Introduction to Deep Learning

Chapter 2: MLP – Matrix Notation

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LECTURE OUTLINE

This chapter introduces a compact way for representing feedforward neural networks: the matrix formalism from linear algebra.

1. First, we explore networks with one hidden layer and one output unit.
2. Next, we investigate networks with one hidden layer but multiple output units.
3. Finally, we focus on multi-layer feedforward networks with an arbitrary number of hidden layers and output units.

Single Hidden Layer Networks for Regression and Binary Classification

SINGLE HIDDEN LAYER NETWORKS

- The input $\mathbf{x}$ is a column vector with dimensions $p \times 1$.
- $\mathbf{W}$ is a weight matrix with dimensions $p \times m$:

$$\mathbf{W} = \begin{pmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,m} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{p,1} & w_{p,2} & \cdots & w_{p,m} \end{pmatrix}$$

- For example, to obtain z_1, we pick the first column of W:

$$\mathbf{W}_1 = \begin{pmatrix} w_{1,1} \\ w_{2,1} \\ \vdots \\ w_{p,1} \end{pmatrix}$$

and compute $z_1 = \sigma(W_1^T \mathbf{x} + b_1)$, where b_1 is the bias of the first hidden neuron and $\sigma : \mathbb{R} \to \mathbb{R}$ is an activation function.

SINGLE HIDDEN LAYER NETWORKS: NOTATION

General notation:

- The network has m hidden neurons $z_1, \dots, z_m$ with

$$z_j = \sigma(\mathbf{W}_j^T \mathbf{x} + b_j)$$

  - $z_{in,j} = \mathbf{W}_j^T \mathbf{x} + b_j$
  - $z_{out,j} = \sigma(z_{in,j}) = \sigma(\mathbf{W}_j^T \mathbf{x} + b_j)$

  for $j \in \{1, \dots, m\}$.

SINGLE HIDDEN LAYER NETWORKS: NOTATION

- Vectorized notation:
  - $\mathbf{z}_{in} = (z_{in,1}, \ldots, z_{in,m})^T = \mathbf{W}^T\mathbf{x} + \mathbf{b}$
    (Note: $\mathbf{W}^T\mathbf{x} = (\mathbf{x}^T\mathbf{W})^T$)
  - $\mathbf{z} = \mathbf{z}_{out} = \sigma(\mathbf{z}_{in}) = \sigma(\mathbf{W}^T\mathbf{x} + \mathbf{b})$, where the (hidden layer) activation function σ is applied element-wise to $\mathbf{z}_{in}$.
- Bias term:
  - We sometimes omit the bias term by adding a constant feature to the input $\tilde{\mathbf{x}} = (1, x_1, \ldots, x_p)$ and by adding the bias term to the weight matrix

$$\tilde{\mathbf{W}} = (\mathbf{b}, \mathbf{W}_1, \ldots, \mathbf{W}_p).$$

  - **Note**: For simplification purposes, we will not explicitly represent the bias term graphically in the following. However, the above “trick” makes it straightforward to represent it graphically.

SINGLE HIDDEN LAYER NETWORKS: NOTATION

General notation:

- For regression or binary classification: one output unit f where
  - $f_{in} = \mathbf{u}^T \mathbf{z} + c$, i.e. a linear combination of derived features plus the bias term c of the output neuron, and
  - $f(\mathbf{x}) = f_{out} = \tau(f_{in}) = \tau(\mathbf{u}^T \mathbf{z} + c)$, where τ is the output activation function.
- For regression τ is the identity function.
- For binary classification, τ is a sigmoid function.
- **Note**: The purpose of the hidden-layer activation function σ is to introduce non-linearities so that the network is able to learn complex functions whereas the purpose of τ is merely to get the final score to the same range as the target.

SINGLE HIDDEN LAYER NETWORKS: NOTATION

General notation: Multiple inputs

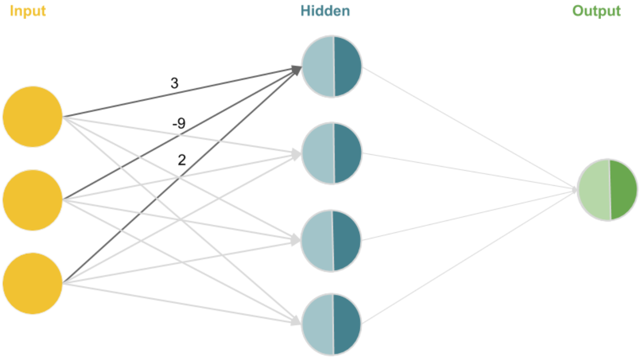
- It is possible to feed multiple inputs to a neural network simultaneously.
- The inputs $\mathbf{x}^{(i)}$, for $i \in \{1, \ldots, n\}$, are arranged as rows in the **design matrix X**.
  - $\mathbf{X}$ is a $(n \times p)$-matrix.
- The weighted sum in the hidden layer is now computed as $\mathbf{XW} + \boldsymbol{B}$, where,
  - $\mathbf{W}$, as usual, is a $(p \times m)$ matrix, and,
  - $\boldsymbol{B}$ is a $(n \times m)$ matrix containing the bias vector $\mathbf{b}$ (duplicated) as the rows of the matrix.
- The *matrix* of hidden activations $\boldsymbol{Z} = \sigma(\mathbf{XW} + \boldsymbol{B})$
  - $\boldsymbol{Z}$ is a $(n \times m)$ matrix.

SINGLE HIDDEN LAYER NETWORKS: NOTATION

- The final output of the network, which contains a prediction for each input, is $\tau(\boldsymbol{Z}\mathbf{u} + \boldsymbol{C})$, where
  - $\mathbf{u}$ is the vector of weights of the output neuron, and,
  - $\boldsymbol{C}$ is a $(n \times 1)$ matrix whose elements are the (scalar) bias c of the output neuron.

SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

- Weights (and biases) of the network.



$$\underbrace{\begin{pmatrix} 3 & -9 & 2 \\ & & \\ & & \end{pmatrix}}_{W^T} \underbrace{\begin{pmatrix} 5 \\ \\ \end{pmatrix}}_{\mathbf{b}} \qquad \underbrace{\begin{pmatrix} \; \end{pmatrix}}_{\mathbf{u}^T} \underbrace{\begin{pmatrix} \; \end{pmatrix}}_{c}$$

SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

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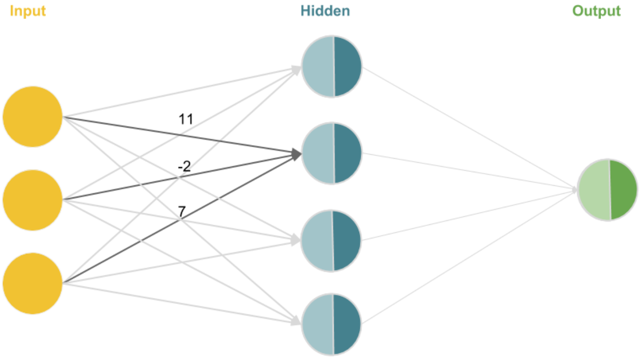

$$\underbrace{\begin{pmatrix} 3 & -9 & 2 \\ \mathbf{11} & \mathbf{-2} & \mathbf{7} \\ & & \\ & & \end{pmatrix}}_{W^T} \underbrace{\begin{bmatrix} 5 \\ \mathbf{2} \\ \\ \\ \end{bmatrix}}_{\mathbf{b}} \qquad \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}}_{\mathbf{u}^T} \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{c}$$

SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

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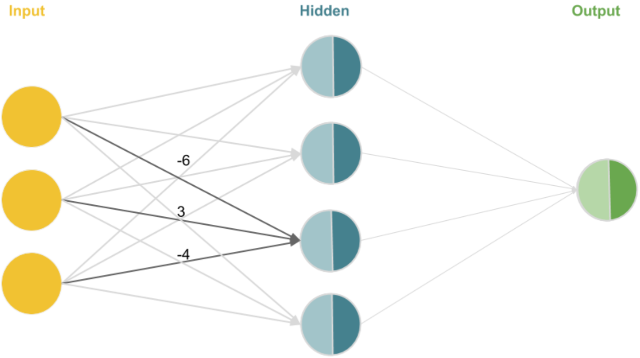

$$\underbrace{\begin{pmatrix} 3 & -9 & 2 \\ 11 & -2 & 7 \\ \mathbf{-6} & \mathbf{3} & \mathbf{-4} \\ & & \end{pmatrix}}_{W^T} \underbrace{\begin{pmatrix} 5 \\ 2 \\ \mathbf{-1} \\ \ \end{pmatrix}}_{\mathbf{b}} \qquad \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{\mathbf{u}^T} \underbrace{\begin{bmatrix} \ \end{bmatrix}}_{c}$$

SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

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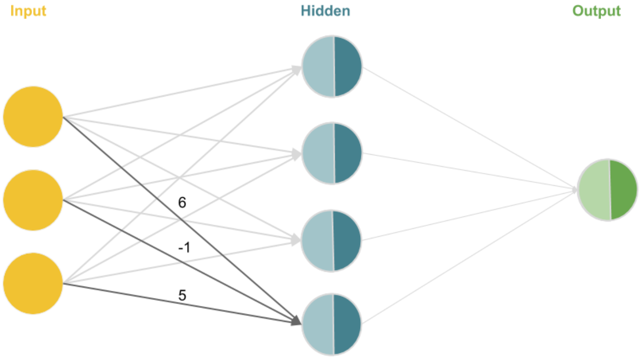
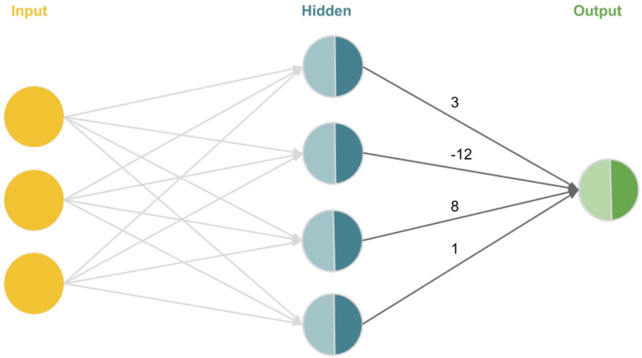


$$\begin{pmatrix} 3 & -9 & 2 \\ 11 & -2 & 7 \\ -6 & 3 & -4 \\ \mathbf{6} & \mathbf{-1} & \mathbf{5} \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ -1 \\ \mathbf{1} \end{pmatrix} \qquad \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \ \end{bmatrix}$$

W^T $\mathbf{b}$ $\mathbf{u}^T$ c

SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

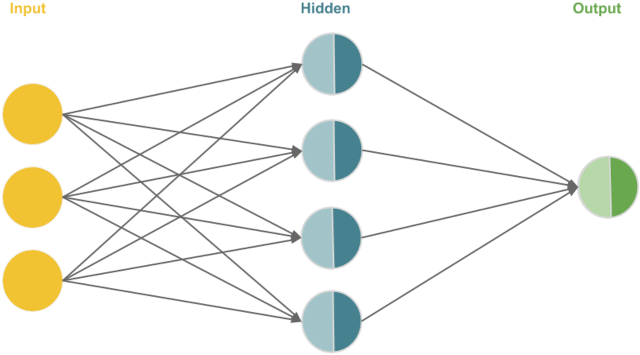
- Weights (and biases) of the network.



$$\underbrace{\begin{pmatrix} 3 & -9 & 2 \\ 11 & -2 & 7 \\ -6 & 3 & -4 \\ 6 & -1 & 5 \end{pmatrix}}_{W^T} \underbrace{\begin{pmatrix} 5 \\ 2 \\ -1 \\ 1 \end{pmatrix}}_{\mathbf{b}} \qquad \underbrace{\begin{pmatrix} 3 & -12 & 8 & 1 \end{pmatrix}}_{\mathbf{u}^T} \underbrace{\begin{pmatrix} 6 \end{pmatrix}}_{c}$$

SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

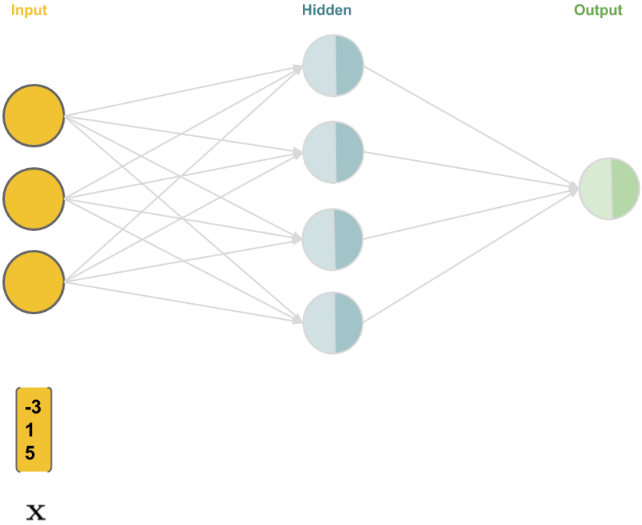
- Weights (and biases) of the network.



$$\underbrace{\begin{pmatrix} 3 & -9 & 2 \\ 11 & -2 & 7 \\ -6 & 3 & -4 \\ 6 & -1 & 5 \end{pmatrix}}_{W^T} \underbrace{\begin{pmatrix} 5 \\ 2 \\ -1 \\ 1 \end{pmatrix}}_{\mathbf{b}} \qquad \underbrace{\begin{bmatrix} 3 & -12 & 8 & 1 \end{bmatrix}}_{\mathbf{u}^T} \underbrace{\begin{bmatrix} 6 \end{bmatrix}}_{c}$$

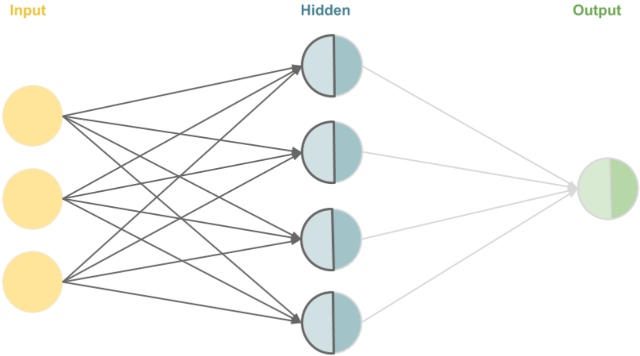
SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

Forward pass through the shallow neural network.



SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

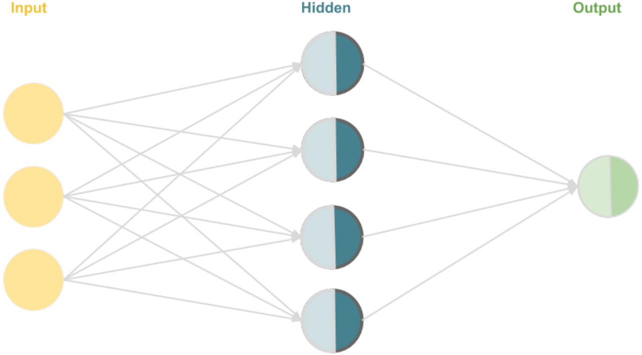
Forward pass through the shallow neural network.



$$\underbrace{\begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}}_{\mathbf{x}} \qquad \underbrace{\begin{pmatrix} (-3)*3 + 1*(-9) + 5*2 + 5 \\ (-3)*11 + 1*(-2) + 5*7 + 2 \\ (-3)*(-6) + 1*3 + 5*(-4) + (-1) \\ (-3)*6 + 1*(-1) + 5*5 + 1 \end{pmatrix}}_{\mathbf{z}_{in} = W^\top \mathbf{x} + \mathbf{b}}$$

SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

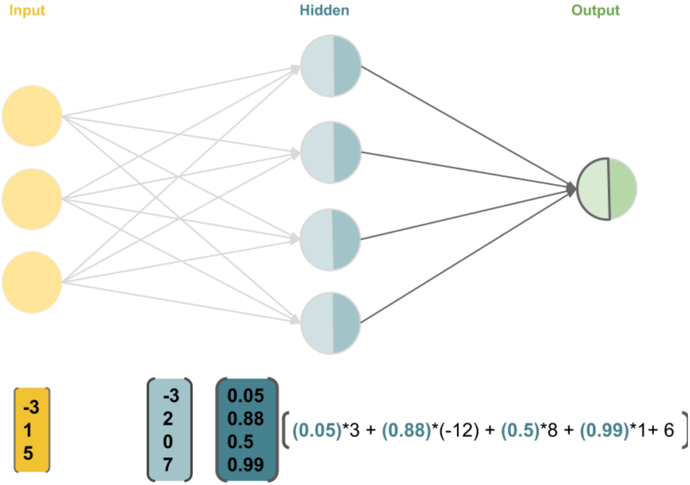
Forward pass through the shallow neural network.



$$\mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \quad \mathbf{z}_{in} = \begin{pmatrix} -3 \\ 2 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{z} = \mathbf{z}_{out} = \sigma(\mathbf{z}_{in}) = \begin{pmatrix} 1/(1+\exp(-(-3))) \\ 1/(1+\exp(-2)) \\ 1/(1+\exp(-0)) \\ 1/(1+\exp(-7)) \end{pmatrix}$$

SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

Forward pass through the shallow neural network.

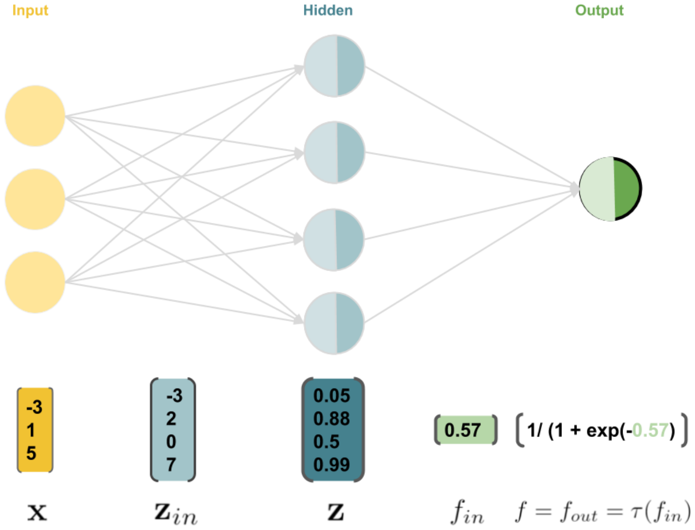


$\mathbf{x}$ $\quad$ $\mathbf{z}_{in}$ $\quad$ $\mathbf{z}$

$$f_{in} = \mathbf{u}^T\mathbf{z} + c$$

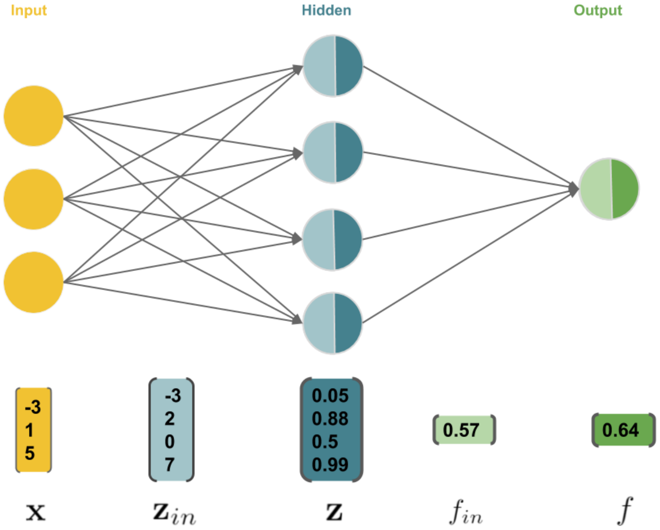
SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

Forward pass through the shallow neural network.



SINGLE HIDDEN LAYER NETWORKS: EXAMPLE

Forward pass through the shallow neural network.



HIDDEN LAYER: ACTIVATION FUNCTION

- It is important to note that if the hidden layer does not have a non-linear activation, the network can only learn linear decision boundaries.
- For simplification purposes, we drop the bias terms in notation and let $\sigma = \text{id}$. Then:

$$
\begin{aligned}
f(\mathbf{x}) &= \tau(\mathbf{u}^T \mathbf{z}) \\
&= \tau(\mathbf{u}^T \sigma(\mathbf{W}^T \mathbf{x})) \\
&= \tau(\mathbf{u}^T \mathbf{W}^T \mathbf{x}) = \tau(\mathbf{v}^T \mathbf{x})
\end{aligned}
$$

  where $\mathbf{v} = \mathbf{W}\mathbf{u}$. It can be seen that $f(\mathbf{x})$ can only yield a linear decision boundary.

Single Hidden Layer Networks for Multi-Class Classification

MULTI-CLASS CLASSIFICATION

- We have only considered regression and binary classification problems so far.
- How can we get a neural network to perform multiclass classification?

MULTI-CLASS CLASSIFICATION

- The first step is to add additional neurons to the output layer.
- Each neuron in the layer will represent a specific class (number of neurons in the output layer = number of classes).

