dfrac{\partial g}{\partial u}(u^*) = 0$

2. $\dfrac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ *(the Hessian is positive definite)*

Definition. $g : \mathbb{R}^m \to \mathbb{R}$ is convex if

$$g(\alpha u_1 + (1-\alpha)u_2) \le \alpha g(u_1) + (1-\alpha) g(u_2) \quad \forall \alpha \in [0,1],\ u_1,u_2 \in \mathbb{R}^m$$



Theorem. *If $\frac{\partial^2 g}{\partial u^2}(u) \succeq 0\ \forall u \in \mathbb{R}^m$, then g is convex. ($\Longleftrightarrow$ for $g \in C^2$)*

Example $\min\limits_u u^{\mathrm{T}} Q u - b^{\mathrm{T}} u$ where $Q = Q^{\mathrm{T}} \succ 0$ (positive definite matrix)

$$\begin{aligned}\frac{\partial g}{\partial u} &= \frac{\partial}{\partial u}(u^{\mathrm{T}} Q u - b^{\mathrm{T}} u) \\ &= u^{\mathrm{T}} Q^{\mathrm{T}} + u^{\mathrm{T}} Q - b^{\mathrm{T}} \\ &= 2u^{\mathrm{T}} Q - b^{\mathrm{T}} \\ \frac{\partial^2 g}{\partial u^2} &= 2Q\end{aligned} \qquad \frac{\partial^2 g}{\partial u^2} = \begin{bmatrix} \frac{\partial^2 g}{\partial u_1^2} & \cdots & \frac{\partial^2 g}{\partial u_1 \partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 g}{\partial u_m^2} \end{bmatrix}$$

From $\frac{\partial g}{\partial u} = 2u^{\mathrm{T}} Q - b^{\mathrm{T}} = 0$,

$$u = \frac{1}{2} Q^{-1} b$$

To see whether this is a minimizer, consider the Hessian. Since $Q \succ 0$, it follows that $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ and $u^* = \frac{1}{2}Q^{-1}b$ is a (local) minimizer. Additionally, since $\frac{\partial^2 g}{\partial u^2} \succ 0$, g is convex and u^* is a global minimizer. In fact, since we have strict convexity ($\succ 0$ rather than $\succeq 0$), it is the unique global minimizer.

In optimal control, *local* is typically all we can ask for. In optimization, we can do better!

But wait, just because we know $\frac{\partial g}{\partial u} = 0$, it doesn't follow that we can actually find u^*...

1.3 Numerical Methods

Idea: $u_{k+1} = u_k + \mathrm{step}_k$. What should step_k be? For small $\mathrm{step}_k = \gamma_k v_k$,

$$g(u_k \cdot \mathrm{step}_k) = g(u_k) + \frac{\partial g}{\partial u}(u_k) \cdot \mathrm{step}_k + o(\|\mathrm{step}_k\|) = g(u_k) + \gamma_k \frac{\partial g}{\partial u}(u_k) v_k + o(\gamma_k)$$

A perfectly reasonable choice of step direction is

$$v_k = -\frac{\partial g^{\mathrm{T}}}{\partial u}(u_k),$$

known as the *steepest descend* direction. This produces

$$g\Big(u_k - \gamma_k \frac{\partial g}{\partial u}(u_k)\Big) = g(u_k) - \gamma_k \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 + o(\gamma_k)$$

Steepest descent

$$u_{k+1} = u_k - \gamma_k \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)$$

Note:

- What should γ_k be?
- This method "pretends" that $g(u)$ is linear. If we pretend $g(u)$ is quadratic, we get

  $$u_{k+1} = u_k - \left(\frac{\partial^2 g}{\partial u^2}(u_k)\right)^{-1} \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k),$$

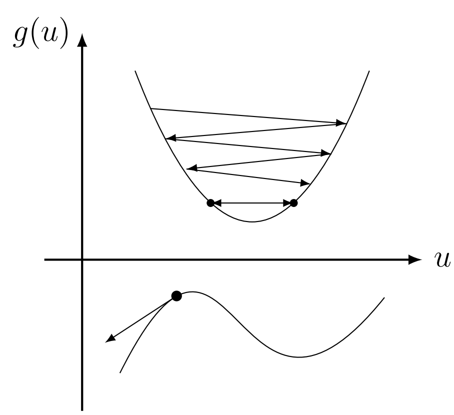
  i.e. Newton's Method

This course: steepest descent

Step-size selection?

- Choice 1: $\gamma_k = \gamma$ "small" $\forall k$; will get close to a minimizer if u_0 is close enough and γ small enough

  Problems:
  - You may not converge! (but you'll get close)
  - You may go off to infinity (diverge)



- Choice 2: Reduce γ_k as a function of k; will get close to a minimizer if u_0 is close enough

  Problem: slow

  Theorem. *If u_0 is close enough to u^* and γ_k satisfies*

  - $\sum_{k=0}^{\infty} \gamma_k = \infty$
  - $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$

  e.g. $\gamma_k = c/k$, then $u_k \to u^*$ as $k \to \infty$.

- Choice 3: **Armijo step-size:** Take as big a step as possible, but no larger

  Pick $\alpha \in (0,1)$, $\beta \in (0,1)$. Let i be the smallest non-negative integer such that

$$g\left(u_k - \beta^i \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)\right) - g(u_k) < -\alpha\beta^i \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2$$

$$u_{k+1} = u_k - \beta^i \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)$$

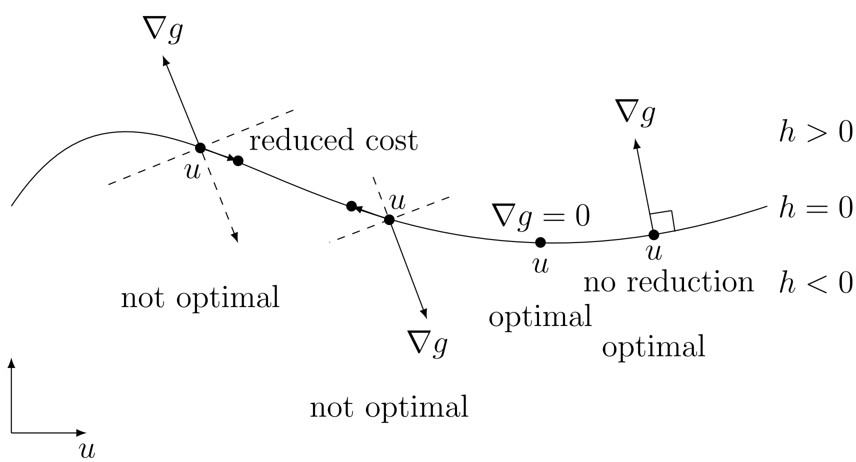
  This will get to a minimizer blazingly fast if u_0 is close enough.

1.4 Constrained Optimization

Equality constraints:

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \ & g(u) \\ \text{s.t.} \ & h(u) = \mathbf{0} \end{aligned}$$

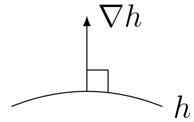
Consider $u \in \mathbb{R}^2$, $h : \mathbb{R}^2 \to \mathbb{R}$



So u is (locally) optimal if $\nabla g \parallel$ (is parallel to) the normal vector to tangent plane to h.

Fact: (HW# 1)

$$\nabla h \perp Th \quad \text{(tangent plane to h)}$$



We need $\nabla g \parallel \nabla h$ at u^* for optimality, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \alpha \frac{\partial h}{\partial u}(u^*), \quad \text{for some } \alpha \in \mathbb{R}$$

or ($\lambda = -\alpha$),

$$\frac{\partial g}{\partial u}(u^*) + \lambda \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}\big(g(u^*) + \lambda h(u^*)\big) = 0, \quad \text{for some } \lambda \in \mathbb{R}$$

More generally,

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \;& g(u) \\ \text{s.t. } \;& h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k \end{aligned}$$

Note that $h(u) = [h_1(u), \ldots, h_k(u)]^{\mathrm{T}}$.

We need $\frac{\partial g}{\partial u}(u^*)$ to be a linear combination of $\frac{\partial h_i}{\partial u}(u^*)$, $i = 1, \ldots, k$, for exactly the same reasons, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \sum_{i=1}^{k} \alpha_i \frac{\partial h_i}{\partial u}(u^*)$$

or ($\lambda = -[\alpha_1, \ldots, \alpha_k]^{\mathrm{T}}$)

$$\frac{\partial g}{\partial u}(u^*) + \lambda^{\mathrm{T}} \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}\big(g(u^*) + \lambda^{\mathrm{T}} h(u^*)\big) = 0, \quad \text{for some } \lambda \in \mathbb{R}^k$$

Theorem. *If u^* is a minimizer to*

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \;& g(u) \\ s.t. \;& h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k \end{aligned}$$

then $\exists \lambda \in \mathbb{R}^k$ s.t.

$$\begin{cases} \dfrac{\partial L}{\partial u}(u^*, \lambda) = 0 \\[2ex] \dfrac{\partial L}{\partial \lambda}(u^*, \lambda) = 0 \end{cases}$$

where the Lagrangian L is given by

$$L(u, \lambda) = g(u) + \lambda^T h(u)$$

Note:

- λ are the Lagrange multipliers
- $\frac{\partial L}{\partial \lambda} = 0$ is fancy speak for $h(u^*) = 0$

Example

$$\begin{aligned}
\min_{u \in \mathbb{R}^m} \ & \frac{1}{2}\|u\|^2 \\
\text{s.t.} \ & Au = b
\end{aligned}$$

where A is $k \times m$, $k \leq m$. Assume $(AA^{\mathrm{T}})^{-1}$ exists (constraints are linearly independent, none of the constraints are "duplicates", all the constraints are essential).

$$\begin{aligned}
L &= \frac{1}{2}u^{\mathrm{T}}u + \lambda^{\mathrm{T}}(Au - b) \\
\frac{\partial L}{\partial u} &= u^{\mathrm{T}} + \lambda^{\mathrm{T}}A = 0 \\
u^* &= -A^{\mathrm{T}}\lambda
\end{aligned}$$

Using the equality constraint,

$$\begin{aligned}
Au^* &= b \\
-AA^{\mathrm{T}}\lambda &= b \\
\lambda &= -(AA^{\mathrm{T}})^{-1}b \\
u^* &= A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b
\end{aligned}$$

Example

$$\begin{aligned}
\min \ & u_1u_2 + u_2u_3 + u_1u_3 \\
\text{s.t.} \ & u_1 + u_2 + u_3 = 3
\end{aligned}$$

$$L = u_1u_2 + u_2u_3 + u_1u_3 + \lambda(u_1 + u_2 + u_3 - 3)$$

$$\begin{cases}
\frac{\partial L}{\partial u_1} = u_2 + u_3 + \lambda = 0 \\
\frac{\partial L}{\partial u_2} = u_1 + u_3 + \lambda = 0 \\
\frac{\partial L}{\partial u_3} = u_2 + u_1 + \lambda = 0 \\
\frac{\partial L}{\partial \lambda} = u_1 + u_2 + u_3 = 3
\end{cases}
\implies
\begin{cases}
u_1^* = 1 \\
u_2^* = 1 \\
u_3^* = 1 \\
\lambda = -2
\end{cases}
\quad \text{optimal solution}$$

Note: This was actually the worst we can do—maximize! Even weirder: no local minimizer!

1.4.1 Equality Constraints

$$\begin{aligned}\min_{u\in\mathcal{R}^m}\ & g(u)\\ \text{s.t. } & h(u)=\mathbf{0}, \quad h:\mathbb{R}^m\to\mathbb{R}^k\end{aligned}$$

Theorem. *If u^* is a minimizer/maximizer then $\exists\lambda\in\mathbb{R}^k$ s.t.*

$$\begin{aligned}&\frac{\partial L}{\partial u}(u^*,\lambda)=0\\ &\frac{\partial L}{\partial \lambda}(u^*,\lambda)=0 \qquad (\Longleftrightarrow h(u^*)=0)\end{aligned}$$

where $L(u,\lambda)=g(u)+\lambda^T h(u)$.

Example [Entropy Maximization]

Given $S=\{x_1,\dots,x_n\}$ and a distribution over S such that it takes the value x_j with probability p_j. The entropy is

$$E(p)=\sum_{j=1}^n(-p_j\ln p_j).$$

The mean is

$$m=\sum_{j=1}^n p_j x_j.$$

Problem: Given m, find p such that E is maximized.

$$\begin{aligned}\min_p\ & -\sum_{j=1}^n p_j\ln p_j\\ \text{s.t. } & \sum_{j=1}^n p_j x_j=m\\ & \sum_{j=1}^n p_j=1\\ & p_j\geq 0,\ j=1,\dots,n \quad \text{(ignore this...)}\end{aligned}$$

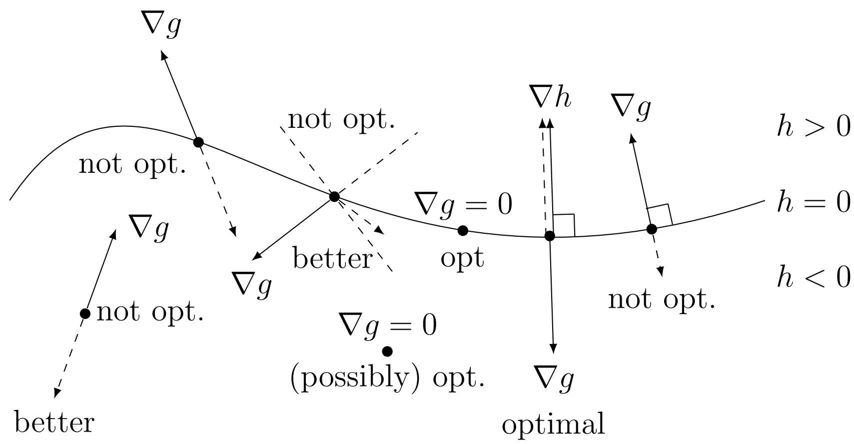
$$\begin{aligned}L&=-\sum p_j\ln p_j+\lambda_1\left[\sum p_j x_j-m\right]+\lambda_2\left[\sum p_j-1\right]\\ \frac{\partial L}{\partial p_j}&=-\ln p_j-1+\lambda_1 x_j+\lambda_2=0\\ p_j&=e^{\lambda_2-1+\lambda_1 x_j}, \quad j=1,\dots,n \qquad (p_j\geq 0 \text{ so we're ok with ignoring that})\end{aligned}$$

$$\begin{aligned}\sum e^{\lambda_2-1+\lambda_1 x_j}x_j&=m \qquad && n+2 \text{ equations and}\\ \sum e^{\lambda_2-1+\lambda_1 x_j}&=1 && n+2 \text{ unknowns...}\end{aligned}$$

No analytical solution, but numerically “solvable”

1.4.2 Inequality Constraints

$$\min_{u \in \mathcal{R}^m} \ g(u)$$
$$\text{s.t. } h(u) \leq \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k$$



We need:

- if $h(u^*) < 0$ then $\frac{\partial g}{\partial u}(u^*) = 0$
- if $h(u^*) = 0$ then we need either

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or

$$\frac{\partial g}{\partial u}(u^*) = -\lambda \frac{\partial h}{\partial u}(u^*) \quad \text{for } \lambda > 0$$

Or, even better,

$$\frac{\partial}{\partial u}\left(g(u^*) + \lambda h(u^*)\right) = 0 \quad \text{for } \lambda \geq 0,$$

where $\lambda h(u^*) = 0$. ($h < 0 \to \lambda = 0$, $h = 0 \to \lambda \geq 0$)

In general, if $u \in \mathbb{R}^m$ and $h : \mathbb{R}^m \to \mathbb{R}^k$, we have that u^*, if optimal, has to satisfy

$$\frac{\partial}{\partial u} L(u^*, \lambda) = 0$$
$$h(u^*) \leq \mathbf{0}$$
$$\lambda^{\mathrm{T}} h(u^*) = 0$$
$$\lambda \geq \mathbf{0}$$

where the Lagrangian is $L(u, \lambda) = g(u) + \lambda^{\mathrm{T}} h(u)$. Note that if we're maximizing, the same holds except we need $\lambda \leq 0$.

Example

$$\min\ 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2$$
$$\text{s.t.}\ \begin{cases} u_1^2 + u_2^2 \leq 5 \\ 3u_1 + u_2 \leq 6 \end{cases}$$

$$L = 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 + \lambda_1(u_1^2 + u_2^2 - 5) + \lambda_2(3u_1 + u_2 - 6)$$

FONC:

i) $\partial L/\partial u_1 = 4u_1 + 2u_2 - 10 + 2\lambda_1 u_1 + 3\lambda_2$

ii) $\partial L/\partial u_2 = 2u_1 + 2u_2 - 10 + 2\lambda_1 u_2 + \lambda_2$

iii) $u_1^2 + u_2^2 \leq 5$

iv) $3u_1 + u_2 \leq 6$

v) $\lambda_1(u_1^2 + u_2^2 - 5) = 0$

vi) $\lambda_2(3u_1 + u_2 - 6) = 0$

vii) $\lambda_1 \geq 0$

viii) $\lambda_2 \geq 0$

To solve, assume different constraints are active/inactive:

1. Both constraints are inactive ($u_1^2 + u_2^2 < 5$, $3u_1 + u_2 < 6$) $\Longrightarrow \lambda_1 = \lambda_2 = 0$

$$\begin{cases} 4u_1 + 2u_2 - 10 = 0 \\ 2u_1 + 2u_2 - 10 = 0 \end{cases} \Longrightarrow \begin{cases} u_1 = 0 \\ u_2 = 5 \end{cases}$$

   Note: iii) $0^2 + 5^2 \not\leq 5$

   Not feasible

2. Assume constraint 1 is active and constraint 2 is inactive ($u_1^2 + u_2^2 = 5$, $\lambda_2 = 0$)

$$\begin{cases} 4u_1 + 2u_2 - 10 + 2\lambda_1 u_1 = 0 \\ 2u_1 + 2u_2 - 10 + 2\lambda_1 u_2 = 0 \\ u_1^2 + u_2^2 = 5 \end{cases} \Longrightarrow \begin{cases} u_1 = 1 \\ u_2 = 2 \\ \lambda_1 = 1 \end{cases}$$

   - ☑ $\lambda_1 \geq 0$
   - ☑ $3 \cdot 1 + 2 \leq 6$

   This is a local minimizer

3. Assume constraint 2 is active and constraint 1 is inactive

4. Assume both constraints are active

Kuhn-Tucker Conditions (KKT conditions, Karush-Kuhn-Tucker)

Problem:

$$\begin{aligned}&\min_{u\in\mathbb{R}^m} \; g(u)\\ &\text{s.t.} \begin{cases} h_1(u) = 0, & h_1 : \mathbb{R}^m \to \mathbb{R}^p \\ h_2(u) \le 0, & h_2 : \mathbb{R}^m \to \mathbb{R}^k \end{cases}\end{aligned} \tag{1.1}$$

Theorem. *Let u^* be feasible ($h_1 = 0$, $h_2 \le 0$). If u^* is a minimizer to (1.1) than there exists vectors $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^k$ with $\mu \ge \mathbf{0}$ such that*

$$\begin{cases} \dfrac{\partial g}{\partial u}(u^*) + \lambda^T \dfrac{\partial h_1}{\partial u}(u^*) + \mu^T \dfrac{\partial h_2}{\partial u}(u^*) = 0 \\ \mu^T h_2(u^*) = 0 \end{cases}$$

Looking ahead: $\min \operatorname{cost}(u(\cdot))$ s.t. $\dot{x} = f(x, u)$ (dynamics), where u is a function. Note the equality constraint.

Question: How do we go from $u \in \mathbb{R}^m$ to $u \in \mathcal{U}$ (function space)?

Note: Function space is a set of functions of a given kind from a set X to a set Y

1. linear function
2. square-integrable functions: $L_2[0, T] : \int_0^T \|u(t)\|^2 \, \mathrm{d}t < \infty$
3. $C^\infty(\mathbb{R})$

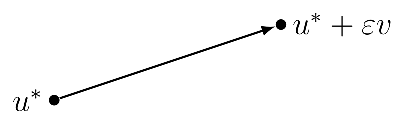
What would ∂"cost"$/\partial u$ mean?

Chapter 2

Calculus of Variations

2.1 Directional Derivatives

Recall: To minimize $g(u)$, let u^* be a candidate minimizer and pitch a perturbation on u^* of εv, where ε is the scale and v is the direction. Taking Taylor's expansion at the perturbation produces



$$
\begin{aligned}
g(u^* + \varepsilon v) &= g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*) v + o(\varepsilon) \\
\text{FONC: } \frac{\partial g}{\partial u}(u^*) &= 0
\end{aligned}
$$

Note: $\frac{\partial g}{\partial u}(u^*)v$ tells us how much $g(u)$ increases/decreases in the direction of v.

Definition. The directional (Gateaux) derivative is given by

$$
\delta g(u; v) = \lim_{\varepsilon \to 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon}
$$

Example

$$
g(u) = \frac{1}{2} u_1^2 - u_1 + 2u_2, \quad g : \mathbb{R}^2 \to \mathbb{R}
$$

Let's consider $e_1 = [1\ 0]^{\mathrm{T}}$, $e_2 = [0\ 1]^{\mathrm{T}}$. What is $\delta g(u; e_i)$, $i = 1, 2$?

$$
\begin{aligned}
\delta g(u; v) &= \lim_{\varepsilon \to 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{g(u) + \varepsilon \frac{\partial g}{\partial u}(u) v + o(\varepsilon) - g(u)}{\varepsilon} \\
&= \frac{\partial g}{\partial u}(u) v
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial g}{\partial u}(u) &= [u_1 - 1 \;\, 2] \\
\delta g(u; e_1) &= [u_1 - 1 \;\, 2]e_1 = u_1 - 1 \\
\delta g(u; e_2) &= [u_1 - 1 \;\, 2]e_2 = 2
\end{aligned}
$$

But the beauty of directional derivatives is that they generalize beyond vectors, $u \in \mathbb{R}^m$, to function spaces ($\mathcal{U}$) or other "objects" like matrices.

Example $M \in \mathbb{R}^{n\times n}$, $F(M) = M^2$

What is $\frac{\partial F}{\partial M}$? (ponder at home...)

We can easily compute $\delta F(M; N)$!

$$
\begin{aligned}
F(M + \varepsilon N) = (M + \varepsilon N)(M + \varepsilon N) &= M^2 + \varepsilon MN + \varepsilon NM + \varepsilon^2 N^2 \\
\delta F(M; N) = \lim_{\varepsilon\to 0} \frac{F(M + \varepsilon N) - F(M)}{\varepsilon} & \\
= \lim_{\varepsilon\to 0} \frac{\varepsilon MN + \varepsilon NM + \varepsilon^2 N^2}{\varepsilon} &= MN + NM
\end{aligned}
$$

Infinite Dimensional Optimization Let $u \in \mathcal{U}$ (function space) and let $J(u)$ be the cost:

$$
\min_{u\in\mathcal{U}} J(u)
$$

Theorem. *If $u^* \in \mathcal{U}$ is a (local) minimizer then*

$$
\delta J(u^*; v) = 0, \quad \forall v \in \mathcal{U}
$$

Example Find minimizer u^* to

$$
J(u) = \int_0^T L(u(t))\,\mathrm{d}t
$$

$$
\begin{aligned}
J(u + \varepsilon v) - J(u) &= \int_0^T L(u(t) + \varepsilon v(t))\,\mathrm{d}t - \int_0^T L(u(t))\,\mathrm{d}t, \quad u, v \in \mathcal{U} \\
&= \int_0^T \left[L(u(t)) + \varepsilon \frac{\partial L}{\partial u}(u(t))v(t) + o(\varepsilon) - L(u(t)) \right] \mathrm{d}t \\
\delta J(u^*; v) &= \lim_{\varepsilon\to 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\int_0^T \varepsilon \frac{\partial L}{\partial u}(u(t))v(t)\,\mathrm{d}t + o(\varepsilon)}{\varepsilon} \\
&= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t)\,\mathrm{d}t
\end{aligned}
$$

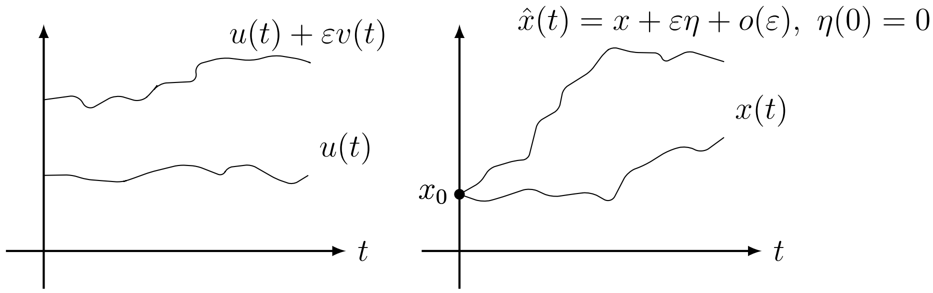


Figure 2.1: Variation in u causes a variation in x.

u^* optimizer:

$$\delta J(u^*; v) = \int_0^T \frac{\partial L}{\partial u}(u(t))v(t)\,\mathrm{d}t = 0 \quad \forall v \in \mathcal{U}$$
$$\Updownarrow$$
$$\frac{\partial L}{\partial u}(u(t)) = 0 \quad \forall t \in [0, T]$$

But, we want *optimal control*! We want our cost to look like

$$\int_0^T L(x(t), u(t))\,\mathrm{d}t$$
$$\dot{x} = f(x, u)$$

2.2 Calculus of Variations

What happens to $x(t)$ when $u(t)$ changes to $u(t) + \varepsilon v(t)$? Let the system be given by

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

After perturbation of u, the new system is

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}, u + \varepsilon v) \\ x(0) = x_0 \end{cases}$$

Consider

$$\tilde{x} = x + \varepsilon\eta,$$

where

$$\begin{aligned} \dot{x} &= f(x, u), & x(0) &= x_0 \\ \dot{\eta} &= \frac{\partial f}{\partial x}(x, u)\eta + \frac{\partial f}{\partial u}(x, u)v, & \eta(0) &= 0 \end{aligned}$$

Theorem. *If f is continuously differentiable in x and u then*

$$\hat{x}(t) = \tilde{x}(t) + o(\varepsilon)$$

Proof.

i) Initial conditions:

$$\begin{aligned}
&\hat{x}(0) = x_0 \\
&\tilde{x}(0) = x(0) + \varepsilon\eta(0) = x_0
\end{aligned}$$

ii) Dynamics:

$$\begin{aligned}
\dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v) \\
\dot{\tilde{x}} &= \dot{x} + \varepsilon\dot{\eta} = f(x,u) + \varepsilon\frac{\partial f}{\partial x}(x,u)\eta + \varepsilon\frac{\partial f}{\partial u}(x,u)v \\
&= f(x + \varepsilon\eta, u + \varepsilon v) + o(\varepsilon) \\
&= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon)
\end{aligned}$$

We can see that the dynamics of $\hat{x}(t)$ are equal to those of $\tilde{x}(t)$ plus higher order terms:

$$\begin{aligned}
\dot{\tilde{x}} &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon) \\
\dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v)
\end{aligned}$$

Therefore, if our perturbation is small enough, we can model $\hat{x}(t)$ as $\tilde{x}(t)$.

