ECE 6553: Optimal Control Notes

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Spring 2017

Chapter 1

Parameter Optimization

1.1 What is optimal control?

Optimal Maximize/minimize cost (subject to constraints): $\min_u g(u)$
With constraints,

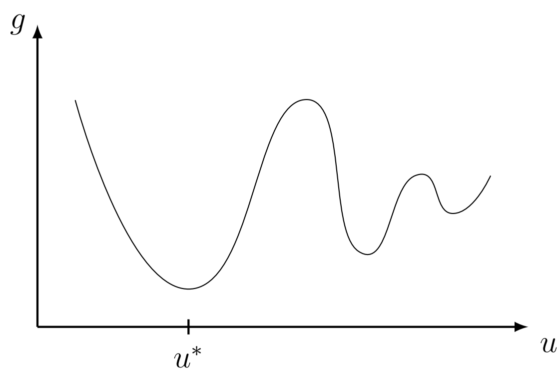
$$\min_u \; g(u)$$
$$\text{s.t.} \; \begin{cases} h_1(u) = 0 \\ h_2(u) \le 0 \end{cases}$$

First-order necessary condition (FONC):

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Optimality can be

- local vs global
- max vs min



Control control design: pick u such that specifications are satisfied:

$$\dot{x} = f(x, u), \qquad \dot{x} = Ax + Bu,$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, and $f(\cdot)$ is the dynamics.

Actually, x and u are signals:

$$x : [0, T] \to \mathbb{R}^n, \qquad u : [0, T] \to \mathbb{R}^m$$

Optimal control find the "best" u!

For "best" to mean anything, we need a cost. The big/deep question is

$$\frac{\partial \text{"cost"}}{\partial u} = 0$$

Example

Suppose we have a car with position p. Its acceleration $\ddot{p}$ is controlled by the gas/brake input u ($\ddot{p} = u$). In order to express the dynamics of the system in the form $\dot{x} = f(x, u)$, we introduce state variables:

$$\begin{aligned} x_1 &= p \\ x_2 &= \dot{p} \end{aligned} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

The task is to move the car from its initial position to a stop at a distance c away.

Minimum energy problem

$$\begin{aligned} \min_u \quad & \int_0^T u^2(t)\, \mathrm{d}t \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0,\ x_2(0) = 0 \\ & x_1(T) = c,\ x_2(T) = 0 \end{aligned}$$

Minimum time problem

$$\begin{aligned} \min_{u,T} \quad & T = \int_0^T \mathrm{d}t \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0,\ x_2(0) = 0 \\ & x_1(T) = c,\ x_2(T) = 0 \\ & u(t) \in [u_{\min}, u_{\max}] \end{aligned}$$

The general optimal control problem we will solve will look like

$$\begin{aligned}
\min_{u,T} \quad & \int_0^T L(x(t), u(t), t)\, \mathrm{d}t + \Psi(x(T)) \\
\text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t), t), \;\; t \in [0, T] \\
& x(0) = x_0 \\
& x(T) \in S \\
& u(t) \in \Omega, \;\; t \in [0, T]
\end{aligned}$$

where $\Psi(\cdot)$ is the terminal cost and S is the terminal manifold. This is a so-called **Bolza Problem**.

What tools do we need to solve this?

1. optimality conditions $\partial \text{cost}/\partial u = 0$
2. some way of representing the optimal signal $u^*(x, t)$
3. some way of actually finding/computing the optimal controllers

1.2 Unconstrained Optimization

Let the decision variable be $u = [u_1, \ldots, u_m]^{\mathrm{T}} \in \mathbb{R}^m$. The cost is $g(u) \in C^1$ (C^k means k times continuously differentiable). The problem is

$$\min_u g(u), \quad g : \mathbb{R}^m \to \mathbb{R}$$

For u^* to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Definition. u^* is a (local) minimizer to g if $\exists \delta > 0$ s.t.

$$\begin{gathered}
g(u^*) \leq g(u) \quad \forall u \in B_\delta(u^*) \\
B_\delta(u^*) = \{u \mid \|u - u^*\| \leq \delta\}
\end{gathered}$$

Note:

- $\dfrac{\partial g}{\partial u}(u^*)\delta u \in \mathbb{R}$ and δu is $m \times 1$, so $\dfrac{\partial g}{\partial u}$ is a $1 \times m$ row vector. For the column vector,

$$\nabla g = \frac{\partial g}{\partial u}^{\mathrm{T}} \in \mathbb{R}^m$$

- $\dfrac{\partial g}{\partial u}\delta u$ is an inner product

$$\langle \nabla g, \delta u \rangle = \left\langle \frac{\partial g}{\partial u}^{\mathrm{T}}, \delta u \right\rangle$$

