Response of a non-linear oscillator

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Abstract

In this note I'm developing a method of processing non-linear oscillator measurements. The idea is to fully characterize the oscillator using only two functions of motion amplitude: nonlinear resonance frequency ω_N and non-linear resonance width δ_N. These functions can be calculated analytically as an integral transform of the non-linear force acting on the oscillator. On the other hand they can be directly measured. The method is based on a usual Van der Pol transformation and averaging over period [1]. This is generalization of the previous note [2] where similar method have been applied to a more specific case of a non-linear dumping of a vibrating wire in superfluid ^{3}He-B. Here I try to focus on more clear and accurate mathematical description and provide a couple of practical examples.

Theory

Let's write equation of motion of a non-linear oscillator driven by external force F at some frequency ω in the following form:

$$\ddot{x} = f(x, \dot{x}) + F\cos(\omega t + \phi), \tag{1}$$

were $x(t)$ is a coordinate and f is a force which can include both linear and non-linear terms. Forces f and F are given per unit mass of the oscillator. We use Van der Pol coordinates (u, v):

$$\begin{aligned} u &= x\,\cos\omega t - \dot{x}/\omega\,\sin\omega t,\\ v &= -x\,\sin\omega t - \dot{x}/\omega\,\cos\omega t, \end{aligned} \tag{2}$$

with inverse transformation given by

$$\begin{aligned} x &= u\,\cos\omega t - v\,\sin\omega t,\\ \dot{x} &= -u\,\omega\,\sin\omega t - v\,\omega\,\cos\omega t. \end{aligned} \tag{3}$$

Note that a harmonic motion $x(t) = A\cos(\omega t + \Phi)$ corresponds to a stationary point in Van der Pol coordinates: $u = A\cos\Phi, v = A\sin\Phi$.

Equation of motion (1) can be written as

$$\begin{aligned} \omega\,\dot{u} &= -\big(\omega^2 x + f + F\cos(\omega t + \phi)\big)\,\sin\omega t,\\ \omega\,\dot{v} &= -\big(\omega^2 x + f + F\cos(\omega t + \phi)\big)\,\cos\omega t. \end{aligned} \tag{4}$$

We introduce averaging over period of the driving force:

$$\bar{X} = \int_0^{2\pi} X(t)\frac{d\omega t}{2\pi}, \tag{5}$$

and average equations of motion (4):

$$\begin{aligned} \omega\,\bar{\dot{u}} &= -\omega^2\,\overline{x\sin\omega t} - \overline{f\sin\omega t} - F\,\overline{\cos(\omega t + \phi)\sin\omega t},\\ \omega\,\bar{\dot{v}} &= -\omega^2\,\overline{x\cos\omega t} - \overline{f\cos\omega t} - F\,\overline{\cos(\omega t + \phi)\cos\omega t}. \end{aligned} \tag{6}$$

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We are interested in the complex response of the oscillator $\mathbf{x}$, which is an average Fourier component of x at frequency ω, which is same as values measured by a lock-in amplifier in a conventional experiment:

$$\mathbf{x} = 2\left(\overline{x\cos\omega t} - i\overline{x\sin\omega t}\right) = 2\,\overline{xe^{-i\omega t}}. \tag{7}$$

The factor 2 is introduced to have $|\mathbf{x}|$ equal to the motion amplitude for a harmonic oscillator. Bold font is used in this note for complex values. The left part of equations (6) is zero for a periodic motion. We combine both equations to have a complex expression for $\mathbf{x}$:

$$\omega^2\mathbf{x} = -2\,\overline{f\,e^{-i\omega t}} - F\,e^{i\phi}, \tag{8}$$

this can be rewritten this in the form

$$\mathbf{x} = \frac{\mathbf{F}}{\mathbf{I}-\omega^2}, \qquad \mathbf{I} = -\frac{2}{\mathbf{x}}\,\overline{f\,e^{-i\omega t}}, \tag{9}$$

where we introduce a complex value $\mathbf{I}$ and use a usual complex notation for the driving force $\mathbf{F} = F\,e^{i\phi}$.

