

# ECE 6553: Optimal Control Notes

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# Chapter 1

## Parameter Optimization

### 1.1 What is optimal control?

**Optimal** Maximize/minimize cost (subject to constraints):  $\min_u g(u)$

With constraints,

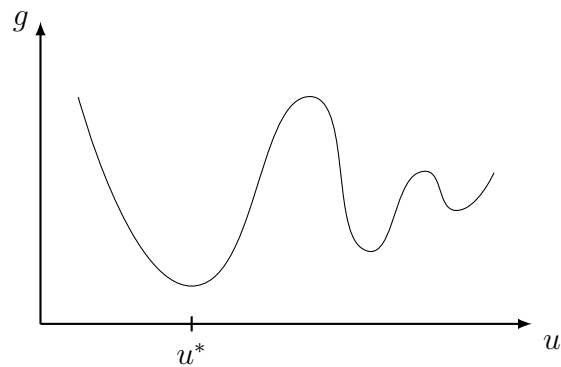
$$\begin{aligned} \min_u \quad & g(u) \\ \text{s.t.} \quad & \begin{cases} h_1(u) = 0 \\ h_2(u) \leq 0 \end{cases} \end{aligned}$$

First-order necessary condition (FONC):

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Optimality can be

- local vs global
- max vs min



**Control** control design: pick  $u$  such that specifications are satisfied:

$$\dot{x} = f(x, u), \quad \dot{x} = Ax + Bu,$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}^m$  is the control, and  $f(\cdot)$  is the dynamics.

Actually,  $x$  and  $u$  are signals:

$$x : [0, T] \rightarrow \mathbb{R}^n, \quad u : [0, T] \rightarrow \mathbb{R}^m$$

**Optimal control** find the “best”  $u$ !

For “best” to mean anything, we need a cost. The big/deep question is

$$\frac{\partial \text{“cost”}}{\partial u} = 0$$

**Example**

Suppose we have a car with position  $p$ . Its acceleration  $\ddot{p}$  is controlled by the gas/brake input  $u$  ( $\ddot{p} = u$ ). In order to express the dynamics of the system in the form  $\dot{x} = f(x, u)$ , we introduce state variables:

$$\begin{aligned} x_1 = p \\ x_2 = \dot{p} \end{aligned} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

The task is to move the car from its initial position to a stop at a distance  $c$  away.

**Minimum energy problem**

$$\begin{aligned} \min_u \quad & \int_0^T u^2(t) dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0, x_2(0) = 0 \\ & x_1(T) = c, x_2(T) = 0 \end{aligned}$$

**Minimum time problem**

$$\begin{aligned} \min_{u, T} \quad & T = \int_0^T dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0, x_2(0) = 0 \\ & x_1(T) = c, x_2(T) = 0 \\ & u(t) \in [u_{\min}, u_{\max}] \end{aligned}$$

The general optimal control problem we will solve will look like

$$\begin{aligned} \min_{u, T} \quad & \int_0^T L(x(t), u(t), t) dt + \Psi(x(T)) \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T] \\ & x(0) = x_0 \\ & x(T) \in S \\ & u(t) \in \Omega, \quad t \in [0, T] \end{aligned}$$

where  $\Psi(\cdot)$  is the terminal cost and  $S$  is the terminal manifold. This is a so-called **Bolza Problem**.

**What tools do we need to solve this?**

1. optimality conditions  $\partial \text{cost} / \partial u = 0$
2. some way of representing the optimal signal  $u^*(x, t)$
3. some way of actually finding/computing the optimal controllers

## 1.2 Unconstrained Optimization

Let the decision variable be  $u = [u_1, \dots, u_m]^T \in \mathbb{R}^m$ . The cost is  $g(u) \in C^1$  ( $C^k$  means  $k$  times continuously differentiable). The problem is

$$\min_u g(u), \quad g : \mathbb{R}^m \rightarrow \mathbb{R}$$

For  $u^*$  to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

**Definition.**  $u^*$  is a (local) minimizer to  $g$  if  $\exists \delta > 0$  s.t.

$$\begin{aligned} g(u^*) &\leq g(u) \quad \forall u \in B_\delta(u^*) \\ B_\delta(u^*) &= \{u \mid \|u - u^*\| \leq \delta\} \end{aligned}$$

**Note:**

- $\frac{\partial g}{\partial u}(u^*)\delta u \in \mathbb{R}$  and  $\delta u$  is  $m \times 1$ , so  $\frac{\partial g}{\partial u}$  is a  $1 \times m$  row vector. For the column vector,

$$\nabla g = \frac{\partial g^T}{\partial u} \in \mathbb{R}^m$$

- $\frac{\partial g}{\partial u} \delta u$  is an inner product

$$\langle \nabla g, \delta u \rangle = \left\langle \frac{\partial g^T}{\partial u}, \delta u \right\rangle$$

- $o(\varepsilon)$  encodes higher-order terms

$$\lim_{\varepsilon \rightarrow 0} \frac{o(\varepsilon)}{\varepsilon} = 0 \quad \text{“faster than linear”}$$

This is opposed to big-O notation:

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{O}(\varepsilon)}{\varepsilon} = c$$

- $\delta u$  has direction and scale so we could write it as

$$\delta u = \varepsilon v, \quad \varepsilon \in \mathbb{R}, \quad v \in \mathbb{R}^m$$

**Theorem.** For  $u^*$  to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or, equivalently,

$$\frac{\partial g}{\partial u}(u^*)v = 0 \quad \forall v \in \mathbb{R}^m$$

*Proof.* Let  $u^*$  be a minimizer. Evaluating the cost  $g(u)$  in the ball and using Taylor’s expansion,

$$g(u^* + \delta u) = g(u^*) + \frac{\partial g}{\partial u}(u^*)\delta u + o(\|\delta u\|) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$

Assume that  $\frac{\partial g}{\partial u} \neq 0$ . Then we could pick  $v = -\frac{\partial g^T}{\partial u}(u^*)$ , i.e.

$$g(u^* + \varepsilon v) = g(u^*) - \varepsilon \left\| \frac{\partial g^T}{\partial u}(u^*) \right\|^2 + o(\varepsilon)$$

Note that the second term is negative per our assumptions. So, for  $\varepsilon$  sufficiently small, we have

$$g\left(u^* - \varepsilon \frac{\partial g^T}{\partial u}(u^*)\right) < g(u^*)$$

This contradicts  $u^*$  being a minimizer.  $\times$  (crossed swords) □

**Definition** (Positive definite).  $M = M^T \succ 0$  if

$$\begin{aligned} z^T M z &> 0 \quad \forall z \neq 0, \quad z \in \mathbb{R}^m \\ \iff M &\text{ has real and positive eigenvalues} \end{aligned}$$

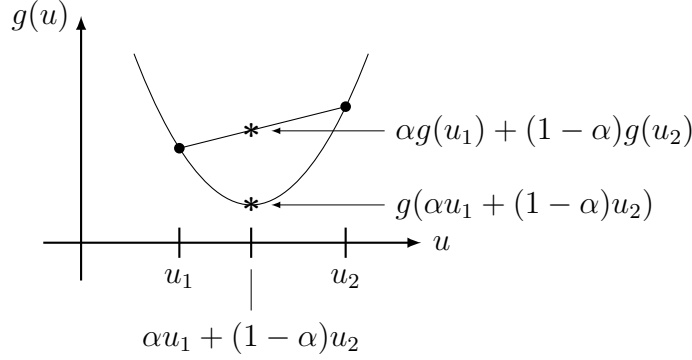
**Theorem.** If  $g \in C^2$ , then a **sufficient** condition for  $u^*$  to be a (local) minimizer is

$$1. \quad \frac{\partial g}{\partial u}(u^*) = 0$$

2.  $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$  (the Hessian is positive definite)

