Lab 1

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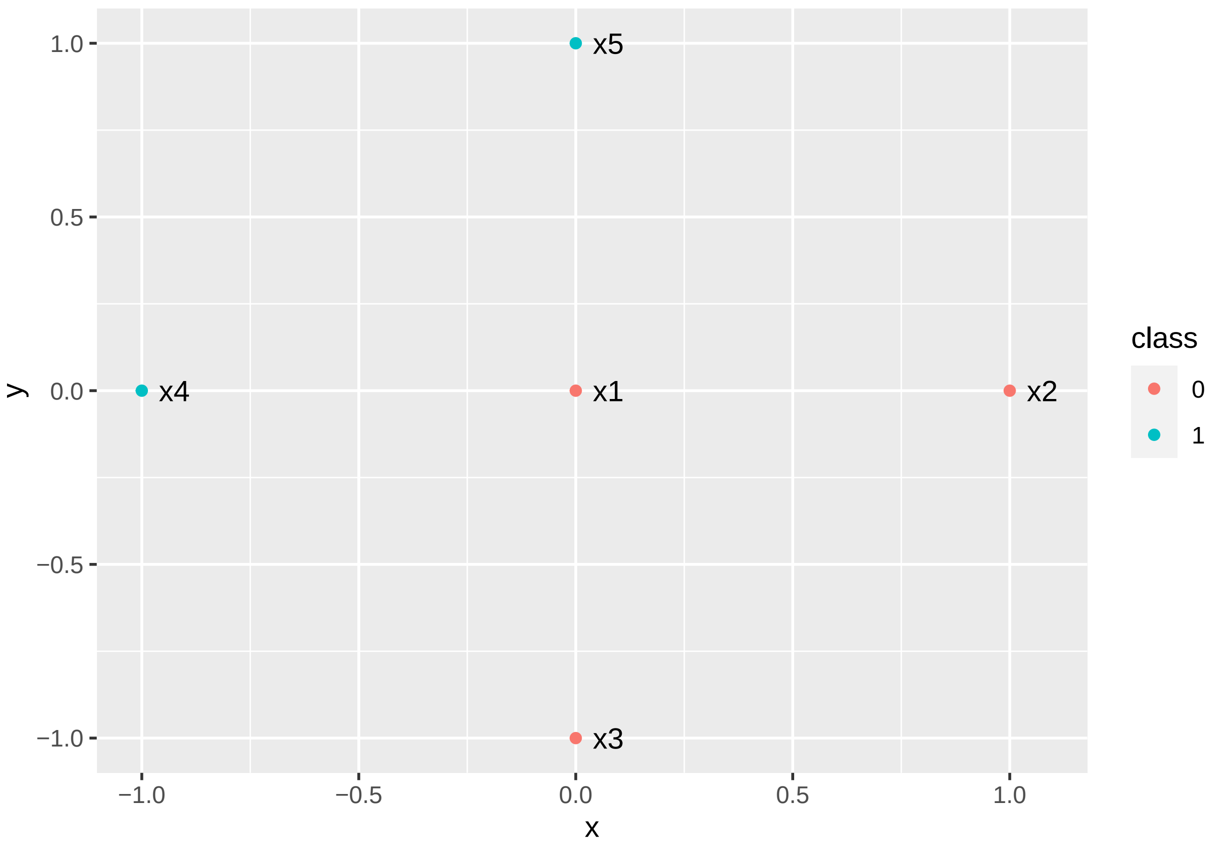
Welcome to the very first lab, in which we will have fun with logistic regression.

Exercise 1

Suppose you have five input points, $\mathbf{x}_1 = |0,0|^T$, $\mathbf{x}_2 = |1,0|^T$, $\mathbf{x}_3 = |0,-1|^T$, $\mathbf{x}_4 = |-1,0|^T$ and $\mathbf{x}_5 = |0,1|^T$, and the corresponding classes are $y_1 = y_2 = y_3 = 0$ and $y_4 = y_5 = 1$:



Consider a logistic regression model $\hat{y}_i = \sigma\left(\alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2}\right)$, with $\sigma(\cdot)$ the sigmoid function, $\sigma(x) = \left(1+e^{-x}\right)^{-1}$. What values for α_0, α_1 and α_2 would result in the correct classification for this dataset? A positive label is predicted when the output of the sigmoid is larger or equal than 0.5.

Note: do not use any formulas or automated methods to find the answer. Think for yourself. A logistic regression classifier is nothing more than a hyper-plane separating points of the two classes. If necessary, review vectors, dot-products and their geometrical interpretation in linear algebra. This applies to the following exercises, too.

