Homework 3

ECE6553 Optimal Control

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Due: March 9, 2017 (and March 16 for DL students – sections Q and QSZ)

1

Consider the Bolza problem

$$\min_u \int_0^T L(x,u)dt + \Psi(x(T)),$$

subject to

$$\dot{x} = f(x,u), \quad x(0) = x_0.$$

The solution involves solving the two-point boundary-value problem

$$\begin{aligned}
\dot{x} &= f(x, u^\star(x,\lambda)) \\
\dot{\lambda} &= -\frac{\partial H(x, u^\star(x,\lambda), \lambda)}{\partial x}^T \\
x(0) &= x_0 \\
\lambda(T) &= \frac{\partial \Psi(x(T))}{\partial x}^T,
\end{aligned}$$

where

$$\frac{\partial H(x, u^\star(x,\lambda), \lambda)}{\partial u} = 0.$$

To solve this, we can pick λ_0 and define a cost $G(\lambda_0) = \|\lambda(T) - \partial\Psi(x(T))/\partial x^T\|^2$ and then numerically evaluate the derivative to this cost and use a steepest descent method to update λ_0 (test-shooting).

However, if we let

$$z = \left[\begin{array}{c} x \\ \lambda \end{array} \right]$$

what we really have here is a new problem

$$\min_{\lambda_0} C(z(T))$$

subject to

$$\dot{z} = F(z), \quad z(0) = [x_0^T, \lambda_0^T]^T$$

and we already know how to solve this! Use what you know about minimization with respect to initial conditions to produce an alternative numerical algorithm to the test shooting algorithm. (Feel free to introduce new variables if it makes your notation easier.)

2

"Brockett's nonholonomic integrator" is given by

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_1u_2 - x_2u_1.\end{aligned}$$

Starting from the origin, $x(0) = 0$, we would like to maximize $x_3(T)$ while expending as little control energy as possible, i.e.,

$$\min_u \frac{1}{2}\int_0^T (u_1(t)^2 + u_2(t)^2)dt - \rho\frac{1}{2}x_3(T)^2.$$

a

What are the optimality conditions to this problem? (Optimal u, costate dynamics, boundary conditions on the costate.)

b

What do the expressions needed to execute your algorithm found in Question **1** look like for this particular problem.

3

a

Solve the two-point boundary value problem associated with your optimality conditions for Question **2a** using the test-shooting method for $T = 1$ and $\rho = 2$, with the initial $\lambda_0 = [0.1, 0.1, 0.1]^T$. Plot the cost $G(\lambda_0)$ as a function of iterations (updates to λ_0).

b

Same question as **3a** but now use your algorithm developed in Question **1** with the expressions from Question **2b** instead. Plot the cost $G(\lambda_0)$ as a function of iterations (updates to λ_0).

c

Which of the two methods do you think is more efficient/accurate? Why is that?

4

Solve the problem

$$\min_u \int_0^T \frac{1}{2}(x^2 + u^2)dt,$$

subject to the constraints that

$$\begin{aligned}&\dot{x} = u\\ &x(0) = 1,\ x(1) = 1.\end{aligned}$$

5

a

Find the optimality conditions for the following problem:

$$\min_p \int_0^T L(x(t))dt,$$

subject to

$$\dot{x} = f(x, p), \ \ x(0) = x_0,$$

where $p \in \mathbb{R}$ is a scalar parameter, i.e., not a time-varying control signal.

b

Based on the observation that, for general functions g,

$$G(t) = \int_0^t g(\tau)d\tau \ \Rightarrow \ \dot{G} = g, \quad G(0) = 0,$$

reformulate your condition in Question **5a** to involve a new "costate-like" variable μ to make the optimality condition look a bit prettier.

6

In the recent midterm, we considered the problem where the dynamics was $f(x, u) = Ax + Bu$ and the instantaneous cost was $L(x, u) = 1/2x^TQx + 1/2u^TRu$, where $Q, R \succ 0$.

By deriving the optimality conditions and showing that if we pick

$$\lambda(t) = P(t)x(t),$$

where

$$\dot{P} = -A^TP - PA - Q + PBR^{-1}B^TP,$$

then this choice of λ does in fact satisfy the costate equation after we plug in the optimal u.

Explain why this is an important observation and why one would use this solution to the optimal control problem rather than the one we find by directly working with λ?