

ECE 6553: Optimal Control Notes

Klaus Z. Okkelberg

Spring 2017

Contents

1	Parameter Optimization	2
1.1	What is optimal control?	2
1.2	Unconstrained Optimization	4
1.3	Numerical Methods	6
1.4	Constrained Optimization	8
1.4.1	Equality Constraints	11
1.4.2	Inequality Constraints	12
1.5	Directional Derivatives	14
1.6	Calculus of Variations	16
1.6.1	An (Almost) Optimal Control Problem	18
1.6.2	Optimal Timing Control	21
1.6.3	The Bolza Problem	24
1.7	Splines	36
1.7.1	Minimum-Energy	37
1.7.2	Generalized Splines	38
1.8	Numerical Methods	39
1.9	Terminal Manifolds	43
1.9.1	Terminal manifold with inequality constraints	48
1.9.2	Initial manifold	48

Chapter 1

Parameter Optimization

1.1 What is optimal control?

Optimal Maximize/minimize cost (subject to constraints): $\min_u g(u)$

With constraints,

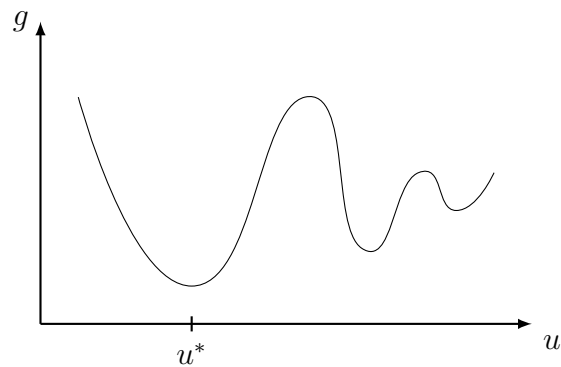
$$\begin{aligned} \min_u \quad & g(u) \\ \text{s.t.} \quad & \begin{cases} h_1(u) = 0 \\ h_2(u) \leq 0 \end{cases} \end{aligned}$$

First-order necessary condition (FONC):

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Optimality can be

- local vs global
- max vs min



Control control design: pick u such that specifications are satisfied:

$$\dot{x} = f(x, u), \quad \dot{x} = Ax + Bu,$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, and $f(\cdot)$ is the dynamics.

Actually, x and u are signals:

$$x : [0, T] \rightarrow \mathbb{R}^n, \quad u : [0, T] \rightarrow \mathbb{R}^m$$

Optimal control find the “best” u !

For “best” to mean anything, we need a cost. The big/deep question is

$$\frac{\partial \text{“cost”}}{\partial u} = 0$$

Example

Suppose we have a car with position p . Its acceleration \ddot{p} is controlled by the gas/brake input u ($\ddot{p} = u$). In order to express the dynamics of the system in the form $\dot{x} = f(x, u)$, we introduce state variables:

$$\begin{aligned} x_1 = p \\ x_2 = \dot{p} \end{aligned} \implies \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

The task is to move the car from its initial position to a stop at a distance c away.

Minimum energy problem

$$\begin{aligned} \min_u \quad & \int_0^T u^2(t) dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0, x_2(0) = 0 \\ & x_1(T) = c, x_2(T) = 0 \end{aligned}$$

Minimum time problem

$$\begin{aligned} \min_{u, T} \quad & T = \int_0^T dt \\ \text{s.t.} \quad & \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases} \\ & x_1(0) = 0, x_2(0) = 0 \\ & x_1(T) = c, x_2(T) = 0 \\ & u(t) \in [u_{\min}, u_{\max}] \end{aligned}$$

The general optimal control problem we will solve will look like

$$\begin{aligned} \min_{u, T} \quad & \int_0^T L(x(t), u(t), t) dt + \Psi(x(T)) \\ \text{s.t.} \quad & \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T] \\ & x(0) = x_0 \\ & x(T) \in S \\ & u(t) \in \Omega, \quad t \in [0, T] \end{aligned}$$

where $\Psi(\cdot)$ is the terminal cost and S is the terminal manifold. This is a so-called **Bolza Problem**.

What tools do we need to solve this?

1. optimality conditions $\partial \text{cost} / \partial u = 0$
2. some way of representing the optimal signal $u^*(x, t)$
3. some way of actually finding/computing the optimal controllers

1.2 Unconstrained Optimization

Let the decision variable be $u = [u_1, \dots, u_m]^T \in \mathbb{R}^m$. The cost is $g(u) \in C^1$ (C^k means k times continuously differentiable). The problem is

$$\min_u g(u), \quad g : \mathbb{R}^m \rightarrow \mathbb{R}$$

For u^* to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

Definition. u^* is a (local) minimizer to g if $\exists \delta > 0$ s.t.

$$\begin{aligned} g(u^*) &\leq g(u) \quad \forall u \in B_\delta(u^*) \\ B_\delta(u^*) &= \{u \mid \|u - u^*\| \leq \delta\} \end{aligned}$$

Note:

- $\frac{\partial g}{\partial u}(u^*) \delta u \in \mathbb{R}$ and δu is $m \times 1$, so $\frac{\partial g}{\partial u}$ is a $1 \times m$ row vector. For the column vector,

$$\nabla g = \frac{\partial g^T}{\partial u} \in \mathbb{R}^m$$

- $\frac{\partial g}{\partial u} \delta u$ is an inner product

$$\langle \nabla g, \delta u \rangle = \left\langle \frac{\partial g^T}{\partial u}, \delta u \right\rangle$$

- $o(\varepsilon)$ encodes higher-order terms

$$\lim_{\varepsilon \rightarrow 0} \frac{o(\varepsilon)}{\varepsilon} = 0 \quad \text{“faster than linear”}$$

This is opposed to big-O notation:

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{O}(\varepsilon)}{\varepsilon} = c$$

- δu has direction and scale so we could write it as

$$\delta u = \varepsilon v, \quad \varepsilon \in \mathbb{R}, \quad v \in \mathbb{R}^m$$

Theorem. For u^* to be a minimizer, we need

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or, equivalently,

$$\frac{\partial g}{\partial u}(u^*)v = 0 \quad \forall v \in \mathbb{R}^m$$

Proof. Let u^* be a minimizer. Evaluating the cost $g(u)$ in the ball and using Taylor’s expansion,

$$g(u^* + \delta u) = g(u^*) + \frac{\partial g}{\partial u}(u^*)\delta u + o(\|\delta u\|) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*)v + o(\varepsilon)$$

Assume that $\frac{\partial g}{\partial u} \neq 0$. Then we could pick $v = -\frac{\partial g}{\partial u}^T(u^*)$, i.e.

$$g(u^* + \varepsilon v) = g(u^*) - \varepsilon \left\| \frac{\partial g}{\partial u}(u^*) \right\|^2 + o(\varepsilon)$$

Note that the second term is negative per our assumptions. So, for ε sufficiently small, we have

$$g\left(u^* - \varepsilon \frac{\partial g}{\partial u}(u^*)\right) < g(u^*)$$

This contradicts u^* being a minimizer. \times (crossed swords) □

Definition (Positive definite). $M = M^T \succ 0$ if

$$\begin{aligned} z^T M z &> 0 \quad \forall z \neq 0, \quad z \in \mathbb{R}^m \\ \iff M &\text{ has real and positive eigenvalues} \end{aligned}$$

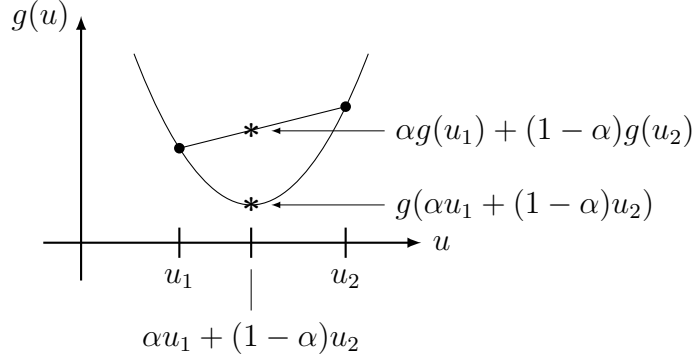
Theorem. If $g \in C^2$, then a **sufficient** condition for u^* to be a (local) minimizer is

$$1. \quad \frac{\partial g}{\partial u}(u^*) = 0$$

2. $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ (the Hessian is positive definite)

Definition. $g : \mathbb{R}^m \rightarrow \mathbb{R}$ is convex if

$$g(\alpha u_1 + (1 - \alpha)u_2) \leq \alpha g(u_1) + (1 - \alpha)g(u_2) \quad \forall \alpha \in [0, 1], \quad u_1, u_2 \in \mathbb{R}^m$$



Theorem. If $\frac{\partial^2 g}{\partial u^2}(u) \succeq 0 \quad \forall u \in \mathbb{R}^m$, then g is convex. (\iff for $g \in C^2$)

Example $\min_u u^T Q u - b^T u$ where $Q = Q^T \succ 0$ (positive definite matrix)

$$\begin{aligned} \frac{\partial g}{\partial u} &= \frac{\partial}{\partial u} (u^T Q u - b^T u) \\ &= u^T Q^T + u^T Q - b^T \\ &= 2u^T Q - b^T \\ \frac{\partial^2 g}{\partial u^2} &= 2Q \end{aligned} \quad \frac{\partial^2 g}{\partial u^2} = \begin{bmatrix} \frac{\partial^2 g}{\partial u_1^2} & \cdots & \frac{\partial^2 g}{\partial u_1 \partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 g}{\partial u_m^2} \end{bmatrix}$$

From $\frac{\partial g}{\partial u} = 2u^T Q - b^T = 0$,

$$u = \frac{1}{2} Q^{-1} b$$

To see whether this is a minimizer, consider the Hessian. Since $Q \succ 0$, it follows that $\frac{\partial^2 g}{\partial u^2}(u^*) \succ 0$ and $u^* = \frac{1}{2} Q^{-1} b$ is a (local) minimizer. Additionally, since $\frac{\partial^2 g}{\partial u^2} \succ 0$, g is convex and u^* is a global minimizer. In fact, since we have strict convexity ($\succ 0$ rather than $\succeq 0$), it is the unique global minimizer.

In optimal control, *local* is typically all we can ask for. In optimization, we can do better!

But wait, just because we know $\frac{\partial g}{\partial u} = 0$, it doesn't follow that we can actually find $u^* \dots$

1.3 Numerical Methods

Idea: $u_{k+1} = u_k + \text{step}_k$. What should step_k be? For small $\text{step}_k = \gamma_k v_k$,

$$g(u_k + \text{step}_k) = g(u_k) + \frac{\partial g}{\partial u}(u_k) \cdot \text{step}_k + o(\|\text{step}_k\|) = g(u_k) + \gamma_k \frac{\partial g}{\partial u}(u_k) v_k + o(\gamma_k)$$

A perfectly reasonable choice of step direction is

$$v_k = -\frac{\partial g^T}{\partial u}(u_k),$$

known as the *steepest descend* direction. This produces

$$g\left(u_k - \gamma_k \frac{\partial g}{\partial u}(u_k)\right) = g(u_k) - \gamma_k \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 + o(\gamma_k)$$

Steepest descent

$$u_{k+1} = u_k - \gamma_k \frac{\partial g^T}{\partial u}(u_k)$$

Note:

- What should γ_k be?
- This method “pretends” that $g(u)$ is linear. If we pretend $g(u)$ is quadratic, we get

$$u_{k+1} = u_k - \left(\frac{\partial^2 g}{\partial u^2}(u_k) \right)^{-1} \frac{\partial g^T}{\partial u}(u_k),$$

i.e. Newton’s Method

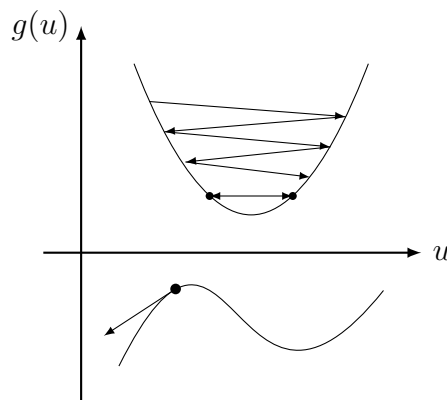
This course: steepest descent

Step-size selection?

- Choice 1: $\gamma_k = \gamma$ “small” $\forall k$; will get close to a minimizer if u_0 is close enough and γ small enough

Problems:

- You may not converge! (but you’ll get close)
- You may go off to infinity (diverge)



- Choice 2: Reduce γ_k as a function of k ; will get close to a minimizer if u_0 is close enough

Problem: slow

Theorem. If u_0 is close enough to u^* and γ_k satisfies

$$\begin{aligned} - \sum_{k=0}^{\infty} \gamma_k &= \infty \\ - \sum_{k=0}^{\infty} \gamma_k^2 &< \infty \end{aligned}$$

e.g. $\gamma_k = c/k$, then $u_k \rightarrow u^*$ as $k \rightarrow \infty$.