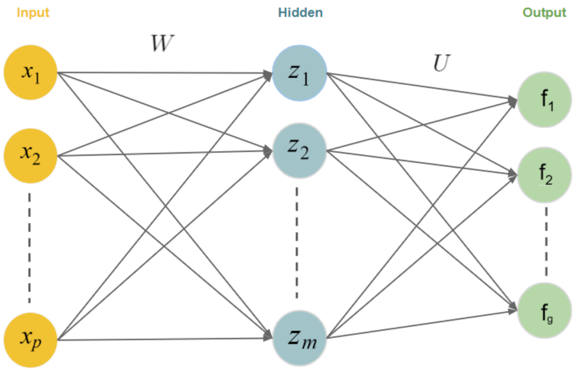


Figure: Structure of a single hidden layer, feed-forward neural network for g-class classification problems (bias term omitted).

MULTI-CLASS CLASSIFICATION

Notation:

- For g-class classification, g output units:

$$\mathbf{f} = (f_1, \dots, f_g)$$

- m hidden neurons $z_1, \dots, z_m$, with

$$z_j = \sigma(\mathbf{W}_j^T \mathbf{x}), \quad j = 1, \dots, m.$$

- Compute linear combinations of derived features z:

$$f_{in,k} = \boldsymbol{U}_k^T \mathbf{z}, \quad \mathbf{z} = (z_1, \dots, z_m)^T, \quad k = 1, \dots, g$$

MULTI-CLASS CLASSIFICATION

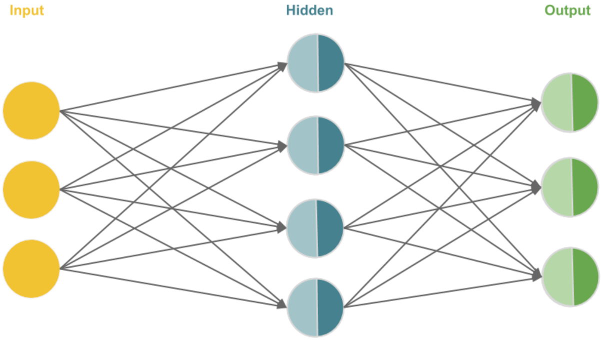
- The second step is to apply a softmax activation function to the output layer.
- This gives us a probability distribution over g different possible classes:

$$f_{out,k} = \tau_k(f_{in,k}) = \frac{\exp(f_{in,k})}{\sum_{k'=1}^{g} \exp(f_{in,k'})}$$

- This is the same transformation used in softmax regression!
- Derivative $\frac{\partial \tau(\mathbf{f}_{in})}{\partial \mathbf{f}_{in}} = \text{diag}(\tau(\mathbf{f}_{in})) - \tau(\mathbf{f}_{in})\tau(\mathbf{f}_{in})^T$
- It is a “smooth” approximation of the argmax operation, so $\tau((1, 1000, 2)^T) \approx (0, 1, 0)^T$ (picks out 2nd element!).

MULTI-CLASS CLASSIFICATION: EXAMPLE

Forward pass (Hidden: Sigmoid, Output: Softmax).

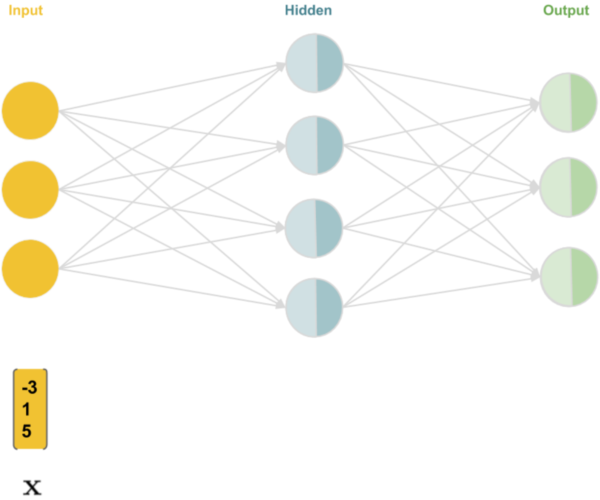


$$\begin{pmatrix} 3 & -9 & 2 \\ 11 & -2 & 7 \\ -6 & 3 & -4 \\ 6 & -1 & 5 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 3 & -12 & 8 & 1 \\ 2 & -3 & 9 & 1 \\ -5 & 1 & -1 & 7 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ -8 \end{pmatrix}$$

$$W^T \quad \mathbf{b} \qquad\qquad U^T \quad \mathbf{c}$$

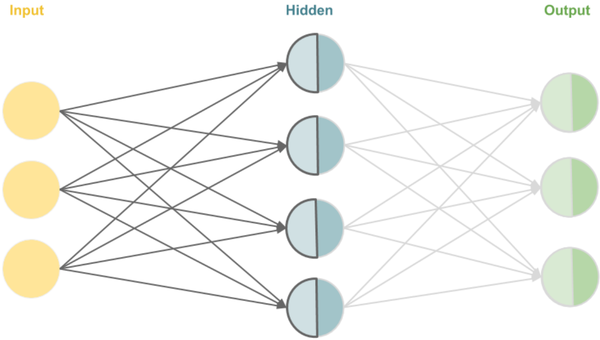
MULTI-CLASS CLASSIFICATION: EXAMPLE

Forward pass (Hidden: Sigmoid, Output: Softmax).



MULTI-CLASS CLASSIFICATION: EXAMPLE

Forward pass (Hidden: Sigmoid, Output: Softmax).

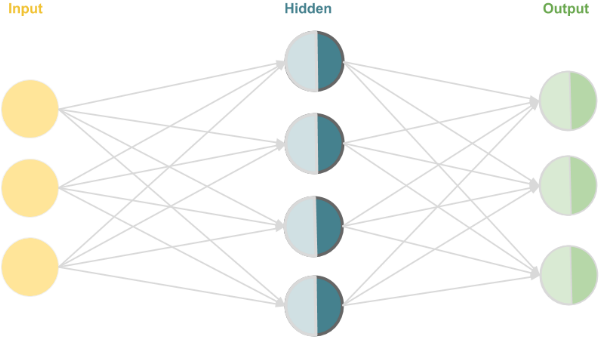


$$\begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} \qquad \begin{pmatrix} (-3)*3 + 1*(-9) + 5*2 + 5 \\ (-3)*11 + 1*(-2) + 5*7 + 2 \\ (-3)*(-6) + 1*3 + 5*(-4) + (-1) \\ (-3)*6 + 1*(-1) + 5*5 + 1 \end{pmatrix}$$

$$\mathbf{x} \qquad \mathbf{z}_{in} = \mathbf{w}^T\mathbf{x} + \mathbf{b}$$

MULTI-CLASS CLASSIFICATION: EXAMPLE

Forward pass (Hidden: Sigmoid, Output: Softmax).

