

Introduction to Deep Learning

Chapter 2: Single Neuron / Preceptron

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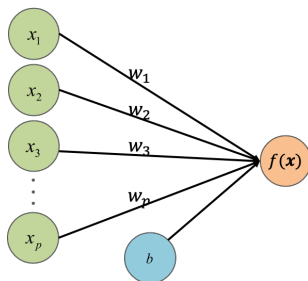
A SINGLE NEURON

- To illustrate the types of functions that neural networks can represent, let us begin with a simple model: logistic regression.
- The hypothesis space of logistic regression can be written as follows, where $\tau(z) = (1 + \exp(-z))^{-1}$ is the logistic sigmoid function:

$$\mathcal{H} = \left\{ f : \mathbb{R}^p \rightarrow [0, 1] \mid f(\mathbf{x}) = \tau \left(\sum_{j=1}^p w_j x_j + b \right), \mathbf{w} \in \mathbb{R}^p, b \in \mathbb{R} \right\},$$

- It is straightforward to represent $f(\mathbf{x})$ graphically as a neuron.
- Note: \mathbf{w} and b together constitute θ .

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Perceptron z , with **input features** x_1, x_2, \dots, x_p , **weights** w_1, w_2, \dots, w_p , **bias term** b and **activation function** τ .

- The perceptron is the basic computational unit for neural networks.
- It is a weighted sum of input values, transformed by τ :

$$y = \tau(w_1x_1 + \dots + w_px_p + b) = \tau(w^T x + b)$$

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Choices for τ : a single neuron can represent different functions if we choose a suitable activation function for it.

- The identity function gives us the simple **linear regression**:

$$y = \tau(w^T x) = w^T x$$

- The logistic function gives us the **logistic regression**:

$$y = \tau(w^T x) = \frac{1}{1 + \exp(-w^T x)}$$

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We consider a logistic regression model for $p = 3$, i.e.

$$f(\mathbf{x}) = \tau(w_1 \mathbf{x}_1 + w_2 \mathbf{x}_2 + w_3 \mathbf{x}_3 + b).$$

- First, features of \mathbf{x} are represented by nodes in the “input layer”.

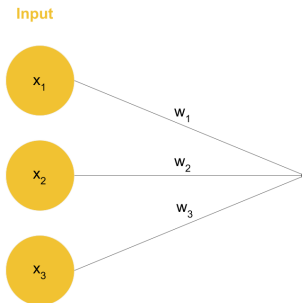
Input



- In general, a p -dimensional input vector \mathbf{x} will be represented by p nodes in the input layer.

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- Next, weights \mathbf{w} are represented by edges from the input layer.



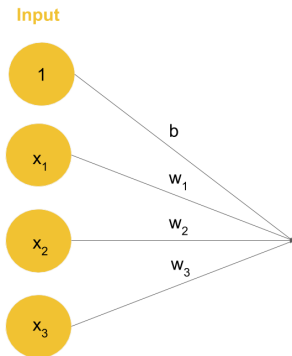
- The bias term b is implicit here. It is often not visualized as a separate node.

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For an explicit graphical representation, we do a simple trick:

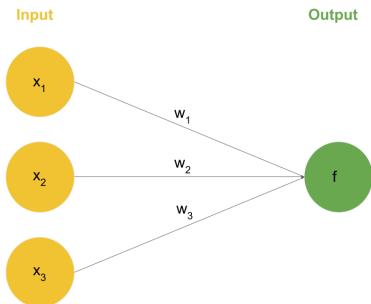
- Add a constant feature to the inputs $\tilde{\mathbf{x}} = (1, x_1, \dots, x_p)^T$
- and absorb the bias into the weight vector $\tilde{\mathbf{w}} = (b, w_1, \dots, w_p)$.

The graphical representation is then:



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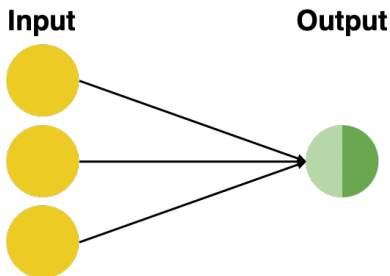
- Finally, the computation $\tau(w_1x_1 + w_2x_2 + w_3x_3 + b)$ is represented by the neuron in the “output layer”.