- Choice 3: **Armijo step-size:** Take as big a step as possible, but no larger
Pick $\alpha \in (0, 1)$, $\beta \in (0, 1)$. Let i be the smallest non-negative integer such that

$$\begin{aligned} g\left(u_k - \beta^i \frac{\partial g^T}{\partial u}(u_k)\right) - g(u_k) &< -\alpha \beta^i \left\| \frac{\partial g}{\partial u}(u_k) \right\|^2 \\ u_{k+1} &= u_k - \beta^i \frac{\partial g^T}{\partial u}(u_k) \end{aligned}$$

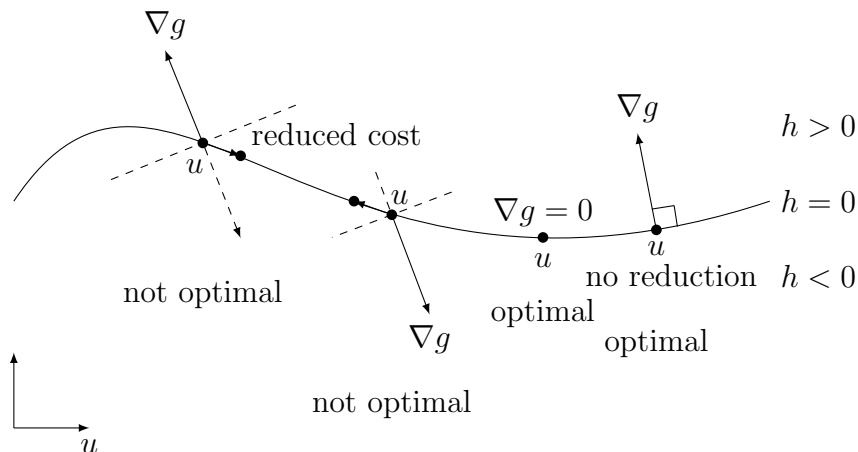
This will get to a minimizer blazingly fast if u_0 is close enough.

1.4 Constrained Optimization

Equality constraints:

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = 0 \end{aligned}$$

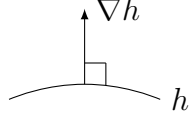
Consider $u \in \mathbb{R}^2$, $h : \mathbb{R}^2 \rightarrow \mathbb{R}$



So u is (locally) optimal if $\nabla g \parallel$ (is parallel to) the normal vector to tangent plane to h .

Fact: (HW# 1)

$$\nabla h \perp Th \quad (\text{tangent plane to } h)$$



We need $\nabla g \parallel \nabla h$ at u^* for optimality, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \alpha \frac{\partial h}{\partial u}(u^*), \quad \text{for some } \alpha \in \mathbb{R}$$

or ($\lambda = -\alpha$),

$$\frac{\partial g}{\partial u}(u^*) + \lambda \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}(g(u^*) + \lambda h(u^*)) = 0, \quad \text{for some } \lambda \in \mathbb{R}$$

More generally,

$$\begin{aligned} \min_{u \in \mathbb{R}^m} g(u) \\ \text{s.t. } h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

Note that $h(u) = [h_1(u), \dots, h_k(u)]^T$.

We need $\frac{\partial g}{\partial u}(u^*)$ to be a linear combination of $\frac{\partial h_i}{\partial u}(u^*)$, $i = 1, \dots, k$, for exactly the same reasons, i.e.

$$\frac{\partial g}{\partial u}(u^*) = \sum_{i=1}^k \alpha_i \frac{\partial h_i}{\partial u}(u^*)$$

or ($\lambda = -[\alpha_1, \dots, \alpha_k]^T$)

$$\frac{\partial g}{\partial u}(u^*) + \lambda^T \frac{\partial h}{\partial u}(u^*) = 0$$

or

$$\frac{\partial}{\partial u}(g(u^*) + \lambda^T h(u^*)) = 0, \quad \text{for some } \lambda \in \mathbb{R}^k$$

Theorem. If u^* is a minimizer to

$$\begin{aligned} \min_{u \in \mathbb{R}^m} g(u) \\ \text{s.t. } h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

then $\exists \lambda \in \mathbb{R}^k$ s.t.

$$\begin{cases} \frac{\partial L}{\partial u}(u^*, \lambda) = 0 \\ \frac{\partial L}{\partial \lambda}(u^*, \lambda) = 0 \end{cases}$$

where the Lagrangian L is given by

$$L(u, \lambda) = g(u) + \lambda^T h(u)$$

Note:

- λ are the Lagrange multipliers
- $\frac{\partial L}{\partial \lambda} = 0$ is fancy speak for $h(u^*) = 0$

Example

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & \frac{1}{2} \|u\|^2 \\ \text{s.t.} \quad & Au = b \end{aligned}$$

where A is $k \times m$, $k \leq m$. Assume $(AA^T)^{-1}$ exists (constraints are linearly independent, none of the constraints are “duplicates”, all the constraints are essential).

$$\begin{aligned} L &= \frac{1}{2} u^T u + \lambda^T (Au - b) \\ \frac{\partial L}{\partial u} &= u^T + \lambda^T A = 0 \\ u^* &= -A^T \lambda \end{aligned}$$

Using the equality constraint,

$$\begin{aligned} Au^* &= b \\ -AA^T \lambda &= b \\ \lambda &= -(AA^T)^{-1} b \\ u^* &= A^T (AA^T)^{-1} b \end{aligned}$$

Example

$$\begin{aligned} \min \quad & u_1 u_2 + u_2 u_3 + u_1 u_3 \\ \text{s.t.} \quad & u_1 + u_2 + u_3 = 3 \end{aligned}$$

$$L = u_1 u_2 + u_2 u_3 + u_1 u_3 + \lambda(u_1 + u_2 + u_3 - 3)$$

$$\begin{cases} \frac{\partial L}{\partial u_1} = u_2 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_2} = u_1 + u_3 + \lambda = 0 \\ \frac{\partial L}{\partial u_3} = u_2 + u_1 + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = u_1 + u_2 + u_3 = 3 \end{cases} \implies \begin{cases} u_1^* = 1 \\ u_2^* = 1 \\ u_3^* = 1 \\ \lambda = -2 \end{cases} \quad \text{optimal solution}$$

Note: This was actually the worst we can do—maximize! Even weirder: no local minimizer!

1.4.1 Equality Constraints

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) = \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$

Theorem. If u^* is a minimizer/maximizer then $\exists \lambda \in \mathbb{R}^k$ s.t.

$$\begin{aligned} \frac{\partial L}{\partial u}(u^*, \lambda) &= 0 \\ \frac{\partial L}{\partial \lambda}(u^*, \lambda) &= 0 \quad (\iff h(u^*) = 0) \end{aligned}$$

where $L(u, \lambda) = g(u) + \lambda^T h(u)$.

Example [Entropy Maximization]

Given $S = \{x_1, \dots, x_n\}$ and a distribution over S such that it takes the value x_j with probability p_j . The entropy is

$$E(p) = \sum_{j=1}^n (-p_j \ln p_j).$$

The mean is

$$m = \sum_{j=1}^n p_j x_j.$$

Problem: Given m , find p such that E is maximized.

$$\begin{aligned} \min_p \quad & - \sum_{j=1}^n p_j \ln p_j \\ \text{s.t.} \quad & \sum_{j=1}^n p_j x_j = m \\ & \sum_{j=1}^n p_j = 1 \\ & p_j \geq 0, \quad j = 1, \dots, n \quad (\text{ignore this...}) \end{aligned}$$

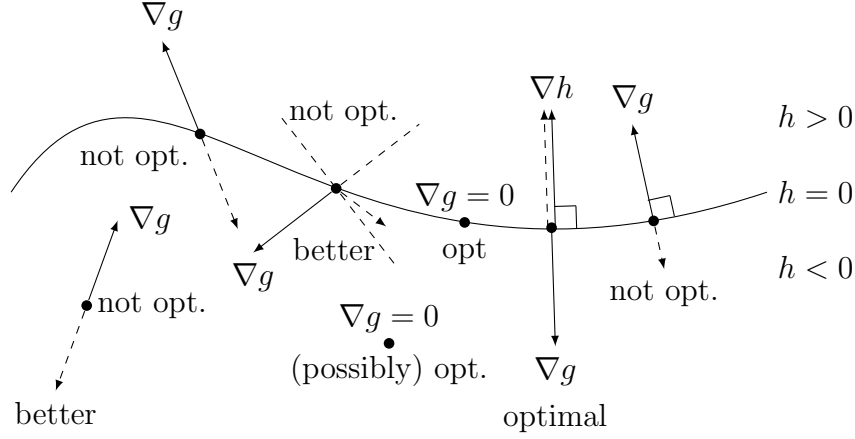
$$\begin{aligned} L &= - \sum p_j \ln p_j + \lambda_1 \left[\sum p_j x_j - m \right] + \lambda_2 \left[\sum p_j - 1 \right] \\ \frac{\partial L}{\partial p_j} &= - \ln p_j - 1 + \lambda_1 x_j + \lambda_2 = 0 \\ p_j &= e^{\lambda_2 - 1 + \lambda_1 x_j}, \quad j = 1, \dots, n \quad (p_j \geq 0 \text{ so we're ok with ignoring that}) \end{aligned}$$

$$\begin{aligned} \sum e^{\lambda_2 - 1 + \lambda_1 x_j} x_j &= m & n + 2 \text{ equations and} \\ \sum e^{\lambda_2 - 1 + \lambda_1 x_j} &= 1 & n + 2 \text{ unknowns...} \end{aligned}$$

No analytical solution, but numerically “solvable”

1.4.2 Inequality Constraints

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & h(u) \leq \mathbf{0}, \quad h : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{aligned}$$



We need:

- if $h(u^*) < 0$ then $\frac{\partial g}{\partial u}(u^*) = 0$
- if $h(u^*) = 0$ then we need either

$$\frac{\partial g}{\partial u}(u^*) = 0$$

or

$$\frac{\partial g}{\partial u}(u^*) = -\lambda \frac{\partial h}{\partial u}(u^*) \quad \text{for } \lambda > 0$$

Or, even better,

$$\frac{\partial}{\partial u} (g(u^*) + \lambda h(u^*)) = 0 \quad \text{for } \lambda \geq 0,$$

where $\lambda h(u^*) = 0$. ($h < 0 \rightarrow \lambda = 0$, $h = 0 \rightarrow \lambda \geq 0$)

In general, if $u \in \mathbb{R}^m$ and $h : \mathbb{R}^m \rightarrow \mathbb{R}^k$, we have that u^* , if optimal, has to satisfy

$$\begin{aligned} \frac{\partial}{\partial u} L(u^*, \lambda) &= 0 \\ h(u^*) &\leq \mathbf{0} \\ \lambda^T h(u^*) &= 0 \\ \lambda &\geq \mathbf{0} \end{aligned}$$

where the Lagrangian is $L(u, \lambda) = g(u) + \lambda^T h(u)$. Note that if we're maximizing, the same holds except we need $\lambda \leq 0$.

Example

$$\begin{aligned} \min \quad & 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 \\ \text{s.t.} \quad & \begin{cases} u_1^2 + u_2^2 \leq 5 \\ 3u_1 + u_2 \leq 6 \end{cases} \end{aligned}$$

$$L = 2u_1^2 + 2u_1u_2 + u_2^2 - 10u_1 - 10u_2 + \lambda_1(u_1^2 + u_2^2 - 5) + \lambda_2(3u_1 + u_2 - 6)$$

FONC:

- i) $\partial L / \partial u_1 = 4u_1 + 2u_2 - 10 + 2\lambda_1u_1 + 3\lambda_2$
- ii) $\partial L / \partial u_2 = 2u_1 + 2u_2 - 10 + 2\lambda_1u_2 + \lambda_2$
- iii) $u_1^2 + u_2^2 \leq 5$
- iv) $3u_1 + u_2 \leq 6$
- v) $\lambda_1(u_1^2 + u_2^2 - 5) = 0$
- vi) $\lambda_2(3u_1 + u_2 - 6) = 0$
- vii) $\lambda_1 \geq 0$
- viii) $\lambda_2 \geq 0$

To solve, assume different constraints are active/inactive:

1. Both constraints are inactive ($u_1^2 + u_2^2 < 5$, $3u_1 + u_2 < 6$) $\implies \lambda_1 = \lambda_2 = 0$

$$\begin{cases} 4u_1 + 2u_2 - 10 = 0 \\ 2u_1 + 2u_2 - 10 = 0 \end{cases} \implies \begin{cases} u_1 = 0 \\ u_2 = 5 \end{cases}$$

Note: iii) $0^2 + 5^2 \not\leq 5$

Not feasible

2. Assume constraint 1 is active and constraint 2 is inactive ($u_1^2 + u_2^2 = 5$, $\lambda_2 = 0$)

$$\begin{cases} 4u_1 + 2u_2 - 10 + 2\lambda_1u_1 = 0 \\ 2u_1 + 2u_2 - 10 + 2\lambda_1u_2 = 0 \\ u_1^2 + u_2^2 = 5 \end{cases} \implies \begin{cases} u_1 = 1 \\ u_2 = 2 \\ \lambda_1 = 1 \end{cases}$$

$$\checkmark \lambda_1 \geq 0$$

$$\checkmark 3 \cdot 1 + 2 \leq 6$$

This is a local minimizer

3. Assume constraint 2 is active and constraint 1 is inactive
4. Assume both constraints are active

Kuhn-Tucker Conditions (KKT conditions, Karush-Kuhn-Tucker)

Problem:

$$\begin{aligned} \min_{u \in \mathbb{R}^m} \quad & g(u) \\ \text{s.t.} \quad & \begin{cases} h_1(u) = 0, & h_1 : \mathbb{R}^m \rightarrow \mathbb{R}^p \\ h_2(u) \leq 0, & h_2 : \mathbb{R}^m \rightarrow \mathbb{R}^k \end{cases} \end{aligned} \quad (1.1)$$

Theorem. Let u^* be feasible ($h_1 = 0$, $h_2 \leq 0$). If u^* is a minimizer to (1.1) then there exists vectors $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^k$ with $\mu \geq 0$ such that

$$\begin{cases} \frac{\partial g}{\partial u}(u^*) + \lambda^T \frac{\partial h_1}{\partial u}(u^*) + \mu^T \frac{\partial h_2}{\partial u}(u^*) = 0 \\ \mu^T h_2(u^*) = 0 \end{cases}$$

Looking ahead: $\min \text{cost}(u(\cdot))$ s.t. $\dot{x} = f(x, u)$ (dynamics), where u is a function. Note the equality constraint.