□

Note: Taylor expansion with two elements is

$$\begin{aligned}
h(w + \varepsilon v, z + \varepsilon y) &= h(w, z + \varepsilon y) + \frac{\partial h}{\partial w}(w, z + \varepsilon y)\varepsilon v + o(\varepsilon) \\
&= \left\{ h(w,z) + \frac{\partial h}{\partial z}(w,z)\varepsilon y + o(\varepsilon) \right\} \\
&\qquad + \left\{ \frac{\partial h}{\partial w}(w,z)\varepsilon v + \underbrace{\frac{\partial^2 h}{\partial z \partial w}\varepsilon v \odot \varepsilon y}_{o(\varepsilon)} + o(\varepsilon) \right\} \\
&= h(w,z) + \frac{\partial h}{\partial z}\varepsilon y + \frac{\partial h}{\partial w}\varepsilon v + o(\varepsilon)
\end{aligned}$$

Last class:

1. $u \in \mathcal{U}$ (space of functions), $J : \mathcal{U} \to \mathbb{R}$ (cost).

   FONC: If u^* is optimal, then

   $$\delta J(u; \nu) = 0 \quad \forall \nu \in \mathcal{U},$$

   where the directional derivative is given by

   $$\delta J(u; \nu) = \lim_{\varepsilon \to 0} \frac{J(u + \varepsilon\nu) - J(u)}{\varepsilon}.$$

2. If

   $$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

   then a variation in u:

   $$u \longmapsto u + \varepsilon\nu$$

   results in a variation in x:

   $$x \longmapsto x + \varepsilon\eta + o(\varepsilon)$$

   See Figure 2.1. Note $\eta(0) = 0$.

2.2.1 An (Almost) Optimal Control Problem

Let $\dot{x} = f(x)$, $x(0) = x_0$. Note we get to pick the initial condition!

Problem

$$\min_{x_0 \in \mathbb{R}^m} J(x_0) = \int_0^T L(x(t))\,\mathrm{d}t$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = f(x(t)) & \text{the } \textit{constraint}\text{! (equality)} \\ x(0) = x_0 \end{cases}$$

Note every constraint needs a Lagrange multiplier. We have infinitely many constraints:

$$\dot{x}(t) = f(x(t)) \quad \forall t \in [0, T]$$

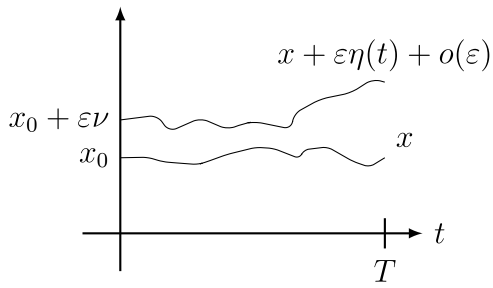
We need $\lambda(t)$ as a function of t. Also, the sum in the Lagrangian has to become an integral. The continuous-time Lagrangian thus becomes

$$\tilde{J}(x_0, \lambda) = \int_0^T \Big[L(x(t)) + \lambda^{\mathrm{T}}(t)(f(x(t)) - \dot{x}(t))\Big]\,\mathrm{d}t$$

The task is to perturb x_0 as $x_0 \longmapsto x_0 + \varepsilon\nu$, $\nu \in \mathbb{R}^m$ and compute

$$\delta\tilde{J}(x_0; \nu) = \lim_{\varepsilon \to 0} \frac{\tilde{J}(x_0 + \varepsilon\nu) - \tilde{J}(x_0)}{\varepsilon}$$

and make this equal to 0 $\forall \nu \in \mathbb{R}^m$. The variation in x is



Note:

x_0	decision variable
ν	variation in x_0
$x(t)$	trajectory starting at x_0
$\eta(t)$	change in trajectory resulting from ν-variation in x_0
$\lambda(t)$	time-varying Lagrange multiplier

$$\begin{aligned}
\tilde{J}(x_0+\varepsilon\nu) &= \int_0^T \Big\{L(x(t)) + \lambda^{\mathrm{T}}(t)[f(x(t)+\varepsilon\eta(t)) - \dot{x}(t) - \varepsilon\dot{\eta}(t)]\Big\}\,\mathrm{d}t + o(\varepsilon)\\
&= \int_0^T \left[L(x) + \varepsilon\frac{\partial L}{\partial x}(x)\eta + \lambda^{\mathrm{T}}\left(f(x) + \varepsilon\frac{\partial f}{\partial x}(x)\eta - \dot{x} - \varepsilon\dot{\eta}\right)\right]\mathrm{d}t + o(\varepsilon)\\
\tilde{J}(x_0+\varepsilon\nu) - \tilde{J}(x_0) &= \int_0^T \left[\varepsilon\frac{\partial L}{\partial x}(x)\eta + \lambda^{\mathrm{T}}\left(\varepsilon\frac{\partial f}{\partial x}\eta - \varepsilon\dot{\eta}\right)\right]\mathrm{d}t + o(\varepsilon)\\
\delta\tilde{J}(x_0;\nu) &= \int_0^T \left[\frac{\partial L}{\partial x}(x)\eta + \lambda^{\mathrm{T}}\left(\frac{\partial f}{\partial x}\eta - \dot{\eta}\right)\right]\mathrm{d}t
\end{aligned}$$

A powerful idea: we want $\delta\tilde{J}(x_0;\nu) = 0\ \forall\nu$. Somehow get this in the form

$$\int_0^T \Big(\text{stuff}(t)\Big)\eta(t)\,\mathrm{d}t = 0$$

We can pick $\text{stuff}(t) = 0\ \forall t \in [0,T]$.

In $\delta\tilde{J}(x_0;\nu)$ we have $\dot{\eta}$ (problem!). We can solve this using *integration by parts*.

$$\int_0^T \lambda^{\mathrm{T}}\dot{\eta}\,\mathrm{d}t = \lambda^{\mathrm{T}}(T)\eta(T) - \lambda^{\mathrm{T}}(0)\eta(0) - \int_0^T \dot{\lambda}^{\mathrm{T}}\eta\,\mathrm{d}t$$

Hence,

$$\delta\tilde{J}(x_0;\nu) = \int_0^T \underbrace{\Big(\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial x} + \dot{\lambda}^{\mathrm{T}}\Big)}_{\text{pick}=0}\eta\,\mathrm{d}t - \underbrace{\lambda^{\mathrm{T}}(T)}_{\text{pick}=0}\eta(T) + \lambda^{\mathrm{T}}(0)\underbrace{\eta(0)}_{\nu}$$

We are free to pick λ freely if it gives $\delta\tilde{J} = 0$.

$$\text{Pick: } \begin{cases}\dot{\lambda}(t) = -\frac{\partial L}{\partial x}^{\mathrm{T}}(x(t)) - \frac{\partial f}{\partial x}^{\mathrm{T}}(x(t))\lambda(t)\\ \lambda(T) = 0 \qquad\qquad\qquad\qquad \text{backwards diff. eq}\end{cases}$$

Under this choice of λ we get

$$\delta \tilde{J}(x_0; \nu) = \lambda^{\mathrm{T}}(0)\nu$$

This is linear in ν so the FONC is $\lambda(0) = 0$.

Moreover, we really have a "normal" optimization problem

$$\min_{x_0 \in \mathbb{R}^m} \tilde{J}(x_0)$$

$$\delta \tilde{J}(x_0; \nu) = \frac{\partial \tilde{J}}{\partial x_0}(x_0)\nu$$

which means that

$$\frac{\partial \tilde{J}}{\partial x_0} = \lambda^{\mathrm{T}}(0)$$

If x_0^* minimizes

$$\int_0^T L(x(t))\,\mathrm{d}t$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0^* \end{cases}$$

then

$$\lambda(0) = \mathbf{0}$$

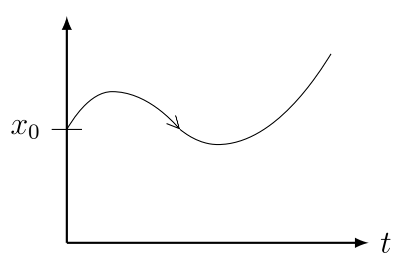
where $\lambda(t)$ satisfies

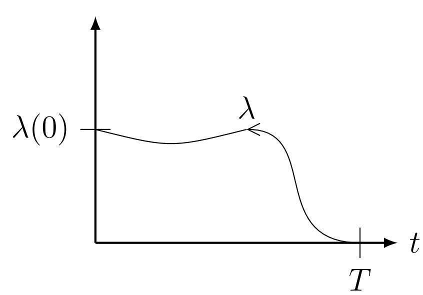
$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial L^{\mathrm{T}}}{\partial x}(x(t)) - \frac{\partial f^{\mathrm{T}}}{\partial x}(x(t))\lambda(t) \\ \lambda(T) = 0 \end{cases}$$

So what? We actually have a two-point boundary value problem.

$$\dot{x} = f(x) \qquad\qquad \dot{\lambda} = -\frac{\partial L^{\mathrm{T}}}{\partial x} - \frac{\partial f^{\mathrm{T}}}{\partial x}\lambda$$

$$x(0) = x_0 \qquad\qquad \lambda(T) = 0$$





We want to find x_0 that gives $f(x)$ such that after solving backwards for $\lambda(t)$, we find that

$$\lambda(0) = \frac{\partial \tilde{J}^{\mathrm{T}}}{\partial x_0} = 0.$$

This leads to the following:

Pick $x_{0,0}$
$k = 1$
repeat
  Simulate $x(t)$ from $x_{0,k}$ over $[0, T]$
  Simulate $\lambda(t)$ from $\lambda(T) = 0$ backwards using $x(t)$
  Update $x_{0,k}$ as $x_{0,k+1} = x_{0,k} - \gamma\lambda(0)$ ▷ $\lambda(0)$ is the gradient
  $k := k + 1$
until $\lambda(0) = 0$

An algorithm

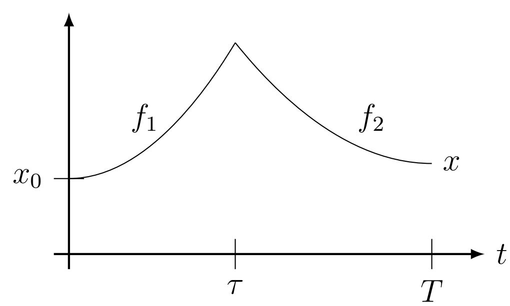
Example: `optinit.m`

$$\dot{x} = Ax, \quad L = x^{\mathrm{T}}Qx - q, \quad Q = Q^{\mathrm{T}} \succ 0$$
$$\dot{\lambda} = -2Qx - A^{\mathrm{T}}\lambda$$
$$\lambda(0) = 0$$

2.2.2 Optimal Timing Control

When to switch between modes?

$$\dot{x} = \begin{cases} f_1(x) & \text{if } t \in [0, \tau) \\ f_2(x) & \text{if } t \in [\tau, T] \end{cases} \tag{2.1}$$
$$x(0) = x_0$$

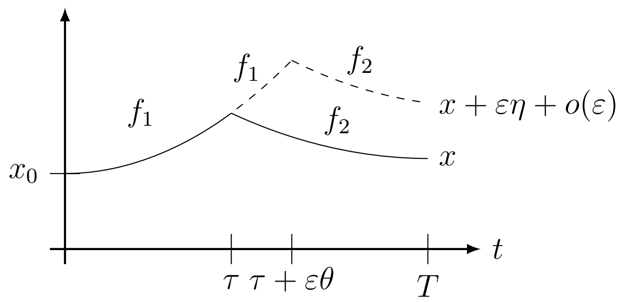


$$\min_{\tau} \int_0^T L(x(t))\,\mathrm{d}t = J(\tau)$$
$$\text{s.t. (2.1) holds}$$

Step 1: Augment cost with constraint

$$\tilde{J} = \int_0^{\tau} \Big[L(x) + \lambda^{\mathrm{T}}(f_1(x) - \dot{x})\Big]\,\mathrm{d}t + \int_{\tau}^{T} \Big[L(x) + \lambda^{\mathrm{T}}(f_2(x) - \dot{x})\Big]\,\mathrm{d}t$$

Step 2: Variation $\tau \longmapsto \tau + \varepsilon\theta$



Step 3: Compute $\delta\tilde{J}(\tau;\theta)$

$$\tilde{J}(\tau+\varepsilon\theta) = \int_0^{\tau+\varepsilon\theta} \Big\{ L(x+\varepsilon\eta) + \lambda^{\mathrm{T}}[f_1(x+\varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \Big\}\,\mathrm{d}t$$
$$+ \int_{\tau+\varepsilon\theta}^{T} \Big\{ L(x+\varepsilon\eta) + \lambda^{\mathrm{T}}[f_2(x+\varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \Big\}\,\mathrm{d}t + o(\varepsilon)$$

Note that $\eta = \dot{\eta} = 0$ on $[0, \tau)$.

$$\tilde{J}(\tau+\varepsilon\theta) = \int_0^{\tau} \Big\{ L(x) + \lambda^{\mathrm{T}}[f_1(x) - \dot{x}] \Big\}\,\mathrm{d}t$$
$$+ \int_{\tau}^{\tau+\varepsilon\theta} \Big\{ \underbrace{L(x+\varepsilon\eta)}_{L(x)+\varepsilon\frac{\partial L}{\partial x}\eta} + \lambda^{\mathrm{T}}[\underbrace{f_1(x+\varepsilon\eta)}_{f_1(x)+\varepsilon\frac{\partial f_1}{\partial x}\eta} - \dot{x} - \varepsilon\dot{\eta}] \Big\}\,\mathrm{d}t$$
$$+ \int_{\tau+\varepsilon\theta}^{T} \Big\{ \underbrace{L(x+\varepsilon\eta)}_{L(x)+\varepsilon\frac{\partial L}{\partial x}\eta} + \lambda^{\mathrm{T}}[\underbrace{f_2(x+\varepsilon\eta)}_{f_2(x)+\varepsilon\frac{\partial f_2}{\partial x}\eta} - \dot{x} - \varepsilon\dot{\eta}] \Big\}\,\mathrm{d}t + o(\varepsilon)$$

$$
\delta\tilde{J}(\tau;\theta) = \lim_{\varepsilon\to 0}\frac{\tilde{J}(\tau+\varepsilon\theta)-\tilde{J}(\tau)}{\varepsilon}
$$

$$
\begin{aligned}
\tilde{J}(\tau+\varepsilon\theta)-\tilde{J}(\tau) = \int_0^{\tau} 0\cdot \mathrm{d}t + \underbrace{\int_{\tau}^{\tau+\varepsilon\theta}\left[\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\Big(f_1(x)+\varepsilon\frac{\partial f_1}{\partial x}\eta - f_2(x) - \varepsilon\dot{\eta}\Big)\right]\mathrm{d}t}_{I_1} \\
+ \underbrace{\int_{\tau+\varepsilon\theta}^{T}\left[\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\Big(\varepsilon\frac{\partial f_2}{\partial x}\eta - \varepsilon\dot{\eta}\Big)\right]\mathrm{d}t}_{I_2} + o(\varepsilon)
\end{aligned}
$$

Theorem (Mean-value theorem)**.**

$$
\int_{t_1}^{t_2} h(t)\,\mathrm{d}t = (t_2-t_1)h(\xi) \quad \textit{for some } \xi\in[t_1,t_2]
$$

The first integral is

$$
\begin{aligned}
I_1 &= \int_{\tau}^{\tau+\varepsilon\theta}\left\{\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\left[f_1(x)+\varepsilon\frac{\partial f_1}{\partial x}\eta - \varepsilon\dot{\eta} - f_2(x)\right]\right\}\mathrm{d}t \\
&= \varepsilon\theta\Big\{\lambda^{\mathrm{T}}(\xi)\big[f_1(x(\xi)) - f_2(x(\xi))\big]\Big\} + o(\varepsilon)
\end{aligned}
$$

Note that as $\varepsilon\to 0$, $\xi\to\tau$. Using integration by parts, the second integral is

$$
\int_{\tau}^{T}\lambda^{\mathrm{T}}\dot{\eta}\,\mathrm{d}t = \lambda^{\mathrm{T}}(T)\eta(T) - \lambda^{\mathrm{T}}(\tau)\underbrace{\eta(\tau)}_{=0} - \int_{\tau}^{T}\dot{\lambda}^{\mathrm{T}}\eta\,\mathrm{d}t
$$

$$
\begin{aligned}
I_2 &= \int_{\tau}^{T}\left[\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\Big(\varepsilon\frac{\partial f_2}{\partial x}\eta - \varepsilon\dot{\eta}\Big)\right]\mathrm{d}t - \underbrace{\int_{\tau}^{\tau+\varepsilon\theta}\left[\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\Big(\varepsilon\frac{\partial f_2}{\partial x}\eta - \varepsilon\dot{\eta}\Big)\right]\mathrm{d}t}_{o(\varepsilon)} \\
&= \varepsilon\int_{\tau}^{T}\left[\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f_2}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right]\eta\,\mathrm{d}t - \varepsilon\lambda^{\mathrm{T}}(T)\eta(T) + o(\varepsilon)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\delta\tilde{J}(\tau;\theta) &= \lim_{\varepsilon\to 0}\frac{\tilde{J}(\tau+\varepsilon\theta)-\tilde{J}(\tau)}{\varepsilon} \\
&= \theta\lambda^{\mathrm{T}}(\tau)\Big[f_1(x(\tau)) - f_2(x(\tau))\Big] + \int_{\tau}^{T}\left[\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f_2}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right]\eta\,\mathrm{d}t - \lambda^{\mathrm{T}}(T)\eta(T)
\end{aligned}
$$

Step 4: Select the *costate* $\lambda(t)$. The key idea is to get rid of any term that has η in it, i.e.

$$
\begin{aligned}
\dot{\lambda} &= -\frac{\partial L^{\mathrm{T}}}{\partial x} - \frac{\partial f_2^{\mathrm{T}}}{\partial x}\lambda \quad \text{on } [\tau,T] \\
\lambda(T) &= 0
\end{aligned}
$$

Step 5: With this choice of $\lambda(t)$, we have

$$\delta \tilde{J}(\tau;\theta) = \theta\lambda^{\mathrm{T}}(\tau)\Big[f_1(x(\tau)) - f_2(x(\tau))\Big] = \frac{\partial \tilde{J}}{\partial \tau}\theta.$$

Therefore,

$$\frac{\partial \tilde{J}}{\partial \tau} = \lambda^{\mathrm{T}}(\tau)\Big[f_1(x(\tau)) - f_2(x(\tau))\Big] = 0 \quad \text{(for optimality)}$$

Algorithm

Pick τ_0
$k = 0$
repeat
  Simulate x forward in time from $x(0) = x_0$
  Simulate λ backwards from $\lambda(T) = 0$
  Update τ_k as $\tau_{k+1} = \tau_k - \gamma\lambda^{\mathrm{T}}(\tau_k)\big[f_1(x(\tau_k)) - f_2(x(\tau_k))\big]$
  $k := k + 1$
until $\|\lambda^{\mathrm{T}}(f_1 - f_2)\| < \varepsilon$

Where are we going? Come up with general principles for $\min_{u\in\mathcal{U}} J(u)$:

- Costate equations
- Optimality conditions
- Algorithms
- Applications

Chapter 3

The Maximum Principle

3.1 The Bolza Problem

Up until now, we have optimized with respect to finite-dimensional parameters. Today, we will minimize with respect to $u \in \mathcal{U}$.

$$
\begin{aligned}
\min_{u\in\mathcal{U}} J(u) &= \int_0^T L(x(t),u(t),t)\,\mathrm{d}t + \underbrace{\Psi(x(T))}_{\substack{\text{terminal cost}\\ \text{(parking cost)}}} \\
\text{s.t.}\quad \dot{x}(t) &= f(x(t),u(t),t) \\
x(0) &= x_0
\end{aligned}
$$

Assume that f and L are C^1 in x, u and piecewise continuous in t. Then, a small change in u causes small changes in f and L. The variation: $u \longmapsto u + \varepsilon v$, $\varepsilon \in \mathbb{R}$, $v \in \mathcal{U}$. See Figure 2.1.

$$
\begin{aligned}
\tilde{J}(u) &= \int_0^T \left[L(x,u,t) + \lambda^{\mathrm{T}}(f(x,u,t) - \dot{x})\right] \mathrm{d}t + \Psi(x(T)) \\
\tilde{J}(u+\varepsilon v) &= \int_0^T \left[L(x+\varepsilon\eta, u+\varepsilon v, t) + \lambda^{\mathrm{T}}(f(x+\varepsilon\eta, u+\varepsilon v, t) - \dot{x} - \varepsilon\dot{\eta})\right] \mathrm{d}t \\
&\quad + \Psi(x(T) + \varepsilon\eta(T)) + o(\varepsilon) \\
\tilde{J}(u+\varepsilon v) - \tilde{J}(u) &= \int_0^T \Bigg[L(x+\varepsilon\eta, u+\varepsilon v, t) - L(x,u,t) \\
&\qquad + \lambda^{\mathrm{T}}\Big(f(x+\varepsilon\eta, u+\varepsilon v, t) - f(x,u,t) - \dot{x} - \varepsilon\dot{\eta} + \dot{x}\Big)\Bigg] \mathrm{d}t \\
&\quad + \Psi(x(T) + \varepsilon\eta(T)) - \Psi(x(T)) + o(\varepsilon) \\
&= \int_0^T \left[\frac{\partial L}{\partial x}\varepsilon\eta + \frac{\partial L}{\partial u}\varepsilon v + \lambda^{\mathrm{T}}\left(\frac{\partial f}{\partial x}\varepsilon\eta + \frac{\partial f}{\partial u}\varepsilon v - \varepsilon\dot{\eta}\right)\right] \mathrm{d}t \\
&\quad + \frac{\partial \Psi}{\partial x}(x(T))\varepsilon\eta(T) + o(\varepsilon)
\end{aligned}
$$

(See Taylor expansion with respect to two variables.)

$$\delta\tilde{J}(u;v) = \int_0^T \left(\frac{\partial L}{\partial u} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial u}\right) v\,\mathrm{d}t + \int_0^T \left[\left(\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial x}\right)\eta - \lambda^{\mathrm{T}}\dot{\eta}\right]\mathrm{d}t$$
$$+ \frac{\partial \Psi}{\partial x}(x(T))\eta(T)$$

Integrating by parts,

$$\begin{aligned}\int_0^T \lambda^{\mathrm{T}}\dot{\eta}\,\mathrm{d}t &= \lambda^{\mathrm{T}}(T)\eta(T) - \lambda^{\mathrm{T}}(0)\eta(0) - \int_0^T \dot{\lambda}^{\mathrm{T}}\eta\,\mathrm{d}t \\ &= \lambda^{\mathrm{T}}(T)\eta(T) - \int_0^T \dot{\lambda}^{\mathrm{T}}\eta\,\mathrm{d}t \\ \delta\tilde{J}(u;v) &= \int_0^T \left(\frac{\partial L}{\partial u} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial u}\right) v\,\mathrm{d}t + \int_0^T \left(\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta\,\mathrm{d}t \\ &\quad + \left(\frac{\partial \Psi}{\partial x}(x(T)) - \lambda^{\mathrm{T}}(T)\right)\eta(T)\end{aligned}$$

For optimality, we need the directional derivative to be zero for every $v \in \mathcal{U}$, where v represents the direction of the derivative. Therefore, the term $(\frac{\partial L}{\partial u} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial u})$ in the first integral has to be identically zero. Thus, we need

$$\begin{cases} \dfrac{\partial L}{\partial u} + \lambda^{\mathrm{T}}\dfrac{\partial f}{\partial u} = 0, & \forall t \in [0,T] \\ \dfrac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\dfrac{\partial f}{\partial x} + \dot{\lambda}^{\mathrm{T}} = 0, & \forall t \in [0,T] \\ \dfrac{\partial \Psi}{\partial x}(x(T)) - \lambda^{\mathrm{T}}(T) = 0 \end{cases}$$

