Lab 01

Hüseyin Anil Gündüz

Welcome to the very first lab, in which we will have fun with logistic regression.

Imports

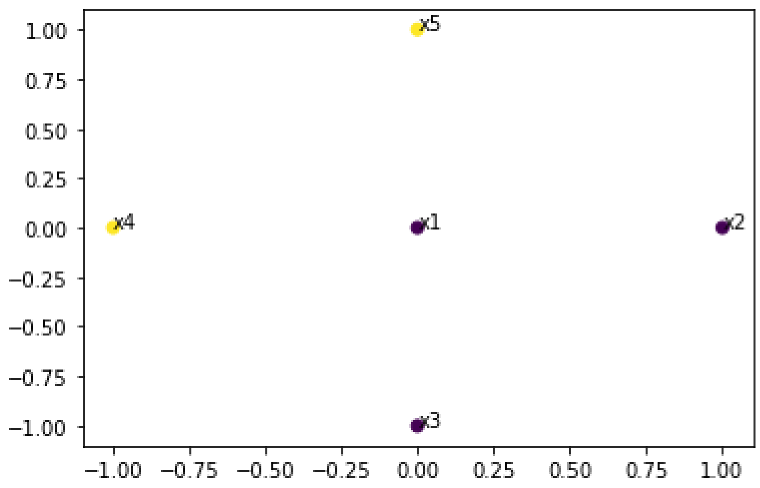
In [1]:
```
import torch
from torch import Tensor

import matplotlib.pyplot as plt
from  matplotlib_inline.backend_inline import set_matplotlib_formats
set_matplotlib_formats('png', 'pdf')
```

Exercise 1

Suppose you have five input points, $\mathbf{x}_1 = |0, 0|^T$, $\mathbf{x}_2 = |1, 0|^T$, $\mathbf{x}_3 = |0, -1|^T$, $\mathbf{x}_4 = |-1, 0|^T$ and $\mathbf{x}_5 = |0, 1|^T$, and the corresponding classes are $y_1 = y_2 = y_3 = 0$ and $y_4 = y_5 = 1$:

In [2]:
```
x = torch.tensor([
    [0, 0],
    [1, 0],
    [0, -1],
    [-1, 0],
    [0, 1]
])
y = torch.tensor([0, 0, 0, 1, 1])
labs = ['x1', 'x2', 'x3', 'x4', 'x5']
```

In [3]:
```
plt.scatter(x[:, 0], x[:, 1], c=y)
for i, lab in enumerate(labs):
    plt.annotate(lab, (x[i, 0], x[i, 1]))
plt.show()
```



Consider a logistic regression model $\hat{y}_i = \sigma(\alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2})$, with $\sigma(\cdot)$ the sigmoid function, $\sigma(x) = (1 + e^{-x})^{-1}$. What values for α_0, α_1 and α_2 would result in the correct classification for this dataset? A positive label is predicted when the output of the sigmoid is larger or equal than 0.5.

Note: do not use any formulas or automated methods to find the answer. Think for yourself. A logistic regression classifier is nothing more than a hyper-plane separating points of the two classes. If necessary, review vectors, dot-products and their geometrical interpretation in linear algebra. This applies to the following exercises, too.

We add a first column of ones, which is used for the 'bias'.

In [4]:
```
x = torch.cat([torch.ones(5, 1), x], dim=1)
```

In [5]:
```
a0 = (
  -5
)

a1 = (
  -10
)

a2 = (
  10
)

a = torch.tensor([a0, a1, a2], dtype=torch.float)

# We define a custom sigmoid function
def sigmoid(x: Tensor) -> Tensor:
    return 1 / (1 + torch.exp(-x))

# Calculate predictions
scores = sigmoid(x @ a)

# Let's investigate the obtained scores.
def print_scores(target: Tensor, scores: Tensor) -> None:
    [print('{}\t{}\t{:.2e}'.format('x' + str(i), int(t), float(s)))
     for i, (t, s) in enumerate(zip(target, scores), start=1)]

print_scores(y, scores)
```

```
x1      0       6.69e-03
x2      0       3.06e-07
x3      0       3.06e-07
x4      1       9.93e-01
x5      1       9.93e-01
```

You should make sure that the last two values are close to one and the others are close to zero.