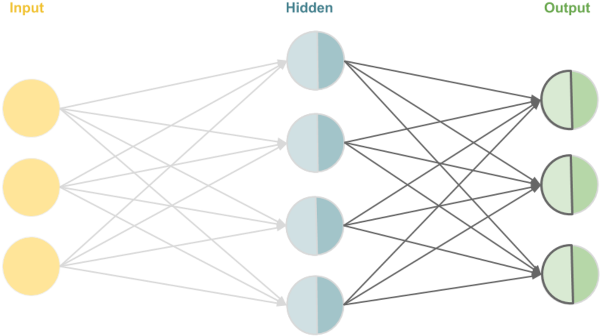


$$\begin{bmatrix}-3\\1\\5\end{bmatrix} \quad \begin{bmatrix}-3\\2\\0\\7\end{bmatrix} \quad \begin{pmatrix}1/\,(1+\exp(-(-3)))\\1/\,(1+\exp(-2))\\1/\,(1+\exp(-0))\\1/\,(1+\exp(-7))\end{pmatrix}$$

$$\mathbf{x} \qquad \mathbf{z}_{in} \qquad \mathbf{z} = \mathbf{z}_{out} = \sigma(\mathbf{z}_{in})$$

MULTI-CLASS CLASSIFICATION: EXAMPLE

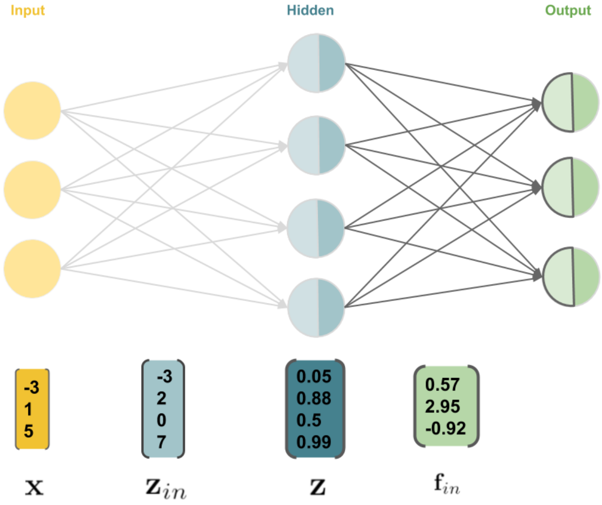
Forward pass (Hidden: Sigmoid, Output: Softmax).



$$\underbrace{\begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}}_{\mathbf{x}} \qquad \underbrace{\begin{bmatrix} -3 \\ 2 \\ 0 \\ 7 \end{bmatrix}}_{\mathbf{z}_{in}} \quad \underbrace{\begin{bmatrix} 0.05 \\ 0.88 \\ 0.5 \\ 0.99 \end{bmatrix}}_{\mathbf{z}} \quad \underbrace{\begin{bmatrix} (0.05)*3 + (0.88)*(-12) + (0.5)*8 + (0.99)*1 + 6 \\ (0.05)*2 + (0.88)*(-3) + (0.5)*9 + (0.99)*1 + 0 \\ (0.05)*(-5) + (0.88)*1 + (0.5)*(-1) + (0.99)*7 + -8 \end{bmatrix}}_{\mathbf{f}_{in} = U^T\mathbf{z} + \mathbf{c}}$$

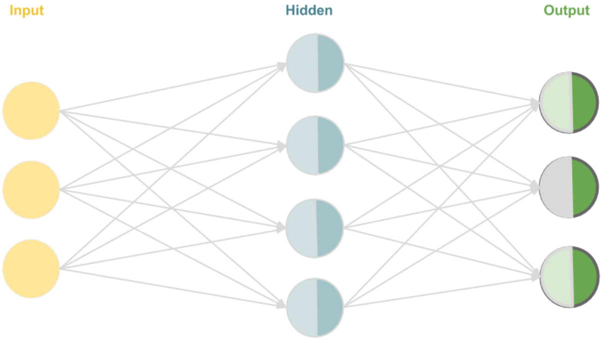
MULTI-CLASS CLASSIFICATION: EXAMPLE

Forward pass (Hidden: Sigmoid, Output: Softmax).



MULTI-CLASS CLASSIFICATION: EXAMPLE

Forward pass (Hidden: Sigmoid, Output: Softmax).

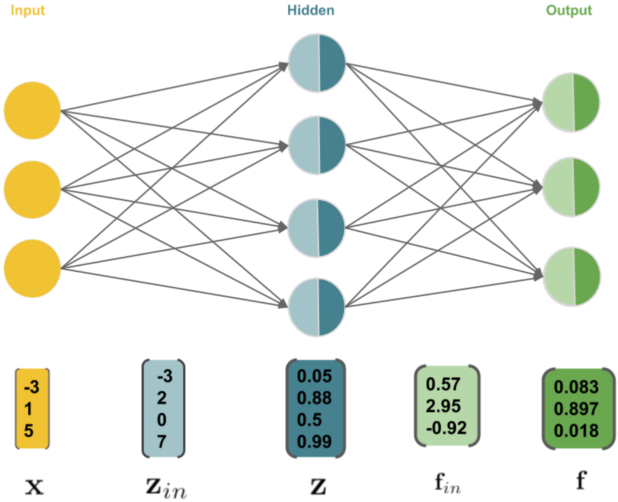


$$\mathbf{f}_{in} = \begin{pmatrix} 0.57 \\ 2.95 \\ -0.92 \end{pmatrix}$$

$$\mathbf{f} = \mathbf{f}_{out} = \tau(\mathbf{f}_{in}) = \begin{pmatrix} \frac{\exp(0.57)}{\exp(0.57)+\exp(2.95)+\exp(-0.92)} \\ \frac{\exp(2.95)}{\exp(0.57)+\exp(2.95)+\exp(-0.92)} \\ \frac{\exp(-0.92)}{\exp(0.57)+\exp(2.95)+\exp(-0.92)} \end{pmatrix}$$

MULTI-CLASS CLASSIFICATION: EXAMPLE

Forward pass (Hidden: Sigmoid, Output: Softmax).



SOFTMAX LOSS

- The loss function for a softmax classifier is

$$L(y, f(\mathbf{x})) = -\sum_{k=1}^{g} [y = k] \log \left(\frac{\exp(f_{in,k})}{\sum_{k'=1}^{g} \exp(f_{in,k'})} \right)$$

where $[y = k] = \begin{cases} 1 & \text{if } y = k \\ 0 & \text{otherwise} \end{cases}$.

- This is equivalent to the cross-entropy loss when the label vector $\boldsymbol{y}$ is one-hot coded (e.g. $\mathbf{y} = (0, 0, 1, 0)^T$).
- Optimization: Again, there is no analytic solution.

Multi-Layer Feedforward Neural Networks

FEEDFORWARD NEURAL NETWORKS

- We will now extend the model class once again, such that we allow an arbitrary amount of l (hidden) layers.
- The general term for this model class is (multi-layer) **feedforward networks** (inputs are passed through the network from left to right, no feedback-loops are allowed)

FEEDFORWARD NEURAL NETWORKS

- We can characterize those models by the following chain structure:

$$f(\mathbf{x}) = \tau \circ \phi \circ \sigma^{(l)} \circ \phi^{(l)} \circ \sigma^{(l-1)} \circ \phi^{(l-1)} \circ \ldots \circ \sigma^{(1)} \circ \phi^{(1)}$$

  where $\sigma^{(i)}$ and $\phi^{(i)}$ are the activation function and the weighted sum of hidden layer i, respectively. τ and ϕ are the corresponding components of the output layer.