**Definition.**  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  is convex if

$$g(\alpha u_1 + (1 - \alpha)u_2) \leq \alpha g(u_1) + (1 - \alpha)g(u_2) \quad \forall \alpha \in [0, 1], u_1, u_2 \in \mathbb{R}^m$$



**Theorem.** If  $\frac{\partial^2 g}{\partial u^2}(u) \succeq 0 \quad \forall u \in \mathbb{R}^m$ , then  $g$  is convex. ( $\Longleftrightarrow$  for  $g \in C^2$ )

**Example**  $\min_u u^T Q u - b^T u$  where  $Q = Q^T \succ 0$  (positive definite matrix)

$$\begin{aligned} \frac{\partial g}{\partial u} &= \frac{\partial}{\partial u} (u^T Q u - b^T u) \\ &= u^T Q^T + u^T Q - b^T \\ &= 2u^T Q - b^T \end{aligned}$$

$$\frac{\partial^2 g}{\partial u^2} = 2Q$$

$$\frac{\partial^2 g}{\partial u^2} = \begin{bmatrix} \frac{\partial^2 g}{\partial u_1^2} & \cdots & \frac{\partial^2 g}{\partial u_1 \partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 g}{\partial u_m^2} \end{bmatrix}$$

From  $\frac{\partial g}{\partial u} = 2u^T Q - b^T = 0$ ,

$$u = \frac{1}{2} Q^{-1} b$$

To see whether this is a minimizer, consider the Hessian. Since  $Q \succ 0$ , it follows that  $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$  and  $u^* = \frac{1}{2} Q^{-1} b$  is a (local) minimizer. Additionally, since  $\frac{\partial^2 g}{\partial u^2} \succ 0$ ,  $g$  is convex and  $u^*$  is a global minimizer. In fact, since we have strict convexity ( $\succ 0$  rather than  $\succeq 0$ ), it is the unique global minimizer.

In optimal control, *local* is typically all we can ask for. In optimization, we can do better!

But wait, just because we know  $\frac{\partial g}{\partial u} = 0$ , it doesn't follow that we can actually find  $u^* \dots$

## 1.3 Numerical Methods

Idea:  $u_{k+1} = u_k + \text{step}_k$ . What should  $\text{step}_k$  be? For small  $\text{step}_k = \gamma_k v_k$ ,

$$g(u_k + \text{step}_k) = g(u_k) + \frac{\partial g}{\partial u}(u_k) \cdot \text{step}_k + o(\|\text{step}_k\|) = g(u_k) + \gamma_k \frac{\partial g}{\partial u}(u_k) v_k + o(\gamma_k)$$

A perfectly reasonable choice of step direction is

$$v_k = -\frac{\partial g^T}{\partial u}(u_k),$$

known as the *steepest descend* direction. This produces

$$g\left(u_k - \gamma_k \frac{\partial g}{\partial u}(u_k)\right) = g(u_k) - \gamma_k \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 + o(\gamma_k)$$

### Steepest descent

$$u_{k+1} = u_k - \gamma_k \frac{\partial g^T}{\partial u}(u_k)$$

**Note:**

- What should  $\gamma_k$  be?
- This method “pretends” that  $g(u)$  is linear. If we pretend  $g(u)$  is quadratic, we get

$$u_{k+1} = u_k - \left( \frac{\partial^2 g}{\partial u^2}(u_k) \right)^{-1} \frac{\partial g^T}{\partial u}(u_k),$$

i.e. Newton’s Method

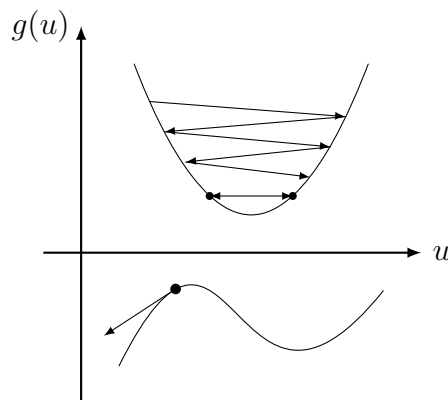
**This course:** steepest descent

### Step-size selection?

- Choice 1:  $\gamma_k = \gamma$  “small”  $\forall k$ ; will get close to a minimizer if  $u_0$  is close enough and  $\gamma$  small enough

Problems:

- You may not converge! (but you’ll get close)
- You may go off to infinity (diverge)



- Choice 2: Reduce  $\gamma_k$  as a function of  $k$ ; will get close to a minimizer if  $u_0$  is close enough

Problem: slow

**Theorem.** If  $u_0$  is close enough to  $u^*$  and  $\gamma_k$  satisfies

$$\begin{aligned} - \sum_{k=0}^{\infty} \gamma_k &= \infty \\ - \sum_{k=0}^{\infty} \gamma_k^2 &< \infty \end{aligned}$$

e.g.  $\gamma_k = c/k$ , then  $u_k \rightarrow u^*$  as  $k \rightarrow \infty$ .

- Choice 3: **Armijo step-size:** Take as big a step as possible, but no larger  
Pick  $\alpha \in (0, 1)$ ,  $\beta \in (0, 1)$ . Let  $i$  be the smallest non-negative integer such that

$$\begin{aligned} g\left(u_k - \beta^i \frac{\partial g^T}{\partial u}(u_k)\right) - g(u_k) &< -\alpha \beta^i \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 \\ u_{k+1} &= u_k - \beta^i \frac{\partial g^T}{\partial u}(u_k) \end{aligned}$$

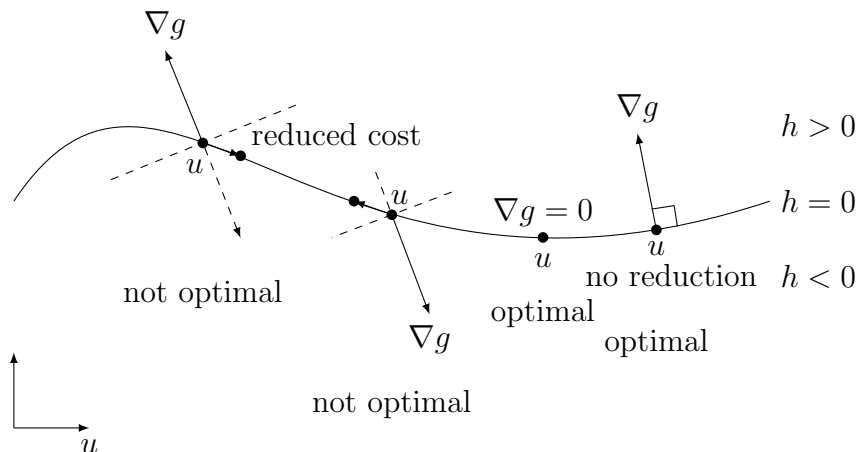
This will get to a minimizer blazingly fast if  $u_0$  is close enough.

## 1.4 Constrained Optimization

Equality constraints:

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = 0 \end{aligned}$$

Consider  $u \in \mathbb{R}^2$ ,  $h: \mathbb{R}^2 \rightarrow \mathbb{R}$

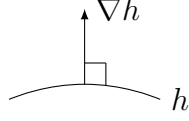




So  $u$  is (locally) optimal if  $\nabla g \parallel$  (is parallel to) the normal vector to tangent plane to  $h$ .