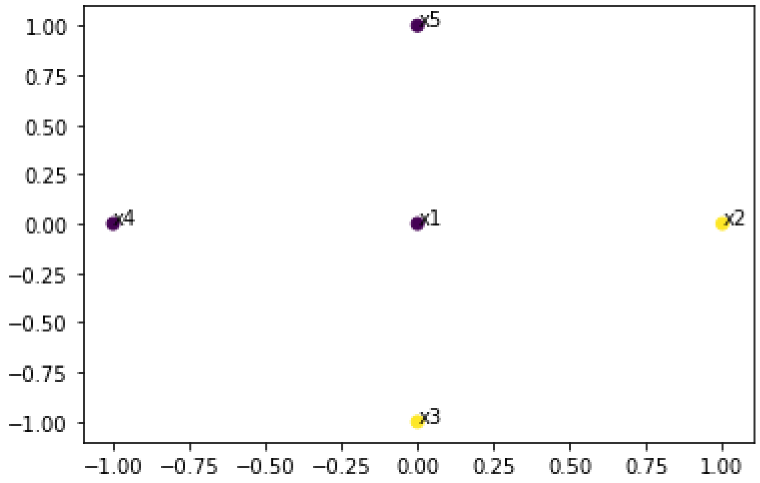
Note: There are many valid parametrization that lead to a separating hyperplane. How would you prioritize between them?

Exercise 2

Continuing from the previous exercise, suppose now that $y_2 = y_3 = 1$ and $y_1 = y_2 = y_5 = 0$.

In [6]:
```
y = torch.tensor([0, 1, 1, 0, 0])
```

In [7]:
```
plt.scatter(x[:, 1], x[:, 2], c=y)
for i, lab in enumerate(labs):
    plt.annotate(lab, (x[i, 1], x[i, 2]))
plt.show()
```



Consider the same logistic regression model above with coefficients β_0, β_1 and β_2, how would you need to set these coefficients to correctly classify this dataset?

In [8]:
```
b0 = (
  -5
)

b1 = (
  10
)

b2 = (
  -10
)

b = torch.tensor([b0, b1, b2], dtype=torch.float)

print_scores(y, sigmoid(x @ b))
```

```
x1      0       6.69e-03
x2      1       9.93e-01
x3      1       9.93e-01
x4      0       3.06e-07
x5      0       3.06e-07
```

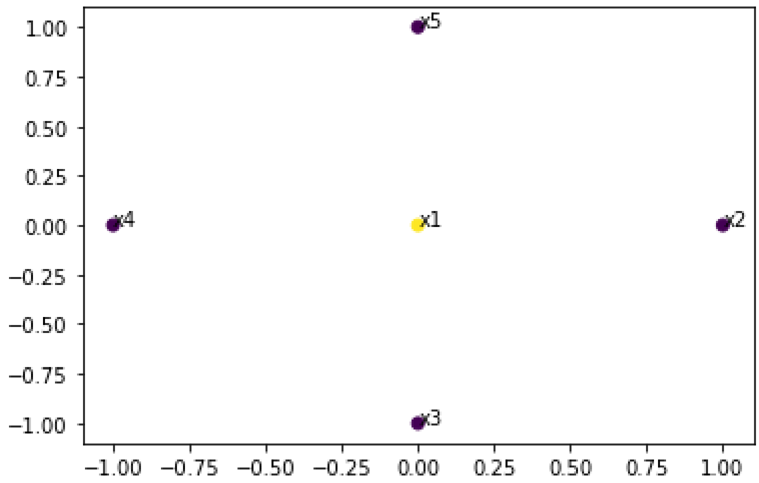
Make sure that the second and third elements are close to one, and the others close to zero.