If non-linearities are small then motion of the oscillator is close to harmonic $x_h(t) = |\mathbf{x}|\cos(\omega t + \arg(\mathbf{x}))$. We can replace x by x_h in arguments of f. More precise description of this operation is given in the averaging theorem in [1]. Then we shift the integration range by $-\arg(\mathbf{x})/\omega$ to cancel out phase of $\mathbf{x}$ in $\mathbf{I}$. As a result $\mathbf{I}$ does not depend on phase of $\mathbf{x}$, which makes it a very useful quantity for characterizing the non-linear oscillator.

Now we have formula for the oscillator response $\mathbf{x}$:

$$\mathbf{x} = \frac{\mathbf{F}}{\mathbf{I}-\omega^2}, \qquad \mathbf{I}(|\mathbf{x}|,\omega) = -\frac{2}{|\mathbf{x}|}\int_0^{2\pi} f(|\mathbf{x}|\cos\omega t, -\omega|\mathbf{x}|\sin\omega t)\,e^{-i\omega t}\,\frac{d\omega t}{2\pi}, \tag{10}$$

This can be also rewritten in terms of complex velocity response $\mathbf{v} = i\omega\mathbf{x}$ if needed. The complex function $\mathbf{I}$ can be obtained analytically by averaging a given non-linear force f, or directly measured by measuring response $\mathbf{x}$ as a function of $\mathbf{F}$ and ω and calculating $\mathbf{I} = \mathbf{F}/\mathbf{x} + \omega^2$.

For the given function $\mathbf{I}$ the response (10) can be solved for $\mathbf{x}$ as a non-linear equation, or calculated iteratively, starting with $\mathbf{I}(|\mathbf{x}| = 0)$ or used for fitting by putting experimental data for $\mathbf{x}$ in both parts and minimizing difference.

In general $\mathbf{I}$ is a complex function of two variables, $|\mathbf{x}|$ and ω (or $|\mathbf{x}|$ and $|\mathbf{v}| = \omega|\mathbf{x}|$). When working with experimental data it's very useful to reduce it to a 1D function. Two assumptions can be used:

- If quality factor of the system is large and all measurements are done in a small frequency range near the resonance, then ω can be treated as a constant.
- If force acting on the the oscillator can be written as a sum $f(x,\dot{x}) = f_x(x) + f_v(\dot{x})$ then $\mathbf{I}$ can be written as a combination of two real 1D functions: "non-linear resonance frequency" $\omega_N(|\mathbf{x}|)$ and "non-linear resonance width" $\delta_N(|\mathbf{v}|)$,

$$\mathbf{I} = \omega_N^2(|\mathbf{x}|) + i\omega\,\delta_N(|\mathbf{v}|), \tag{11}$$

$$\omega_N^2 = -\frac{2}{|\mathbf{x}|}\int_0^{2\pi} f_x(|\mathbf{x}|\cos\omega t)\cos\omega t\,\frac{d\omega t}{2\pi}, \quad \delta_N = \frac{2}{|\mathbf{v}|}\int_0^{2\pi} f_v(-|\mathbf{v}|\sin\omega t)\sin\omega t\,\frac{d\omega t}{2\pi}. \tag{12}$$

One can check that both integrals are exactly same.

We came to a linear integral transform:

$$f(x) \quad \xrightarrow{\text{tr}} \quad g(x) = -\frac{2}{|x|}\int_0^{2\pi} f(|x|\cos\phi)\cos\phi\,\frac{d\phi}{2\pi} \tag{13}$$

It looks similar to Abel integral transform, with a similar way of finding the inverse transform:

$$g(x) \quad \xrightarrow{\text{inv}} \quad f(x) = -\mathrm{sign}(x)\int_0^{\pi/2}\left(g(|x|\cos\phi)\,|x|\cos\phi + g'(|x|\cos\phi)\,\frac{1}{2}|x|^2\cos^2\phi\right)\,d\phi. \tag{14}$$

Note that we can find only odd part of the function $f(x)$, because even functions transform to $g(x) = 0$.

