ECE 6553: Optimal Control Notes

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Chapter 1

Parameter Optimization

1.1 What is optimal control?

Optimal Maximize/minimize cost (subject to constraints): $\min_u g(u)$
With constraints,

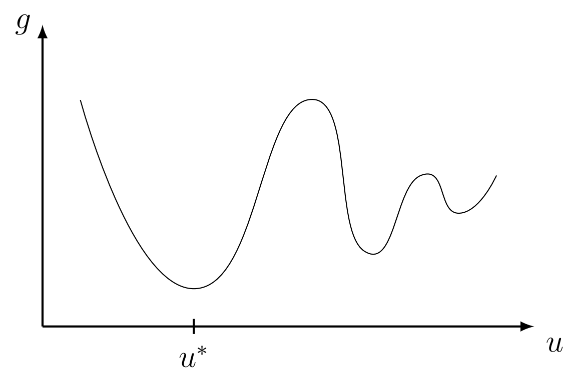
$$\min_u \; g(u)$$
$$\text{s.t.} \; \begin{cases} h_1(u) = 0 \\ h_2(u) \leq 0 \end{cases}$$

First-order necessary condition (FONC):

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Optimality can be

- local vs global
- max vs min



Control control design: pick u such that specifications are satisfied:

$$\dot{x} = f(x,u), \qquad \dot{x} = Ax + Bu,$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, and $f(\cdot)$ is the dynamics.

Actually, x and u are signals:

$$x : [0,T] \to \mathbb{R}^n, \qquad u : [0,T] \to \mathbb{R}^m$$

Optimal control find the "best" u!

For "best" to mean anything, we need a cost. The big/deep question is

$$\frac{\partial \text{“cost”}}{\partial u} = 0$$

Example

Suppose we have a car with position p. Its acceleration $\ddot{p}$ is controlled by the gas/brake input u ($\ddot{p} = u$). In order to express the dynamics of the system in the form $\dot{x} = f(x,u)$, we introduce state variables:

$$\begin{aligned} x_1 &= p \\ x_2 &= \dot{p} \end{aligned} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

The task is to move the car from its initial position to a stop at a distance c away.

Minimum energy problem

$$\begin{aligned} \min_u \quad & \int_0^T u^2(t)\,\mathrm{d}t \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0,\ x_2(0) = 0 \\ & x_1(T) = c,\ x_2(T) = 0 \end{aligned}$$

Minimum time problem

$$\begin{aligned} \min_{u,T} \quad & T = \int_0^T \mathrm{d}t \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0,\ x_2(0) = 0 \\ & x_1(T) = c,\ x_2(T) = 0 \\ & u(t) \in [u_{\min}, u_{\max}] \end{aligned}$$

The general optimal control problem we will solve will look like

$$
\begin{aligned}
\min_{u,T} \quad & \int_0^T L(x(t),u(t),t)\,\mathrm{d}t + \Psi(x(T)) \\
\text{s.t.} \quad & \dot{x}(t) = f(x(t),u(t),t),\ t \in [0,T] \\
& x(0) = x_0 \\
& x(T) \in S \\
& u(t) \in \Omega,\ t \in [0,T]
\end{aligned}
$$

where $\Psi(\cdot)$ is the terminal cost and S is the terminal manifold. This is a so-called **Bolza Problem**.

What tools do we need to solve this?

1. optimality conditions $\partial\text{cost}/\partial u = 0$
2. some way of representing the optimal signal $u^*(x,t)$
3. some way of actually finding/computing the optimal controllers

1.2 Unconstrained Optimization

Let the decision variable be $u = [u_1,\ldots,u_m]^{\mathrm{T}} \in \mathbb{R}^m$. The cost is $g(u) \in C^1$ (C^k means k times continuously differentiable). The problem is

$$\min_u g(u), \quad g : \mathbb{R}^m \to \mathbb{R}$$

For u^* to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Definition. u^* is a (local) minimizer to g if $\exists \delta > 0$ s.t.

$$
\begin{gathered}
g(u^*) \le g(u) \quad \forall u \in B_\delta(u^*) \\
B_\delta(u^*) = \{u \mid \|u - u^*\| \le \delta\}
\end{gathered}
$$

Note:

- $\dfrac{\partial g}{\partial u}(u^*)\delta u \in \mathbb{R}$ and δu is $m \times 1$, so $\dfrac{\partial g}{\partial u}$ is a $1 \times m$ row vector. For the column vector,

$$\nabla g = \frac{\partial g}{\partial u}^{\mathrm{T}} \in \mathbb{R}^m$$

- $\dfrac{\partial g}{\partial u}\,\delta u$ is an inner product

$$\langle \nabla g, \delta u \rangle = \left\langle \frac{\partial g}{\partial u}^{\mathrm{T}}, \delta u \right\rangle$$

- $o(\varepsilon)$ encodes higher-order terms

$$\lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0 \qquad \text{“faster than linear”}$$

This is opposed to big-O notation:

$$\lim_{\varepsilon \to 0} \frac{\mathcal{O}(\varepsilon)}{\varepsilon} = c$$

- δu has direction and scale so we could write it as

$$\delta u = \varepsilon v, \quad \varepsilon \in \mathbb{R},\ v \in \mathbb{R}^m$$

Theorem. *For u^* to be a minimizer, we need*

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or, equivalently,

$$\frac{\partial g}{\partial u}(u^*)v = 0 \quad \forall v \in \mathbb{R}^m$$

Proof. Let u^* be a minimizer. Evaluating the cost $g(u)$ in the ball and using Taylor's expansion,

$$g(u^* + \delta u) = g(u^*) + \frac{\partial g}{\partial u}(u^*)\delta u + o(\|\delta u\|) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$

Assume that $\frac{\partial g}{\partial u} \neq 0$. Then we could pick $v = -\frac{\partial g^{\mathrm{T}}}{\partial u}(u^*)$, i.e.

$$g(u^* + \varepsilon v) = g(u^*) - \varepsilon \left\| \frac{\partial g^{\mathrm{T}}}{\partial u}(u^*) \right\|^2 + o(\varepsilon)$$

Note that the second term is negative per our assumptions. So, for ε sufficiently small, we have

$$g\Big(u^* - \varepsilon \frac{\partial g^{\mathrm{T}}}{\partial u}(u^*)\Big) < g(u^*)$$

This contradicts u^* being a minimizer. ⚔ (crossed swords) □

Definition (Positive definite)**.** $M = M^{\mathrm{T}} \succ 0$ if

$$z^{\mathrm{T}} M z > 0 \quad \forall z \neq 0,\ z \in \mathbb{R}^m$$
$$\iff M \text{ has real and positive eigenvalues}$$

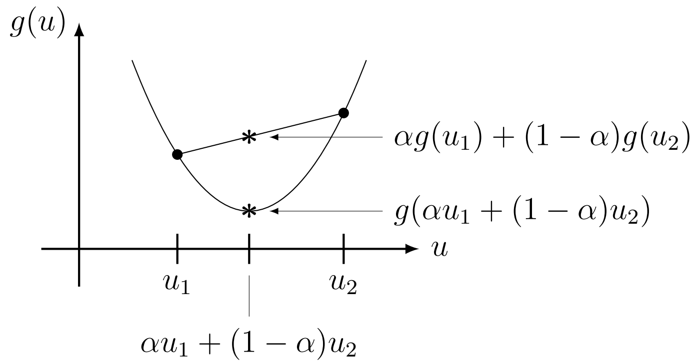
Theorem. *If $g \in C^2$, then a **sufficient** condition for u^* to be a (local) minimizer is*

1. $\dfrac{\partial g}{\partial u}(u^*) = 0$

2. $\dfrac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ *(the Hessian is positive definite)*

Definition. $g : \mathbb{R}^m \to \mathbb{R}$ is convex if

$$g(\alpha u_1 + (1-\alpha)u_2) \le \alpha g(u_1) + (1-\alpha)g(u_2) \quad \forall \alpha \in [0,1],\ u_1,u_2 \in \mathbb{R}^m$$



Theorem. *If $\frac{\partial^2 g}{\partial u^2}(u) \succeq 0\ \forall u \in \mathbb{R}^m$, then g is convex. ($\iff$ for $g \in C^2$)*

Example $\min\limits_u u^{\mathrm{T}}Qu - b^{\mathrm{T}}u$ where $Q = Q^{\mathrm{T}} \succ 0$ (positive definite matrix)

$$\begin{aligned}\frac{\partial g}{\partial u} &= \frac{\partial}{\partial u}(u^{\mathrm{T}}Qu - b^{\mathrm{T}}u)\\ &= u^{\mathrm{T}}Q^{\mathrm{T}} + u^{\mathrm{T}}Q - b^{\mathrm{T}}\\ &= 2u^{\mathrm{T}}Q - b^{\mathrm{T}}\\ \frac{\partial^2 g}{\partial u^2} &= 2Q\end{aligned} \qquad \frac{\partial^2 g}{\partial u^2} = \begin{bmatrix} \frac{\partial^2 g}{\partial u_1^2} & \cdots & \frac{\partial^2 g}{\partial u_1 \partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 g}{\partial u_m^2} \end{bmatrix}$$

From $\frac{\partial g}{\partial u} = 2u^{\mathrm{T}}Q - b^{\mathrm{T}} = 0$,

$$u = \frac{1}{2}Q^{-1}b$$

To see whether this is a minimizer, consider the Hessian. Since $Q \succ 0$, it follows that $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ and $u^* = \frac{1}{2}Q^{-1}b$ is a (local) minimizer. Additionally, since $\frac{\partial^2 g}{\partial u^2} \succ 0$, g is convex and u^* is a global minimizer. In fact, since we have strict convexity ($\succ 0$ rather than $\succeq 0$), it is the unique global minimizer.

In optimal control, *local* is typically all we can ask for. In optimization, we can do better!

But wait, just because we know $\frac{\partial g}{\partial u} = 0$, it doesn't follow that we can actually find u^*...

1.3 Numerical Methods

Idea: $u_{k+1} = u_k + \text{step}_k$. What should step_k be? For small $\text{step}_k = \gamma_k v_k$,

$$g(u_k \cdot \text{step}_k) = g(u_k) + \frac{\partial g}{\partial u}(u_k) \cdot \text{step}_k + o(\|\text{step}_k\|) = g(u_k) + \gamma_k \frac{\partial g}{\partial u}(u_k) v_k + o(\gamma_k)$$

A perfectly reasonable choice of step direction is

$$v_k = -\frac{\partial g^{\mathrm{T}}}{\partial u}(u_k),$$

known as the *steepest descend* direction. This produces

$$g\Big(u_k - \gamma_k \frac{\partial g}{\partial u}(u_k)\Big) = g(u_k) - \gamma_k \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 + o(\gamma_k)$$

Steepest descent

$$u_{k+1} = u_k - \gamma_k \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)$$

Note:

- What should γ_k be?
- This method "pretends" that $g(u)$ is linear. If we pretend $g(u)$ is quadratic, we get

  $$u_{k+1} = u_k - \left(\frac{\partial^2 g}{\partial u^2}(u_k)\right)^{-1} \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k),$$

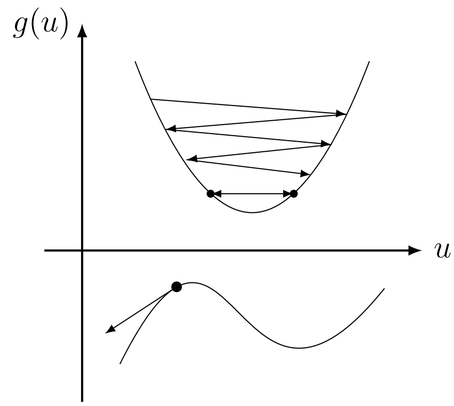
  i.e. Newton's Method

This course: steepest descent

Step-size selection?

- Choice 1: $\gamma_k = \gamma$ "small" $\forall k$; will get close to a minimizer if u_0 is close enough and γ small enough

  Problems:
  - You may not converge! (but you'll get close)
  - You may go off to infinity (diverge)



- Choice 2: Reduce γ_k as a function of k; will get close to a minimizer if u_0 is close enough

  Problem: slow

  Theorem. *If u_0 is close enough to u^* and γ_k satisfies*

  - $\displaystyle\sum_{k=0}^{\infty} \gamma_k = \infty$
  - $\displaystyle\sum_{k=0}^{\infty} \gamma_k^2 < \infty$

  e.g. $\gamma_k = c/k$, then $u_k \to u^*$ as $k \to \infty$.