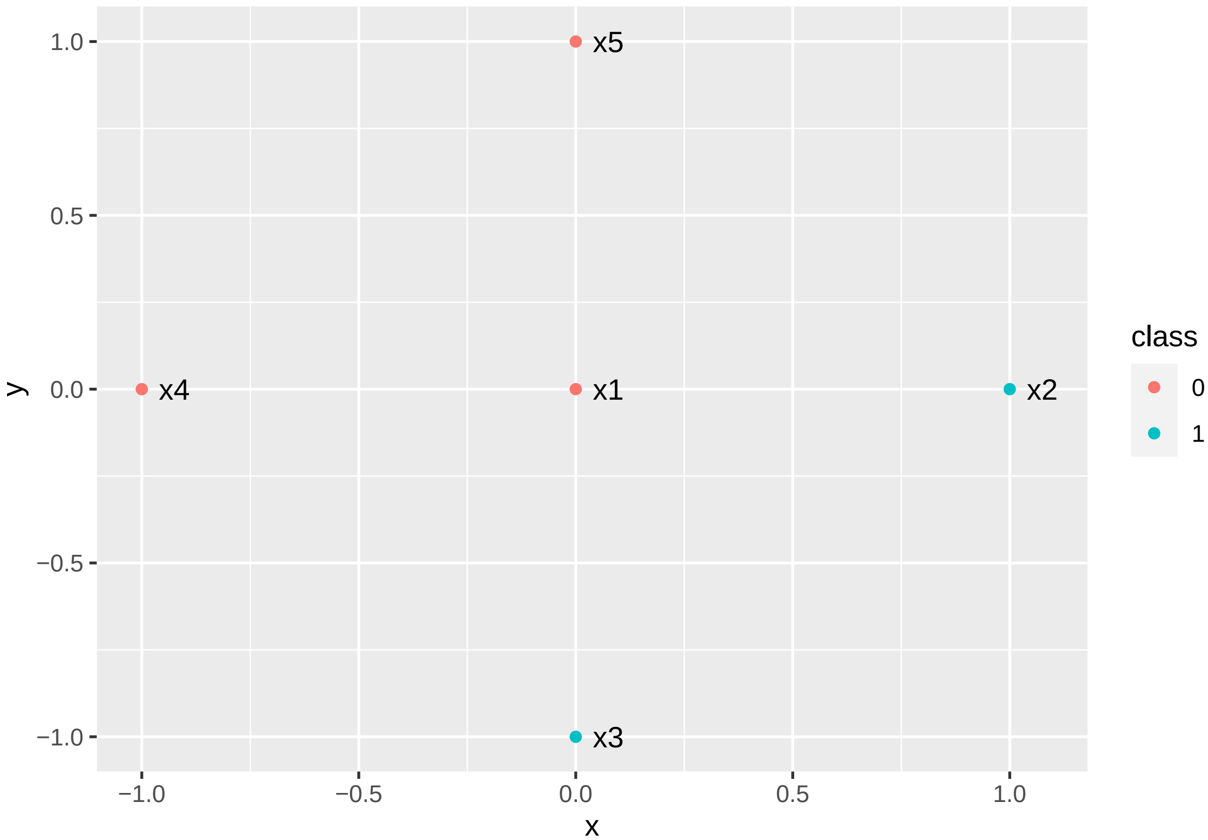
```r
a0 = (
  -5
)

a1 = (
  -10
)

a2 = (
  10
)

# the first column is always one and is used for the "bias"
xs = matrix(c(
  1, 0, 0,
  1, 1, 0,
  1, 0, -1,
  1, -1, 0,
  1, 0, 1
), ncol = 3, byrow = T)

sigmoid = function(x) {
  1 / (1 + exp(-x))
}

sigmoid(xs %*% c(a0, a1, a2))
```

```
##              [,1]
## [1,] 6.692851e-03
## [2,] 3.059022e-07
## [3,] 3.059022e-07
## [4,] 9.933071e-01
## [5,] 9.933071e-01
```

You should make sure that the last two values are close to one, and the others are close to zero.

Exercise 2

Continuing from the previous exercise, suppose now that $y_2 = y_3 = 1$ and $y_1 = y_2 = y_5 = 0$:



Consider the same logistic regression model above with coefficients β_0, β_1 and β_2, how would you need to set these coefficients to correctly classify this dataset?

