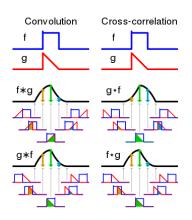
Deep Learning

Convolutions - Mathematical Perspective



Learning goals

Convolution vs.
 Cross-Correlation

CONVOLUTIONS: A DEEPER LOOK

- CNNs borrow their name from a mathematical operation termed convolution that originates in Signal Processing.
- Basic understanding of this concept and related operations improves the understanding of the CNN functionality.

CONVOLUTIONS: A DEEPER LOOK

Definition:

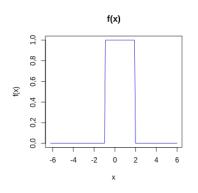
$$h(i) = (f * g)(i) = \int f(x)g(i - x)dx$$

where $f(x)$: input function

and g(x): weighting function, kernel

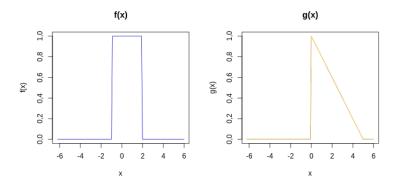
and h(i): output function, feature map elements

- Intuition 1: weighted smoothing of f(x) with weighting function g(x).
- Intuition 2: filter function g(x) filters features h(i) from input signal f(x).



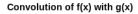
$$f(x) = \begin{cases} 1, & \text{if } x \in [-1, 2] \\ 0, & \text{otherwise} \end{cases}$$

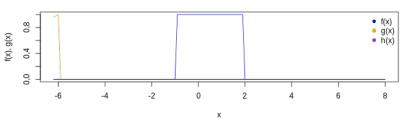
$$g(x) = \begin{cases} 1 - 0.2 * |x|, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

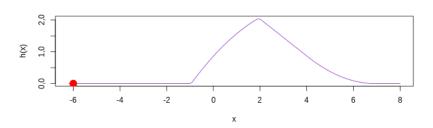


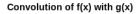
Kernel is flipped due to the negative iterator in

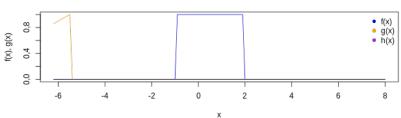
$$h(i) = \int_{x = -\infty}^{\infty} f(x)g(i - x)$$