Definition. Let the *Hamiltonian* $H(x,u,t,\lambda)$ be given by

$$H(x,u,t,\lambda) = L(x,u,t) + \lambda^{\mathrm{T}} f(x,u,t)$$

Theorem. *For u to solve the Bolza problem, it has to satisfy*

$$\frac{\partial H}{\partial u}(x,u,t,\lambda) = 0,$$

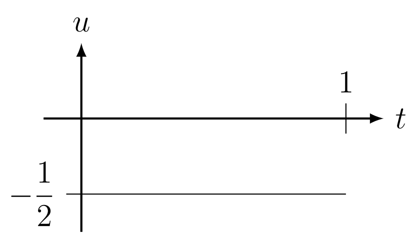
where the costate *satisfies*

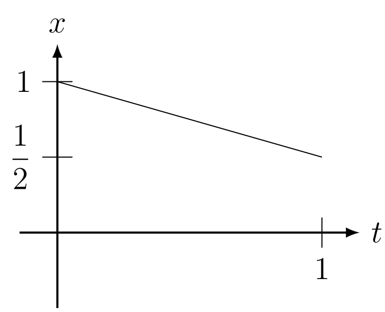
$$\begin{cases} \dot{\lambda} = -\dfrac{\partial H}{\partial x}^T(x,u,t,\lambda) \\ \lambda(T) = \dfrac{\partial \Psi}{\partial x}^T(x(T)) \end{cases}$$

Example

$$\min_u \int_0^1 \frac{1}{2}u^2(t)\,\mathrm{d}t + \frac{1}{2}x^2(1)$$
$$\text{s.t.} \begin{cases} \dot{x} = u, & x, u \in \mathbb{R} \\ x(0) = 1 \end{cases}$$

$$H = \frac{1}{2}u^2 + \lambda u$$
$$\frac{\partial H}{\partial u} = u + \lambda = 0 \Longrightarrow u = -\lambda$$
$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0 \Longrightarrow \lambda(t) = c$$
$$\lambda(T) = c = \frac{\partial \Psi}{\partial x}(x(1)) = x(1)$$
$$\dot{x} = u = -c \Longrightarrow x(t) = -ct + x(0) = -ct + 1$$
$$x(1) = -c + 1$$
$$\lambda(1) = c = x(1) = -c + 1 \Longrightarrow c = \frac{1}{2}$$
$$\boxed{u^* = -\frac{1}{2}}$$





We really used five different equations to solve this!

i) $\dfrac{\partial H}{\partial u} = 0$

ii) $\dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x}$

iii) $\lambda(T) = \dfrac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T))$

iv) $\dot{x} = f(x,u,t)$

v) $x(0) = x_0$

There is a sixth condition that is pretty useful if L and f do not depend on t ($L(x,u)$, $f(x,u)$). This is called a *conservative system*. Then, along optimal trajectories (equations i-v are satisfied), the total time derivative of the Hamiltonian is

$$\frac{\mathrm{d}}{\mathrm{d}t}H = \underbrace{\frac{\partial H}{\partial t}}_{\substack{0\\ H(x,u,\lambda)}} + \underbrace{\frac{\partial H}{\partial x}}_{-\dot{\lambda}^{\mathrm{T}}}\dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{\substack{0\\ u \text{ is optimal}}}\dot{u} + \underbrace{\frac{\partial H}{\partial \lambda}}_{f^{\mathrm{T}}=\dot{x}^{\mathrm{T}}}\dot{\lambda} = -\dot{\lambda}^{\mathrm{T}}\dot{x} + \dot{x}^{\mathrm{T}}\dot{\lambda} = 0$$

Therefore, for conservative systems,

vi) H is constant along optimal trajectories. (Hamilton's Principle in analytical mechanics)

Back to the example,

$$H = \frac{1}{2}u^2 + \lambda u = \frac{1}{2}c^2 - c^2 = -\frac{1}{2}c^2 = -\frac{1}{8}$$

The Hamiltonian

$$H(x,u,t,\lambda) = L(x,u,t) + \lambda^{\mathrm{T}} f(x,u,t)$$

lets us write the Lagrangian as

$$\tilde{J}(u) = \int_0^T \left[L + \lambda^{\mathrm{T}}(f - \dot{x})\right] \mathrm{d}t + \Psi = \int_0^T \left(H - \lambda^{\mathrm{T}}\dot{x}\right) \mathrm{d}t + \Psi$$

The optimality conditions are

$$\frac{\partial H}{\partial u} = 0, \tag{3.1}$$

where

$$\begin{cases} \dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x} \\ \lambda(T) = \dfrac{\partial \Psi}{\partial x}(x(T)) \end{cases} \tag{3.2}$$

Example Hamilton's Principle

Let q be the generalized coordinates (positions and angles). Then, $\dot{q} = u$ are generalized velocities, which we assume we can control. Let $T(q,u) = u^{\mathrm{T}} M(q) u$, $M \succ 0$, be the kinetic energy and $V(q)$ be the potential energy.

For conservative systems, the following quantity is minimized:

$$\int_0^T \underbrace{\big[T(q,u) - V(q)\big]}_{\substack{L(q,u)= \\ \text{Lagrange's ``action function''}}} \mathrm{d}t$$

The Hamiltonian is

$$H(q,u,\lambda) = L(q,u) + \lambda^{\mathrm{T}} f(q,u) = L(q,u) + \lambda^{\mathrm{T}} u$$

In mechanics, λ is called a generalized momentum, satisfying

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial H}{\partial q}^{\mathrm{T}} = -\frac{\partial L}{\partial q}^{\mathrm{T}} + 0 \\
0 &= \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^{\mathrm{T}} \implies \lambda = -\frac{\partial L}{\partial u}^{\mathrm{T}} \\
\dot{\lambda} &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial u}^{\mathrm{T}} = -\frac{\partial L}{\partial q}^{\mathrm{T}}
\end{aligned}$$

This produces the Euler-Lagrange Equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0}$$

Recall, along optimal trajectories

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \underbrace{\frac{\partial H}{\partial t}}_{\substack{=0 \text{ if } L \text{ and } f \text{ do not} \\ \text{depend explicitly on } t}} + \overbrace{\frac{\partial H}{\partial x}}^{-\dot{\lambda}^{\mathrm{T}}\dot{x}}\dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{=0}\dot{u}\ \underbrace{\frac{\partial H}{\partial \lambda}}_{f^{\mathrm{T}}=\dot{x}^{\mathrm{T}}}\ \dot{\lambda} = -\dot{\lambda}^{\mathrm{T}}\dot{x} + \dot{x}^{\mathrm{T}}\dot{\lambda} = 0$$

Therefore, along optimal trajectories, the Hamiltonian is constant!

We had

$$\begin{aligned}
H &= L + \lambda^{\mathrm{T}} u \\
\frac{\partial H}{\partial u} &= \lambda^{\mathrm{T}} + \frac{\partial L}{\partial u} = 0
\end{aligned}$$

Along optimal trajectories,

$$H = L - \frac{\partial L}{\partial u} u$$

Recall, $L(q,u) = T(q,u) - V(q)$.

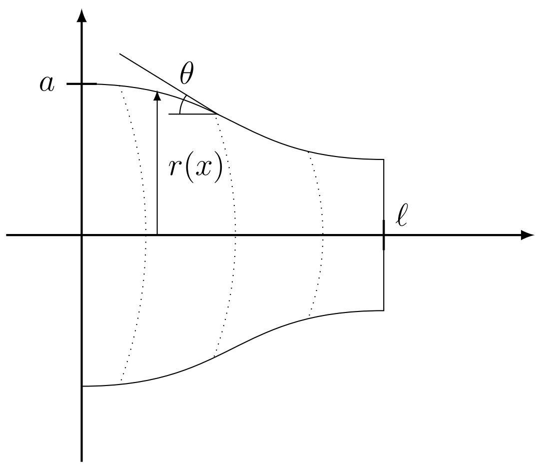
$$\begin{aligned}
\frac{\partial L}{\partial u} &= \frac{\partial T}{\partial u} - 0 \\
T(q,u) &= u^{\mathrm{T}} M(q) u \\
\frac{\partial T}{\partial u} &= 2u^{\mathrm{T}} M
\end{aligned}$$

So,

$$H = \underbrace{T}_{u^{\mathrm{T}}Mu} - V - 2u^{\mathrm{T}}Mu = -(V + u^{\mathrm{T}}Mu) = -(V+T)$$

Therefore, the total energy (kinetic plus potential energy) remains constant for conservative systems.

Example minimum drag nose shape (Newton 1686)



The drag is

$$D = -2\pi q \int_{x=0}^{\ell} C_p(\theta) r \, \mathrm{d}r,$$

where q is a pressure constant and $C_p(\theta) = 2\sin^2\theta$ is Newton's pressure formula.

Geometry tells us

$$\frac{\mathrm{d}r}{\mathrm{d}x} = -\tan\theta = -u$$

Choose the control as $\tan\theta$. Manipulating the drag,

$$\frac{D}{4\pi q} = \int_0^{\ell} \frac{ru^3}{1+u^2} \, \mathrm{d}x + \frac{1}{2} r(\ell)^2$$

The optimal control problem is

$$\begin{aligned}
&\min_u \int_0^{\ell} \frac{ru^3}{1+u^2} \, \mathrm{d}x + \frac{1}{2} r(\ell)^2 \\
&\text{s.t. } \frac{\mathrm{d}r}{\mathrm{d}x} = -u
\end{aligned}$$

This is in the standard form with the following changes of variables:

$$
\begin{aligned}
\ell &\longleftarrow T \\
x &\longleftarrow t \\
r &\longleftarrow x
\end{aligned}
$$

Refer to (3.1) and (3.2) for the following steps.

$$
\begin{aligned}
H &= \frac{ru^3}{1+u^2} - \lambda u \\
\frac{\partial H}{\partial u} &= \frac{3ru^2(1+u^2) - ru^3 \cdot 2u}{(1+u^2)^2} - \lambda \\
&= \frac{ru^4 + 3ru^2}{(1+u^2)^2} - \lambda = 0 \\
\lambda &= \frac{ru^2(u^2+3)}{(1+u^2)^2} \qquad (3.3) \\
\frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2} \\
\lambda(\ell) &= r(\ell)
\end{aligned}
$$

Right now, we know

$$
\begin{cases}
\dfrac{\mathrm{d}r}{\mathrm{d}x} = -u \\
r(0) = a \\
\dfrac{\mathrm{d}\lambda}{\mathrm{d}x} = -\dfrac{u^3}{1+u^2} \\
\lambda(\ell) = r(\ell)
\end{cases}
$$

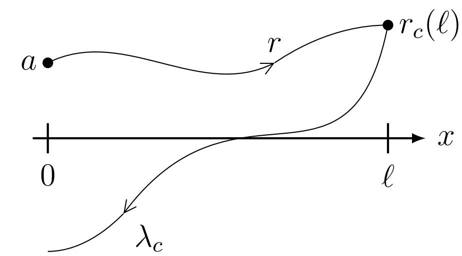
We need to remove u and get a function of r and λ instead. However, it is difficult to solve (3.3). Maybe $H = \text{const.}$ gives us something nicer?

$$
\begin{aligned}
H &= \frac{ru^3}{1+u^2} - \lambda u \\
&= \frac{ru^3}{1+u^2} - \frac{ru^2(u^2+3)}{(1+u^2)^2} u \\
&= -\frac{2ru^3}{(1+u^2)^2} = c
\end{aligned}
$$

Assume we can find $u = G(r, c)$, either numerically or some other way. So, now we have

$$
\begin{cases}
\dfrac{\mathrm{d}r}{\mathrm{d}x} = -G(r, c) \\
r(0) = a \\
\dfrac{\mathrm{d}\lambda}{\mathrm{d}x} = -\dfrac{G^3(r, c)}{1 + G^2(r, c)} \\
\lambda(\ell) = r(\ell)
\end{cases}
$$

We do not know c, but we can guess c and simulate r forward in "time" (x) from $r(0) = a$. Then, we simulate λ backwards from $r(\ell)$.

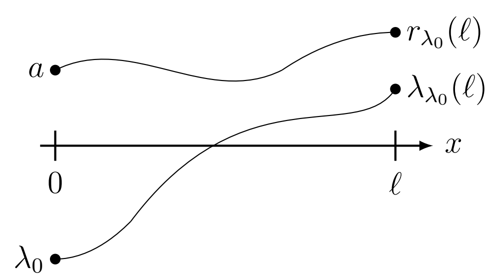


Problem: we can do this for any c. Which c is it? *Last 15 minutes was a dead end!*

Back to $u = F(r, \lambda)$. Assume we have F (numerically).

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}x} &= -F(r, \lambda) \\
r(0) &= a \\
\frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{F^3(r, \lambda)}{1 + F^2(r, \lambda)} \\
\lambda(\ell) &= r(\ell)
\end{aligned}$$

The mistake before was that the simulation forward from a depends on λ.



Therefore, we should guess λ_0 and simulate both r and λ to get $r_{\lambda_0}(\ell)$ and $\lambda_{\lambda_0}(\ell)$. We need

$$r_{\lambda_0}(\ell) = \lambda_{\lambda_0}$$

for optimality. To do this, we need numerics.

Terminal Constraints

Let $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ and solve

$$\begin{aligned}
\min_{u \in \mathcal{U}} \int_0^T & L(x, u, t)\,\mathrm{d}t + \Psi(x(T)) \\
\text{s.t.} \qquad \dot{x} &= f(x, u, t) \\
x(0) &= x_0 \\
x_i(T) &= x_{iT} \quad \text{given for } i \in \mathcal{T} \subset \{1, \ldots, n\}
\end{aligned}$$

First, we augment the cost:

$$
\begin{aligned}
\tilde{J}(u) &= \int_0^T \left[L + \lambda^{\mathrm{T}}(f - \dot{x})\right] \mathrm{d}t + \Psi \\
&= \int_0^T (H - \lambda^{\mathrm{T}}\dot{x}) \, \mathrm{d}t + \Psi \\
\tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \left(\varepsilon \frac{\partial H}{\partial u} v + \varepsilon \frac{\partial H}{\partial x} \eta - \varepsilon \lambda^{\mathrm{T}} \dot{\eta}\right) \mathrm{d}t + \varepsilon \frac{\partial \Psi}{\partial x}(x(T))\eta(T) + o(\varepsilon) \\
\delta \tilde{J}(u; v) &= \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right) \eta \, \mathrm{d}t + \int_0^T \frac{\partial H}{\partial u} v \, \mathrm{d}t \\
&\quad + \lambda^{\mathrm{T}}(0)\eta(0) - \lambda^{\mathrm{T}}(T)\eta(T) + \frac{\partial \Psi}{\partial x}(x(T))\eta(T)
\end{aligned}
$$

As always,

$$
\begin{aligned}
\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} \\
\frac{\partial H}{\partial u} &= 0 \quad \text{(FONC)}
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
\eta(0) &= 0 \\
\eta_i(T) &= 0 \quad \text{for } i \in \mathcal{T}
\end{aligned}
$$

Note that if $x(T) = x_T$ is given, then $x(T) = x(T) + \varepsilon\eta(T) + o(\varepsilon)$, so $\eta(T) = 0$. Here, we have $x_i(T) = x_{iT}$ fixed for $i \in \mathcal{T}$ so $\eta_i(T) = 0$ for $i \in \mathcal{T}$.

For optimality, we want

$$
\left[-\lambda^{\mathrm{T}}(T) + \frac{\partial \Psi}{\partial x}(x(T))\right] \eta(T) = 0 \quad \text{for all } \textit{admissible} \text{ variations}
$$

$$
\begin{bmatrix} \dfrac{\partial \Psi}{\partial x_1} - \lambda_1, & \cdots, & \dfrac{\partial \Psi}{\partial x_n} - \lambda_n \end{bmatrix} \begin{bmatrix} \eta_1(T) \\ \vdots \\ \eta_n(T) \end{bmatrix} = 0
$$

Hence, we need

$$
\begin{aligned}
\lambda_j(T) &= \frac{\partial \Psi}{\partial x_j}(x(T)) &&\text{if } j \notin \mathcal{T} \\
\lambda_i(T) &= \text{free} &&\text{if } i \in \mathcal{T}
\end{aligned}
$$

So we have

$$
\begin{cases}
\dot{x} = f \\
\dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x},
\end{cases}
$$

an ODE with $2n$ variables. We need $2n$ boundary conditions for this ODE to be well-posed.

At $t = 0$		At $t = T$	
$x(0) = x_0$	$[n]$	$x_i(T) = x_{iT},\ i \in \mathcal{T}$ $\lvert\mathcal{T}\rvert = q$	$[q]$
		$x_j(T)$ free, $j \notin \mathcal{T}$	$[0]$
$\lambda(0)$ free	$[0]$	$\lambda_i(T)$ free, $i \in \mathcal{T}$	$[0]$
		$\lambda_j(T) = \dfrac{\partial \Psi}{\partial x_j}(x(T)),\ j \notin \mathcal{T}$	$[n-q]$

So we have $n + q + (n - q) = 2n$ boundary conditions.

We could even fix some but not all of $x(0)$, i.e.

$$\begin{aligned} x_i(0) &= x_{i0} \quad && \text{if } i \in \mathcal{I} \\ x_j(0) &= \text{free} && \text{if } j \notin \mathcal{I} \end{aligned}$$

Recall,

$$\delta \tilde{J}(u; v) = \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}} \right) \eta \,\mathrm{d}t + \int_0^T \frac{\partial H}{\partial u} v \,\mathrm{d}t + \lambda^{\mathrm{T}}(0)\eta(0) + \left[\lambda^{\mathrm{T}}(T) - \frac{\partial \Psi}{\partial x}(x(T)) \right] \eta(T)$$

For $x_i(0) = x_{i0}$ fixed, we have $\eta_i(0) = 0$ and $\lambda_i(0)$ free. For $x_j(0)$ free, we have $\eta_j(0)$ free and $\lambda_j(0) = 0$.

To ponder, what if $J = \int L \,\mathrm{d}t + \Psi(x(T)) + \Theta(x(0))$?

To summarize, the minimizer to

$$\begin{aligned} &\min_{u \in \mathcal{U}} \int_0^T L(x, u, t) \,\mathrm{d}t + \Psi(x(T)) \\ &\text{s.t.} \qquad \dot{x} = f(x, u, t) \\ &\qquad\quad x_i(0) = x_{i0}, \quad i \in \mathcal{I} \\ &\qquad\quad x_j(T) = x_{jT} \quad j \in \mathcal{T} \end{aligned}$$

has to satisfy

$$\begin{aligned} &\frac{\partial H}{\partial u} = 0 \\ &\dot{\lambda} = -\frac{\partial H^{\mathrm{T}}}{\partial x} \\ &\lambda_i(0) = 0, \quad i \notin \mathcal{I} \\ &\lambda_j(T) = \frac{\partial \Psi}{\partial x_j}(x(T)), \quad j \notin \mathcal{T} \end{aligned}$$

Example

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = f(x_1, x_2, x_3, x_4)$$

$$x_1(0) = 1,\ x_3(0) = 7,\ x_4(0) = 0,\ x_1(1) = 2$$

$$\mathcal{I} = \{1, 3, 4\},\ \mathcal{T} = \{1\}$$

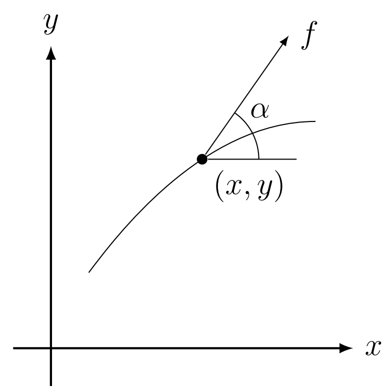
$$\min \int_0^1 L(x, u)\, \mathrm{d}t + \left(x_2^2(1) - x_3^2(1) + 7x_1(1) + 14\right)$$

Note there are 4 boundary conditions on x so there must be 4 boundary conditions on λ:

$\lambda_1(0)$ free/unspecified	$\lambda_1(1)$ free
$\lambda_2(0) = 0$	$\lambda_2(1) = 2x_2(1)$
$\lambda_3(0)$ free	$\lambda_3(1) = -2x_3(1)$
$\lambda_4(0)$ free	$\lambda_4(1) = 0$

Example

A force f acts on a particle at position (x, y) (mass = 1).



$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= |f| \cos\alpha \\ \dot{v}_y &= |f| \sin\alpha \\ \alpha &= \text{control variable} \end{aligned}$$

Assume we only care about where the particle ends up (to be specified later), i.e. $L = 0$.

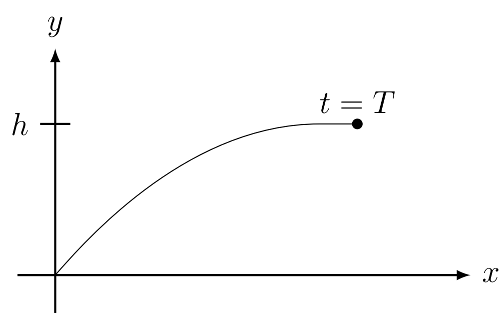
$$H = \begin{bmatrix} \lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f| \cos\alpha \\ |f| \sin\alpha \end{bmatrix}$$

$$
\begin{aligned}
\dot{\lambda}_x &= -\frac{\partial H}{\partial x} = 0 &&\implies \lambda_x(t) = c_1 \\
\dot{\lambda}_y &= -\frac{\partial H}{\partial y} = 0 &&\implies \lambda_y(t) = c_2 \\
\dot{\lambda}_{v_x} &= -\frac{\partial H}{\partial v_x} = -\lambda_x &&\implies \lambda_{v_x}(t) = -c_1 t + c_3 \\
\dot{\lambda}_{v_y} &= -\frac{\partial H}{\partial v_y} = -\lambda_y &&\implies \lambda_{v_y}(t) = -c_2 t + c_4
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{\partial H}{\partial \alpha} &= -\lambda_{v_x}|f|\sin\alpha + \lambda_{v_y}|f|\cos\alpha = 0 \\
\tan\alpha &= \frac{\lambda_{v_y}}{\lambda_{v_x}} = \frac{-c_2 t + c_4}{-c_1 t + c_3}
\end{aligned}
$$

We want to drive the particle from $[0,0,0,0]^{\mathrm{T}}$ to a path parallel to the x-axis with $y(T) = h$.



Choose $\Psi = -v_x$,

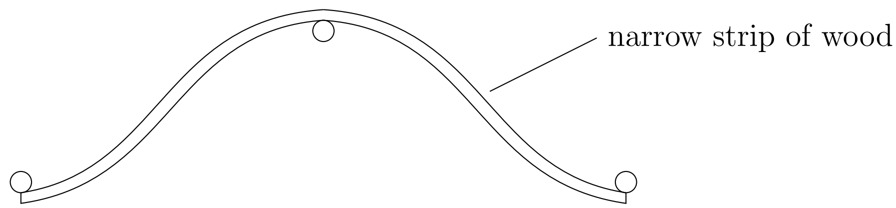
$$
\begin{array}{rl}
y(T) = h & v_y(T) = 0 \\
x(T) \text{ free} & v_x(T) \text{ free, but costs} \\
\lambda_i(0) \text{ free} & \\
\lambda_y(T) \text{ free} & \lambda_{v_y}(T) \text{ free} \\
\lambda_x(T) = 0 & \lambda_{v_x}(T) = -1
\end{array}
$$

$$
\begin{aligned}
c_1 &= \lambda_x(t) = 0 \\
\implies \lambda_{v_x} &= -c_1 t + c_3 = c_3 = -1 \\
\implies \tan\alpha &= -\frac{-c_2 t + c_4}{-1} = c_2 t + c_4
\end{aligned}
$$

How do we find c_2 and c_4? Plug into $\dot{x}$ and $\dot{\lambda}$ and try to satisfy the remaining boundary conditions. (This is hard=numerics.)

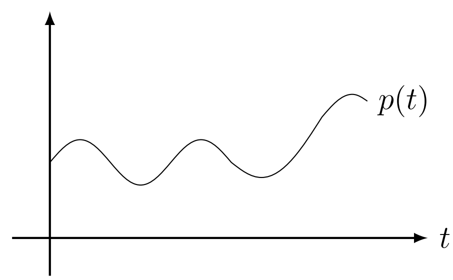
3.2 Splines

From ship building. Splines are used a lot in path-planning, e.g. cubic splines.



But, they are solutions to optimal control problems.

Let $p(t)$ be a curve we'd like to shape.



We want to minimize the "energy" put into the curve, a.k.a acceleration. Let $x_1 = p$ and $x_2 = \dot{p}$, so

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

3.2.1 Minimum-Energy

$$\begin{aligned}
\min_{u \in \mathcal{U}} \frac{1}{2} \int_0^T u^2(t)\,\mathrm{d}t \quad &+ \text{Boundary conditions on } x \\
H = L + \lambda^{\mathrm{T}} f &= \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u \\
\frac{\partial H}{\partial u} &= u + \lambda_2 = 0 \implies u = -\lambda_2 \\
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = c_1 \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -c_1 t + c_2 \\
u &= -\lambda_2 = c_1 t - c_2 \\
\dot{x}_2 &= u = c_1 t - c_2 \implies x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3 \\
\dot{x}_1 &= x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3 \\
&\implies x_1 = \frac{c_1}{6} t^3 - \frac{c_2}{2} t^2 + c_3 t + c_4
\end{aligned}$$

$p(t)$ is a cubic polynomial!

What about boundary conditions?

Let $T = 1$, $p(0)$ given, $p(1)$ given, $\dot{p}(0) = 0$, $\dot{p}(1) = 0$, e.g. $p(0) = 0$, $p(1) = 1$. Since the boundary conditions for x are all specified, those for the costate are free.

$$\left.\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= 0 \\ x_1(1) &= 1 \\ x_2(1) &= 0 \end{aligned}\right\} \implies \left\{\begin{aligned} &\lambda_1(0) \\ &\lambda_2(0) \\ &\lambda_1(1) \\ &\lambda_2(1) \end{aligned}\right. \quad \text{free/unspecified}$$

$$\begin{aligned} x_2(0) &= c_3 = 0 & x_1(1) &= \frac{2c_2}{6} - \frac{c_2}{2} = 1 \\ x_1(0) &= c_4 = 0 & c_2 &= -6 \\ x_2(1) &= \frac{c_1}{2} - c_2 + \underbrace{c_3}_{0} = 0 & c_1 &= -12 \\ c_1 &= 2c_2 & & \end{aligned}$$

$$\begin{aligned} \implies p(t) &= -2t^3 + 3t^2 \\ u(t) &= -12t + 6 \end{aligned}$$

Or, what if $\dot{p}(0)$, $\dot{p}(1)$ are not specified?

$$\left.\begin{aligned} &x_1(0) = 0 \\ &x_2(0) \text{ unspec.} \\ &x_1(1) = 1 \\ &x_2(1) \text{ unspec.} \end{aligned}\right\} \implies \left\{\begin{aligned} &\lambda_1(0) \text{ unspec.} \\ &\lambda_2(0) = 0 \\ &\lambda_1(1) \text{ unspec.} \\ &\lambda_2(1) = 0 \end{aligned}\right.$$

$$\left.\begin{aligned} \lambda_2(0) &= c_2 = 0 \\ \lambda_2(1) &= -c_1 + c_2 = 0 \end{aligned}\right\} \implies u = c_1 t - c_2 = 0$$

$$\left.\begin{aligned} x_1(0) &= c_4 = 0 \\ x_1(1) &= c_3 = 1 \end{aligned}\right\} \implies p(t) = t$$

What did we do?