- $o(\epsilon)$ encodes higher-order terms

$$\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0 \qquad \text{“faster than linear”}$$

  This is opposed to big-O notation:

$$\lim_{\epsilon \to 0} \frac{\mathcal{O}(\epsilon)}{\epsilon} = c$$

- δu has direction and scale so we could write it as

$$\delta u = \epsilon v, \quad \epsilon \in \mathbb{R},\ v \in \mathbb{R}^m$$

Theorem. *For u^* to be a minimizer, we need*

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or, equivalently,

$$\frac{\partial g}{\partial u}(u^*)v = 0 \quad \forall v \in \mathbb{R}^m$$

Proof. Let u^* be a minimizer. Evaluating the cost $g(u)$ in the ball and using Taylor's expansion,

$$g(u^* + \delta u) = g(u^*) + \frac{\partial g}{\partial u}(u^*)\delta u + o(\|\delta u\|) = g(u^*) + \epsilon \frac{\partial g}{\partial u}(u^*)v + o(\epsilon)$$

Assume that $\frac{\partial g}{\partial u} \neq 0$. Then we could pick $v = -\frac{\partial g^{\mathrm{T}}}{\partial u}(u^*)$, i.e.

$$g(u^* + \epsilon v) = g(u^*) - \epsilon \left\| \frac{\partial g^{\mathrm{T}}}{\partial u}(u^*) \right\|^2 + o(\epsilon)$$

Note that the second term is negative per our assumptions. So, for ϵ sufficiently small, we have

$$g\Big(u^* - \epsilon \frac{\partial g^{\mathrm{T}}}{\partial u}(u^*)\Big) < g(u^*)$$

This contradicts u^* being a minimizer. ⚔ (crossed swords) □

Definition (Positive definite)**.** $M = M^{\mathrm{T}} \succ 0$ if

$$z^{\mathrm{T}} M z > 0 \quad \forall z \neq 0,\ z \in \mathbb{R}^m$$
$$\iff M \text{ has real and positive eigenvalues}$$

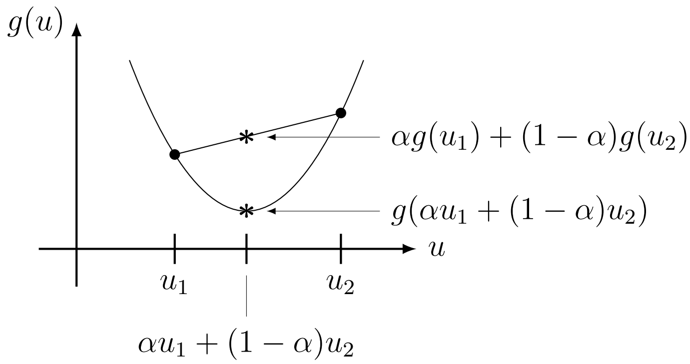
Theorem. *If $g \in C^2$, then a* ***sufficient*** *condition for u^* to be a (local) minimizer is*

1. $\dfrac{\partial g}{\partial u}(u^*) = 0$

2. $\dfrac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ *(the Hessian is positive definite)*

Definition. $g : \mathbb{R}^m \to \mathbb{R}$ is convex if

$$g(\alpha u_1 + (1-\alpha)u_2) \leq \alpha g(u_1) + (1-\alpha)g(u_2) \quad \forall \alpha \in [0,1],\ u_1,u_2 \in \mathbb{R}^m$$



Theorem. *If $\frac{\partial^2 g}{\partial u^2}(u) \succeq 0\ \forall u \in \mathbb{R}^m$, then g is convex. ($\Longleftrightarrow$ for $g \in C^2$)*

Example $\min\limits_u u^{\mathrm{T}}Qu - b^{\mathrm{T}}u$ where $Q = Q^{\mathrm{T}} \succ 0$ (positive definite matrix)

$$\begin{aligned}\frac{\partial g}{\partial u} &= \frac{\partial}{\partial u}(u^{\mathrm{T}}Qu - b^{\mathrm{T}}u)\\ &= u^{\mathrm{T}}Q^{\mathrm{T}} + u^{\mathrm{T}}Q - b^{\mathrm{T}}\\ &= 2u^{\mathrm{T}}Q - b^{\mathrm{T}}\\ \frac{\partial^2 g}{\partial u^2} &= 2Q\end{aligned} \qquad \frac{\partial^2 g}{\partial u^2} = \begin{bmatrix} \frac{\partial^2 g}{\partial u_1^2} & \cdots & \frac{\partial^2 g}{\partial u_1 \partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 g}{\partial u_m^2}\end{bmatrix}$$