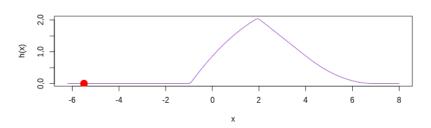


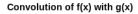


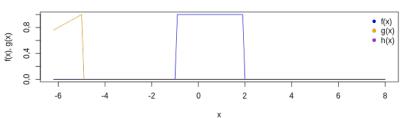


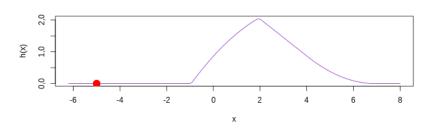


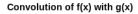


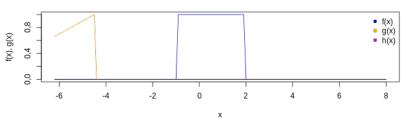


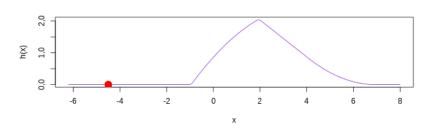


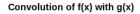


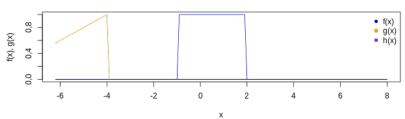


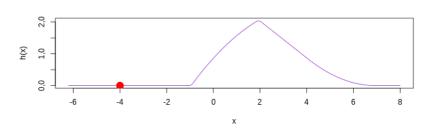


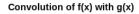


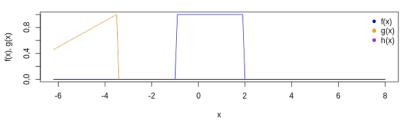


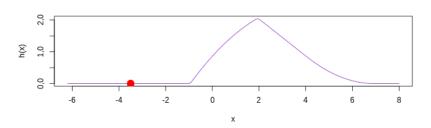


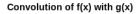


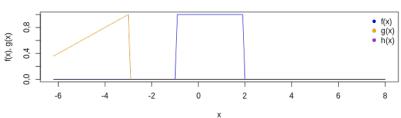


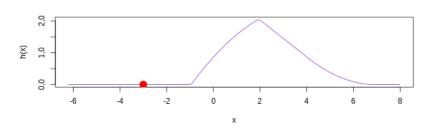


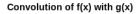


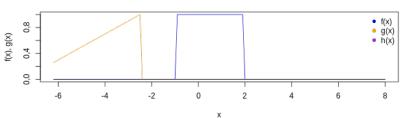


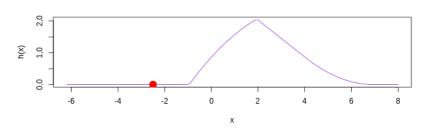




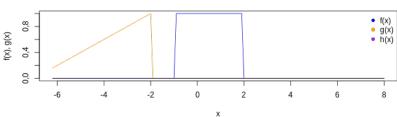


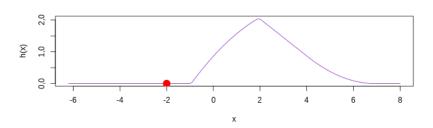


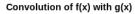


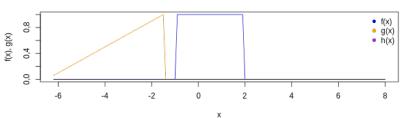


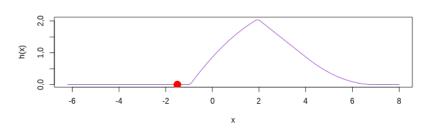


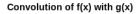


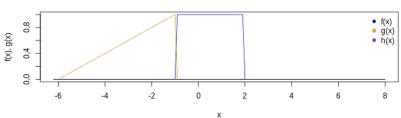


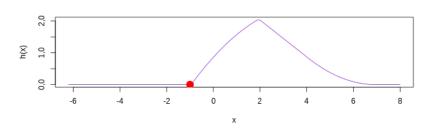