- Because this single neuron represents exactly the same hypothesis space as logistic regression, it can only learn linear decision boundaries.

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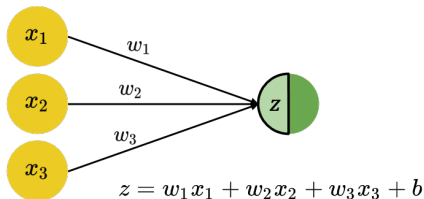
- A nice thing about this graphical representation of functions is that you can picture the input vector being "fed" to the neuron on the left followed by a sequence of computations being performed from left to right. This is called a **forward pass**.



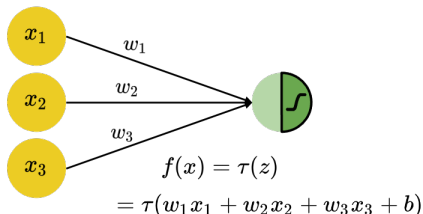
A SINGLE NEURON

Therefore, a neuron performs a 2-step computation:

- 1 **Affine Transformation:** weighted sum of inputs plus bias.

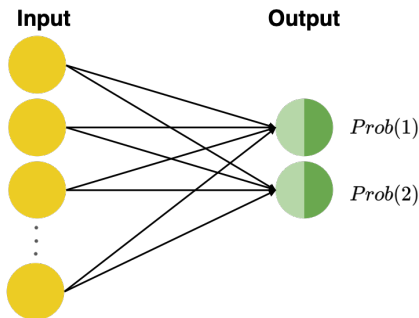


- 2 **Non-linear Activation:** a non-linear transformation applied to the weighted sum.



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- Even though all neurons compute a weighted sum in the first step, there is considerable flexibility in the type of activation function used in the second step.
- For example, setting the activation function to logistic sigmoid function allows a neuron to represent logistic regression. The following neuron represents a logistic regression with two outputs.



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- The hypothesis space that is formed by single neuron architectures is

$$\mathcal{H} = \left\{ f : \mathbb{R}^p \rightarrow \mathbb{R} \mid f(\mathbf{x}) = \tau \left(\sum_{j=1}^p w_j x_j + b \right), \mathbf{w} \in \mathbb{R}^p, b \in \mathbb{R} \right\}.$$

- Both logistic regression and linear regression are subspaces of \mathcal{H} (if τ is the logistic sigmoid / identity function).

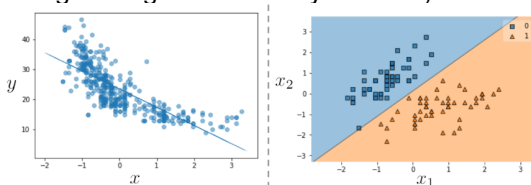


Figure: *Left:* A regression line learned by a single neuron. *Right:* A decision-boundary learned by a single neuron in a binary classification task.

A SINGLE NEURON: OPTIMIZATION

- To optimize this model, we minimize the empirical risk

$$\mathcal{R}_{\text{emp}} = \sum_{i=1}^n L\left(y^{(i)}, f\left(\mathbf{x}^{(i)}\right)\right),$$

where $L(y, f(\mathbf{x}))$ is a loss function. It compares the network's predictions $f(\mathbf{x})$ to the ground truth y .

- For regression, we typically use the L2 loss (rarely L1):

$$L(y, f(\mathbf{x})) = \frac{1}{2}(y - f(\mathbf{x}))^2$$

- For binary classification, we typically apply the cross entropy loss (also known as bernoulli loss):

$$L(y, f(\mathbf{x})) = y \log f(\mathbf{x}) + (1 - y) \log(1 - f(\mathbf{x}))$$

A SINGLE NEURON: OPTIMIZATION

- For a single neuron, in both cases, the loss function is convex and the global optimum can be found with an iterative algorithm like gradient descent.
- In fact, a single neuron with logistic sigmoid function trained with the bernoulli loss does not only have the same hypothesis space as a logistic regression and is therefore the same model, but will also yield to the very same result when trained until convergence.
- Note: In the case of regression and the L2-loss, the solution can also be found analytically using the “normal equations”. However, a closed-form solution is usually not available.