Case 1:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1/2 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_2(0) \\ x_2(1) \end{bmatrix}$$

Case 2:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ \lambda_2(0) \\ \lambda_2(1) \end{bmatrix}$$

3.2.2 Generalized Splines

We had $\dot{x} = Ax + Bu$ with
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
This A is nilpotent ($A^k = 0$ for some $k \in \mathbb{Z}^+$). This means e^{At} is a polynomial in t. (This e^{At} is cubic.)

In general, e^{At} is a mix of polynomials, exponentials, and trignometric terms. The eigenvalues of A determine the form of $x(t)$.

$$\begin{aligned} \dot{x} &= Ax && \Longrightarrow x(t) = e^{At}x(0) \\ \dot{x} &= Ax + Bu && \Longrightarrow x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\,\mathrm{d}\tau \end{aligned}$$

The general problem to solve is

$$\begin{aligned} &\min_{u \in \mathcal{U}} \int_0^T \frac{1}{2}\|u\|^2 \,\mathrm{d}t \\ &\text{s.t. } \dot{x} = Ax + Bu \\ &\quad + \text{Boundary conditions} \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2}\|u\|^2 + \lambda^{\mathrm{T}}(Ax + Bu) \\ \frac{\partial H}{\partial u} &= u^{\mathrm{T}} + \lambda^{\mathrm{T}} B = 0 \\ &\Rightarrow u = -B^{\mathrm{T}}\lambda \\ \dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} = -A^{\mathrm{T}}\lambda \end{aligned}$$

We have the Hamiltonian Dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BB^{\mathrm{T}} \\ 0 & -A^{\mathrm{T}} \end{bmatrix}}_{M} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

Where we used $\dot{x} = Ax + Bu = Ax - BB^{\mathrm{T}}\lambda$. Then,

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

Suppose we want to drive from $x(0) = x_0$ to $x(T) = x_T$.

$$\begin{gathered} \begin{bmatrix} x_T \\ \lambda(T) \end{bmatrix} = e^{MT} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix} = \begin{bmatrix} N_{xx} & N_{x\lambda} \\ N_{\lambda x} & N_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix} \\ x_T = N_{xx}x_0 + N_{x\lambda}\lambda(0) \end{gathered}$$

$N_{x\lambda}$ is invertible if (A, B) is completely controllable. Assume it is.

$$\begin{gathered}
\lambda(0) = N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \\
\implies \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} x_0 \\ N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \end{bmatrix} \\
\implies u(t) = -B^{\mathrm{T}}\lambda(t)
\end{gathered}$$

This is the optimal trajectory, but there is no feedback. We will consider closed-loop systems after the midterm.

As a preview, we need to find λ as a function of x. For example, $u = -R^{-1}B^{\mathrm{T}}Px$ minimizes $u^{\mathrm{T}}Ru$, so $\lambda = Px$ where P is the solution to the Riccati equation.

3.3 Numerical Methods

Optimal control boils down to solving two sets of differential equations:

$$\begin{aligned}
\dot{x} &= f(x, u) & \frac{\partial H}{\partial u}(x, u, \lambda) &= 0 \\
\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x}(x, u, \lambda) & u &= F(x, \lambda)
\end{aligned}$$

$$\implies \begin{cases} \dot{x} = f(x, F(x, \lambda)) \\ \dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x}(x, F(x, \lambda), \lambda) \end{cases}$$

The equations are functions of x and λ. They are completely determined by the boundary conditions on $x(0)$, $x(T)$, $\lambda(0)$, $\lambda(T)$. This is known as the *Boundary Value Problem*. This is solved using *test shooting*:

1. Guess initial conditions
2. Simulate forward in time
3. Update the guess (cleverly…)

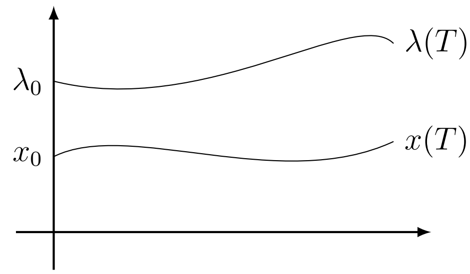
Exmaple: Bolza problem

$$\begin{aligned}
&\min_{u \in \mathcal{U}} \int_0^T L(x, u)\,\mathrm{d}t + \Psi(x(T)) \\
&\text{s.t. } \begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases} \\
&H(x, u, \lambda) = L(x, u) + \lambda^{\mathrm{T}} f(x, u) \\
&u^*(x, \lambda) \text{ satisfies } \frac{\partial H}{\partial u} = 0
\end{aligned}$$

The optimal control satisfies

$$\begin{cases} x = f(x, u^*(x, \lambda)) \\ x(0) = x_0 \\ \lambda = -\dfrac{\partial H^{\mathrm{T}}}{\partial x}(x, u^*(x, \lambda), \lambda) \\ \lambda(T) = \dfrac{\partial \Psi}{\partial x}(x(T)) \end{cases}$$

Algorithm Guess λ_0 and solve for $x(t)$, $\lambda(t)$.



Let's define a cost:

$$\left\| \lambda(T) - \frac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) \right\|^2 = g(\lambda_0)$$

Update λ_0 through

$$\lambda_0 \coloneqq \lambda_0 - \underset{\substack{\uparrow \\ \text{any choice of step size works}}}{\gamma} \frac{\partial g^{\mathrm{T}}}{\partial \lambda_0}(\lambda_0)$$

Repeat

Problem: What is $\partial g / \partial \lambda_0$? We estimate $\partial g / \partial \lambda_0$ numerically. This is where "test shooting" comes into play.

Let e_i be the ith unit vector, $i = 1, \ldots, n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \; \ldots, \; e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\frac{\partial g}{\partial \lambda_0} = \left(\frac{\partial g}{\partial \lambda_{0,1}}, \frac{\partial g}{\partial \lambda_{0,2}}, \ldots, \frac{\partial g}{\partial \lambda_{o,n}} \right)$$

The ith component of $\partial g / \partial \lambda_0$ is given by the directional derivative

$$\frac{\partial g}{\partial \lambda_{0,i}} = \frac{\partial g}{\partial \lambda_0} \cdot e_i = \delta g(\lambda_0; e_i) = \lim_{\varepsilon \to 0} \frac{g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)}{\varepsilon}$$

So, if $x \in \mathbb{R}^n$ (and thus so is λ_0), we have to do this n times (with a small ε) and get the full derivative $\partial g / \partial \lambda_0$.

Given λ_0, $g(\lambda_0)$

for $i = 1$ to n **do**

  Compute $g(\lambda_0 + \varepsilon e_i)$

  $dg_i = \dfrac{1}{\varepsilon}[g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)]$

end for

$\dfrac{\partial g}{\partial \lambda_0} = [dg_1, \ldots, dg_n]$

Algorithm

Example LQ

$$
\begin{aligned}
&\min_u \frac{1}{2}\int_0^1 (x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)\,\mathrm{d}t + \frac{1}{2}x^{\mathrm{T}}(1)Sx(1) \\
&\text{s.t. } \begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases} \\
&\qquad Q, R, S \succ 0
\end{aligned}
$$

$$
\begin{aligned}
H &= \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + \lambda^{\mathrm{T}}(Ax + Bu) \\
\frac{\partial H}{\partial u} &= u^{\mathrm{T}}R + \lambda^{\mathrm{T}}B = 0 \\
u^* &= -R^{-1}B^{\mathrm{T}}\lambda \\
\dot{\lambda} &= -\frac{\partial H}{\partial x}^{\mathrm{T}} = -Qx - A^{\mathrm{T}}\lambda \\
\lambda(1) &= \frac{\partial \Psi}{\partial x}^{\mathrm{T}}(x(1)) = Sx(1)
\end{aligned}
$$

So putting it all together,

$$
\begin{aligned}
\dot{x} &= Ax - BR^{-1}B^{\mathrm{T}}\lambda \qquad & x(0) &= x_0 \\
\dot{\lambda} &= -Qx - A^{\mathrm{T}}\lambda & \lambda(1) &= Sx(1)
\end{aligned}
$$

Example Newton's nose shape problem (revisited, see previous)

$$
\begin{aligned}
&\min_u \int_0^\ell \frac{ru^3}{1+u^2}\,\mathrm{d}x + \frac{1}{2}r(\ell)^2 \\
&\text{s.t. } \frac{\mathrm{d}r}{\mathrm{d}x} = -u \qquad r(0) = a
\end{aligned}
$$

$$
\begin{aligned}
H &= \frac{ru^3}{1+u^2} + \lambda(-u) \\
\frac{\partial H}{\partial u} &= \frac{ru^2(3+u^2)}{(1+u^2)^2} - \lambda = 0
\end{aligned}
$$

We solve the above numerically to get $u^*(r,\lambda)$.

$$
\begin{aligned}
\frac{\partial \lambda}{\partial x} &= -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2} \\
\lambda(\ell) &= r(\ell)
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}x} &= -u & r(0) &= a & u &= F(x,\lambda) \\
\frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{u^3}{1+u^2} & \lambda(\ell) &= r(\ell)
\end{aligned}
$$

Example Fixed terminal constraints (revisited, see previous)

$$
\begin{aligned}
&\min_{\alpha} -v_x(T) \qquad \alpha = \text{control} \\
&\text{s.t.} \qquad \dot{x} = v_x \qquad\qquad x(0) = 0 \\
&\qquad\qquad \dot{y} = v_y \qquad\qquad y(0) = 0 \\
&\qquad\qquad \dot{v}_x = |f|\cos\alpha \quad v_x(0) = 0 \\
&\qquad\qquad \dot{v}_y = |f|\sin\alpha \quad v_y(0) = 0 \\
&\qquad\quad y(T) = h \\
&\qquad\quad v_y(T) = 0
\end{aligned}
$$

$$
\begin{aligned}
H &= -v_x(T) + \begin{bmatrix} \lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f|\cos\alpha \\ |f|\sin\alpha \end{bmatrix} \\
\frac{\mathrm{d}H}{\mathrm{d}\alpha} &= 0 \Rightarrow \tan\alpha = \frac{\lambda_{v_y}}{\lambda_{v_x}} \\
\dot{\lambda}_x &= 0 \\
\dot{\lambda}_y &= 0 \\
\dot{\lambda}_{v_x} &= -\lambda_x \\
\dot{\lambda}_{v_y} &= -\lambda_y \\
\boldsymbol{\lambda}(0) &\text{ unspecified} \\
\lambda_x(T) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) = 0 \\
\lambda_y(T) &\text{ unspecified} \\
\lambda_{v_x}(T) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial v_x}(v_x(T)) = -1 \\
\lambda_{v_y}(T) &\text{ unspecified}
\end{aligned}
$$

Again, we guess λ_0 and solve forward in time. But, we have terminal constraints on y and v_y as well.

$$
g(\lambda_0) = \frac{1}{2}\left[(y(T) - h)^2 + (v_y(T))^2 + (\lambda_x(T))^2 + (\lambda_{v_x} + 1)^2\right]
$$

3.4 Terminal Manifolds

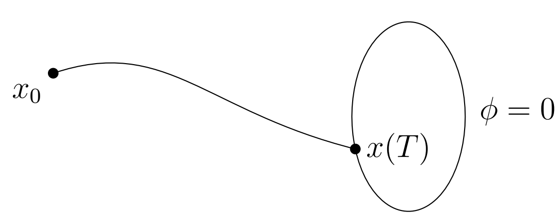
We can solve

$$
\begin{aligned}
&\min_{u \in \mathcal{U}} \int_0^T L(x, u, t)\,\mathrm{d}t + \Psi(x(T)) \\
&\text{s.t. } \dot{x} = f(x, u, t)
\end{aligned}
$$

with all sorts of boundary conditions on x:

- $x(0) = x_0$, $x(T)$ free (typical)
- $x_i(0) = x_{i0}$, $i \in \mathcal{I}$ and $x_j(T) = x_{jT}$, $j \in \mathcal{T}$

But what if we want $x(T)$ to belong to a set?



Problem

$$\begin{aligned}
&\min_{u\in\mathcal{U}} \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T)) \\
&\text{s.t. } \dot{x} = f(x,u,t), \quad x \in \mathbb{R}^n \\
&\qquad x(0) = x_0 \\
&\qquad \phi(x(T)) = 0, \quad \phi : \mathbb{R}^n \to \mathbb{R}^q,\ q \le n
\end{aligned}$$

The augmented cost is

$$\tilde{J} = \int_0^T [H(x,u,t,\lambda) - \lambda^{\mathrm{T}}\dot{x}]\,\mathrm{d}t + \Psi(x(T)) + \underset{\substack{\uparrow \\ \text{q-dimensional} \\ \text{Lagrange multiplier}}}{\nu^{\mathrm{T}}} \phi(x(T))$$

Let $\Phi(x(T),\nu) = \Psi(x(T)) + \nu^{\mathrm{T}}\phi(x(T))$. Then,

$$\tilde{J} = \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta\,\mathrm{d}t + \Phi(x(T),\nu)$$

We know how to solve this! With $u \mapsto u + \varepsilon v$, $x \mapsto x + \varepsilon\eta + o(\varepsilon)$,

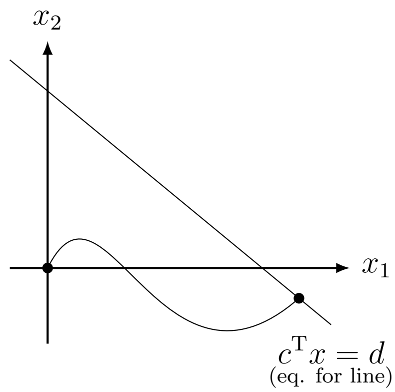
$$\begin{aligned}
\delta\tilde{J} = &\int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta\,\mathrm{d}t + \int_0^T \frac{\partial H}{\partial u} v\,\mathrm{d}t + \frac{\partial \Phi}{\partial x}(x(T),\nu)\eta(T) \\
&- \lambda^{\mathrm{T}}(T)\eta(T) + \lambda^{\mathrm{T}}(0)\underbrace{\eta(0)}_{=0}
\end{aligned}$$

$$\begin{cases}
\dfrac{\partial H}{\partial u} = 0 \\
\dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x} \\
\lambda(T) = \dfrac{\partial \Phi^{\mathrm{T}}}{\partial x}(x(T), \overset{\substack{q \text{ new variables} \\ \downarrow}}{\nu}) \\
\phi(x(T)) = 0 \ \leftarrow q \text{ new equations}
\end{cases}$$

$$\Longrightarrow u^*$$

Spline to line

$$\begin{aligned}
&\min_u \frac{1}{2}\int_0^1 u^2(t)\,\mathrm{d}t \\
&\text{s.t. } \begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = u \\
x_1(0) = 0,\ x_2(0) = 0 \\
c_1 x_1(1) + c_2 x_2(1) = d
\end{cases}
\end{aligned}$$



$$
\begin{aligned}
H &= \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u \\
\frac{\partial H}{\partial u} &= u + \lambda_2 \Longrightarrow u = -\lambda_2 \\
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \Longrightarrow \lambda_1 = k_1 \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \Longrightarrow \lambda_2 = -k_1 t + k_2 \\
\phi(x(1)) &= c_1 x_1(1) + c_2 x_2(1) - d \\
\Psi &= 0 \Longrightarrow \Phi = \nu(c_1 x_1(1) + c_2 x_2(1) - d) \\
\lambda_1(1) &= \frac{\partial \Phi}{\partial x_1} = \nu c_1 \\
\lambda_2(1) &= \frac{\partial \Phi}{\partial x_2} = \nu c_2
\end{aligned}
$$

So,

$$
\begin{aligned}
\lambda_1(1) &= \nu c_1 = k_1 \\
\lambda_2(1) &= \nu c_2 = -k_1 + k_2 \\
k_2 &= \nu(c_1 + c_2) \\
\dot{x}_2 &= u = -\lambda_2 = k_1 t - k_2 \\
x_2 &= \frac{k_1}{2}t^2 - k_2 t + 0 \\
\dot{x}_1 &= x_2 \\
x_1 &= \frac{k_1}{6}t^3 - \frac{k_2}{2}t^2 + 0
\end{aligned}
$$

Substituting k_1 and k_2 into $c_1 x_1(1) + c_2 x_2(1) = d$,

$$
\begin{aligned}
\nu\left(-\frac{c_1^2}{3} - c_1 c_2 - c_2^2\right) &= d \\
\nu &= -\frac{d}{c_1^2/3 + c_1 c_2 + c_2^2}
\end{aligned}
$$

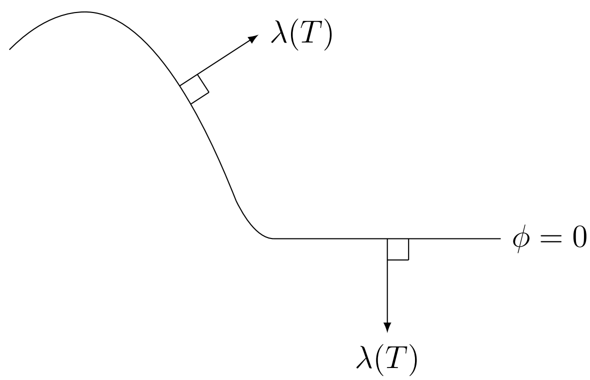
And finally

$$\boxed{u = k_1 t - k_2 = \frac{d}{c_1^2/3 + c_1 c_2 + c_2^2}(c_1 + c_2 - c_1 t)}$$

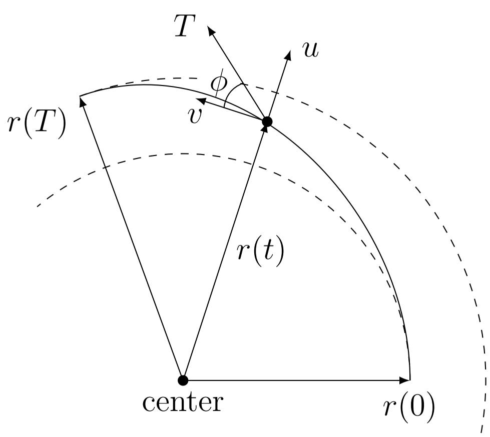
As a final observation if $\Psi = 0$ then

$$\lambda(T) = \nu^{\mathrm{T}} \frac{\partial \phi}{\partial x}(x(T)),$$

which means $\lambda(T)$ is orthogonal to the tangent plane to $\phi(x(T))$.



Example Maximum orbit transform (e.g. Hidden Figures)



r = radial distance from spacecraft to center
u = radial velocity
v = tangential velocity
m = mass of spacecraft
$\dot{m}$ = $-$fuel consumption rate
ϕ = thrust angle (control input)
T = thrust

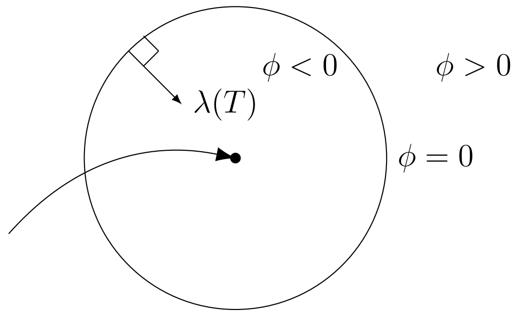
$$
\begin{aligned}
&\max_{\phi} r(T) \Longleftrightarrow \min_{\phi} -r(T) \\
&\text{s.t.} \quad \begin{cases}
\dot{r} = u \\
\dot{u} = \dfrac{v^2}{r} - \dfrac{g}{r^2} + \dfrac{T\sin\phi}{m_0 - |\dot{m}|t} \\
\dot{v} = -\dfrac{uv}{r} + \dfrac{T\cos\phi}{m_0 - |\dot{m}|t} \\
r(0) = r_0 \\
u(0) = 0 \\
v(0) = \sqrt{\dfrac{g}{r_0}} \\
u(T) = 0 = \phi_1 \\
v(T) = \sqrt{\dfrac{g}{r(T)}} = \phi_2
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
H &= \lambda_r u + \lambda_u \left(\frac{v^2}{r} - \frac{g}{r^2} + \frac{T\sin\phi}{m_0 - |\dot{m}|t} \right) + \lambda_v \left(-\frac{uv}{r} + \frac{T\cos\phi}{m_0 - |\dot{m}|t} \right) \\
\Phi &= \underbrace{\nu_1 u(T) + \nu_2 \left(v(T) - \sqrt{\frac{g}{r(T)}} \right)}_{\nu^{\mathrm{T}}\phi} \underbrace{-r(T)}_{\Psi} \\
\frac{\partial H}{\partial \phi} &= \frac{\lambda_u T\cos\phi - \lambda_v T\sin\phi}{m_0 - |\dot{m}|t} = 0 \\
&\Rightarrow \tan\phi = \frac{\lambda_u}{\lambda_v} \\
\dot{\lambda}_r &= -\frac{\partial H}{\partial r} = -\lambda_u \left(-\frac{v^2}{r^2} + \frac{2g}{r^3} \right) - \lambda_v \cdot \frac{uv}{r^2} \\
\dot{\lambda}_u &= -\frac{\partial H}{\partial u} = -\lambda_r + \lambda_v \cdot \frac{v}{r} \\
\dot{\lambda}_v &= -\frac{\partial H}{\partial v} = -\lambda_u \cdot \frac{2v}{r} + \lambda_v \cdot \frac{u}{r} \\
&\begin{cases}
\lambda_r(T) = \dfrac{\partial \Phi}{\partial r} = -1 + \dfrac{\nu_2 \sqrt{g}}{2(r(T))^{3/2}} \\
\lambda_u(T) = \dfrac{\partial \Phi}{\partial u} = \nu_1 \\
\lambda_v(T) = \dfrac{\partial \Phi}{\partial v} = \nu_2 \\
u(T) = 0 \\
v(T) = \sqrt{\dfrac{g}{r(T)}}
\end{cases}
\end{aligned}
$$

This needs numerics to solve.

