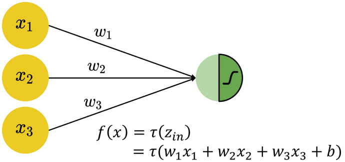
Deep Learning

Single Neuron / Perceptron



Learning goals

- Graphical representation of a single neuron
- Affine transformations and non-linear activation functions
- Hypothesis spaces of single neuron architectures
- Typical loss functions

A SINGLE NEURON

- To illustrate the types of functions that neural networks can represent, let us begin with a simple model: logistic regression.

- The hypothesis space of logistic regression can be written as follows, where $\tau(z) = (1 + \exp(-z))^{-1}$ is the logistic sigmoid function:

$$\mathcal{H} = \left\{ f : \mathbb{R}^p \to [0, 1] \;\middle|\; f(\mathbf{x}) = \tau\left(\sum_{j=1}^{p} w_j x_j + b\right), \mathbf{w} \in \mathbb{R}^p, b \in \mathbb{R} \right\},$$

- It is straightforward to represent $f(\mathbf{x})$ graphically as a neuron.

- Note: $\mathbf{w}$ and b together constitute $\boldsymbol{\theta}$.

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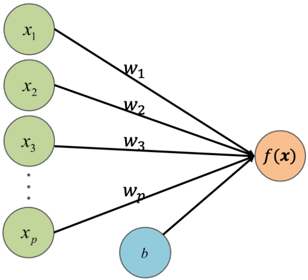

Perceptron z, with **input features** $x_1, x_2, \ldots, x_p$, **weights** $w_1, w_2, \ldots, w_p$, **bias term** b and **activation function** τ.

- The perceptron is the basic computational unit for neural networks.

- It is a weighted sum of input values, transformed by τ:

$$y = \tau(w_1 x_1 + \ldots + w_p x_p + b) = \tau(w^T x + b)$$

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Choices for τ: a single neuron can represent different functions if we choose a suitable activation function for it.

- The identity function gives us the simple **linear regression**:

$$y = \tau(w^T x) = w^T x$$

- The logistic function gives us the **logistic regression**:

$$y = \tau(w^T x) = \frac{1}{1 + \exp(-w^T x)}$$

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We consider a logistic regression model for $p = 3$, i.e.
$f(\mathbf{x}) = \tau(w_1 x_1 + w_2 x_2 + w_3 x_3 + b)$.

- First, features of $\mathbf{x}$ are represented by nodes in the "input layer".

- In general, a p-dimensional input vector $\mathbf{x}$ will be represented by p nodes in the input layer.

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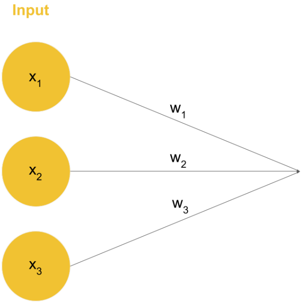
- Next, weights **w** are represented by edges from the input layer.



- The bias term b is implicit here. It is often not visualized as a separate node.

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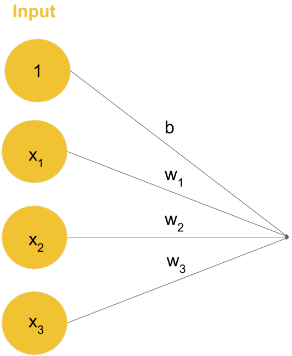
For an explicit graphical representation, we do a simple trick:

- Add a constant feature to the inputs $\tilde{\mathbf{x}} = (1, x_1, ..., x_p)^T$
- and absorb the bias into the weight vector $\tilde{\boldsymbol{w}} = (b, w_1, ..., w_p)$.

The graphical representation is then:



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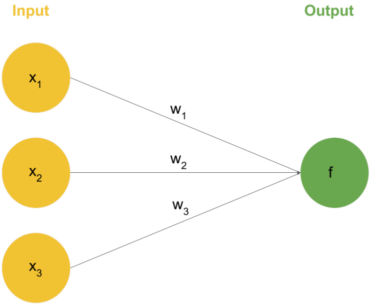
- Finally, the computation $\tau(w_1 x_1 + w_2 x_2 + w_3 x_3 + b)$ is represented by the neuron in the "output layer".



- Because this single neuron represents exactly the same hypothesis space as logistic regression, it can only learn linear decision boundaries.

