Lab 7

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Welcome to the seventh lab, which is focused on convolutions and convolutional neural networks. The first exercise shows how to train CNNs with Keras, while the second exercise is about implementing convolutions for black-and-white images. The third exercise is about computing the gradients of the convolution operator.

Imports

In [1]:

```
from io import BytesIO
from math import ceil

import matplotlib.pyplot as plt
import requests
import torch
import torch.nn.functional as F
from PIL import Image
from matplotlib_inline.backend_inline import set_matplotlib_formats
from torch import nn, Tensor
from torch.optim import RMSprop
from torch.utils.data import DataLoader
from torchsummary import summary
from torchvision.datasets import MNIST
from torchvision.transforms import ToTensor
from torchvision.utils import make_grid

set_matplotlib_formats('png', 'pdf')
```

Exercise 1

In this exercise, we will learn how to build CNNs in PyTorch to classify images.

CNNs are a special type of neural network inspired by the structure of the visual cortex in animals. They can be applied to a wide range of tasks such as image recognition, time-series analysis, sentence classification, etc. Two key features that differentiate CNNs from fully connected nets are:

1. Local connections: Each neuron in a convolutional layer is only connected to a subset of the neurons in the previous layer.
2. Shared weights: Each convolutional layer consists of multiple filters and each filter consists of multiple neurons. All the neurons in a given filter share the same weights but each of these neurons is connected to a different subset of the neurons in the previous layer.

CNNs consistently outperform all other models in machine vision tasks such as image recognition, object detection, etc.

Classifying hand-written digits

We will be working with is the MNIST dataset (included in `torchvision`). It consists of 28x28 pixels, grayscale images of hand-written digits and their associated labels (0 to 9). The training set contains 60,000 images and the test set contains 10,000.

The `torchvision` implementation of MNIST directly gives us a `Dataset` that we can use for the `Dataloader`. However, in its plain form the dataset will return a PIL image in which the elements take values between 0 and 255. It is standard practice to scale the inputs so that the elements take values between 0 and 1. This typically helps the network train better and we need a tensor anyway. This can be quickly done by specifying the `transform` argument with the `ToTensor` transformation, which will convert the PIL images on the fly. There are a lot of transformations available and they are arbitrarily composable to any complexity. This is usually done in the context of data augmentation. You can check some other transforms [here (https://pytorch.org/vision/stable/transforms.html)](https://pytorch.org/vision/stable/transforms.html).

In [2]:

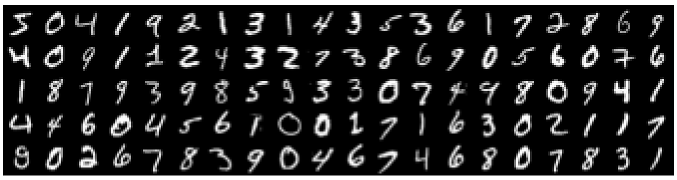
```
train_dataset = MNIST(root='.data', train=True, download=True, transform=ToTensor()
test_dataset = MNIST(root='.data', train=False, download=True, transform=ToTensor()
```

```
/home/edo/phd/repos/lecture_i2dl/exercises/python/.venv/lib/python3.7/
site-packages/torchvision/datasets/mnist.py:498: UserWarning: The give
n NumPy array is not writeable, and PyTorch does not support non-write
able tensors. This means you can write to the underlying (supposedly n
on-writeable) NumPy array using the tensor. You may want to copy the a
rray to protect its data or make it writeable before converting it to
a tensor. This type of warning will be suppressed for the rest of this
program. (Triggered internally at  ../torch/csrc/utils/tensor_numpy.cp
p:180.)
  return torch.from_numpy(parsed.astype(m[2], copy=False)).view(*s)
```

Let's visualize a few images with the `make_grid` utility. We need to permute the dimensions as tensors are of shape `channel x height x width` but the matplotlib convention follows `height x width x channel`.

In [3]:

```
image_batch = torch.stack([train_dataset[i][0] for i in range(100)])
image_grid = make_grid(image_batch, nrow=20)
plt.imshow(image_grid.permute(1, 2, 0))
plt.axis('off')
plt.show()
```



Build the network

A CNN typically consists of a series of convolutional and pooling layers followed by a few fully-connected layers. The convolutional layers detect important visual patterns in the input, and the fully-connected layers then classify the input based on the activations in the final convolutional/pooling layer. Each convolutional layer consists of multiple filters. When the CNN is trained, each filter in a layer specializes in identifying patterns in the image that downstream layers can use.