- Choice 3: **Armijo step-size:** Take as big a step as possible, but no larger

  Pick $\alpha \in (0,1)$, $\beta \in (0,1)$. Let i be the smallest non-negative integer such that

$$g\left(u_k - \beta^i \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)\right) - g(u_k) < -\alpha\beta^i \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2$$
$$u_{k+1} = u_k - \beta^i \frac{\partial g^{\mathrm{T}}}{\partial u}(u_k)$$

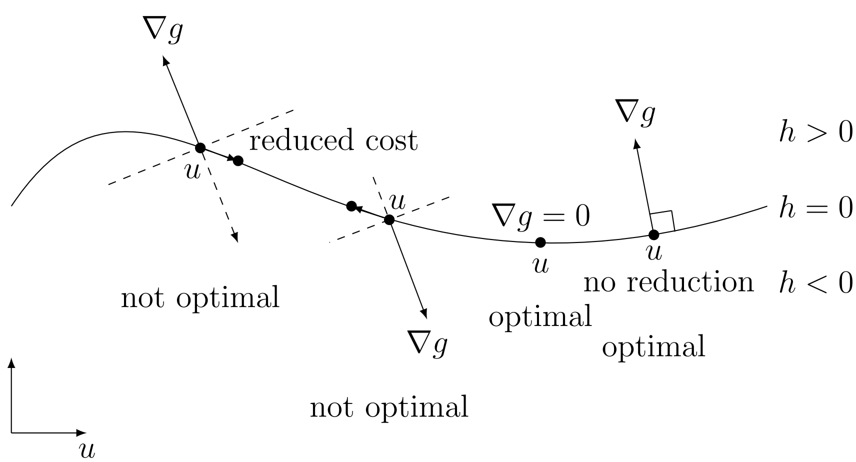
  This will get to a minimizer blazingly fast if u_0 is close enough.

1.4 Constrained Optimization

Equality constraints:

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \ & g(u) \\ \text{s.t.} \ & h(u) = \mathbf{0} \end{aligned}$$

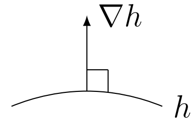
Consider $u \in \mathbb{R}^2$, $h : \mathbb{R}^2 \to \mathbb{R}$



So u is (locally) optimal if $\nabla g \parallel$ (is parallel to) the normal vector to tangent plane to h.

Fact: (HW# 1)

$$\nabla h \perp Th \quad \text{(tangent plane to h)}$$



We need $\nabla g \parallel \nabla h$ at u^* for optimality, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \alpha \frac{\partial h}{\partial u}(u^*), \quad \text{for some } \alpha \in \mathbb{R}$$

or ($\lambda = -\alpha$),

$$\frac{\partial g}{\partial u}(u^*) + \lambda \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}\big(g(u^*) + \lambda h(u^*)\big) = 0, \quad \text{for some } \lambda \in \mathbb{R}$$

More generally,

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \ & g(u) \\ \text{s.t. } & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k \end{aligned}$$

Note that $h(u) = [h_1(u), \dots, h_k(u)]^{\mathrm{T}}$.

We need $\frac{\partial g}{\partial u}(u^*)$ to be a linear combination of $\frac{\partial h_i}{\partial u}(u^*)$, $i = 1, \dots, k$, for exactly the same reasons, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \sum_{i=1}^{k} \alpha_i \frac{\partial h_i}{\partial u}(u^*)$$

or ($\lambda = -[\alpha_1, \dots, \alpha_k]^{\mathrm{T}}$)

$$\frac{\partial g}{\partial u}(u^*) + \lambda^{\mathrm{T}} \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}\big(g(u^*) + \lambda^{\mathrm{T}} h(u^*)\big) = 0, \quad \text{for some } \lambda \in \mathbb{R}^k$$

Theorem. *If u^* is a minimizer to*

$$\begin{aligned} \min_{u \in \mathcal{R}^m} \ & g(u) \\ s.t. \ & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k \end{aligned}$$

then $\exists \lambda \in \mathbb{R}^k$ s.t.

$$\begin{cases} \dfrac{\partial L}{\partial u}(u^*, \lambda) = 0 \\[2ex] \dfrac{\partial L}{\partial \lambda}(u^*, \lambda) = 0 \end{cases}$$

where the Lagrangian L is given by

$$L(u, \lambda) = g(u) + \lambda^T h(u)$$

Note:

- λ are the Lagrange multipliers
- $\frac{\partial L}{\partial \lambda} = 0$ is fancy speak for $h(u^*) = 0$

Example

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \ & \frac{1}{2}\|u\|^2 \\ \text{s.t.} \ & Au = b \end{aligned}$$

where A is $k \times m$, $k \leq m$. Assume $(AA^{\mathrm{T}})^{-1}$ exists (constraints are linearly independent, none of the constraints are "duplicates", all the constraints are essential).

$$\begin{aligned} L &= \frac{1}{2}u^{\mathrm{T}}u + \lambda^{\mathrm{T}}(Au - b) \\ \frac{\partial L}{\partial u} &= u^{\mathrm{T}} + \lambda^{\mathrm{T}}A = 0 \\ u^* &= -A^{\mathrm{T}}\lambda \end{aligned}$$

Using the equality constraint,

$$\begin{aligned} Au^* &= b \\ -AA^{\mathrm{T}}\lambda &= b \\ \lambda &= -(AA^{\mathrm{T}})^{-1}b \\ u^* &= A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}b \end{aligned}$$

Example

$$\begin{aligned} \min \ & u_1u_2 + u_2u_3 + u_1u_3 \\ \text{s.t.} \ & u_1 + u_2 + u_3 = 3 \end{aligned}$$

$$L = u_1u_2 + u_2u_3 + u_1u_3 + \lambda(u_1 + u_2 + u_3 - 3)$$

$$\begin{cases} \frac{\partial L}{\partial u_1} = u_2 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_2} = u_1 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_3} = u_2 + u_1 + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = u_1 + u_2 + u_3 = 3 \end{cases} \implies \begin{cases} u_1^* = 1 \\ u_2^* = 1 \\ u_3^* = 1 \\ \lambda = -2 \end{cases} \quad \text{optimal solution}$$

Note: This was actually the worst we can do—maximize! Even weirder: no local minimizer!

1.4.1 Equality Constraints

$$\begin{aligned}\min_{u\in\mathcal{R}^m} \quad & g(u)\\ \text{s.t.} \quad & h(u)=\mathbf{0}, \quad h:\mathbb{R}^m\to\mathbb{R}^k\end{aligned}$$

Theorem. *If u^* is a minimizer/maximizer then $\exists\lambda\in\mathbb{R}^k$ s.t.*

$$\begin{aligned}&\frac{\partial L}{\partial u}(u^*,\lambda)=0\\ &\frac{\partial L}{\partial \lambda}(u^*,\lambda)=0 \qquad (\Longleftrightarrow h(u^*)=0)\end{aligned}$$

where $L(u,\lambda)=g(u)+\lambda^T h(u)$.

Example [Entropy Maximization]

Given $S=\{x_1,\ldots,x_n\}$ and a distribution over S such that it takes the value x_j with probability p_j. The entropy is

$$E(p)=\sum_{j=1}^n(-p_j\ln p_j).$$

The mean is

$$m=\sum_{j=1}^n p_jx_j.$$

Problem: Given m, find p such that E is maximized.

$$\begin{aligned}\min_p \quad & -\sum_{j=1}^n p_j\ln p_j\\ \text{s.t.} \quad & \sum_{j=1}^n p_jx_j=m\\ & \sum_{j=1}^n p_j=1\\ & p_j\ge 0,\ j=1,\ldots,n \quad \text{(ignore this...)}\end{aligned}$$

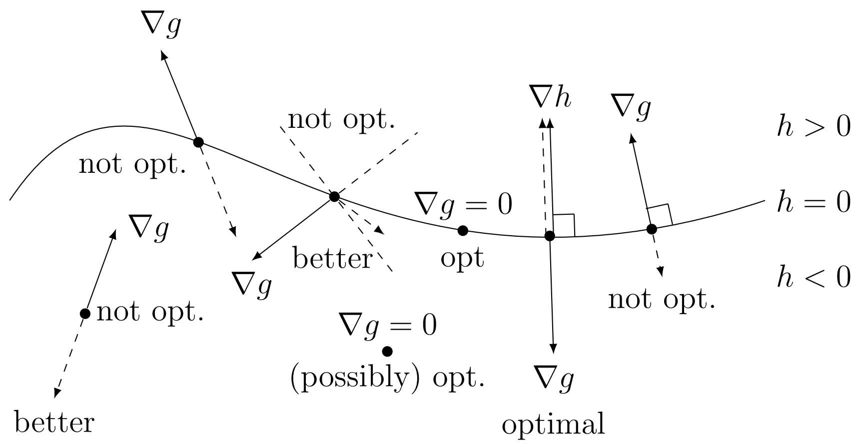
$$\begin{aligned}L&=-\sum p_j\ln p_j+\lambda_1\left[\sum p_jx_j-m\right]+\lambda_2\left[\sum p_j-1\right]\\ \frac{\partial L}{\partial p_j}&=-\ln p_j-1+\lambda_1x_j+\lambda_2=0\\ p_j&=e^{\lambda_2-1+\lambda_1x_j}, \quad j=1,\ldots,n \qquad (p_j\ge 0 \text{ so we're ok with ignoring that})\end{aligned}$$

$$\begin{aligned}\sum e^{\lambda_2-1+\lambda_1x_j}x_j&=m \qquad\qquad & n+2 \text{ equations and}\\ \sum e^{\lambda_2-1+\lambda_1x_j}&=1 & n+2 \text{ unknowns...}\end{aligned}$$

No analytical solution, but numerically "solvable"

1.4.2 Inequality Constraints

$$
\begin{aligned}
\min_{u \in \mathcal{R}^m} \quad & g(u) \\
\text{s.t.} \quad & h(u) \leq \mathbf{0}, \quad h : \mathbb{R}^m \to \mathbb{R}^k
\end{aligned}
$$



We need:

- if $h(u^*) < 0$ then $\frac{\partial g}{\partial u}(u^*) = 0$
- if $h(u^*) = 0$ then we need either

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or

$$\frac{\partial g}{\partial u}(u^*) = -\lambda \frac{\partial h}{\partial u}(u^*) \quad \text{for } \lambda > 0$$

Or, even better,

$$\frac{\partial}{\partial u}\left(g(u^*) + \lambda h(u^*)\right) = 0 \quad \text{for } \lambda \geq 0,$$

where $\lambda h(u^*) = 0$. ($h < 0 \to \lambda = 0$, $h = 0 \to \lambda \geq 0$)

In general, if $u \in \mathbb{R}^m$ and $h : \mathbb{R}^m \to \mathbb{R}^k$, we have that u^*, if optimal, has to satisfy

$$
\begin{aligned}
\frac{\partial}{\partial u} L(u^*, \lambda) &= 0 \\
h(u^*) &\leq \mathbf{0} \\
\lambda^{\mathrm{T}} h(u^*) &= 0 \\
\lambda &\geq \mathbf{0}
\end{aligned}
$$

where the Lagrangian is $L(u, \lambda) = g(u) + \lambda^{\mathrm{T}} h(u)$. Note that if we're maximizing, the same holds except we need $\lambda \leq 0$.

