ECE 6553: Homework #5

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1. (a)
   - We need $R = R^{\mathrm{T}} \succ 0$ so that the cost term $u^{\mathrm{T}}Ru$ is non-negative (so cost can't go to negative infinity) and that the term is zero if and only if $u = 0$ (so the control can't be infinite and while having zero cost).
   - We need $Q = Q^{\mathrm{T}} \succcurlyeq 0$ and $S = S^{\mathrm{T}} \succcurlyeq 0$ so the terms $x^{\mathrm{T}}Qx$ and $x^{\mathrm{T}}(T)Sx(T)$ are non-negative (so cost can't go to negative infinity).
   - There are no conditions on A and B since this is a finite-horizon problem. The observability and controllability of the system does not matter since the cost and the norm of the state are guaranteed to be finite (unless the initial state has infinite norm) with the finite terminal time.

   (b)
   - We need $R = R^{\mathrm{T}} \succ 0$ and $Q = Q^{\mathrm{T}} \succcurlyeq 0$ for the same reasons as in the previous question.
   - We need (A, B) to be completely controllable so that there exists u such that $x \to 0$ as $t \to \infty$ (so the cost is finite). Specifically, completely controllable means that there exists u that takes the system between any two points.
   - We need $(A, \sqrt{Q})$ to be completely observable so that, over time, x affects $x^{\mathrm{T}}Qx$, meaning that $\|x\| \to \infty$ results in infinite cost.

   (c) Assume that a solution to the Riccati equation exists. We will show it has to be symmetric. First, note that S is symmetric so $P(T) = S = S^{\mathrm{T}} = P^{\mathrm{T}}(T)$. Furthermore, by symmetry of Q and R,

$$\begin{aligned}\dot{P}^{\mathrm{T}} &= -[A^{\mathrm{T}}P]^{\mathrm{T}} - [PA]^{\mathrm{T}} - Q^{\mathrm{T}} + [PBR^{-1}B^{\mathrm{T}}P]^{\mathrm{T}} \\ &= -P^{\mathrm{T}}A - A^{\mathrm{T}}P^{\mathrm{T}} - Q + P^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}P^{\mathrm{T}}\end{aligned}$$

   Therefore, for symmetric $P(t)$, $\dot{P}(t)$ is also symmetric. Furthermore, since the boundary value $P(T)$ is symmetric, the solution $P(t)$ will stay in the set of symmetric matrices, i.e. $P(t) = P^{\mathrm{T}}(t)\ \forall t \in [0, T]$.

2. For the claimed solution to be the optimal solution, it has to satisfy the HJB equation

$$-\frac{\partial J^*}{\partial t} = \min_u \left\{ \frac{1}{2}\rho\big[x(t) - \sigma(t)\big]^2 + \frac{1}{2}u^2(t) + \frac{\partial J^*}{\partial x}\big[ax(t) + bu(t)\big] \right\},$$

where $J^*(x, T) = \Psi(x(T)) = 0$. Assume that the form of the optimal cost-to-go is correct. We start by minimizing the RHS with respect to u:

$$\begin{aligned}\frac{\partial\{\cdot\}}{\partial u} &= u + b\frac{\partial J^*}{\partial x} = 0 \\ u &= -b\frac{\partial J^*}{\partial x} = -b\frac{\partial}{\partial x}\left[\frac{1}{2}x^2(t)p(t) + w(t)x(t) + v(t)\right] \\ u &= -b\big[p(t)x(t) + w(t)\big] \qquad (1)\end{aligned}$$

Substituting the optimal u into the HJB equation produces

$$
\begin{aligned}
-\frac{\partial J^*}{\partial t} &= \frac{1}{2}\rho(x-\sigma)^2 + \frac{1}{2}b^2(px+w)^2 + (px+w)[ax - b^2(px+w)] \\
-\frac{\partial}{\partial t}\left[\frac{1}{2}x^2(t)p(t) + w(t)x(t) + v(t)\right] &= \frac{1}{2}\rho(x^2 - 2x\sigma + \sigma^2) + \frac{1}{2}b^2(p^2x^2 + 2pwx + w^2) \\
&\quad + (a-b^2p)px^2 + [-b^2pw + (a-b^2p)w]x - b^2w^2 \\
-\frac{1}{2}\dot{p}x^2 - \dot{w}x - \dot{v} &= \frac{1}{2}\Big[\rho + b^2p^2 + 2(a-b^2p)p\Big]x^2 + \Big[-\rho\sigma + b^2pw - 2b^2pw + aw\Big]x \\
&\quad + \frac{1}{2}\rho\sigma^2 + \frac{1}{2}b^2w^2 - b^2w^2 \\
\frac{1}{2}\dot{p}x^2 + \dot{w}x + \dot{v} &= \frac{1}{2}\Big(-\rho - 2ap + b^2p^2\Big)x^2 + \Big(\rho\sigma - aw + b^2pw\Big)x - \frac{1}{2}\rho\sigma^2 + \frac{1}{2}b^2w^2
\end{aligned}
$$