Question: How do we go from $u \in \mathbb{R}^m$ to $u \in \mathcal{U}$ (function space)?

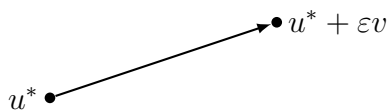
Note: Function space is a set of functions of a given kind from a set X to a set Y

1. linear function
2. square-integrable functions: $L_2[0, T] : \int_0^T \|u(t)\|^2 dt < \infty$
3. $C^\infty(\mathbb{R})$

What would $\partial \text{"cost"} / \partial u$ mean?

1.5 Directional Derivatives

Recall: To minimize $g(u)$, let u^* be a candidate minimizer and pitch a perturbation on u^* of εv , where ε is the scale and v is the direction. Taking Taylor's expansion at the perturbation produces



$$g(u^* + \varepsilon v) = g(u^*) + \varepsilon \frac{\partial g}{\partial u}(u^*) v + o(\varepsilon)$$

$$\text{FONC: } \frac{\partial g}{\partial u}(u^*) = 0$$

Note: $\frac{\partial g}{\partial u}(u^*)v$ tells us how much $g(u)$ increases/decreases in the direction of v .

Definition. The directional (Gateaux) derivative is given by

$$\delta g(u; v) = \lim_{\varepsilon \rightarrow 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon}$$

Example

$$g(u) = \frac{1}{2}u_1^2 - u_1 + 2u_2, \quad g : \mathbb{R}^2 \rightarrow \mathbb{R}$$

Let's consider $e_1 = [1 \ 0]^T$, $e_2 = [0 \ 1]^T$. What is $\delta g(u; e_i)$, $i = 1, 2$?

$$\begin{aligned} \delta g(u; v) &= \lim_{\varepsilon \rightarrow 0} \frac{g(u + \varepsilon v) - g(u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{g(u) + \varepsilon \frac{\partial g}{\partial u}(u)v + o(\varepsilon) - g(u)}{\varepsilon} \\ &= \frac{\partial g}{\partial u}(u)v \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial u}(u) &= [u_1 - 1 \ 2] \\ \delta g(u; e_1) &= [u_1 - 1 \ 2]e_1 = u_1 - 1 \\ \delta g(u; e_2) &= [u_1 - 1 \ 2]e_2 = 2 \end{aligned}$$

But the beauty of directional derivatives is that they generalize beyond vectors, $u \in \mathbb{R}^m$, to function spaces (\mathcal{U}) or other “objects” like matrices.

Example $M \in \mathbb{R}^{n \times n}$, $F(M) = M^2$

What is $\frac{\partial F}{\partial M}$? (ponder at home...)

We can easily compute $\delta F(M; N)$!

$$\begin{aligned} F(M + \varepsilon N) &= (M + \varepsilon N)(M + \varepsilon N) = M^2 + \varepsilon MN + \varepsilon NM + \varepsilon^2 N^2 \\ \delta F(M; N) &= \lim_{\varepsilon \rightarrow 0} \frac{F(M + \varepsilon N) - F(M)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon MN + \varepsilon NM + \varepsilon^2 N^2}{\varepsilon} = MN + NM \end{aligned}$$

Infinite Dimensional Optimization Let $u \in \mathcal{U}$ (function space) and let $J(u)$ be the cost:

$$\min_{u \in \mathcal{U}} J(u)$$

Theorem. If $u^* \in \mathcal{U}$ is a (local) minimizer then

$$\delta J(u^*; v) = 0, \quad \forall v \in \mathcal{U}$$

Example Find minimizer u^* to

$$J(u) = \int_0^T L(u(t)) \, dt$$

$$\begin{aligned} J(u + \varepsilon v) - J(u) &= \int_0^T L(u(t) + \varepsilon v(t)) \, dt - \int_0^T L(u(t)) \, dt, \quad u, v \in \mathcal{U} \\ &= \int_0^T \left[L(u(t)) + \varepsilon \frac{\partial L}{\partial u}(u(t))v(t) + o(\varepsilon) - L(u(t)) \right] \, dt \\ \delta J(u^*; v) &= \lim_{\varepsilon \rightarrow 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\int_0^T \varepsilon \frac{\partial L}{\partial u}(u(t))v(t) \, dt + o(\varepsilon)}{\varepsilon} \\ &= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t) \, dt \end{aligned}$$

u^* optimizer:

$$\begin{aligned} \delta J(u^*; v) &= \int_0^T \frac{\partial L}{\partial u}(u(t))v(t) \, dt = 0 \quad \forall v \in \mathcal{U} \\ &\quad \Updownarrow \\ \frac{\partial L}{\partial u}(u(t)) &= 0 \quad \forall t \in [0, T] \end{aligned}$$

But, we want *optimal control*! We want our cost to look like

$$\begin{aligned} &\int_0^T L(x(t), u(t)) \, dt \\ &\dot{x} = f(x, u) \end{aligned}$$

1.6 Calculus of Variations

What happens to $x(t)$ when $u(t)$ changes to $u(t) + \varepsilon v(t)$? Let the system be given by

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

After perturbation of u , the new system is

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}, u + \varepsilon v) \\ \hat{x}(0) = x_0 \end{cases}$$

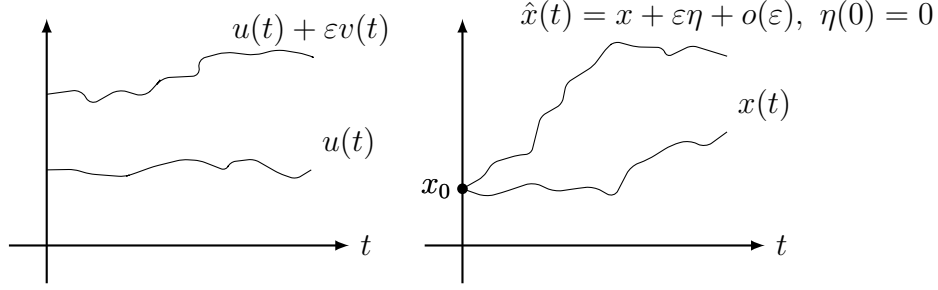


Figure 1.1: Variation in u causes a variation in x .

Consider

$$\tilde{x} = x + \varepsilon\eta,$$

where

$$\begin{aligned} \dot{x} &= f(x, u), & x(0) &= x_0 \\ \dot{\eta} &= \frac{\partial f}{\partial x}(x, u)\eta + \frac{\partial f}{\partial u}(x, u)v, & \eta(0) &= 0 \end{aligned}$$

Theorem. *If f is continuously differentiable in x and u then*

$$\hat{x}(t) = \tilde{x}(t) + o(\varepsilon)$$

Proof.

i) Initial conditions:

$$\begin{aligned} \hat{x}(0) &= x_0 \\ \tilde{x}(0) &= x(0) + \varepsilon\eta(0) = x_0 \end{aligned}$$

ii) Dynamics:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v) \\ \dot{\tilde{x}} &= \dot{x} + \varepsilon\dot{\eta} = f(x, u) + \varepsilon \frac{\partial f}{\partial x}(x, u)\eta + \varepsilon \frac{\partial f}{\partial u}(x, u)v \\ &= f(x + \varepsilon\eta, u + \varepsilon v) + o(\varepsilon) \\ &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon) \end{aligned}$$

We can see that the dynamics of $\hat{x}(t)$ are equal to those of $\tilde{x}(t)$ plus higher order terms:

$$\begin{aligned} \dot{\tilde{x}} &= f(\tilde{x}, u + \varepsilon v) + o(\varepsilon) \\ \dot{\hat{x}} &= f(\hat{x}, u + \varepsilon v) \end{aligned}$$

Therefore, if our perturbation is small enough, we can model $\hat{x}(t)$ as $\tilde{x}(t)$.

□

Note: Taylor expansion with two elements is

$$\begin{aligned}
 h(w + \varepsilon v, z + \varepsilon y) &= h(w, z + \varepsilon y) + \frac{\partial h}{\partial w}(w, z + \varepsilon y)\varepsilon v + o(\varepsilon) \\
 &= \left\{ h(w, z) + \frac{\partial h}{\partial z}(w, z)\varepsilon y + o(\varepsilon) \right\} \\
 &\quad + \left\{ \frac{\partial h}{\partial w}(w, z)\varepsilon v + \underbrace{\frac{\partial^2 h}{\partial z \partial w}\varepsilon v \odot \varepsilon y}_{o(\varepsilon)} + o(\varepsilon) \right\} \\
 &= h(w, z) + \frac{\partial h}{\partial z}\varepsilon y + \frac{\partial h}{\partial w}\varepsilon v + o(\varepsilon)
 \end{aligned}$$

Last class:

1. $u \in \mathcal{U}$ (space of functions), $J : \mathcal{U} \rightarrow \mathbb{R}$ (cost).

FONC: If u^* is optimal, then

$$\delta J(u; \nu) = 0 \quad \forall \nu \in \mathcal{U},$$

where the directional derivative is given by

$$\delta J(u; \nu) = \lim_{\varepsilon \rightarrow 0} \frac{J(u + \varepsilon \nu) - J(u)}{\varepsilon}.$$

2. If

$$\begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases}$$

then a variation in u :

$$u \longmapsto u + \varepsilon \nu$$

results in a variation in x :

$$x \longmapsto x + \varepsilon \eta + o(\varepsilon)$$

See Figure 1.1. Note $\eta(0) = 0$.

1.6.1 An (Almost) Optimal Control Problem

Let $\dot{x} = f(x)$, $x(0) = x_0$. Note we get to pick the initial condition!

Problem

$$\begin{aligned} \min_{x_0 \in \mathbb{R}^m} J(x_0) &= \int_0^T L(x(t)) dt \\ \text{s.t. } \begin{cases} \dot{x}(t) = f(x(t)) & \text{the constraint! (equality)} \\ x(0) = x_0 \end{cases} \end{aligned}$$

Note every constraint needs a Lagrange multiplier. We have infinitely many constraints:

$$\dot{x}(t) = f(x(t)) \quad \forall t \in [0, T]$$

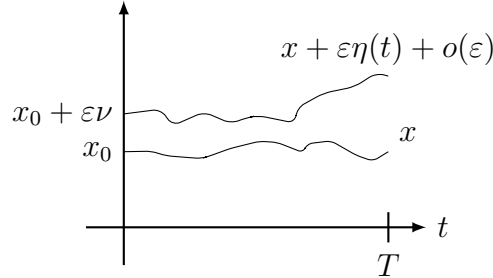
We need $\lambda(t)$ as a function of t . Also, the sum in the Lagrangian has to become an integral. The continuous-time Lagrangian thus becomes

$$\tilde{J}(x_0, \lambda) = \int_0^T \left[L(x(t)) + \lambda^T(t)(f(x(t)) - \dot{x}(t)) \right] dt$$

The task is to perturb x_0 as $x_0 \mapsto x_0 + \varepsilon\nu$, $\nu \in \mathbb{R}^m$ and compute

$$\delta \tilde{J}(x_0; \nu) = \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(x_0 + \varepsilon\nu) - \tilde{J}(x_0)}{\varepsilon}$$

and make this equal to 0 $\forall \nu \in \mathbb{R}^m$. The variation in x is



Note:

- x_0 decision variable
- ν variation in x_0
- $x(t)$ trajectory starting at x_0
- $\eta(t)$ change in trajectory resulting from ν -variation in x_0
- $\lambda(t)$ time-varying Lagrange multiplier

$$\begin{aligned} \tilde{J}(x_0 + \varepsilon\nu) &= \int_0^T \left\{ L(x(t)) + \lambda^T(t)[f(x(t) + \varepsilon\eta(t)) - \dot{x}(t) - \varepsilon\dot{\eta}(t)] \right\} dt + o(\varepsilon) \\ &= \int_0^T \left[L(x) + \varepsilon \frac{\partial L}{\partial x}(x)\eta + \lambda^T \left(f(x) + \varepsilon \frac{\partial f}{\partial x}(x)\eta - \dot{x} - \varepsilon\dot{\eta} \right) \right] dt + o(\varepsilon) \\ \tilde{J}(x_0 + \varepsilon\nu) - \tilde{J}(x_0) &= \int_0^T \left[\varepsilon \frac{\partial L}{\partial x}(x)\eta + \lambda^T \left(\varepsilon \frac{\partial f}{\partial x}\eta - \varepsilon\dot{\eta} \right) \right] dt + o(\varepsilon) \\ \delta \tilde{J}(x_0; \nu) &= \int_0^T \left[\frac{\partial L}{\partial x}(x)\eta + \lambda^T \left(\frac{\partial f}{\partial x}\eta - \dot{\eta} \right) \right] dt \end{aligned}$$

A powerful idea: we want $\delta\tilde{J}(x_0; \nu) = 0 \forall \nu$. Somehow get this in the form

$$\int_0^T (\text{stuff}(t)) \eta(t) dt = 0$$

We can pick $\text{stuff}(t) = 0 \forall t \in [0, T]$.