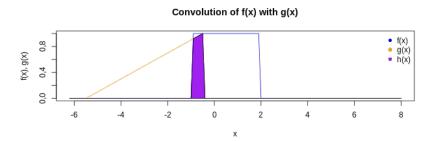


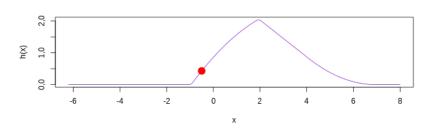


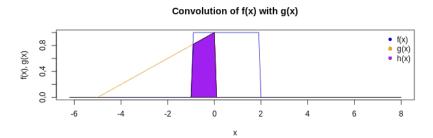


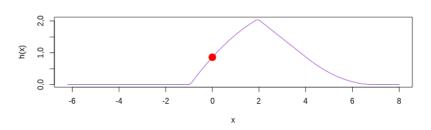


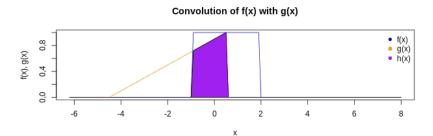


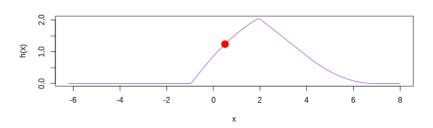


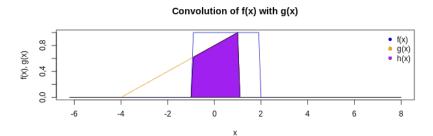


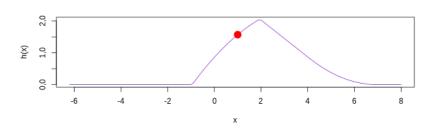


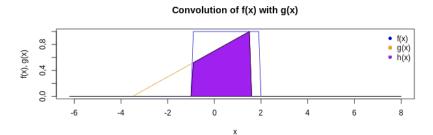


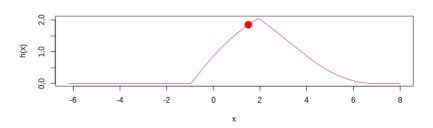


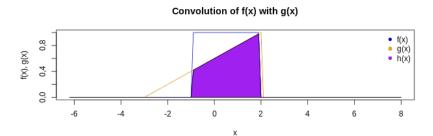


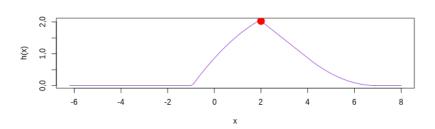


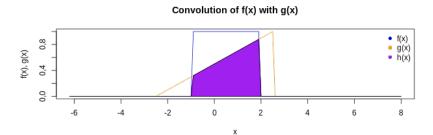


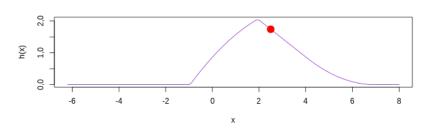


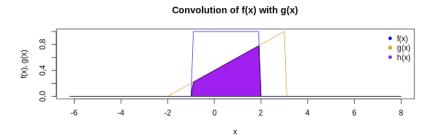


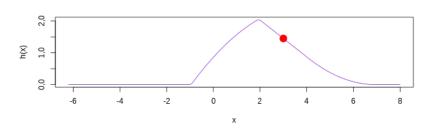


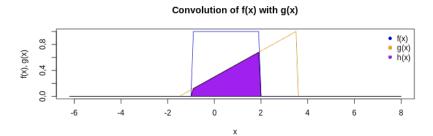


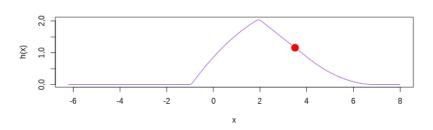






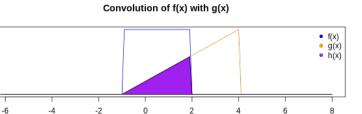




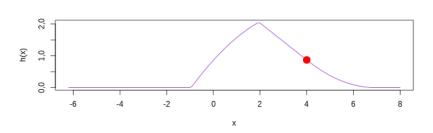


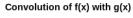
8,0

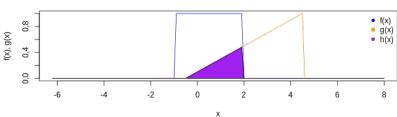
f(x), g(x)

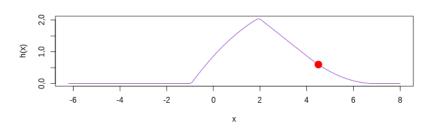


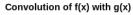
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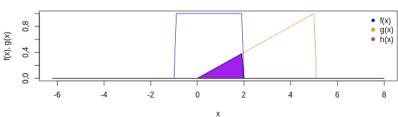


