# FTL FOR ONLINE LINEAR OPTIMIZATION

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- Let $\mathcal{A} = [-1, 1]$ and suppose that $z_t = \begin{cases} -\frac{1}{2}, & t = 1, \\ 1, & t \text{ is even}, \\ -1, & t \text{ is odd}. \end{cases}$

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- No matter how we choose the first action $a_1^{\mathtt{FTL}}$, it will hold that FTL has a cumulative loss greater than (or equal) $T - 3/2$, while the best action in hindsight has a cumulative loss of $-1/2$.
- Thus, FTL's cumulative regret is at least $T - 1$, which is linearly growing in $T$.

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- Indeed, note that

$$
\begin{aligned}
a_{t+1}^{\texttt{FTL}} = \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{t} (a, z_s) &= \arg\min_{a \in [-1,1]} a \sum_{s=1}^{t} z_s \\
&= \begin{cases} -1, & \text{if } \sum_{s=1}^{t} z_s > 0, \\ 1, & \text{if } \sum_{s=1}^{t} z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^{t} z_s = 0. \end{cases}
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| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
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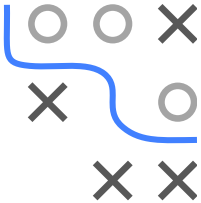
- The best action has cumulative loss

$$
\inf_{a \in \mathcal{A}} \sum_{s=1}^{T} (a, z_s) = \inf_{a \in [-1,1]} a \underbrace{\sum_{s=1}^{T} z_s}_{=(-1/2)^T} = -1/2.
$$

# FTL FOR ONLINE LINEAR OPTIMIZATION



- Thus, we see: FTL can fail for online linear optimization problems, although it is well suited for online quadratic optimization problems!
- The reason is that the action selection of FTL is not stable enough (caused by the loss function), which is fine for the latter problem, but problematic for the former.
- One has to note that the online linear optimization problem example above, where FTL fails, is in fact an adversarial learning setting: The environmental data is generated in such a way that the FTL learner is fooled in each time step.