From $\frac{\partial g}{\partial u} = 2u^{\mathrm{T}}Q - b^{\mathrm{T}} = 0$,

$$u = \frac{1}{2}Q^{-1}b$$

To see whether this is a minimizer, consider the Hessian. Since $Q \succ 0$, it follows that $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ and $u^* = \frac{1}{2}Q^{-1}b$ is a (local) minimizer. Additionally, since $\frac{\partial^2 g}{\partial u^2} \succ 0$, g is convex and u^* is a global minimizer. In fact, since we have strict convexity ($\succ 0$ rather than $\succeq 0$), it is the unique global minimizer.

In optimal control, *local* is typically all we can ask for. In optimization, we can do better!

But wait, just because we know $\frac{\partial g}{\partial u} = 0$, it doesn't follow that we can actually find u^*...

1.3 Numerical Methods

Idea: $u_{k+1} = u_k + \text{step}_k$. What should step_k be? For small $\text{step}_k = \gamma_k v_k$,

$$g(u_k \cdot \text{step}_k) = g(u_k) + \frac{\partial g}{\partial u}(u_k) \cdot \text{step}_k + o(\|\text{step}_k\|) = g(u_k) + \gamma_k \frac{\partial g}{\partial u}(u_k) v_k + o(\gamma_k)$$

A perfectly reasonable choice of step direction is

$$v_k = -\frac{\partial g^{\mathrm{T}}}{\partial u}(u_k),$$

known as the *steepest descend* direction. This produces

$$g\Big(u_k - \gamma_k \frac{\partial g}{\partial u}(u_k)\Big) = g(u_k) - \gamma_k \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 + o(\gamma_k)$$

Steepest descent

$$u_{k+1} = u_k - \gamma_k \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)$$

Note:

- What should γ_k be?
- This method "pretends" that $g(u)$ is linear. If we pretend $g(u)$ is quadratic, we get

  $$u_{k+1} = u_k - \left(\frac{\partial^2 g}{\partial u^2}(u_k)\right)^{-1} \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k),$$

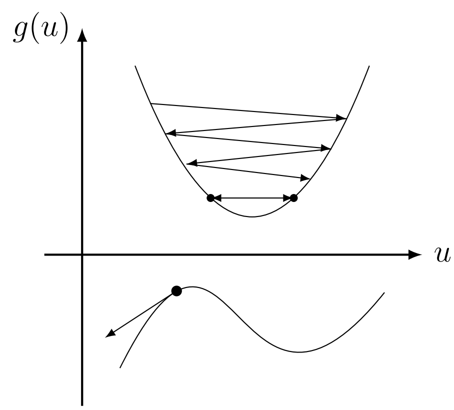
  i.e. Newton's Method

This course: steepest descent

Step-size selection?

- Choice 1: $\gamma_k = \gamma$ "small" $\forall k$; will get close to a minimizer if u_0 is close enough and γ small enough

  Problems:
  - You may not converge! (but you'll get close)
  - You may go off to infinity (diverge)



- Choice 2: Reduce γ_k as a function of k; will get close to a minimizer if u_0 is close enough

  Problem: slow

  Theorem. *If u_0 is close enough to u^* and γ_k satisfies*

  - $\sum_{k=0}^{\infty} \gamma_k = \infty$
  - $\sum_{k=0}^{\infty} \gamma_k^2 < \infty$

  e.g. $\gamma_k = c/k$, then $u_k \to u^*$ as $k \to \infty$.

- Choice 3: **Armijo step-size:** Take as big a step as possible, but no larger

  Pick $\alpha \in (0,1)$, $\beta \in (0,1)$. Let i be the smallest non-negative integer such that

$$g\left(u_k - \beta^i \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)\right) - g(u_k) < -\alpha\beta^i \left\|\frac{\partial g}{\partial u}(u_k)\right\|^2$$

$$u_{k+1} = u_k - \beta^i \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)$$