Properties of the transform:

- Nonlinear resonance frequency and width are:

$$f(x) \quad \xrightarrow{\text{tr}} \quad \omega_N^2(|\mathbf{x}|), \qquad f(\dot{x}) \quad \xrightarrow{\text{tr}} \quad \delta_N(|\mathbf{v}|), \tag{15}$$

- Functions containing powers of x:

$$\begin{aligned} f_n(x) = -x^n \quad \xrightarrow{\text{tr}} \quad & g_n(x) = \frac{n}{n+1}\,|x|^2\,g_{n-2}(x) \quad \text{for} \quad n = 3,5,7\ldots \quad \text{with} \quad g_1(x) = 1. \\ & g_n(x) = 0 \quad \text{for} \quad n = 0,2,4\ldots \end{aligned} \tag{16}$$

  In particular, for linear oscillator: $f(x,\dot{x}) = -\omega_0^2 x - \delta\dot{x} \quad \xrightarrow{\text{tr}} \quad \omega_N^2 = \omega_0^2, \quad \delta_N = \delta.$
  for Duffing force: $f(x) = -\alpha x^3 \quad \xrightarrow{\text{tr}} \quad \omega_N^2 = \frac{3}{4}\,\alpha|\mathbf{x}|^2.$

- Odd functions with even powers of x:

$$f_n(x) = -\text{sign}(x)\,x^n \quad \xrightarrow{\text{tr}} \quad g_n(x) = \frac{n}{n+1}\,|x|^2\,g_{n-2}(x) \quad \text{for} \quad n = 2,4,6\ldots \quad \text{with} \quad g_0(x) = \frac{4}{\pi|x|} \tag{17}$$

- Simple 1D ballistic scattering model in the B phase of ^{3}He [3]:

$$f(\dot{x}) = -\delta v_0\,\text{sign}(\dot{x})(1 - \exp(-|\dot{x}|/v_0)) \quad \xrightarrow{\text{tr}} \quad \delta_N(|\mathbf{v}|) = 2\delta\frac{v_0}{|\mathbf{v}|}\left(I_1\left(\frac{|\mathbf{v}|}{v_0}\right) - L_{-1}\left(\frac{|\mathbf{v}|}{v_0}\right) + \frac{2}{\pi}\right) \tag{18}$$

  where $I_n(x)$ is modified Bessel function of first kind and $L_n(x)$ is modified Struve function.
  A good practical approximation is $\delta_N = \delta/(1{+}0.477(|\mathbf{v}|/v_0)^{1.16})$ (within 1.5% accuracy for $|\mathbf{v}|/v_0 < 12$).

The response formula (10) can be written in terms of ω_N and δ_N as

$$\mathbf{x} = \frac{\mathbf{F}}{\omega_N^2(|\mathbf{x}|) + i\omega\,\delta_N(|\mathbf{v}|) - \omega^2}. \tag{19}$$

In usual measurements the oscillator response $\mathbf{x}$ is measured as a function of ω and $\mathbf{F}$. From these data one can extract both functions:

$$\omega_N^2(|\mathbf{x}|) = \Re\frac{\mathbf{F}}{\mathbf{x}} + \omega^2, \qquad \delta_N(|\mathbf{v}|) = \frac{1}{\omega}\,\Im\frac{\mathbf{F}}{\mathbf{v}}. \tag{20}$$

Obtained functions should have no dependence on ω as far as one of assumptions which allowed us to switch from $\mathbf{I}(|\mathbf{x}|,\omega)$ to ω_N and δ_N is valid.

Example 1. Duffing oscillator

For a Duffing oscillator the non-linear force is

$$f(x,\dot{x}) = -\omega_0^2 x - \alpha x^3 - \delta\dot{x} \tag{21}$$

and the non-linear resonance frequency and width (12) are

$$\omega_N^2(|\mathbf{x}|) = \omega_0^2 + 3/4\,\alpha|\mathbf{x}|^2, \qquad \delta_N(|\mathbf{v}|) = \delta, \tag{22}$$

which gives the coordinate response

$$\mathbf{x} = \frac{\mathbf{F}}{\omega_0^2 - \omega^2 + i\omega\delta + 3/4\,\alpha|\mathbf{x}|^2}. \tag{23}$$

By calculating absolute value of this expression we get a well-known cubic equation for $|\mathbf{x}|^2$:

$$9/16\,\alpha^2|\mathbf{x}|^6 + 3/2\,\alpha(\omega_0^2 - \omega^2)|\mathbf{x}|^4 + \{(\omega_0^2 - \omega^2)^2 + (\omega\delta)^2\}|\mathbf{x}|^2 - F^2 = 0. \tag{24}$$

After finding roots one can insert amplitude $|\mathbf{x}|$ to the right side of (23) to obtain phase of $\mathbf{x}$.