- Each hidden layer has:
  - an associated weight matrix $\mathbf{W}^{(i)}$, bias $\mathbf{b}^{(i)}$ and activations $\mathbf{z}^{(i)}$ for $i \in \{1 \ldots l\}$
  - $\mathbf{z}^{(i)} = \sigma^{(i)}(\phi^{(i)}) = \sigma^{(i)}(\mathbf{W}^{(i)T}\mathbf{z}^{(i-1)} + \mathbf{b}^{(i)})$, where $\mathbf{z}^{(0)} = \mathbf{x}$.

- Again, without non-linear activations in the hidden layers, the network can only learn linear decision boundaries.

FEEDFORWARD NEURAL NETWORKS

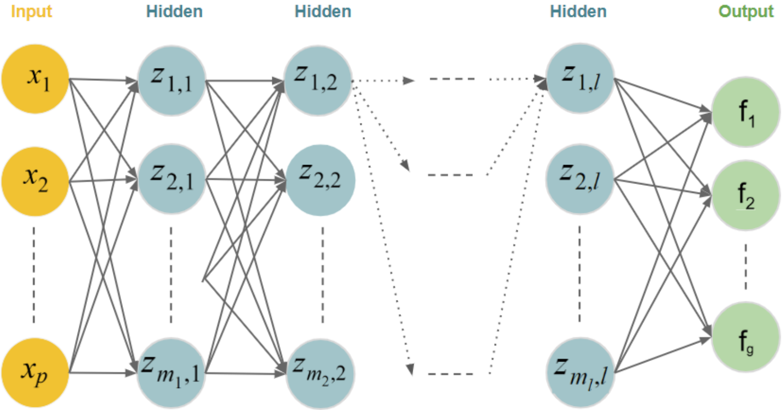
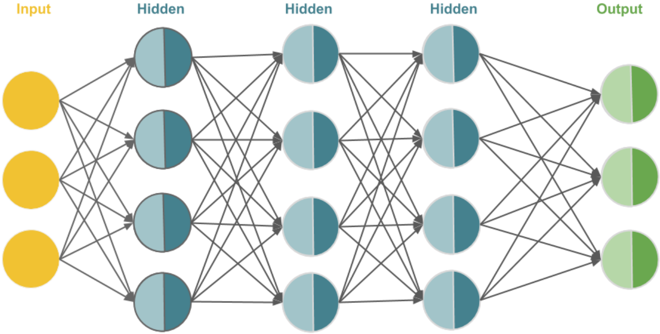


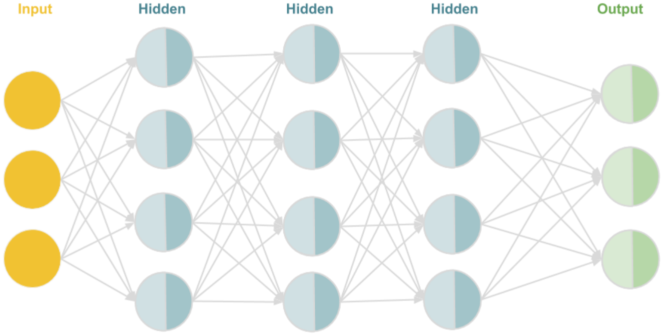
Figure: Structure of a deep neural network with l hidden layers (bias terms omitted).

FEEDFORWARD NEURAL NETWORKS: EXAMPLE



$$
\underbrace{\begin{pmatrix} 13 & -9 & 2 \\ -8 & 0 & 3 \\ 4 & -1 & 5 \\ -3 & 12 & 7 \end{pmatrix}}_{(W^{(1)})^T}
\underbrace{\begin{bmatrix} 5 \\ -2 \\ 2 \\ 11 \end{bmatrix}}_{\mathbf{b}^{(1)}}
\underbrace{\begin{pmatrix} 1 & 0 & -4 & 1 \\ 0 & 11 & 2 & -14 \\ -1 & 5 & -2 & 16 \\ 0 & -9 & -3 & 4 \end{pmatrix}}_{(W^{(2)})^T}
\underbrace{\begin{bmatrix} -5 \\ 3 \\ 1 \\ -8 \end{bmatrix}}_{\mathbf{b}^{(2)}}
\underbrace{\begin{pmatrix} 1 & -2 & -18 & -7 \\ 3 & -4 & 8 & 0 \\ -2 & 1 & 21 & 5 \\ 2 & -2 & 11 & -13 \end{pmatrix}}_{(W^{(3)})^T}
\underbrace{\begin{bmatrix} 4 \\ -6 \\ 1 \\ -17 \end{bmatrix}}_{\mathbf{b}^{(3)}}
\underbrace{\begin{pmatrix} 9 & 3 & -1 & -4 \\ -8 & -2 & 14 & 3 \\ 13 & 2 & -9 & -1 \end{pmatrix}}_{U^T}
\underbrace{\begin{bmatrix} -1 \\ -4 \\ -30 \end{bmatrix}}_{\mathbf{c}}
$$

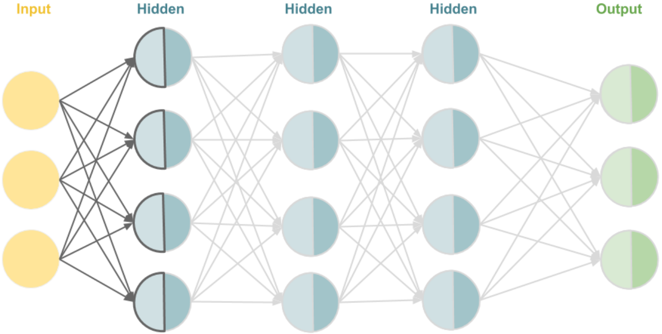
FEEDFORWARD NEURAL NETWORKS: EXAMPLE



$$\begin{bmatrix} 7 \\ 1 \\ -4 \end{bmatrix}$$

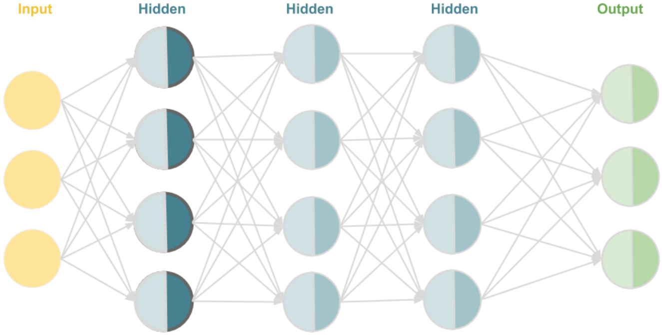
$\mathbf{x}$

FEEDFORWARD NEURAL NETWORKS: EXAMPLE



$$\underbrace{\begin{pmatrix} 7 \\ 1 \\ -4 \end{pmatrix}}_{\mathbf{x}} \quad \underbrace{\begin{pmatrix} 7*13 + 1*(-9) + (-4)*2 + 5 \\ 7*(-8) + 1*0 + (-4)*3 + (-2) \\ 7*4 + 1*(-1) + (-4)*5 + 2 \\ 7*(-3) + 1*12 + (-4)*7 + 11 \end{pmatrix}}_{\mathbf{z}_{in}^{(1)} = W^{(1)T}\mathbf{x} + \mathbf{b}^{(1)}}$$

