

DISCRETE FUNCTIONS

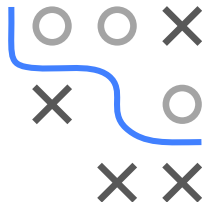
For simplicity, let us consider functions with finite domains first.

Let $\mathcal{X} = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\}$ be a finite set of elements and \mathcal{H} the set of all functions from $\mathcal{X} \rightarrow \mathbb{R}$.

Remark: \mathcal{X} does not mean the training data here but means the “real” domain of the functions.

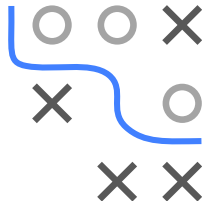
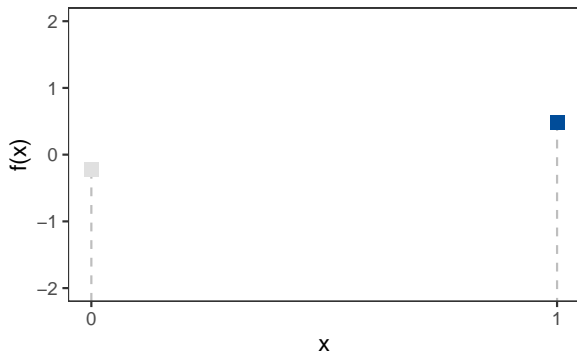
Since the domain of any $f(\cdot) \in \mathcal{H}$ has only n elements, we can represent the function $f(\cdot)$ compactly as a n -dimensional vector

$$\mathbf{f} = \left[f\left(\mathbf{x}^{(1)}\right), \dots, f\left(\mathbf{x}^{(n)}\right) \right].$$



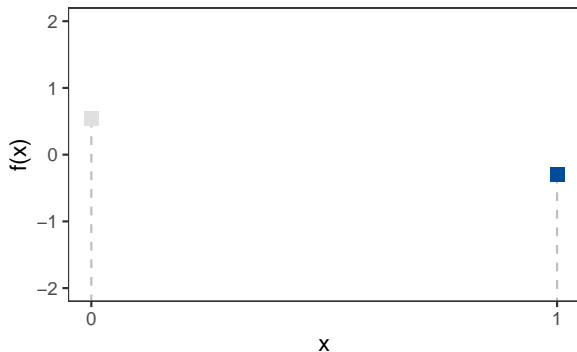
DISCRETE FUNCTIONS

Some examples $f : \mathcal{X} \rightarrow \mathbb{R}$ where \mathcal{X} is univariate and finite:



