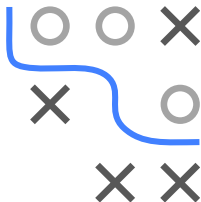


ONLINE GRADIENT DESCENT

- The *Online Gradient Descent* (OGD) algorithm with step size $\eta > 0$ chooses its action by

$$a_{t+1}^{\text{OGD}} = a_t^{\text{OGD}} - \eta \nabla_a(a_t^{\text{OGD}}, z_t), \quad t = 1, \dots, T. \quad (1)$$

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- We have the following connection between FTRL and OGD:
 - Let $\tilde{z}_t^{\text{OGD}} := \nabla_a(a_t^{\text{OGD}}, z_t)$ for any $t = 1, \dots, T$.
 - The update formula for FTRL with $_2$ norm regularization for the linear loss L^{lin} and the environmental data \tilde{z}_t^{OGD} is

$$a_{t+1}^{\text{FTRL}} = a_t^{\text{FTRL}} - \eta \tilde{z}_t^{\text{OGD}} = a_t^{\text{FTRL}} - \eta \nabla_a(a_t^{\text{OGD}}, z_t).$$

- If we have that $a_1^{\text{FTRL}} = a_1^{\text{OGD}}$, then it iteratively follows that $a_{t+1}^{\text{FTRL}} = a_{t+1}^{\text{OGD}}$ for any $t = 1, \dots, T$ in this case.



ONLINE GRADIENT DESCENT: DEFINITION AND PROPERTIES

- With the deliberations above we can infer that

$$\begin{aligned} R_{T,}^{\text{OGD}}(\tilde{a} \mid (z_t)_t) &= \sum_{t=1}^T (a_t^{\text{OGD}}, z_t) - (\tilde{a}, z_t) \\ &\leq \sum_{t=1}^T L^{\text{lin}}(a_t^{\text{OGD}}, \tilde{z}_t^{\text{OGD}}) - L^{\text{lin}}(\tilde{a}, \tilde{z}_t^{\text{OGD}}) \\ &\quad (\text{if } a_1^{\text{OGD}} = a_1^{\text{FTRL}}) \sum_{t=1}^T L^{\text{lin}}(a_t^{\text{FTRL}}, \tilde{z}_t^{\text{OGD}}) - L^{\text{lin}}(\tilde{a}, \tilde{z}_t^{\text{OGD}}) \\ &= R_{T, L^{\text{lin}}}^{\text{FTRL}}(\tilde{a} \mid (\tilde{z}_t^{\text{OGD}})_t), \end{aligned}$$

where we write in the subscripts of the regret the corresponding loss function and also include the corresponding environmental data as a second argument in order to emphasize the connections.



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where we write in the subscripts of the regret the corresponding loss function and also include the corresponding environmental data as a second argument in order to emphasize the connections.

- *Interpretation:* The regret of the FTRL algorithm (with $_2$ norm regularization) for the online linear optimization problem (characterized by the linear loss L^{lin}) with environmental data \tilde{z}_t^{OGD} is an upper bound for the OGD algorithm for the online convex problem (characterized by a differentiable convex loss) with the original environmental data z_t .



ONLINE GRADIENT DESCENT: REGRET

- Due to this connection we immediately obtain a similar decomposition of the regret upper bound into a bias term and a variance term as for the FTRL algorithm for OLO problems.
- **Corollary.** Using the OGD algorithm on any online convex optimization problem (with differentiable loss function) leads to a regret of OGD with respect to any action $\tilde{a} \in \mathcal{A}$ of

$$\begin{aligned} R_T^{\text{OGD}}(\tilde{a}) &\leq \frac{1}{2\eta} \|\tilde{a}\|_2^2 + \eta \sum_{t=1}^T \|\tilde{z}_t^{\text{OGD}}\|_2^2 \\ &= \frac{1}{2\eta} \|\tilde{a}\|_2^2 + \eta \sum_{t=1}^T \|\nabla_a(a_t^{\text{OGD}}, z_t)\|_2^2. \end{aligned}$$

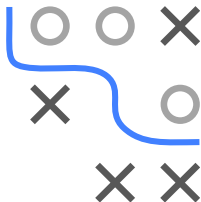


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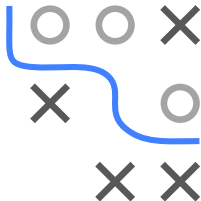
- Note that the step size $\eta > 0$ of OGD has the same role as the regularization magnitude of FTRL: It should balance the trade-off between the bias- and the variance-term.



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- As a consequence, we can also derive a similar order of the regret for the OGD algorithm on OCO problems as for the FTRL on OLO problems by imposing a slightly different assumption on the (new) “variance” term

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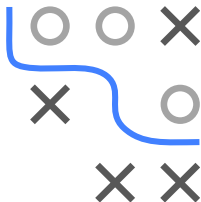
$$\sum_{t=1}^T \|\nabla_a(a_t^{\text{OGD}}, z_t)\|_2^2.$$

- Corollary:** Suppose we use the OGD algorithm on an online convex optimization problem with a convex action space $\mathcal{A} \subset \mathbb{R}^d$ such that

- $\sup_{\tilde{a} \in \mathcal{A}} \|\tilde{a}\|_2 \leq B$ for some finite constant $B > 0$
- $\sup_{a \in \mathcal{A}, z \in \mathcal{Z}} \|\nabla_a(a, z)\|_2 \leq V$ for some finite constant $V > 0$.

Then, by choosing the step size η for OGD as $\eta = \frac{B}{V\sqrt{2T}}$ we get

$$R_T^{\text{OGD}} \leq BV\sqrt{2T}.$$



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↪ This result shows that the Online Gradient Descent is *optimal* regarding its order of its regret with respect to the time horizon T .