Fact: (HW# 1)

$$\nabla h \perp Th \quad (\text{tangent plane to } h)$$



We need  $\nabla g \parallel \nabla h$  at  $u^*$  for optimality, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \alpha \frac{\partial h}{\partial u}(u^*), \quad \text{for some } \alpha \in \mathbb{R}$$

or ( $\lambda = -\alpha$ ),

$$\frac{\partial g}{\partial u}(u^*) + \lambda \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}(g(u^*) + \lambda h(u^*)) = 0, \quad \text{for some } \lambda \in \mathbb{R}$$

More generally,

$$\begin{aligned} \min_{u \in \mathbb{R}^m} g(u) \\ \text{s.t. } h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

Note that  $h(u) = [h_1(u), \dots, h_k(u)]^T$ .

We need  $\frac{\partial g}{\partial u}(u^*)$  to be a linear combination of  $\frac{\partial h_i}{\partial u}(u^*)$ ,  $i = 1, \dots, k$ , for exactly the same reasons, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \sum_{i=1}^k \alpha_i \frac{\partial h_i}{\partial u}(u^*)$$

or ( $\lambda = -[\alpha_1, \dots, \alpha_k]^T$ )

$$\frac{\partial g}{\partial u}(u^*) + \lambda^T \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}(g(u^*) + \lambda^T h(u^*)) = 0, \quad \text{for some } \lambda \in \mathbb{R}^k$$

**Theorem.** If  $u^*$  is a minimizer to

$$\begin{aligned} \min_{u \in \mathbb{R}^m} g(u) \\ \text{s.t. } h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

then  $\exists \lambda \in \mathbb{R}^k$  s.t.

$$\begin{cases} \frac{\partial L}{\partial u}(u^*, \lambda) = 0 \\ \frac{\partial L}{\partial \lambda}(u^*, \lambda) = 0 \end{cases}$$

where the Lagrangian  $L$  is given by

$$L(u, \lambda) = g(u) + \lambda^T h(u)$$

**Note:**

- $\lambda$  are the Lagrange multipliers
- $\frac{\partial L}{\partial \lambda} = 0$  is fancy speak for  $h(u^*) = 0$

**Example**

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & \frac{1}{2} \|u\|^2 \\ \text{s.t.} \quad & Au = b \end{aligned}$$

where  $A$  is  $k \times m$ ,  $k \leq m$ . Assume  $(AA^T)^{-1}$  exists (constraints are linearly independent, none of the constraints are “duplicates”, all the constraints are essential).

$$\begin{aligned} L &= \frac{1}{2} u^T u + \lambda^T (Au - b) \\ \frac{\partial L}{\partial u} &= u^T + \lambda^T A = 0 \\ u^* &= -A^T \lambda \end{aligned}$$

Using the equality constraint,

$$\begin{aligned} Au^* &= b \\ -AA^T \lambda &= b \\ \lambda &= -(AA^T)^{-1} b \\ u^* &= A^T (AA^T)^{-1} b \end{aligned}$$

**Example**

$$\begin{aligned} \min \quad & u_1 u_2 + u_2 u_3 + u_1 u_3 \\ \text{s.t.} \quad & u_1 + u_2 + u_3 = 3 \end{aligned}$$

$$L = u_1 u_2 + u_2 u_3 + u_1 u_3 + \lambda(u_1 + u_2 + u_3 - 3)$$

$$\begin{cases} \frac{\partial L}{\partial u_1} = u_2 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_2} = u_1 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_3} = u_2 + u_1 + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = u_1 + u_2 + u_3 = 3 \end{cases} \implies \begin{cases} u_1^* = 1 \\ u_2^* = 1 \\ u_3^* = 1 \\ \lambda = -2 \end{cases} \quad \text{optimal solution}$$

Note: This was actually the worst we can do—maximize! Even weirder: no local minimizer!

### 1.4.1 Equality Constraints

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

**Theorem.** If  $u^*$  is a minimizer/maximizer then  $\exists \lambda \in \mathbb{R}^k$  s.t.

$$\begin{aligned} \frac{\partial L}{\partial u}(u^*, \lambda) &= 0 \\ \frac{\partial L}{\partial \lambda}(u^*, \lambda) &= 0 \quad (\iff h(u^*) = 0) \end{aligned}$$

where  $L(u, \lambda) = g(u) + \lambda^T h(u)$ .

**Example** [Entropy Maximization]

Given  $S = \{x_1, \dots, x_n\}$  and a distribution over  $S$  such that it takes the value  $x_j$  with probability  $p_j$ . The entropy is

$$E(p) = \sum_{j=1}^n (-p_j \ln p_j).$$

The mean is

$$m = \sum_{j=1}^n p_j x_j.$$

Problem: Given  $m$ , find  $p$  such that  $E$  is maximized.

$$\begin{aligned} \min_p \quad & - \sum_{j=1}^n p_j \ln p_j \\ \text{s.t.} \quad & \sum_{j=1}^n p_j x_j = m \\ & \sum_{j=1}^n p_j = 1 \\ & p_j \geq 0, \quad j = 1, \dots, n \quad (\text{ignore this...}) \end{aligned}$$

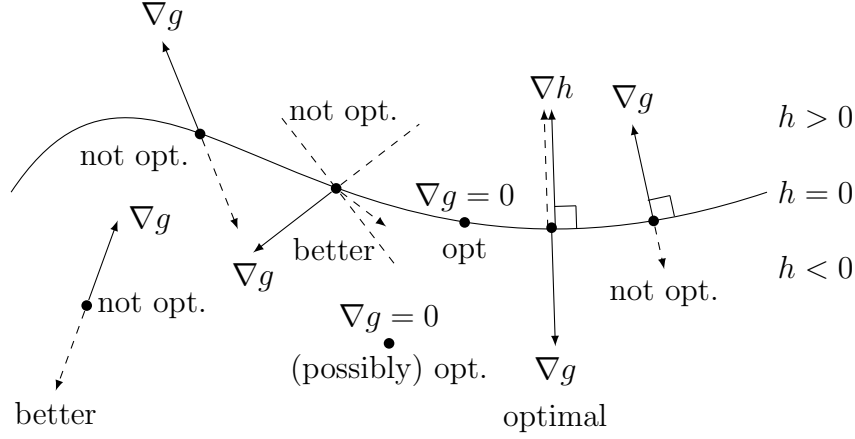
$$\begin{aligned} L &= - \sum p_j \ln p_j + \lambda_1 \left[ \sum p_j x_j - m \right] + \lambda_2 \left[ \sum p_j - 1 \right] \\ \frac{\partial L}{\partial p_j} &= - \ln p_j - 1 + \lambda_1 x_j + \lambda_2 = 0 \\ p_j &= e^{\lambda_2 - 1 + \lambda_1 x_j}, \quad j = 1, \dots, n \quad (p_j \geq 0 \text{ so we're ok with ignoring that}) \end{aligned}$$

$$\begin{aligned} \sum e^{\lambda_2 - 1 + \lambda_1 x_j} x_j &= m & n + 2 \text{ equations and} \\ \sum e^{\lambda_2 - 1 + \lambda_1 x_j} &= 1 & n + 2 \text{ unknowns...} \end{aligned}$$

No analytical solution, but numerically “solvable”

## 1.4.2 Inequality Constraints

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) \leq \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$



We need:

- if  $h(u^*) < 0$  then  $\frac{\partial g}{\partial u}(u^*) = 0$
- if  $h(u^*) = 0$  then we need either

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or

$$\frac{\partial g}{\partial u}(u^*) = -\lambda \frac{\partial h}{\partial u}(u^*) \quad \text{for } \lambda > 0$$

Or, even better,

$$\frac{\partial}{\partial u} (g(u^*) + \lambda h(u^*)) = 0 \quad \text{for } \lambda \geq 0,$$

where  $\lambda h(u^*) = 0$ . ( $h < 0 \rightarrow \lambda = 0$ ,  $h = 0 \rightarrow \lambda \geq 0$ )

In general, if  $u \in \mathbb{R}^m$  and  $h : \mathbb{R}^m \rightarrow \mathbb{R}^k$ , we have that  $u^*$ , if optimal, has to satisfy

$$\begin{aligned} \frac{\partial}{\partial u} L(u^*, \lambda) &= 0 \\ h(u^*) &\leq \mathbf{0} \\ \lambda^T h(u^*) &= 0 \\ \lambda &\geq \mathbf{0} \end{aligned}$$

where the Lagrangian is  $L(u, \lambda) = g(u) + \lambda^T h(u)$ . Note that if we're maximizing, the same holds except we need  $\lambda \leq 0$ .