To create a convolutional layer in PyTorch, call the [`Conv2D` (https://pytorch.org/docs/stable/generated/torch.nn.Conv2d.html)](https://pytorch.org/docs/stable/generated/torch.nn.Conv2d.html) module, and specify the the number of filters in the layer (`in_channels` and 'out_channels' parameter) and the size of the filters (`kernel_size` parameter).

Pooling layers are used to downsample intermediate feature maps in the CNN. PyTorch has multiple options for pooling layers, but today we will only use `MaxPool2D` [(https://pytorch.org/docs/stable/nn.html#pooling-layers)](https://pytorch.org/docs/stable/nn.html#pooling-layers) which takes a `kernel_size` argument for the size of the pooling window.

In [4]:

```
model = nn.Sequential(
    nn.Conv2d(
        in_channels=1,
        out_channels=32,
        kernel_size=(3, 3),
    ),
    nn.ReLU(),
    nn.MaxPool2d(kernel_size=2),
    nn.Conv2d(
        in_channels=32,
        out_channels=64,
        kernel_size=(3, 3),
    ),
    nn.ReLU(),
    nn.MaxPool2d(kernel_size=2),
    nn.Conv2d(
        in_channels=64,
        out_channels=64,
        kernel_size=(3, 3),
    ),
    nn.Flatten(),
    nn.Linear(in_features=576, out_features=64),
    nn.ReLU(),
    nn.Linear(in_features=64, out_features=10)
)
```

Now, we define the device that we would like to use for handling tensors. Here, it will use the GPU for computation if it is available.

In [5]:

```
device = torch.device('cuda' if torch.cuda.is_available() else 'cpu')
```

Don't forget to send the model to the correct device!

In [6]:

```
model = model.to(device)
```

Let's take a look at what we've built so far. The `summary` function from `torchsummary` is able to visualize the created network a bit better. We can also inspect the number of parameters and the memory consumption of a forward & backward pass.

In [7]:

```
summary(model, input_size=(1, 28, 28), device=str(device))
```

```
----------------------------------------------------------------
        Layer (type)               Output Shape         Param #
================================================================
            Conv2d-1           [-1, 32, 26, 26]             320
              ReLU-2           [-1, 32, 26, 26]               0
         MaxPool2d-3           [-1, 32, 13, 13]               0
            Conv2d-4           [-1, 64, 11, 11]          18,496
              ReLU-5           [-1, 64, 11, 11]               0
         MaxPool2d-6             [-1, 64, 5, 5]               0
            Conv2d-7             [-1, 64, 3, 3]          36,928
           Flatten-8                  [-1, 576]               0
            Linear-9                   [-1, 64]          36,928
             ReLU-10                   [-1, 64]               0
           Linear-11                   [-1, 10]             650
================================================================
Total params: 93,322
Trainable params: 93,322
Non-trainable params: 0
----------------------------------------------------------------
Input size (MB): 0.00
Forward/backward pass size (MB): 0.51
Params size (MB): 0.36
Estimated Total Size (MB): 0.87
----------------------------------------------------------------
```

You can see that the output of every `Conv2D` and `MaxPool2D` is a 3D tensor of shape `(channels, height, width)`. For example, the output of the first layer is a tensor of shape `(32, 26, 26)`. Note that the width and height dimensions shrink as you go deeper in the network. The number of channels is controlled by the `filters` parameter of the convolutional layers.

In the model we specified a 576-dimensional fully connected kind of as a magic number, but now we can see why this is done.

After the 3 convolutional layers, the output tensor has a size of $64 \times 3 \times 3$. As a linear layer only accepts a one dimensional input, we flatten the tensor and arrive at $64 \cdot 3 \cdot 3 = 576$

Train and evaluate the model

Similar as in the exercise before, we will define our training loop. You may notice some differences:

- The dataloader now has a `num_workers` argument. This specifies the number of parallel workers that will fetch data from the dataset. A typical heuristic is to set this parameter to the total number of your CPU cores. This can be a critical bottleneck, when a lot of stuff like augmentations is happening in the `__get_item__` method of your `Dataset`. In our case we only apply the `ToTensor` transformation, so the overhead should be pretty low.
- Our dataset is currently on the CPU by default. Thus, we still need to push the `x` and `y` tensors from the dataloader to the correct device.
- We don't have any softmax function in our workflow. That's right! The cross entropy loss of PyTorch works directly on the raw scores and so we can save unnecessary computations. The predicted labels of the model can be obtained by applying `argmax` to the `y_hat` tensor, which returns the index with the largest score.

