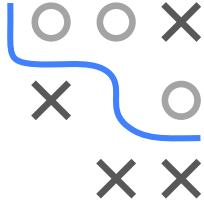
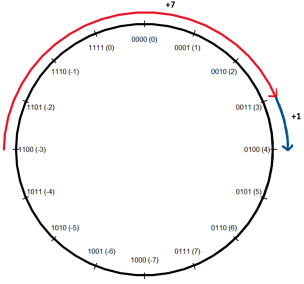
Algorithms and Data Structures



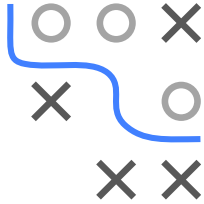
Encoding
Machine numbers for $\mathbb{Z}$



Learning goals

- Signed magnitude representation
- One's complement
- Two's complement
- Integer overflow

MACHINE NUMBERS FOR $\mathbb{Z}$



There are different options to represent positive and negative integers ($\mathbb{Z}$) by a computer:

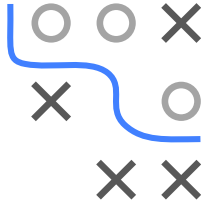
- Signed magnitude representation
- Excess encoding
- One's complement
- Two's complement

Each representation has advantages and disadvantages regarding:

- Symmetry of the representable value range
- Uniqueness of representation
- Execution of arithmetic operations

SIGNED MAGNITUDE REPRESENTATION

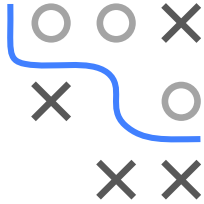
If the 32nd bit is reserved for the sign on a 32-bit computer, 31 bits are available for encoding the absolute value of the number.



	Bit u_i										
	sign	31	...	8	7	6	5	4	3	2	1
-1	1	0	...	0	0	0	0	0	0	0	1
0	1/0	0	...	0	0	0	0	0	0	0	0
1	0	0	...	0	0	0	0	0	0	0	1
51	0	0	...	0	0	1	1	0	0	1	1
		2^{30}	...	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

The number is then given by: $x = (-1)^{u_{32}} \sum_{i=1}^{31} u_i 2^{i-1}$.
The sign bit $u_{32} = 1$ indicates negative numbers.

SIGNED MAGNITUDE REPRESENTATION / 2



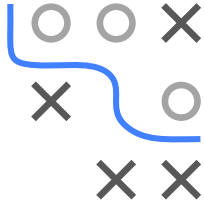
- Covered number range in 32-bit: $-2^{31}+1$ to $2^{31}-1$
- Very good readability
- Representation of zero not unique (e.g. problem with equality check: $-0 \neq 0$)
- Addition/subtraction is cumbersome, since the sign bit must be handled separately. You cannot simply write and add two numbers below each other (but this would be desirable!).

Example $7-3$ in 4-bit system:

	0111	|(7)
+	1011	|(-3)
	(1)0010	|(2)

But $0010_2 \neq 4_{10}$. Implementation of addition is complicated.

MACHINE NUMBERS FOR $\mathbb{Z}$: EXCESS CODE



An option without a sign bit can be achieved by shifting the value ranges: All values are shifted by a bias (so that they are not negative).

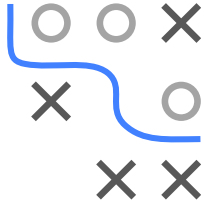
	Bit u_i										
	32	31	...	8	7	6	5	4	3	2	1
-2^{31}	0	0	...	0	0	0	0	0	0	0	0
-1	0	1	...	1	1	1	1	1	1	1	1
0	1	0	...	0	0	0	0	0	0	0	0
1	1	0	...	0	0	0	0	0	0	0	1
$2^{31}-1$	1	1	...	1	1	1	1	1	1	1	1
	2^{31}	2^{30}	...	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

The coded number is calculated according to: $x = \sum_{i=1}^{32} u_i 2^{i-1} - 2^{31}$.

- Covered number range in 32-bit: -2^{31} to $2^{31}-1$
- Uniqueness of zero
- No simple addition/subtraction of binary numbers

ONE'S COMPLEMENT

A negative number $-z$ is represented by the bitwise complement of the corresponding positive number z.

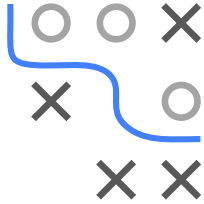


	Bit u_i										
	32	31	...	8	7	6	5	4	3	2	1
-51	1	1	...	1	1	0	0	1	1	0	0
-1	1	1	...	1	1	1	1	1	1	1	0
-0	1	1	...	1	1	1	1	1	1	1	1
0	0	0	...	0	0	0	0	0	0	0	0
1	0	0	...	0	0	0	0	0	0	0	1
51	0	0	...	0	0	1	1	0	0	1	1
	$-(2^{31}-1)$	2^{30}	...	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

The 32nd bit marks negative numbers again. The coded number is given by: $x = \sum_{i=1}^{31} u_i 2^{i-1} - u_{32}(2^{31}-1)$.