Example

$$\min \; 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2$$
$$\text{s.t.} \; \begin{cases} u_1^2 + u_2^2 \leq 5 \\ 3u_1 + u_2 \leq 6 \end{cases}$$

$$L = 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 + \lambda_1(u_1^2 + u_2^2 - 5) + \lambda_2(3u_1 + u_2 - 6)$$

FONC:

i) $\partial L/\partial u_1 = 4u_1 + 2u_2 - 10 + 2\lambda_1 u_1 + 3\lambda_2$

ii) $\partial L/\partial u_2 = 2u_1 + 2u_2 - 10 + 2\lambda_1 u_2 + \lambda_2$

iii) $u_1^2 + u_2^2 \leq 5$

iv) $3u_1 + u_2 \leq 6$

v) $\lambda_1(u_1^2 + u_2^2 - 5) = 0$

vi) $\lambda_2(3u_1 + u_2 - 6) = 0$

vii) $\lambda_1 \geq 0$

viii) $\lambda_2 \geq 0$

To solve, assume different constraints are active/inactive:

1. Both constraints are inactive ($u_1^2 + u_2^2 < 5$, $3u_1 + u_2 < 6$) $\Longrightarrow \lambda_1 = \lambda_2 = 0$

$$\begin{cases} 4u_1 + 2u_2 - 10 = 0 \\ 2u_1 + 2u_2 - 10 = 0 \end{cases} \Longrightarrow \begin{cases} u_1 = 0 \\ u_2 = 5 \end{cases}$$

   Note: iii) $0^2 + 5^2 \not\leq 5$

   Not feasible

2. Assume constraint 1 is active and constraint 2 is inactive ($u_1^2 + u_2^2 = 5$, $\lambda_2 = 0$)

$$\begin{cases} 4u_1 + 2u_2 - 10 + 2\lambda_1 u_1 = 0 \\ 2u_1 + 2u_2 - 10 + 2\lambda_1 u_2 = 0 \\ u_1^2 + u_2^2 = 5 \end{cases} \Longrightarrow \begin{cases} u_1 = 1 \\ u_2 = 2 \\ \lambda_1 = 1 \end{cases}$$

   ☑ $\lambda_1 \geq 0$

   ☑ $3 \cdot 1 + 2 \leq 6$

   This is a local minimizer

3. Assume constraint 2 is active and constraint 1 is inactive

4. Assume both constraints are active

Kuhn-Tucker Conditions (KKT conditions, Karush-Kuhn-Tucker)

Problem:

$$\begin{aligned} &\min_{u\in\mathbb{R}^m} \; g(u) \\ &\text{s.t.} \; \begin{cases} h_1(u) = 0, & h_1 : \mathbb{R}^m \to \mathbb{R}^p \\ h_2(u) \le 0, & h_2 : \mathbb{R}^m \to \mathbb{R}^k \end{cases} \end{aligned} \tag{1.1}$$

Theorem. *Let u^* be feasible ($h_1 = 0$, $h_2 \le 0$). If u^* is a minimizer to (1.1) than there exists vectors $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^k$ with $\mu \ge \mathbf{0}$ such that*

$$\begin{cases} \dfrac{\partial g}{\partial u}(u^*) + \lambda^T \dfrac{\partial h_1}{\partial u}(u^*) + \mu^T \dfrac{\partial h_2}{\partial u}(u^*) = 0 \\ \mu^T h_2(u^*) = 0 \end{cases}$$

Looking ahead: $\min \text{cost}(u(\cdot))$ s.t. $\dot{x} = f(x,u)$ (dynamics), where u is a function. Note the equality constraint.

Question: How do we go from $u \in \mathbb{R}^m$ to $u \in \mathcal{U}$ (function space)?

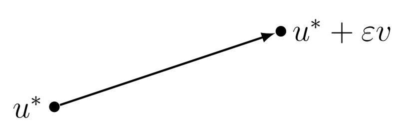
Note: Function space is a set of functions of a given kind from a set X to a set Y

1. linear function
2. square-integrable functions: $L_2[0,T] : \int_0^T \|u(t)\|^2 \, \mathrm{d}t < \infty$
3. $C^\infty(\mathbb{R})$

What would ∂"cost"$/\partial u$ mean?

1.5 Directional Derivatives

Recall: To minimize $g(u)$, let u^* be a candidate minimizer and pitch a perturbation on u^* of εv, where ε is the scale and v is the direction. Taking Taylor's expansion at the perturbation produces



$$g(u^* + \varepsilon v) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*) v + o(\varepsilon)$$

$$\text{FONC: } \frac{\partial g}{\partial u}(u^*) = 0$$

Note: $\frac{\partial g}{\partial u}(u^*)v$ tells us how much $g(u)$ increases/decreases in the direction of v.

Definition. The directional (Gateaux) derivative is given by

$$\delta g(u; v) = \lim_{\varepsilon \to 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon}$$

Example

$$g(u) = \frac{1}{2}u_1^2 - u_1 + 2u_2, \quad g : \mathbb{R}^2 \to \mathbb{R}$$

Let's consider $e_1 = [1\ 0]^{\mathrm{T}}$, $e_2 = [0\ 1]^{\mathrm{T}}$. What is $\delta g(u; e_i)$, $i = 1, 2$?

$$\begin{aligned}
\delta g(u; v) &= \lim_{\varepsilon \to 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{g(u) + \varepsilon \frac{\partial g}{\partial u}(u)v + o(\varepsilon) - g(u)}{\varepsilon} \\
&= \frac{\partial g}{\partial u}(u)v
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g}{\partial u}(u) &= [u_1 - 1\ 2] \\
\delta g(u; e_1) &= [u_1 - 1\ 2]e_1 = u_1 - 1 \\
\delta g(u; e_2) &= [u_1 - 1\ 2]e_2 = 2
\end{aligned}$$

But the beauty of directional derivatives is that they generalize beyond vectors, $u \in \mathbb{R}^m$, to function spaces ($\mathcal{U}$) or other "objects" like matrices.

Example $M \in \mathbb{R}^{n \times n}$, $F(M) = M^2$
What is $\frac{\partial F}{\partial M}$? (ponder at home...)
We can easily compute $\delta F(M; N)$!

$$\begin{aligned}
F(M + \varepsilon N) = (M + \varepsilon N)(M + \varepsilon N) &= M^2 + \varepsilon MN + \varepsilon NM + \varepsilon^2 N^2 \\
\delta F(M; N) = \lim_{\varepsilon \to 0} \frac{F(M + \varepsilon N) - F(M)}{\varepsilon}& \\
= \lim_{\varepsilon \to 0} \frac{\varepsilon MN + \varepsilon NM + \varepsilon^2 N^2}{\varepsilon} &= MN + NM
\end{aligned}$$

Infinite Dimensional Optimization Let $u \in \mathcal{U}$ (function space) and let $J(u)$ be the cost:

$$\min_{u \in \mathcal{U}} J(u)$$

Theorem. *If $u^* \in \mathcal{U}$ is a (local) minimizer then*

$$\delta J(u^*; v) = 0, \quad \forall v \in \mathcal{U}$$

Example Find minimizer u^* to

$$J(u) = \int_0^T L(u(t))\,\mathrm{d}t$$

$$\begin{aligned}
J(u+\varepsilon v) - J(u) &= \int_0^T L(u(t)+\varepsilon v(t))\,\mathrm{d}t - \int_0^T L(u(t))\,\mathrm{d}t, \quad u, v \in \mathcal{U} \\
&= \int_0^T \left[L(u(t)) + \varepsilon \frac{\partial L}{\partial u}(u(t))v(t) + o(\varepsilon) - L(u(t)) \right] \mathrm{d}t \\
\delta J(u^*; v) &= \lim_{\varepsilon \to 0} \frac{J(u+\varepsilon v) - J(u)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\int_0^T \varepsilon \frac{\partial L}{\partial u}(u(t))v(t)\,\mathrm{d}t + o(\varepsilon)}{\varepsilon} \\
&= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t)\,\mathrm{d}t
\end{aligned}$$

u^* optimizer:

$$\delta J(u^*; v) = \int_0^T \frac{\partial L}{\partial u}(u(t))v(t)\,\mathrm{d}t = 0 \quad \forall v \in \mathcal{U}$$

$$\Updownarrow$$

$$\frac{\partial L}{\partial u}(u(t)) = 0 \quad \forall t \in [0, T]$$

But, we want *optimal control*! We want our cost to look like

$$\int_0^T L(x(t), u(t))\,\mathrm{d}t$$

$$\dot{x} = f(x, u)$$

1.6 Calculus of Variations

What happens to $x(t)$ when $u(t)$ changes to $u(t) + \varepsilon v(t)$? Let the system be given by

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

After perturbation of u, the new system is

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}, u + \varepsilon v) \\ x(0) = x_0 \end{cases}$$

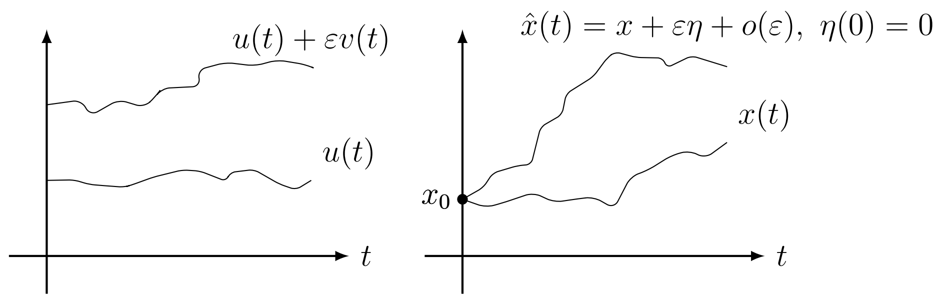


Figure 1.1: Variation in u causes a variation in x.

Consider

$$\tilde{x} = x + \varepsilon\eta,$$

where

$$\begin{aligned} \dot{x} &= f(x,u), && x(0) = x_0 \\ \dot{\eta} &= \frac{\partial f}{\partial x}(x,u)\eta + \frac{\partial f}{\partial u}(x,u)v, && \eta(0) = 0 \end{aligned}$$

Theorem. *If f is continuously differentiable in x and u then*

$$\hat{x}(t) = \tilde{x}(t) + o(\varepsilon)$$

Proof.

i) Initial conditions:

$$\begin{aligned} &\hat{x}(0) = x_0 \\ &\tilde{x}(0) = x(0) + \varepsilon\eta(0) = x_0 \end{aligned}$$

ii) Dynamics:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v) \\ \dot{\tilde{x}} &= \dot{x} + \varepsilon\dot{\eta} = f(x,u) + \varepsilon\frac{\partial f}{\partial x}(x,u)\eta + \varepsilon\frac{\partial f}{\partial u}(x,u)v \\ &= f(x + \varepsilon\eta, u + \varepsilon v) + o(\varepsilon) \\ &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon) \end{aligned}$$

We can see that the dynamics of $\hat{x}(t)$ are equal to those of $\tilde{x}(t)$ plus higher order terms:

$$\begin{aligned} \dot{\tilde{x}} &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon) \\ \dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v) \end{aligned}$$

Therefore, if our perturbation is small enough, we can model $\hat{x}(t)$ as $\tilde{x}(t)$.

□

Note: Taylor expansion with two elements is

$$
\begin{aligned}
h(w+\varepsilon v, z+\varepsilon y) &= h(w, z+\varepsilon y) + \frac{\partial h}{\partial w}(w, z+\varepsilon y)\varepsilon v + o(\varepsilon) \\
&= \left\{ h(w,z) + \frac{\partial h}{\partial z}(w,z)\varepsilon y + o(\varepsilon) \right\} \\
&\quad + \left\{ \frac{\partial h}{\partial w}(w,z)\varepsilon v + \underbrace{\frac{\partial^2 h}{\partial z \partial w}\varepsilon v \odot \varepsilon y}_{o(\varepsilon)} + o(\varepsilon) \right\} \\
&= h(w,z) + \frac{\partial h}{\partial z}\varepsilon y + \frac{\partial h}{\partial w}\varepsilon v + o(\varepsilon)
\end{aligned}
$$

Last class:

1. $u \in \mathcal{U}$ (space of functions), $J : \mathcal{U} \to \mathbb{R}$ (cost).

   FONC: If u^* is optimal, then

   $$\delta J(u; \nu) = 0 \quad \forall \nu \in \mathcal{U},$$

   where the directional derivative is given by

   $$\delta J(u; \nu) = \lim_{\varepsilon \to 0} \frac{J(u+\varepsilon\nu) - J(u)}{\varepsilon}.$$

2. If

   $$\begin{cases} \dot{x} = f(x,u) \\ x(0) = x_0 \end{cases}$$

   then a variation in u:

   $$u \longmapsto u + \varepsilon \nu$$

   results in a variation in x:

   $$x \longmapsto x + \varepsilon \eta + o(\varepsilon)$$

   See Figure 1.1. Note $\eta(0) = 0$.

1.6.1 An (Almost) Optimal Control Problem

Let $\dot{x} = f(x)$, $x(0) = x_0$. Note we get to pick the initial condition!