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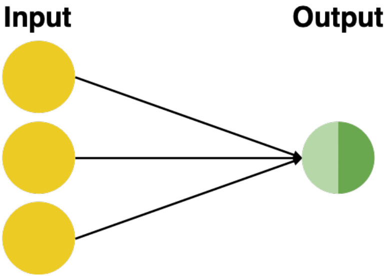
- A nice thing about this graphical representation of functions is that you can picture the input vector being "fed" to the neuron on the left followed by a sequence of computations being performed from left to right. This is called a **forward pass**.



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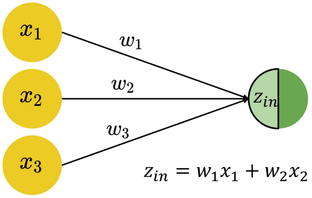
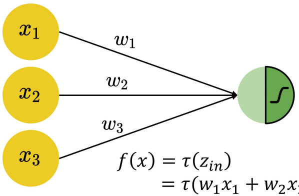
Therefore, a neuron performs a 2-step computation:

1. **Affine Transformation:** weighted sum of inputs plus bias.



$$z_{in} = w_1x_1 + w_2x_2 + w_3x_3 + b$$

2. **Non-linear Activation:** a non-linear transformation applied to the weighted sum.



$$\begin{aligned} f(x) &= \tau(z_{in}) \\ &= \tau(w_1x_1 + w_2x_2 + w_3x_3 + b) \end{aligned}$$

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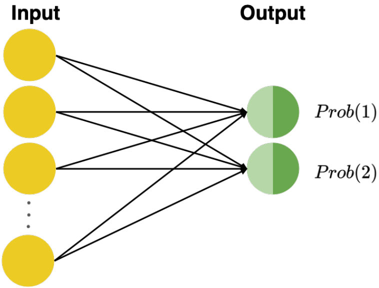
- Even though all neurons compute a weighted sum in the first step, there is considerable flexibility in the type of activation function used in the second step.
- For example, setting the activation function to the logistic sigmoid function allows a neuron to represent logistic regression. The following architecture with two neurons represents two logistic regressions using the same input features:



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- The hypothesis space that is formed by single neuron architectures is

$$\mathcal{H} = \left\{ f : \mathbb{R}^p \to \mathbb{R} \;\middle|\; f(\mathbf{x}) = \tau\left(\sum_{j=1}^{p} w_j x_j + b\right), \mathbf{w} \in \mathbb{R}^p, b \in \mathbb{R} \right\}.$$

- Both logistic regression and linear regression are subspaces of $\mathcal{H}$ (if τ is the logistic sigmoid / identity function).

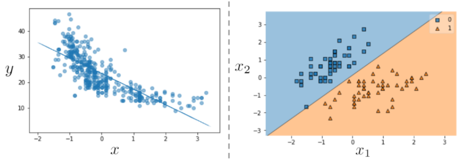

Figure: *Left*: A regression line learned by a single neuron. *Right*: A decision-boundary learned by a single neuron in a binary classification task.

A SINGLE NEURON: OPTIMIZATION

- To optimize this model, we minimize the empirical risk

$$\mathcal{R}_{\text{emp}} = \frac{1}{n}\sum_{i=1}^{n} L\left(y^{(i)}, f\left(\mathbf{x}^{(i)}\right)\right),$$

  where $L(y, f(\mathbf{x}))$ is a loss function. It compares the network's predictions $f(\mathbf{x})$ to the ground truth y.

- For regression, we typically use the L2 loss (rarely L1):

$$L(y, f(\mathbf{x})) = \frac{1}{2}(y - f(\mathbf{x}))^2$$

- For binary classification, we typically apply the cross entropy loss (also known as bernoulli loss):

$$L(y, f(\mathbf{x})) = -(y \log f(\mathbf{x}) + (1 - y)\log(1 - f(\mathbf{x})))$$

A SINGLE NEURON: OPTIMIZATION

- For a single neuron, in both cases, the loss function is convex and the global optimum can be found with an iterative algorithm like gradient descent.
- In fact, a single neuron with logistic sigmoid function trained with the bernoulli loss does not only have the same hypothesis space as a logistic regression and is therefore the same model, but will also yield to the very same result when trained until convergence.
- Note: In the case of regression and the L2-loss, the solution can also be found analytically using the “normal equations”. However, a closed-form solution is usually not available.