FOLLOW THE REGULARIZED LEADER

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- To be more precise, let for $t \geq 1$
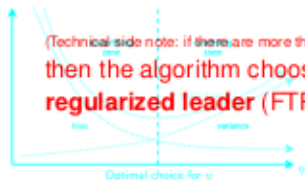
$$a_t^{\mathrm{FTRL}} \in \arg\min_{a \in \mathcal{A}} \left(\psi(a) + \sum_{s=1}^{t-1} (a, z_s) \right),$$

(Technical side note: if there are more than one minimum, then one of them is chosen.)

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REGRET ANALYSIS OF FTRL: A HELPFUL LEMMA

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⇒ We have seen an analogous result for FTL!

(The proof is similar.)

FTRL FOR ONLINE LINEAR OPTIMIZATION

- In the following, we analyze the FTRL algorithm for the linear loss $(a, z) \mapsto a^\top z$ for online linear optimization (OLO) problems.
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- For the more general case, where $\mathcal{A}$ is a strict subset of $\mathbb{R}^d$, we need a slight modification of the update formula above:

$$a_t^{\text{FTRL}} = \Pi_{\mathcal{A}}\left(-\eta \sum_{i=1}^{t-1} z_i\right) = \underset{a \in \mathcal{A}}{\arg\min} \left\|a - \eta \sum_{i=1}^{t-1} z_i\right\|_2^2.$$

In words, the action of the FTRL algorithm has to be projected onto the set $\mathcal{A}$. Here, $\Pi_{\mathcal{A}} : \mathbb{R}^d \to \mathcal{A}$ is the projection onto $\mathcal{A}$.

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  Reminder (1): $R_T^{\text{FTRL}}(\tilde{a}) \leq \psi(\tilde{a}) - \psi(a_1^{\text{FTRL}}) + \sum_{t=1}^{T} \left(\langle a_t^{\text{FTRL}}, z_t \rangle - \langle a_{t+1}^{\text{FTRL}}, z_t \rangle \right).$

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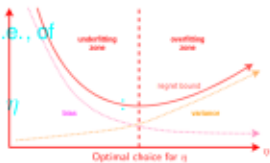- Interpretation of the terms in the proposition, i.e., of

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FTRL FOR OLO: THEORETICAL GUARANTEES

- Under certain assumptions we can balance the trade-off induced by the bias and the variance by choosing η appropriately.
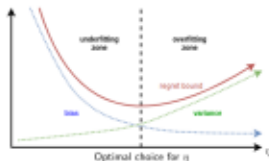

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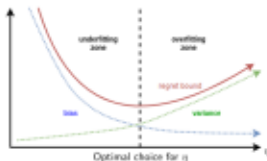

- **Corollary:** Suppose we use the FTRL algorithm with the squared L2-norm regularization on an online linear optimization problem with $\mathcal{A} \subset \mathbb{R}^d$ such that
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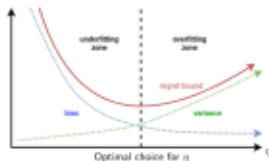 - $\sup_{z \in \mathcal{Z}} \|z\|_2 \leq V$ for some finite constant $V > 0$.

Then, by choosing the step size η for FTRL as $\eta = \frac{B}{V\sqrt{2\,T}}$ it holds that

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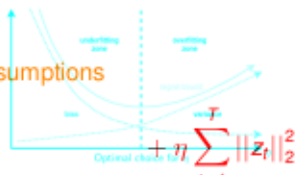
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FTRL FOR OLO: THEORETICAL GUARANTEES

- **Proof:**

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THEORETICAL GUARANTEES

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  - online quadratic optimization (OQO) problems by using no regularity ($\psi \equiv 0$). In this case, we have satisfactory regret guarantees and also a quick update rule for a_{t+1}^{FTRL} (It is just the empirical average over all data points seen till t),

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