Exercise 3

Finally, with the same data as before, suppose that $y_1 = 1$ and $y_2 = y_3 = y_4 = y_5 = 0$:

In [9]:
```
y = torch.tensor([1, 0, 0, 0, 0])
```

In [10]:
```
plt.scatter(x[:, 1], x[:, 2], c=y, label=y)
for i, lab in enumerate(labs):
    plt.annotate(lab, (x[i, 1], x[i, 2]))
plt.show()
```



Clearly, logistic regression cannot correctly classify this dataset, since the two classes are not linearly separable (optional: prove it, see solution at the bottom).

However, as we have shown in the previous exercises, it is possible to separate x_2 and x_3 from the rest, and x_4 and x_5 from the rest.

Can these two simple classifiers be composed into one that is powerful enough to separate x_1 from the rest?

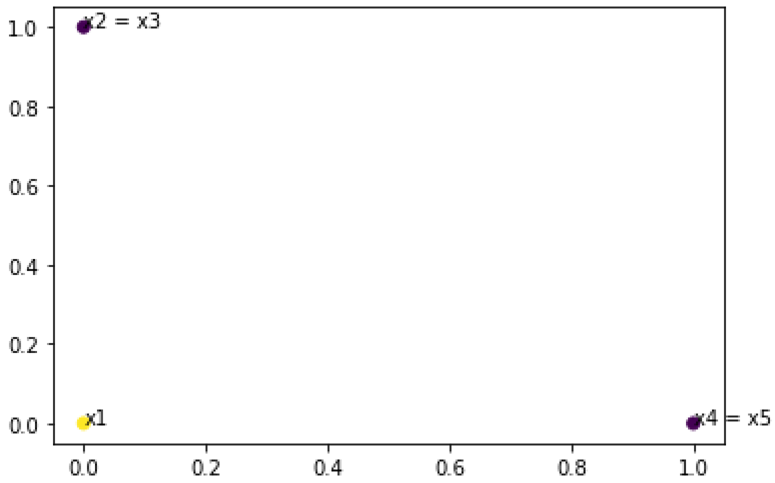
Can we use their predictions as input for another logistic regression classifier?

Let $z_{i1} = \sigma(\alpha_0 + \alpha_1 x_{i1} + \alpha_2 x_{i2})$ and $z_{i2} = \sigma(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2})$ be the output of the two logistic regression classifiers for point i. Then, the dataset would become:

i	z_{i1}	z_{i2}	y
1	0	0	1
2	0	1	0
3	0	1	0
4	1	0	0
5	1	0	0

In graphical form:

In [11]:
```
x_axis = [0, 0, 1]
y_axis = [0, 1, 0]
plt.scatter(x_axis, y_axis, c=[1, 0, 0])
for i, lab in enumerate(['x1', 'x2 = x3', 'x4 = x5']):
    plt.annotate(lab, (x_axis[i], y_axis[i]))
plt.show()
```



This sure looks linearly separable! As before, find the coefficients for a linear classifier $\hat{y}_i = \sigma\left(\gamma_0 + \gamma_1 z_{i1} + \gamma_2 z_{i2}\right)$:

In [12]:
```
g0 = (
  5
)

g1 = (
  -10
)

g2 = (
  -10
)

g = torch.tensor([g0, g1, g2], dtype=torch.float)

z = torch.tensor([
  [1, 0, 0,],
  [1, 0, 1,],
  [1, 0, 1,],
  [1, 1, 0,],
  [1, 1, 0]
], dtype=torch.float)

print_scores(y, sigmoid(z @ g))
```

```
x1      1       9.93e-01
x2      0       6.69e-03
x3      0       6.69e-03
x4      0       6.69e-03
x5      0       6.69e-03
```

Make sure that the first element is close to one, and the others close to zero.

This big classifier can be summarized as follows:

In [13]:
```
z1 = sigmoid(x @ a)
z2 = sigmoid(x @ b)

print_scores(y, sigmoid(g0 + g1 * z1 + g2 * z2))
```

```
x1      1       9.92e-01
x2      0       7.15e-03
x3      0       7.15e-03
x4      0       7.15e-03
x5      0       7.15e-03
```

And this is just what a neural network looks like! Each neuron is a simple linear classifier, and we just stack linear classifiers on top of linear classifiers. And we could go on and on, with many layers of linear classifiers.