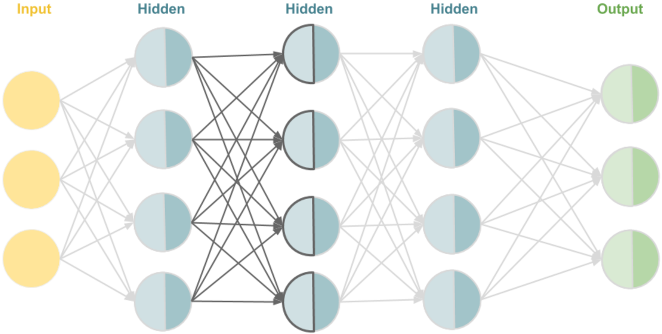
FEEDFORWARD NEURAL NETWORKS: EXAMPLE



$$\begin{bmatrix} 7 \\ 1 \\ -4 \end{bmatrix} \quad \begin{bmatrix} 79 \\ -70 \\ 9 \\ -26 \end{bmatrix} \quad \begin{pmatrix} \max(0, 79) \\ \max(0, -70) \\ \max(0, 9) \\ \max(0, -26) \end{pmatrix}$$

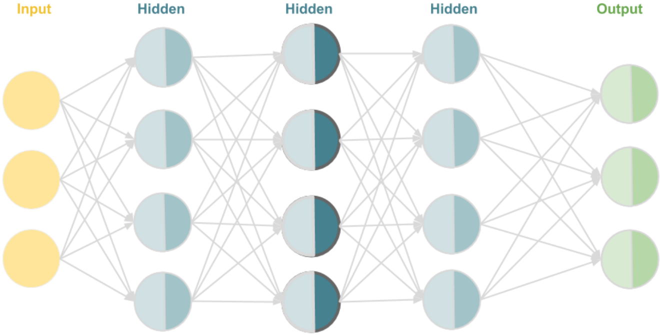
$$\mathbf{x} \quad \mathbf{z}_{in}^{(1)} \quad \mathbf{z}^{(1)} = \mathbf{z}_{out}^{(1)} = \sigma(\mathbf{z}_{in}^{(1)})$$

FEEDFORWARD NEURAL NETWORKS: EXAMPLE



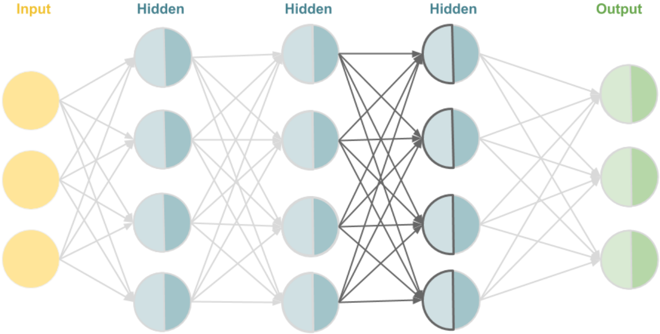
$$
\underbrace{\begin{bmatrix} 7 \\ 1 \\ -4 \end{bmatrix}}_{\mathbf{x}} \quad
\underbrace{\begin{bmatrix} 79 \\ -70 \\ 9 \\ -26 \end{bmatrix}}_{\mathbf{z}_{in}^{(1)}} \quad
\underbrace{\begin{bmatrix} 79 \\ 0 \\ 9 \\ 0 \end{bmatrix}}_{\mathbf{z}^{(1)}} \quad
\underbrace{\begin{pmatrix} 79*1 + 0*0 + 9*(-4) + 0*1 + (-5) \\ 79*0 + 0*11 + 9*2 + 0*(-14) + 3 \\ 79*(-1) + 0*5 + 9*(-2) + 0*16 + 1 \\ 79*0 + 0*(-9) + 9*(-3) + 0*4 + (-8) \end{pmatrix}}_{\mathbf{z}_{in}^{(2)} = W^{(2)T}\mathbf{z}^{(1)} + \mathbf{b}^{(2)}}
$$

FEEDFORWARD NEURAL NETWORKS: EXAMPLE



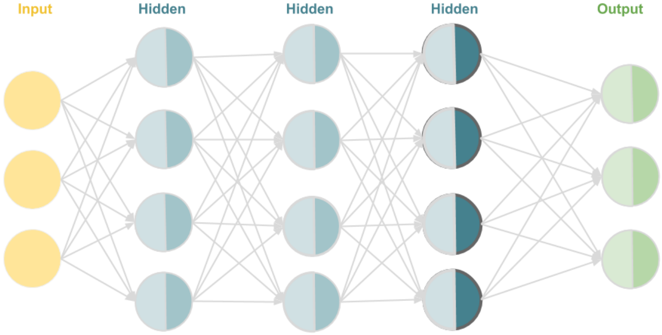
$$\underset{\mathbf{x}}{\begin{bmatrix} 7 \\ 1 \\ -4 \end{bmatrix}} \quad \underset{\mathbf{z}_{in}^{(1)}}{\begin{bmatrix} 79 \\ -70 \\ 9 \\ -26 \end{bmatrix}} \quad \underset{\mathbf{z}^{(1)}}{\begin{bmatrix} 79 \\ 0 \\ 9 \\ 0 \end{bmatrix}} \quad \underset{\mathbf{z}_{in}^{(2)}}{\begin{bmatrix} 38 \\ 21 \\ -96 \\ -35 \end{bmatrix}} \quad \underset{\mathbf{z}^{(2)} = \mathbf{z}_{out}^{(2)} = \sigma(\mathbf{z}_{in}^{(2)})}{\begin{pmatrix} \max(0, 38) \\ \max(0, 21) \\ \max(0, -96) \\ \max(0, -36) \end{pmatrix}}$$

FEEDFORWARD NEURAL NETWORKS: EXAMPLE



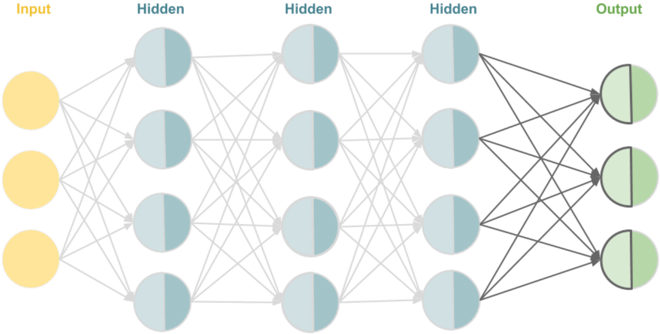
$$\underbrace{\begin{pmatrix} 7 \\ 1 \\ -4 \end{pmatrix}}_{\mathbf{x}} \quad \underbrace{\begin{pmatrix} 79 \\ -70 \\ 9 \\ -26 \end{pmatrix}}_{\mathbf{z}_{in}^{(1)}} \quad \underbrace{\begin{pmatrix} 79 \\ 0 \\ 9 \\ 0 \end{pmatrix}}_{\mathbf{z}^{(1)}} \quad \underbrace{\begin{pmatrix} 38 \\ 21 \\ -96 \\ -35 \end{pmatrix}}_{\mathbf{z}_{in}^{(2)}} \quad \underbrace{\begin{pmatrix} 38 \\ 21 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{z}^{(2)}} \quad \underbrace{\begin{pmatrix} 38*1 + 21*(-2) + 0*(-18) + 0*(-7) + 4 \\ 38*3 + 21*(-4) + 0*8 + 0*0 + (-6) \\ 38*(-2) + 21*1 + 0*21 + 0*5 + 1 \\ 38*2 + 21*(-2) + 0*11 + 0*(-13) + (-17) \end{pmatrix}}_{\mathbf{z}_{in}^{(3)} = W^{(3)T}\mathbf{z}^{(2)} + \mathbf{b}^{(3)}}$$