Equating like powers of x, we find

$$
\begin{aligned}
\dot{p} &= -2ap + p^2b^2 - \rho && (2) \\
\dot{w} &= -(a-b^2p)w + \rho\sigma && (3) \\
\dot{v} &= \frac{1}{2}\big(w^2b^2 - \sigma^2\rho\big) && (4)
\end{aligned}
$$

Furthermore, since $J^*(x,T) = 0\ \forall x$, we need

$$p(T) = w(T) = v(T) = 0. \tag{5}$$

Since equations (1), (2), (3), (4), and (5) match the claimed solution, it is indeed the optimal solution. ■

3. (a) Note that $x \in \mathbb{R}^5$ since A is 5×5. In Matlab, we can see that the controllability matrix has full rank (`rank(ctrb(A,B)) = 5`). This means the system is completely controllable.

   For the uncontrolled system, using Matlab we find the eigenvalues of A are

$$\text{eig}(A) = \begin{bmatrix} -24.1755 \\ -16.4772 \\ -5.4012 \\ 20.0907 \\ 15.3074 \end{bmatrix}$$

   Since there are positive eigenvalues, the uncontrolled system $\dot{x} = Ax$ is unstable.

   (b) I set $Q = qI$ and $R = I$, where $q > 0$, and varied q. I tried q in the form of 10^k, $k \in \mathbb{Z}$. I found that $k \le 4$ produced similar shapes for x and u; for increasing k, x converges slightly faster to 0 and initial u is slightly larger, see Figs. 1 and 2. For $k \ge 5$, x converges at about the same rate but the initial u is significantly increased from $k < 5$, see Figs. 3 and 4.

   Based on these findings, I choose

$$Q = 10^4 I, \quad R = I,$$

   because this has the best trade-off between fast convergence of x without expending too much control energy $\|u\|^2$.