The Fig. 1A shows an example of response $\mathbf{x}$ (amplitude and phase) calculated as a function of frequency for $\omega_0 = 1$, $\delta = 0.2$, $\alpha = 0.1$, and a few values of the driving force $F = 0.4$, 0.8, 1.2. In the Fig. 1B another common representation of response is shown: as a function of driving force F either at a constant frequency $\omega = 0.8, 1.0, 1.2$ or with frequency adjusted to $\omega_N(|\mathbf{x}|)$ using (20) on each step.

Finally, the Fig. 1C shows that all data from figures 1A and 1B represented in term of $\omega_N(|\mathbf{x}|)$ and $\delta_N(|\mathbf{v}|)$ using (20) collapse into simple functions (22).

Similarly, in an experiment, measured response data can be used to obtain $\omega_N(|\mathbf{x}|)$ and $\delta_N(|\mathbf{v}|)$ by using (20) which gives a complete model for calculating response and finding the oscillator parameters.

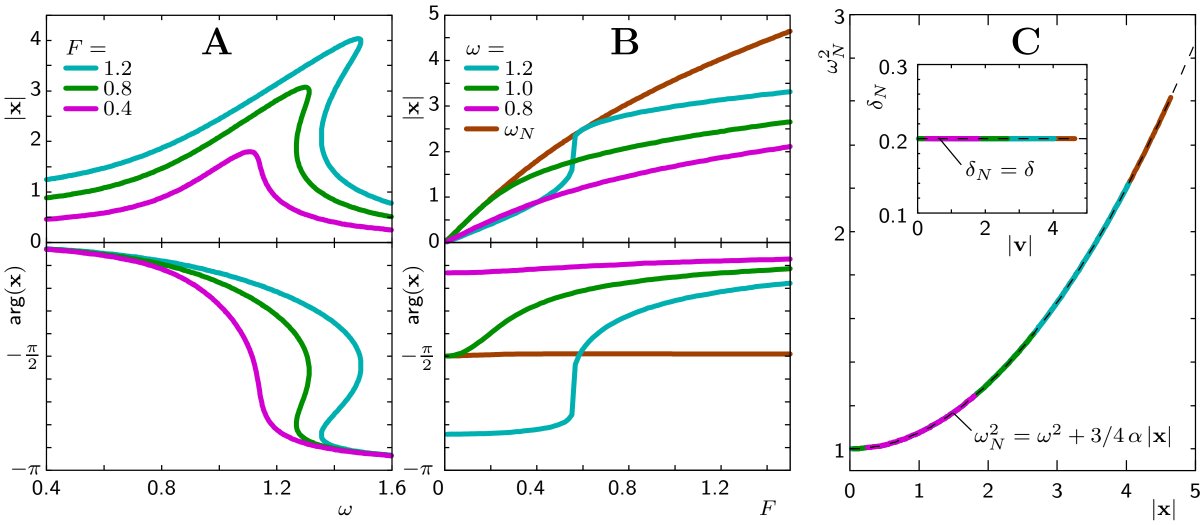

Figure 1: **A**. Response of a Duffing oscillator calculated using (23), absolute value and phase of $\mathbf{x}$ as a function of ω for a few different values of the driving force F. Parameters: $\omega_0 = 1$, $\delta = 0.2$, $\alpha = 0.1$.
B. Response of the same oscillator calculated as a function of the driving force F for a few different values of ω. The curve marked ω_N calculated by adjusting frequency to $\omega_N(|\mathbf{x}|)$ using (20) on each step.
C. Data from plots **A** and **B** converted to functions $\omega_N(|\mathbf{x}|)$ and $\delta_N(|\mathbf{v}|)$ using equations (20). All data collapse to functions (22).