3.4.1 Terminal manifold with inequality constraints

$$
\begin{aligned}
&\min_u \int_0^T L\,\mathrm{d}t + \Psi \\
&\dot{x} = f(x,u) \\
&\phi(x(T)) \leq 0
\end{aligned}
$$



Repeat process: $\tilde{J} = \int (H - \lambda^{\mathrm{T}}\dot{x})\,\mathrm{d}t + \Psi + \nu^{\mathrm{T}}\phi$. The optimality conditions are

$$
\begin{cases}
\dfrac{\partial H}{\partial u} = 0 \\
\dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x} \\
\lambda(T) = \dfrac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) + \nu^{\mathrm{T}}\dfrac{\partial \phi^{\mathrm{T}}}{\partial x}(x(T)) \\
\nu \geq 0 \\
\phi(x(T)) \leq 0 \\
\nu^{\mathrm{T}}\phi(x(T)) = 0 \quad \text{(KKT)}
\end{cases}
$$

3.4.2 Initial manifold

$$
\begin{aligned}
&\min_{x_0,u} \int L + \Psi(x(T)) + \Theta(x(0)) \\
\text{s.t.}\;\; &\dot{x} = f(x,u) \\
&\phi(x(T)) = 0 \\
&\xi(x(0)) = 0
\end{aligned}
$$

$$
\begin{aligned}
\tilde{J} &= \int (H - \lambda^{\mathrm{T}}\dot{x})\,\mathrm{d}t + \Psi(x(T)) + \Theta(x(0)) + \nu_\phi^{\mathrm{T}}\phi(x(T)) + \nu_\xi^{\mathrm{T}}\xi(x(0)) \\
\delta\tilde{J} &= \int \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta + \frac{\partial H}{\partial u}v\right]\mathrm{d}t + \left[\frac{\partial \Psi}{\partial x}(x(T)) + \nu_\phi^{\mathrm{T}}\frac{\partial \phi}{\partial x}(x(T)) - \lambda^{\mathrm{T}}(T)\right]\eta(T) \\
&\quad + \left[\frac{\partial \Theta}{\partial x}(x(0)) + \nu_\xi^{\mathrm{T}}\frac{\partial \xi}{\partial x}(x(0)) + \lambda^{\mathrm{T}}(0)\right]\eta(0)
\end{aligned}
$$

The optimality conditions are

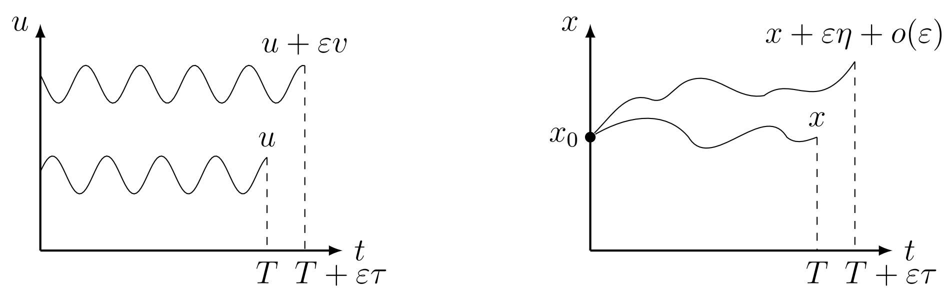
$$
\begin{cases}
\dfrac{\partial H}{\partial u} = 0 \\
\dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x} \\
\lambda(T) = \dfrac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) + \nu_\phi^{\mathrm{T}} \dfrac{\partial \phi^{\mathrm{T}}}{\partial x}(x(T)) \\
\lambda(0) = -\dfrac{\partial \Theta^{\mathrm{T}}}{\partial x}(x(0)) - \nu_\xi^{\mathrm{T}} \dfrac{\partial \xi^{\mathrm{T}}}{\partial x}(x(0))
\end{cases}
$$

3.4.3 Unspecified Terminal Times

For example, instead of driving to the moon using minimum fuel, we want to get there as soon as possible:

$$
\min_{u,T} \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T),T).
$$

The variations are $u \mapsto u + \varepsilon v$ and $T \mapsto T + \varepsilon\tau$



$$
\begin{aligned}
\tilde{J}(u,T) &= \int_0^T \left[L(x,u,t) + \lambda^{\mathrm{T}}(f(x) - \dot{x})\right] \mathrm{d}t + \Psi(x(T),T) \\
&= \int_0^T \left[H - \lambda^{\mathrm{T}}\dot{x}\right] \mathrm{d}t + \Psi \\
\tilde{J}(u+\varepsilon v, T+\varepsilon\tau) &= \int_0^T \left[H(x+\varepsilon\eta, u+\varepsilon v, t, \lambda) - \lambda^{\mathrm{T}}(\dot{x} + \varepsilon\dot{\eta})\right] \mathrm{d}t \\
&\quad + \int_T^{T+\varepsilon\tau} \left[H(x+\varepsilon\eta, u+\varepsilon v, t, \lambda) - \lambda^{\mathrm{T}}(\dot{x} + \varepsilon\dot{\eta})\right] \mathrm{d}t \\
&\quad + \Psi(x(T+\varepsilon\tau) + \varepsilon\eta(T+\varepsilon\tau), T+\varepsilon\tau)
\end{aligned}
$$

$$\tilde{J}(u+\varepsilon v, T+\varepsilon\tau) - \tilde{J}(u,T) = \varepsilon \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta\,\mathrm{d}t + \varepsilon\int_0^T \frac{\partial H}{\partial u} v\,\mathrm{d}t \\ - \varepsilon\lambda^{\mathrm{T}}(T)\eta(T) + \varepsilon\lambda^{\mathrm{T}}(0)\eta(0) + o(\varepsilon) \\ + \underbrace{\int_T^{T+\varepsilon\tau} [H(x+\varepsilon\eta, u+\varepsilon v, t, \lambda) - \lambda^{\mathrm{T}}(\dot{x}+\varepsilon\dot{\eta})]\,\mathrm{d}t}_{\text{(I)}} \\ + \underbrace{\Psi(x(T+\varepsilon\tau) + \varepsilon\eta(T+\varepsilon\tau), T+\varepsilon\tau) - \Psi(x(T),T)}_{\text{(II)}} \tag{3.4}$$

For term I, use the mean value theorem to get rid of terms inside the integral that have a ε before them:

$$\int_T^{T+\varepsilon\tau} [L + \lambda^{\mathrm{T}}(f - \dot{x} - \varepsilon\dot{\eta})]\,\mathrm{d}t \\ = \int_T^{T+\varepsilon\tau} \left[L(x,u,t) + \varepsilon\frac{\partial L}{\partial x}\eta + \varepsilon\frac{\partial L}{\partial u}v + \lambda^{\mathrm{T}}\left(f + \varepsilon\frac{\partial f}{\partial x}\eta + \varepsilon\frac{\partial f}{\partial u}v - \dot{x} - \varepsilon\dot{\eta}\right)\right]\mathrm{d}t + o(\varepsilon) \\ = \varepsilon\tau\big[L + \lambda^{\mathrm{T}}(f-\dot{x})\big]\big|_{t=\xi} + o(\varepsilon) = \varepsilon\tau L\big|_{t=\xi} + o(\varepsilon) \\ = \varepsilon\tau L(x(\xi), u(\xi), \xi) + o(\varepsilon), \quad \xi \in [T, T+\varepsilon\xi] \tag{3.5}$$

Note that as $\varepsilon \to 0$, $\xi \to T$.

For term II, we further split it into two parts:

$$\Psi(x+\varepsilon\eta, T+\varepsilon\tau) - \Psi(x,T) = \underbrace{\Psi(x, T+\varepsilon\tau)}_{\text{(II.a)}} + \underbrace{\varepsilon\frac{\partial\Psi}{\partial x}(x, T+\varepsilon\tau)\eta(T+\varepsilon\tau)}_{\text{(II.b)}} - \Psi(x,T)$$

$$\text{(II.a)} \implies \Psi(x, T+\varepsilon\tau) = \Psi(x(T), T+\varepsilon\tau) + \varepsilon\frac{\partial\Psi}{\partial x}(x(T), T+\varepsilon\tau)\dot{x}(T)\tau + o(\varepsilon) \\ = \Psi(x(T),T) + \varepsilon\frac{\partial\Psi}{\partial x}(x(T),T)\dot{x}(T)\tau + \varepsilon\frac{\partial\Psi}{\partial T}(x(T),T)\tau + o(\varepsilon)$$

$$\text{(II.b)} \implies \varepsilon\frac{\partial\Psi}{\partial x}(x, T+\varepsilon\tau)\eta(T+\varepsilon\tau) \\ = \varepsilon\left[\frac{\partial\Psi}{\partial x}(x(T),T) + \varepsilon\frac{\partial^2\Psi}{\partial x^2}\dot{x}\tau + \varepsilon\frac{\partial^2\Psi}{\partial T\partial x}\tau + o(\varepsilon)\right] \\ \times\left[\eta(T) + \varepsilon\dot{\eta}(T)\tau + o(\varepsilon)\right] \\ = \varepsilon\frac{\partial\Psi}{\partial x}(x(T),T)\eta(T) + o(\varepsilon)$$

$$\text{(II)} \implies \Psi(x+\varepsilon\eta, T+\varepsilon\tau) - \Psi(x,T) \\ = \varepsilon\frac{\partial\Psi}{\partial x}(x(T),T)[\dot{x}(T)\tau + \eta(T)] + \varepsilon\frac{\partial\Psi}{\partial T}(x(T),T)\tau + o(\varepsilon) \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4) and taking the directional derivative,

$$
\begin{aligned}
\delta \tilde{J} = &\int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right) \eta \,\mathrm{d}t + \int_0^T \frac{\partial H}{\partial u} v \,\mathrm{d}t + \lambda^{\mathrm{T}}(0)\eta(0) \\
&+ \left[L + \frac{\partial \Psi}{\partial T} + \frac{\partial \Psi}{\partial x} f\right] \tau \Big|_{t=T} + \left(\frac{\partial \Psi}{\partial x} - \lambda^{\mathrm{T}}\right) \eta \Big|_{t=T}
\end{aligned}
$$

So we have a mix of old and new:

$$
\begin{aligned}
\text{old: } \frac{\partial H}{\partial u} &= 0 \\
\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} \\
\lambda(T) &= \frac{\partial \Psi}{\partial x}\Big|_T \\
\text{new: } L + \frac{\partial \Psi}{\partial T} + \lambda^{\mathrm{T}} f\Big|_T &= 0
\end{aligned}
$$

This last condition is known as the *Transversality condition.*

Example Pure minimum time question

$$
\begin{aligned}
\min_{u,T} \int_0^T \mathrm{d}t& \\
\dot{x} &= f(x,u) \\
x(0) &= x_0 \\
x(T) &= x_T \\
H = L + \lambda^{\mathrm{T}} f &= 1 + \lambda^{\mathrm{T}} f
\end{aligned}
$$

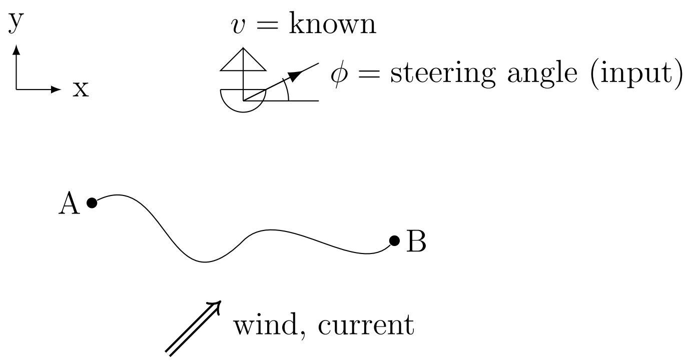
The transversality condition is

$$
\begin{aligned}
L + \frac{\partial \Psi}{\partial T} + \lambda^{\mathrm{T}} f\Big|_T &= 0 \\
\lambda^{\mathrm{T}} f\big|_T &= -1 \\
H(T) = 1 + \lambda^{\mathrm{T}} f\big|_T &= 1 - 1 = 0
\end{aligned}
$$

But this is a conservative system, so H is a constant. Therefore,

$$H(t) = 0 \quad \forall t \in [0, T]$$

Example Zermelo's problem: sail from A to B as quickly as possible in the presence of known winds and currents.



The dynamics are

$$\begin{aligned} \dot{x} &= v\cos\phi + c_1(x,y) \\ \dot{y} &= v\sin\phi + c_2(x,y) \end{aligned} \qquad \lambda = \begin{bmatrix} \lambda_x \\ \lambda_y \end{bmatrix}$$

For minimum time, $L = 1$.

$$\begin{aligned} H &= 1 + \lambda_x(v\cos\phi + c_1) + \lambda_y(v\sin\phi + c_2) \\ 0 &= \frac{\partial H}{\partial \phi} = -v\lambda_x \sin\phi + v\lambda_y\cos\phi \\ \phi &= \tan^{-1}\left(\frac{\lambda_y}{\lambda_x}\right) \end{aligned}$$

Since this is a conservative system and $\partial\Psi/\partial T = 0$, then $H(t) = H(T) = 0$.

$$\begin{aligned} -1 &= \lambda_x(v\cos\phi + c_1) + \lambda_y(v\sin\phi + c_2) \\ \lambda_x &= -\frac{\cos\phi}{v + c_1\cos\phi + c_2\sin\phi} \\ \lambda_y &= -\frac{\sin\phi}{v + c_1\cos\phi + c_2\sin\phi} \\ \dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} \\ \dot{\lambda}_x &= -\lambda_x\frac{\partial c_1}{\partial x} - \lambda_y\frac{\partial c_2}{\partial x} \\ \dot{\lambda}_y &= -\lambda_x\frac{\partial c_1}{\partial y} - \lambda_y\frac{\partial c_2}{\partial y} \\ \dot{\phi} &= \sin^2\phi\frac{\partial c_2}{\partial x} + \sin\phi\cos\phi\left(\frac{\partial c_1}{\partial x} - \frac{\partial c_2}{\partial y}\right) - \cos^2\phi\frac{\partial c_1}{\partial y} \end{aligned}$$

This is an ODE that completely determines ϕ if we just had ϕ_0.

Example We want to drive a car and stop at a stop sign as quickly as possible. Assume that the stop sign is at the origin, and our control is the acceleration ($\ddot{x} = u$).

$$\min_{u,T} \int_0^T \mathrm{d}t$$
$$\text{s.t.} \quad \begin{cases} \dot{x}_1 = x_2, & x(0) = x_0 \\ \dot{x}_2 = u, & x(T) = 0 \end{cases}$$

Recall the transversality condition:

$$H + \left.\frac{\partial \Psi}{\partial T}\right|_{t=T} = 0.$$

For minimum-time problems, $L = 1$ and $\Psi = 0$, so $\lambda^{\mathrm{T}} f|_{t=T} = -1$.

$$H = 1 + \lambda_1 x_2 + \lambda_2 u$$
$$\lambda_1(T) \underbrace{x_2(T)}_{=0 \text{ (rest)}} + \lambda_2(T) u(T) = -1$$
$$\boxed{\lambda_2(T) u(T) = -1}$$
$$\frac{\partial H}{\partial u} = \boxed{\lambda_2 = 0},$$

i.e. $0 \cdot u(T) = -1$!? This problem is ill-posed; we need to go infinitely fast…

Idea 1: Constrain u. We don't know how to do this.

Idea 2: Pay for gas. This is a design choice.

For the second idea,

$$\min_{u,T} \int_0^T \frac{1}{2} u^2(t)\,\mathrm{d}t$$
$$\text{s.t.} \quad \begin{cases} \dot{x}_1 = x_2, & x(0) = x_0 \\ \dot{x}_2 = u, & x(T) = 0 \end{cases}$$

$$H = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u$$
$$\frac{1}{2}u^2(T) + \lambda_1(T) x_2(T) + \lambda_2(T) u(T) = 0$$
$$\boxed{\frac{1}{2}u^2(T) + \lambda_2(T) u(T) = 0}$$
$$\frac{\partial H}{\partial u} = u + \lambda_2 = 0 \Longrightarrow u = -\lambda_2$$
$$\frac{1}{2}\lambda_2^2(T) - \lambda_2^2(T) = 0$$
$$\boxed{\lambda_2(T) = 0}$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \Longrightarrow \lambda_1 = c$$
$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \Longrightarrow \lambda_2 = -ct + d$$
$$\lambda_2(T) = -cT + d = 0 \Longrightarrow T = \frac{d}{c}$$
$$\dot{x}_2 = u = -\lambda_2 = ct - d$$
$$x_2 = c\frac{t^2}{2} - dt + x_{2,0}$$
$$\dot{x}_1 = x_2 \Longrightarrow x_1 = c\frac{t^3}{6} - d\frac{t^2}{2} + x_{2,0}t + x_{1,0}$$
$$\begin{cases} x_1(T) = c\frac{T^3}{6} - d\frac{T^2}{2} + x_{2,0}T + x_{1,0} = 0 \\ x_2(T) = c\frac{T^2}{2} - dT + x_{2,0} = 0 \\ T = \frac{d}{c} \end{cases}$$
$$\Longrightarrow \begin{cases} c = -\frac{2}{3}\frac{x_{2,0}^2}{x_{1,0}} \\ d = \sqrt{-\frac{4}{3}\frac{x_{2,0}^3}{x_{1,0}}} \\ T = \frac{d}{c} \\ u = ct - d \end{cases}$$

Fine, but we really want to get there as quickly as possible! We have to constrain u, e.g. $u(t) \in [-1, 1]$, $\forall t \in [0, T]$. How do we deal with the constraints on u?

3.5 Hamilton's Minor "Mistake"

$$\begin{aligned} \min_{u \in \mathcal{U}_{\text{constr.}}} \ & \int_0^T L(x, u, t)\,\mathrm{d}t + \Psi(x(T)) \\ \text{s.t. } & \dot{x} = f(x, u, t) \\ & x(0) = x_0 \\ & (u(t) \in U) \end{aligned}$$

Augment the cost:

$$\tilde{J}(u) = \int_0^T \left(H(x, u, t, \lambda) - \lambda^{\mathrm{T}}\dot{x}\right)\mathrm{d}t + \Psi(x(T))$$

Vary $u \mapsto u + \varepsilon v$ s.t. $u + \varepsilon v \in \mathcal{U}_{\text{constr.}} \Rightarrow x \mapsto x + \varepsilon\eta + o(\varepsilon)$:

$$\tilde{J}(u+\varepsilon v) = \int_0^T \left(H(x+\varepsilon\eta, u+\varepsilon v, t, \lambda) - \lambda^{\mathrm{T}}\dot{x} - \lambda^{\mathrm{T}}\varepsilon\dot{\eta} \right) \mathrm{d}t + \Psi(x(T) + \varepsilon\eta(T)) + o(\varepsilon)$$

Instead of computing $\delta\tilde{J}(u;v)$, let's check $\Delta\tilde{J} = \tilde{J}(u+\varepsilon v) - \tilde{J}(u)$. If $\Delta\tilde{J} \geq 0$ $\forall v$ s.t. $u + \varepsilon v \in \mathcal{U}_{\text{constr.}}$ for ε small enough, then u is a local minimum!

$$\begin{aligned}\Delta\tilde{J} = \int_0^T &\left[H(x+\varepsilon\eta, u+\varepsilon v, t, \lambda) - H(x,u,t,\lambda) - \lambda^{\mathrm{T}}(\dot{x} + \varepsilon\dot{\eta} - \dot{x}) \right] \mathrm{d}t \\ &+ \Psi(x(T) + \varepsilon\eta(T)) - \Psi(x(T)) + o(\varepsilon)\end{aligned}$$

Only Taylor expanding w.r.t. x:

$$\begin{aligned}\Delta\tilde{J} &= \int_0^T \left[\varepsilon\frac{\partial H}{\partial x}(x,u,t,\lambda)\eta - \varepsilon\lambda^{\mathrm{T}}\dot{\eta} \right] \mathrm{d}t + \int_0^T \left[H(x,u+\varepsilon v,t,\lambda) - H(x,u,t,\lambda) \right] \mathrm{d}t \\ &\quad + \varepsilon\frac{\partial\Psi}{\partial x}(x(T))\eta(T) + o(\varepsilon) \\ &= \varepsilon\int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}} \right) \eta \,\mathrm{d}t + \varepsilon\lambda^{\mathrm{T}}(0)\eta(0) - \varepsilon\lambda^{\mathrm{T}}(T)\eta(T) + \varepsilon\frac{\partial\Psi}{\partial x}(x(T))\eta(T) \\ &\quad + \int_0^T \left[H(x,u+\varepsilon v,t,\lambda) - H(x,u,t,\lambda) \right] \mathrm{d}t + o(\varepsilon)\end{aligned}$$

With $\dot{\lambda} = -\partial H^{\mathrm{T}}/\partial x$ and $\lambda(T) = \partial\Psi(x(T))/\partial x$,

$$\Delta\tilde{J} = \int_0^T \left[H(x,u+\varepsilon v,t,\lambda) - H(x,u,t,\lambda) \right] \mathrm{d}t + o(\varepsilon)$$

Here, Hamilton did Taylor's expansion and set $\partial H/\partial u = 0$. Instead, Pontryagin desired $\Delta\tilde{J} \geq 0$ $\forall v | u + \varepsilon v \in \mathcal{U}_{\text{constr.}}$, ε small enough, i.e. we need

$$H(x, u^* + \varepsilon v, t, \lambda) \geq H(x, u^*, t, \lambda)$$

$\forall t \in [0,t]$, $\forall v | u + \varepsilon v \in \mathcal{U}_{\text{constr.}}$, ε small enough. That is, we need

$$u^* = \arg\min_u H(x,u,t,\lambda)$$

In summary,

$$\text{Hamilton: } \frac{\partial H}{\partial u} = 0$$
$$\text{Pontryagin: } \min_u H$$

Theorem (Pontryagin's Maximum Principle (PMP))**.** *Consider the problem:*

$$\begin{aligned}\min_{u,T} \quad & \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T),T) \\ \textit{s.t.} \quad & \dot{x} = f(x,u,t) \\ & u(t) \in U(x,t), \quad \forall t \in [0,T] \\ & x_i(0) = x_{i0}, \qquad i \in \mathcal{I} \\ & x_j(T) = x_{jT}, \qquad j \in \mathcal{T}\end{aligned}$$

The necessary condition for optimality is

$$
\begin{aligned}
&H = L + \lambda^T f \\
&\dot{\lambda} = -\frac{\partial H}{\partial x}^T \\
&\lambda_j(0) = 0, \quad j \notin \mathcal{I} \\
&\lambda_i(T) = \frac{\partial \Psi}{\partial x_i}(x(T)), \quad i \notin \mathcal{T} \\
&H + \left.\frac{\partial \Psi}{\partial T}\right|_{t=T} = 0 \\
&u^*(x,t,\lambda) = \arg\min_{u \in U(x,t)} H(x,u,t,\lambda)
\end{aligned}
$$

We have two paths to solve optimality problems: we always start with the Hamiltonian, find the costate dynamics and boundary conditions, and apply the transversality condition; then, we can either apply calculus of variations (COV) or Pontryagin's Maximum Principle (PMP). COV only works for unconstrained problems, while with PMP we can deal with constraints.

3.6 Bang-Bang Control

Return to the car problem:

$$
\begin{aligned}
&\min_{u,T} \int_0^T \mathrm{d}t \\
&\text{s.t.} \begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{1,0}, \quad x_1(T) = 0 \\ \dot{x}_2 = u, & x_2(0) = x_{2,0}, \quad x_2(T) = 0 \\ u(t) \in [-1,1] & \forall t \in [0,T] \end{cases}
\end{aligned}
$$

How do we minimize H w.r.t. u?

$$H = 1 + \lambda_1 x_2 + \lambda_2 u$$

Clearly, we minimize H by letting

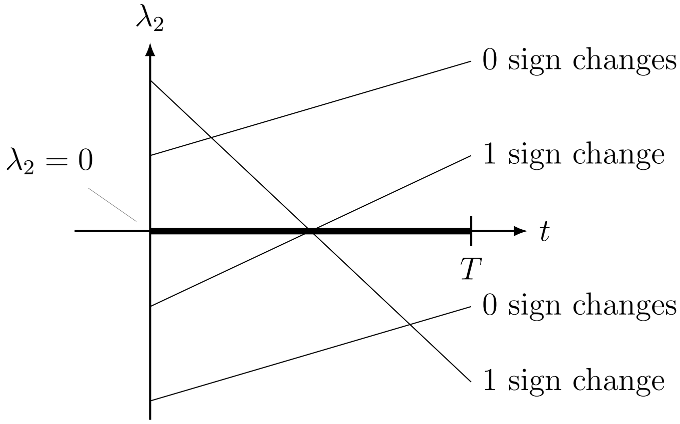
$$u = \begin{cases} -1, & \lambda_2 > 0 \\ +1, & \lambda_2 < 0 = -\operatorname{sign}(\lambda_2) \\ ??, & \lambda_2 = 0 \end{cases}$$

Therefore, the optimal u switches between -1 and $+1$ (bang-bang control).

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \Longrightarrow \lambda_1 = c$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \Longrightarrow \lambda_2 = -ct + d$$

Notice that $\lambda_2(t)$ is a line, so it has at most one sign change. Thus, u also changes sign (from ± 1 to ∓ 1) at most one time.



Let's solve this for all x_0!

i) Assume $\lambda_2 > 0 \; \forall t \in [0, T]$, $\therefore u = -1 \quad \forall t \in [0, T]$

$$
\begin{aligned}
\dot{x}_2 &= -1 \Longrightarrow x_2 = -t + k_1 \\
x_2(T) &= 0 = -T + k_1 \Longrightarrow k_1 = T \\
x_2(t) &= T - t \Longrightarrow x_2 > 0, \; t \in [0, T) \\
\dot{x}_1 &= x_2 = T - t \Longrightarrow x_1 = -\frac{t^2}{2} + Tt + k_2 \\
x_1(T) &= 0 = -\frac{T^2}{2} + T^2 + k_2 \Longrightarrow k_2 = -\frac{T^2}{2} \\
x_1(t) &= -\frac{t^2}{2} + Tt - \frac{T^2}{2} = -\frac{(T-t)^2}{2} \quad (< 0, \; t \in [0, T)) \\
&= -\frac{x_2^2(t)}{2}
\end{aligned}
$$

Let's consider the curve

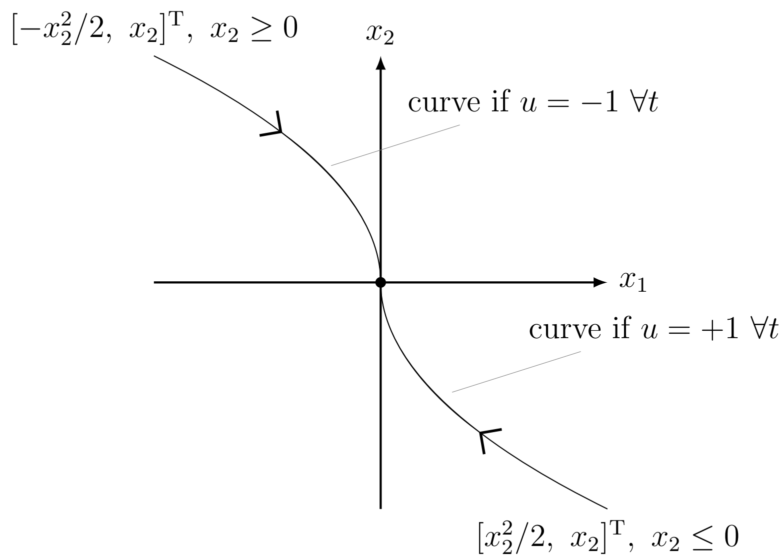
$$
\begin{bmatrix} -x_2^2/2 \\ x_2 \end{bmatrix}
$$

for $x_2 \geq 0$. If x_0 lies on this curve, use $u = -1$ and drive to the origin.