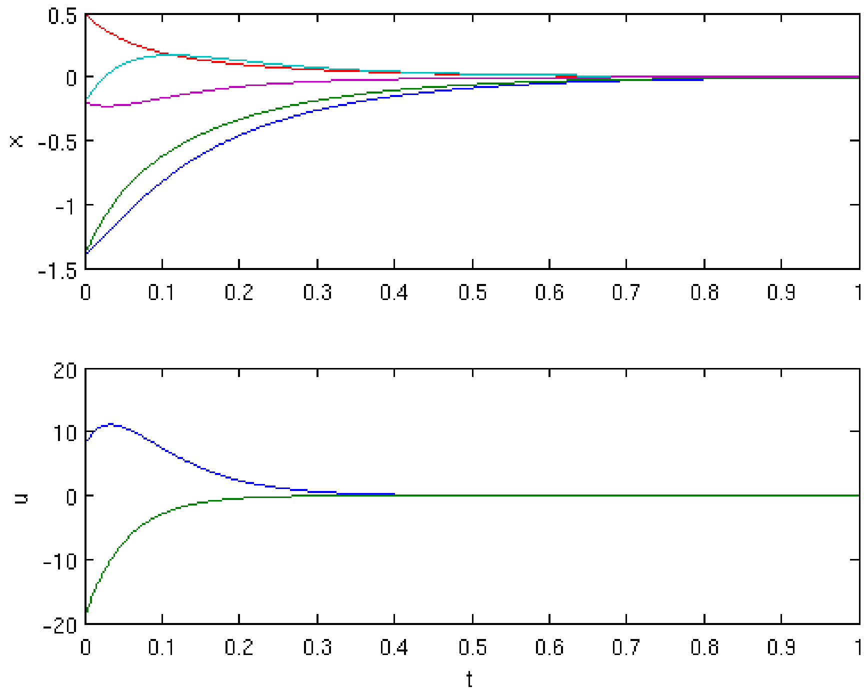

Figure 1: $Q = 10^{-4}I$, $R = I$

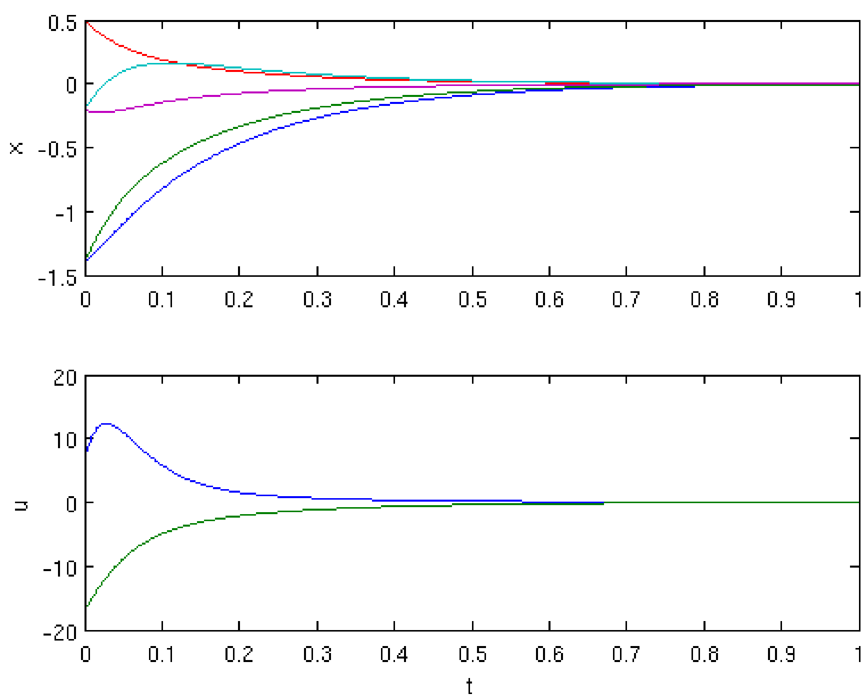

Figure 2: $Q = 10^{4}I$, $R = I$

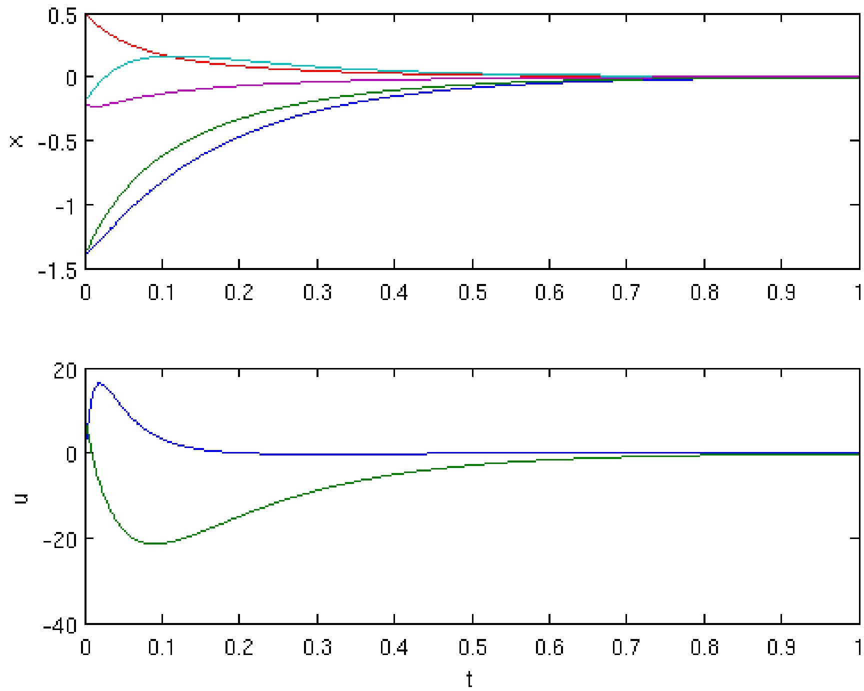

Figure 3: $Q = 10^5 I,\ R = I$

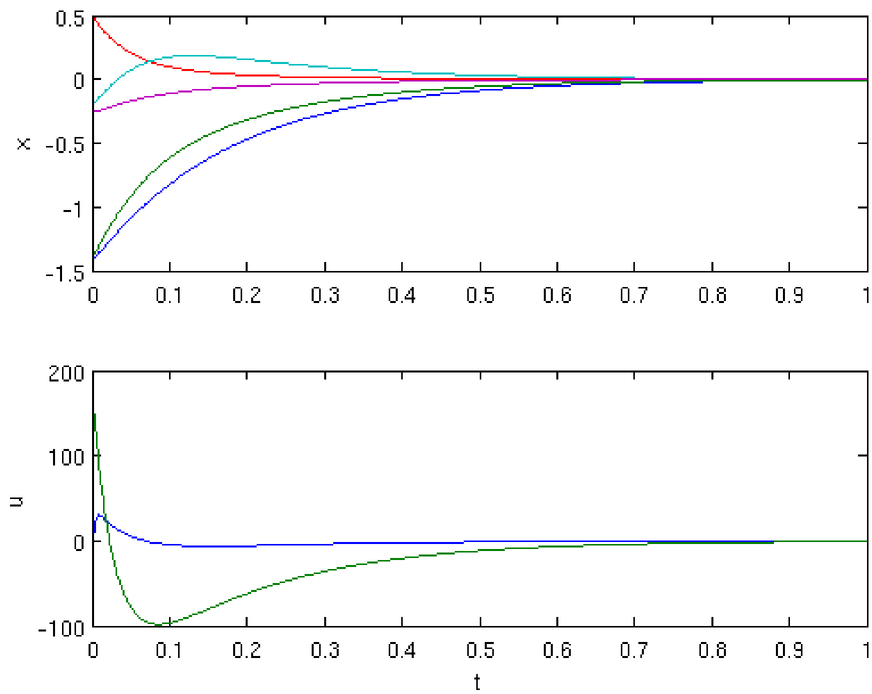

Figure 4: $Q = 10^6 I,\ R = I$

4. (a) For this problem, we have

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ R = 1, \ A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We have $\operatorname{eig}(Q) = 0, 2$ and $\operatorname{eig}(R) = 1$, so $Q \succcurlyeq 0$ and $R \succ 0$. Additionally,

$$\sqrt{Q} = \frac{1}{\sqrt{2}} Q.$$

The controllability matrix is

$$\Gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with rank 2, so the system is completely controllable. Furthermore, the observability matrix is

$$\Omega = \begin{bmatrix} \sqrt{Q} \\ \sqrt{Q}A \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}$$

with rank 2, so the system is completely observable. Since all the conditions for infinite-horizon LQ are met, the problem is well-defined.

(b) In Matlab, `care(A,B,Q,R)` produces

$$P = \begin{bmatrix} 4.8284 & 2.4142 \\ 2.4142 & 2.4142 \end{bmatrix}.$$

The optimal controller is given by