Problem

$$\min_{x_0\in\mathbb{R}^m} J(x_0) = \int_0^T L(x(t))\,\mathrm{d}t$$

$$\text{s.t.}\quad \begin{cases} \dot{x}(t) = f(x(t)) & \text{the } \textit{constraint}! \text{ (equality)} \\ x(0) = x_0 \end{cases}$$

Note every constraint needs a Lagrange multiplier. We have infinitely many constraints:

$$\dot{x}(t) = f(x(t)) \quad \forall t \in [0, T]$$

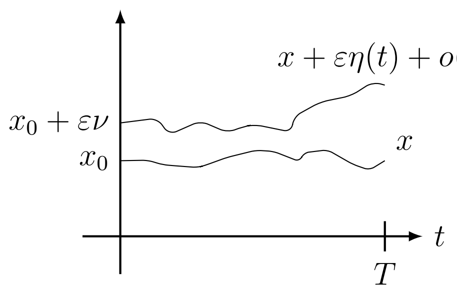
We need $\lambda(t)$ as a function of t. Also, the sum in the Lagrangian has to become an integral. The continuous-time Lagrangian thus becomes

$$\tilde{J}(x_0, \lambda) = \int_0^T \Big[L(x(t)) + \lambda^{\mathrm{T}}(t)(f(x(t)) - \dot{x}(t))\Big]\,\mathrm{d}t$$

The task is to perturb x_0 as $x_0 \longmapsto x_0 + \varepsilon\nu$, $\nu \in \mathbb{R}^m$ and compute

$$\delta\tilde{J}(x_0;\nu) = \lim_{\varepsilon\to 0} \frac{\tilde{J}(x_0+\varepsilon\nu) - \tilde{J}(x_0)}{\varepsilon}$$

and make this equal to 0 $\forall \nu \in \mathbb{R}^m$. The variation in x is



Note:

x_0	decision variable
ν	variation in x_0
$x(t)$	trajectory starting at x_0
$\eta(t)$	change in trajectory resulting from ν-variation in x_0
$\lambda(t)$	time-varying Lagrange multiplier

$$\begin{aligned}
\tilde{J}(x_0+\varepsilon\nu) &= \int_0^T \Big\{L(x(t)) + \lambda^{\mathrm{T}}(t)[f(x(t)+\varepsilon\eta(t)) - \dot{x}(t) - \varepsilon\dot{\eta}(t)]\Big\}\,\mathrm{d}t + o(\varepsilon) \\
&= \int_0^T \left[L(x) + \varepsilon\frac{\partial L}{\partial x}(x)\eta + \lambda^{\mathrm{T}}\left(f(x) + \varepsilon\frac{\partial f}{\partial x}(x)\eta - \dot{x} - \varepsilon\dot{\eta}\right)\right]\mathrm{d}t + o(\varepsilon) \\
\tilde{J}(x_0+\varepsilon\nu) - \tilde{J}(x_0) &= \int_0^T \left[\varepsilon\frac{\partial L}{\partial x}(x)\eta + \lambda^{\mathrm{T}}\left(\varepsilon\frac{\partial f}{\partial x}\eta - \varepsilon\dot{\eta}\right)\right]\mathrm{d}t + o(\varepsilon) \\
\delta\tilde{J}(x_0;\nu) &= \int_0^T \left[\frac{\partial L}{\partial x}(x)\eta + \lambda^{\mathrm{T}}\left(\frac{\partial f}{\partial x}\eta - \dot{\eta}\right)\right]\mathrm{d}t
\end{aligned}$$

A powerful idea: we want $\delta\tilde{J}(x_0;\nu) = 0\ \forall \nu$. Somehow get this in the form

$$\int_0^T \Big(\text{stuff}(t)\Big)\eta(t)\,\mathrm{d}t = 0$$

We can pick $\text{stuff}(t) = 0\ \forall t \in [0,T]$.

In $\delta\tilde{J}(x_0;\nu)$ we have $\dot{\eta}$ (problem!). We can solve this using *integration by parts.*

$$\int_0^T \lambda^{\mathrm{T}}\dot{\eta}\,\mathrm{d}t = \lambda^{\mathrm{T}}(T)\eta(T) - \lambda^{\mathrm{T}}(0)\eta(0) - \int_0^T \dot{\lambda}^{\mathrm{T}}\eta\,\mathrm{d}t$$

Hence,

$$\delta\tilde{J}(x_0;\nu) = \int_0^T \underbrace{\Big(\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial x} + \dot{\lambda}^{\mathrm{T}}\Big)}_{\text{pick}=0}\eta\,\mathrm{d}t - \underbrace{\lambda^{\mathrm{T}}(T)}_{\text{pick}=0}\eta(T) + \lambda^{\mathrm{T}}(0)\underbrace{\eta(0)}_{\nu}$$

We are free to pick λ freely if it gives $\delta\tilde{J} = 0$.

$$\text{Pick: } \begin{cases} \dot{\lambda}(t) = -\frac{\partial L^{\mathrm{T}}}{\partial x}(x(t)) - \frac{\partial f^{\mathrm{T}}}{\partial x}(x(t))\lambda(t) \\ \lambda(T) = 0 \end{cases} \qquad \text{backwards diff. eq}$$

Under this choice of λ we get

$$\delta\tilde{J}(x_0;\nu) = \lambda^{\mathrm{T}}(0)\nu$$

This is linear in ν so the FONC is $\lambda(0) = 0$.

Moreover, we really have a "normal" optimization problem

$$\min_{x_0\in\mathbb{R}^m} \tilde{J}(x_0)$$

$$\delta\tilde{J}(x_0;\nu) = \frac{\partial\tilde{J}}{\partial x_0}(x_0)\nu$$

which means that

$$\frac{\partial\tilde{J}}{\partial x_0} = \lambda^{\mathrm{T}}(0)$$

If x_0^* minimizes

$$\int_0^T L(x(t))\,\mathrm{d}t$$

$$\text{s.t. } \begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0^* \end{cases}$$

then

$$\lambda(0) = \mathbf{0}$$

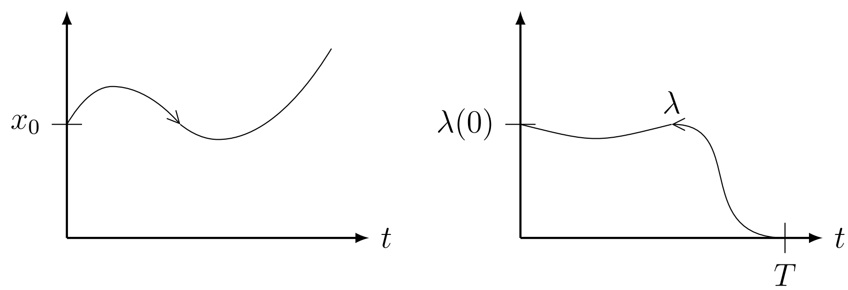
where $\lambda(t)$ satisfies

$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial L^{\mathrm{T}}}{\partial x}(x(t)) - \frac{\partial f^{\mathrm{T}}}{\partial x}(x(t))\lambda(t) \\ \lambda(T) = 0 \end{cases}$$

So what? We actually have a two-point boundary value problem.

$$\dot{x} = f(x) \qquad \dot{\lambda} = -\frac{\partial L^{\mathrm{T}}}{\partial x} - \frac{\partial f^{\mathrm{T}}}{\partial x}\lambda$$

$$x(0) = x_0 \qquad \lambda(T) = 0$$



We want to find x_0 that gives $f(x)$ such that after solving backwards for $\lambda(t)$, we find that

$$\lambda(0) = \frac{\partial \tilde{J}^{\mathrm{T}}}{\partial x_0} = 0.$$

This leads to the following:

An algorithm

Pick $x_{0,0}$
$k = 1$
repeat
  Simulate $x(t)$ from $x_{0,k}$ over $[0, T]$
  Simulate $\lambda(t)$ from $\lambda(T) = 0$ backwards using $x(t)$
  Update $x_{0,k}$ as $x_{0,k+1} = x_{0,k} - \gamma\lambda(0)$ ▷ $\lambda(0)$ is the gradient
  $k := k + 1$
until $\lambda(0) = 0$

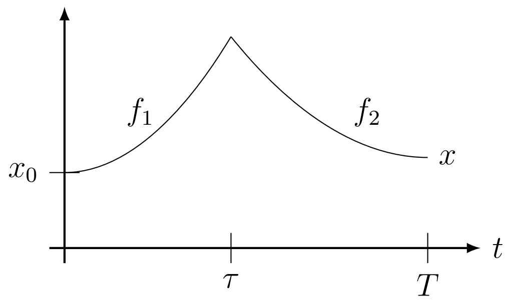
Example: `optinit.m`

$$\dot{x} = Ax, \quad L = x^{\mathrm{T}}Qx - q, \quad Q = Q^{\mathrm{T}} \succ 0$$
$$\dot{\lambda} = -2Qx - A^{\mathrm{T}}\lambda$$
$$\lambda(0) = 0$$

1.6.2 Optimal Timing Control

When to switch between modes?

$$\dot{x} = \begin{cases} f_1(x) & \text{if } t \in [0, \tau) \\ f_2(x) & \text{if } t \in [\tau, T] \end{cases} \tag{1.2}$$
$$x(0) = x_0$$

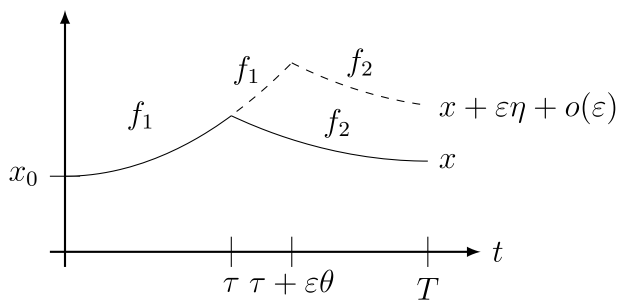


$$\min_{\tau} \int_0^T L(x(t))\,\mathrm{d}t = J(\tau)$$
$$\text{s.t. (1.2) holds}$$

Step 1: Augment cost with constraint

$$\tilde{J} = \int_0^{\tau} \Big[L(x) + \lambda^{\mathrm{T}}(f_1(x) - \dot{x})\Big]\,\mathrm{d}t + \int_{\tau}^{T} \Big[L(x) + \lambda^{\mathrm{T}}(f_2(x) - \dot{x})\Big]\,\mathrm{d}t$$

Step 2: Variation $\tau \longmapsto \tau + \varepsilon\theta$



Step 3: Compute $\delta\tilde{J}(\tau;\theta)$

$$\begin{aligned}\tilde{J}(\tau + \varepsilon\theta) = &\int_0^{\tau+\varepsilon\theta} \Big\{L(x+\varepsilon\eta) + \lambda^{\mathrm{T}}[f_1(x+\varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}]\Big\}\,\mathrm{d}t \\ &+ \int_{\tau+\varepsilon\theta}^{T} \Big\{L(x+\varepsilon\eta) + \lambda^{\mathrm{T}}[f_2(x+\varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}]\Big\}\,\mathrm{d}t + o(\varepsilon)\end{aligned}$$

Note that $\eta = \dot{\eta} = 0$ on $[0, \tau)$.

$$\begin{aligned}\tilde{J}(\tau + \varepsilon\theta) = &\int_0^{\tau} \Big\{L(x) + \lambda^{\mathrm{T}}[f_1(x) - \dot{x}]\Big\}\,\mathrm{d}t \\ &+ \int_{\tau}^{\tau+\varepsilon\theta} \Big\{\underbrace{L(x+\varepsilon\eta)}_{L(x)+\varepsilon\frac{\partial L}{\partial x}\eta} + \lambda^{\mathrm{T}}[\underbrace{f_1(x+\varepsilon\eta)}_{f_1(x)+\varepsilon\frac{\partial f_1}{\partial x}\eta} - \dot{x} - \varepsilon\dot{\eta}]\Big\}\,\mathrm{d}t \\ &+ \int_{\tau+\varepsilon\theta}^{T} \Big\{\underbrace{L(x+\varepsilon\eta)}_{L(x)+\varepsilon\frac{\partial L}{\partial x}\eta} + \lambda^{\mathrm{T}}[\underbrace{f_2(x+\varepsilon\eta)}_{f_2(x)+\varepsilon\frac{\partial f_2}{\partial x}\eta} - \dot{x} - \varepsilon\dot{\eta}]\Big\}\,\mathrm{d}t + o(\varepsilon)\end{aligned}$$

$$
\begin{aligned}
\delta\tilde{J}(\tau;\theta) &= \lim_{\varepsilon\to 0}\frac{\tilde{J}(\tau+\varepsilon\theta)-\tilde{J}(\tau)}{\varepsilon} \\
\tilde{J}(\tau+\varepsilon\theta)-\tilde{J}(\tau) &= \int_0^\tau 0\cdot \mathrm{d}t + \underbrace{\int_\tau^{\tau+\varepsilon\theta}\left[\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\Big(f_1(x)+\varepsilon\frac{\partial f_1}{\partial x}\eta - f_2(x)-\varepsilon\dot{\eta}\Big)\right]\mathrm{d}t}_{I_1} \\
&\quad + \underbrace{\int_{\tau+\varepsilon\theta}^{T}\left[\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\Big(\varepsilon\frac{\partial f_2}{\partial x}\eta-\varepsilon\dot{\eta}\Big)\right]\mathrm{d}t}_{I_2} + o(\varepsilon)
\end{aligned}
$$