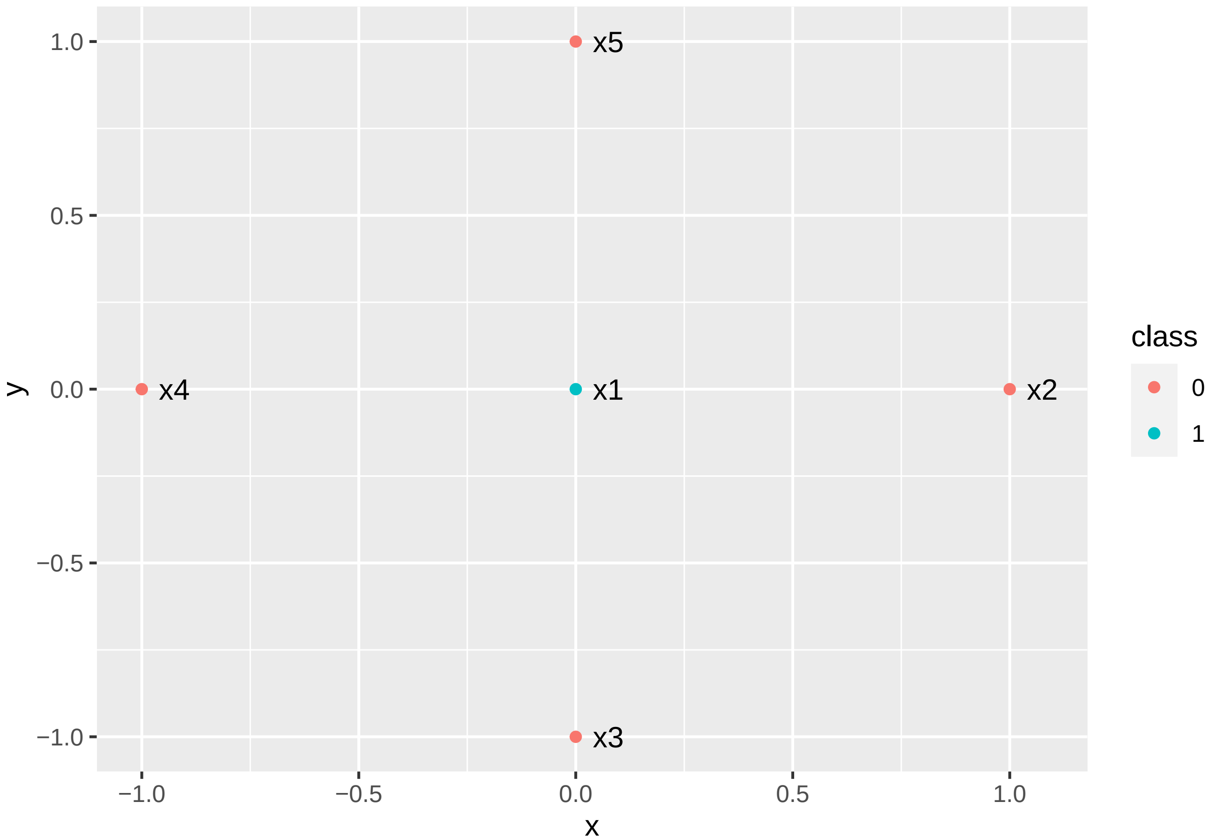
```
b0 = (
  -5
)

b1 = (
  10
)

b2 = (
  -10
)

sigmoid(xs %*% c(b0, b1, b2))
```

```
##              [,1]
## [1,] 6.692851e-03
## [2,] 9.933071e-01
## [3,] 9.933071e-01
## [4,] 3.059022e-07
## [5,] 3.059022e-07
```

Make sure that the second and third elements are close to one, and the others close to zero.

Exercise 3

Finally, with the same data as before, suppose that $y_1 = 1$ and $y_2 = y_3 = y_4 = y_5 = 0$:



Clearly, logistic regression cannot correctly classify this dataset, since the two classes are not linearly separable (optional: prove it, see solution at the bottom).

However, as we have shown in the previous exercises, it is possible to separate x_2 and x_3 from the rest, and x_4 and x_5 from the rest.