In $\delta\tilde{J}(x_0; \nu)$ we have $\dot{\eta}$ (problem!). We can solve this using *integration by parts*.

$$\int_0^T \lambda^T \dot{\eta} dt = \lambda^T(T) \eta(T) - \lambda^T(0) \eta(0) - \int_0^T \dot{\lambda}^T \eta dt$$

Hence,

$$\delta\tilde{J}(x_0; \nu) = \int_0^T \underbrace{\left(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T \right)}_{\text{pick}=0} \eta dt - \underbrace{\lambda^T(T) \eta(T)}_{\text{pick}=0} + \lambda^T(0) \underbrace{\eta(0)}_{\nu}$$

We are free to pick λ freely if it gives $\delta\tilde{J} = 0$.

$$\text{Pick: } \begin{cases} \dot{\lambda}(t) = -\frac{\partial L^T}{\partial x}(x(t)) - \frac{\partial f^T}{\partial x}(x(t)) \lambda(t) \\ \lambda(T) = 0 \end{cases} \quad \text{backwards diff. eq}$$

Under this choice of λ we get

$$\delta\tilde{J}(x_0; \nu) = \lambda^T(0) \nu$$

This is linear in ν so the FONC is $\lambda(0) = 0$.

Moreover, we really have a “normal” optimization problem

$$\begin{aligned} \min_{x_0 \in \mathbb{R}^m} \tilde{J}(x_0) \\ \delta\tilde{J}(x_0; \nu) = \frac{\partial \tilde{J}}{\partial x_0}(x_0) \nu \end{aligned}$$

which means that

$$\frac{\partial \tilde{J}}{\partial x_0} = \lambda^T(0)$$

If x_0^* minimizes

$$\begin{aligned} & \int_0^T L(x(t)) dt \\ \text{s.t. } & \begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = x_0^* \end{cases} \end{aligned}$$

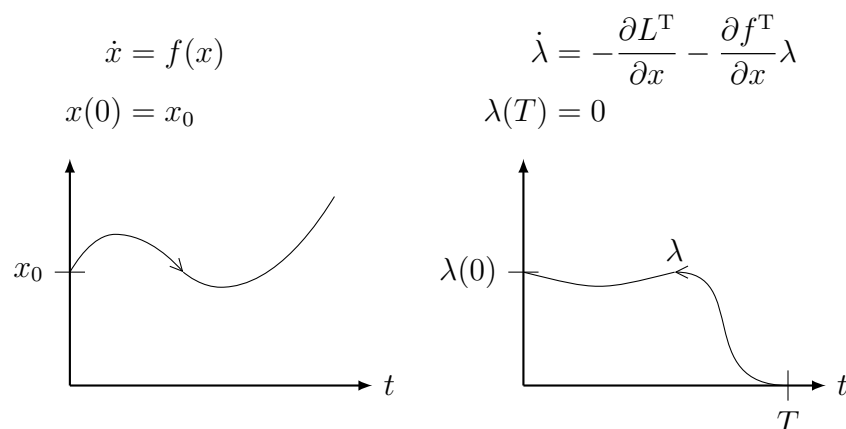
then

$$\lambda(0) = \mathbf{0}$$

where $\lambda(t)$ satisfies

$$\begin{cases} \dot{\lambda}(t) = -\frac{\partial L^T}{\partial x}(x(t)) - \frac{\partial f^T}{\partial x}(x(t)) \lambda(t) \\ \lambda(T) = 0 \end{cases}$$

So what? We actually have a two-point boundary value problem.



We want to find x_0 that gives $f(x)$ such that after solving backwards for $\lambda(t)$, we find that

$$\lambda(0) = \frac{\partial \tilde{J}^T}{\partial x_0} = 0.$$

This leads to the following:

An algorithm

```

Pick  $x_{0,0}$ 
 $k = 1$ 
repeat
    Simulate  $x(t)$  from  $x_{0,k}$  over  $[0, T]$ 
    Simulate  $\lambda(t)$  from  $\lambda(T) = 0$  backwards using  $x(t)$ 
    Update  $x_{0,k}$  as  $x_{0,k+1} = x_{0,k} - \gamma \lambda(0)$ 
     $k := k + 1$ 
until  $\lambda(0) = 0$ 

```

▷ $\lambda(0)$ is the gradient

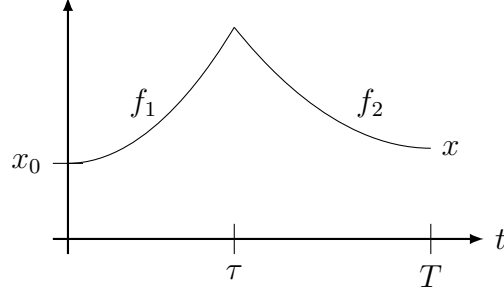
Example: `optinit.m`

$$\begin{aligned} \dot{x} &= Ax, & L &= x^T Q x - q, & Q &= Q^T \succ 0 \\ \dot{\lambda} &= -2Qx - A^T \lambda \\ \lambda(0) &= 0 \end{aligned}$$

1.6.2 Optimal Timing Control

When to switch between modes?

$$\begin{aligned} \dot{x} &= \begin{cases} f_1(x) & \text{if } t \in [0, \tau) \\ f_2(x) & \text{if } t \in [\tau, T] \end{cases} \\ x(0) &= x_0 \end{aligned} \tag{1.2}$$

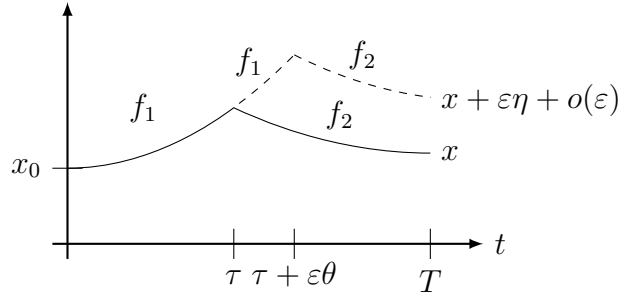


$$\begin{aligned} \min_{\tau} \int_0^T L(x(t)) dt &= J(\tau) \\ \text{s.t. (1.2) holds} \end{aligned}$$

Step 1: Augment cost with constraint

$$\tilde{J} = \int_0^{\tau} \left[L(x) + \lambda^T (f_1(x) - \dot{x}) \right] dt + \int_{\tau}^T \left[L(x) + \lambda^T (f_2(x) - \dot{x}) \right] dt$$

Step 2: Variation $\tau \mapsto \tau + \varepsilon\theta$



Step 3: Compute $\delta\tilde{J}(\tau; \theta)$

$$\begin{aligned} \tilde{J}(\tau + \varepsilon\theta) &= \int_0^{\tau + \varepsilon\theta} \left\{ L(x + \varepsilon\eta) + \lambda^T [f_1(x + \varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \right\} dt \\ &\quad + \int_{\tau + \varepsilon\theta}^T \left\{ L(x + \varepsilon\eta) + \lambda^T [f_2(x + \varepsilon\eta) - \dot{x} - \varepsilon\dot{\eta}] \right\} dt + o(\varepsilon) \end{aligned}$$

Note that $\eta = \dot{\eta} = 0$ on $[0, \tau)$.

$$\begin{aligned} \tilde{J}(\tau + \varepsilon\theta) &= \int_0^{\tau} \left\{ L(x) + \lambda^T [f_1(x) - \dot{x}] \right\} dt \\ &\quad + \int_{\tau}^{\tau + \varepsilon\theta} \left\{ \underbrace{L(x + \varepsilon\eta)}_{L(x) + \varepsilon \frac{\partial L}{\partial x} \eta} + \lambda^T \left[\underbrace{f_1(x + \varepsilon\eta)}_{f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta} - \dot{x} - \varepsilon\dot{\eta} \right] \right\} dt \\ &\quad + \int_{\tau + \varepsilon\theta}^T \left\{ \underbrace{L(x + \varepsilon\eta)}_{L(x) + \varepsilon \frac{\partial L}{\partial x} \eta} + \lambda^T \left[\underbrace{f_2(x + \varepsilon\eta)}_{f_2(x) + \varepsilon \frac{\partial f_2}{\partial x} \eta} - \dot{x} - \varepsilon\dot{\eta} \right] \right\} dt + o(\varepsilon) \end{aligned}$$

$$\begin{aligned}
\delta \tilde{J}(\tau; \theta) &= \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau)}{\varepsilon} \\
\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau) &= \int_0^\tau 0 \cdot dt + \underbrace{\int_\tau^{\tau+\varepsilon\theta} \left[\varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left(f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta - f_2(x) - \varepsilon \dot{\eta} \right) \right] dt}_{I_1} \\
&\quad + \underbrace{\int_{\tau+\varepsilon\theta}^T \left[\varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left(\varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt}_{I_2} + o(\varepsilon)
\end{aligned}$$

Theorem (Mean-value theorem).

$$\int_{t_1}^{t_2} h(t) dt = (t_2 - t_1)h(\xi) \quad \text{for some } \xi \in [t_1, t_2]$$

The first integral is

$$\begin{aligned}
I_1 &= \int_\tau^{\tau+\varepsilon\theta} \left\{ \varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left[f_1(x) + \varepsilon \frac{\partial f_1}{\partial x} \eta - \varepsilon \dot{\eta} - f_2(x) \right] \right\} dt \\
&= \varepsilon \theta \left\{ \lambda^\top(\xi) [f_1(x(\xi)) - f_2(x(\xi))] \right\} + o(\varepsilon)
\end{aligned}$$

Note that as $\varepsilon \rightarrow 0$, $\xi \rightarrow \tau$. Using integration by parts, the second integral is

$$\begin{aligned}
\int_\tau^T \lambda^\top \dot{\eta} dt &= \lambda^\top(T) \eta(T) - \lambda^\top(\tau) \underbrace{\eta(\tau)}_{=0} - \int_\tau^T \dot{\lambda}^\top \eta dt \\
I_2 &= \int_\tau^T \left[\varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left(\varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt - \underbrace{\int_\tau^{\tau+\varepsilon\theta} \left[\varepsilon \frac{\partial L}{\partial x} \eta + \lambda^\top \left(\varepsilon \frac{\partial f_2}{\partial x} \eta - \varepsilon \dot{\eta} \right) \right] dt}_{o(\varepsilon)} \\
&= \varepsilon \int_\tau^T \left[\frac{\partial L}{\partial x} + \lambda^\top \frac{\partial f_2}{\partial x} + \dot{\lambda}^\top \right] \eta dt - \varepsilon \lambda^\top(T) \eta(T) + o(\varepsilon)
\end{aligned}$$

Hence,

$$\begin{aligned}
\delta \tilde{J}(\tau; \theta) &= \lim_{\varepsilon \rightarrow 0} \frac{\tilde{J}(\tau + \varepsilon\theta) - \tilde{J}(\tau)}{\varepsilon} \\
&= \theta \lambda^\top(\tau) [f_1(x(\tau)) - f_2(x(\tau))] + \int_\tau^T \left[\frac{\partial L}{\partial x} + \lambda^\top \frac{\partial f_2}{\partial x} + \dot{\lambda}^\top \right] \eta dt - \lambda^\top(T) \eta(T)
\end{aligned}$$

Step 4: Select the *costate* $\lambda(t)$. The key idea is to get rid of any term that has η in it, i.e.