In [8]:

```
# Set some constants
epochs = 5
batch_size = 64
num_workers = 4

max_batches = ceil(len(train_dataset) / batch_size)

loss = (
    nn.CrossEntropyLoss()
)

optimizer = (
    RMSprop(model.parameters())
)


train_loader = DataLoader(
    dataset=train_dataset,
    batch_size=batch_size,
    shuffle=True,
    num_workers=4,
)

for ep in range(1, epochs + 1):

    total_loss = 0
    num_correct = 0

    for batch_idx, (x, y) in enumerate(train_loader):

        # Push tensors to device
        x = x.to(device)
        y = y.to(device)


        # Forward pass through the models
        y_hat = model(x).squeeze()
        # Obtain the loss
        batch_loss = loss(y_hat, y)

        # Set all parameter gradients to zero
        optimizer.zero_grad()
        # Backpropagate the error
        batch_loss.backward()
        # Apply gradients
        optimizer.step()


        # Print progress every 10 batches
        if batch_idx % 10 == 0:
            print('BATCH:\t({:5} / {:5})\tLOSS:\t{:.3f}'
                  .format(batch_idx, max_batches, float(batch_loss) / batch_size),

        total_loss += float(batch_loss)
        num_correct += int(torch.sum(torch.argmax(y_hat, dim=1) == y))

    print('EPOCH:\t{:5}\tLOSS:\t{:.3f}\tACCURACY:\t{:.3f}'
          .format(ep, total_loss / len(train_dataset), num_correct / len(train_data
```

```
                    end='\r'))
```

```
EPOCH:      1   LOSS:   0.024   ACCURACY:       0.774
EPOCH:      2   LOSS:   0.002   ACCURACY:       0.954
EPOCH:      3   LOSS:   0.002   ACCURACY:       0.961
EPOCH:      4   LOSS:   0.002   ACCURACY:       0.965
EPOCH:      5   LOSS:   0.003   ACCURACY:       0.963
```

After we trained the model, we evaluate it on the test dataset.

In [9]:

```python
# We can choose a much larger batch size,
# as we don't need to compute any gradients and only do inference.
test_loader = DataLoader(
    dataset=train_dataset,
    batch_size=1024,
    num_workers=4,
)

total_loss = 0
num_correct = 0

for batch_idx, (x, y) in enumerate(train_loader):
    # Push tensors to device
    x = x.to(device)
    y = y.to(device)

    with torch.no_grad():
        # Forward pass through the model
        y_hat = model(x).squeeze()
        # Obtain the loss
        batch_loss = loss(y_hat, y)

    total_loss += float(batch_loss)
    num_correct += int(torch.sum(torch.argmax(y_hat, dim=1) == y))

print('EVALUATION LOSS:\t{:.3f}\tEVALUATION: ACCURACY:\t{:.3f}'
          .format(total_loss / len(train_dataset), num_correct / len(train_dataset)
                  end='\r'))
```

```
EVALUATION LOSS:        0.002   EVALUATION: ACCURACY:   0.971
```

The accuracy of our model on MNIST is quite good!

Exercise 2

In this exercise we are going to implement convolution on images, without worrying about stride and padding, and test it with the Sobel filter. There are two Sobel filters: G_x detects horizontal edges and G_y detects vertical edges.

$$G_x = \begin{vmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{vmatrix} \qquad G_y = \begin{vmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{vmatrix} = {G_x}^T$$

Can you explain why and how these filters work?

In order to get the image E with the edges, we convolve G_x and G_y with the input image I, to obtain the degree of horizontal and vertical "borderness" of each pixel. We then combine these values (separately for each pixel) with an L2 norm:

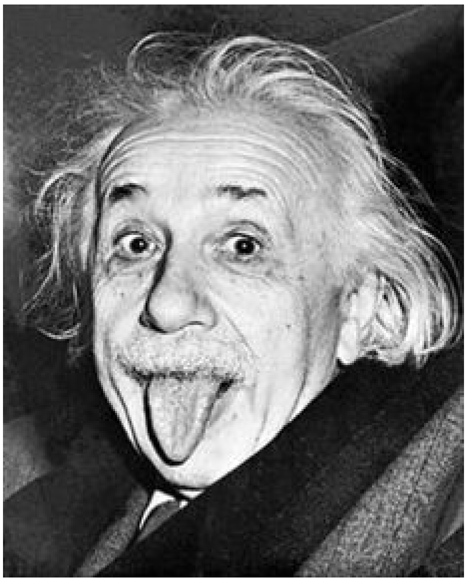
$$E = \sqrt{(G_x * I)^2 + (G_y * I)^2}$$

Let's load an example image:

In [10]:

```
url = 'https://upload.wikimedia.org/wikipedia/en/8/86/Einstein_tongue.jpg'
response = requests.get(url)
img_raw = Image.open(BytesIO(response.content))

img_raw
```

Out[10]:



We will further use this image as a tensor and drop the channel dimension.