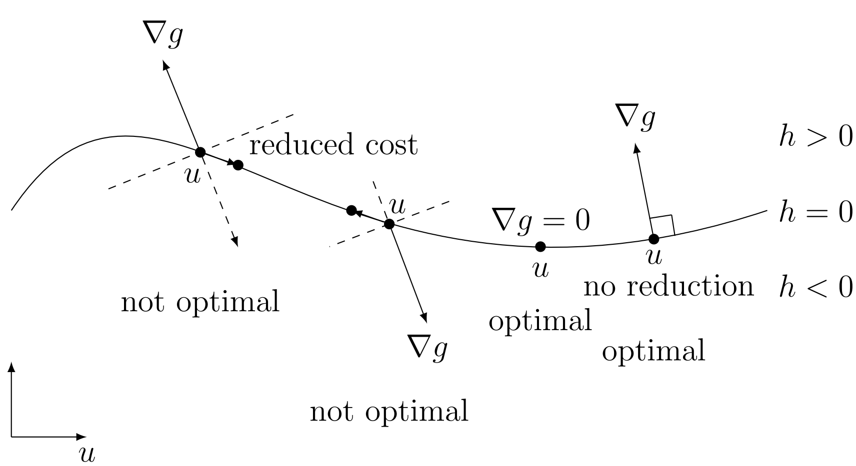
  This will get to a minimizer blazingly fast if u_0 is close enough.

1.4 Constrained Optimization

Equality constraints:

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \ & g(u) \\ \text{s.t. } & h(u) = \mathbf{0} \end{aligned}$$

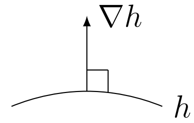
Consider $u \in \mathbb{R}^2$, $h : \mathbb{R}^2 \to \mathbb{R}$



So u is (locally) optimal if $\nabla g \parallel$ (is parallel to) the normal vector to tangent plane to h.

Fact: (HW# 1)

$$\nabla h \perp Th \quad \text{(tangent plane to h)}$$



We need $\nabla g \parallel \nabla h$ at u^* for optimality, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \alpha \frac{\partial h}{\partial u}(u^*), \quad \text{for some } \alpha \in \mathbb{R}$$

or ($\lambda = -\alpha$),

$$\frac{\partial g}{\partial u}(u^*) + \lambda \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}\big(g(u^*) + \lambda h(u^*)\big) = 0, \quad \text{for some } \lambda \in \mathbb{R}$$

More generally,

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \; & g(u) \\ \text{s.t. } & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k \end{aligned}$$

Note that $h(u) = [h_1(u), \ldots, h_k(u)]^{\mathrm{T}}$.

We need $\frac{\partial g}{\partial u}(u^*)$ to be a linear combination of $\frac{\partial h_i}{\partial u}(u^*)$, $i = 1, \ldots, k$, for exactly the same reasons, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \sum_{i=1}^{k} \alpha_i \frac{\partial h_i}{\partial u}(u^*)$$

or ($\lambda = -[\alpha_1, \ldots, \alpha_k]^{\mathrm{T}}$)

$$\frac{\partial g}{\partial u}(u^*) + \lambda^{\mathrm{T}} \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}\big(g(u^*) + \lambda^{\mathrm{T}} h(u^*)\big) = 0, \quad \text{for some } \lambda \in \mathbb{R}^k$$

Theorem. *If u^* is a minimizer to*

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \; & g(u) \\ s.t. \; & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k \end{aligned}$$

then $\exists \lambda \in \mathbb{R}^k$ s.t.

$$\begin{cases} \dfrac{\partial L}{\partial u}(u^*, \lambda) = 0 \\ \dfrac{\partial L}{\partial \lambda}(u^*, \lambda) = 0 \end{cases}$$

where the Lagrangian L is given by

$$L(u, \lambda) = g(u) + \lambda^T h(u)$$

Note:

- λ are the Lagrange multipliers
- $\frac{\partial L}{\partial \lambda} = 0$ is fancy speak for $h(u^*) = 0$

Example

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \ & \frac{1}{2}\|u\|^2 \\ \text{s.t.} \ & Au = b \end{aligned}$$

where A is $k \times m$, $k \leq m$. Assume $(AA^{\mathrm{T}})^{-1}$ exists (constraints are linearly independent, none of the constraints are "duplicates", all the constraints are essential).

$$\begin{aligned} L &= \frac{1}{2}u^{\mathrm{T}}u + \lambda^{\mathrm{T}}(Au - b) \\ \frac{\partial L}{\partial u} &= u^{\mathrm{T}} + \lambda^{\mathrm{T}}A = 0 \\ u^* &= -A^{\mathrm{T}}\lambda \end{aligned}$$

Using the equality constraint,

$$\begin{aligned} Au^* &= b \\ -AA^{\mathrm{T}}\lambda &= b \\ \lambda &= -(AA^{\mathrm{T}})^{-1}b \\ u^* &= A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b \end{aligned}$$

