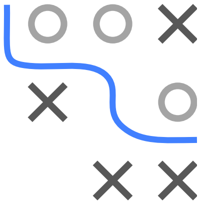
BATCH LEARNING



- The conventional machine learning is rooted in the *statistical learning theory* and is sometimes referred to as the *batch learning scenario*:
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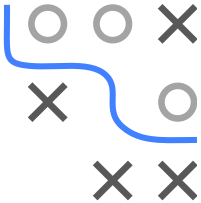

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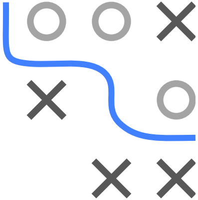

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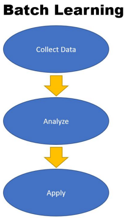
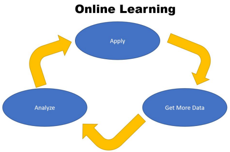


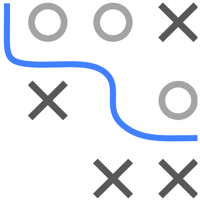
- The learning task on the available data beforehand is called the *training phase* and the prediction on the unseen data is called the *testing phase.* Both phases are **separated**.

ONLINE LEARNING

- However, many real-world problems are *dynamic* with the following aspects:
  - *Sequential order* — data is generated only bit by bit;
  - *On-the-fly decisions* — decisions or predictions have to be made during the data generating process;
  - *Unforeseeable consequences* — decisions can have a drastic influence on the data generating process;
  - *Constraints* — there is a specific time limit or computational limit for the decision.



ONLINE LEARNING



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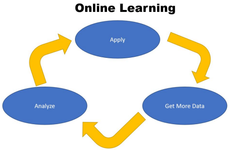
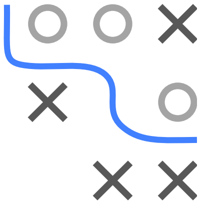


- These dynamic aspects outline the framework where **online learning** is settled.
- Characteristically: In the online learning scenario the training phase and the testing phase are **interleaved**.

ONLINE LEARNING: EXAMPLES



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- *Weather Forecasting* — Observe meteorological data as data streams by satellites for instance and keep the current weather prediction up to date.

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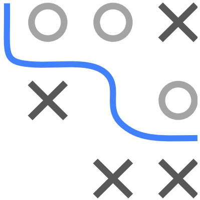

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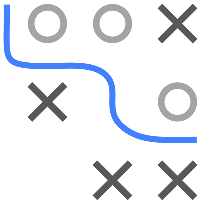

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- *Navigation systems* — Find the best path from A to B given the current traffic situation.
- *Autonomous driving systems* — Steer the automotive, while constantly monitoring the environment and react appropriately to any changes.
- . . .

ONLINE LEARNING: ILLUSTRATION

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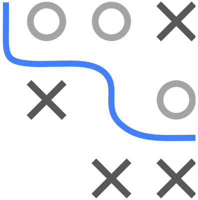

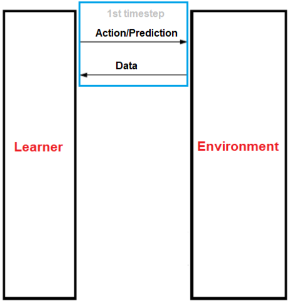

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The data is available only in a **sequential order** generated by the environment and the learner's actions/predictions have to be made **on-the-fly.**
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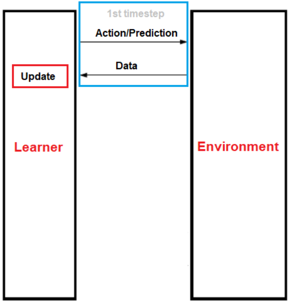

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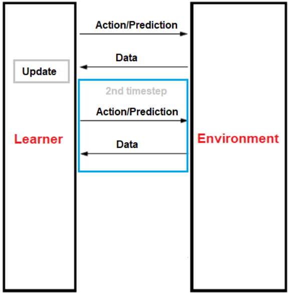

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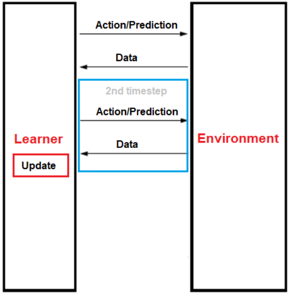

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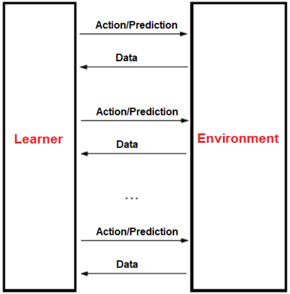

$\Rightarrow$ The learner and the environment are alternately performing their actions.

THE BASIC ONLINE LEARNING PROTOCOL

Formally, an online learning problem consists of:

- a learner (forecaster, agent resp. decision maker), an environment (user resp. adversary, system resp. nature),
- time steps $1, 2, \ldots, T$ (may be infinite),
- available actions $\mathcal{A}$ for the learner (may be infinite),
- environmental data space $\mathcal{Z}$,
- a loss function $L : \mathcal{A} \times \mathcal{Z} \to \mathbb{R}$.

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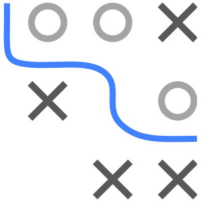

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- learner chooses an action $a_t \in \mathcal{A}$,
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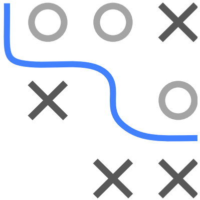

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Typically $\mathcal{A} = \mathcal{Z} = \mathcal{Y}$, so that

- the learner's chosen action $a_t = \hat{y}_t$ corresponds to a prediction,
- the generated data point $z_t = y_t$ is the revealed outcome.