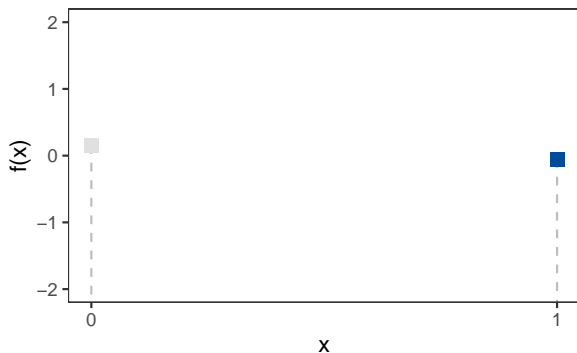
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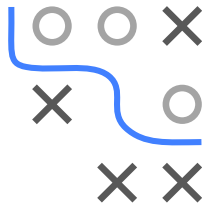
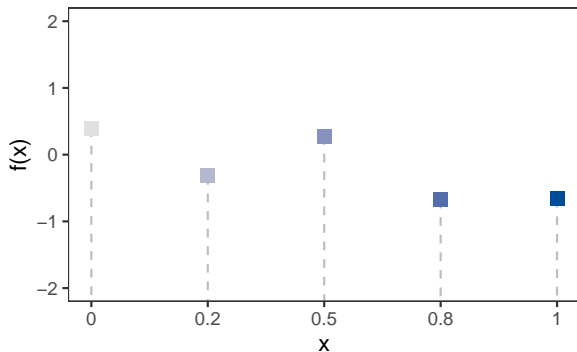
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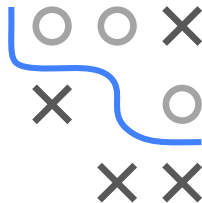
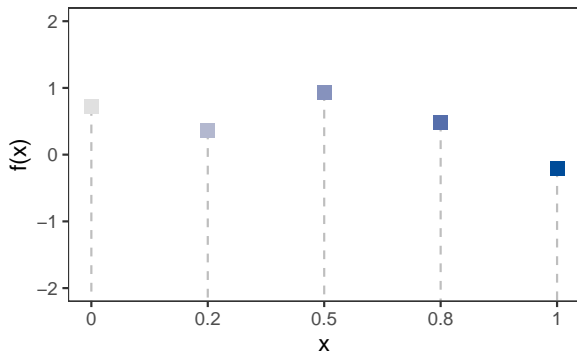
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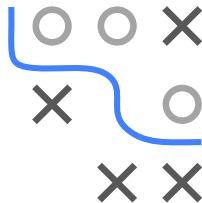
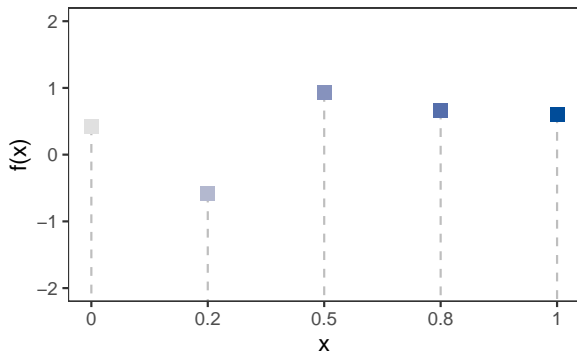
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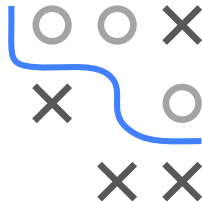
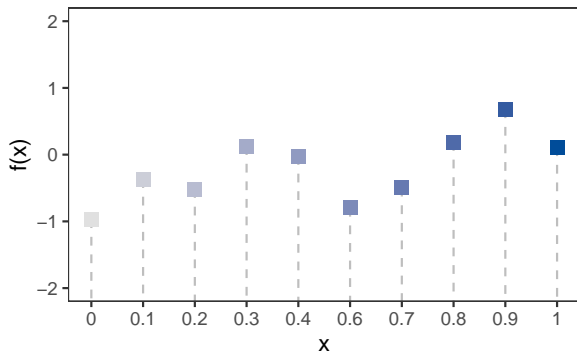
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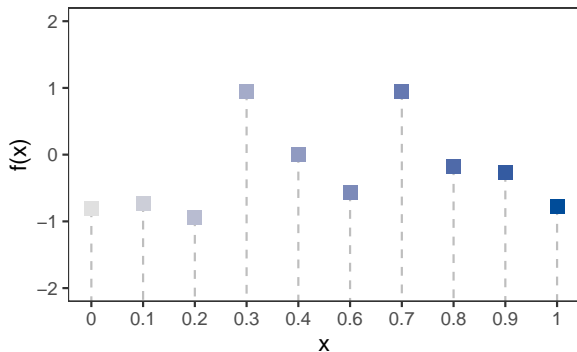
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DISTRIBUTIONS ON DISCRETE FUNCTIONS

One natural way to specify a probability function on a discrete function $f \in \mathcal{H}$ is to use the vector representation

$$\mathbf{f} = \left[f(\mathbf{x}^{(1)}), f(\mathbf{x}^{(2)}), \dots, f(\mathbf{x}^{(n)}) \right]$$

of the function.

Let us see \mathbf{f} as a n -dimensional random variable. We will further assume the following normal distribution:

$$\mathbf{f} \sim \mathcal{N}(\mathbf{m}, \mathbf{K}).$$

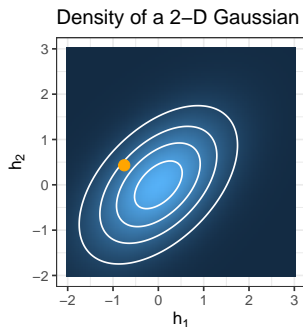
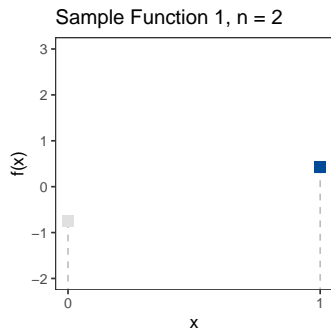
Note: For now, we set $\mathbf{m} = \mathbf{0}$ and take the covariance matrix \mathbf{K} as given. We will see later how they are chosen / estimated.