Theorem (Mean-value theorem)**.**

$$
\int_{t_1}^{t_2} h(t)\,\mathrm{d}t = (t_2-t_1)h(\xi) \quad \textit{for some } \xi\in[t_1,t_2]
$$

The first integral is

$$
\begin{aligned}
I_1 &= \int_\tau^{\tau+\varepsilon\theta}\left\{\varepsilon\frac{\partial L}{\partial x}\eta + \lambda^{\mathrm{T}}\left[f_1(x)+\varepsilon\frac{\partial f_1}{\partial x}\eta-\varepsilon\dot{\eta}-f_2(x)\right]\right\}\mathrm{d}t \\
&= \varepsilon\theta\Big\{\lambda^{\mathrm{T}}(\xi)\big[f_1(x(\xi))-f_2(x(\xi))\big]\Big\}+o(\varepsilon)
\end{aligned}
$$

Note that as $\varepsilon\to 0$, $\xi\to\tau$. Using integration by parts, the second integral is

$$
\int_\tau^T \lambda^{\mathrm{T}}\dot{\eta}\,\mathrm{d}t = \lambda^{\mathrm{T}}(T)\eta(T)-\lambda^{\mathrm{T}}(\tau)\underbrace{\eta(\tau)}_{=0}-\int_\tau^T\dot{\lambda}^{\mathrm{T}}\eta\,\mathrm{d}t
$$

$$
\begin{aligned}
I_2 &= \int_\tau^T\left[\varepsilon\frac{\partial L}{\partial x}\eta+\lambda^{\mathrm{T}}\Big(\varepsilon\frac{\partial f_2}{\partial x}\eta-\varepsilon\dot{\eta}\Big)\right]\mathrm{d}t - \underbrace{\int_\tau^{\tau+\varepsilon\theta}\left[\varepsilon\frac{\partial L}{\partial x}\eta+\lambda^{\mathrm{T}}\Big(\varepsilon\frac{\partial f_2}{\partial x}\eta-\varepsilon\dot{\eta}\Big)\right]\mathrm{d}t}_{o(\varepsilon)} \\
&= \varepsilon\int_\tau^T\left[\frac{\partial L}{\partial x}+\lambda^{\mathrm{T}}\frac{\partial f_2}{\partial x}+\dot{\lambda}^{\mathrm{T}}\right]\eta\,\mathrm{d}t-\varepsilon\lambda^{\mathrm{T}}(T)\eta(T)+o(\varepsilon)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\delta\tilde{J}(\tau;\theta) &= \lim_{\varepsilon\to 0}\frac{\tilde{J}(\tau+\varepsilon\theta)-\tilde{J}(\tau)}{\varepsilon} \\
&= \theta\lambda^{\mathrm{T}}(\tau)\Big[f_1(x(\tau))-f_2(x(\tau))\Big]+\int_\tau^T\left[\frac{\partial L}{\partial x}+\lambda^{\mathrm{T}}\frac{\partial f_2}{\partial x}+\dot{\lambda}^{\mathrm{T}}\right]\eta\,\mathrm{d}t-\lambda^{\mathrm{T}}(T)\eta(T)
\end{aligned}
$$

Step 4: Select the *costate* $\lambda(t)$. The key idea is to get rid of any term that has η in it, i.e.

$$
\begin{aligned}
\dot{\lambda} &= -\frac{\partial L^{\mathrm{T}}}{\partial x}-\frac{\partial f_2^{\mathrm{T}}}{\partial x}\lambda \quad \text{on } [\tau,T] \\
\lambda(T) &= 0
\end{aligned}
$$

Step 5: With this choice of $\lambda(t)$, we have

$$\delta \tilde{J}(\tau;\theta) = \theta \lambda^{\mathrm{T}}(\tau)\Big[f_1(x(\tau)) - f_2(x(\tau))\Big] = \frac{\partial \tilde{J}}{\partial \tau}\theta.$$

Therefore,

$$\frac{\partial \tilde{J}}{\partial \tau} = \lambda^{\mathrm{T}}(\tau)\Big[f_1(x(\tau)) - f_2(x(\tau))\Big] = 0 \quad \text{(for optimality)}$$

Algorithm

Pick τ_0
$k = 0$
repeat
  Simulate x forward in time from $x(0) = x_0$
  Simulate λ backwards from $\lambda(T) = 0$
  Update τ_k as $\tau_{k+1} = \tau_k - \gamma \lambda^{\mathrm{T}}(\tau_k)\big[f_1(x(\tau_k)) - f_2(x(\tau_k))\big]$
  $k := k + 1$
until $\|\lambda^{\mathrm{T}}(f_1 - f_2)\| < \varepsilon$

Where are we going? Come up with general principles for $\min_{u\in\mathcal{U}} J(u)$:

- Costate equations
- Optimality conditions
- Algorithms
- Applications

1.6.3 The Bolza Problem

Up until now, we have optimized with respect to finite-dimensional parameters. Today, we will minimize with respect to $u \in \mathcal{U}$.

$$\min_{u\in\mathcal{U}} J(u) = \int_0^T L(x(t), u(t), t)\,\mathrm{d}t + \underbrace{\Psi(x(T))}_{\substack{\text{terminal cost}\\ \text{(parking cost)}}}$$

$$\text{s.t.} \quad \dot{x}(t) = f(x(t), u(t), t)$$
$$x(0) = x_0$$

Assume that f and L are C^1 in x, u and piecewise continuous in t. Then, a small change in u causes small changes in f and L. The variation: $u \longmapsto u + \varepsilon v$, $\varepsilon \in \mathbb{R}$, $v \in \mathcal{U}$. See Figure 1.1.

$$
\begin{aligned}
\tilde{J}(u) &= \int_0^T \big[L(x,u,t) + \lambda^{\mathrm{T}}(f(x,u,t) - \dot{x})\big]\,\mathrm{d}t + \Psi(x(T)) \\
\tilde{J}(u+\varepsilon v) &= \int_0^T \big[L(x+\varepsilon\eta, u+\varepsilon v, t) + \lambda^{\mathrm{T}}(f(x+\varepsilon\eta, u+\varepsilon v, t) - \dot{x} - \varepsilon\dot{\eta})\big]\,\mathrm{d}t \\
&\quad + \Psi(x(T) + \varepsilon\eta(T)) + o(\varepsilon) \\
\tilde{J}(u+\varepsilon v) - \tilde{J}(u) &= \int_0^T \Bigg[L(x+\varepsilon\eta, u+\varepsilon v, t) - L(x,u,t) \\
&\qquad + \lambda^{\mathrm{T}}\Big(f(x+\varepsilon\eta, u+\varepsilon v, t) - f(x,u,t) - \dot{x} - \varepsilon\dot{\eta} + \dot{x}\Big)\Bigg]\,\mathrm{d}t \\
&\quad + \Psi(x(T) + \varepsilon\eta(T)) - \Psi(x(T)) + o(\varepsilon) \\
&= \int_0^T \left[\frac{\partial L}{\partial x}\varepsilon\eta + \frac{\partial L}{\partial u}\varepsilon v + \lambda^{\mathrm{T}}\left(\frac{\partial f}{\partial x}\varepsilon\eta + \frac{\partial f}{\partial u}\varepsilon v - \varepsilon\dot{\eta}\right)\right]\mathrm{d}t \\
&\quad + \frac{\partial \Psi}{\partial x}(x(T))\varepsilon\eta(T) + o(\varepsilon)
\end{aligned}
$$

(See Taylor expansion with respect to two variables.)

$$
\begin{aligned}
\delta\tilde{J}(u;v) &= \int_0^T \left(\frac{\partial L}{\partial u} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial u}\right) v\,\mathrm{d}t + \int_0^T \left[\left(\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial x}\right)\eta - \lambda^{\mathrm{T}}\dot{\eta}\right]\mathrm{d}t \\
&\quad + \frac{\partial \Psi}{\partial x}(x(T))\eta(T)
\end{aligned}
$$

Integrating by parts,

$$
\begin{aligned}
\int_0^T \lambda^{\mathrm{T}}\dot{\eta}\,\mathrm{d}t &= \lambda^{\mathrm{T}}(T)\eta(T) - \lambda^{\mathrm{T}}(0)\eta(0) - \int_0^T \dot{\lambda}^{\mathrm{T}}\eta\,\mathrm{d}t \\
&= \lambda^{\mathrm{T}}(T)\eta(T) - \int_0^T \dot{\lambda}^{\mathrm{T}}\eta\,\mathrm{d}t \\
\delta\tilde{J}(u;v) &= \int_0^T \left(\frac{\partial L}{\partial u} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial u}\right) v\,\mathrm{d}t + \int_0^T \left(\frac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta\,\mathrm{d}t \\
&\quad + \left(\frac{\partial \Psi}{\partial x}(x(T)) - \lambda^{\mathrm{T}}(T)\right)\eta(T)
\end{aligned}
$$

For optimality, we need the directional derivative to be zero for every $v \in \mathcal{U}$, where v represents the direction of the derivative. Therefore, the term $(\frac{\partial L}{\partial u} + \lambda^{\mathrm{T}}\frac{\partial f}{\partial u})$ in the first integral has to be identically zero. Thus, we need

$$
\begin{cases}
\dfrac{\partial L}{\partial u} + \lambda^{\mathrm{T}}\dfrac{\partial f}{\partial u} = 0, & \forall t \in [0,T] \\
\dfrac{\partial L}{\partial x} + \lambda^{\mathrm{T}}\dfrac{\partial f}{\partial x} + \dot{\lambda}^{\mathrm{T}} = 0, & \forall t \in [0,T] \\
\dfrac{\partial \Psi}{\partial x}(x(T)) - \lambda^{\mathrm{T}}(T) = 0
\end{cases}
$$

Definition. Let the *Hamiltonian* $H(x,u,t,\lambda)$ be given by

$$H(x,u,t,\lambda) = L(x,u,t) + \lambda^{\mathrm{T}} f(x,u,t)$$

Theorem. *For u to solve the Bolza problem, it has to satisfy*

$$\frac{\partial H}{\partial u}(x,u,t,\lambda) = 0,$$

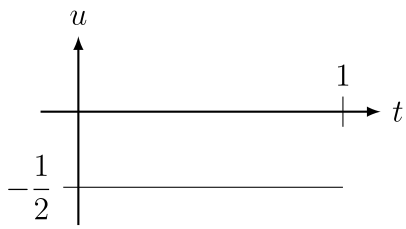
where the costate *satisfies*

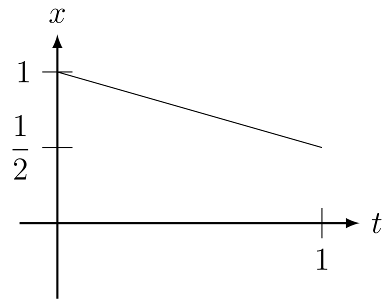
$$\begin{cases} \dot{\lambda} = -\dfrac{\partial H}{\partial x}^T(x,u,t,\lambda) \\ \lambda(T) = \dfrac{\partial \Psi}{\partial x}^T(x(T)) \end{cases}$$

Example

$$\min_u \int_0^1 \frac{1}{2}u^2(t)\,\mathrm{d}t + \frac{1}{2}x^2(1)$$
$$\text{s.t.} \begin{cases} \dot{x} = u, & x,u \in \mathbb{R} \\ x(0) = 1 \end{cases}$$

$$\begin{aligned}
H &= \frac{1}{2}u^2 + \lambda u \\
\frac{\partial H}{\partial u} &= u + \lambda = 0 \Longrightarrow u = -\lambda \\
\dot{\lambda} &= -\frac{\partial H}{\partial x} = 0 \Longrightarrow \lambda(t) = c \\
\lambda(T) &= c = \frac{\partial \Psi}{\partial x}(x(1)) = x(1) \\
\dot{x} &= u = -c \Longrightarrow x(t) = -ct + x(0) = -ct + 1 \\
x(1) &= -c + 1 \\
\lambda(1) &= c = x(1) = -c + 1 \Longrightarrow c = \frac{1}{2} \\
&\boxed{u^* = -\frac{1}{2}}
\end{aligned}$$





We really used five different equations to solve this!