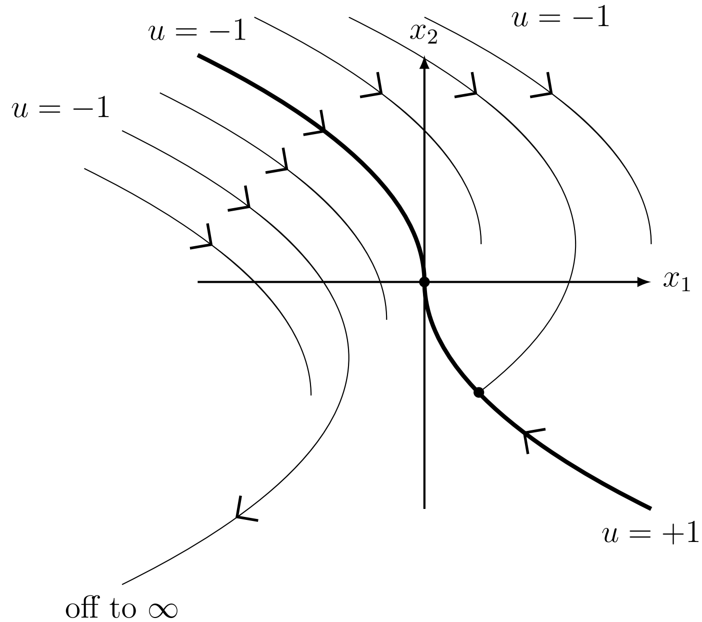
ii) Assume $u = +1 \quad \forall t \in [0, T]$

$$
\begin{aligned}
x_2 &= t - T \quad (\leq 0 \text{ on } [0, T]) \\
x_1 &= \frac{x_2^2}{2} \quad (\geq 0)
\end{aligned}
$$

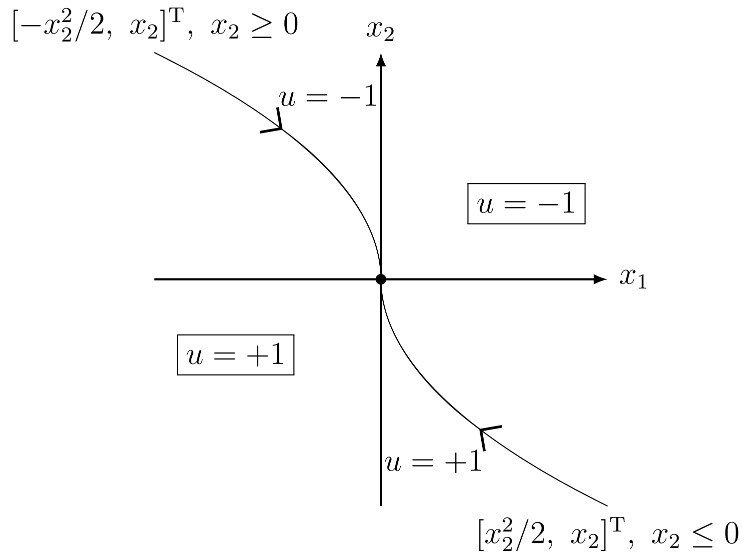
For this curve, use $u = +1$.



What happens when we do not start on the curves? We start with a certain u depending on x_0 and perform a single switch of u when we encounter one of the initial curves that travel to the origin. Note that for the case $\lambda_2 = 0 \forall t$, we start at the stop sign at rest, so the control does not matter.



The optimal solution is given by the following *switching curve*.



Note 1: Bang-bang control typically involves

a) finding the number of switches
b) find the switching surfaces

Note 2: This is a feedback law! (u depends on x!!)

3.6.1 Linear Systems (scalar input)

$$\begin{aligned}
&\min_{u,T} \int_0^T \mathrm{d}t \\
&\text{s.t. } \dot{x} = Ax + Bu \\
&\qquad x(0) = x_0, \quad x(T) = 0 \\
&\qquad u \in [-1, 1] \\
&\quad H = 1 + \lambda^{\mathrm{T}}(Ax + Bu) \\
&\quad u = -\operatorname{sign}(\lambda^{\mathrm{T}} B) \quad \text{(bang-bang)}
\end{aligned}$$

Aside…

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u \quad \text{(control affine)} \\
H &= 1 + \lambda^{\mathrm{T}} f + \lambda^{\mathrm{T}} g u \\
u &= -\operatorname{sign}(\lambda^{\mathrm{T}} g(x)) \quad \text{(bang-bang)}
\end{aligned}$$

Back to linear…

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} = -A^{\mathrm{T}}\lambda \\
\lambda(t) &= e^{-A^{\mathrm{T}}t}\lambda_0 \\
u(t) &= -\operatorname{sign}\left(\lambda_0^{\mathrm{T}} e^{-At} B\right)
\end{aligned}$$

How do we find λ_0?

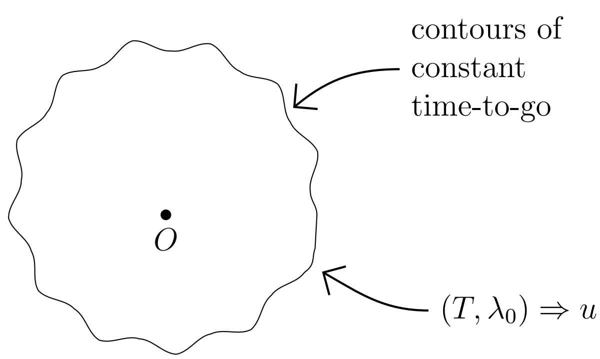
$$
\begin{aligned}
\dot{x} &= Ax + Bu, \quad x(0) = x_0 \\
x(T) &= e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)\,\mathrm{d}\tau \\
x(T) &= 0 = e^{AT}x_0 - \int_0^T e^{A(T-t)}B\,\mathrm{sign}\left(\lambda_0^{\mathrm{T}}e^{-At}B\right)\mathrm{d}t \qquad (3.7)
\end{aligned}
$$

Problem 1: Given x_0, figure out λ_0 from (3.7). Then, $u = -\,\mathrm{sign}\left(\lambda_0^{\mathrm{T}}e^{-At}B\right)$. This has to be done numerically in general (not super simple...).

Problem 2: Find all x_0s from which it takes the same amount of time to get to $x(T) = 0$.

$$
\begin{aligned}
e^{At}x_0 &= \int_0^T e^{A(T-t)}B\,\mathrm{sign}\left(\lambda_0^{\mathrm{T}}e^{-At}B\right)\mathrm{d}t \\
x_0 &= \int_0^T e^{-At}B\,\mathrm{sign}\left(\lambda_0^{\mathrm{T}}e^{-At}B\right)\mathrm{d}t
\end{aligned}
$$

Fix T. By varying λ_0, we will get the x_0s that take time T to go to $x(T) = 0$ optimally.



So by solving problem 2, we find λ_0 associated with all x_0, i.e. we have "solved" problem 1 as well.

3.7 Integral Constraints (Isoperimetric)

Recall PMP is

$$\min_{u \in U(x,t)} H(x, u, \lambda, t)$$

We have see $U = [-1, 1]$ in the context of bang-bang control. Now, we consider integral constraints of the form

$$C = \int_0^T N(x, u, t)\,\mathrm{d}t \quad (\in \mathbb{R}^p)$$

Let $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$. Introduce p new states $\hat{x} = [x_{n+1}, \ldots, x_{n+p}]^{\mathrm{T}}$, where

$$\hat{x}(t) = \int_0^t N(x(\tau), u(\tau), \tau)\, \mathrm{d}\tau$$

and $\dot{\hat{x}}(t) = N(x, u, t)$. Its boundary conditions are $\hat{x}(0) = 0$ and $\hat{x}(T) = C$. The Hamiltonian is

$$\begin{aligned}
H(x, \hat{x}, u, t, \lambda) &= L(x, u, t) + \lambda^{\mathrm{T}} f(x, u, t) + \widehat{\lambda}^{\mathrm{T}} N(x, u, t) \\
\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} = -\frac{\partial L^{\mathrm{T}}}{\partial x} - \frac{\partial f^{\mathrm{T}}}{\partial x}\lambda - \frac{\partial N^{\mathrm{T}}}{\partial x}\widehat{\lambda} \\
\dot{\widehat{\lambda}} &= -\frac{\partial H^{\mathrm{T}}}{\partial \hat{x}} = 0 \Longrightarrow \widehat{\lambda} \text{ is constant}
\end{aligned}$$

Moreover, this is now an unconstrained problem, i.e.

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial u} + \widehat{\lambda}^{\mathrm{T}}\frac{\partial N}{\partial u} = 0$$

Going back to the car problem of stopping at the origin, suppose we want to use up exactly the “energy”

$$E = \int_0^T u^2(t)\, \mathrm{d}t.$$

If possible, it is better to transform an inequality constraint to an equality constraint.

$$\begin{aligned}
&\min_{u,T} \int_0^T \mathrm{d}t \\
&\text{s.t.} \begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{10}, \quad x_1(T) = 0 \\ \dot{x}_2 = u, & x_2(0) = x_{20}, \quad x_2(T) = 0 \\ \dot{x}_3 = u^2, & x_3(0) = 0, \quad\;\; x_3(T) = E \end{cases}
\end{aligned}$$

As we have seen, without the energy constraint this is an ill-posed problem.

$$\begin{aligned}
H &= 1 + \lambda_1 x_2 + \lambda_2 u + \lambda_3 u^2 \\
\lambda_3 &= \text{constant} \\
\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \Longrightarrow \lambda_1 = c \\
\dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \Longrightarrow \lambda_2 = -ct + d \\
\frac{\partial H}{\partial u} &= \lambda_2 + 2\lambda_3 u = 0 \\
\Rightarrow u &= -\frac{\lambda_2}{2\lambda_3} = \frac{c}{2\lambda_3}t - \frac{d}{2\lambda_3} \quad \text{(linear in time)} \\
\dot{x}_2 = u &\Longrightarrow x_2 = \frac{c}{4\lambda_3}t^2 - \frac{d}{2\lambda_3}t + x_{20}
\end{aligned}$$

$$\dot{x}_1 = x_2 \Longrightarrow x_1 = \frac{c}{12\lambda_3}t^3 - \frac{d}{4\lambda_3}t^2 + x_{20}t + x_{10}$$

$$\dot{x}_3 = u^2 \Longrightarrow x_3 = \frac{c^2}{12\lambda_3^2}t^3 + \frac{d^2}{4\lambda_3^2}t - \frac{cd}{4\lambda_3^2}t^2$$

$$H + \left.\frac{\partial \Psi}{\partial T}\right|_T = 0$$

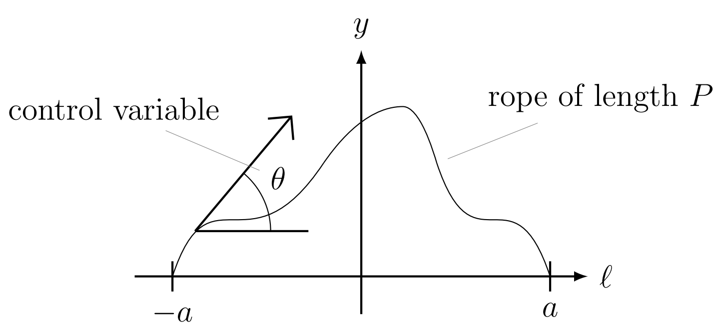
$$1 + \lambda_1 x_2 + \lambda_2 u + \lambda_3 u^2 + 0\Big|_T = 0$$

$$1 + (d - cT)\left(\frac{c}{2\lambda_3}T - \frac{d}{2\lambda_3}\right) + \lambda_3\left(\frac{c}{2\lambda_3}T - \frac{d}{2\lambda_3}\right) = 0$$

The boundary conditions ($x_1(T) = 0$, $x_2(T) = 0$, $x_3(T) = E$) and the transversality condition give four equations for four unknowns.

$$\begin{cases} T = \left(\frac{3}{E}\right)^{1/3} \\ c = -\frac{2}{3}T \\ d = -\frac{T^2}{3} \\ \lambda_3 = \frac{T^4}{18} \end{cases} \qquad \Longrightarrow u = \ldots$$

Dido's Problem Given a strip of oxhide, enclose the most area along the Mediterranean Sea. This region has a fixed width and is bounded to the south by the ℓ axis (the sea). Historically, this became the city Carthage.



The area of this region is

$$\int_{-a}^{a} y \, \mathrm{d}\ell.$$

The dynamics are

$$\frac{\mathrm{d}y}{\mathrm{d}\ell} = \tan\theta.$$

The constraint is

$$P = \int_{-a}^{a} \frac{1}{\cos\theta}\,\mathrm{d}\ell.$$

The problem becomes

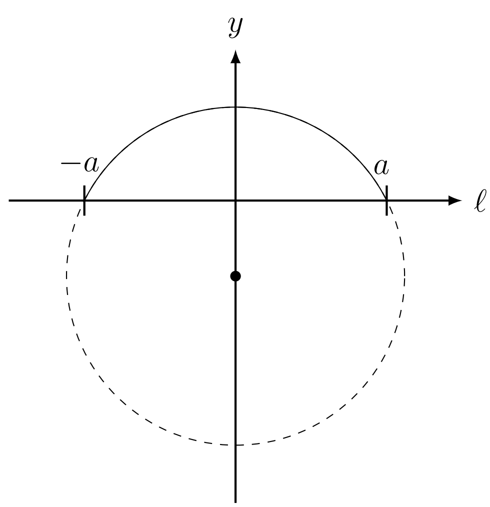
$$\begin{aligned}
&\min_{\theta} \; -\int_{-a}^{a} y(\ell)\,\mathrm{d}\ell \\
&\text{s.t. } \frac{\mathrm{d}y}{\mathrm{d}\ell} = \tan\theta, \quad y(-a) = 0, \quad y(a) = 0 \\
&\phantom{\text{s.t. }} \frac{\mathrm{d}\hat{y}}{\mathrm{d}\ell} = \frac{1}{\cos\theta}, \quad \hat{y}(-a) = 0, \quad \hat{y}(a) = P
\end{aligned}$$

$$\begin{aligned}
H &= -y + \lambda\tan\theta + \widehat{\theta}\frac{1}{\cos\theta} \\
\widehat{\lambda} &= \text{constant} \\
\frac{\mathrm{d}\lambda}{\mathrm{d}\ell} &= -\frac{\partial H}{\partial y} = 1 \Longrightarrow \lambda(\ell) = \ell + c \\
\frac{\partial H}{\partial \theta} &= 0 = \lambda(1 + \tan^2\theta) + \widehat{\lambda}\,\frac{\tan\theta}{\cos\theta} \\
\sin\theta(\ell) &= -\frac{\ell + c}{\widehat{\lambda}}
\end{aligned}$$

Let $\sin\alpha/\alpha = 2a/P$. The optimal shape is a circular arc centered at $\ell = 0$ and

$$y = -\frac{P\cos\alpha}{2\alpha},$$

with radius $P/2\alpha$. (This produces the semi-circular city of Carthage!)



Note that this formulation cannot handle $P > \pi a$. In reality, a is also undefined and chosen so that the solution is exacty a semicircle with $P = \pi a$.

The punchline is integral constraints are no big deal. What about other constraints?

3.8 Control Constraints

Suppose the control constraint is $u(t) \in U(t)$, e.g. $h(u,t) = 0$ or $h(u,t) \le 0$.

$$\begin{aligned} &\min_u H(x,u,\lambda,t) \\ &\text{s.t. } h(u,t) = 0 \end{aligned}$$

Introduce a Lagrange multiplier:

$$\begin{aligned} &\tilde{H} = H + \mu^{\mathrm{T}} h \\ &\left.\begin{aligned} &\frac{\partial \tilde{H}}{\partial u} = 0 \\ &h = 0 \end{aligned}\right\} \Longrightarrow u^*(x,t,\lambda) \end{aligned}$$

We still have

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x}(x,t,\lambda,u^*(x,t,\lambda)) \\ \dot{x} &= f(x,u,t) = f(x,u^*(x,t,\lambda),t) \\ &\quad + \text{Boundary cond. on } x \text{ and } \lambda \end{aligned}$$

The only change from the unconstrained control version is the method by which $u^*(x,t,\lambda)$ is found.

Example

$$\begin{aligned} &\min_u \frac{1}{2}\int_0^T u^2(t)\,\mathrm{d}t + \frac{1}{2}\|x(T)\|^2 \\ &\text{s.t. } \dot{x} = g(t)u, \quad g(t) \in \mathbb{R}^n \\ &\quad\ |u(t)| \le 1\ \forall t \\ &\quad\ \Rightarrow \begin{cases} u(t) - 1 \le 0 \\ -u(t) - 1 \le 0 \end{cases} \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2}u^2 + \lambda^{\mathrm{T}} g u \\ \widetilde{H} &= \frac{1}{2}u^2 + \lambda^{\mathrm{T}} g u + \mu_1(u-1) + \mu_2(-u-1) \\ \dot{\lambda} &= -\frac{\partial \widetilde{H}^{\mathrm{T}}}{\partial x} = 0 \Longrightarrow \lambda = \text{const} \\ \lambda(T) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial x} = x(T) \Longrightarrow \lambda(t) = x(T)\ \forall t \end{aligned}$$

Now, let's find u by minimizing H. Assume $|u| < 1$ (no constraints active), so $\mu_1 = \mu_2 = 0$. Then,

$$\frac{\partial \widetilde{H}}{\partial u} = u + \lambda^{\mathrm{T}} g = 0 \Longrightarrow u(t) = -x^{\mathrm{T}}(T)g(t),$$

as long as $|x^{\mathrm{T}}(T)g(t)| < 1$. Assume $u = -1$, so $\mu_1 = 0$ and $\mu_2 \geq 0$. Then,

$$\begin{aligned}\frac{\partial \widetilde{H}}{\partial u} &= u + \lambda^{\mathrm{T}} g - \mu_2 = 0 \\ x^{\mathrm{T}}(T)g(t) &= \mu_2 + 1 \geq 1\end{aligned}$$

We get a similar results assuming $u = 1$. The optimal control law is

$$\begin{aligned}u(t) &= \begin{cases} -x^{\mathrm{T}}(T)g(t), & |x^{\mathrm{T}}(T)g(t)| < 1 \\ -1, & x^{\mathrm{T}}(T)g(t) \geq 1 \\ +1, & x^{\mathrm{T}}(T)g(t) \leq -1 \end{cases} \\ u(t) &= -\operatorname{Sat}(x^{\mathrm{T}}(T)g(t))\end{aligned}$$

where

$$\operatorname{Sat}(\xi) = \begin{cases} \xi, & |\xi| \leq 1 \\ \operatorname{sign}(\xi), & \text{otherwise} \end{cases}$$

Problem: we don't know $x(T)$! We have to solve this numerically through

$$x(t) = x(0) + \int_0^t \dot{x}(\tau)\,\mathrm{d}\tau$$

$$x(T) = x_0 - \int_0^T g(t) \operatorname{Sat}\big(x^{\mathrm{T}}(T)g(t)\big)\,\mathrm{d}t$$

Example

$$\begin{aligned}&\min_u \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T)) \\ &\text{s.t. } \dot{x} = f(x,u,t), \quad x(0) = x_0 \\ &\qquad h(x,u,t) = 0 \ \forall t\end{aligned}$$

$$\begin{aligned}\widetilde{H} &= L + \lambda^{\mathrm{T}} f + \mu^{\mathrm{T}} h \\ \dot{\lambda} &= -\frac{\partial \widetilde{H}^{\mathrm{T}}}{\partial x} = -\frac{\partial L^{\mathrm{T}}}{\partial x} - \frac{\partial f^{\mathrm{T}}}{\partial x}\lambda - \frac{\partial h^{\mathrm{T}}}{\partial x}\mu \\ \lambda(T) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) \\ \frac{\partial \widetilde{H}}{\partial u} &= 0 \\ h &= 0\end{aligned}$$

Example

$$\begin{aligned}&\min_u \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T)) \\ &\text{s.t. } \dot{x} = f(x,u), \quad x(0) = x_0 \\ &\qquad h(x) = 0\end{aligned}$$

Problem: We need a constraint involving u. First, we need $h(x_0) = 0$; otherwise we have no chance. Then, if

$$\frac{\mathrm{d}}{\mathrm{d}t} h(x(t)) = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x} f(x, u) = 0,$$

we have $h(x(t)) = 0 \ \forall t$. This derivative is the Lie derivative of h along f ($L_f h = (\partial h/\partial x) f$).

$$\widetilde{H} = L + \lambda^{\mathrm{T}} f + \mu^{\mathrm{T}} \frac{\partial h}{\partial x} f$$

Problem: $(\partial h/\partial x) f$ is not guaranteed to have u in it, e.g.

$$h = x_1, \quad f = \begin{bmatrix} 17x_2 \\ \sin(x_1)u \end{bmatrix}$$
$$\frac{\partial h}{\partial x} f = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 17x_2 \\ \sin(x_1)u \end{bmatrix} = 17x_2$$

So, we keep taking derivatives until u shows up. (If u never shows up, then the control has no effect on the state.)

3.9 A Look Forward

So far, we found $u(t)$ over the horizon $[0, T]$. This is, in general, not robust. We need to know f exactly. We also need to know $x(0)$. What to do?

There are three paths forward:

1. If we're super lucky, we get $u(x, t)$ directly from PMP, like in the bang-bang example with switching surfaces.
2. Go from PMP to LQ (linear system, quadratic cost). This is used a lot.
3. Use Model-Predictive Control (MPC). In this, at time t_c (current time), we are at state x_c. We solve an optimal control problem:

$$\begin{aligned} \min_u \ & \int_{t_c}^{t_c + \Delta T} L(x, u, t)\,\mathrm{d}t + \Psi(x(t_c + \Delta T)) \\ \text{s.t. } & \dot{x} = f(x, u, t) \\ & x(t_c) = x_c \end{aligned}$$

   where ΔT is the prediction horizon. This problem can be solved using PMP, producing $u(t)$, $t \in [t_c, t_c + \Delta T]$. Instead of using $u(t)$, only use $u(t_c)$ at time t_c. This control solution depends on x_c, so we really have a feedback law $u(x_c, t_c)$. (In practice, we use $u(x_c, t_c)$ over a small interval of length δ.) Then, we resolve the optimal control problem.

   The features of MPC are

(a) Turns open-loop into closed-loop
(b) Used a lot
(c) Requires computation, but once a solution is found, it can be reused as initial conditions…
(d) Use with caution! A solution may be optimal over $[t_c, t_c + \Delta T]$ but it may still be bad (unstable) over $[t_c, \infty)$.

Chapter 4

Linear-Quadratic Control

4.1 Towards Global Optimal Control

Consider a discrete-time system

$$x_{k+1} = F(x_k, u_k),$$

where x_k is the state at time k and u_k is the input at time k.

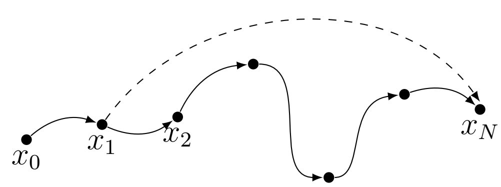
Let $c(x_k, u_k) \in \mathbb{R}$ be the cost associated with doing u_k at x_k.

Let $u = u_0, u_1, \ldots, u_{N-1}$ and assume x_0 is given. The total cost over N steps using u is

$$V_N^u(x_0) = \sum_{k=0}^{N-1} c(x_k, u_k) + \Theta(x_N),$$

where $\Theta(x_N)$ is the terminal cost.

Assume we've found the *globally* minimizing u^*. The best path over N steps is represented by the figure below.



Consider the dashed path. There is no way this path is better from x_1 to x_N using $N-1$ steps. Therefore, the solid path from x_1 to x_N is the best path over $N-1$ steps.

Definition (Bellman's Principle of Optimality)**.** Let u^* be optimal, with corresponding state sequence x^*.

$$\begin{aligned}
V_N^*(x_0) &= \sum_{k=0}^{N-1} c(x_k^*, u_k^*) + \Theta(x_N^*) \\
&= c(x_0, u_0^*) + \sum_{k=1}^{N-1} c(x_k^*, u_k^*) + \Theta(x_N^*) \\
&= c(x_0, u_0^*) + V_{N-1}^*(x_1^*) \\
V_N^*(x) &= c(x, u_0^*) + V_{N-1}^*\big(F(x, u_0^*)\big)
\end{aligned}$$

Equivalently,

$$V_N^*(x) = \min_u \Big\{ c(x,u) + V_{N-1}^*\big(F(x,u)\big) \Big\}$$

Theorem (Bellman's Equation)**.** *The optimal cost-to-go satisfies*

$$\begin{cases} V_k^*(x) = \min_u \Big\{ c(x,u) + V_{k-1}^*\big(F(x,u)\big) \Big\}, \quad k = 1, \dots, N \\ V_0^*(x) = \Theta(x) \end{cases}$$

What does this have to do with optimal control? We need to reformulate the cost function J in an analogous manner. Let

$$J^*(x_t, t) = \int_t^T L(x^*(s), u^*(s)) \, \mathrm{d}s + \Psi(x^*(T)),$$

where $x^*(t) = x_t$, u^* is *globally* optimal, and $\dot{x}^* = f(x^*, u^*)$. $J^*(x_t, t)$ is the optimal cost-to-go over $[t, T]$ starting at x_t. Let's discretize time with sample time Δt.

$$\begin{aligned}
J^*(x_t, t) &= \int_t^{t+\Delta t} L(x^*(s), u^*(s)) \, \mathrm{d}s + \int_{t+\Delta t}^T L(x^*(s), u^*(s)) \, \mathrm{d}s + \Psi(x^*(T)) \\
&= \int_t^{t+\Delta t} L(x^*(s), u^*(s)) \, \mathrm{d}s + J^*(x_{t+\Delta t}^*, t + \Delta t)
\end{aligned}$$

Note $x_{t+\Delta t}^* = x_t + f(x_t, u^*(t))\Delta t + o(\Delta t)$. Also, assume u^* is constant over $[t, t + \Delta t]$.

$$\begin{aligned}
\int_t^{t+\Delta t} L(x^*(s), u_t^*) \, \mathrm{d}s &= \Delta t L(x_t, u_t^*) + o(\Delta t) \\
\because J^*(x_t, t) &= \Delta t L(x_t, u_t^*) + J^*\big(x_t + \Delta t f(x_t, u_t^*), t + \Delta t\big) + o(\Delta t) \\
J^*(x, t) &= \min_u \Big\{ \Delta t L(x, u) + J^*\big(x + \Delta t f(x, u), t + \Delta t\big) \Big\} + o(\Delta t)
\end{aligned}$$

Hence $J^*(x,t) \sim V_k^*(x)$ and $\Delta t L(x,u) \sim c(x,u)$. Also, $J^*(x,T) = \Psi(x)$, so $\Psi \sim \Theta$.