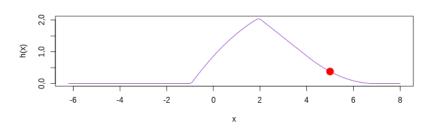


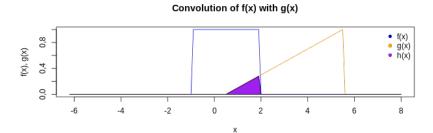


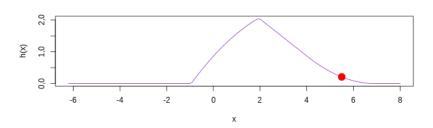




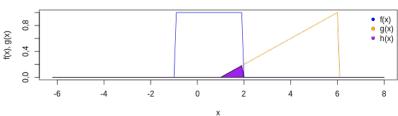


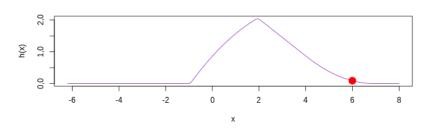


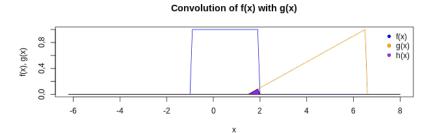


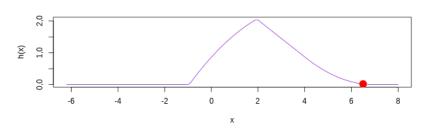


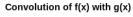


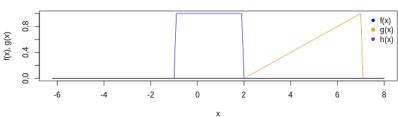


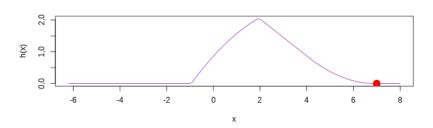


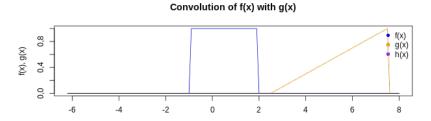




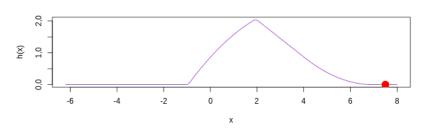


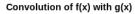


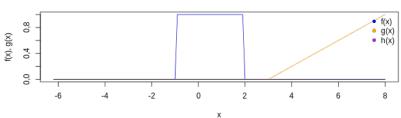


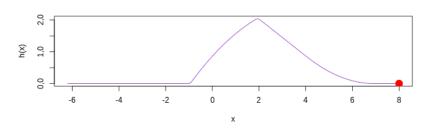


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DISCRETIZATION

Discretization for one-dimensional input:

$$h(i) = (f * g)(i) = \sum_{x} f(x)g(i - x)$$

- Discretization for 2D images:
 - $\mathcal{I} \in \mathcal{R}^2$ contains two dimensions
 - Use 2D Kernel \mathcal{G} as well to yield feature map \mathcal{H} :

$$H(i,j) = (\mathcal{I} * \mathcal{G})(i,j) = \sum_{x} \sum_{y} \mathcal{I}(x,y) \mathcal{G}(i-x,j-y)$$

where $x, y := \text{indices } \mathcal{I} \text{ and } \mathcal{G}$

and i, j := indices elements in \mathcal{H}

PROPERTIES OF THE CONVOLUTION

Commutativity:

$$f * g = g * f$$

Associativity:

$$(f*g)*h=f*(g*h)$$

Distributivity:

$$f*(g+h) = f*g+f*h$$

 $\alpha(f*g) = (\alpha f)*g$ for scalar α

Differentiability:

$$\frac{\partial (f * g)(x)}{\partial x_i} = \frac{\partial f(x)}{\partial x_i} * g(x) = \frac{\partial g(x)}{\partial x_i} * f(x)$$

 \rightarrow (f * g)(x) is as many times differentiable as the max of g(x) and f(x).

RELATED OPERATIONS

- Convolution is strongly related to two other mathematical operators:
 - Fourier transform via the Convolution Theorem
 - Cross correlation

CONVOLUTION THEOREM

 Fourier transform of the convolution of two functions can be expressed as the product of their Fourier transforms:

$$\mathcal{F}\{f*g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}$$

- Transformation of a signal from time to frequency domain.
- Convolution in the time domain is equivalent to multiplication in frequency domain.
- The computationally fastest way to compute a convolution is therefore to take the Fourier inverse of the multiplication of the Fourier-transformed input and filter function:

$$(f * g)(t) = \mathcal{F}^{-1} \{ \mathcal{F} \{ f(t) \} \mathcal{F} \{ g(t) \} \}$$