THE EXTENDED ONLINE LEARNING PROTOCOL

- In some applications, the environmental data consists of two parts: $z_t = (z_t^{(1)}, z_t^{(2)})$, where the first part of the data, $z_t^{(1)}$, is revealed to the learner **before** the action is made. After the learner carries out its action, the remaining part of the environmental data is revealed, that is, $z_t^{(2)}$.

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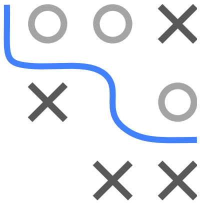
- We call this setting the *extended online learning protocol.*
- Typically $\mathcal{A} = \mathcal{Y}$ and $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, so that
  - the first part $z_t^{(1)} = \mathbf{x}_t$ is some feature information,
  - the learner's chosen action $a_t = \hat{y}_t$ corresponds to a prediction (dep. on $\mathbf{x}_t$),
  - the second part $z_t^{(2)} = y_t$ is the corresponding outcome.

DATA GENERATION IN ONLINE LEARNING



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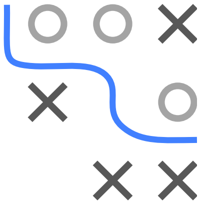
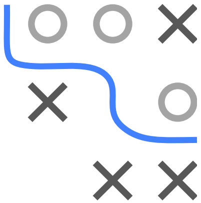


- Typically for the online learning setting is that **no** statistical assumptions is made on how the sequence of environmental data is generated.
- In particular, the environmental data are not necessarily generated by a probability distribution!
- This also covers the area of *adversarial learning*: the data can even be generated by an adversary trying to fool the learner.
- However, the online learning setting can of course also be considered in a statistical setting.

ONLINE LEARNING: REQUIREMENTS



- The dynamical aspects have to be incorporated for the design of efficient learning algorithms.
- The online learner has to cope with the sequential availability of the data and to cope with time as well as computational constraints.
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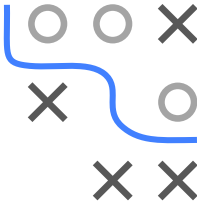

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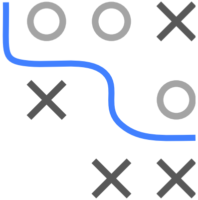
- Roughly speaking, one seeks to construct an online learning algorithm which is adaptive to the environment and allows incremental as well as preferably cheap updates over time.
- Although consideration of time and memory constraints is important for practical purposes, we will only implicitly consider these constraints in this lecture.
- We will mainly focus our theoretical analysis on the performance of the learner in terms of its (cumulative) loss, which, however, will usually ignore computational aspects of the learner.

MEASURE OF QUALITY IN ONLINE LEARNING



- In order to measure the quality of an online learner one can compute the difference between the cumulative loss of the learner and the cumulative loss by taking some competing action $a \in \mathcal{A}$:

$$R_T(a) = \sum_{t=1}^{T} L(a_t, z_t) - \sum_{t=1}^{T} L(a, z_t).$$

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- Here,
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  - $\sum_{t=1}^{T} L(a, z_t)$ is the cumulative loss of the competing action a.

MEASURE OF QUALITY IN ONLINE LEARNING

- It seems natural to compare the incurred cumulative loss of the learner with the *best action(s) in hindsight*:

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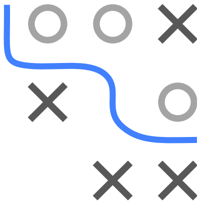

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- We refer to R_T as the *(cumulative) regret* of the online learner. It is easy to see that $R_T = \sup_{a \in \mathcal{A}} R_T(a)$.

MEASURE OF QUALITY IN ONLINE LEARNING

- The objective of the online learner is to minimize the cumulative regret R_T.

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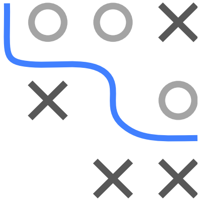

- The objective of the online learner is to minimize the cumulative regret R_T.
- Note that the cumulative regret can be in principle negative as the action sequence could be such that $L(a_s, z_s) < L(a^*, z_s)$ holds for specific time steps s, where $a^* \in \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{T} L(a, z_s)$ is one of the best actions in hindsight (may be unique).

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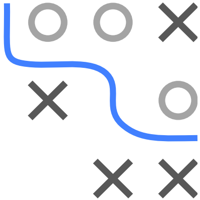

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- Formally, the following should hold

$$R_T = o(T).$$

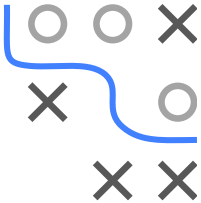
Interpretation: The average regret per time step (or per example) goes to zero:

$$\frac{1}{T}\Big(\sum_{t=1}^{T} L(a_t, z_t) - \inf_{a \in \mathcal{A}} \sum_{t=1}^{T} L(a, z_t)\Big) = \frac{R_T}{T} = o(1).$$

DYNAMIC REGRET

- One might ask why one compares only with a fixed best action in hindsight, say a^*, instead of a sequence of actions $a_1^*, a_2^*, \ldots, a_T^*$?

DYNAMIC REGRET



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- We will cover only the static regret in this lecture.