Bellman's equation produces

$$
\begin{aligned}
J^*(x,t) &= \min_u \Big\{ \Delta t L(x,u) + J^*\big(x + \Delta t f(x,u), t + \Delta t\big) \Big\} + o(\Delta t), \\
&\qquad\qquad\qquad\qquad\qquad\qquad t = 0, \Delta t, 2\Delta t, \dots, T - \Delta t \\
J^*(x,T) &= \Psi(x)
\end{aligned}
$$

But we need this in continuous time. Taylor expansion produces

$$
\begin{aligned}
J^*\big(x + \Delta t f(x,u), t + \Delta t\big) &= J^*(x,t) + \frac{\partial J^*(x,t)}{\partial x} \Delta t f(x,u) + \frac{\partial J^*(x,t)}{\partial t} \Delta t + o(\Delta t) \\
J^*(x,t) &= \min_u \Big\{ \Delta t L(x,u) + J^*(x,t) + \frac{\partial J^*(x,t)}{\partial x} \Delta t f(x,u) + \frac{\partial J^*(x,t)}{\partial t} \Delta t \Big\} + o(\Delta t) \\
J^*(x,t) - J^*(x,t) - \frac{\partial J^*(x,t)}{\partial t} \Delta t &= \min_u \Big\{ \Delta t L(x,u) + \frac{\partial J^*(x,t)}{\partial x} \Delta t f(x,u) \Big\} + o(\Delta t)
\end{aligned}
$$

Dividing both sides by Δt and taking the limit as $\Delta t \to 0$,

$$
-\frac{\partial J^*(x,t)}{\partial t} = \min_u \Big\{ L(x,u) + \frac{\partial J^*(x,t)}{\partial x} f(x,u) \Big\}
$$

This is known as the Hamilton-Jacobi-Bellman (HJB) equation.

Theorem. *u^* is a global minimizer to*

$$
\begin{aligned}
&\min_u \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T)) \\
&s.t.\ \dot{x} = f(x,u)
\end{aligned}
$$

if and only if u^* solves the HJB equation

$$
-\frac{\partial J^*(x,t)}{\partial t} = \min_u \Big\{ L(x,u) + \frac{\partial J^*(x,t)}{\partial x} f(x,u) \Big\}, \quad t \in [0,T),
$$

where $J^*(x,T) = \Psi(T)$,

$$
J^*(x_t, t) = \int_t^T L(x^*(s), u^*(s), s)\,\mathrm{d}s + \Psi(x^*(T)),
$$

$x^*(t) = x_t$, and $\dot{x}^* = f(x^*, u^*, t)$.

Note:

1. The HJB equation is a partial differential equation (PDE) rather than an ODE (hard to solve in general).
2. It is solvable when we have linear dynamics and quadratic costs (LQ).

4.2 Linear-Quadratic Problems

$$
\begin{aligned}
\min_u \frac{1}{2}\int_0^T & \Big[x^{\mathrm{T}}(t)Q(t)x(t) + u^{\mathrm{T}}(t)R(t)u(t)\Big]\,\mathrm{d}t + \frac{1}{2}x^{\mathrm{T}}(T)Sx(T), \\
& \qquad\qquad Q(t) = Q^{\mathrm{T}}(t) \succeq 0,\ S = S^{\mathrm{T}} \succeq 0,\ R(t) = R^{\mathrm{T}}(t) \succ 0 \\
\text{s.t. } & \dot{x}(t) = A(t)x(t) + B(t)u(t) \\
& x(0) = x_0
\end{aligned}
$$

HJB states

$$
\begin{aligned}
-\frac{\partial J^*}{\partial t} &= \min_u \left\{\frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + \frac{\partial J^*}{\partial x}(Ax + Bu)\right\} \\
J^*(x,T) &= \frac{1}{2}x^{\mathrm{T}}Sx
\end{aligned}
$$

Minimizing the first equation with respect to u produces

$$
\begin{aligned}
\frac{\partial\{\cdot\}}{\partial u} = u^{\mathrm{T}}R + \frac{\partial J^*}{\partial x}B &= 0 \\
Ru + B^{\mathrm{T}}\frac{\partial J^{*\mathrm{T}}}{\partial x} &= 0 \\
u &= -R^{-1}B^{\mathrm{T}}\frac{\partial J^{*\mathrm{T}}}{\partial x} \\
\frac{\partial^2\{\cdot\}}{\partial u^2} = R \succ 0 &\Rightarrow u^* \text{ is the global minimizer}
\end{aligned}
$$

Going back to HJB,

$$
\begin{aligned}
-\frac{\partial J^*}{\partial t} &= \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}\frac{\partial J^*}{\partial x}BR^{-1}RR^{-1}B^{\mathrm{T}}\frac{\partial J^{*\mathrm{T}}}{\partial x} + \frac{\partial J^*}{\partial x}Ax - \frac{\partial J^*}{\partial x}BR^{-1}B^{\mathrm{T}}\frac{\partial J^{*\mathrm{T}}}{\partial x} \\
&= \frac{1}{2}x^{\mathrm{T}}Qx + \frac{\partial J^*}{\partial x}Ax - \frac{1}{2}\frac{\partial J^*}{\partial x}BR^{-1}B^{\mathrm{T}}\frac{\partial J^{*\mathrm{T}}}{\partial x}
\end{aligned}
$$

We still have a PDE to solve. Note $J^*(x,T) = \frac{1}{2}x^{\mathrm{T}}Sx$. Maybe $J^*(x,t) = \frac{1}{2}x^{\mathrm{T}}P(t)x$ for some $P(t) = P^{\mathrm{T}}(t) \succeq 0$. Let's try:

$$
\begin{aligned}
\frac{\partial J^*}{\partial t} &= \frac{1}{2}x^{\mathrm{T}}\dot{P}x \\
\frac{\partial J^*}{\partial x} &= x^{\mathrm{T}}P \\
-\frac{1}{2}x^{\mathrm{T}}\dot{P}x &= \frac{1}{2}x^{\mathrm{T}}Qx + x^{\mathrm{T}}PAx - \frac{1}{2}x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px \\
&= \frac{1}{2}x^{\mathrm{T}}\Big(Q + 2PA - PBR^{-1}B^{\mathrm{T}}P\Big)x
\end{aligned}
$$

Note $x^{\mathrm{T}}PAx \in \mathbb{R}$ so $x^{\mathrm{T}}PAx = x^{\mathrm{T}}A^{\mathrm{T}}Px = \frac{1}{2}x^{\mathrm{T}}A^{\mathrm{T}}Px + \frac{1}{2}x^{\mathrm{T}}PAx = \frac{1}{2}x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x$.

$$
\Longrightarrow -\frac{1}{2}x^{\mathrm{T}}\dot{P}x = \frac{1}{2}x^{\mathrm{T}}\Big(Q + PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P\Big)x
$$

This has to hold for all x, i.e. P satisfies

$$\begin{cases} \dot{P} = -Q - PA - A^{\mathrm{T}}P + PBR^{-1}B^{\mathrm{T}}P \\ P(T) = S \end{cases}$$

This is known as the differential Riccati equation (RE/DRE). Luckily for us, we can actually solve RE "analytically" (almost if A, B, R, Q depend on t, and completely if they do not).

Theorem. *The optimal control is* $u^* = -R^{-1}B^T P(t)x$, *where* $P(t) = P^T(t) \succeq 0$ *solves the RE.*

Example Scalar example posted on T-square:

$$\begin{aligned} &\min \int_0^1 (qx^2 + ru^2)\,\mathrm{d}t + sx^2(1), \quad q, s \geq 0,\ r > 0 \\ &\text{s.t. } \dot{x} = ax + bu, \quad x, u \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} u &= -R^{-1}B^{\mathrm{T}}Px = -\frac{bp}{r}x \\ \dot{p} &= -q - 2ap + \frac{b^2}{r}p^2 \\ p(1) &= s \end{aligned}$$

This example compares the numerical solution to the following analytical solution.

How do we solve the RE?

$$\begin{aligned} \dot{x} &= Ax + Bu = \underbrace{(A - BR^{-1}B^{\mathrm{T}}P)}_{N(t)}x \\ x(t) &= \Phi(t,0)x(0) = \Phi(t,T)x(T), \end{aligned}$$

where Φ is the state transition matrix. Note this is also the zero-input response.

Let $X(t) = \Phi(t,T) \in \mathbb{R}^{n\times n}$. We know from ECE 6550 that

$$\begin{aligned} \dot{X} &= (A - BR^{-1}B^{\mathrm{T}}P)X \\ X(T) &= \mathrm{I} \end{aligned}$$

Let $Y = PX$. Then,

$$\begin{aligned} \dot{Y} &= \dot{P}X + P\dot{X} \\ &= \Big(-Q - A^{\mathrm{T}}P - PA + PBR^{-1}B^{\mathrm{T}}P\Big)X + P\Big(A - BR^{-1}B^{\mathrm{T}}P\Big)X \\ &= -QX - A^{\mathrm{T}}Y \\ Y(T) &= S \\ &\Longrightarrow \begin{cases} \dot{X} = AX - BR^{-1}B^{\mathrm{T}}Y \\ \dot{Y} = -QX - A^{\mathrm{T}}Y \\ X(T) = \mathrm{I} \\ Y(T) = S \end{cases} \end{aligned}$$

Note that $P = YX^{-1}$, where X is always invertible since it is a state transition matrix.

Assume that A, B, Q, R do not depend on time. Then,

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^{\mathrm{T}} \\ -Q & -A^{\mathrm{T}} \end{bmatrix}}_{M \in \mathbb{R}^{2n \times 2n}} \begin{bmatrix} X \\ Y \end{bmatrix} \tag{4.1}$$

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{M(t-T)} \begin{bmatrix} \mathrm{I} \\ S \end{bmatrix} \tag{4.2}$$

We've traded a quadratic $n \times n$ ODE for a linear $2n \times 2n$ ODE!

In summary, the LQ problem is

$$\begin{aligned} \min_u \; & \int_0^T \left[x^{\mathrm{T}}(t)Q(t)x(t) + u^{\mathrm{T}}(t)R(t)u(t) \right] \mathrm{d}t + x^{\mathrm{T}}(T)Sx(T) \\ \text{s.t. } \; & \dot{x}(t) = A(t)x(t) + B(t)u(t), \end{aligned}$$

where R is positive definite and Q and S are positive semi-definite. Positive semi-definiteness of Q and S are needed so that the cost terms are nonnegative. This ensures that the problem is solvable, because otherwise the cost can go to negative infinity. Positive definiteness of R is needed so the cost term is nonnegative as before; also, this ensures the cost term is only zero if the control u is zero, so the control can't be infinite and still have zero cost. These conditions make the problem well-defined.

The globally optimal solution is

$$u = -R^{-1}(t)B^{\mathrm{T}}(t)P(t)x,$$

where $P(t)$ solves the Riccati equation

$$\begin{cases} \dot{P} = -A^{\mathrm{T}}P - PA - Q + PBR^{-1}B^{\mathrm{T}}P \\ P(T) = S \end{cases}$$

Generally this is solved numerically. But if Q, R, A, B do not depend on t then P can be found analytically through (4.1) and (4.2).

Example

$$\begin{aligned} \min_u \; & \int_0^T u^2(t)\,\mathrm{d}t + x_1^2(T) \\ \text{s.t. } \; & \dot{x}_1 = x_2 \\ & \dot{x}_2 = u \end{aligned}$$

$$\begin{aligned} x &= [x_1 \; x_2]^{\mathrm{T}} \\ A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ Q &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1 \\ S &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Recall that a symmetric matrix is positive definite iff its eigenvalues are nonnegative and real.

Use the X, Y method to find P:

$$\begin{aligned}
\dot{X} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Y \\
X(T) &= \mathrm{I} \\
\dot{Y} &= -\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} X - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} Y \\
Y(T) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&\dot{y}_{11} = 0, y_{11}(T) = 1 && y_{11}(t) = 1 \\
&\dot{y}_{12} = 0, y_{12}(T) = 0 && y_{12}(t) = 0 \\
&\dot{y}_{21} = -y_{11} = -1, y_{21}(T) = 0 && y_{21}(t) = T - t \\
&\dot{y}_{22} = -y_{12} = 0, y_{22}(T) = 0 && y_{22}(t) = 0 \\
&\dot{x}_{21} = -y_{21} = t - T, x_{21}(T) = 0 && x_{21}(t) = \frac{(t-T)^2}{2} \\
&\dot{x}_{22} = -y_{22} = 0, x_{22}(T) = 1 && x_{22}(t) = 1 \\
&\dot{x}_{12} = x_{22} = 1, x_{12} = 0 && x_{12} = t - T \\
&\dot{x}_{11} = x_{21} = \frac{(t-T)^2}{2}, x_{11}(T) = 1 && x_{11}(t) = \frac{(t-T)^3}{6} + 1
\end{aligned}$$

Recall that the inverse of a 2×2 matrix is

$$\begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}^{-1} = \frac{1}{\alpha\gamma - \beta\delta} \begin{bmatrix} \gamma & -\beta \\ -\delta & \alpha \end{bmatrix}.$$

Therefore

$$\begin{aligned}
X(t) &= \begin{bmatrix} \frac{(t-T)^3}{6} + 1 & t - T \\ \frac{(t-T)^2}{2} & 1 \end{bmatrix} \\
X^{-1}(t) &= \begin{bmatrix} 1 & T - t \\ -\frac{(t-T)^2}{2} & \frac{(t-T)^3}{6} + 1 \end{bmatrix} \cdot \frac{1}{\frac{(t-T)^3}{6} + 1 - \frac{(t-T)^3}{2}}
\end{aligned}$$

and

$$\begin{aligned}
P(t) &= \begin{bmatrix} 1 & 0 \\ T - t & 0 \end{bmatrix} \begin{bmatrix} 1 & T - t \\ -\frac{(t-T)^2}{2} & \frac{(t-T)^3}{6} + 1 \end{bmatrix} \cdot \frac{1}{\frac{(t-T)^3}{6} + 1 - \frac{(t-T)^3}{2}} \\
P(t) &= \frac{1}{1 + (T-t)^3/3} \begin{bmatrix} 1 & T - t \\ T - t & (T-t)^2 \end{bmatrix}
\end{aligned}$$

The control is

$$\begin{aligned}
u &= -R^{-1} B^{\mathrm{T}} P x = -1^{-1} [0 \ 1] P(t) x \\
u &= -\frac{1}{1 + (T-t)^3/3} \Big[(T-t) x_1 + (T-t)^2 x_2 \Big]
\end{aligned}$$

Note:

1. If A, B, Q, R are not time-varying, then $u(t)$ is a function of $T - t$.
2. Initial conditions do not matter. This is optimal for any x. (Feedback!)

4.2.1 (Some) Rocket Science

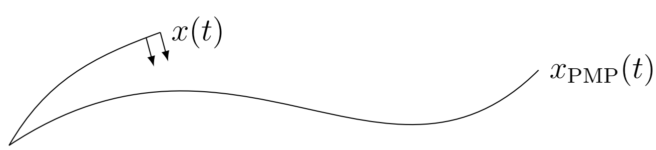
Problem: Send a space ship to the moon.

$$\begin{aligned} \min_u & \int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T)) \\ \text{s.t. } & \dot{x} = f(x,u,t) \\ & x(0) = x_0 \end{aligned}$$

Solve this using PMP:

$$\left.\begin{aligned} & H = L + \lambda^{\mathrm{T}} f \\ & u = \arg\min H \\ & \dot{\lambda} = -\frac{\partial H}{\partial x}^{\mathrm{T}} \\ & \lambda(T) = \frac{\partial \Psi}{\partial x}(x(T)) \end{aligned}\right\} \Longrightarrow \begin{aligned} & u_{\mathrm{PMP}}(t) \\ & x_{\mathrm{PMP}}(t) \end{aligned}$$

Solving numerically produces lookup tables for u and x. This is not robust, and MPC was too slow at that time. Kalman came up with the idea to try to steer the actual trajectory to the optimal PMP trajectory using feedback, linearizing the system around $x_{\mathrm{PMP}}(t)$.



Let $x(t)$ be the actual trajectory obtained using controller $u(t)$.

$$\begin{aligned} \delta x &= x - x_{\mathrm{PMP}} \\ \delta u &= u - u_{\mathrm{PMP}} \end{aligned}$$

We want both δx and δu to be small.

$$\begin{aligned}
\delta\dot{x} &= \dot{x} - \dot{x}_{\mathrm{PMP}} = f(x,u) - f(x_{\mathrm{PMP}}, u_{\mathrm{PMP}}) \\
&= f(\delta x + x_{\mathrm{PMP}}, \delta u + u_{\mathrm{PMP}}) - f(x_{\mathrm{PMP}}, u_{\mathrm{PMP}}) \\
&= f(x_{\mathrm{PMP}}, u_{\mathrm{PMP}}) + \frac{\partial f(x_{\mathrm{PMP}}, u_{\mathrm{PMP}})}{\partial x}\delta x + \frac{\partial f(x_{\mathrm{PMP}}, u_{\mathrm{PMP}})}{\partial u}\delta u - f(x_{\mathrm{PMP}}, u_{\mathrm{PMP}}) + \text{H.O.T.} \\
&= \underbrace{\frac{\partial f(x_{\mathrm{PMP}}, u_{\mathrm{PMP}})}{\partial x}}_{\text{t.-v. } n\times n \text{ matrix}}\delta x + \underbrace{\frac{\partial f(x_{\mathrm{PMP}}, u_{\mathrm{PMP}})}{\partial u}}_{\text{t.-v. } n\times m \text{ matrix}}\delta u + \text{H.O.T.}
\end{aligned}$$

Therefore, set

$$\begin{aligned}
A(t) &= \frac{\partial f(x_{\mathrm{PMP}}(t), u_{\mathrm{PMP}}(t))}{\partial x} \\
B(t) &= \frac{\partial f(x_{\mathrm{PMP}}(t), u_{\mathrm{PMP}}(t))}{\partial u} \\
\delta\dot{x} &= A(t)\delta x + B(t)\delta u
\end{aligned}$$

Now, solve the LQ problem:

$$\min_{\delta u} \int_0^T \left[\delta x^{\mathrm{T}} Q \delta x + \delta u^{\mathrm{T}} R \delta u\right] \mathrm{d}t + \delta x^{\mathrm{T}}(T) S \delta x(T)$$

$$\delta u = -R^{-1} B^{\mathrm{T}}(t) P(t) \delta x$$

$$\boxed{u(t) = u_{\mathrm{PMP}}(t) - R^{-1}\frac{\partial f^{\mathrm{T}}(x_{\mathrm{PMP}}(t), u_{\mathrm{PMP}}(t))}{\partial u} P(t)(x(t) - x_{\mathrm{PMP}}(t))}$$

4.3 Infinite Horizon LQ Control

Recall for the LQ problem, the optimal cost-to-go

$$J^*(x_t, t) = \int_t^T \left(x^{*\mathrm{T}} Q x^* + u^{*\mathrm{T}} R u^*\right) \mathrm{d}t + x^{*\mathrm{T}}(T) S x^*(T),$$

where $x^*(t) = x_t$, is given by

$$J^*(x_t, t) = x_t^{\mathrm{T}} P(t) x_t.$$

The total cost is $x_0^{\mathrm{T}} P(0) x_0$. We would like to make the control time-invariant, because

1. We want $u = -Kx$ (static feedback).
2. A system typically spends most of its time in steady-state ($t \to \infty$).

The cost is

$$\int_0^\infty \left(x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru\right) \mathrm{d}t,$$

where Q, R are static. There is no terminal cost, so either $x(\infty) = 0$ or $J = \infty$. The dynamics are $\dot{x} = Ax + Bu$, where A, B are static. As before, $Q \succeq 0$ and $R \succ 0$.

Problems/Questions:

1. Do we have enough "control authority" to make $x \to 0$?
2. Can we ensure that x cannot "hide" in the $x^{\mathrm{T}}Qx$ term so we don't end up with a finite cost yet $\|x\| \to \infty$?

A ECE 6550 Refresher

Question 1: Can we always find a u that takes the system between any two points?

This is possible iff (A, B) is a completely controllable (CC) system/pair. Therefore, we assume (require!) (A, B) is CC.

Definition. For $x \in \mathbb{R}^n$, (A, B) is CC iff $\operatorname{rank}(\Gamma) = n$, where Γ is the controllability matrix

$$\Gamma = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times nm}.$$

Question 2: Over time, will x show up in $x^{\mathrm{T}}Qx$?

For positive semi-definite Q, there exists $\sqrt{Q} \succeq 0$. Then, the term can be written as

$$x^{\mathrm{T}}Qx = \left(\sqrt{Q}x\right)^{\mathrm{T}}\left(\sqrt{Q}x\right) = \left\|\sqrt{Q}x\right\|^2.$$

Let the output be $y = \sqrt{Q}x$. Can we infer $x(t)$ from the output trajectory $y(s)$, $s \in [0, t]$? This is equivalent to saying given x_0, x_0', can we ensure that $y(t)$ and $y'(t)$ differ at some point.

This is possible if the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= \sqrt{Q}x \end{aligned}$$

is completely observable (CO). Therefore, we require $(A, \sqrt{Q})$ is CO.

Definition. For $y = Cx \in \mathbb{R}^p$, (A, C) is CO iff $\operatorname{rank}(\Omega) = n$, where Ω is the observability matrix

$$\Omega = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Thus, the problem is

$$\begin{aligned}
&\min_u \int_0^\infty (x^\mathrm{T} Q x + u^\mathrm{T} R u)\,\mathrm{d}t \\
&\text{s.t. } \dot{x} = Ax + Bu,
\end{aligned}$$

where $Q = Q^\mathrm{T} \succeq 0$, $R = R^\mathrm{T} \succ 0$, (A, B) is CC, and $(A, \sqrt{Q})$ is CO.

Note that

$$\begin{gathered}
\min_u \int_0^T (x^\mathrm{T} Q x + u^\mathrm{T} R u)\,\mathrm{d}t \le \min_u \int_0^{T+\Delta T} (x^\mathrm{T} Q x + u^\mathrm{T} R u)\,\mathrm{d}t \\
x_0^\mathrm{T} P_T(0) x_0 \le x_0^\mathrm{T} P_{T+\Delta T}(0) x_0
\end{gathered}$$

The optimal cost is a non-decreasing function of T, so either the cost increases indefinitely or it reaches a steady-state cost, i.e. $P_T(0)$ reaches a steady-state value as T increases:

$$\lim_{T\to\infty} P_T(0) = P_\infty$$

Moreover, $\dot{P}_\infty = 0$, i.e.

$$\dot{P}_\infty = -A^\mathrm{T} P_\infty - P_\infty A - Q + P_\infty B R^{-1} B^\mathrm{T} P_\infty = 0.$$

The P in which we are interested is static and satisfies the algebraic Riccati equation (ARE)

$$A^\mathrm{T} P + PA + Q - PBR^{-1}B^\mathrm{T} P = 0.$$

In fact, P is the *unique* positive definite solution to the ARE.

Theorem. *If (A, B) is CC, $(A, \sqrt{Q})$ is CO, $Q = Q^T \succeq 0$, and $R = R^T \succ 0$ then the ARE has a unique positive definite solution P and the optimal controller is $u = -R^{-1}B^TPx = -Kx$.*

Remarks:

1. $u = -R^{-1}B^\mathrm{T} P x$ is a static feedback law.
2. This is an alternative to pole-placement that can be more intuitive. This changes the design choice from the closed-loop eigenvalues $\lambda_1, \ldots, \lambda_n$ to Q, R.

Example

$$\begin{aligned}
&\min_u \int_0^\infty (4x_1^2 + 5x_2^2 + u^2)\,\mathrm{d}t \\
&\text{s.t. } \dot{x}_1 = x_2 \\
&\qquad \dot{x}_2 = u
\end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad R = 1$$

Is (A, B) CC? The controllability matrix is

$$\Gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\text{rank}(\Gamma) = 2 \Rightarrow \text{CC}$$

In Matlab, `Gamma=ctrb(A,B); rank(Gamma)`.

Is $(A, \sqrt{Q})$ CO? In this case, this is trivially true since $Q \succ 0$ so $x = (\sqrt{Q})^{-1}y$ and we immediately get the state from the output. The observability matrix is

$$\Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} \sqrt{Q} \\ \sqrt{Q}A \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{5} \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$
$$\text{rank}(\Omega) = 2 \Rightarrow \text{CO}$$

In Matlab, `C=sqrtm(Q); Omega=obsv(A,C); rank(Omega)`.

The ARE is

$$A^{\mathrm{T}}P + PA + Q - PBR^{-1}B^{\mathrm{T}}P = 0$$
$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} P + P \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} - P \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} P = 0$$
$$\Rightarrow \begin{cases} 4 - p_{12}^2 = 0 \\ p_{11} - p_{12}p_{22} = 0 \\ 2p_{12} - p_{22}^2 + 5 = 0 \end{cases}$$

If $p_{12} = -2$,

$$p_{22}^2 = 5 - 4 = 1, \qquad p_{22} = \pm 1$$
$$p_{11} = p_{12}p_{22} = \mp 2$$

If $p_{12} = 2$,

$$p_{22}^2 = 5 + 4 = 9, \qquad p_{22} = \pm 3$$
$$p_{11} = p_{12}p_{22} = \pm 6$$

We have four possible solutions to the ARE:

$$P_1 = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \qquad P_4 = \begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix}$$

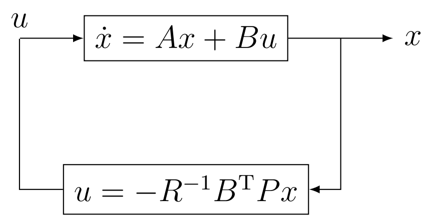
We need $P \succ 0$.

$$\text{eig}(P_1) = -3, 2$$
$$\text{eig}(P_2) = 3, -2$$
$$\text{eig}(P_3) = 7, 2$$
$$\text{eig}(P_4) = -7, -2$$

Only P_3 is positive definite. The optimal control is

$$u = -R^{-1}B^{\mathrm{T}}Px = -1^{-1}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}6 & 2\\ 2 & 3\end{bmatrix}x$$
$$u = -\begin{bmatrix}2 & 3\end{bmatrix}x$$

The closed-loop system using the feedback law is shown below.