Example

$$\begin{aligned} \min \ & u_1u_2 + u_2u_3 + u_1u_3 \\ \text{s.t.} \ & u_1 + u_2 + u_3 = 3 \end{aligned}$$

$$L = u_1u_2 + u_2u_3 + u_1u_3 + \lambda(u_1 + u_2 + u_3 - 3)$$

$$\begin{cases} \frac{\partial L}{\partial u_1} = u_2 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_2} = u_1 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_3} = u_2 + u_1 + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = u_1 + u_2 + u_3 = 3 \end{cases} \implies \begin{cases} u_1^* = 1 \\ u_2^* = 1 \\ u_3^* = 1 \\ \lambda = -2 \end{cases} \quad \text{optimal solution}$$

Note: This was actually the worst we can do—maximize! Even weirder: no local minimizer!

1.4.1 Equality Constraints

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k \end{aligned}$$

Theorem. *If u^* is a minimizer/maximizer then $\exists \lambda \in \mathbb{R}^k$ s.t.*

$$\begin{aligned} &\frac{\partial L}{\partial u}(u^*, \lambda) = 0 \\ &\frac{\partial L}{\partial \lambda}(u^*, \lambda) = 0 \qquad (\Longleftrightarrow h(u^*) = 0) \end{aligned}$$

where $L(u, \lambda) = g(u) + \lambda^T h(u)$.

Example [Entropy Maximization]

Given $S = \{x_1, \ldots, x_n\}$ and a distribution over S such that it takes the value x_j with probability p_j. The entropy is

$$E(p) = \sum_{j=1}^{n} (-p_j \ln p_j).$$

The mean is

$$m = \sum_{j=1}^{n} p_j x_j.$$

Problem: Given m, find p such that E is maximized.

$$\begin{aligned} \min_{p} \quad & -\sum_{j=1}^{n} p_j \ln p_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} p_j x_j = m \\ & \sum_{j=1}^{n} p_j = 1 \\ & p_j \geq 0, \; j = 1, \ldots, n \quad \text{(ignore this...)} \end{aligned}$$

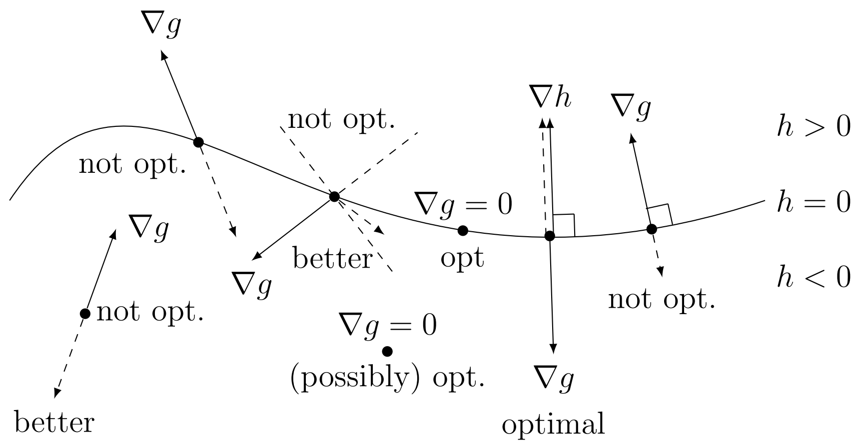
$$\begin{aligned} L &= -\sum p_j \ln p_j + \lambda_1 \left[\sum p_j x_j - m\right] + \lambda_2 \left[\sum p_j - 1\right] \\ \frac{\partial L}{\partial p_j} &= -\ln p_j - 1 + \lambda_1 x_j + \lambda_2 = 0 \\ p_j &= e^{\lambda_2 - 1 + \lambda_1 x_j}, \quad j = 1, \ldots, n \qquad (p_j \geq 0 \text{ so we're ok with ignoring that}) \end{aligned}$$

$$\begin{aligned} \sum e^{\lambda_2 - 1 + \lambda_1 x_j} x_j &= m & \qquad & n + 2 \text{ equations and} \\ \sum e^{\lambda_2 - 1 + \lambda_1 x_j} &= 1 & & n + 2 \text{ unknowns...} \end{aligned}$$

No analytical solution, but numerically "solvable"

1.4.2 Inequality Constraints

$$\min_{u \in \mathcal{R}^m} \; g(u)$$
$$\text{s.t. } h(u) \leq \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k$$



We need:

- if $h(u^*) < 0$ then $\frac{\partial g}{\partial u}(u^*) = 0$
- if $h(u^*) = 0$ then we need either