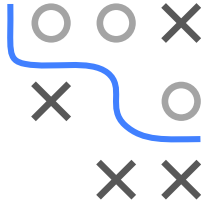
ONE'S COMPLEMENT / 2

Let $\tilde{x}$ be the bitwise complement of x. We check the correctness of the formula:



$$
\begin{aligned}
\tilde{x} &= \sum_{i=1}^{31} \tilde{u}_i 2^{i-1} - \tilde{u}_{32}(2^{31}-1) \\
&= \sum_{i=1}^{31} \underbrace{(1-u_i)}_{\text{complement}} 2^{i-1} - \underbrace{(1-u_{32})}_{\text{complement}} (2^{31}-1) \\
&= \sum_{i=1}^{31} 2^{i-1} - (2^{31}-1) - \sum_{i=1}^{31} u_i 2^{i-1} + u_{32}(2^{31}-1) \\
&= -1 + 2^{31} - (2^{31}-1) - (\sum_{i=1}^{31} u_i 2^{i-1} - u_{32}(2^{31}-1)) = -x
\end{aligned}
$$

ONE'S COMPLEMENT / 3



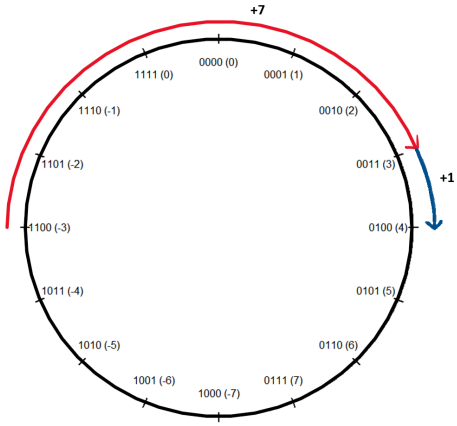
- Covered number range in 32-bit: $-2^{31}+1$ to $2^{31}-1$.
- Very easy conversion from positive to negative and vice versa by inverting all bits.
- The representation of zero is not unique.
- Addition / subtraction works better here than in the signed magnitude representation. But it is still not trivial, since the sum needs to be corrected (by subsequently adding the carry bit as a 1).

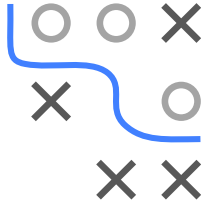
Example: $7-3$ in 4-bit system:

	0111	|(7)
+	1100	|(-3)
	(1)0011	| Carry-Bit
+	0001	| add 1
	0100	|(4)

ONE'S COMPLEMENT / 4

Why adding a 1?

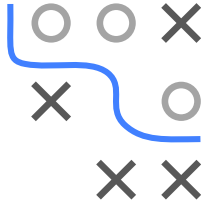




- Because of the 0 being represented twice, a 1 must be added when overflowing the 0, so that the result is correct.

TWO'S COMPLEMENT

With the two's complement, negative numbers are formed by determining the one's complement and adding an additional 1.

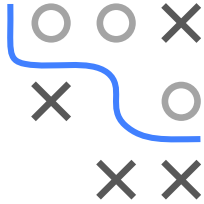


Example: conversion of -51_{10}:

	Bit u_i										
	32	31	...	8	7	6	5	4	3	2	1
$\lvert -51_{10} \rvert$	0	0	...	0	0	1	1	0	0	1	1
invert	1	1	...	1	1	0	0	1	1	0	0
add 1	1	1	...	1	1	0	0	1	1	0	1
	-2^{31}	2^{30}	...	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

TWO'S COMPLEMENT / 2

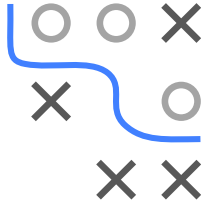
Example: two's complement



	Bit u_i										
	32	31	...	8	7	6	5	4	3	2	1
-2^{31}	1	0	...	0	0	0	0	0	0	0	0
-51	1	1	...	1	1	0	0	1	1	0	1
-1	1	1	...	1	1	1	1	1	1	1	1
0	0	0	...	0	0	0	0	0	0	0	0
1	0	0	...	0	0	0	0	0	0	0	1
51	0	0	...	0	0	1	1	0	0	1	1
$2^{31}-1$	0	1	...	1	1	1	1	1	1	1	1
	-2^{31}	2^{30}	...	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

The coded number is then: $x = \sum_{i=1}^{31} u_i 2^{i-1} - u_{32} 2^{31}$.

TWO'S COMPLEMENT / 3



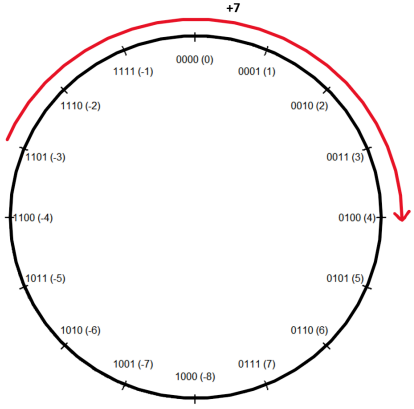
- Covered number range in 32-bit: -2^{31} to $2^{31} - 1$
- Not easy to read anymore. But big advantages for the computer.
- Unique representation of 0.
- Addition / subtraction works as desired. As long as you stay in the number range, you can simply write and add two numbers below each other (the carry bit is ignored).