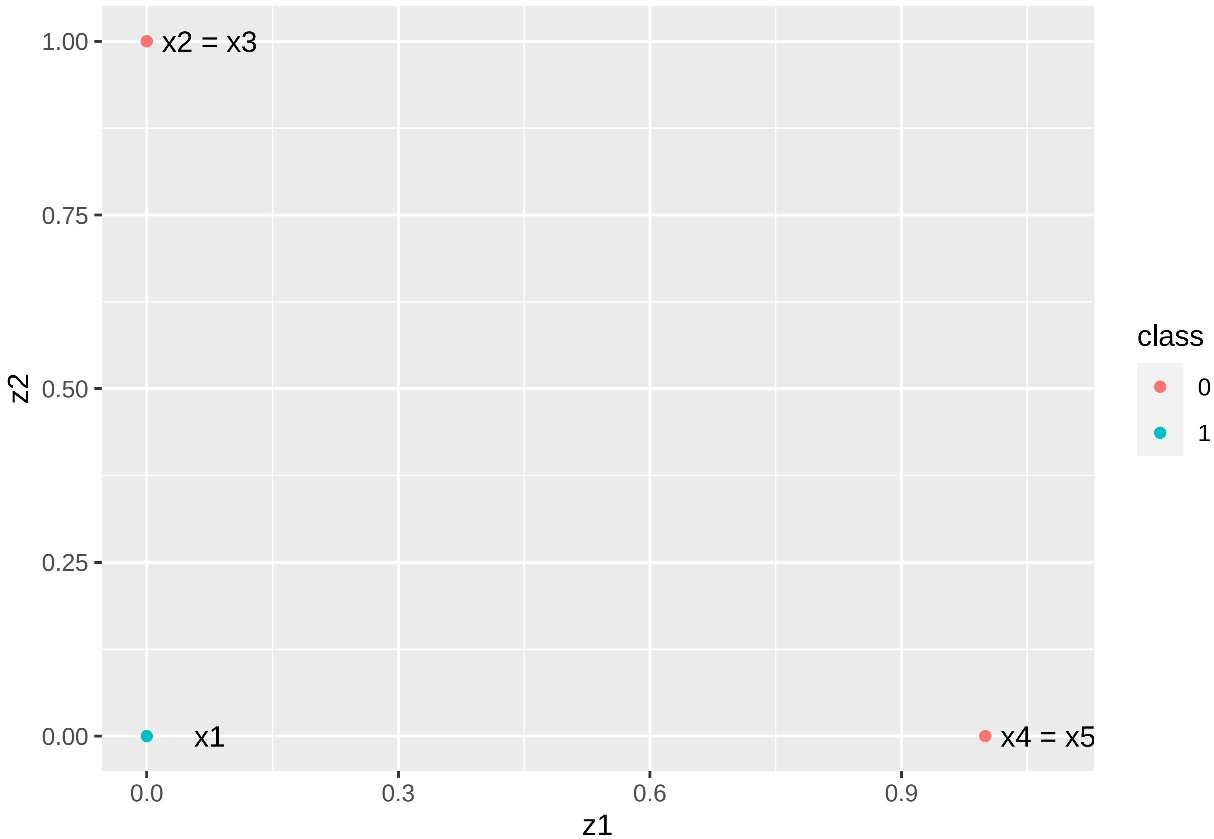
Can these two simple classifiers be composed into one that is powerful enough to separate x_1 from the rest?

Can we use their predictions as input for another logistic regression classifier?

Let $z_{i1} = \sigma(\alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2})$ and $z_{i2} = \sigma(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2})$ be the output of the two logistic regression classifiers for point i. Then, the dataset would become:

i	z_{i1}	z_{i2}	y
1	0	0	1
2	0	1	0
3	0	1	0
4	1	0	0
5	1	0	0

In graphical form:



This sure looks linearly separable! As before, find the coefficients for a linear classifier $\hat{y}_i = \sigma(\gamma_0 + \gamma_1 z_{i1} + \gamma_2 z_{i2})$:

```
g0 = (
  5
)

g1 = (
  -10
)

g2 = (
  -10
)

zs = matrix(c(
  1, 0, 0,
  1, 0, 1,
  1, 0, 1,
  1, 1, 0,
  1, 1, 0
), ncol = 3, byrow = T)

sigmoid(zs %*% c(g0, g1, g2))
```

```
##             [,1]
## [1,] 0.993307149
## [2,] 0.006692851
```

```
## [3,] 0.006692851
## [4,] 0.006692851
## [5,] 0.006692851
```

Make sure that the first element is close to one, and the others close to zero.

This big classifier can be summarized as follows:

```
z1 = sigmoid(xs %*% c(a0, a1, a2))
z2 = sigmoid(xs %*% c(b0, b1, b2))

yhat = sigmoid(g0 + g1 * z1 + g2 * z2)
yhat
```

```
##             [,1]
## [1,] 0.992355864
## [2,] 0.007152788
## [3,] 0.007152788
## [4,] 0.007152788
## [5,] 0.007152788
```

And this is just what a neural network looks like! Each neuron is a simple linear classifier, and we just stack linear classifiers on top of linear classifiers. And we could go on and on, with many layers of linear classifiers.

Proof sketch that this dataset is not linearly separable

The *convex hull* of a set of points is the smallest convex polygon that encloses them. In our case, the convex hull of $x_2, \ldots, x_5$ is a square with those points as vertices.