i) $\dfrac{\partial H}{\partial u} = 0$

ii) $\dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x}$

iii) $\lambda(T) = \dfrac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T))$

iv) $\dot{x} = f(x, u, t)$

v) $x(0) = x_0$

There is a sixth condition that is pretty useful if L and f do not depend on t ($L(x, u)$, $f(x, u)$). This is called a *conservative system.* Then, along optimal trajectories (equations i-v are satisfied), the total time derivative of the Hamiltonian is

$$\frac{\mathrm{d}}{\mathrm{d}t}H = \underbrace{\frac{\partial H}{\partial t}}_{\substack{0 \\ H(x,u,\lambda)}} + \underbrace{\frac{\partial H}{\partial x}}_{-\dot{\lambda}^{\mathrm{T}}}\dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{\substack{0 \\ u \text{ is optimal}}}\dot{u} + \underbrace{\frac{\partial H}{\partial \lambda}}_{f^{\mathrm{T}}=\dot{x}^{\mathrm{T}}}\dot{\lambda} = -\dot{\lambda}^{\mathrm{T}}\dot{x} + \dot{x}^{\mathrm{T}}\dot{\lambda} = 0$$

Therefore, for conservative systems,

vi) H is constant along optimal trajectories. (Hamilton's Principle in analytical mechanics)

Back to the example,

$$H = \frac{1}{2}u^2 + \lambda u = \frac{1}{2}c^2 - c^2 = -\frac{1}{2}c^2 = -\frac{1}{8}$$

The Hamiltonian

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^{\mathrm{T}} f(x, u, t)$$

lets us write the Lagrangian as

$$\tilde{J}(u) = \int_0^T \left[L + \lambda^{\mathrm{T}}(f - \dot{x})\right] \mathrm{d}t + \Psi = \int_0^T \left(H - \lambda^{\mathrm{T}}\dot{x}\right) \mathrm{d}t + \Psi$$

The optimality conditions are

$$\frac{\partial H}{\partial u} = 0, \tag{1.3}$$

where

$$\begin{cases} \dot{\lambda} = -\dfrac{\partial H}{\partial x}^{\mathrm{T}} \\ \lambda(T) = \dfrac{\partial \Psi}{\partial x}(x(T)) \end{cases} \tag{1.4}$$

Example Hamilton's Principle

Let q be the generalized coordinates (positions and angles). Then, $\dot{q} = u$ are generalized velocities, which we assume we can control. Let $T(q,u) = u^{\mathrm{T}}M(q)u$, $M \succ 0$, be the kinetic energy and $V(q)$ be the potential energy.

For conservative systems, the following quantity is minimized:

$$\int_0^T \underbrace{\left[T(q,u) - V(q)\right]}_{\substack{L(q,u)= \\ \text{Lagrange's ``action function''}}} \mathrm{d}t$$

The Hamiltonian is

$$H(q,u,\lambda) = L(q,u) + \lambda^{\mathrm{T}} f(q,u) = L(q,u) + \lambda^{\mathrm{T}} u$$

In mechanics, λ is called a generalized momentum, satisfying

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H}{\partial q}^{\mathrm{T}} = -\frac{\partial L}{\partial q}^{\mathrm{T}} + 0 \\ 0 &= \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^{\mathrm{T}} \Longrightarrow \lambda = -\frac{\partial L}{\partial u}^{\mathrm{T}} \\ \dot{\lambda} &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial u}^{\mathrm{T}} = -\frac{\partial L}{\partial q}^{\mathrm{T}} \end{aligned}$$

This produces the Euler-Lagrange Equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0}$$

Recall, along optimal trajectories

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \underbrace{\frac{\partial H}{\partial t}}_{\substack{=0 \text{ if } L \text{ and } f \text{ do not} \\ \text{depend explicitly on } t}} + \overbrace{\frac{\partial H}{\partial x}}^{-\dot{\lambda}^{\mathrm{T}}\dot{x}}\dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{=0}\dot{u}\ \underbrace{\frac{\partial H}{\partial \lambda}}_{f^{\mathrm{T}}=\dot{x}^{\mathrm{T}}}\ \dot{\lambda} = -\dot{\lambda}^{\mathrm{T}}\dot{x} + \dot{x}^{\mathrm{T}}\dot{\lambda} = 0$$

Therefore, along optimal trajectories, the Hamiltonian is constant!

We had

$$H = L + \lambda^{\mathrm{T}} u$$
$$\frac{\partial H}{\partial u} = \lambda^{\mathrm{T}} + \frac{\partial L}{\partial u} = 0$$

Along optimal trajectories,

$$H = L - \frac{\partial L}{\partial u} u$$

Recall, $L(q,u) = T(q,u) - V(q)$.

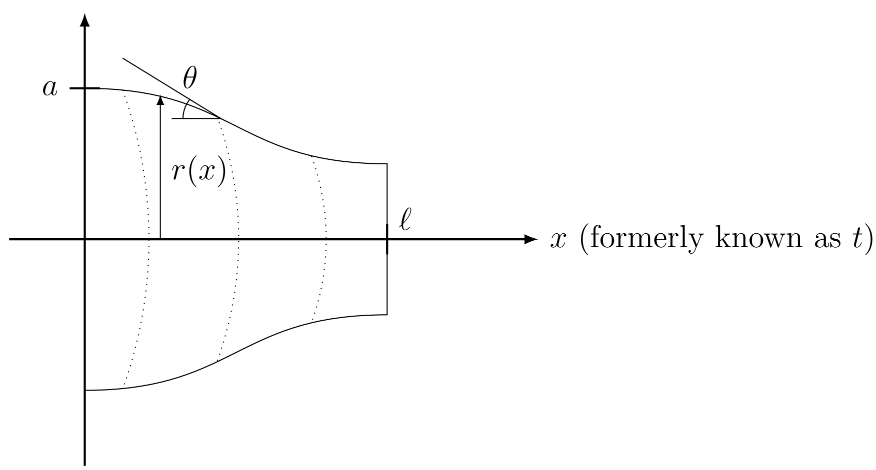
$$\frac{\partial L}{\partial u} = \frac{\partial T}{\partial u} - 0$$
$$T(q,u) = u^{\mathrm{T}} M(q) u$$
$$\frac{\partial T}{\partial u} = 2u^{\mathrm{T}} M$$

So,

$$H = \underbrace{T}_{u^{\mathrm{T}}Mu} - V - 2u^{\mathrm{T}}Mu = -(V + u^{\mathrm{T}}Mu) = -(V + T)$$

Therefore, the total energy (kinetic plus potential energy) remains constant for conservative systems.

Example minimum drag nose shape (Newton 1686)



The drag is

$$D = -2\pi q \int_{x=0}^{\ell} C_p(\theta) r \,\mathrm{d}r,$$

where q is a pressure constant and $C_p(\theta) = 2\sin^2\theta$ is Newton's pressure formula.

Geometry tells us

$$\frac{\mathrm{d}r}{\mathrm{d}x} = -\tan\theta = -u$$

Choose the control as $\tan\theta$. Manipulating the drag,

$$\frac{D}{4\pi q} = \int_0^{\ell} \frac{ru^3}{1+u^2}\,\mathrm{d}x + \frac{1}{2} r(\ell)^2$$

The optimal control problem is

$$\begin{aligned} \min_u \ & \int_0^{\ell} \frac{ru^3}{1+u^2}\,\mathrm{d}x + \frac{1}{2} r(\ell)^2 \\ \text{s.t. } & \frac{\mathrm{d}r}{\mathrm{d}x} = -u \end{aligned}$$

This is in the standard form with the following changes of variables:

$$\begin{aligned} \ell &\longleftarrow T \\ x &\longleftarrow t \\ r &\longleftarrow x \end{aligned}$$

Refer to (1.3) and (1.4) for the following steps.

$$\begin{aligned} H &= \frac{ru^3}{1+u^2} - \lambda u \\ \frac{\partial H}{\partial u} &= \frac{3ru^2(1+u^2) - ru^3 \cdot 2u}{(1+u^2)^2} - \lambda \\ &= \frac{ru^4 + 3ru^2}{(1+u^2)^2} - \lambda = 0 \\ \lambda &= \frac{ru^2(u^2+3)}{(1+u^2)^2} \qquad (1.5) \\ \frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2} \\ \lambda(\ell) &= r(\ell) \end{aligned}$$

Right now, we know

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}x} = -u \\ r(0) = a \\ \dfrac{\mathrm{d}\lambda}{\mathrm{d}x} = -\dfrac{u^3}{1+u^2} \\ \lambda(\ell) = r(\ell) \end{cases}$$

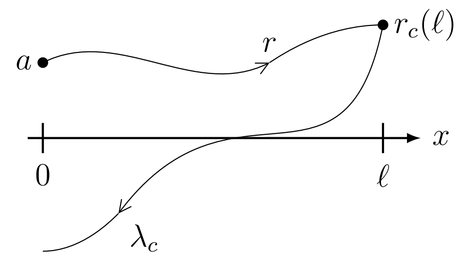
We need to remove u and get a function of r and λ instead. However, it is difficult to solve (1.5). Maybe $H = \text{const.}$ gives us something nicer?

$$\begin{aligned} H &= \frac{ru^3}{1+u^2} - \lambda u \\ &= \frac{ru^3}{1+u^2} - \frac{ru^2(u^2+3)}{(1+u^2)^2} u \\ &= -\frac{2ru^3}{(1+u^2)^2} = c \end{aligned}$$

Assume we can find $u = G(r, c)$, either numerically or some other way. So, now we have

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}x} = -G(r,c) \\ r(0) = a \\ \dfrac{\mathrm{d}\lambda}{\mathrm{d}x} = -\dfrac{G^3(r,c)}{1+G^2(r,c)} \\ \lambda(\ell) = r(\ell) \end{cases}$$

We do not know c, but we can guess c and simulate r forward in "time" (x) from $r(0) = a$. Then, we simulate λ backwards from $r(\ell)$.

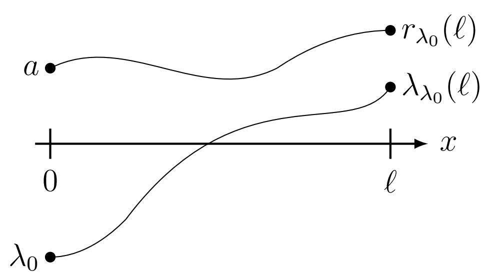


Problem: we can do this for any c. Which c is it? *Last 15 minutes was a dead end!*

Back to $u = F(r, \lambda)$. Assume we have F (numerically).

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}x} &= -F(r,\lambda) \\ r(0) &= a \\ \frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{F^3(r,\lambda)}{1+F^2(r,\lambda)} \\ \lambda(\ell) &= r(\ell) \end{aligned}$$

The mistake before was that the simulation forward from a depends on λ.



Therefore, we should guess λ_0 and simulate both r and λ to get $r_{\lambda_0}(\ell)$ and $\lambda_{\lambda_0}(\ell)$. We need

$$r_{\lambda_0}(\ell) = \lambda_{\lambda_0}$$

for optimality. To do this, we need numerics.

Terminal Constraints

Let $x = [x_1, \dots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ and solve

$$\begin{aligned} \min_{u \in \mathcal{U}} &\int_0^T L(x,u,t)\,\mathrm{d}t + \Psi(x(T)) \\ \text{s.t.} \qquad &\dot{x} = f(x,u,t) \\ x(0) &= x_0 \\ x_i(T) &= x_{iT} \quad \text{given for } i \in \mathcal{T} \subset \{1, \dots, n\} \end{aligned}$$

First, we augment the cost:

$$\begin{aligned} \tilde{J}(u) &= \int_0^T \left[L + \lambda^{\mathrm{T}}(f - \dot{x})\right] \mathrm{d}t + \Psi \\ &= \int_0^T (H - \lambda^{\mathrm{T}}\dot{x})\,\mathrm{d}t + \Psi \\ \tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \left(\varepsilon \frac{\partial H}{\partial u} v + \varepsilon \frac{\partial H}{\partial x}\eta - \varepsilon \lambda^{\mathrm{T}}\dot{\eta}\right) \mathrm{d}t + \varepsilon \frac{\partial \Psi}{\partial x}(x(T))\eta(T) + o(\varepsilon) \\ \delta \tilde{J}(u; v) &= \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta\,\mathrm{d}t + \int_0^T \frac{\partial H}{\partial u} v\,\mathrm{d}t \\ &\quad + \lambda^{\mathrm{T}}(0)\eta(0) - \lambda^{\mathrm{T}}(T)\eta(T) + \frac{\partial \Psi}{\partial x}(x(T))\eta(T) \end{aligned}$$

As always,

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H}{\partial x}^{\mathrm{T}} \\ \frac{\partial H}{\partial u} &= 0 \quad \text{(FONC)} \end{aligned}$$

Additionally,

$$\begin{aligned} \eta(0) &= 0 \\ \eta_i(T) &= 0 \quad \text{for } i \in \mathcal{T} \end{aligned}$$

Note that if $x(T) = x_T$ is given, then $x(T) = x(T) + \varepsilon\eta(T) + o(\varepsilon)$, so $\eta(T) = 0$. Here, we have $x_i(T) = x_{iT}$ fixed for $i \in \mathcal{T}$ so $\eta_i(T) = 0$ for $i \in \mathcal{T}$.