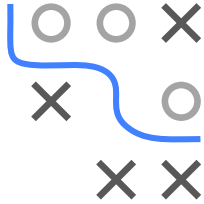
Example: $7 - 3$ in a 4-bit system:

	0111	|(7)
+	1101	|(-3)
	(1)0100	|(4)

TWO'S COMPLEMENT / 4

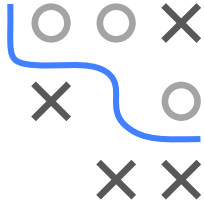
The carry bit can simply be ignored here, since the representation of the 0 is unique.





- Caution when leaving the number range:
- Example: 0011 (3) + 0101 (5) = 1000 (-8)

INTEGER OVERFLOW



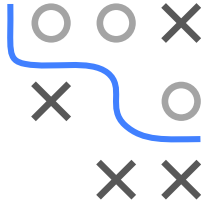
Caution: Arithmetic operations can cause an **overflow**. This is a common programming error in languages like `C` and can lead to undefined behavior (e.g. wrap around).

Example: $(2^{31} - 1) + 1$.
In a 32-bits two's complement representation $(2^{31} - 1) + 1$ would be outside of the covered number range, since $(2^{31} - 1)$ is the largest possible number that can be represented. Adding 1 results in -2^{31} due to an integer overflow.

$(2^{31} - 1)$	01111111	11111111	11111111	11111111
$+1$	00000000	00000000	00000000	00000001
(-2^{31})	10000000	00000000	00000000	00000000

INTEGER OVERFLOW / 2

Excerpt from Wikipedia "Integer Overflow":

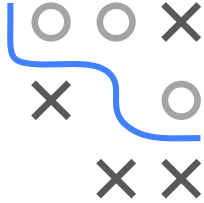


On 30 April 2015, the Federal Aviation Authority announced it will order Boeing 787 operators to reset its electrical system periodically, to avoid an integer overflow which could lead to loss of electrical power and ram air turbine deployment, and Boeing is going to deploy a software update in the fourth quarter.

When Donkey Kong breaks on level 22 it is because of an integer overflow in its time/bonus. Donkey Kong takes the level number you're on, multiplies it by 10 and adds 40. When you reach level 22 the time/bonus number is 260 which is too large for its 8-bit 256 value register so it resets itself to 0 and gives the remaining 4 as the time/bonus - not long enough to complete the level.

MACHINE NUMBERS FOR $\mathbb{Z}$ IN $\mathbb{R}$

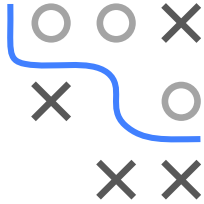
In R, integers are encoded in 32-bit (also in 64-bit R!) by the two's complement.



```
.Machine$integer.max
## [1] 2147483647
```

```
2^31 - 1
## [1] 2147483647
```

MACHINE NUMBERS FOR $\mathbb{Z}$ IN $\mathbb{R}$ / 2



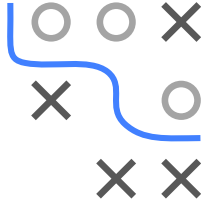
```
intToBits(-51L)
## [1] 01 00 01 01 00 00 01 01 01 01 01 01 01 01 01 01 01 01
## [19] 01 01 01 01 01 01 01 01 01 01 01 01 01 01
```

```
intToBits(51L)
## [1] 01 01 00 00 01 01 00 00 00 00 00 00 00 00 00 00 00 00
## [19] 00 00 00 00 00 00 00 00 00 00 00 00 00 00
```

```
# Caution: In R the operation x^y always results in the type 'numeric'!
intToBits(2L^31L - 1L)
## [1] 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01 01
## [19] 01 01 01 01 01 01 01 01 01 01 01 01 01 01 00
```

MACHINE NUMBERS FOR $\mathbb{Z}$ IN R / 3

In R, integer overflows are caught and set to NA.

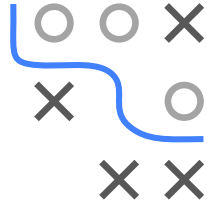


```
.Machine$integer.max + 1
## [1] 2147483648
```

```
str(.Machine$integer.max + 1)
## num 2.15e+09
```

```
str(.Machine$integer.max + 1L)
## Warning in .Machine$integer.max + 1L: NAs produced by
   integer overflow
## int NA
```

R IN 64-BIT SYSTEMS



- In 2010, the 64-bit version of R was released.
- **But:** integers are still encoded in 32-bit.
- The largest integer in R is thus about 2 billion.
- When indexing vectors longer than about 2 billion, R uses a trick:
  - By using floating point numbers in double precision, integers can be represented reliably within the value range $(-2^{53}, 2^{53})$.
  - Beyond that, not all integers are covered!
  - In R you can index with floating point numbers:

```
c(1,2,3)[1.7]
## [1] 1
```