Example 2. Non-linear resonances of a vibrating wire

As an example of experimental data we use measurements of a superconducting suspended wire with 4.2 μm diameter and about 1.4 mm length, driven by AC current in a constant magnetic field. We will not discuss any related physics and measurement details but only show processing of the non-linear resonance measurements. There are two data sets (Fig. 2 and Fig. 3), the first one has been measured in vacuum, at temperature about 20 mK and magnetic field 132 mT, the second one has been measured in superfluid ^{3}He-B at temperature about 0.15 mK, pressure 2 bar, and magnetic field 80 mT.

Wire is driven by AC current I, velocity response $\mathbf{v}$ is proportional to the measured voltage across the wire and presented in Volts. The response is measured as a complex function of frequency at a few magnitudes of the driving force and presented by color dots in figure panels **A**, **B** and **C**. Resonance tails ($|\mathbf{v}| < 0.8\,\mu$V for the first data set and $|\mathbf{v}| < 0.09\,\mu$V for the second one) are fitted with a linear oscillator model to obtain a voltage background and a complex factor between applied current I and force $\mathbf{F}$ (phase and amplitude). Note that the background can have component proportional to the driving current as well as a constant component. It can also depend on frequency.

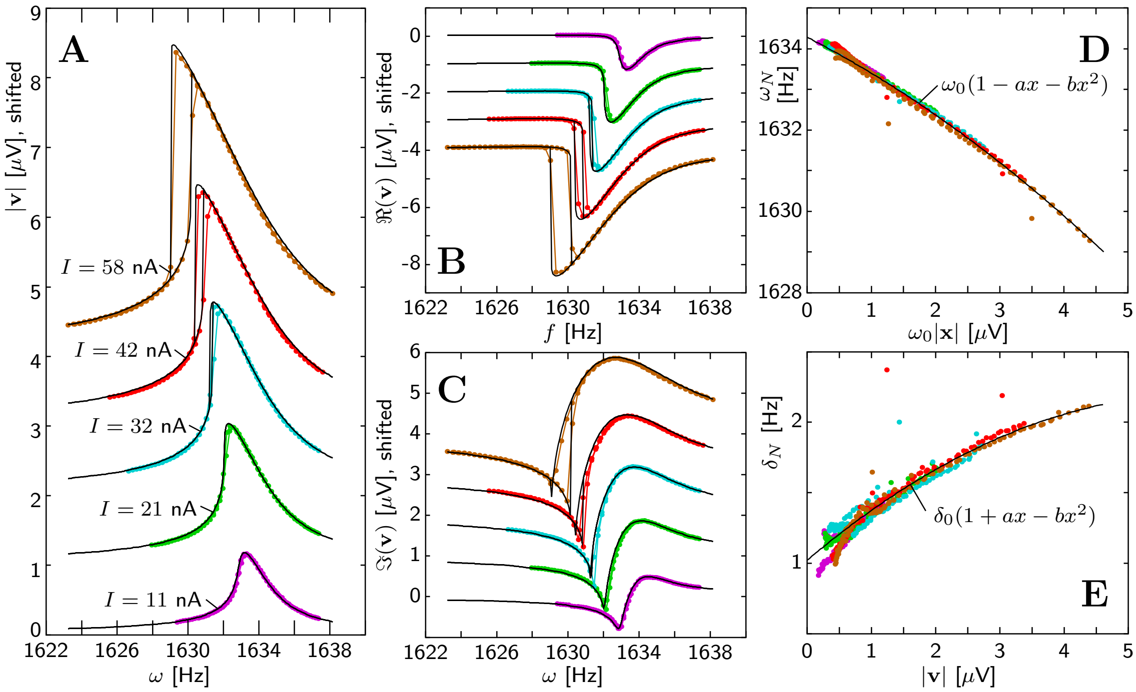

Figure 2: Data set 1. Resonance of the vibrating wire in vacuum. Measured velocity response is shown by color dots in panels **A-C**. Data and approximation models for non-linear resonance frequency ω_N and width δ_N are plotted in panels **D** and **E**. These models are used to calculate response shown by black lines in panels **A-C**.