For optimality, we want

$$\left[-\lambda^{\mathrm{T}}(T) + \frac{\partial \Psi}{\partial x}(x(T))\right]\eta(T) = 0 \quad \text{for all } \textit{admissible} \text{ variations}$$

$$\begin{bmatrix} \dfrac{\partial \Psi}{\partial x_1} - \lambda_1, & \cdots, & \dfrac{\partial \Psi}{\partial x_n} - \lambda_n \end{bmatrix} \begin{bmatrix} \eta_1(T) \\ \vdots \\ \eta_n(T) \end{bmatrix} = 0$$

Hence, we need

$$\begin{aligned}
\lambda_j(T) &= \frac{\partial \Psi}{\partial x_j}(x(T)) && \text{if } j \notin \mathcal{T} \\
\lambda_i(T) &= \text{free} && \text{if } i \in \mathcal{T}
\end{aligned}$$

So we have

$$\begin{cases} \dot{x} = f \\ \dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x}, \end{cases}$$

an ODE with $2n$ variables. We need $2n$ boundary conditions for this ODE to be well-posed.

At $t = 0$		At $t = T$	
$x(0) = x_0$	$[n]$	$x_i(T) = x_{iT}$, $i \in \mathcal{T}$ $\lvert\mathcal{T}\rvert = q$	$[q]$
		$x_j(T)$ free, $j \notin \mathcal{T}$	$[0]$
$\lambda(0)$ free	$[0]$	$\lambda_i(T)$ free, $i \in \mathcal{T}$	$[0]$
		$\lambda_j(T) = \dfrac{\partial \Psi}{\partial x_j}(x(T))$, $j \notin \mathcal{T}$	$[n-q]$

So we have $n + q + (n - q) = 2n$ boundary conditions.

We could even fix some but not all of $x(0)$, i.e.

$$\begin{aligned}
x_i(0) &= x_{i0} && \text{if } i \in \mathcal{I} \\
x_j(0) &= \text{free} && \text{if } j \notin \mathcal{I}
\end{aligned}$$

Recall,

$$\delta \tilde{J}(u; v) = \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}} \right) \eta \, \mathrm{d}t + \int_0^T \frac{\partial H}{\partial u} v \, \mathrm{d}t + \lambda^{\mathrm{T}}(0)\eta(0) + \left[\lambda^{\mathrm{T}}(T) - \frac{\partial \Psi}{\partial x}(x(T)) \right] \eta(T)$$

For $x_i(0) = x_{i0}$ fixed, we have $\eta_i(0) = 0$ and $\lambda_i(0)$ free. For $x_j(0)$ free, we have $\eta_j(0)$ free and $\lambda_j(0) = 0$.

To ponder, what if $J = \int L \, \mathrm{d}t + \Psi(x(T)) + \Theta(x(0))$?

To summarize, the minimizer to

$$\begin{aligned}
\min_{u \in \mathcal{U}} & \int_0^T L(x, u, t) \, \mathrm{d}t + \Psi(x(T)) \\
\text{s.t.} \quad & \dot{x} = f(x, u, t) \\
& x_i(0) = x_{i0}, \quad i \in \mathcal{I} \\
& x_j(T) = x_{jT} \quad j \in \mathcal{T}
\end{aligned}$$

has to satisfy

$$\begin{aligned}
&\frac{\partial H}{\partial u} = 0 \\
&\dot{\lambda} = -\frac{\partial H^{\mathrm{T}}}{\partial x} \\
&\lambda_i(0) = 0, \quad i \notin \mathcal{I} \\
&\lambda_j(T) = \frac{\partial \Psi}{\partial x_j}(x(T)), \quad j \notin \mathcal{T}
\end{aligned}$$

Example

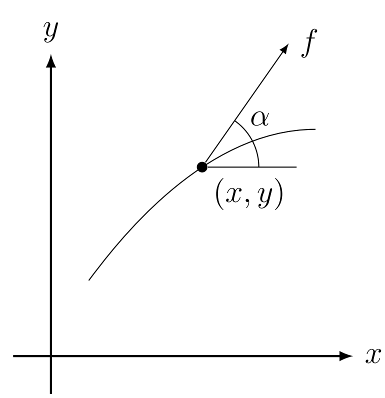
$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = f(x_1, x_2, x_3, x_4) \\
&x_1(0) = 1,\ x_3(0) = 7,\ x_4(0) = 0,\ x_1(1) = 2 \\
&\mathcal{I} = \{1, 3, 4\},\ \mathcal{T} = \{1\} \\
&\min \int_0^1 L(x, u)\,\mathrm{d}t + \big(x_2^2(1) - x_3^2(1) + 7x_1(1) + 14\big)
\end{aligned}$$

Note there are 4 boundary conditions on x so there must be 4 boundary conditions on λ:

$\lambda_1(0)$ free/unspecified	$\lambda_1(1)$ free
$\lambda_2(0) = 0$	$\lambda_2(1) = 2x_2(1)$
$\lambda_3(0)$ free	$\lambda_3(1) = -2x_3(1)$
$\lambda_4(0)$ free	$\lambda_4(1) = 0$

Example

A force f acts on a particle at position (x, y) (mass = 1).



$$
\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= |f|\cos\alpha \\
\dot{v}_y &= |f|\sin\alpha \\
\alpha &= \text{control variable}
\end{aligned}
$$

Assume we only care about where the particle ends up (to be specified later), i.e. $L = 0$.

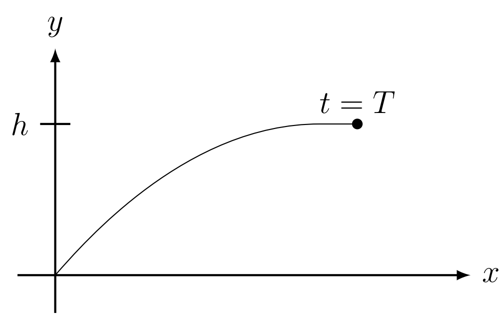
$$
H = \begin{bmatrix} \lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f|\cos\alpha \\ |f|\sin\alpha \end{bmatrix}
$$

$$
\begin{aligned}
\dot{\lambda}_x &= -\frac{\partial H}{\partial x} = 0 && \implies && \lambda_x(t) = c_1 \\
\dot{\lambda}_y &= -\frac{\partial H}{\partial y} = 0 && \implies && \lambda_y(t) = c_2 \\
\dot{\lambda}_{v_x} &= -\frac{\partial H}{\partial v_x} = -\lambda_x && \implies && \lambda_{v_x}(t) = -c_1 t + c_3 \\
\dot{\lambda}_{v_y} &= -\frac{\partial H}{\partial v_y} = -\lambda_y && \implies && \lambda_{v_y}(t) = -c_2 t + c_4
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{\partial H}{\partial \alpha} &= -\lambda_{v_x}|f|\sin\alpha + \lambda_{v_y}|f|\cos\alpha = 0 \\
\tan\alpha &= \frac{\lambda_{v_y}}{\lambda_{v_x}} = \frac{-c_2 t + c_4}{-c_1 t + c_3}
\end{aligned}
$$

We want to drive the particle from $[0,0,0,0]^{\mathrm{T}}$ to a path parallel to the x-axis with $y(T) = h$.



Choose $\Psi = -v_x$,

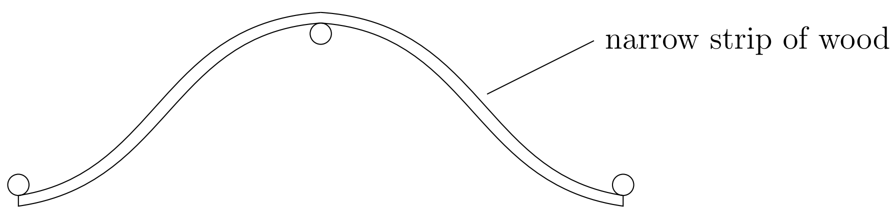
$$
\begin{aligned}
y(T) &= h & v_y(T) &= 0 \\
x(T) &\text{ free} & v_x(T) &\text{ free, but costs} \\
\lambda_i(0) &\text{ free} \\
\lambda_y(T) &\text{ free} & \lambda_{v_y}(T) &\text{ free} \\
\lambda_x(T) &= 0 & \lambda_{v_x}(T) &= -1
\end{aligned}
$$

$$
\begin{aligned}
c_1 = \lambda_x(t) = 0 \\
\implies \lambda_{v_x} = -c_1 t + c_3 = c_3 = -1 \\
\implies \tan\alpha = -\frac{-c_2 t + c_4}{-1} = c_2 t + c_4
\end{aligned}
$$

How do we find c_2 and c_4? Plug into $\dot{x}$ and $\dot{\lambda}$ and try to satisfy the remaining boundary conditions. (This is hard=numerics.)

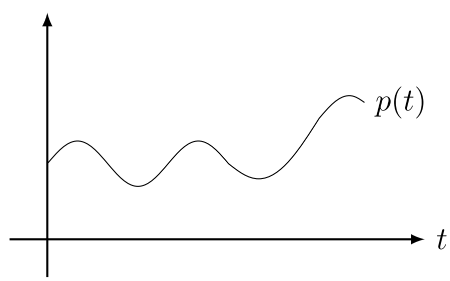
1.7 Splines

From ship building. Splines are used a lot in path-planning, e.g. cubic splines.



But, they are solutions to optimal control problems.

Let $p(t)$ be a curve we'd like to shape.



We want to minimize the "energy" put into the curve, a.k.a acceleration. Let $x_1 = p$ and $x_2 = \dot{p}$, so

$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = u
\end{cases}
$$

1.7.1 Minimum-Energy

$$
\begin{aligned}
&\min_{u\in\mathcal{U}} \frac{1}{2}\int_0^T u^2(t)\,\mathrm{d}t \quad + \text{Boundary conditions on } x \\
&H = L + \lambda^{\mathrm{T}} f = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u \\
&\frac{\partial H}{\partial u} = u + \lambda_2 = 0 \Longrightarrow u = -\lambda_2 \\
&\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \Longrightarrow \lambda_1 = c_1 \\
&\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \Longrightarrow \lambda_2 = -c_1 t + c_2 \\
&u = -\lambda_2 = c_1 t - c_2 \\
&\dot{x}_2 = u = c_1 t - c_2 \Longrightarrow x_2 = c_1\frac{t^2}{2} - c_2 t + c_3 \\
&\dot{x}_1 = x_2 = c_1\frac{t^2}{2} - c_2 t + c_3 \\
&\quad \Longrightarrow x_1 = \frac{c_1}{6}t^3 - \frac{c_2}{2}t^2 + c_3 t + c_4
\end{aligned}
$$

$p(t)$ is a cubic polynomial!

What about boundary conditions?

