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- Indeed, a learner within the basic online learning protocol, say `Algo`, is a function

$$A : \bigcup_{t=1}^T (\mathcal{Z} \times \mathcal{A})^t \rightarrow \mathcal{A}$$

that returns the current action based on (the loss and) the full history of information so far:

$$a_{t+1}^{\text{Algo}} = A(z_1, a_1^{\text{Algo}}, z_2, a_2^{\text{Algo}}, \dots, z_t, a_t^{\text{Algo}}; ).$$



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- In the extended online learning scenario, where the environmental data consists of two parts,  $z_t = (z_t^{(1)}, z_t^{(2)})$ , and the **first part is revealed before the action in  $t$  is performed**, we have that

$$a_{t+1}^{\text{Algo}} = A(z_1, a_1^{\text{Algo}}, z_2, a_2^{\text{Algo}}, \dots, z_t, a_t^{\text{Algo}}, \mathbf{z_{t+1}^{(1)}}; )$$



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- It will be desired that the online learner admits a *cheap update formula*, which is incremental, i.e., only a portion of the previous data is necessary to determine the next action.
- For instance, there exists a function  $u : \mathcal{Z} \times \mathcal{A} \rightarrow \mathcal{A}$  such that

$$A(z_1, a_1^{\text{Algo}}, z_2, a_2^{\text{Algo}}, \dots, z_t, a_t^{\text{Algo}}; ) = u(z_t, a_t^{\text{Algo}}).$$



# FOLLOW THE LEADER ALGORITHM

- A simple algorithm to tackle online learning problems is the **Follow the leader** (FTL) algorithm.
- The algorithm takes as its action  $a_t^{\text{FTL}} \in \mathcal{A}$  in time step  $t \geq 2$ , the element which has the minimal cumulative loss so far over the previous  $t - 1$  time periods:

$$a_t^{\text{FTL}} \in \arg \min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} (a, z_s).$$

(Technical side note: if there are more than one minimum, then one of them is chosen. Moreover,  $a_1^{\text{FTL}}$  is arbitrary. )



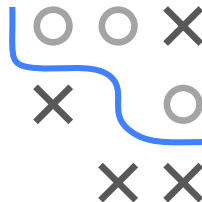
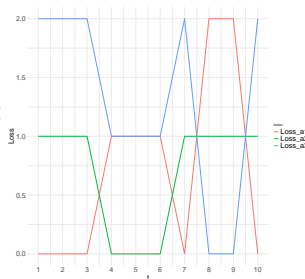
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- *Interpretation:* The action  $a_t^{\text{FTL}}$  is the current "leader" of the actions in  $\mathcal{A}$  in time step  $t$ , as it has the smallest cumulative loss (error) so far.



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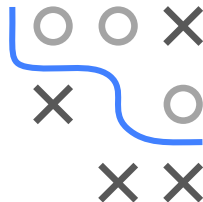
$$a_t^{\text{FTL}} \in \arg \min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} (a, z_s).$$

- Note that the action selection rule of FTL is natural and has much in common with the classical batch learning approaches based on empirical risk minimization.
- This results in a first issue regarding the computation time for the action, because the longer we run this algorithm, the slower it becomes (in general) due to the growth of the seen data.



# FTL: A HELPFUL LEMMA

**Lemma:** Let  $a_1^{\text{FTL}}, a_2^{\text{FTL}}, \dots$  be the sequence of actions used by the FTL algorithm for the environmental data sequence  $z_1, z_2, \dots$





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Then, for all  $\tilde{a} \in \mathcal{A}$  it holds that

$$\begin{aligned} R_T^{\text{FTL}}(\tilde{a}) &= \sum_{t=1}^T ((a_t^{\text{FTL}}, z_t) - (\tilde{a}, z_t)) \\ &\leq \sum_{t=1}^T ((a_t^{\text{FTL}}, z_t) - (a_{t+1}^{\text{FTL}}, z_t)) \\ &= \sum_{t=1}^T (a_t^{\text{FTL}}, z_t) - \sum_{t=1}^T (a_{t+1}^{\text{FTL}}, z_t). \end{aligned}$$



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In particular,

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*Interpretation:* the regret of the FTL algorithm is **bounded** by **the difference of cumulated losses of itself** compared to **its one-step lookahead cheater version**.