$$u = -R^{-1}B^{\mathrm{T}}Px = -1^{-1} \times \begin{bmatrix} 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4.8284 & 2.4142 \\ 2.4142 & 2.4142 \end{bmatrix} x = \boxed{-\begin{bmatrix} 2.4142 & 2.4142 \end{bmatrix} x}$$

5. Note that by the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{A^{\mathrm{T}}t}Qe^{At}\right] = \frac{\mathrm{d}e^{A^{\mathrm{T}}t}}{\mathrm{d}t}Qe^{At} + e^{A^{\mathrm{T}}t}Q\frac{\mathrm{d}e^{At}}{\mathrm{d}t} = A^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At} + e^{A^{\mathrm{T}}t}Qe^{At}A.$$

Additionally, for asymptotically stable $\dot{x} = Ax$

$$\lim_{t\to\infty} e^{At} = \lim_{t\to\infty} e^{A^{\mathrm{T}}t} = 0.$$

Substituting the proposed solution into the ARE produces

$$\begin{aligned}
0 &= A^{\mathrm{T}}P + PA + Q \\
&= A^{\mathrm{T}}\int_0^\infty e^{A^{\mathrm{T}}t}Qe^{At}\,\mathrm{d}t + \int_0^\infty e^{A^{\mathrm{T}}t}Qe^{At}\,\mathrm{d}t \times A + Q \\
&= \int_0^\infty \left[A^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At} + e^{A^{\mathrm{T}}t}Qe^{At}A\right]\mathrm{d}t + Q \\
&= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{A^{\mathrm{T}}t}Qe^{At}\right]\mathrm{d}t + Q \\
&= \left[e^{A^{\mathrm{T}}t}Qe^{At}\right]_0^\infty + Q \\
&= 0 \times Q \times 0 - I \times Q \times I + Q = -Q + Q = 0 \quad \blacksquare
\end{aligned}$$

6. There were 24 questions.