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial L}{\partial x} - \frac{\partial f_2^\top}{\partial x} \lambda \quad \text{on } [\tau, T] \\
\lambda(T) &= 0
\end{aligned}$$

Step 5: With this choice of $\lambda(t)$, we have

$$\delta \tilde{J}(\tau; \theta) = \theta \lambda^T(\tau) [f_1(x(\tau)) - f_2(x(\tau))] = \frac{\partial \tilde{J}}{\partial \tau} \theta.$$

Therefore,

$$\frac{\partial \tilde{J}}{\partial \tau} = \lambda^T(\tau) [f_1(x(\tau)) - f_2(x(\tau))] = 0 \quad (\text{for optimality})$$

Algorithm

```

Pick  $\tau_0$ 
 $k = 0$ 
repeat
    Simulate  $x$  forward in time from  $x(0) = x_0$ 
    Simulate  $\lambda$  backwards from  $\lambda(T) = 0$ 
    Update  $\tau_k$  as  $\tau_{k+1} = \tau_k - \gamma \lambda^T(\tau_k) [f_1(x(\tau_k)) - f_2(x(\tau_k))]$ 
     $k := k + 1$ 
until  $\|\lambda^T(f_1 - f_2)\| < \varepsilon$ 

```

Where are we going? Come up with general principles for $\min_{u \in \mathcal{U}} J(u)$:

- Costate equations
- Optimality conditions
- Algorithms
- Applications

1.6.3 The Bolza Problem

Up until now, we have optimized with respect to finite-dimensional parameters. Today, we will minimize with respect to $u \in \mathcal{U}$.

$$\begin{aligned}
 \min_{u \in \mathcal{U}} J(u) &= \int_0^T L(x(t), u(t), t) dt + \underbrace{\Psi(x(T))}_{\substack{\text{terminal cost} \\ \text{(parking cost)}}} \\
 \text{s.t. } \quad &\dot{x}(t) = f(x(t), u(t), t) \\
 &x(0) = x_0
 \end{aligned}$$

Assume that f and L are C^1 in x, u and piecewise continuous in t . Then, a small change in u causes small changes in f and L . The variation: $u \mapsto u + \varepsilon v$, $\varepsilon \in \mathbb{R}$, $v \in \mathcal{U}$. See Figure 1.1.

$$\begin{aligned}
\tilde{J}(u) &= \int_0^T [L(x, u, t) + \lambda^T(f(x, u, t) - \dot{x})] dt + \Psi(x(T)) \\
\tilde{J}(u + \varepsilon v) &= \int_0^T [L(x + \varepsilon \eta, u + \varepsilon v, t) + \lambda^T(f(x + \varepsilon \eta, u + \varepsilon v, t) - \dot{x} - \varepsilon \dot{\eta})] dt \\
&\quad + \Psi(x(T) + \varepsilon \eta(T)) + o(\varepsilon) \\
\tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \left[L(x + \varepsilon \eta, u + \varepsilon v, t) - L(x, u, t) \right. \\
&\quad \left. + \lambda^T(f(x + \varepsilon \eta, u + \varepsilon v, t) - f(x, u, t) - \dot{x} - \varepsilon \dot{\eta} + \dot{x}) \right] dt \\
&\quad + \Psi(x(T) + \varepsilon \eta(T)) - \Psi(x(T)) + o(\varepsilon) \\
&= \int_0^T \left[\frac{\partial L}{\partial x} \varepsilon \eta + \frac{\partial L}{\partial u} \varepsilon v + \lambda^T \left(\frac{\partial f}{\partial x} \varepsilon \eta + \frac{\partial f}{\partial u} \varepsilon v - \varepsilon \dot{\eta} \right) \right] dt \\
&\quad + \frac{\partial \Psi}{\partial x}(x(T)) \varepsilon \eta(T) + o(\varepsilon)
\end{aligned}$$

(See Taylor expansion with respect to two variables.)

$$\begin{aligned}
\delta \tilde{J}(u; v) &= \int_0^T \left(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) v dt + \int_0^T \left[\left(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} \right) \eta - \lambda^T \dot{\eta} \right] dt \\
&\quad + \frac{\partial \Psi}{\partial x}(x(T)) \eta(T)
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
\int_0^T \lambda^T \dot{\eta} dt &= \lambda^T(T) \eta(T) - \lambda^T(0) \eta(0) - \int_0^T \dot{\lambda}^T \eta dt \\
&= \lambda^T(T) \eta(T) - \int_0^T \dot{\lambda}^T \eta dt \\
\delta \tilde{J}(u; v) &= \int_0^T \left(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \right) v dt + \int_0^T \left(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T \right) \eta dt \\
&\quad + \left(\frac{\partial \Psi}{\partial x}(x(T)) - \lambda^T(T) \right) \eta(T)
\end{aligned}$$

For optimality, we need the directional derivative to be zero for every $v \in \mathcal{U}$, where v represents the direction of the derivative. Therefore, the term $(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u})$ in the first integral has to be identically zero. Thus, we need

$$\begin{cases} \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0, & \forall t \in [0, T] \\ \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T = 0, & \forall t \in [0, T] \\ \frac{\partial \Psi}{\partial x}(x(T)) - \lambda^T(T) = 0 \end{cases}$$

Definition. Let the *Hamiltonian* $H(x, u, t, \lambda)$ be given by

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^T f(x, u, t)$$

Theorem. For u to solve the Bolza problem, it has to satisfy

$$\frac{\partial H}{\partial u}(x, u, t, \lambda) = 0,$$

where the costate satisfies

$$\begin{cases} \dot{\lambda} = -\frac{\partial H^T}{\partial x}(x, u, t, \lambda) \\ \lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T)) \end{cases}$$

Example

$$\begin{aligned} & \min_u \int_0^1 \frac{1}{2} u^2(t) dt + \frac{1}{2} x^2(1) \\ & \text{s.t.} \quad \begin{cases} \dot{x} = u, & x, u \in \mathbb{R} \\ x(0) = 1 \end{cases} \end{aligned}$$

$$H = \frac{1}{2} u^2 + \lambda u$$

$$\frac{\partial H}{\partial u} = u + \lambda = 0 \implies u = -\lambda$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0 \implies \lambda(t) = c$$

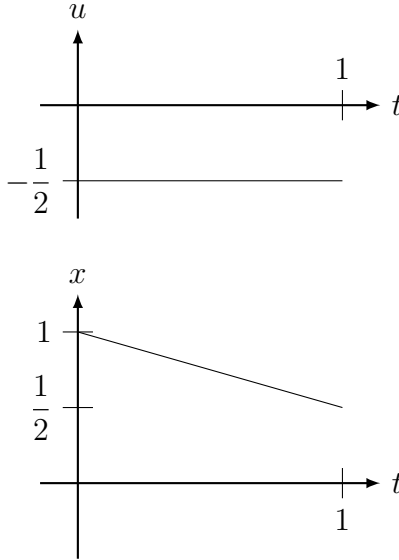
$$\lambda(T) = c = \frac{\partial \Psi}{\partial x}(x(1)) = x(1)$$

$$\dot{x} = u = -c \implies x(t) = -ct + x(0) = -ct + 1$$

$$x(1) = -c + 1$$

$$\lambda(1) = c = x(1) = -c + 1 \implies c = \frac{1}{2}$$

$$\boxed{u^* = -\frac{1}{2}}$$



We really used five different equations to solve this!

- i) $\frac{\partial H}{\partial u} = 0$
- ii) $\dot{\lambda} = -\frac{\partial H^T}{\partial x}$
- iii) $\lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T))$
- iv) $\dot{x} = f(x, u, t)$
- v) $x(0) = x_0$

There is a sixth condition that is pretty useful if L and f do not depend on t ($L(x, u)$, $f(x, u)$). This is called a *conservative system*. Then, along optimal trajectories (equations i-v are satisfied), the total time derivative of the Hamiltonian is

$$\frac{d}{dt}H = \underbrace{\frac{\partial H}{\partial t}}_{0 \text{ } H(x,u,\lambda)} + \underbrace{\frac{\partial H}{\partial x}}_{-\dot{\lambda}^T} \dot{x} + \underbrace{\frac{\partial H}{\partial u}}_{0 \text{ } u \text{ is optimal}} \dot{u} + \underbrace{\frac{\partial H}{\partial \lambda}}_{f^T = \dot{x}^T} \dot{\lambda} = -\dot{\lambda}^T \dot{x} + \dot{x}^T \dot{\lambda} = 0$$

Therefore, for conservative systems,

- vi) H is constant along optimal trajectories. (Hamilton's Principle in analytical mechanics)

Back to the example,

$$H = \frac{1}{2}u^2 + \lambda u = \frac{1}{2}c^2 - c^2 = -\frac{1}{2}c^2 = -\frac{1}{8}$$

The Hamiltonian

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^T f(x, u, t)$$

lets us write the Lagrangian as

$$\tilde{J}(u) = \int_0^T [L + \lambda^T(f - \dot{x})] dt + \Psi = \int_0^T (H - \lambda^T \dot{x}) dt + \Psi$$

The optimality conditions are

$$\frac{\partial H}{\partial u} = 0, \quad (1.3)$$

where

$$\begin{cases} \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = \frac{\partial \Psi}{\partial x}(x(T)) \end{cases} \quad (1.4)$$

Example Hamilton's Principle

Let q be the generalized coordinates (positions and angles). Then, $\dot{q} = u$ are generalized velocities, which we assume we can control. Let $T(q, u) = u^T M(q)u$, $M \succ 0$, be the kinetic energy and $V(q)$ be the potential energy.

For conservative systems, the following quantity is minimized:

$$\int_0^T \underbrace{[T(q, u) - V(q)]}_{L(q, u) = \text{Lagrange's "action function"}} dt$$

The Hamiltonian is

$$H(q, u, \lambda) = L(q, u) + \lambda^T f(q, u) = L(q, u) + \lambda^T u$$

In mechanics, λ is called a generalized momentum, satisfying

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H^T}{\partial q} = -\frac{\partial L^T}{\partial q} + 0 \\ 0 &= \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \implies \lambda = -\frac{\partial L^T}{\partial u} \\ \dot{\lambda} &= -\frac{d}{dt} \frac{\partial L^T}{\partial u} = -\frac{\partial L^T}{\partial q} \end{aligned}$$

This produces the Euler-Lagrange Equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

Recall, along optimal trajectories

$$\frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial t}}_{=0 \text{ if } L \text{ and } f \text{ do not depend explicitly on } t} + \overbrace{\frac{\partial H}{\partial x} \dot{x}}^{-\dot{\lambda}^T \dot{x}} + \underbrace{\frac{\partial H}{\partial u} \dot{u}}_{=0} + \underbrace{\frac{\partial H}{\partial \lambda} \dot{\lambda}}_{f^T = \dot{x}^T} = -\dot{\lambda}^T \dot{x} + \dot{x}^T \dot{\lambda} = 0$$

Therefore, along optimal trajectories, the Hamiltonian is constant!

We had

$$H = L + \lambda^T u$$

$$\frac{\partial H}{\partial u} = \lambda^T + \frac{\partial L}{\partial u} = 0$$

Along optimal trajectories,

$$H = L - \frac{\partial L}{\partial u} u$$

Recall, $L(q, u) = T(q, u) - V(q)$.

$$\frac{\partial L}{\partial u} = \frac{\partial T}{\partial u} - 0$$

$$T(q, u) = u^T M(q) u$$

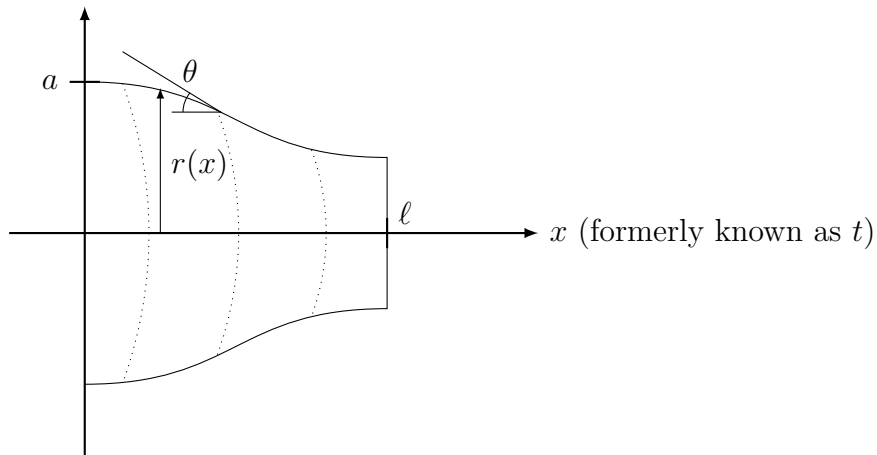
$$\frac{\partial T}{\partial u} = 2u^T M$$

So,

$$H = \underbrace{T}_{u^T M u} - V - 2u^T M u = -(V + u^T M u) = -(V + T)$$

Therefore, the total energy (kinetic plus potential energy) remains constant for conservative systems.

Example minimum drag nose shape (Newton 1686)



The drag is

$$D = -2\pi q \int_{x=0}^{\ell} C_p(\theta) r \, dr,$$

where q is a pressure constant and $C_p(\theta) = 2 \sin^2 \theta$ is Newton's pressure formula.

Geometry tells us

$$\frac{dr}{dx} = -\tan \theta = -u$$

Choose the control as $\tan \theta$. Manipulating the drag,

$$\frac{D}{4\pi q} = \int_0^{\ell} \frac{ru^3}{1+u^2} dx + \frac{1}{2}r(\ell)^2$$

The optimal control problem is

$$\begin{aligned} \min_u \quad & \int_0^{\ell} \frac{ru^3}{1+u^2} dx + \frac{1}{2}r(\ell)^2 \\ \text{s.t.} \quad & \frac{dr}{dx} = -u \end{aligned}$$

This is in the standard form with the following changes of variables:

$$\begin{aligned} \ell &\longleftarrow T \\ x &\longleftarrow t \\ r &\longleftarrow x \end{aligned}$$

Refer to (1.3) and (1.4) for the following steps.

$$\begin{aligned} H &= \frac{ru^3}{1+u^2} - \lambda u \\ \frac{\partial H}{\partial u} &= \frac{3ru^2(1+u^2) - ru^3 \cdot 2u}{(1+u^2)^2} - \lambda \\ &= \frac{ru^4 + 3ru^2}{(1+u^2)^2} - \lambda = 0 \\ \lambda &= \frac{ru^2(u^2 + 3)}{(1+u^2)^2} \\ \frac{d\lambda}{dx} &= -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2} \\ \lambda(\ell) &= r(\ell) \end{aligned} \tag{1.5}$$

Right now, we know

$$\begin{cases} \frac{dr}{dx} = -u \\ r(0) = a \\ \frac{d\lambda}{dx} = -\frac{u^3}{1+u^2} \\ \lambda(\ell) = r(\ell) \end{cases}$$

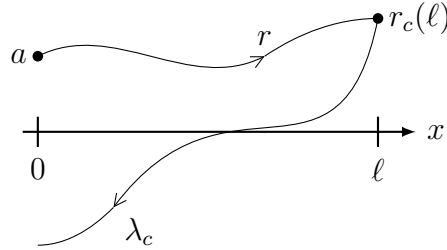
We need to remove u and get a function of r and λ instead. However, it is difficult to solve (1.5). Maybe $H = \text{const.}$ gives us something nicer?

$$\begin{aligned} H &= \frac{ru^3}{1+u^2} - \lambda u \\ &= \frac{ru^3}{1+u^2} - \frac{ru^2(u^2+3)}{(1+u^2)^2}u \\ &= -\frac{2ru^3}{(1+u^2)^2} = c \end{aligned}$$

Assume we can find $u = G(r, c)$, either numerically or some other way. So, now we have

$$\begin{cases} \frac{dr}{dx} = -G(r, c) \\ r(0) = a \\ \frac{d\lambda}{dx} = -\frac{G^3(r, c)}{1+G^2(r, c)} \\ \lambda(\ell) = r(\ell) \end{cases}$$

We do not know c , but we can guess c and simulate r forward in “time” (x) from $r(0) = a$. Then, we simulate λ backwards from $r(\ell)$.