CONVOLUTION THEOREM - PROOF

$$\widehat{(f * g)(t)} = \int_{-\infty}^{\infty} \exp(-2\pi i\omega t) \Big[\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \Big]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-2\pi i\omega t)f(\tau)g(t-\tau)d\tau dt$$

$$\stackrel{Fubini}{=} \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} \exp(-2\pi i\omega t)f(\tau)g(t-\tau)dt \Big]d\tau$$

$$\stackrel{f(\tau)\perp t}{=} \int_{-\infty}^{\infty} f(\tau) \Big[\int_{-\infty}^{\infty} \exp(-2\pi i\omega t)g(t-\tau)dt \Big]d\tau$$

$$\stackrel{u=t-\tau}{=} \int_{-\infty}^{\infty} f(\tau) \Big[\int_{-\infty}^{\infty} \exp(-2\pi i\omega \tau)\exp(-2\pi i\omega u)g(u)du \Big]d\tau$$

$$= \int_{-\infty}^{\infty} \exp(-2\pi i\omega \tau)f(\tau) \Big[\int_{-\infty}^{\infty} \exp(-2\pi i\omega u)g(u)du \Big]d\tau$$
Fubini

CONVOLUTION THEOREM - PROOF

...
$$\int_{-\infty}^{\infty} \exp(-2\pi i\omega \tau) f(\tau) d\tau \int_{-\infty}^{\infty} \exp(-2\pi i\omega u) g(u) du$$
$$= f(\hat{t}) g(\hat{t})$$

- Measurement for similarity of two functions f(x), g(x).
- More specifically, at which position are the two functions most similar to each other? Where does the pattern of g(x) match f(x) the best?
- Intuition:
 - Slide with g(x) over f(x) and at each discrete step compute the sum of the product of their elements.
 - When peaks of both functions are aligned, the product of high (positive or negative) values will lead to high sums.
 - Thus, both functions are most similar at points with equal peaks.

Definition:

$$h(i)=(f\star g)(i)=\int_{-\infty}^{\infty}f(x)g(i+x)dx$$
 where $f(x)$: input function and $g(x)$: weighting function, kernel and $h(i)$: output function, feature map elements for $f,g\in\mathcal{R}^d\mapsto h\in\mathcal{R}^d$

Discrete formulation:

$$h(i) = (f \star g)(i) = \sum_{x=-\infty}^{\infty} f(x)g(i+x)$$

Thus:

$$f(i) \star g(i) = f(-i) * g(i)$$

- Remember: * is used for convolution and * for cross correlation.
- Similar formulation as the convolution despite the flipped filter function in the convolutional kernel.

- This operation also works in 2 dimensions
- The difference w.r.t. the convolution are the positive iterators in the sum:

$$H(i,j) = (\mathcal{I} \star \mathcal{G})(i,j) = \sum_{x} \sum_{y} \mathcal{I}(x,y) \mathcal{G}(i+x,j+y)$$

where $x, y := \text{indices } \mathcal{I} \text{ and } \mathcal{G}$

and i,j:= indices elements in \mathcal{H}

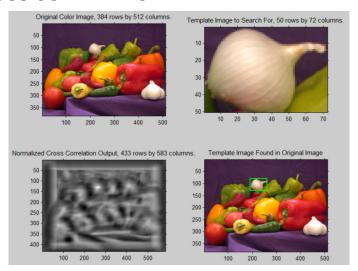
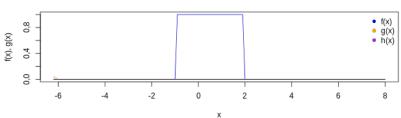
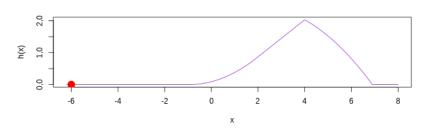


Figure: Cross-correlation used to detect a template (onion) in an image. Cross correlation peaks (white) at the position where template and input match best.

