

FTL FOR ONLINE LINEAR OPTIMIZATION

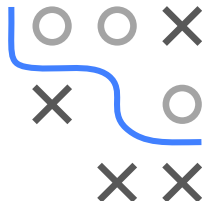
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- Let $\mathcal{A} = [-1, 1]$ and suppose that $z_t = \begin{cases} -\frac{1}{2}, & t = 1, \\ 1, & t \text{ is even,} \\ -1, & t \text{ is odd.} \end{cases}$



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- Indeed, note that

$$\begin{aligned} a_{t+1}^{\text{FTL}} &= \arg \min_{a \in \mathcal{A}} \sum_{s=1}^t (a, z_s) = \arg \min_{a \in [-1, 1]} a \sum_{s=1}^t z_s \\ &= \begin{cases} -1, & \text{if } \sum_{s=1}^t z_s > 0, \\ 1, & \text{if } \sum_{s=1}^t z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^t z_s = 0. \end{cases} \end{aligned}$$

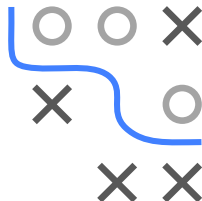


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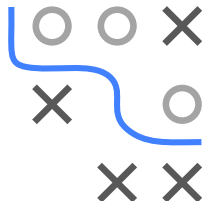


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| 3 | -1 | -1 | 1 | $2 - 1/2$ | $-1/2$ |

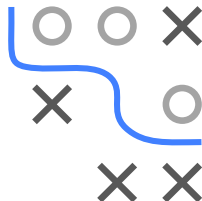


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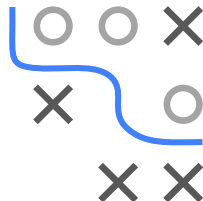
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| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
| T | $(-1)^T$ | $(-1)^T$ | 1 | $T - 1 - 1/2$ | $(-1/2)^T$ |



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| 1 | 1 | -1/2 | -1/2 | -1/2 | -1/2 |
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| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
| T | $(-1)^T$ | $(-1)^T$ | 1 | $T - 1 - 1/2$ | $(-1/2)^T$ |

- The best action has cumulative loss

$$\inf_{a \in \mathcal{A}} \sum_{s=1}^T (a, z_s) = \inf_{a \in [-1,1]} a \underbrace{\sum_{s=1}^T z_s}_{=(-1/2)^T} = -1/2.$$

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- Thus, we see: FTL can fail for **online linear optimization problems**, although it is well suited for **online quadratic optimization problems**!
- The reason is that the action selection of FTL is not stable enough (caused by the loss function), which is fine for **the latter problem**, but problematic for **the former**.
- One has to note that the online linear optimization problem example above, where FTL fails, is in fact an adversarial learning setting: The environmental data is generated in such a way that the FTL learner is fooled in each time step.