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or

$$\frac{\partial g}{\partial u}(u^*) = -\lambda \frac{\partial h}{\partial u}(u^*) \quad \text{for } \lambda > 0$$

Or, even better,

$$\frac{\partial}{\partial u}\left(g(u^*) + \lambda h(u^*)\right) = 0 \quad \text{for } \lambda \geq 0,$$

where $\lambda h(u^*) = 0$. ($h < 0 \to \lambda = 0$, $h = 0 \to \lambda \geq 0$)

In general, if $u \in \mathbb{R}^m$ and $h : \mathbb{R}^m \to \mathbb{R}^k$, we have that u^*, if optimal, has to satisfy

$$\frac{\partial}{\partial u} L(u^*, \lambda) = 0$$
$$h(u^*) \leq \mathbf{0}$$
$$\lambda^{\mathrm{T}} h(u^*) = 0$$
$$\lambda \geq \mathbf{0}$$

where the Lagrangian is $L(u, \lambda) = g(u) + \lambda^{\mathrm{T}} h(u)$. Note that if we're maximizing, the same holds except we need $\lambda \leq 0$.

Example

$$\min\ 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2$$
$$\text{s.t.}\ \begin{cases} u_1^2 + u_2^2 \le 5 \\ 3u_1 + u_2 \le 6 \end{cases}$$

$$L = 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 + \lambda_1(u_1^2 + u_2^2 - 5) + \lambda_2(3u_1 + u_2 - 6)$$

FONC:

i) $\partial L/\partial u_1 = 4u_1 + 2u_2 - 10 + 2\lambda_1 u_1 + 3\lambda_2$

ii) $\partial L/\partial u_2 = 2u_1 + 2u_2 - 10 + 2\lambda_1 u_2 + \lambda_2$

iii) $u_1^2 + u_2^2 \le 5$

iv) $3u_1 + u_2 \le 6$

v) $\lambda_1(u_1^2 + u_2^2 - 5) = 0$

vi) $\lambda_2(3u_1 + u_2 - 6) = 0$

vii) $\lambda_1 \ge 0$

viii) $\lambda_2 \ge 0$

To solve, assume different constraints are active/inactive:

1. Both constraints are inactive ($u_1^2 + u_2^2 < 5$, $3u_1 + u_2 < 6$) $\implies \lambda_1 = \lambda_2 = 0$

$$\begin{cases} 4u_1 + 2u_2 - 10 = 0 \\ 2u_1 + 2u_2 - 10 = 0 \end{cases} \implies \begin{cases} u_1 = 0 \\ u_2 = 5 \end{cases}$$

   Note: iii) $0^2 + 5^2 \not\le 5$

   Not feasible

2. Assume constraint 1 is active and constraint 2 is inactive ($u_1^2 + u_2^2 = 5$, $\lambda_2 = 0$)

$$\begin{cases} 4u_1 + 2u_2 - 10 + 2\lambda_1 u_1 = 0 \\ 2u_1 + 2u_2 - 10 + 2\lambda_1 u_2 = 0 \\ u_1^2 + u_2^2 = 5 \end{cases} \implies \begin{cases} u_1 = 1 \\ u_2 = 2 \\ \lambda_1 = 1 \end{cases}$$

   ☑ $\lambda_1 \ge 0$

   ☑ $3 \cdot 1 + 2 \le 6$

   This is a local minimizer

3. Assume constraint 2 is active and constraint 1 is inactive

4. Assume both constraints are active

Kuhn-Tucker Conditions (KKT conditions, Karush-Kuhn-Tucker)

Problem:

$$\begin{aligned} &\min_{u \in \mathbb{R}^m} \; g(u) \\ &\text{s.t.} \; \begin{cases} h_1(u) = 0, & h_1 : \mathbb{R}^m \to \mathbb{R}^p \\ h_2(u) \leq 0, & h_2 : \mathbb{R}^m \to \mathbb{R}^k \end{cases} \end{aligned} \tag{1.1}$$

Theorem. *Let u^* be feasible ($h_1 = 0$, $h_2 \leq 0$). If u^* is a minimizer to (1.1) than there exists vectors $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^k$ with $\mu \geq \mathbf{0}$ such that*

$$\begin{cases} \dfrac{\partial g}{\partial u}(u^*) + \lambda^T \dfrac{\partial h_1}{\partial u}(u^*) + \mu^T \dfrac{\partial h_2}{\partial u}(u^*) = 0 \\ \mu^T h_2(u^*) = 0 \end{cases}$$

Looking ahead: $\min \text{cost}(u(\cdot))$ s.t. $\dot{x} = f(x, u)$ (dynamics), where u is a function. Note the equality constraint.