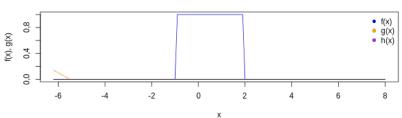
- From the following animation we see that
 - Kernel is not flipped as opposed to the convolution.
 - Cross-Correlation peaks, where the filter matches the signal the most.
- In some frameworks, Cross-Correlation is implemented instead of the convolution due to
 - better computational performance.
 - similar properties, as the kernel weights are learned throughout the training process.

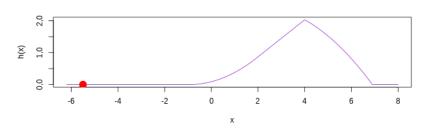




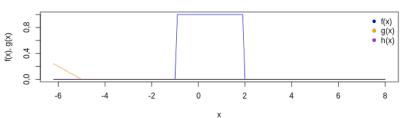


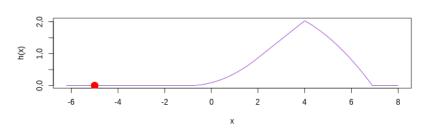




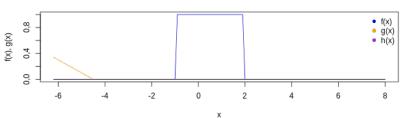


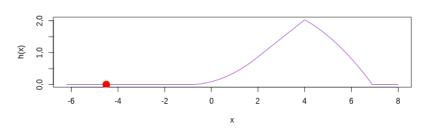




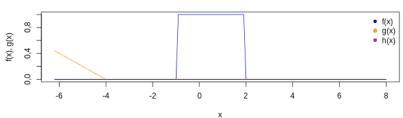


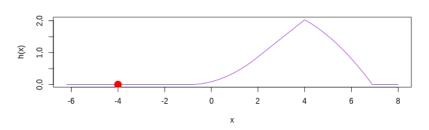




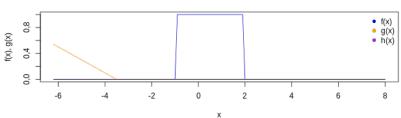


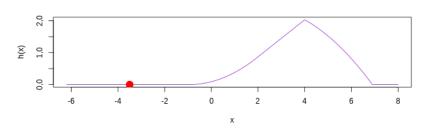




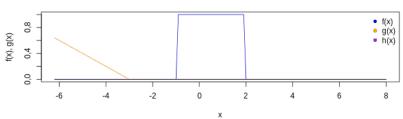


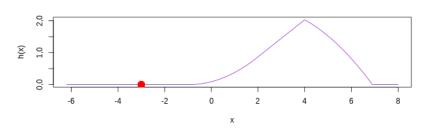




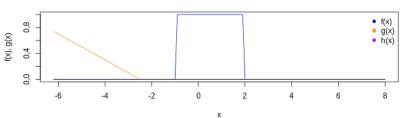


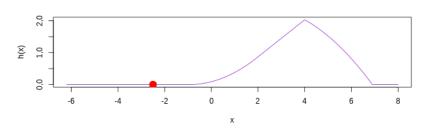




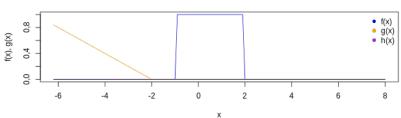


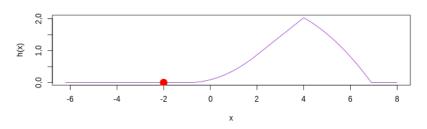