### Example

$$\begin{aligned} \min \quad & 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 \\ \text{s.t.} \quad & \begin{cases} u_1^2 + u_2^2 \leq 5 \\ 3u_1 + u_2 \leq 6 \end{cases} \end{aligned}$$

$$L = 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 + \lambda_1(u_1^2 + u_2^2 - 5) + \lambda_2(3u_1 + u_2 - 6)$$

FONC:

- i)  $\partial L / \partial u_1 = 4u_1 + 2u_2 - 10 + 2\lambda_1u_1 + 3\lambda_2$
- ii)  $\partial L / \partial u_2 = 2u_1 + 2u_2 - 10 + 2\lambda_1u_2 + \lambda_2$
- iii)  $u_1^2 + u_2^2 \leq 5$
- iv)  $3u_1 + u_2 \leq 6$
- v)  $\lambda_1(u_1^2 + u_2^2 - 5) = 0$
- vi)  $\lambda_2(3u_1 + u_2 - 6) = 0$
- vii)  $\lambda_1 \geq 0$
- viii)  $\lambda_2 \geq 0$

To solve, assume different constraints are active/inactive:

1. Both constraints are inactive ( $u_1^2 + u_2^2 < 5$ ,  $3u_1 + u_2 < 6$ )  $\implies \lambda_1 = \lambda_2 = 0$

$$\begin{cases} 4u_1 + 2u_2 - 10 = 0 \\ 2u_1 + 2u_2 - 10 = 0 \end{cases} \implies \begin{cases} u_1 = 0 \\ u_2 = 5 \end{cases}$$

Note: iii)  $0^2 + 5^2 \not\leq 5$

Not feasible

2. Assume constraint 1 is active and constraint 2 is inactive ( $u_1^2 + u_2^2 = 5$ ,  $\lambda_2 = 0$ )

$$\begin{cases} 4u_1 + 2u_2 - 10 + 2\lambda_1u_1 = 0 \\ 2u_1 + 2u_2 - 10 + 2\lambda_1u_2 = 0 \\ u_1^2 + u_2^2 = 5 \end{cases} \implies \begin{cases} u_1 = 1 \\ u_2 = 2 \\ \lambda_1 = 1 \end{cases}$$

$$\checkmark \lambda_1 \geq 0$$

$$\checkmark 3 \cdot 1 + 2 \leq 6$$

This is a local minimizer

3. Assume constraint 2 is active and constraint 1 is inactive
4. Assume both constraints are active

## Kuhn-Tucker Conditions (KKT conditions, Karush-Kuhn-Tucker)

**Problem:**

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & \begin{cases} h_1(u) = 0, & h_1 : \mathbb{R}^m \rightarrow \mathbb{R}^p \\ h_2(u) \leq 0, & h_2 : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{cases} \end{aligned} \quad (1.1)$$

**Theorem.** Let  $u^*$  be feasible ( $h_1 = 0$ ,  $h_2 \leq 0$ ). If  $u^*$  is a minimizer to (1.1) then there exists vectors  $\lambda \in \mathbb{R}^p$ ,  $\mu \in \mathbb{R}^k$  with  $\mu \geq \mathbf{0}$  such that

$$\begin{cases} \frac{\partial g}{\partial u}(u^*) + \lambda^T \frac{\partial h_1}{\partial u}(u^*) + \mu^T \frac{\partial h_2}{\partial u}(u^*) = 0 \\ \mu^T h_2(u^*) = 0 \end{cases}$$

Looking ahead:  $\min \text{cost}(u(\cdot))$  s.t.  $\dot{x} = f(x, u)$  (dynamics), where  $u$  is a function. Note the equality constraint.

**Question:** How do we go from  $u \in \mathbb{R}^m$  to  $u \in \mathcal{U}$  (function space)?

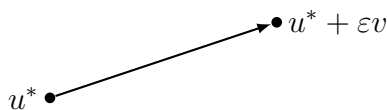
**Note:** Function space is a set of functions of a given kind from a set  $X$  to a set  $Y$

1. linear function
2. square-integrable functions:  $L_2[0, T] : \int_0^T \|u(t)\|^2 dt < \infty$
3.  $C^\infty(\mathbb{R})$

What would  $\partial \text{"cost"} / \partial u$  mean?

## 1.5 Directional Derivatives

**Recall:** To minimize  $g(u)$ , let  $u^*$  be a candidate minimizer and pitch a perturbation on  $u^*$  of  $\varepsilon v$ , where  $\varepsilon$  is the scale and  $v$  is the direction. Taking Taylor's expansion at the perturbation produces



$$g(u^* + \varepsilon v) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$

$$\text{FONC: } \frac{\partial g}{\partial u}(u^*) = 0$$

**Note:**  $\frac{\partial g}{\partial u}(u^*)v$  tells us how much  $g(u)$  increases/decreases in the direction of  $v$ .

**Definition.** The directional (Gateaux) derivative is given by

$$\delta g(u; v) = \lim_{\varepsilon \rightarrow 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon}$$

**Example**

$$g(u) = \frac{1}{2}u_1^2 - u_1 + 2u_2, \quad g : \mathbb{R}^2 \rightarrow \mathbb{R}$$

Let's consider  $e_1 = [1 \ 0]^T$ ,  $e_2 = [0 \ 1]^T$ . What is  $\delta g(u; e_i)$ ,  $i = 1, 2$ ?

$$\begin{aligned} \delta g(u; v) &= \lim_{\varepsilon \rightarrow 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{g(u) + \varepsilon \frac{\partial g}{\partial u}(u)v + o(\varepsilon) - g(u)}{\varepsilon} \\ &= \frac{\partial g}{\partial u}(u)v \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial u}(u) &= [u_1 - 1 \ 2] \\ \delta g(u; e_1) &= [u_1 - 1 \ 2]e_1 = u_1 - 1 \\ \delta g(u; e_2) &= [u_1 - 1 \ 2]e_2 = 2 \end{aligned}$$

But the beauty of directional derivatives is that they generalize beyond vectors,  $u \in \mathbb{R}^m$ , to function spaces ( $\mathcal{U}$ ) or other “objects” like matrices.

**Example**  $M \in \mathbb{R}^{n \times n}$ ,  $F(M) = M^2$

What is  $\frac{\partial F}{\partial M}$ ? (ponder at home...)