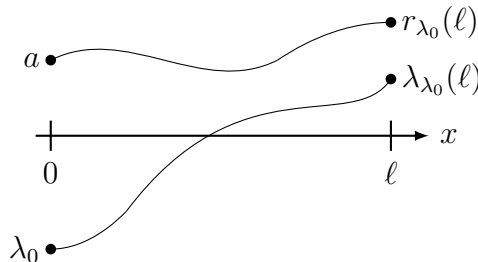


Problem: we can do this for any c . Which c is it? *Last 15 minutes was a dead end!*

Back to $u = F(r, \lambda)$. Assume we have F (numerically).

$$\begin{aligned} \frac{dr}{dx} &= -F(r, \lambda) \\ r(0) &= a \\ \frac{d\lambda}{dx} &= -\frac{F^3(r, \lambda)}{1+F^2(r, \lambda)} \\ \lambda(\ell) &= r(\ell) \end{aligned}$$

The mistake before was that the simulation forward from a depends on λ .



Therefore, we should guess λ_0 and simulate both r and λ to get $r_{\lambda_0}(\ell)$ and $\lambda_{\lambda_0}(\ell)$. We need

$$r_{\lambda_0}(\ell) = \lambda_{\lambda_0}$$

for optimality. To do this, we need numerics.

Terminal Constraints

Let $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ and solve

$$\begin{aligned} \min_{u \in \mathcal{U}} \quad & \int_0^T L(x, u, t) dt + \Psi(x(T)) \\ \text{s.t.} \quad & \dot{x} = f(x, u, t) \\ & x(0) = x_0 \\ & x_i(T) = x_{iT} \quad \text{given for } i \in \mathcal{T} \subset \{1, \dots, n\} \end{aligned}$$

First, we augment the cost:

$$\begin{aligned} \tilde{J}(u) &= \int_0^T [L + \lambda^T(f - \dot{x})] dt + \Psi \\ &= \int_0^T (H - \lambda^T \dot{x}) dt + \Psi \\ \tilde{J}(u + \varepsilon v) - \tilde{J}(u) &= \int_0^T \left(\varepsilon \frac{\partial H}{\partial u} v + \varepsilon \frac{\partial H}{\partial x} \eta - \varepsilon \lambda^T \dot{\eta} \right) dt + \varepsilon \frac{\partial \Psi}{\partial x}(x(T)) \eta(T) + o(\varepsilon) \\ \delta \tilde{J}(u; v) &= \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \int_0^T \frac{\partial H}{\partial u} v dt \\ &\quad + \lambda^T(0) \eta(0) - \lambda^T(T) \eta(T) + \frac{\partial \Psi}{\partial x}(x(T)) \eta(T) \end{aligned}$$

As always,

$$\begin{aligned} \dot{\lambda} &= -\frac{\partial H^T}{\partial x} \\ \frac{\partial H}{\partial u} &= 0 \quad (\text{FONC}) \end{aligned}$$

Additionally,

$$\begin{aligned} \eta(0) &= 0 \\ \eta_i(T) &= 0 \quad \text{for } i \in \mathcal{T} \end{aligned}$$

Note that if $x(T) = x_T$ is given, then $x(T) = x(T) + \varepsilon \eta(T) + o(\varepsilon)$, so $\eta(T) = 0$. Here, we have $x_i(T) = x_{iT}$ fixed for $i \in \mathcal{T}$ so $\eta_i(T) = 0$ for $i \in \mathcal{T}$.

For optimality, we want

$$\begin{aligned} \left[-\lambda^T(T) + \frac{\partial \Psi}{\partial x}(x(T)) \right] \eta(T) &= 0 \quad \text{for all admissible variations} \\ \left[\frac{\partial \Psi}{\partial x_1} - \lambda_1, \quad \dots, \quad \frac{\partial \Psi}{\partial x_n} - \lambda_n \right] \begin{bmatrix} \eta_1(T) \\ \vdots \\ \eta_n(T) \end{bmatrix} &= 0 \end{aligned}$$

Hence, we need

$$\begin{aligned}\lambda_j(T) &= \frac{\partial \Psi}{\partial x_j}(x(T)) & \text{if } j \notin \mathcal{T} \\ \lambda_i(T) &= \text{free} & \text{if } i \in \mathcal{T}\end{aligned}$$

So we have

$$\begin{cases} \dot{x} = f \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x}, \end{cases}$$

an ODE with $2n$ variables. We need $2n$ boundary conditions for this ODE to be well-posed.

At $t = 0$		At $t = T$	
$x(0) = x_0$	$[n]$	$x_i(T) = x_{iT}, i \in \mathcal{T}$	$[q]$
		$ \mathcal{T} = q$	
		$x_j(T) \text{ free}, j \notin \mathcal{T}$	$[0]$
$\lambda(0) \text{ free}$	$[0]$	$\lambda_i(T) \text{ free}, i \in \mathcal{T}$	$[0]$
		$\lambda_j(T) = \frac{\partial \Psi}{\partial x_j}(x(T)), j \notin \mathcal{T}$	$[n - q]$

So we have $n + q + (n - q) = 2n$ boundary conditions.

We could even fix some but not all of $x(0)$, i.e.

$$\begin{aligned}x_i(0) &= x_{i0} & \text{if } i \in \mathcal{I} \\ x_j(0) &= \text{free} & \text{if } j \notin \mathcal{I}\end{aligned}$$

Recall,

$$\delta \tilde{J}(u; v) = \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta \, dt + \int_0^T \frac{\partial H}{\partial u} v \, dt + \lambda^T(0) \eta(0) + \left[\lambda^T(T) - \frac{\partial \Psi}{\partial x}(x(T)) \right] \eta(T)$$

For $x_i(0) = x_{i0}$ fixed, we have $\eta_i(0) = 0$ and $\lambda_i(0)$ free. For $x_j(0)$ free, we have $\eta_j(0)$ free and $\lambda_j(0) = 0$.

To ponder, what if $J = \int L \, dt + \Psi(x(T)) + \Theta(x(0))$?

To summarize, the minimizer to

$$\begin{aligned} \min_{u \in \mathcal{U}} & \int_0^T L(x, u, t) \, dt + \Psi(x(T)) \\ \text{s.t.} & \quad \dot{x} = f(x, u, t) \\ & \quad x_i(0) = x_{i0}, \quad i \in \mathcal{I} \\ & \quad x_j(T) = x_{jT} \quad j \in \mathcal{T} \end{aligned}$$

has to satisfy

$$\begin{aligned}\frac{\partial H}{\partial u} &= 0 \\ \dot{\lambda} &= -\frac{\partial H^T}{\partial x} \\ \lambda_i(0) &= 0, \quad i \notin \mathcal{I} \\ \lambda_j(T) &= \frac{\partial \Psi}{\partial x_j}(x(T)), \quad j \notin \mathcal{T}\end{aligned}$$

Example

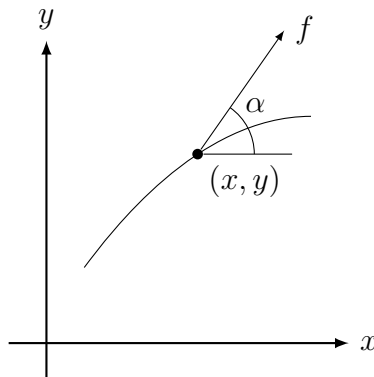
$$\begin{aligned}\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= f(x_1, x_2, x_3, x_4) \\ x_1(0) &= 1, \quad x_3(0) = 7, \quad x_4(0) = 0, \quad x_1(1) = 2 \\ \mathcal{I} &= \{1, 3, 4\}, \quad \mathcal{T} = \{1\} \\ \min \int_0^1 L(x, u) dt &+ (x_2^2(1) - x_3^2(1) + 7x_1(1) + 14)\end{aligned}$$

Note there are 4 boundary conditions on x so there must be 4 boundary conditions on λ :

$\lambda_1(0)$ free/unspecified	$\lambda_1(1)$ free
$\lambda_2(0) = 0$	$\lambda_2(1) = 2x_2(1)$
$\lambda_3(0)$ free	$\lambda_3(1) = -2x_3(1)$
$\lambda_4(0)$ free	$\lambda_4(1) = 0$

Example

A force f acts on a particle at position (x, y) (mass = 1).



$$\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= |f| \cos \alpha \\
\dot{v}_y &= |f| \sin \alpha \\
\alpha &= \text{control variable}
\end{aligned}$$

Assume we only care about where the particle ends up (to be specified later), i.e. $L = 0$.

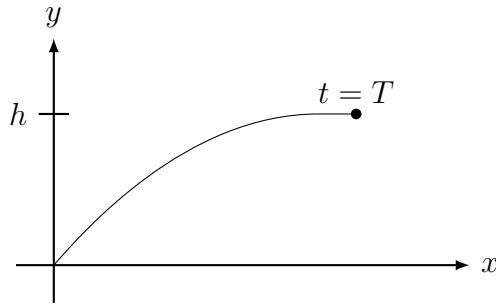
$$H = [\lambda_x \quad \lambda_y \quad \lambda_{v_x} \quad \lambda_{v_y}] \begin{bmatrix} v_x \\ v_y \\ |f| \cos \alpha \\ |f| \sin \alpha \end{bmatrix}$$

$$\begin{aligned}
\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0 &\implies \lambda_x(t) = c_1 \\
\dot{\lambda}_y = -\frac{\partial H}{\partial y} = 0 &\implies \lambda_y(t) = c_2 \\
\dot{\lambda}_{v_x} = -\frac{\partial H}{\partial v_x} = -\lambda_x &\implies \lambda_{v_x}(t) = -c_1 t + c_3 \\
\dot{\lambda}_{v_y} = -\frac{\partial H}{\partial v_y} = -\lambda_y &\implies \lambda_{v_y}(t) = -c_2 t + c_4
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{\partial H}{\partial \alpha} &= -\lambda_{v_x} |f| \sin \alpha + \lambda_{v_y} |f| \cos \alpha = 0 \\
\tan \alpha &= \frac{\lambda_{v_y}}{\lambda_{v_x}} = \frac{-c_2 t + c_4}{-c_1 t + c_3}
\end{aligned}$$

We want to drive the particle from $[0, 0, 0, 0]^T$ to a path parallel to the x-axis with $y(T) = h$.



Choose $\Psi = -v_x$,

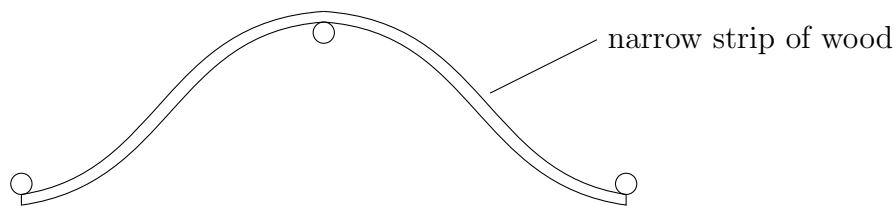
$$\begin{aligned}
y(T) &= h & v_y(T) &= 0 \\
x(T) &\text{ free} & v_x(T) &\text{ free, but costs} \\
\lambda_i(0) &\text{ free} \\
\lambda_y(T) &\text{ free} & \lambda_{v_y}(T) &\text{ free} \\
\lambda_x(T) &= 0 & \lambda_{v_x}(T) &= -1
\end{aligned}$$

$$\begin{aligned}
c_1 &= \lambda_x(t) = 0 \\
\Rightarrow \lambda_{v_x} &= -c_1 t + c_3 = c_3 = -1 \\
\Rightarrow \tan \alpha &= -\frac{-c_2 t + c_4}{-1} = c_2 t + c_4
\end{aligned}$$

How do we find c_2 and c_4 ? Plug into \dot{x} and $\dot{\lambda}$ and try to satisfy the remaining boundary conditions. (This is hard=numerics.)

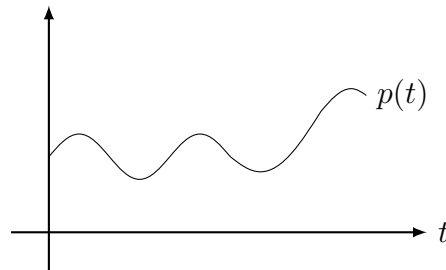
1.7 Splines

From ship building. Splines are used a lot in path-planning, e.g. cubic splines.