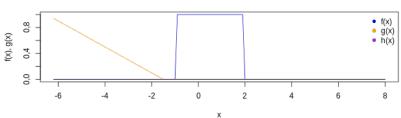


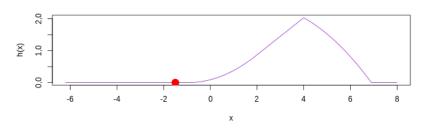




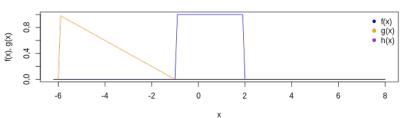


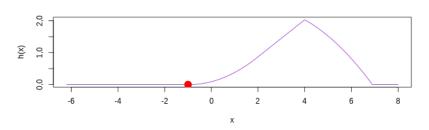




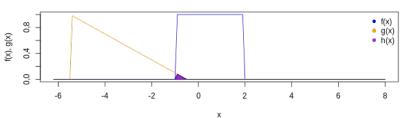


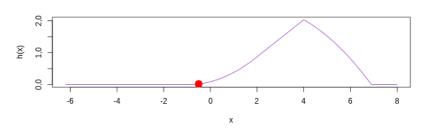




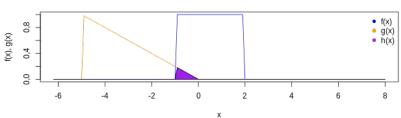


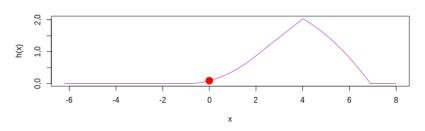




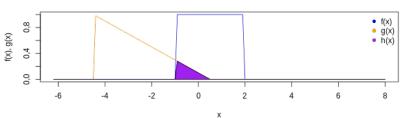


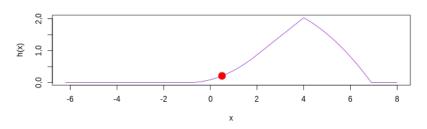




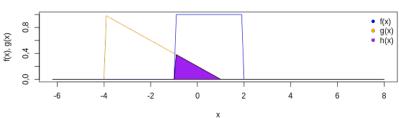


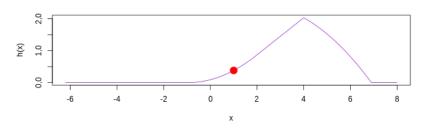




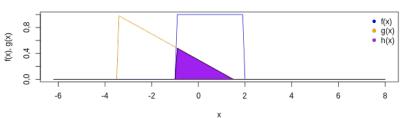


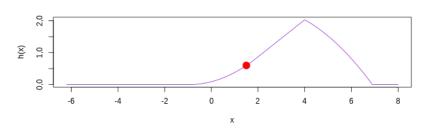




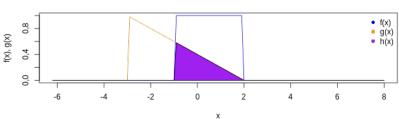


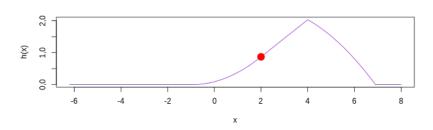




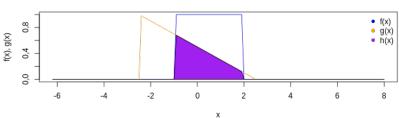


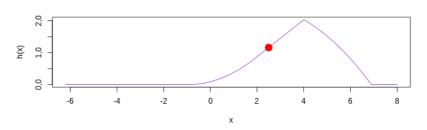




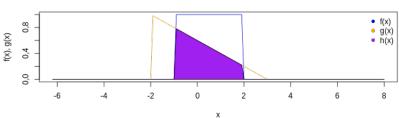


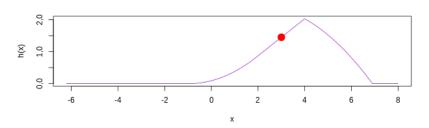




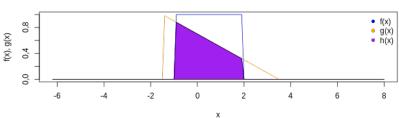


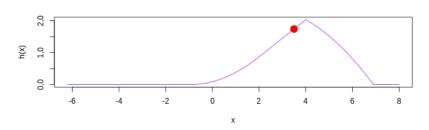




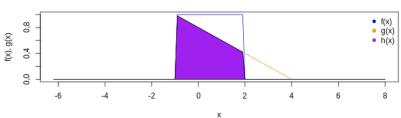


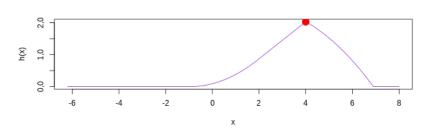




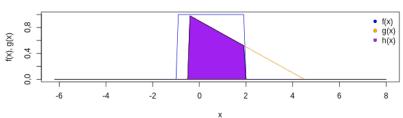