DISCRETE FUNCTIONS

Let $f : \mathcal{X} \rightarrow \mathbb{R}$. Sample functions by sampling from a two-dimensional normal variable.

$$\mathbf{f} = [f(1), f(2)] \sim \mathcal{N}(\mathbf{m}, \mathbf{K})$$



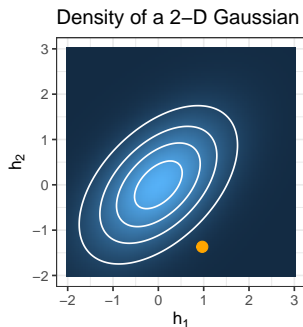
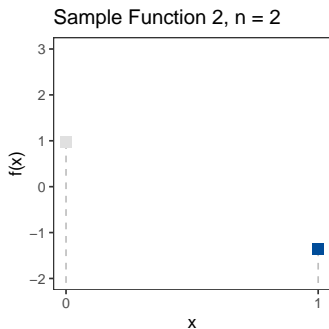
In this example, $\mathbf{m} = (0, 0)$ and $\mathbf{K} = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$.



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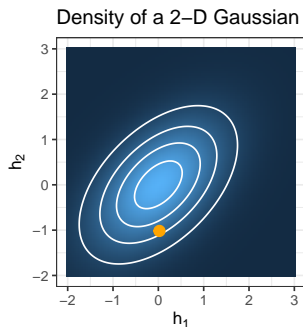
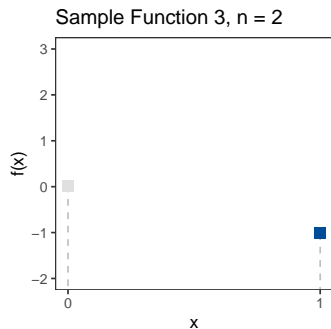
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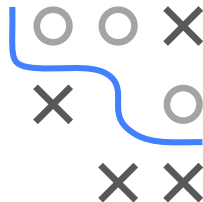
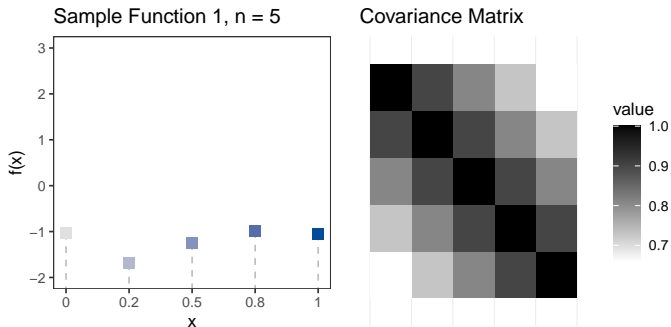
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Let $f : \mathcal{X} \rightarrow \mathbb{R}$. Sample functions by sampling from a five-dimensional normal variable.

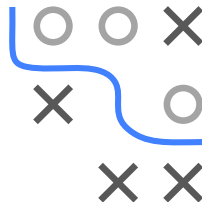
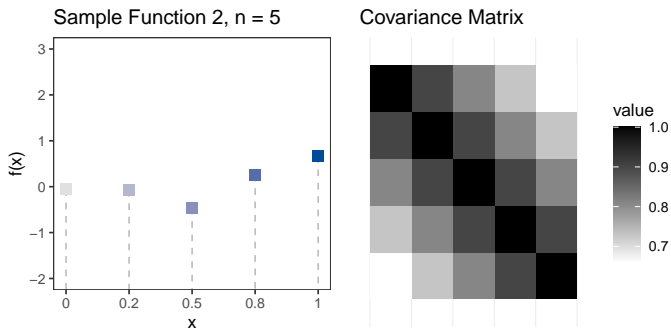
$$\mathbf{f} = [f(1), f(2), f(3), f(4), f(5)] \sim \mathcal{N}(\mathbf{m}, \mathbf{K})$$