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**Proof:** In the following, we denote  $a_1^{\text{FTL}}, a_2^{\text{FTL}}, \dots$  simply by  $a_1, a_2, \dots$

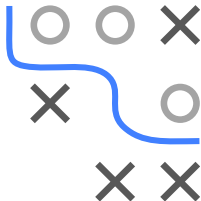


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$$\begin{aligned} R_T^{\text{FTL}}(\tilde{a}) &= \sum_{t=1}^T ((a_t, z_t) - (\tilde{a}, z_t)) \leq \sum_{t=1}^T ((a_t, z_t) - (a_{t+1}, z_t)) \\ &\Leftrightarrow \sum_{t=1}^T (a_{t+1}, z_t) \leq \sum_{t=1}^T (\tilde{a}, z_t). \end{aligned}$$



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Hence, we will verify the inequality  $\sum_{t=1}^T (a_{t+1}, z_t) \leq \sum_{t=1}^T (\tilde{a}, z_t)$ , which implies the assertion.

$\rightsquigarrow$  This will be done by induction over  $T$ .



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**Reminder:** 
$$a_t^{\text{FTL}} \in \arg \min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} (a, z_s).$$

**Initial step:**  $T = 1$ . It holds that

$$\begin{aligned} \sum_{t=1}^T (a_{t+1}, z_t) &= (a_2, z_1) = \left( \arg \min_{a \in \mathcal{A}} (a, z_1), z_1 \right) \\ &= \min_{a \in \mathcal{A}} (a, z_1) \leq (\tilde{a}, z_1) \quad \left( = \sum_{t=1}^T (\tilde{a}, z_t) \right) \end{aligned}$$

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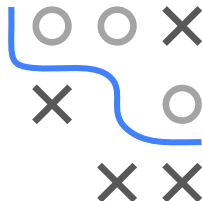
for all  $\tilde{a} \in \mathcal{A}$ .

**Induction Step:**  $T - 1 \rightarrow T$ . Assume that for any  $\tilde{a} \in \mathcal{A}$  it holds that

$$\sum_{t=1}^{T-1} (a_{t+1}, z_t) \leq \sum_{t=1}^{T-1} (\tilde{a}, z_t).$$

Then, the following holds as well (adding  $(a_{T+1}, z_T)$  on both sides)

$$\sum_{t=1}^T (a_{t+1}, z_t) \leq (a_{T+1}, z_T) + \sum_{t=1}^{T-1} (\tilde{a}, z_t), \quad \forall \tilde{a} \in \mathcal{A}.$$





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**Reminder (1):**  $\sum_{t=1}^T (a_{t+1}, z_t) \leq (a_{T+1}, z_T) + \sum_{t=1}^{T-1} (\tilde{a}, z_t).$

**Reminder (2):**  $a_t^{\text{FTL}} \in \arg \min_{a \in \mathcal{A}} \sum_{s=1}^{t-1} (a, z_s).$



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Using (1) with  $\tilde{a} = a_{T+1}$  yields

$$\begin{aligned} \sum_{t=1}^T (a_{t+1}, z_t) &\leq \sum_{t=1}^T (a_{T+1}, z_t) = \sum_{t=1}^T \left( \arg \min_{a \in \mathcal{A}} \sum_{s=1}^T (a, z_s), z_t \right) \\ &= \min_{a \in \mathcal{A}} \sum_{t=1}^T (a, z_t) \leq \sum_{t=1}^T (\tilde{a}, z_t) \end{aligned}$$

for all  $\tilde{a} \in \mathcal{A}$ .

□



# FTL FOR OQO PROBLEMS

- One popular instantiation of the online learning problem is the problem of *online quadratic optimization* (OQO).
- In its most general form, the loss function is thereby defined as

$$(a_t, z_t) = \frac{1}{2} \|a_t - z_t\|_2^2,$$

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- **Proposition:** Using FTL on any online quadratic optimization problem with  $\mathcal{A} = \mathbb{R}^d$  and  $V = \sup_{z \in \mathcal{Z}} \|z\|_2$ , leads to a regret of

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- This result is satisfactory for three reasons:
  - 1 The regret is definitely sublinear, that is,  $R_T^{\text{FTL}} = o(T)$ .
  - 2 We just have a mild constraint on the online quadratic optimization problem, namely that  $\|z\|_2 \leq V$  holds for any possible environmental data instance  $z \in \mathcal{Z}$ .
  - 3 The action  $a_t^{\text{FTL}}$  is simply the empirical average of the environmental data seen so far:  $a_t^{\text{FTL}} = \frac{1}{t-1} \sum_{s=1}^{t-1} z_s$ .