Let $T = 1$, $p(0)$ given, $p(1)$ given, $\dot{p}(0) = 0$, $\dot{p}(1) = 0$, e.g. $p(0) = 0$, $p(1) = 1$. Since the boundary conditions for x are all specified, those for the costate are free.

$$
\left.\begin{aligned}
x_1(0) &= 0 \\
x_2(0) &= 0 \\
x_1(1) &= 1 \\
x_2(1) &= 0
\end{aligned}\right\} \Longrightarrow \left\{\begin{aligned}
&\lambda_1(0) \\
&\lambda_2(0) \\
&\lambda_1(1) \\
&\lambda_2(1)
\end{aligned}\right. \quad \text{free/unspecified}
$$

$$
\begin{aligned}
x_2(0) &= c_3 = 0 & x_1(1) &= \frac{2c_2}{6} - \frac{c_2}{2} = 1 \\
x_1(0) &= c_4 = 0 & c_2 &= -6 \\
x_2(1) &= \frac{c_1}{2} - c_2 + \underbrace{c_3}_{0} = 0 & c_1 &= -12 \\
c_1 &= 2c_2
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow p(t) &= -2t^3 + 3t^2 \\
u(t) &= -12t + 6
\end{aligned}
$$

Or, what if $\dot{p}(0)$, $\dot{p}(1)$ are not specified?

$$
\left.\begin{aligned}
&x_1(0) = 0 \\
&x_2(0) \text{ unspec.} \\
&x_1(1) = 1 \\
&x_2(1) \text{ unspec.}
\end{aligned}\right\} \Longrightarrow \left\{\begin{aligned}
&\lambda_1(0) \text{ unspec.} \\
&\lambda_2(0) = 0 \\
&\lambda_1(1) \text{ unspec.} \\
&\lambda_2(1) = 0
\end{aligned}\right.
$$

$$\left.\begin{aligned} \lambda_2(0) &= c_2 = 0 \\ \lambda_2(1) &= -c_1 + c_2 = 0 \end{aligned}\right\} \Longrightarrow u = c_1 t - c_2 = 0$$

$$\left.\begin{aligned} x_1(0) &= c_4 = 0 \\ x_1(1) &= c_3 = 1 \end{aligned}\right\} \Longrightarrow p(t) = t$$

What did we do?

Case 1:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1/2 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_2(0) \\ x_2(1) \end{bmatrix}$$

Case 2:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ \lambda_2(0) \\ \lambda_2(1) \end{bmatrix}$$

1.7.2 Generalized Splines

We had $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This A is nilpotent ($A^k = 0$ for some $k \in \mathbb{Z}^+$). This means e^{At} is a polynomial in t. (This e^{At} is cubic.)

In general, e^{At} is a mix of polynomials, exponentials, and trignometric terms. The eigenvalues of A determine the form of $x(t)$.

$$\begin{aligned} \dot{x} &= Ax && \Longrightarrow x(t) = e^{At}x(0) \\ \dot{x} &= Ax + Bu && \Longrightarrow x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau \end{aligned}$$

The general problem to solve is

$$\begin{aligned} &\min_{u \in \mathcal{U}} \int_0^T \frac{1}{2}\|u\|^2 \,\mathrm{d}t \\ &\text{s.t. } \dot{x} = Ax + Bu \\ &\quad + \text{Boundary conditions} \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2}\|u\|^2 + \lambda^{\mathrm{T}}(Ax + Bu) \\ \frac{\partial H}{\partial u} &= u^{\mathrm{T}} + \lambda^{\mathrm{T}}B = 0 \\ &\Rightarrow u = -B^{\mathrm{T}}\lambda \\ \dot{\lambda} &= -\frac{\partial H}{\partial x}^{\mathrm{T}} = -A^{\mathrm{T}}\lambda \end{aligned}$$

We have the Hamiltonian Dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BB^{\mathrm{T}} \\ 0 & -A^{\mathrm{T}} \end{bmatrix}}_{M} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

Where we used $\dot{x} = Ax + Bu = Ax - BB^{\mathrm{T}}\lambda$. Then,

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

Suppose we want to drive from $x(0) = x_0$ to $x(T) = x_T$.

$$\begin{bmatrix} x_T \\ \lambda(T) \end{bmatrix} = e^{MT} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix} = \begin{bmatrix} N_{xx} & N_{x\lambda} \\ N_{\lambda x} & N_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix}$$
$$x_T = N_{xx}x_0 + N_{x\lambda}\lambda(0)$$

$N_{x\lambda}$ is invertible if (A, B) is completely controllable. Assume it is.

$$\lambda(0) = N_{x\lambda}^{-1}(x_T - N_{xx}x_0)$$
$$\implies \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} x_0 \\ N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \end{bmatrix}$$
$$\implies u(t) = -B^{\mathrm{T}}\lambda(t)$$

This is the optimal trajectory, but there is no feedback. We will consider closed-loop systems after the midterm.

As a preview, we need to find λ as a function of x. For example, $u = -R^{-1}B^{\mathrm{T}}Px$ minimizes $u^{\mathrm{T}}Ru$, so $\lambda = Px$ where P is the solution to the Riccati equation.

1.8 Numerical Methods

Optimal control boils down to solving two sets of differential equations:

$$\begin{aligned} \dot{x} &= f(x, u) & \frac{\partial H}{\partial u}(x, u, \lambda) &= 0 \\ \dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x}(x, u, \lambda) & u &= F(x, \lambda) \end{aligned}$$
$$\implies \begin{cases} \dot{x} = f(x, F(x, \lambda)) \\ \dot{\lambda} = -\dfrac{\partial H^{\mathrm{T}}}{\partial x}(x, F(x, \lambda), \lambda) \end{cases}$$

The equations are functions of x and λ. They are completely determined by the boundary conditions on $x(0)$, $x(T)$, $\lambda(0)$, $\lambda(T)$. This is known as the *Boundary Value Problem*. This is solved using *test shooting*:

1. Guess initial conditions
2. Simulate forward in time
3. Update the guess (cleverly…)

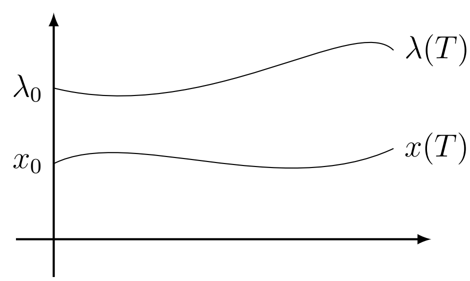
Exmaple: Bolza problem

$$\min_{u \in \mathcal{U}} \int_0^T L(x,u)\,\mathrm{d}t + \Psi(x(T))$$
$$\text{s.t.} \quad \begin{cases} \dot{x} = f(x,u) \\ x(0) = x_0 \end{cases}$$
$$H(x,u,\lambda) = L(x,u) + \lambda^{\mathrm{T}} f(x,u)$$
$$u^*(x,\lambda) \text{ satisfies } \frac{\partial H}{\partial u} = 0$$

The optimal control satisfies

$$\begin{cases} x = f(x, u^*(x,\lambda)) \\ x(0) = x_0 \\ \lambda = -\dfrac{\partial H^{\mathrm{T}}}{\partial x}(x, u^*(x,\lambda), \lambda) \\ \lambda(T) = \dfrac{\partial \Psi}{\partial x}(x(T)) \end{cases}$$

Algorithm Guess λ_0 and solve for $x(t)$, $\lambda(t)$.



Let's define a cost:

$$\left\| \lambda(T) - \frac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) \right\|^2 = g(\lambda_0)$$

Update λ_0 through

$$\lambda_0 \coloneqq \lambda_0 - \underset{\substack{\uparrow \\ \text{any choice of step size works}}}{\gamma} \frac{\partial g^{\mathrm{T}}}{\partial \lambda_0}(\lambda_0)$$

Repeat

Problem: What is $\partial g / \partial \lambda_0$? We estimate $\partial g / \partial \lambda_0$ numerically. This is where "test shooting" comes into play.

Let e_i be the ith unit vector, $i = 1, \dots, n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \; \dots, \; e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\frac{\partial g}{\partial \lambda_0} = \left(\frac{\partial g}{\partial \lambda_{0,1}}, \frac{\partial g}{\partial \lambda_{0,2}}, \dots, \frac{\partial g}{\partial \lambda_{o,n}} \right)$$

The ith component of $\partial g / \partial \lambda_0$ is given by the directional derivative

$$\frac{\partial g}{\partial \lambda_{0,i}} = \frac{\partial g}{\partial \lambda_0} \cdot e_i = \delta g(\lambda_0; e_i) = \lim_{\varepsilon \to 0} \frac{g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)}{\varepsilon}$$

So, if $x \in \mathbb{R}^n$ (and thus so is λ_0), we have to do this n times (with a small ε) and get the full derivative $\partial g / \partial \lambda_0$.

Algorithm

Given λ_0, $g(\lambda_0)$
for $i = 1$ to n **do**
  Compute $g(\lambda_0 + \varepsilon e_i)$
  $dg_i = \dfrac{1}{\varepsilon}[g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)]$
end for
$\dfrac{\partial g}{\partial \lambda_0} = [dg_1, \dots, dg_n]$

Example LQ

$$\begin{aligned} &\min_u \frac{1}{2} \int_0^1 (x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u) \, \mathrm{d}t + \frac{1}{2} x^{\mathrm{T}}(1) S x(1) \\ &\text{s.t.} \quad \begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases} \\ &\qquad Q, R, S \succ 0 \end{aligned}$$

$$\begin{aligned} H &= \frac{1}{2} x^{\mathrm{T}} Q x + \frac{1}{2} u^{\mathrm{T}} R u + \lambda^{\mathrm{T}} (Ax + Bu) \\ \frac{\partial H}{\partial u} &= u^{\mathrm{T}} R + \lambda^{\mathrm{T}} B = 0 \\ u^* &= -R^{-1} B^{\mathrm{T}} \lambda \\ \dot{\lambda} &= -\frac{\partial H^{\mathrm{T}}}{\partial x} = -Qx - A^{\mathrm{T}} \lambda \\ \lambda(1) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(1)) = Sx(1) \end{aligned}$$

So putting it all together,

$$\begin{aligned}
\dot{x} &= Ax - BR^{-1}B^{\mathrm{T}}\lambda && x(0) = x_0 \\
\dot{\lambda} &= -Qx - A^{\mathrm{T}}\lambda && \lambda(1) = Sx(1)
\end{aligned}$$

Example Newton's nose shape problem (revisited, see previous)

$$\begin{aligned}
&\min_u \int_0^\ell \frac{ru^3}{1+u^2}\,\mathrm{d}x + \frac{1}{2}r(\ell)^2 \\
&\text{s.t. } \frac{\mathrm{d}r}{\mathrm{d}x} = -u \qquad r(0) = a
\end{aligned}$$

$$\begin{aligned}
H &= \frac{ru^3}{1+u^2} + \lambda(-u) \\
\frac{\partial H}{\partial u} &= \frac{ru^2(3+u^2)}{(1+u^2)^2} - \lambda = 0
\end{aligned}$$

We solve the above numerically to get $u^*(r, \lambda)$.

$$\begin{aligned}
\frac{\partial \lambda}{\partial x} &= -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2} \\
\lambda(\ell) &= r(\ell)
\end{aligned}$$

So, we have

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}x} &= -u && r(0) = a && u = F(x, \lambda) \\
\frac{\mathrm{d}\lambda}{\mathrm{d}x} &= -\frac{u^3}{1+u^2} && \lambda(\ell) = r(\ell)
\end{aligned}$$

Example Fixed terminal constraints (revisited, see previous)

$$\begin{aligned}
&\min_\alpha -v_x(T) && \alpha = \text{control} \\
&\text{s.t.} \quad \dot{x} = v_x && x(0) = 0 \\
&\qquad \dot{y} = v_y && y(0) = 0 \\
&\qquad \dot{v}_x = |f|\cos\alpha && v_x(0) = 0 \\
&\qquad \dot{v}_y = |f|\sin\alpha && v_y(0) = 0 \\
&\qquad y(T) = h \\
&\qquad v_y(T) = 0
\end{aligned}$$

$$
\begin{aligned}
H &= -v_x(T) + \begin{bmatrix}\lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y}\end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f|\cos\alpha \\ |f|\sin\alpha \end{bmatrix} \\
\frac{\mathrm{d}H}{\mathrm{d}\alpha} &= 0 \Rightarrow \tan\alpha = \frac{\lambda_{v_y}}{\lambda_{v_x}} \\
\dot{\lambda}_x &= 0 \\
\dot{\lambda}_y &= 0 \\
\dot{\lambda}_{v_x} &= -\lambda_x \\
\dot{\lambda}_{v_y} &= -\lambda_y \\
\boldsymbol{\lambda}(0) &\text{ unspecified} \\
\lambda_x(T) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial x}(x(T)) = 0 \\
\lambda_y(T) &\text{ unspecified} \\
\lambda_{v_x}(T) &= \frac{\partial \Psi^{\mathrm{T}}}{\partial v_x}(v_x(T)) = -1 \\
\lambda_{v_y}(T) &\text{ unspecified}
\end{aligned}
$$

Again, we guess λ_0 and solve forward in time. But, we have terminal constraints on y and v_y as well.

$$
q(\lambda_0) = \frac{1}{2}\left[(y(T) - h)^2 + (v_y(T))^2 + (\lambda_x(T))^2 + (\lambda_{v_x} + 1)^2\right]
$$