The closed-loop dynamics are

$$\dot{x} = Ax - BR^{-1}B^{\mathrm{T}}Px = \underbrace{(A - BR^{-1}B^{\mathrm{T}}P)}_{A_{\mathrm{CL}}}x$$

The "optimal" closed-loop eigenvalues (poles) are $\mathrm{eig}(A - BR^{-1}B^{\mathrm{T}}P) = -1, -2$. In Matlab, `K=place(A,B,[-1 -2])=[2 3]`, which are the same.

In Matlab, the ARE is solved by `P=are(A,B*R^-1*B',Q)`.

4.3.1 Linear Quadratic Regulation (LQR)

Now we have an "output", a part of the state we care about given by

$$y = Cx, \quad y \in \mathbb{R}^m \ (m \le n).$$

We want to drive $y \to 0$.

$$\begin{aligned}\min_u \ & \int_0^\infty (y^{\mathrm{T}}Qy + u^{\mathrm{T}}Ru)\,\mathrm{d}t\\ \text{s.t. } & \dot{x} = Ax + Bu\\ & y = Cx,\end{aligned}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^m$, $Q \in \mathbb{R}^{m\times m} \succ 0$, and $R \in \mathbb{R}^{p\times p} \succ 0$. We still need (A, B) to be CC and $(A, \sqrt{Q}C)$ to be CO. Typically, $Q = \mathrm{I}$, so (A, C) needs to be CO.

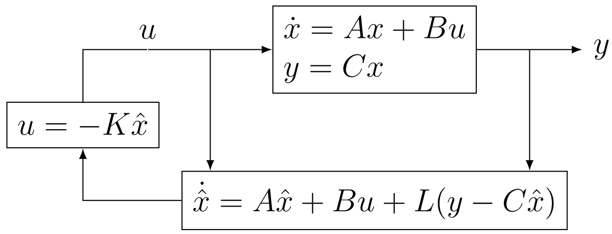
To solve this, we turn it into a LQ problem:

$$\min_u \int_0^\infty (u^{\mathrm{T}}Ru + x^{\mathrm{T}}C^{\mathrm{T}}QCx)\,\mathrm{d}t$$

The optimal control is $u = -R^{-1}B^{\mathrm{T}}Px$, where $P = P^{\mathrm{T}} \succ 0$ uniquely solves

$$A^{\mathrm{T}}P + PA + C^{\mathrm{T}}QC - PBR^{-1}B^{\mathrm{T}}P = 0.$$

But what if y is not just an "output" but the actual measured output of the system? We can no longer directly compute the control since we don't know x. The solution is to estimate $\hat{x}$ from y and use $\hat{x}$ instead (observer design). The following figure shows an Luenberger observer.



Here, pole placement can be used to place the eigenvalues of $A - LC$ larger than those of the closed-loop system so the observer $\hat{x}$ converges quickly to the actual x.

4.4 Connections Between Machine Learning and Optimal Control[1]

Recall the discrete-time derivation of Bellman's equation. The system was

$$x_{k+1} = F(x_k, u_k)$$

with cost $c(x_k, u_k)$ for doing u_k at x_k. Using $u = u_0, \ldots, u_{N-1}$ from x_0, the total cost was

$$V_N^u(x_0) = \sum_{k=0}^{N-1} c(x_k, u_k) + \Theta(x_N).$$

Bellman's equation for the optimal cost-to-go is

$$\begin{aligned} V_k^*(x) &= \min_u \Big\{ c(x,u) + V_{k-1}^*\big(F(x,u)\big) \Big\} \\ V_0^*(x) &= \Theta(x), \end{aligned}$$

for $k = 1, \ldots, N$. If we do an infinite-horizon version instead, we would get a *policy* $u = \Pi(x)$, where the control action depends solely on the state and not time. The infinite-horizon cost is

$$V^{\Pi}(x) = \sum_{k=0}^{\infty} c(x_k, \Pi(x_k))\gamma^k,$$

where $\gamma \in (0, 1)$ is the discount. The discount ensures the cost converges. Note there is no step count in the notation V^{Π} and no terminal cost. Bellman's equation for the infinite-horizon case is

$$V^*(x) = \min_u \left\{ c(x,u) + \sum_{k=1}^{\infty} \gamma^k c\big(x_k, \Pi^*(x_k)\big) \right\}.$$

[1] Not on final.

Note that

$$\begin{aligned}\sum_{k=1}^{\infty}\gamma^k c\big(x_k,\Pi^*(x_k)\big) &= \gamma\sum_{k=0}^{\infty}\gamma^k c\big(x_{k+1},\Pi^*(x_{k+1})\big)\\ &= \gamma\sum_{k=0}^{\infty}\gamma^k c\Big(F\big(x_k,\Pi^*(x_k)\big),\Pi^*\big(F(x_k,\Pi^*(x_k))\big)\Big)\\ &= \gamma V^*\big(F(x_0,\Pi^*(x_0))\big)\end{aligned}$$

Then, Bellman's infinite horizon equation is

$$V^*(x) = \min_u \Big\{c(x,u) + \gamma V^*\big(F(x,u)\big)\Big\}.$$

Let's assume we magically had $V^*(x)$. Then

$$\Pi^*(x) = \arg\min_u \Big\{c(x,u) + \gamma V^*\big(F(x,u)\big)\Big\}.$$

This works only if we somehow also know $c(x,u)$ and $F(x,u)$ for all x, u. The problem is we may not know c and F so we cannot evaluate $\arg\min\{\cdot\}$ even if we knew $V^*(x)\ \forall x$.

What if we were instead given (using magic)

$$W^*(x,u) = c(x,u) + \gamma\min_v W^*\big(F(x,u),v\big).$$

We can rewrite this as

1. $\min_u W^*(x,u) = V^*(x)$
2. $\Pi^*(x) = \arg\min_u W^*(x,u)$

Note that this is just Bellman's equation. So knowing $W^*(x,u)\ \forall x,u$ (do not need to know c, F) is enough for us to know $\Pi^*(x)$! Instead of using magic, we could instead

1. use optimal control (last few weeks)
2. move around in the "world", "experiencing" the costs as they show up; this is known as *reinforcement learning*

4.4.1 Reinforcement Learning

The idea is to move around (randomly) and build up the values in the Q-table. This is a form of reinforcement learning known as Q-learning.

Let's assume (for now) that X ($x \in X$) and U ($u \in U$) are both finite. Let's call $W^*(x,u) = Q(x,u)$.

Algorithm 1 Q-learning

$Q_0(x,u) = q_0\ \forall x,u$
$k = 0$
repeat
Pick (x,u) "randomly" and update the Q-table through

$$Q_k(x,u) = Q_{k-1}(x,u) + \alpha_k\Big[c(x,u) + \gamma\min_v\Big\{Q_{k-1}\big(F(x,u),v\big) - Q_{k-1}(x,u)\Big\}\Big]$$

$k = k+1$
until "done"

Note:

1. $c(x, u)$ and $F(x, u)$ are "experienced", i.e. there is no need to know these in advance
2. α_k is the "learning rate" and "needs" to satisfy

$$\sum \alpha_k^2 < \infty, \qquad \sum \alpha_k = \infty$$

3. If all (x, u) pairs are visited infinitely many times, then $Q_k(x, u) \to Q(x, u)\ \forall x, u$ as $k \to \infty$.

A extension (even for non-finite X, U) is to choose a "fatter" basis function instead of points. A basis function should have compact support. Examples include sigmoids, wavelets, B-splines, and kernels.

How many basis functions? How to update their amplitudes from the data?

Matlab example `qlearn` on T-Square.

Chapter 5

Hamilton-Jacobi Theory

5.1 Hamilton-Jacobi: Relating HJB to PMP

Recall the HJB theorem: u^* is the global minimizer to

$$\int_0^T L(x,u)\,\mathrm{d}t + \Psi(x(T))$$

s.t. $\dot{x} = f(x,u)$ if it solves

$$\begin{aligned} -\frac{\partial J^*}{\partial t} &= \min_u \left\{ L(x,u) + \frac{\partial J^*}{\partial x} f(x,u) \right\} \\ J^*(x,T) &= \Psi(x) \end{aligned} \tag{5.1}$$

where $J^*(x,t)$ is the optimal cost-to-go from t to T starting at x.

Note that the term within the curly brackets on the right-hand side of (5.1) is a Hamiltonian:

$$H(x,u,\lambda) = L(x,u) + \lambda^{\mathrm{T}} f(x,u)$$

with the odd choice of $\lambda = \partial J^{*\mathrm{T}}(x,t)/\partial x$.

From Pontryagin, $u^* = \min_u H(x,u,\lambda)$ (locally). However, the minimization in (5.1) is global. What if we insisted on a global minimizer u^*?

Definition. The Hamiltonian $H(x,u,\lambda)$ is *regular* if, for every x, λ, there is a unique global minimizer $u^*(x,\lambda)$, i.e.

$$H(x, u^*(x,\lambda), \lambda) < H(x,u,\lambda) \quad \forall u \neq u^*(x,\lambda).$$

From HJB, we know that we should pick $\lambda = \partial J^{*\mathrm{T}}(x,t)/\partial x$. Assuming H is regular, we should use

$$u^*\left(x, \frac{\partial J^{*\mathrm{T}}}{\partial x}\right).$$

PMP states we should find λ from

$$\begin{aligned}\dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x}, \\ \lambda(T) &= \frac{\partial \Psi}{\partial x}(x(T)).\end{aligned}$$

Hamilton-Jacobi (HJ) states we should get our new "costate" $\partial J^*/\partial x$ from a "modified" HJB:

$$\min_u \left\{ L(x,u) + \frac{\partial J^*}{\partial x} f(x,u) \right\} = \min_u \left\{ H\left(x, u, \frac{\partial J^{*\mathrm{T}}}{\partial x}\right) \right\} = H\left(x, u^*\left(x, \frac{\partial J^{*\mathrm{T}}}{\partial x}\right), \frac{\partial J^{*\mathrm{T}}}{\partial x}\right).$$

HJ is preferable to HJB because it makes statements about u^* (directly) rather than J^*.

Theorem (Hamilton-Jacobi)**.** *If H is regular and $J^*(x,t)$ satisfies the HJ equation*

$$\begin{cases} -\dfrac{\partial J^*}{\partial t} = H\left(x, u^*\left(x, \dfrac{\partial J^{*T}}{\partial x}\right), \dfrac{\partial J^{*T}}{\partial x}\right) \\ J^*(x,T) = \Psi(x) \end{cases}$$

then

$$u^*\left(x, \frac{\partial J^{*T}}{\partial x}\right).$$

is the global optimal solution.

Example

$$\begin{aligned}&\min_u \int_0^1 2u^2(t)\,\mathrm{d}t + (x(1)-1)^2 \\ &\text{s.t. } \dot{x} = u, \quad x, u \in \mathbb{R}\end{aligned}$$

This is almost LQ except for the wrong terminal condition.

$$\begin{aligned} H &= 2u^2 + \lambda u \\ \frac{\partial H}{\partial u} &= 4u + \lambda \\ \frac{\partial^2 H}{\partial u^2} &= 4 > 0, \end{aligned}$$

so H is strictly convex in u, and H is regular. By HJ,

$$u^*(x,\lambda) = -\frac{\lambda}{4}$$

Replace λ with $\partial J^*/\partial x$ and plug into the HJ equation:

$$\begin{aligned}
-\frac{\partial J^*}{\partial t} &= H\left(x, u^*\left(x, \frac{\partial J^{*\mathrm{T}}}{\partial x}\right), \frac{\partial J^{*\mathrm{T}}}{\partial x}\right) \\
&= 2\underbrace{\left(-\frac{\partial J^*/\partial x}{4}\right)^2}_{u^2} + \underbrace{\frac{\partial J^*}{\partial x}}_{\lambda} \cdot \underbrace{\left(-\frac{\partial J^*/\partial x}{4}\right)}_{u} \\
&= \frac{(\partial J^*/\partial x)^2}{8} - \frac{(\partial J^*/\partial x)^2}{4} = -\frac{1}{8}\left(\frac{\partial J^*}{\partial x}\right)^2 \\
\frac{\partial J^*}{\partial t} &= \frac{1}{8}\left(\frac{\partial J^*}{\partial x}\right)^2 \\
J^*(x,1) &= (x-1)^2
\end{aligned}$$

What is $J^*(x,t)$? A standard trick is to assume separability, i.e.

$$J^*(x,t) = F_1(x)F_2(t).$$

Then,

$$F_1(x)F_2'(t) = \frac{1}{8}[F_1'(x)F_2(t)]^2 = \frac{1}{8}[F_1'(x)]^2F_2^2(t).$$

We have $[F_1'(x)]^2 \sim F_1(x)$, so $F_1(x)$ is probably quadratic in x. We also have $F_2'(t) \sim F_2^2(t)$, so $F_2(t)$ is probably $\sim 1/t$. Let's try

$$J^*(x,t) = \frac{ax^2+bx+c}{\alpha t+\beta}.$$

Then,

$$\begin{aligned}
\frac{\partial J^*}{\partial x} &= \frac{2ax+b}{\alpha t+\beta} \\
\left(\frac{\partial J^*}{\partial x}\right)^2 &= \frac{4a^2x^2+4abx+b^2}{(\alpha t+\beta)^2} \\
\frac{\partial J^*}{\partial t} &= -\alpha\frac{ax^2+bx+c}{(\alpha t+\beta)^2}
\end{aligned}$$

The HJ equation is then

$$\begin{aligned}
0 &= \frac{\partial J^*}{\partial t} - \frac{1}{8}\left(\frac{\partial J^*}{\partial x}\right)^2 \\
&= \frac{-\alpha(ax^2+bx+c)}{(\alpha t+\beta)^2} - \frac{1}{8}\frac{4a^2x^2+4abx+b^2}{(\alpha t+\beta)^2} \\
&= \left(\alpha a+\frac{1}{2}a^2\right)x^2 + \left(\alpha b+\frac{1}{2}ab\right)x + \left(\alpha c+\frac{1}{8}b^2\right)
\end{aligned}$$

This gives us two equations and five unknowns:

$$\begin{aligned}
x^2 &: \alpha a + \frac{1}{2}a^2 = 0 \qquad \alpha = -\frac{1}{2}a \quad (\text{or } a = 0)\\
x^1 &: \alpha b + \frac{1}{2}ab = 0 \qquad \alpha = -\frac{1}{2}a \quad (\text{or } b = 0)\\
x^0 &: \alpha c + \frac{1}{8}b^2 = 0
\end{aligned}$$

We also need to consider the boundary conditions:

$$\begin{gathered}
J^*(x,1) = (x-1)^2\\
\frac{ax^2 + bx + c}{\alpha + \beta} = x^2 - 2x + 1\\
x^2 : \frac{a}{\alpha + \beta} = 1\\
x^1 : \frac{b}{\alpha + \beta} = -2\\
x^0 : \frac{c}{\alpha + \beta} = 1
\end{gathered}$$

This gives us three additional equations, making for five equations and five unknowns. The solution is

$$\begin{cases}
a = 2\\
b = -4\\
c = 2\\
\alpha = -1\\
\beta = 3
\end{cases}$$

$$\begin{gathered}
J^*(x,t) = \frac{2x^2 - 4x + 2}{-t + 3} = 2\frac{(x-1)^2}{3-t}\\
u^* = -\frac{\partial J^*/\partial x}{4}\\
\boxed{u^* = \frac{1-x}{3-t}}
\end{gathered}$$

This is the globally optimal solution!

This problem could have been solved by turning it into an LQ problem. We need to turn the terminal cost from $(x(1)-1)^2$ into $\hat{x}(1)^2$. Let $\hat{x} = x - 1$, so $\dot{\hat{x}} = \dot{x} = u$. Then, the problem becomes

$$\begin{aligned}
\min_u \; & \int_0^1 2u^2 \, \mathrm{d}t + \hat{x}^2(1)\\
\text{s.t. } \; & \dot{\hat{x}} = u\\
& \hat{x}(0) = x_0 - 1
\end{aligned}$$

The optimal controller is

$$u = -R^{-1}B^{\mathrm{T}}P\hat{x} = -\frac{1}{2}(x-1)p(t),$$

where

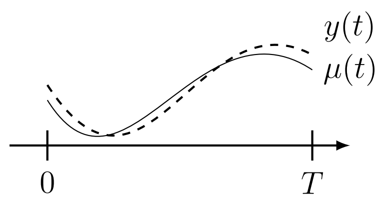
$$\begin{aligned}\dot{p} &= -A^{\mathrm{T}}P - PA - Q + PBR^{-1}B^{\mathrm{T}}P \\ \dot{p} &= \frac{p^2}{2} \\ p(1) &= 1\end{aligned}$$

5.2 The Tracking Problem

Given a linear system

$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y &= Cx, \quad y \in \mathbb{R}^p\end{aligned}$$

We would like $y(t) = Cx(t)$ to track a reference trajectory $\mu(t)$.



Let the cost be

$$\min_u \frac{1}{2}\int_0^T \Big[(y-\mu)^{\mathrm{T}}Q(y-\mu) + u^{\mathrm{T}}Ru\Big]\,\mathrm{d}t + \frac{1}{2}\big(y(T)-\mu(T)\big)^{\mathrm{T}}S\big(y(T)-\mu(T)\big),$$

where $Q = Q^{\mathrm{T}} \succeq 0$, $R = R^{\mathrm{T}} \succ 0$, and $S = S^{\mathrm{T}} \succeq 0$. This looks like a LQ problem, but it is not. (**Check:** To be a LQ problem, the B matrix needs to be invertible.)

This is a "classical" control problem. This can be solved using the HJ equation. Typically, it is difficult to solve the PDE, so instead we will verify a claimed solution.

We claim the optimal solution is

$$u(t) = -R^{-1}B^{\mathrm{T}}[P(t)x(t) + w(t)],$$

where $P(t)$ solves the Riccati equation

$$\begin{aligned}\dot{P} &= -A^{\mathrm{T}}P - PA - C^{\mathrm{T}}QC + PBR^{-1}B^{\mathrm{T}}P \\ P(T) &= C^{\mathrm{T}}SC\end{aligned}$$

and $w(t)$ satisfies

$$\begin{aligned}\dot{w} &= -(A - BR^{-1}B^{\mathrm{T}}P)^{\mathrm{T}}w + C^{\mathrm{T}}Q\mu \\ w(T) &= -C^{\mathrm{T}}S\mu(T)\end{aligned}$$

The optimal cost-to-go is given as

$$J^*(x,t) = \frac{1}{2}x^{\mathrm{T}}P(t)x + w^{\mathrm{T}}(t)x + v(t)$$

where

$$\begin{aligned}\dot{v} &= \frac{1}{2}(w^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}w - \mu^{\mathrm{T}}Q\mu) \\ v(T) &= \frac{1}{2}\mu^{\mathrm{T}}(T)S\mu(T)\end{aligned}$$

Let's verify if the claim is correct using the HJ theorem. Step 1 is to check that the Hamiltonian is regular:

$$\begin{aligned}H &= \frac{1}{2}\Big[(Cx-\mu)^{\mathrm{T}}Q(Cx-\mu) + u^{\mathrm{T}}Ru\Big] + \lambda^{\mathrm{T}}(Ax+Bu) \\ \frac{\partial H}{\partial u} &= u^{\mathrm{T}}R + \lambda^{\mathrm{T}}B = 0 \\ &\boxed{u^* = -R^{-1}B^{\mathrm{T}}\lambda} \\ \frac{\partial^2 H}{\partial u^2} &= R \succ 0,\end{aligned}$$

so u^* is a unique global minimizer (H is regular). Step 2 is to substitute $\lambda^{\mathrm{T}} = \partial J^*/\partial x$:

$$\begin{aligned}\frac{\partial J^*}{\partial x} &= x^{\mathrm{T}}P + w^{\mathrm{T}} \\ \lambda &= Px + w \\ u^* &= -R^{-1}B^{\mathrm{T}}(Px+w),\end{aligned}$$

which matches the claim. Step 3 is to show that this choice of J^* satisfies the theorem. We need to check both the boundary condition and the differential equation. For the boundary condition,

$$\begin{aligned}J^*(x,T) &= \frac{1}{2}x^{\mathrm{T}}P(T)x + w^{\mathrm{T}}(T)x + v(T) \\ &= \frac{1}{2}x^{\mathrm{T}}C^{\mathrm{T}}SCx - \mu^{\mathrm{T}}(T)SCx + \frac{1}{2}\mu^{\mathrm{T}}(T)S\mu(T) \\ &= \frac{1}{2}x^{\mathrm{T}}C^{\mathrm{T}}SCx - \frac{1}{2}\mu^{\mathrm{T}}(T)SCx - \frac{1}{2}x^{\mathrm{T}}C^{\mathrm{T}}S\mu(T) + \frac{1}{2}\mu^{\mathrm{T}}(T)S\mu(T) \\ &= \frac{1}{2}(Cx-\mu)^{\mathrm{T}}S(Cx-\mu) = \Psi(x),\end{aligned}$$

which matches the claim. For the PDE,

$$-\frac{\partial J^*}{\partial t} = H\Big(x, u^*\Big(x, \frac{\partial J^{*\mathrm{T}}}{\partial x}\Big), \frac{\partial J^{*\mathrm{T}}}{\partial x}\Big)$$

$$\begin{aligned}
\frac{\partial J^*}{\partial t} &= \frac{1}{2}x^{\mathrm{T}}\dot{P}x + \dot{w}^{\mathrm{T}}x + \dot{v} \\
&= \frac{1}{2}x^{\mathrm{T}}\Big(-A^{\mathrm{T}}P - PA - C^{\mathrm{T}}QC + PBR^{-1}B^{\mathrm{T}}P\Big)x \\
&\qquad - w^{\mathrm{T}}\Big(A - BR^{-1}B^{\mathrm{T}}P\Big)x + \mu^{\mathrm{T}}QCx + \frac{1}{2}\Big(w^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}w - \mu^{\mathrm{T}}Q\mu\Big)
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial J^*}{\partial t} &= \frac{1}{2}x^{\mathrm{T}}\Big(A^{\mathrm{T}}P + PA + C^{\mathrm{T}}QC - PBR^{-1}B^{\mathrm{T}}P\Big)x \\
&\qquad + w^{\mathrm{T}}\Big(A - BR^{-1}B^{\mathrm{T}}P\Big)x - \mu^{\mathrm{T}}QCx - \frac{1}{2}\Big(w^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}w - \mu^{\mathrm{T}}Q\mu\Big)
\end{aligned} \tag{5.2}$$

The Hamiltonian is

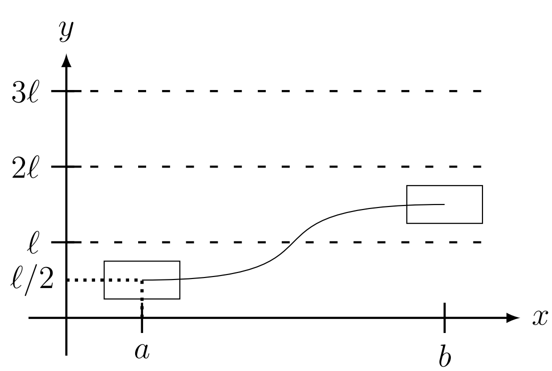
$$\begin{aligned}
H(x,u,\lambda) &= \frac{1}{2}(Cx-\mu)^{\mathrm{T}}Q(Cx-\mu) + \frac{1}{2}u^{\mathrm{T}}Ru + \lambda^{\mathrm{T}}(Ax+Bu) \\
&= \frac{1}{2}x^{\mathrm{T}}C^{\mathrm{T}}QCx - \mu^{\mathrm{T}}QCx + \frac{1}{2}\mu^{\mathrm{T}}Q\mu + \frac{1}{2}\big(w^{\mathrm{T}} + x^{\mathrm{T}}P\big)BR^{-1}RR^{-1}B^{\mathrm{T}}\big(Px+w\big) \\
&\qquad + \big(w^{\mathrm{T}} + x^{\mathrm{T}}P\big)\big(Ax - BR^{-1}B^{\mathrm{T}}(Px+w)\big)
\end{aligned}$$

Expanding and simplifying the last two terms,

$$\begin{aligned}
H(x,u,\lambda) &= \frac{1}{2}x^{\mathrm{T}}C^{\mathrm{T}}QCx - \mu^{\mathrm{T}}QCx + \frac{1}{2}\mu^{\mathrm{T}}Q\mu - \frac{1}{2}w^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}w \\
&\qquad - \frac{1}{2}x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px + w^{\mathrm{T}}Ax + \underbrace{x^{\mathrm{T}}PAx}_{(x^{\mathrm{T}}PAx + x^{\mathrm{T}}A^{\mathrm{T}}Px)/2} - x^{\mathrm{T}}PBR^{-1}Bw \\
&= \frac{1}{2}x^{\mathrm{T}}\Big(C^{\mathrm{T}}QC + PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P\Big)x \\
&\qquad w^{\mathrm{T}}\Big(A - BR^{-1}B^{\mathrm{T}}P\Big)x - \mu^{\mathrm{T}}QCx + \frac{1}{2}\mu^{\mathrm{T}}Q\mu - \frac{1}{2}w^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}w,
\end{aligned}$$

which matches (5.2), as desired.

Example Self-driving car—lane changes



Let the state of the system be

$$x = \begin{bmatrix} p_x \\ p_y \\ v_x \\ v_y \end{bmatrix}.$$

With input $[u_x, u_y]^{\mathrm{T}}$, the dynamics are

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{v}_x \\ \dot{v}_y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ v_x \\ v_y \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$\begin{bmatrix} y_{p_x} \\ y_{p_y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ v_x \\ v_y \end{bmatrix}$$

On T-Square, the Matlab file `self_driving_car.m` is posted.