We can easily compute  $\delta F(M; N)$ !

$$\begin{aligned} F(M + \varepsilon N) &= (M + \varepsilon N)(M + \varepsilon N) = M^2 + \varepsilon MN + \varepsilon NM + \varepsilon^2 N^2 \\ \delta F(M; N) &= \lim_{\varepsilon \rightarrow 0} \frac{F(M + \varepsilon N) - F(M)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon MN + \varepsilon NM + \varepsilon^2 N^2}{\varepsilon} = MN + NM \end{aligned}$$

**Infinite Dimensional Optimization** Let  $u \in \mathcal{U}$  (function space) and let  $J(u)$  be the cost:

$$\min_{u \in \mathcal{U}} J(u)$$

**Theorem.** If  $u^* \in \mathcal{U}$  is a (local) minimizer then

$$\delta J(u^*; v) = 0, \quad \forall v \in \mathcal{U}$$

**Example** Find minimizer  $u^*$  to

$$J(u) = \int_0^T L(u(t)) \, dt$$

$$\begin{aligned} J(u + \varepsilon v) - J(u) &= \int_0^T L(u(t) + \varepsilon v(t)) \, dt - \int_0^T L(u(t)) \, dt, \quad u, v \in \mathcal{U} \\ &= \int_0^T \left[ L(u(t)) + \varepsilon \frac{\partial L}{\partial u}(u(t))v(t) + o(\varepsilon) - L(u(t)) \right] \, dt \\ \delta J(u^*; v) &= \lim_{\varepsilon \rightarrow 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\int_0^T \varepsilon \frac{\partial L}{\partial u}(u(t))v(t) \, dt + o(\varepsilon)}{\varepsilon} \\ &= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t) \, dt \end{aligned}$$

$u^*$  optimizer:

$$\begin{aligned} \delta J(u^*; v) &= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t) \, dt = 0 \quad \forall v \in \mathcal{U} \\ &\quad \Updownarrow \\ \frac{\partial L}{\partial u}(u(t)) &= 0 \quad \forall t \in [0, T] \end{aligned}$$

But, we want *optimal control*! We want our cost to look like

$$\begin{aligned} &\int_0^T L(x(t), u(t)) \, dt \\ &\dot{x} = f(x, u) \end{aligned}$$

## 1.6 Calculus of Variations

What happens to  $x(t)$  when  $u(t)$  changes to  $u(t) + \varepsilon v(t)$ ? Let the system be given by

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

After perturbation of  $u$ , the new system is

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}, u + \varepsilon v) \\ \hat{x}(0) = x_0 \end{cases}$$



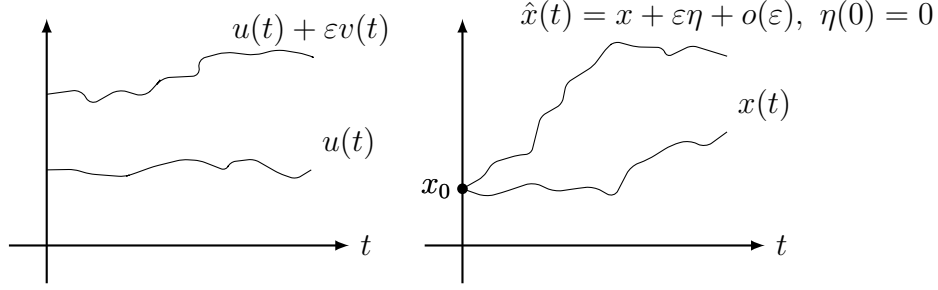


Figure 1.1: Variation in  $u$  causes a variation in  $x$ .

Consider

$$\tilde{x} = x + \varepsilon\eta,$$

where

$$\begin{aligned} \dot{x} &= f(x, u), & x(0) &= x_0 \\ \dot{\eta} &= \frac{\partial f}{\partial x}(x, u)\eta + \frac{\partial f}{\partial u}(x, u)v, & \eta(0) &= 0 \end{aligned}$$

**Theorem.** *If  $f$  is continuously differentiable in  $x$  and  $u$  then*

$$\hat{x}(t) = \tilde{x}(t) + o(\varepsilon)$$

*Proof.*

i) Initial conditions:

$$\begin{aligned} \hat{x}(0) &= x_0 \\ \tilde{x}(0) &= x(0) + \varepsilon\eta(0) = x_0 \end{aligned}$$

ii) Dynamics:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v) \\ \dot{\tilde{x}} &= \dot{x} + \varepsilon\dot{\eta} = f(x, u) + \varepsilon \frac{\partial f}{\partial x}(x, u)\eta + \varepsilon \frac{\partial f}{\partial u}(x, u)v \\ &= f(x + \varepsilon\eta, u + \varepsilon v) + o(\varepsilon) \\ &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon) \end{aligned}$$

We can see that the dynamics of  $\hat{x}(t)$  are equal to those of  $\tilde{x}(t)$  plus higher order terms:

$$\begin{aligned} \dot{\tilde{x}} &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon) \\ \dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v) \end{aligned}$$

Therefore, if our perturbation is small enough, we can model  $\hat{x}(t)$  as  $\tilde{x}(t)$ .

□

Note: Taylor expansion with two elements is

$$\begin{aligned}
h(w + \varepsilon v, z + \varepsilon y) &= h(w, z + \varepsilon y) + \frac{\partial h}{\partial w}(w, z + \varepsilon y)\varepsilon v + o(\varepsilon) \\
&= \left\{ h(w, z) + \frac{\partial h}{\partial z}(w, z)\varepsilon y + o(\varepsilon) \right\} \\
&\quad + \left\{ \frac{\partial h}{\partial w}(w, z)\varepsilon v + \underbrace{\frac{\partial^2 h}{\partial z \partial w}\varepsilon v \odot \varepsilon y}_{o(\varepsilon)} + o(\varepsilon) \right\} \\
&= h(w, z) + \frac{\partial h}{\partial z}\varepsilon y + \frac{\partial h}{\partial w}\varepsilon v + o(\varepsilon)
\end{aligned}$$

**Last class:**

1.  $u \in \mathcal{U}$  (space of functions),  $J : \mathcal{U} \rightarrow \mathbb{R}$  (cost).

FONC: If  $u^*$  is optimal, then

$$\delta J(u; \nu) = 0 \quad \forall \nu \in \mathcal{U},$$

where the directional derivative is given by

$$\delta J(u; \nu) = \lim_{\varepsilon \rightarrow 0} \frac{J(u + \varepsilon \nu) - J(u)}{\varepsilon}.$$

2. If

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

then a variation in  $u$ :

$$u \mapsto u + \varepsilon \nu$$

results in a variation in  $x$ :

$$x \mapsto x + \varepsilon \eta + o(\varepsilon)$$

See Figure 1.1. Note  $\eta(0) = 0$ .

### 1.6.1 An (Almost) Optimal Control Problem

Let  $\dot{x} = f(x)$ ,  $x(0) = x_0$ . Note we get to pick the initial condition!