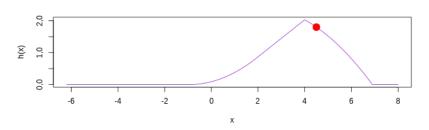


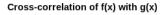


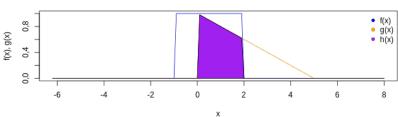


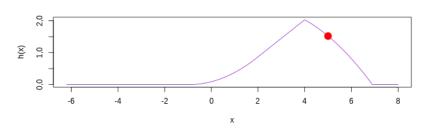


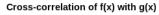


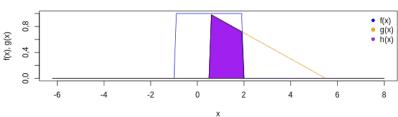


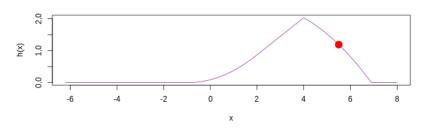


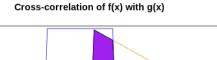


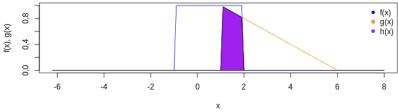


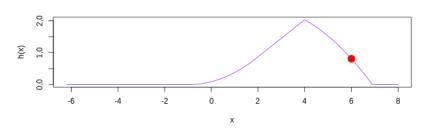


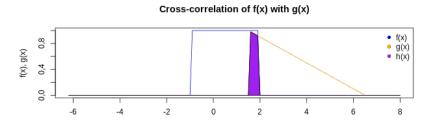




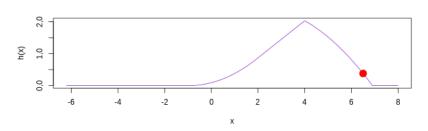




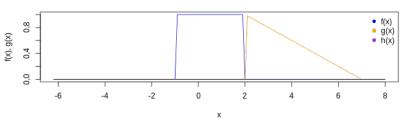


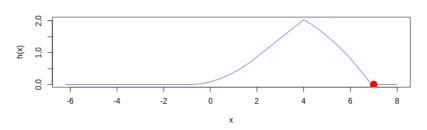


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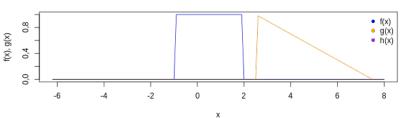


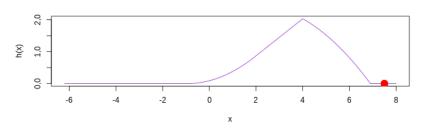




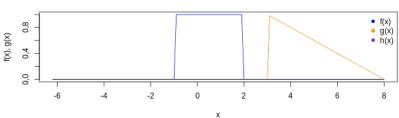


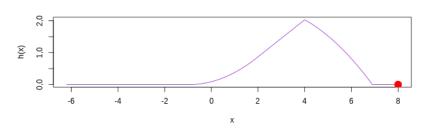












CROSS CORRELATION VS. CONVOLUTION

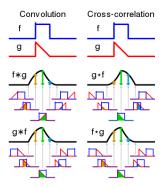


Figure: Comparison of convolution and cross-correlation

- Cross correlation is not commutative
- but often implemented instead of convolution in the practice.