But, they are solutions to optimal control problems.

Let $p(t)$ be a curve we'd like to shape.



We want to minimize the “energy” put into the curve, a.k.a acceleration. Let $x_1 = p$ and $x_2 = \dot{p}$, so

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \end{cases}$$

1.7.1 Minimum-Energy

$$\begin{aligned}
 & \min_{u \in \mathcal{U}} \frac{1}{2} \int_0^T u^2(t) dt \quad + \text{Boundary conditions on } x \\
 & H = L + \lambda^T f = \frac{1}{2} u^2 + \lambda_1 x_2 + \lambda_2 u \\
 & \frac{\partial H}{\partial u} = u + \lambda_2 = 0 \implies u = -\lambda_2 \\
 & \dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = c_1 \\
 & \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -c_1 t + c_2 \\
 & u = -\lambda_2 = c_1 t - c_2 \\
 & \dot{x}_2 = u = c_1 t - c_2 \implies x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3 \\
 & \dot{x}_1 = x_2 = c_1 \frac{t^2}{2} - c_2 t + c_3 \\
 & \implies x_1 = \frac{c_1}{6} t^3 - \frac{c_2}{2} t^2 + c_3 t + c_4
 \end{aligned}$$

$p(t)$ is a cubic polynomial!

What about boundary conditions?

Let $T = 1$, $p(0)$ given, $p(1)$ given, $\dot{p}(0) = 0$, $\dot{p}(1) = 0$, e.g. $p(0) = 0$, $p(1) = 1$. Since the boundary conditions for x are all specified, those for the costate are free.

$$\begin{aligned}
 & \left. \begin{aligned} x_1(0) &= 0 \\ x_2(0) &= 0 \\ x_1(1) &= 1 \\ x_2(1) &= 0 \end{aligned} \right\} \implies \left\{ \begin{aligned} \lambda_1(0) \\ \lambda_2(0) \\ \lambda_1(1) \\ \lambda_2(1) \end{aligned} \right. \quad \text{free/unspecified} \\
 & x_2(0) = c_3 = 0 \qquad x_1(1) = \frac{2c_2}{6} - \frac{c_2}{2} = 1 \\
 & x_1(0) = c_4 = 0 \qquad c_2 = -6 \\
 & x_2(1) = \frac{c_1}{2} - c_2 + \underbrace{c_3}_0 = 0 \quad c_1 = -12 \\
 & c_1 = 2c_2 \\
 & \implies p(t) = -2t^3 + 3t^2 \\
 & u(t) = -12t + 6
 \end{aligned}$$

Or, what if $\dot{p}(0)$, $\dot{p}(1)$ are not specified?

$$\left. \begin{aligned} x_1(0) &= 0 \\ x_2(0) &\text{ unspec.} \\ x_1(1) &= 1 \\ x_2(1) &\text{ unspec.} \end{aligned} \right\} \implies \left\{ \begin{aligned} \lambda_1(0) &\text{ unspec.} \\ \lambda_2(0) &= 0 \\ \lambda_1(1) &\text{ unspec.} \\ \lambda_2(1) &= 0 \end{aligned} \right.$$

$$\left. \begin{aligned} \lambda_2(0) &= c_2 = 0 \\ \lambda_2(1) &= -c_1 + c_2 = 0 \end{aligned} \right\} \implies u = c_1 t - c_2 = 0$$

$$\left. \begin{aligned} x_1(0) &= c_4 = 0 \\ x_1(1) &= c_3 = 1 \end{aligned} \right\} \implies p(t) = t$$

What did we do?

Case 1:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1/2 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_2(0) \\ x_2(1) \end{bmatrix}$$

Case 2:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1/6 & -1/2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ \lambda_2(0) \\ \lambda_2(1) \end{bmatrix}$$

1.7.2 Generalized Splines

We had $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This A is nilpotent ($A^k = 0$ for some $k \in \mathbb{Z}^+$). This means e^{At} is a polynomial in t . (This e^{At} is cubic.)

In general, e^{At} is a mix of polynomials, exponentials, and trigonometric terms. The eigenvalues of A determine the form of $x(t)$.

$$\dot{x} = Ax \implies x(t) = e^{At}x(0)$$

$$\dot{x} = Ax + Bu \implies x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

The general problem to solve is

$$\min_{u \in \mathcal{U}} \int_0^T \frac{1}{2} \|u\|^2 dt$$

$$\text{s.t. } \dot{x} = Ax + Bu$$

+ Boundary conditions

$$H = \frac{1}{2} \|u\|^2 + \lambda^T (Ax + Bu)$$

$$\frac{\partial H}{\partial u} = u^T + \lambda^T B = 0$$

$$\implies u = -B^T \lambda$$

$$\dot{\lambda} = -\frac{\partial H^T}{\partial x} = -A^T \lambda$$

We have the Hamiltonian Dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BB^T \\ 0 & -A^T \end{bmatrix}}_M \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

Where we used $\dot{x} = Ax + Bu = Ax - BB^T\lambda$. Then,

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = e^{Mt} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix}$$

Suppose we want to drive from $x(0) = x_0$ to $x(T) = x_T$.

$$\begin{bmatrix} x_T \\ \lambda(T) \end{bmatrix} = e^{MT} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix} = \begin{bmatrix} N_{xx} & N_{x\lambda} \\ N_{\lambda x} & N_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda(0) \end{bmatrix}$$

$$x_T = N_{xx}x_0 + N_{x\lambda}\lambda(0)$$

$N_{x\lambda}$ is invertible if (A, B) is completely controllable. Assume it is.

$$\begin{aligned} \lambda(0) &= N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \\ \implies \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} &= e^{Mt} \begin{bmatrix} x_0 \\ N_{x\lambda}^{-1}(x_T - N_{xx}x_0) \end{bmatrix} \\ \implies u(t) &= -B^T\lambda(t) \end{aligned}$$

This is the optimal trajectory, but there is no feedback. We will consider closed-loop systems after the midterm.

As a preview, we need to find λ as a function of x . For example, $u = -R^{-1}B^TPx$ minimizes u^TRu , so $\lambda = Px$ where P is the solution to the Riccati equation.

1.8 Numerical Methods

Optimal control boils down to solving two sets of differential equations:

$$\begin{aligned} \dot{x} &= f(x, u) & \frac{\partial H}{\partial u}(x, u, \lambda) &= 0 \\ \dot{\lambda} &= -\frac{\partial H^T}{\partial x}(x, u, \lambda) & u &= F(x, \lambda) \\ \implies & \begin{cases} \dot{x} = f(x, F(x, \lambda)) \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x}(x, F(x, \lambda), \lambda) \end{cases} \end{aligned}$$

The equations are functions of x and λ . They are completely determined by the boundary conditions on $x(0)$, $x(T)$, $\lambda(0)$, $\lambda(T)$. This is known as the *Boundary Value Problem*. This is solved using *test shooting*:

1. Guess initial conditions
2. Simulate forward in time
3. Update the guess (cleverly...)

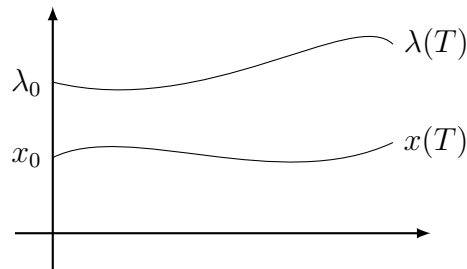
Exmaple: Bolza problem

$$\begin{aligned} \min_{u \in \mathcal{U}} & \int_0^T L(x, u) \, dt + \Psi(x(T)) \\ \text{s.t.} & \begin{cases} \dot{x} = f(x, u) \\ x(0) = x_0 \end{cases} \\ & H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u) \\ & u^*(x, \lambda) \text{ satisfies } \frac{\partial H}{\partial u} = 0 \end{aligned}$$

The optimal control satisfies

$$\begin{cases} x = f(x, u^*(x, \lambda)) \\ x(0) = x_0 \\ \lambda = -\frac{\partial H^T}{\partial x}(x, u^*(x, \lambda), \lambda) \\ \lambda(T) = \frac{\partial \Psi}{\partial x}(x(T)) \end{cases}$$

Algorithm Guess λ_0 and solve for $x(t)$, $\lambda(t)$.



Let's define a cost:

$$\left\| \lambda(T) - \frac{\partial \Psi^T}{\partial x}(x(T)) \right\|^2 = g(\lambda_0)$$

Update λ_0 through

$$\lambda_0 := \lambda_0 - \gamma \frac{\partial g^T}{\partial \lambda_0}(\lambda_0)$$

↑
any choice of step size works

Repeat

Problem: What is $\partial g / \partial \lambda_0$? We estimate $\partial g / \partial \lambda_0$ numerically. This is where “test shooting” comes into play.

Let e_i be the i th unit vector, $i = 1, \dots, n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\frac{\partial g}{\partial \lambda_0} = \left(\frac{\partial g}{\partial \lambda_{0,1}}, \frac{\partial g}{\partial \lambda_{0,2}}, \dots, \frac{\partial g}{\partial \lambda_{0,n}} \right)$$

The i th component of $\partial g / \partial \lambda_0$ is given by the directional derivative

$$\frac{\partial g}{\partial \lambda_{0,i}} = \frac{\partial g}{\partial \lambda_0} \cdot e_i = \delta g(\lambda_0; e_i) = \lim_{\varepsilon \rightarrow 0} \frac{g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)}{\varepsilon}$$

So, if $x \in \mathbb{R}^n$ (and thus so is λ_0), we have to do this n times (with a small ε) and get the full derivative $\partial g / \partial \lambda_0$.

Algorithm

```

Given  $\lambda_0, g(\lambda_0)$ 
for  $i = 1$  to  $n$  do
    Compute  $g(\lambda_0 + \varepsilon e_i)$ 
     $dg_i = \frac{1}{\varepsilon} [g(\lambda_0 + \varepsilon e_i) - g(\lambda_0)]$ 
end for
 $\frac{\partial g}{\partial \lambda_0} = [dg_1, \dots, dg_n]$ 

```

Example LQ

$$\min_u \frac{1}{2} \int_0^1 (x^T Q x + u^T R u) dt + \frac{1}{2} x^T(1) S x(1)$$

$$\text{s.t. } \begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_0 \end{cases}$$

$$Q, R, S \succ 0$$

$$H = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \lambda^T (Ax + Bu)$$

$$\frac{\partial H}{\partial u} = u^T R + \lambda^T B = 0$$

$$u^* = -R^{-1} B^T \lambda$$

$$\dot{\lambda} = -\frac{\partial H^T}{\partial x} = -Qx - A^T \lambda$$

$$\lambda(1) = \frac{\partial \Psi^T}{\partial x}(x(1)) = Sx(1)$$

So putting it all together,

$$\begin{aligned} \dot{x} &= Ax - BR^{-1}B^T\lambda & x(0) &= x_0 \\ \dot{\lambda} &= -Qx - A^T\lambda & \lambda(1) &= Sx(1) \end{aligned}$$

Example Newton's nose shape problem (revisited, see previous)

$$\begin{aligned} \min_u & \int_0^\ell \frac{ru^3}{1+u^2} dx + \frac{1}{2}r(\ell)^2 \\ \text{s.t.} & \frac{dr}{dx} = -u & r(0) &= a \end{aligned}$$

$$\begin{aligned} H &= \frac{ru^3}{1+u^2} + \lambda(-u) \\ \frac{\partial H}{\partial u} &= \frac{ru^2(3+u^2)}{(1+u^2)^2} - \lambda = 0 \end{aligned}$$

We solve the above numerically to get $u^*(r, \lambda)$.

$$\begin{aligned} \frac{\partial \lambda}{\partial x} &= -\frac{\partial H}{\partial r} = -\frac{u^3}{1+u^2} \\ \lambda(\ell) &= r(\ell) \end{aligned}$$

So, we have

$$\begin{aligned} \frac{dr}{dx} &= -u & r(0) &= a & u &= F(x, \lambda) \\ \frac{d\lambda}{dx} &= -\frac{u^3}{1+u^2} & \lambda(\ell) &= r(\ell) \end{aligned}$$

Example Fixed terminal constraints (revisited, see previous)

$$\begin{aligned} \min_{\alpha} & -v_x(T) & \alpha &= \text{control} \\ \text{s.t.} & \dot{x} = v_x & x(0) &= 0 \\ & \dot{y} = v_y & y(0) &= 0 \\ & \dot{v}_x = |f| \cos \alpha & v_x(0) &= 0 \\ & \dot{v}_y = |f| \sin \alpha & v_y(0) &= 0 \\ & y(T) &= h \\ & v_y(T) &= 0 \end{aligned}$$

$$H = -v_x(T) + \begin{bmatrix} \lambda_x & \lambda_y & \lambda_{v_x} & \lambda_{v_y} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ |f| \cos \alpha \\ |f| \sin \alpha \end{bmatrix}$$

$$\frac{dH}{d\alpha} = 0 \Rightarrow \tan \alpha = \frac{\lambda_{v_y}}{\lambda_{v_x}}$$

$$\dot{\lambda}_x = 0$$

$$\dot{\lambda}_y = 0$$

$$\dot{\lambda}_{v_x} = -\lambda_x$$

$$\dot{\lambda}_{v_y} = -\lambda_y$$

$$\boldsymbol{\lambda}(0) \text{ unspecified}$$

$$\lambda_x(T) = \frac{\partial \Psi^T}{\partial x}(x(T)) = 0$$

$$\lambda_y(T) \text{ unspecified}$$

$$\lambda_{v_x}(T) = \frac{\partial \Psi^T}{\partial v_x}(v_x(T)) = -1$$

$$\lambda_{v_y}(T) \text{ unspecified}$$

Again, we guess λ_0 and solve forward in time. But, we have terminal constraints on y and v_y as well.

$$g(\lambda_0) = \frac{1}{2} \left[(y(T) - h)^2 + (v_y(T))^2 + (\lambda_x(T))^2 + (\lambda_{v_x} + 1)^2 \right]$$

1.9 Terminal Manifolds

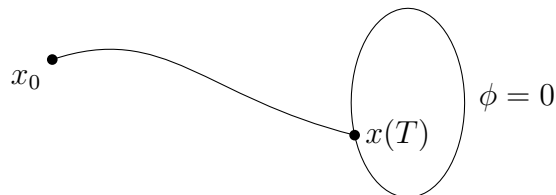
We can solve

$$\begin{aligned} \min_{u \in \mathcal{U}} \int_0^T L(x, u, t) dt + \Psi(x(T)) \\ \text{s.t. } \dot{x} = f(x, u, t) \end{aligned}$$

with all sorts of boundary conditions on x :

- $x(0) = x_0$, $x(T)$ free (typical)
- $x_i(0) = x_{i0}$, $i \in \mathcal{I}$ and $x_j(T) = x_{jT}$, $j \in \mathcal{T}$

But what if we want $x(T)$ to belong to a set?