## Problem

$$\begin{aligned} \min_{x_0 \in \mathbb{R}^m} J(x_0) &= \int_0^T L(x(t)) dt \\ \text{s.t. } \begin{cases} \dot{x}(t) = f(x(t)) & \text{the constraint! (equality)} \\ x(0) = x_0 \end{cases} \end{aligned}$$

Note every constraint needs a Lagrange multiplier. We have infinitely many constraints:

$$\dot{x}(t) = f(x(t)) \quad \forall t \in [0, T]$$

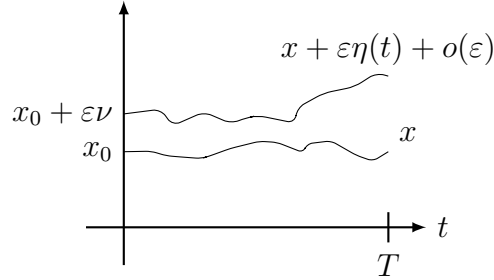
We need  $\lambda(t)$  as a function of  $t$ . Also, the sum in the Lagrangian has to become an integral. The continuous-time Lagrangian thus becomes

$$\tilde{J}(x_0, \lambda) = \int_0^T \left[ L(x(t)) + \lambda^T(t)(f(x(t)) - \dot{x}(t)) \right] dt$$

The task is to perturb  $x_0$  as  $x_0 \mapsto x_0 + \varepsilon\nu$ ,  $\nu \in \mathbb{R}^m$  and compute

$$\delta \tilde{J}(x_0; \nu) = \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(x_0 + \varepsilon\nu) - \tilde{J}(x_0)}{\varepsilon}$$

and make this equal to 0  $\forall \nu \in \mathbb{R}^m$ . The variation in  $x$  is



Note:

- $x_0$  decision variable
- $\eta$  variation in  $x_0$
- $x(t)$  trajectory starting at  $x_0$
- $\eta(t)$  change in trajectory resulting from  $\nu$ -variation in  $x_0$
- $\lambda(t)$  time-varying Lagrange multiplier

$$\begin{aligned} \tilde{J}(x_0 + \varepsilon\nu) &= \int_0^T \left\{ L(x(t)) + \lambda^T(t)[f(x(t) + \varepsilon\eta(t)) - \dot{x}(t) - \varepsilon\dot{\eta}(t)] \right\} dt + o(\varepsilon) \\ &= \int_0^T \left[ L(x) + \varepsilon \frac{\partial L}{\partial x}(x)\eta + \lambda^T \left( f(x) + \varepsilon \frac{\partial f}{\partial x}(x)\eta - \dot{x} - \varepsilon\dot{\eta} \right) \right] dt + o(\varepsilon) \\ \tilde{J}(x_0 + \varepsilon\nu) - \tilde{J}(x_0) &= \int_0^T \left[ \varepsilon \frac{\partial L}{\partial x}(x)\eta + \lambda^T \left( \varepsilon \frac{\partial f}{\partial x}\eta - \varepsilon\dot{\eta} \right) \right] dt + o(\varepsilon) \\ \delta \tilde{J}(x_0; \nu) &= \int_0^T \left[ \frac{\partial L}{\partial x}(x)\eta + \lambda^T \left( \frac{\partial f}{\partial x}\eta - \dot{\eta} \right) \right] dt \end{aligned}$$

A powerful idea: we want  $\delta\tilde{J}(x_0; \nu) = 0 \forall \nu$ . Somehow get this in the form

$$\int_0^T (\text{stuff}(t)) \eta(t) dt = 0$$

We can pick  $\text{stuff}(t) = 0 \forall t \in [0, T]$ .

In  $\delta\tilde{J}(x_0; \nu)$  we have  $\dot{\eta}$  (problem!). We can solve this using *integration by parts*.

$$\int_0^T \lambda^T \dot{\eta} dt = \lambda^T(T) \eta(T) - \lambda^T(0) \eta(0) - \int_0^T \dot{\lambda}^T \eta dt$$

Hence,

$$\delta\tilde{J}(x_0; \nu) = \int_0^T \underbrace{\left( \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T \right)}_{\text{pick}=0} \eta dt - \underbrace{\lambda^T(T) \eta(T)}_{\text{pick}=0} + \lambda^T(0) \underbrace{\eta(0)}_{\nu}$$

We are free to pick  $\lambda$  freely if it gives  $\delta\tilde{J} = 0$ .

$$\text{Pick: } \begin{cases} \dot{\lambda}(t) = -\frac{\partial L^T}{\partial x}(x(t)) - \frac{\partial f^T}{\partial x}(x(t)) \lambda(t) \\ \lambda(T) = 0 \end{cases} \quad \text{backwards diff. eq:}$$

Under this choice of  $\lambda$  we get

$$\delta\tilde{J}(x_0; \nu) = \lambda^T(0) \nu$$

This is linear in  $\nu$  so the FONC is  $\lambda(0) = 0$ .

Moreover, we really have a “normal” optimization problem

$$\begin{aligned} \min_{x_0 \in \mathbb{R}^m} \tilde{J}(x_0) \\ \delta\tilde{J}(x_0; \nu) = \frac{\partial \tilde{J}}{\partial x_0}(x_0) \nu \end{aligned}$$

which means that

$$\frac{\partial \tilde{J}}{\partial x_0} = \lambda^T(0)$$

If  $x_0^*$  minimizes

$$\begin{aligned} & \int_0^T L(x(t)) dt \\ \text{s.t. } & \begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0^* \end{cases} \end{aligned}$$

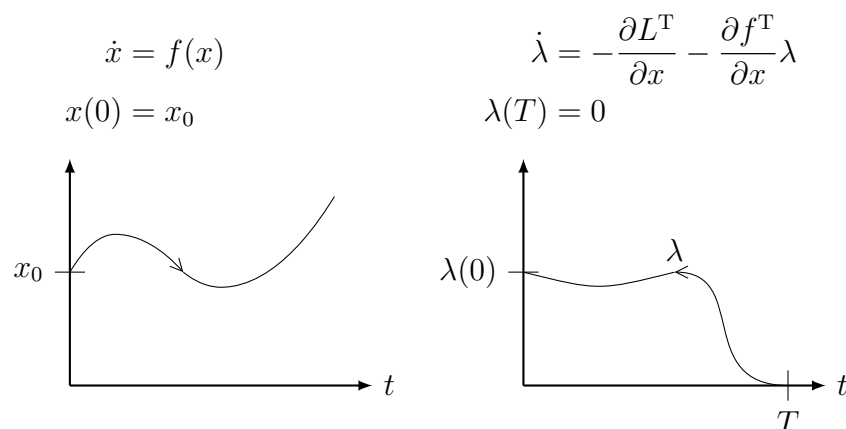
then

$$\lambda(0) = \mathbf{0}$$

where  $\lambda(t)$  satisfies

$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial L^T}{\partial x}(x(t)) - \frac{\partial f^T}{\partial x}(x(t)) \lambda(t) \\ \lambda(T) = 0 \end{cases}$$

**So what?** We actually have a two-point boundary value problem.



We want to find  $x_0$  that gives  $f(x)$  such that after solving backwards for  $\lambda(t)$ , we find that

$$\lambda(0) = \frac{\partial \tilde{J}^T}{\partial x_0} = 0.$$

This leads to the following:

### An algorithm

---

```

Pick  $x_{0,0}$ 
 $k = 1$ 
repeat
    Simulate  $x(t)$  from  $x_{0,k}$  over  $[0, T]$ 
    Simulate  $\lambda(t)$  from  $\lambda(T) = 0$  backwards using  $x(t)$ 
    Update  $x_{0,k}$  as  $x_{0,k+1} = x_{0,k} - \gamma \lambda(0)$ 
     $k := k + 1$ 
until  $\lambda(0) = 0$ 

```

---

▷  $\lambda(0)$  is the gradient

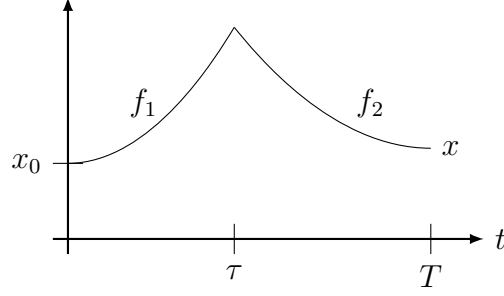
**Example:** `optinit.m`

$$\begin{aligned} \dot{x} &= Ax, & L &= x^T Q x - q, & Q &= Q^T \succ 0 \\ \dot{\lambda} &= -2Qx - A^T \lambda \\ \lambda(0) &= 0 \end{aligned}$$