Question: How do we go from $u \in \mathbb{R}^m$ to $u \in \mathcal{U}$ (function space)?

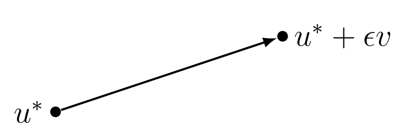
Note: Function space is a set of functions of a given kind from a set X to a set Y

1. linear function
2. square-integrable functions: $L_2[0, T] : \int_0^T \|u(t)\|^2 \, \mathrm{d}t < \infty$
3. $C^\infty(\mathbb{R})$

What would ∂"cost"$/\partial u$ mean?

1.5 Directional Derivatives

Recall: To minimize $g(u)$, let u^* be a candidate minimizer and pitch a perturbation on u^* of ϵv, where ϵ is the scale and v is the direction. Taking Taylor's expansion at the perturbation produces



$$g(u^* + \epsilon v) = g(u^*) + \epsilon \frac{\partial g}{\partial u}(u^*) v + o(\epsilon)$$

$$\text{FONC: } \frac{\partial g}{\partial u}(u^*) = 0$$

Note: $\frac{\partial g}{\partial u}(u^*)v$ tells us how much $g(u)$ increases/decreases in the direction of v.

Definition. The directional (Gateaux) derivative is given by

$$\delta g(u; v) = \lim_{\epsilon \to 0} \frac{g(u + \epsilon v) - g(u)}{\epsilon}$$

Example

$$g(u) = \frac{1}{2}u_1^2 - u_1 + 2u_2, \quad g : \mathbb{R}^2 \to \mathbb{R}$$

Let's consider $e_1 = [1\ 0]^{\mathrm{T}}$, $e_2 = [0\ 1]^{\mathrm{T}}$. What is $\delta g(u; e_i)$, $i = 1, 2$?

$$\begin{aligned}
\delta g(u; v) &= \lim_{\epsilon \to 0} \frac{g(u + \epsilon v) - g(u)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{g(u) + \epsilon \frac{\partial g}{\partial u}(u)v + o(\epsilon) - g(u)}{\epsilon} \\
&= \frac{\partial g}{\partial u}(u)v
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g}{\partial u}(u) &= [u_1 - 1\ 2] \\
\delta g(u; e_1) &= [u_1 - 1\ 2]e_1 = u_1 - 1 \\
\delta g(u; e_2) &= [u_1 - 1\ 2]e_2 = 2
\end{aligned}$$

But the beauty of directional derivatives is that they generalize beyond vectors, $u \in \mathbb{R}^m$, to function spaces ($\mathcal{U}$) or other "objects" like matrices.

Example $M \in \mathbb{R}^{n \times n}$, $F(M) = M^2$

What is $\frac{\partial F}{\partial M}$? (ponder at home...)

We can easily compute $\delta F(M; N)$!

$$\begin{aligned}
F(M + \epsilon N) = (M + \epsilon N)(M + \epsilon N) &= M^2 + \epsilon MN + \epsilon NM + \epsilon^2 N^2 \\
\delta F(M; N) = \lim_{\epsilon \to 0} \frac{F(M + \epsilon N) - F(M)}{\epsilon}& \\
= \lim_{\epsilon \to 0} \frac{\epsilon MN + \epsilon NM + \epsilon^2 N^2}{\epsilon} &= MN + NM
\end{aligned}$$

Infinite Dimensional Optimization Let $u \in \mathcal{U}$ (function space) and let $J(u)$ be the cost:

$$\min_{u \in \mathcal{U}} J(u)$$

Theorem. *If $u^* \in \mathcal{U}$ is a (local) minimizer then*

$$\delta J(u^*; v) = 0, \quad \forall v \in \mathcal{U}$$

Example Find minimizer u^* to

$$J(u) = \int_0^T L(u(t))\,\mathrm{d}t$$

$$\begin{aligned}
J(u+\epsilon v) - J(u) &= \int_0^T L(u(t)+\epsilon v(t))\,\mathrm{d}t - \int_0^T L(u(t))\,\mathrm{d}t, \quad u, v \in \mathcal{U} \\
&= \int_0^T \left[L(u(t)) + \epsilon \frac{\partial L}{\partial u}(u(t))v(t) + o(\epsilon) - L(u(t)) \right] \mathrm{d}t \\
\delta J(u^*; v) &= \lim_{\epsilon \to 0} \frac{J(u+\epsilon v) - J(u)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\int_0^T \epsilon \frac{\partial L}{\partial u}(u(t))v(t)\,\mathrm{d}t + o(\epsilon)}{\epsilon} \\
&= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t)\,\mathrm{d}t
\end{aligned}$$