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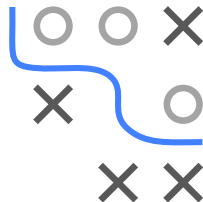
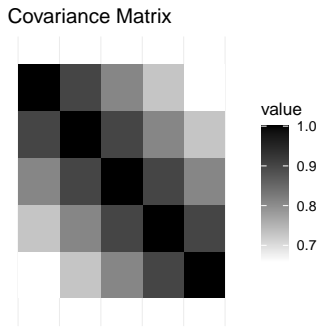
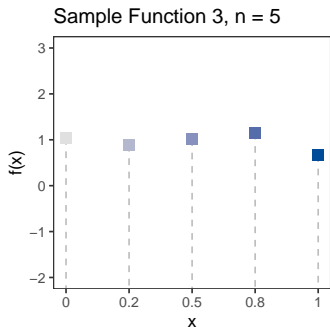
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ROLE OF THE COVARIANCE FUNCTION

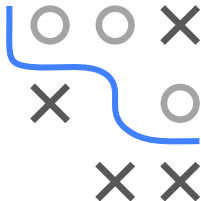
- “Meaningful” functions (on a numeric space \mathcal{X}) may be characterized by a spatial property:

If two points $\mathbf{x}^{(i)}, \mathbf{x}^{(j)}$ are close in \mathcal{X} -space, their function values $f(\mathbf{x}^{(i)}), f(\mathbf{x}^{(j)})$ should be close in \mathcal{Y} -space.

In other words: If they are close in \mathcal{X} -space, their functions values should be **correlated**!

- We can enforce that by choosing a covariance function with

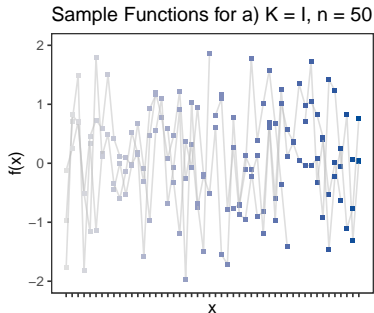
K_{ij} high, if $\mathbf{x}^{(i)}, \mathbf{x}^{(j)}$ close.



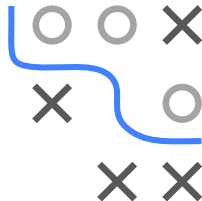
ROLE OF THE COVARIANCE FUNCTION / 2

Covariance controls the “shape” of the drawn function. Consider cases of varying correlation structure

a) uncorrelated: $K = I$.

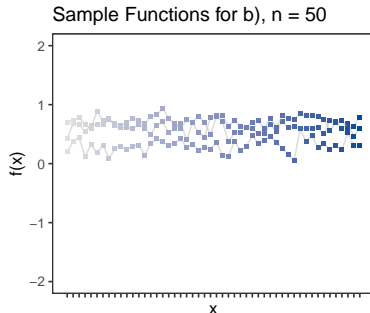


Points are uncorrelated. We sample white noise.



ROLE OF THE COVARIANCE FUNCTION

b) Correlation almost 1: $\mathbf{K} = \begin{pmatrix} 1 & 0.99 & \dots & 0.99 \\ 0.99 & 1 & \dots & 0.99 \\ 0.99 & 0.99 & \ddots & 0.99 \\ 0.99 & \dots & 0.99 & 1 \end{pmatrix}.$

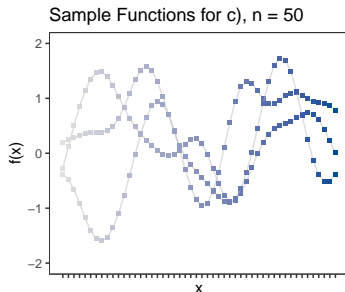


Points are highly correlated. Functions become very smooth and flat.

ROLE OF THE COVARIANCE FUNCTION / 2

- We can compute the entries of the covariance matrix by a function that is based on the distance between $\mathbf{x}^{(i)}, \mathbf{x}^{(j)}$, for example:

c) Spatial correlation: $K_{ij} = k(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) = \exp\left(-\frac{1}{2} \|\mathbf{x}^{(i)} - \mathbf{x}^{(j)}\|^2\right)$



Function exhibit interesting, variable shape.

NB: $k(\cdot, \cdot)$ is called **covar. function** or **kernel**, we will study it more later.