## 1.6.2 Optimal Timing Control

When to switch between modes?

$$\begin{aligned} \dot{x} &= \begin{cases} f_1(x) & \text{if } t \in [0, \tau) \\ f_2(x) & \text{if } t \in [\tau, T] \end{cases} \\ x(0) &= x_0 \end{aligned} \tag{1.2}$$



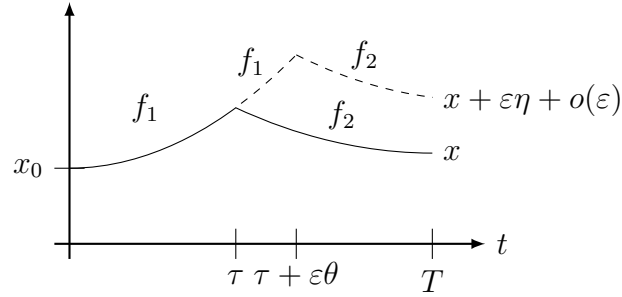
$$\min_{\tau} \int_0^T L(x(t)) dt = J(\tau)$$

s.t. (1.2) holds

Step 1: Augment cost with constraint

$$\tilde{J} = \int_0^{\tau} \left[ L(x) + \lambda^T (f_1(x) - \dot{x}) \right] dt + \int_{\tau}^T \left[ L(x) + \lambda^T (f_2(x) - \dot{x}) \right] dt$$

Step 2: Variation  $\tau \mapsto \tau + \varepsilon\theta$



Step 3: Compute  $\delta\tilde{J}(\tau; \theta)$

$$\begin{aligned} \tilde{J}(\tau + \varepsilon\theta) &= \int_0^{\tau + \varepsilon\theta} \left\{ L(x + \varepsilon\eta) + \lambda^T [f_1(x + \varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \right\} dt \\ &\quad + \int_{\tau + \varepsilon\theta}^T \left\{ L(x + \varepsilon\eta) + \lambda^T [f_2(x + \varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \right\} dt + o(\varepsilon) \end{aligned}$$

Note that  $\eta = \dot{\eta} = 0$  on  $[0, \tau)$ .

$$\begin{aligned} \tilde{J}(\tau + \varepsilon\theta) &= \int_0^{\tau} \left\{ L(x) + \lambda^T [f_1(x) - \dot{x}] \right\} dt \\ &\quad + \int_{\tau}^{\tau + \varepsilon\theta} \left\{ \underbrace{L(x + \varepsilon\eta)}_{L(x) + \varepsilon \frac{\partial L}{\partial x} \eta} + \lambda^T \left[ \underbrace{f_1(x + \varepsilon\eta)}_{f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta} - \dot{x} - \varepsilon\dot{\eta} \right] \right\} dt \\ &\quad + \int_{\tau + \varepsilon\theta}^T \left\{ \underbrace{L(x + \varepsilon\eta)}_{L(x) + \varepsilon \frac{\partial L}{\partial x} \eta} + \lambda^T \left[ \underbrace{f_2(x + \varepsilon\eta)}_{f_2(x) + \varepsilon \frac{\partial f_2}{\partial x} \eta} - \dot{x} - \varepsilon\dot{\eta} \right] \right\} dt + o(\varepsilon) \end{aligned}$$

$$\begin{aligned}
\delta \tilde{J}(\tau; \theta) &= \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau)}{\varepsilon} \\
\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau) &= \int_0^\tau 0 \cdot dt + \underbrace{\int_\tau^{\tau+\varepsilon\theta} \left[ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left( f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta - f_2(x) - \varepsilon \dot{\eta} \right) \right] dt}_{I_1} \\
&\quad + \underbrace{\int_{\tau+\varepsilon\theta}^T \left[ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left( \varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt}_{I_2} + o(\varepsilon)
\end{aligned}$$

**Theorem** (Mean-value theorem).

$$\int_{t_1}^{t_2} h(t) dt = (t_2 - t_1)h(\xi) \quad \text{for some } \xi \in [t_1, t_2]$$

The first integral is

$$\begin{aligned}
I_1 &= \int_\tau^{\tau+\varepsilon\theta} \left[ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top (f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta - \varepsilon \dot{\eta} - f_x(x)) \right] dt \\
&= \varepsilon \theta \left\{ \lambda^\top(\xi) [f_1(x(\xi)) - f_2(x(\xi))] \right\} + o(\varepsilon)
\end{aligned}$$

Note that as  $\varepsilon \rightarrow 0$ ,  $\xi \rightarrow \tau$ . Using integration by parts, the second integral is

$$\begin{aligned}
\int_\tau^T \lambda^\top \dot{\eta} dt &= \lambda^\top(T) \eta(T) - \lambda^\top(\tau) \underbrace{\eta(\tau)}_{=0} - \int_\tau^T \dot{\lambda}^\top \eta dt \\
I_2 &= \int_\tau^T \left[ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left( \varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt - \underbrace{\int_\tau^{\tau+\varepsilon\theta} \left[ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left( \varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt}_{o(\varepsilon)} \\
&= \varepsilon \int_\tau^T \left[ \frac{\partial L}{\partial x} + \lambda^\top \frac{\partial f_2}{\partial x} + \dot{\lambda}^\top \right] \eta dt - \varepsilon \lambda^\top(T) \eta(T) + o(\varepsilon)
\end{aligned}$$

Hence,

$$\begin{aligned}
\delta \tilde{J}(\tau; \theta) &= \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau)}{\varepsilon} \\
&= \theta \lambda^\top(\tau) [f_1(x(\tau)) - f_2(x(\tau))] + \int_\tau^T \left[ \frac{\partial L}{\partial x} + \lambda^\top \frac{\partial f_2}{\partial x} + \dot{\lambda}^\top \right] \eta dt - \lambda^\top(T) \eta(T)
\end{aligned}$$

Step 4: Select the *costate*  $\lambda(t)$ . The key idea is to get rid of any term that has  $\eta$  in it, i.e.

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial L}{\partial x} - \frac{\partial f_2^\top}{\partial x} \lambda \quad \text{on } [\tau, T] \\
\lambda(T) &= 0
\end{aligned}$$

Step 5: With this choice of  $\lambda(t)$ , we have

$$\delta \tilde{J}(\tau; \theta) = \theta \lambda^T(\tau) [f_1(x(\tau)) - f_2(x(\tau))] = \frac{\partial \tilde{J}}{\partial \tau} \theta.$$

Therefore,

$$\frac{\partial \tilde{J}}{\partial \tau} = \lambda^T(\tau) [f_1(x(\tau)) - f_2(x(\tau))] = 0 \quad (\text{for optimality})$$

## Algorithm

---

Pick  $\tau_0$   
 $k = 0$   
**repeat**  
    Simulate  $x$  forward in time from  $x(0) = x_0$   
    Simulate  $\lambda$  backwards from  $\lambda(T) = 0$   
    Update  $\tau_k$  as  $\tau_{k+1} = \tau_k - \gamma \lambda^T(\tau_k) [f_1(x(\tau_k)) - f_2(x(\tau_k))]$   
     $k := k + 1$   
**until**  $\|\lambda^T(f_1 - f_2)\| < \varepsilon$

---

Where are we going? Come up with general principles for  $\min_{u \in \mathcal{U}} J(u)$ :