u^* optimizer:

$$\delta J(u^*; v) = \int_0^T \frac{\partial L}{\partial u}(u(t))v(t)\,\mathrm{d}t = 0 \quad \forall v \in \mathcal{U}$$

$$\Updownarrow$$

$$\frac{\partial L}{\partial u}(u(t)) = 0 \quad \forall t \in [0, T]$$

But, we want *optimal control*! We want our cost to look like

$$\int_0^T L(x(t), u(t))\,\mathrm{d}t$$
$$\dot{x} = f(x, u)$$

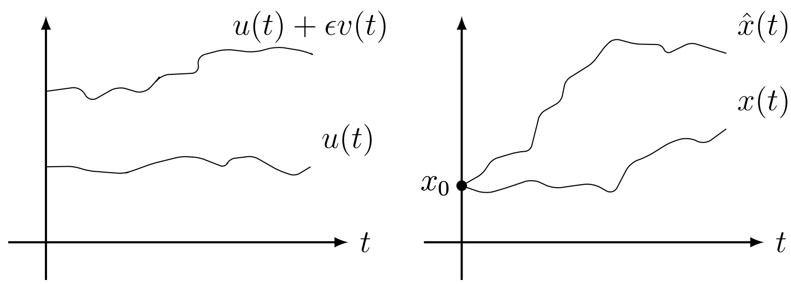
1.6 Calculus of Variations

What happens to $x(t)$ when $u(t)$ changes to $u(t) + \epsilon v(t)$? Let the system be given by

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

After perturbation of u, the new system is

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}, u + \epsilon v) \\ x(0) = x_0 \end{cases}$$



Consider

$$\tilde{x} = x + \epsilon\eta,$$

where

$$\begin{aligned}
\dot{x} &= f(x,u), & x(0) &= x_0 \\
\dot{\eta} &= \frac{\partial f}{\partial x}(x,u)\eta + \frac{\partial f}{\partial u}(x,u)v, & \eta(0) &= 0
\end{aligned}$$

Theorem. *If f is continuously differentiable in x and u then*

$$\hat{x}(t) = \tilde{x}(t) + o(\epsilon)$$

Proof.

i) Initial conditions:

$$\begin{aligned}
\hat{x}(0) &= x_0 \\
\tilde{x}(0) &= x(0) + \epsilon\eta(0) = x_0
\end{aligned}$$

ii) Dynamics:

$$\begin{aligned}
\dot{\hat{x}} &= f(\hat{x}, u + \epsilon v) \\
\dot{\tilde{x}} &= \dot{x} + \epsilon\dot{\eta} = f(x,u) + \epsilon\frac{\partial f}{\partial x}(x,u)\eta + \epsilon\frac{\partial f}{\partial u}(x,u)v \\
&= f(x + \epsilon\eta, u + \epsilon v) + o(\epsilon) \\
&= f(\tilde{x}, u + \epsilon v) + o(\epsilon)
\end{aligned}$$

We can see that the dynamics of $\hat{x}(t)$ are equal to those of $\tilde{x}(t)$ plus higher order terms:

$$\begin{aligned}
\dot{\tilde{x}} &= f(\tilde{x}, u + \epsilon v) + o(\epsilon) \\
\dot{\hat{x}} &= f(\hat{x}, u + \epsilon v)
\end{aligned}$$

Therefore, if our perturbation is small enough, we can model $\hat{x}(t)$ as $\tilde{x}(t)$.

$\square$

Note: Taylor expansion with two elements is

$$\begin{aligned}
g(z_1 + \epsilon w_1, z_2 + \epsilon w_2) &= g(z_1, z_2 + \epsilon w_2) + \epsilon \frac{\partial g}{\partial z_1}(z_1, z_2 + \epsilon w_2) w_1 + o(\epsilon) \\
&= g(z_1, z_2) + \epsilon \frac{\partial g}{\partial z_2}(z_1, z_2) w_2 + \epsilon \frac{\partial g}{\partial z_1}(z_1, z_2) w_1 \\
&\quad + \epsilon^2 \frac{\partial^2 g}{\partial z_2 \partial z_1}(z_1, z_2) w_1 w_2 + o(\epsilon) \\
&= g(z_1, z_2) + \epsilon \frac{\partial g}{\partial z_1}(z_1, z_2) w_1 + \epsilon \frac{\partial g}{\partial z_2}(z_1, z_2) w_2 + o(\epsilon)
\end{aligned}$$