In [11]:

```
img = ToTensor()(img_raw)[0]
print(img.shape)
```

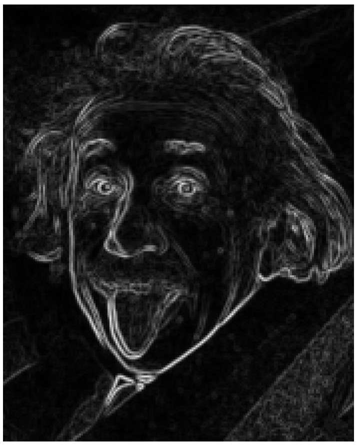
```
torch.Size([286, 230])
```

As a reference, this is the result we want to obtain:

In [12]:

```
def apply_sobel(img: Tensor) -> Tensor:
    # Define the two filters
    sobel_x = torch.tensor([
        [-1, 0, 1],
        [-2, 0, 2],
        [-1, 0, 1]],
        dtype=torch.float)

    sobel_y = torch.tensor([
        [-1, -2, -1],
        [0, 0, 0],
        [1, 2, 1]],
        dtype=torch.float)

    # We can use the functional API of torch to perform convolution manually
    conv_x = F.conv2d(img.unsqueeze(0).unsqueeze(0), sobel_x.view(1, 1, 3, 3))
    conv_y = F.conv2d(img.unsqueeze(0).unsqueeze(0), sobel_y.view(1, 1, 3, 3))

    # Combine the two convolutions
    conv = torch.sqrt(conv_x**2 + conv_y**2)

    # Normalize maximum value to 1
    return (conv / torch.max(conv)).squeeze(0).squeeze(0)

img_sobel = apply_sobel(img)
print(img_sobel.shape)
plt.imshow(img_sobel, cmap='gray')
plt.axis('off')
plt.show()
```

```
torch.Size([284, 228])
```



We now implement our version of convolutions. For an input matrix $\mathbf{X}$ of size $r(\mathbf{X}) \times c(\mathbf{X})$ and a kernel $\mathbf{K}$ of size $r(\mathbf{K}) \times c(\mathbf{K})$, the result of the convolution is $\mathbf{Y} = \mathbf{K} * \mathbf{X}$ with $r(\mathbf{Y}) = r(\mathbf{X}) - r(\mathbf{K}) + 1$, $c(\mathbf{Y}) = c(\mathbf{X}) - c(\mathbf{K}) + 1$, and elements:

$$y_{ij} = \sum_{k=1}^{r(\mathbf{K})} \sum_{l=1}^{c(\mathbf{K})} x_{i+k-1,j+l-1} \cdot k_{kl}$$

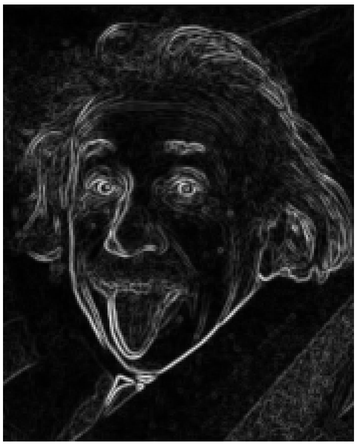
for $1 \leq i \leq r(\mathbf{Y})$ and $1 \leq j \leq c(\mathbf{Y})$.

You now have to implement a function that computes y_{ij} given the image, the kernel, i and j.

In [13]:

```python
def compute_convolution_at_position(i: int, j: int, img: Tensor, kernel: Tensor) ->
    result_ij = 0
    num_rows_kernel, num_cols_kernel = kernel.shape
    for k in range(num_rows_kernel):
        for l in range(num_cols_kernel):
            result_ij += img[i + k - 1, j + l - 1] * kernel[k, l]
    return result_ij


def apply_convolution(img: Tensor, kernel:Tensor) -> Tensor:
    height, width = img.shape
    img_out = torch.zeros(height - 2, width - 2)

    for i in range(1, height - 1):
        for j in range(1, width - 1):
            img_out[i - 1, j - 1] = compute_convolution_at_position(i, j, img, kern

    return img_out

def apply_custom_sobel(img: Tensor) -> Tensor:
    # Define the two filters
    sobel_x = torch.tensor([
        [-1, 0, 1],
        [-2, 0, 2],
        [-1, 0, 1]],
        dtype=torch.float)

    sobel_y = torch.tensor([
        [-1, -2, -1],
        [0, 0, 0],
        [1, 2, 1]],
        dtype=torch.float)

    # We apply our own convolution
    conv_x = apply_convolution(img, sobel_x)
    conv_y = apply_convolution(img, sobel_y)

    # Combine the two convolutions
    conv = torch.sqrt(conv_x**2 + conv_y**2)

    # Normalize maximum value to 1
    return (conv / torch.max(conv))

img_sobel = apply_custom_sobel(img)
print(img_sobel.shape)
plt.imshow(img_sobel, cmap='gray')
plt.axis('off')
plt.show()
```

```
torch.Size([284, 228])
```



If you did everything correctly, this image should match the image above.