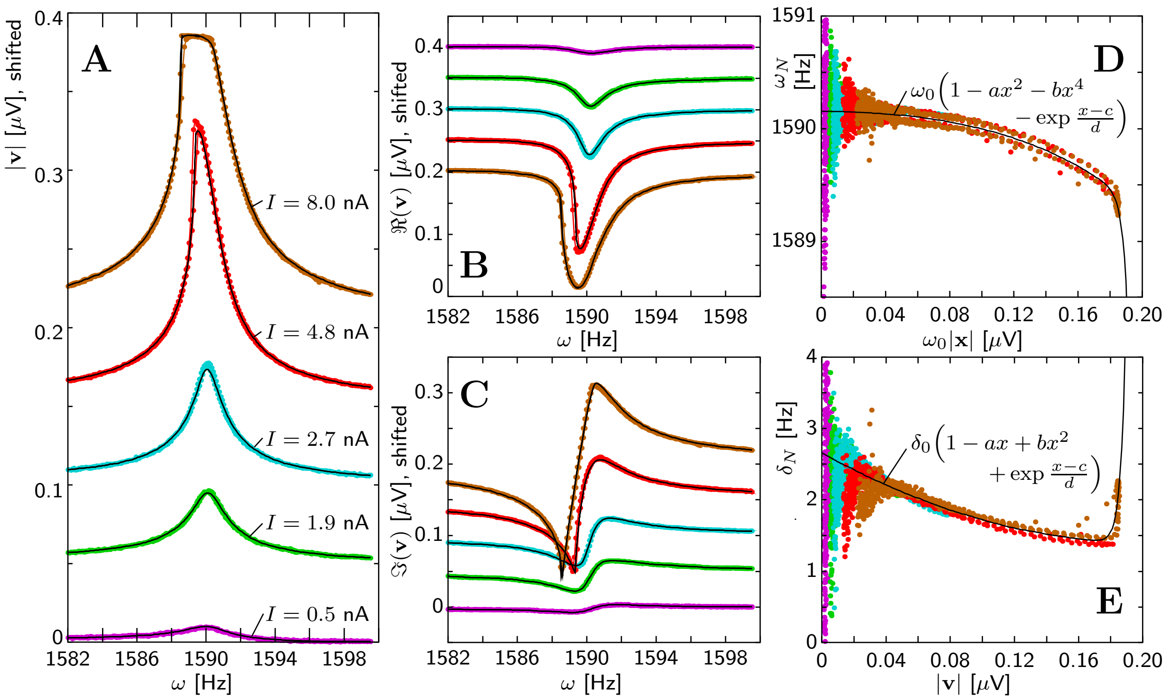

Figure 3: Data set 2. Resonance of the vibrating wire in superfluid ^{3}He-B. Data is presented in the same way as in Fig 2.

After removing the background and converting current to the driving force with correct phase, we calculate non-linear resonance frequency ω_N and width δ_N using equations (20). Result is shown in figure panels **D** and **E** with color dots (in Hz units). We make simple approximation models for these quantities (see black lines). Using these models we can calculate response with equation (19) and repeat data fitting for getting more accurate background, amplitude and phase. After a few of such steps we have a final model which is shown by black lines and have quite a good agreement with measured data.

Using models for ω_N and δ_N one can reconstruct the non-linear force $f(x, \dot{x})$ using inverse transform (14). Note the linear component in both ω_N and δ_N near $|x| = 0$ in the first dataset. It means presence of terms with discontinuous second derivative in the non-linear force: $\mathrm{sign}(x)\, x^2$ and $\mathrm{sign}(\dot{x})\, \dot{x}^2$ which looks quite puzzling. For the second dataset ω_N at small amplitudes is close to a quadratic function (Duffing non-linearity) while δ_N looks very similar to the B-phase non-linear damping (18). In the second dataset a critical velocity can be seen at high amplitudes where emission of ^{3}He quasiparticles from the surface of the wire into bulk helium starts.

A Python library for resonance fitting, together with presented examples can be found in [4].

References

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