FEEDFORWARD NEURAL NETWORKS: EXAMPLE



$$\underset{\mathbf{x}}{\begin{pmatrix}7\\1\\-4\end{pmatrix}} \quad \underset{\mathbf{z}_{in}^{(1)}}{\begin{pmatrix}79\\-70\\9\\-26\end{pmatrix}} \quad \underset{\mathbf{z}^{(1)}}{\begin{pmatrix}79\\0\\9\\0\end{pmatrix}} \quad \underset{\mathbf{z}_{in}^{(2)}}{\begin{pmatrix}38\\21\\-96\\-35\end{pmatrix}} \quad \underset{\mathbf{z}^{(2)}}{\begin{pmatrix}38\\21\\0\\0\end{pmatrix}} \quad \underset{\mathbf{z}_{in}^{(3)}}{\begin{pmatrix}0\\24\\-54\\17\end{pmatrix}} \quad \underset{\mathbf{z}^{(3)} = \mathbf{z}_{out}^{(3)} = \sigma(\mathbf{z}_{in}^{(3)})}{\begin{pmatrix}\max(0, 0)\\\max(0, 24)\\\max(0, -54)\\\max(0, 17)\end{pmatrix}}$$

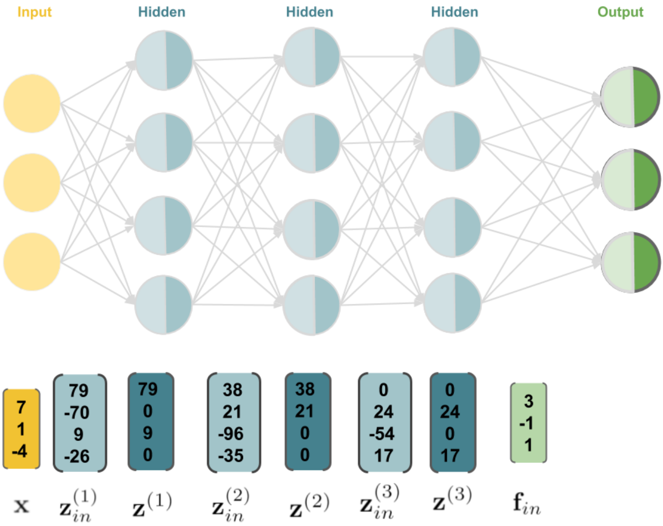
FEEDFORWARD NEURAL NETWORKS: EXAMPLE



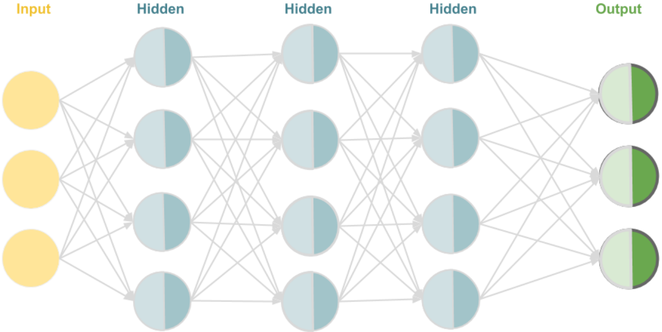
$$\mathbf{x} = \begin{bmatrix}7\\1\\-4\end{bmatrix} \quad \mathbf{z}_{in}^{(1)} = \begin{bmatrix}79\\-70\\9\\-26\end{bmatrix} \quad \mathbf{z}^{(1)} = \begin{bmatrix}79\\0\\9\\0\end{bmatrix} \quad \mathbf{z}_{in}^{(2)} = \begin{bmatrix}38\\21\\-96\\-35\end{bmatrix} \quad \mathbf{z}^{(2)} = \begin{bmatrix}38\\21\\0\\0\end{bmatrix} \quad \mathbf{z}_{in}^{(3)} = \begin{bmatrix}0\\24\\-54\\17\end{bmatrix} \quad \mathbf{z}^{(3)} = \begin{bmatrix}0\\24\\0\\17\end{bmatrix}$$

$$\mathbf{f}_{in} = U^T\mathbf{z}^{(3)} + \mathbf{c} = \begin{pmatrix}24*3 + 17*(-4) + (-1)\\ 24*(-2) + 17*3 + (-4)\\ 24*2 + 17*(-1) + (-30)\end{pmatrix}$$

FEEDFORWARD NEURAL NETWORKS: EXAMPLE



FEEDFORWARD NEURAL NETWORKS: EXAMPLE



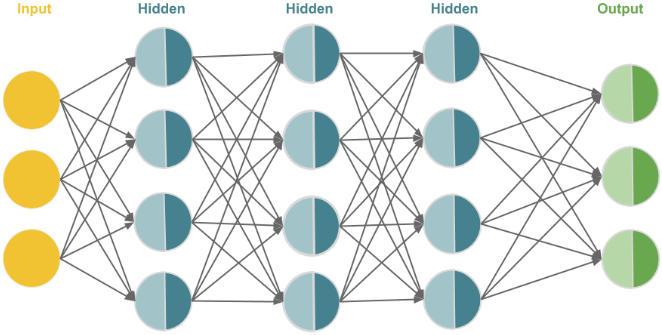
$$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$\mathbf{f}_{in}$

$$\begin{pmatrix} \frac{\exp(3)}{\exp 3 + \exp(-1) + \exp(1)} \\ \frac{\exp(-1)}{\exp 3 + \exp(-1) + \exp(1)} \\ \frac{\exp(1)}{\exp 3 + \exp(-1) + \exp(1)} \end{pmatrix}$$

$$\mathbf{f} = \mathbf{f}_{out} = \tau(\mathbf{f}_{in})$$

FEEDFORWARD NEURAL NETWORKS: EXAMPLE



$\mathbf{x}$	$\mathbf{z}_{in}^{(1)}$	$\mathbf{z}^{(1)}$	$\mathbf{z}_{in}^{(2)}$	$\mathbf{z}^{(2)}$	$\mathbf{z}_{in}^{(3)}$	$\mathbf{z}^{(3)}$	$\mathbf{f}_{in}$	$\mathbf{f}$
7	79	79	38	38	0	0	3	0.866
1	-70	0	21	21	24	24	-1	0.015
-4	9	9	-96	0	-54	0	1	0.117
	-26	0	-35	0	17	17		