- Costate equations
- Optimality conditions
- Algorithms
- Applications

### 1.6.3 The Bolza Problem

Up until now, we have optimized with respect to finite-dimensional parameters. Today, we will minimize with respect to  $u \in \mathcal{U}$ .

$$\begin{aligned} \min_{u \in \mathcal{U}} \quad & \int_0^T L(x(t), u(t), t) \, dt + \underbrace{\Psi(x(T))}_{\substack{\text{terminal cost} \\ \text{(parking cost)}}} \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t), t) \\ & x(0) = x_0 \end{aligned}$$



Assume that  $f$  and  $L$  are  $C^1$  in  $x, u$  and piecewise continuous in  $t$ . Then, a small change in  $u$  causes small changes in  $f$  and  $L$ . The variation:  $u \mapsto u + \varepsilon v$ ,  $\varepsilon \in \mathbb{R}$ ,  $v \in \mathcal{U}$ . See Figure 1.1.

$$\begin{aligned}
\tilde{J}(u) &= \int_0^T [L(x, u, t) + \lambda^T(f(x, u, t) - \dot{x})] dt + \Psi(x(T)) \\
\tilde{J}(u + \varepsilon v) &= \int_0^T [L(x + \varepsilon \eta, u + \varepsilon v, t) + \lambda^T(f(x + \varepsilon \eta, u + \varepsilon v, t) - \dot{x} - \varepsilon \dot{\eta})] dt \\
&\quad + \Psi(x(T) + \varepsilon \eta(T)) + o(\varepsilon) \\
\tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \left[ L(x + \varepsilon \eta, u + \varepsilon v, t) - L(x, u, t) \right. \\
&\quad \left. + \lambda^T(f(x + \varepsilon \eta, u + \varepsilon v, t) - f(x, u, t) - \dot{x} - \varepsilon \dot{\eta} + \dot{x}) \right] dt \\
&\quad + \Psi(x(T) + \varepsilon \eta(T)) - \Psi(x(T)) + o(\varepsilon) \\
&= \int_0^T \left[ \frac{\partial L}{\partial x} \varepsilon \eta + \frac{\partial L}{\partial u} \varepsilon v + \lambda^T \left( \frac{\partial f}{\partial x} \varepsilon \eta + \frac{\partial f}{\partial u} \varepsilon v - \varepsilon \dot{\eta} \right) \right] dt \\
&\quad + \frac{\partial \Psi}{\partial x}(x(T)) \varepsilon \eta(T) + o(\varepsilon)
\end{aligned}$$

(See Taylor expansion with respect to two variables.)

$$\begin{aligned}
\delta \tilde{J}(u; v) &= \int_0^T \left( \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) v dt + \int_0^T \left[ \left( \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} \right) \eta - \lambda^T \dot{\eta} \right] dt \\
&\quad + \frac{\partial \Psi}{\partial x}(x(T)) \eta(T)
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
\int_0^T \lambda^T \dot{\eta} dt &= \lambda^T(T) \eta(T) - \lambda^T(0) \eta(0) - \int_0^T \dot{\lambda}^T \eta dt \\
&= \lambda^T(T) \eta(T) - \int_0^T \dot{\lambda}^T \eta dt \\
\delta \tilde{J}(u; v) &= \int_0^T \left( \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) v dt + \int_0^T \left( \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T \right) \eta dt \\
&\quad + \left( \frac{\partial \Psi}{\partial x}(x(T)) - \lambda^T(T) \right) \eta(T)
\end{aligned}$$

For optimality, we need the directional derivative to be zero for every  $v \in \mathcal{U}$ , where  $v$  represents the direction of the derivative. Therefore, the term  $(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u})$  in the first integral has to be identically zero. Thus, we need

$$\begin{cases} \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0, & \forall t \in [0, T] \\ \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T = 0, & \forall t \in [0, T] \\ \frac{\partial \Psi}{\partial x}(x(T)) - \lambda^T(T) = 0 \end{cases}$$

**Definition.** Let the *Hamiltonian*  $H(x, u, t, \lambda)$  be given by

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^T f(x, u, t)$$

**Theorem.** For  $u$  to solve the Bolza problem, it has to satisfy

$$\frac{\partial H}{\partial u}(x, u, t, \lambda) = 0,$$

where the costate satisfies

$$\begin{cases} \dot{\lambda} = -\frac{\partial H^T}{\partial x}(x, u, t, \lambda) \\ \lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T)) \end{cases}$$

**Example**

$$\begin{aligned} & \min_u \int_0^1 \frac{1}{2} u^2(t) dt + \frac{1}{2} x^2(1) \\ & \text{s.t.} \quad \begin{cases} \dot{x} = u, & x, u \in \mathbb{R} \\ x(0) = 1 \end{cases} \end{aligned}$$

$$H = \frac{1}{2} u^2 + \lambda u$$

$$\frac{\partial H}{\partial u} = u + \lambda = 0 \implies u = -\lambda$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0 \implies \lambda(t) = c$$

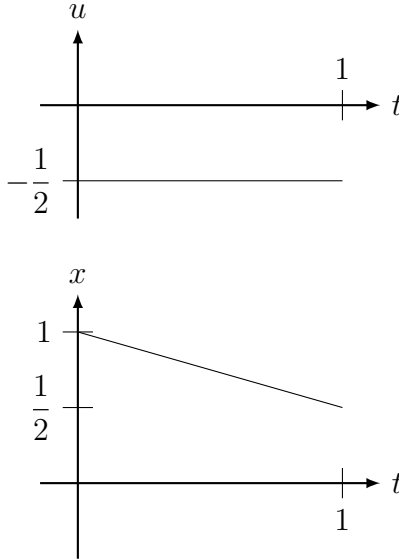
$$\lambda(T) = c = \frac{\partial \Psi}{\partial x}(x(1)) = x(1)$$

$$\dot{x} = u = -c \implies x(t) = -ct + x(0) = -ct + 1$$

$$x(1) = -c + 1$$

$$\lambda(1) = c = x(1) = -c + 1 \implies c = \frac{1}{2}$$

$$\boxed{u^* = -\frac{1}{2}}$$



We really used five different equations to solve this!

- i)  $\frac{\partial H}{\partial u} = 0$
- ii)  $\dot{\lambda} = -\frac{\partial H^T}{\partial x}$
- iii)  $\lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T))$
- iv)  $\dot{x} = f(x, u, t)$
- v)  $x(0) = x_0$

There is a sixth condition that is pretty useful if  $L$  and  $f$  do not depend on  $t$  ( $L(x, u)$ ,  $f(x, u)$ ). This is called a *conservative system*. Then, along optimal trajectories (equations i-v are satisfied), the total time derivative of the Hamiltonian is

$$\frac{d}{dt}H = \underbrace{\frac{\partial H}{\partial t}}_{\substack{0 \\ H(x,u,\lambda)}} + \underbrace{\frac{\partial H}{\partial x}}_{-\dot{\lambda}^T} \dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{\substack{0 \\ u \text{ is optimal}}} \dot{u} + \underbrace{\frac{\partial H}{\partial \lambda}}_{f^T = \dot{x}^T} \dot{\lambda} = -\dot{\lambda}^T \dot{x} + \dot{x}^T \dot{\lambda} = 0$$

Therefore, for conservative systems,

- vi)  $H$  is constant along optimal trajectories. (Hamilton's Principle in analytical mechanics)

Back to the example,

$$H = \frac{1}{2}u^2 + \lambda u = \frac{1}{2}c^2 - c^2 = -\frac{1}{2}c^2 = -\frac{1}{8}$$