Problem

$$\begin{aligned}
& \min_{u \in \mathcal{U}} \int_0^T L(x, u, t) dt + \Psi(x(T)) \\
& \text{s.t. } \dot{x} = f(x, u, t), \quad x \in \mathbb{R}^n \\
& \quad x(0) = x_0 \\
& \quad \phi(x(T)) = 0, \quad \phi : \mathbb{R}^n \rightarrow \mathbb{R}^q, \quad q \leq n
\end{aligned}$$

The augmented cost is

$$\tilde{J} = \int_0^T [H(x, u, t, \lambda) - \lambda^T \dot{x}] dt + \Psi(x(T)) + \underbrace{\nu^T \phi(x(T))}_{\substack{\text{q-dimensional} \\ \text{Lagrange multiplier}}}$$

Let $\Phi(x(T), \nu) = \Psi(x(T)) + \nu^T \phi(x(T))$. Then,

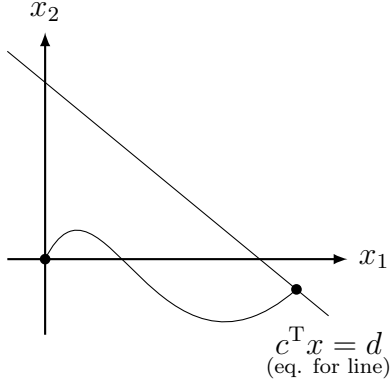
$$\tilde{J} = \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \Phi(x(T), \nu)$$

We know how to solve this! With $u \mapsto u + \varepsilon v$, $x \mapsto x + \varepsilon \eta + o(\varepsilon)$,

$$\begin{aligned}
\delta \tilde{J} &= \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta dt + \int_0^T \frac{\partial H}{\partial u} v dt + \frac{\partial \Phi}{\partial x}(x(T), \nu) \eta(T) \\
&\quad - \lambda^T(T) \eta(T) + \underbrace{\lambda^T(0) \eta(0)}_{=0} \\
&\left\{ \begin{array}{l} \frac{\partial H}{\partial u} = 0 \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = \frac{\partial \Phi^T}{\partial x}(x(T), \nu) \\ \phi(x(T)) = 0 \end{array} \right. \quad \begin{array}{l} q \text{ new variables} \\ \downarrow \\ q \text{ new equations} \end{array} \\
&\implies u^*
\end{aligned}$$

Spring to line

$$\begin{aligned}
& \min_u \frac{1}{2} \int_0^1 u^2(t) dt \\
& \text{s.t. } \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u \\ x_1(0) = 0, \quad x_2(0) = 0 \\ c_1 x_1(1) + c_2 x_2(1) = d \end{cases}
\end{aligned}$$



$$H = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u$$

$$\frac{\partial H}{\partial u} = u + \lambda_2 \implies u = -\lambda_2$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \implies \lambda_1 = k_1$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 \implies \lambda_2 = -k_1 t + k_2$$

$$\phi(x(1)) = c_1 x_1(1) + c_2 x_2(1) - d$$

$$\Psi = 0 \implies \Phi = \nu(c_1 x_1(1) + c_2 x_2(1) - d)$$

$$\lambda_1(1) = \frac{\partial \Phi}{\partial x_1} = \nu c_1$$

$$\lambda_2(1) = \frac{\partial \Phi}{\partial x_2} = \nu c_2$$

So,

$$\lambda_1(1) = \nu c_1 = k_1$$

$$\lambda_2(1) = \nu c_2 = -k_1 + k_2$$

$$k_2 = \nu(c_1 + c_2)$$

$$\dot{x}_2 = u = -\lambda_2 = k_1 t - k_2$$

$$x_2 = \frac{k_1}{2}t^2 - k_2 t + 0$$

$$\dot{x}_1 = x_2$$

$$x_1 = \frac{k_1}{6}t^3 - \frac{k_2}{2}t^2 + 0$$

Substituting k_1 and k_2 into $c_1 x_1(1) + c_2 x_2(1) = d$,

$$\nu \left(-\frac{c_1^2}{3} - c_1 c_2 - c_2^2 \right) = d$$

$$\nu = -\frac{d}{c_1^2/3 + c_1 c_2 + c_2^2}$$

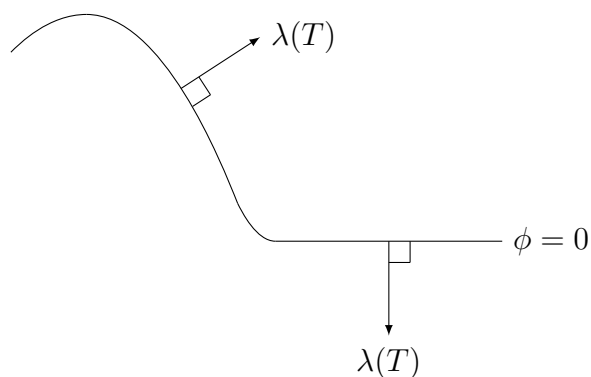
And finally

$$u = k_1 t - k_2 = \frac{d}{c_1^2/3 + c_1 c_2 + c_2^2} (c_1 + c_2 - c_1 t)$$

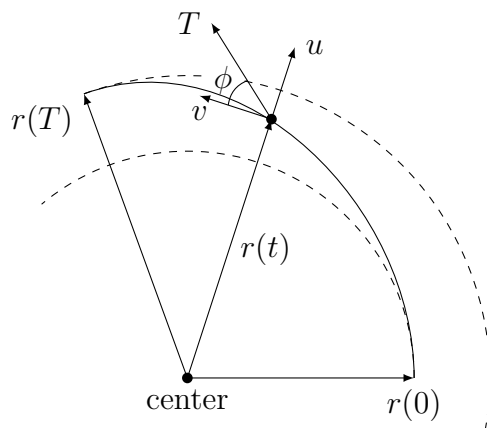
As a final observation if $\Psi = 0$ then

$$\lambda(T) = \nu^T \frac{\partial \phi}{\partial x}(x(T)),$$

which means $\lambda(T)$ is orthogonal to the tangent plane to $\phi(x(T))$.



Example Maximum orbit transform (e.g. Hidden Figures)



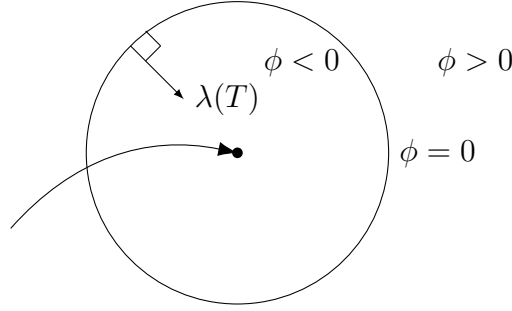
- r = radial distance from spacecraft to center
- u = radial velocity
- v = tangential velocity
- m = mass of spacecraft
- \dot{m} = $-$ fuel consumption rate
- ϕ = thrust angle (control input)
- T = thrust

$$\begin{aligned}
& \max_{\phi} r(T) \iff \min_{\phi} -r(T) \\
& \text{s.t.} \quad \begin{cases} \dot{r} = u \\ \dot{u} = \frac{v^2}{r} - \frac{g}{r^2} + \frac{T \sin \phi}{m_0 - |\dot{m}|t} \\ \dot{v} = -\frac{uv}{r} + \frac{T \cos \phi}{m_0 - |\dot{m}|t} \\ r(0) = r_0 \\ u(0) = 0 \\ v(0) = \sqrt{\frac{g}{r_0}} \\ u(T) = 0 = \phi_1 \\ v(T) = \sqrt{\frac{g}{r(T)}} = \phi_2 \end{cases} \\
\\
& H = \lambda_r u + \lambda_u \left(\frac{v^2}{r} - \frac{g}{r^2} + \frac{T \sin \phi}{m_0 - |\dot{m}|t} \right) + \lambda_v \left(-\frac{uv}{r} + \frac{T \cos \phi}{m_0 - |\dot{m}|t} \right) \\
& \Phi = \underbrace{\nu_1 u(T) + \nu_2 \left(v(T) - \sqrt{\frac{g}{r(T)}} \right)}_{\nu^T \phi} \underbrace{-r(T)}_{\Psi} \\
\\
& \frac{\partial H}{\partial \phi} = \frac{\lambda_u T \cos \phi - \lambda_v T \sin \phi}{m_0 - |\dot{m}|t} = 0 \\
& \Rightarrow \tan \phi = \frac{\lambda_u}{\lambda_v} \\
\\
& \dot{\lambda}_r = -\frac{\partial H}{\partial r} = -\lambda_u \left(-\frac{v^2}{r^2} + \frac{2g}{r^3} \right) - \lambda_v \cdot \frac{uv}{r^2} \\
& \dot{\lambda}_u = -\frac{\partial H}{\partial u} = -\lambda_r + \lambda_v \cdot \frac{v}{r} \\
& \dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_u \cdot \frac{2v}{r} + \lambda_v \cdot \frac{u}{r} \\
\\
& \begin{cases} \lambda_r(T) = \frac{\partial \Phi}{\partial r} = -1 + \frac{\nu_2 \sqrt{g}}{2(r(T))^{3/2}} \\ \lambda_u(T) = \frac{\partial \Phi}{\partial u} = \nu_1 \\ \lambda_v(T) = \frac{\partial \Phi}{\partial v} = \nu_2 \\ u(T) = 0 \\ v(T) = \sqrt{\frac{g}{r(T)}} \end{cases}
\end{aligned}$$

This needs numerics to solve.

1.9.1 Terminal manifold with inequality constraints

$$\begin{aligned} \min_u \int_0^T L \, dt + \Psi \\ \dot{x} &= f(x, u) \\ \phi(x(T)) &\leq 0 \end{aligned}$$



Repeat process: $\tilde{J} = \int (H - \lambda^T \dot{x}) \, dt + \Psi + \nu^T \phi$. The optimality conditions are

$$\begin{cases} \frac{\partial H}{\partial u} = 0 \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = \frac{\partial \Psi^T}{\partial x}(x(T)) + \nu^T \frac{\partial \phi^T}{\partial x}(x(T)) \\ \nu \geq 0 \\ \phi(x(T)) \leq 0 \\ \nu^T \phi(x(T)) = 0 \quad (\text{KKT}) \end{cases}$$

1.9.2 Initial manifold

$$\begin{aligned} \min_{x_0, u} \int L + \Psi(x(T)) + \Theta(x(0)) \\ \text{s.t. } \dot{x} &= f(x, u) \\ \phi(x(T)) &= 0 \\ \xi(x(0)) &= 0 \end{aligned}$$

$$\begin{aligned} \tilde{J} &= \int (H - \lambda^T \dot{x}) \, dt + \Psi(x(T)) + \Theta(x(0)) + \nu_\phi^T \phi(x(T)) + \nu_\xi^T \xi(x(0)) \\ \delta \tilde{J} &= \int \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \eta + \frac{\partial H}{\partial u} v \right] \, dt + \left[\frac{\partial \Psi}{\partial x}(x(T)) + \nu_\phi^T \frac{\partial \phi}{\partial x}(x(T)) - \lambda^T(T) \right] \eta(T) \\ &\quad + \left[\frac{\partial \Theta}{\partial x}(x(0)) + \nu_\xi^T \frac{\partial \xi}{\partial x}(x(0)) + \lambda^T(0) \right] \eta(0) \end{aligned}$$

The optimality conditions are

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial u} = 0 \\ \dot{\lambda} = -\frac{\partial H^T}{\partial x} \\ \lambda(T) = -\frac{\partial \Psi^T}{\partial x}(x(T)) - \nu_\phi^T \frac{\partial \phi^T}{\partial x}(x(T)) \\ \lambda(0) = -\frac{\partial \Theta^T}{\partial x}(x(0)) - \nu_\xi^T \frac{\partial \xi^T}{\partial x}(x(0)) \end{array} \right.$$