Exercise 3

Recall that the convolution $\mathbf{Y} = \mathbf{K} * \mathbf{X}$ has elements

$$y_{ij} = \sum_{k=1}^{r(\mathbf{K})} \sum_{l=1}^{c(\mathbf{K})} x_{i+k-1,j+l-1} \cdot k_{kl}$$

Now consider $\mathbf{X}$ and $\mathbf{Y}$ to be the input and output of a convolutional layer with filter $\mathbf{K}$. For simplicity, we focus on a single channel; actual convolution layers in CNN perform this operation several times with different learnable filters.

Imagine this convolution is a hidden layer of the neural network, with $\mathbf{X}$ being the input from the previous layer, and $\mathbf{Y}$ the pre-activation output to the next layer. Then, we can define the loss function in terms of $\mathbf{Y}$, i.e. $\mathcal{L} = f(\mathbf{Y})$, where f includes the activation, all the following layers, and the classification/regression loss.

Show that:

$$\frac{\partial \mathcal{L}}{\partial k_{kl}} = \sum_{i=1}^{r(\mathbf{Y})} \sum_{j=1}^{c(\mathbf{Y})} \frac{\partial \mathcal{L}}{\partial y_{ij}} \cdot x_{i+k-1,j+l-1}$$

Then show that

$$\frac{\partial \mathcal{L}}{\partial x_{ij}} = \sum_{k=L_k}^{U_k} \sum_{l=L_l}^{U_l} \frac{\partial \mathcal{L}}{\partial y_{ab}} k_{kl}$$

With

$$\begin{aligned}
a &= i - k + 1 \\
b &= j - l + 1 \\
L_k &= \max(1, i - r(\mathbf{X}) + r(\mathbf{K})) \\
L_l &= \max(1, j - c(\mathbf{X}) + c(\mathbf{K})) \\
U_k &= \min(r(\mathbf{K}), i) \\
U_l &= \min(c(\mathbf{K}), j)
\end{aligned}$$

As you can see, the gradient of the input is obtained by convolving the same filter with the gradient of the output, with some care at the borders.

Hint: it is easier to analyze convolutions in one dimension with a small example, then generalize the result to two dimensions and arbitrary filter/image size.

Now, write a function that computes $\partial\mathcal{L}/\partial x_{ij}$, with $\mathcal{L} = \sum_{i,j} {y_{ij}}^2$ and $\mathbf{K} = G_x$.

Solution

We start by showing that

$$\frac{\partial\mathcal{L}}{\partial k_{kl}} = \sum_{i=1}^{r(\mathbf{Y})}\sum_{j=1}^{c(\mathbf{Y})} \frac{\partial\mathcal{L}}{\partial y_{ij}} \cdot x_{i+k-1,j+l-1}$$

The loss is a generic function of every y_{ij}, therefore by the chain rule we have

$$\frac{\partial\mathcal{L}}{\partial k_{kl}} = \sum_{i=1}^{r(\mathbf{Y})}\sum_{j=1}^{c(\mathbf{Y})} \frac{\partial\mathcal{L}}{\partial y_{ij}} \cdot \frac{\partial y_{ij}}{\partial k_{kl}}$$

The element of the kernel is fixed (because k and l are given), and each y_{ij} is influenced only once by k_{kl}. This means that we have to find which element of $\mathbf{X}$ is $\partial y_{ij}/\partial k_{kl}$. In the image below, $\mathbf{X}$ is a vector of $c(\mathbf{X}) = 6$ elements, and $\mathbf{Y}$ was obtained with a kernel of size $c(\mathbf{K}) = 3$, which result in $c(\mathbf{Y}) = 4$. The arrows indicate which elements of $\mathbf{X}$ were used to compute which element of $\mathbf{Y}$, and the red arrows highlight the third element of the kernel, i.e. k_3.

We can see that $\partial y_i/\partial k_3 = x_{i+3-1}$, e.g. $\partial y_2/\partial k_3 = x_4$, so that:

$$\frac{\partial\mathcal{L}}{\partial k_i} = \sum_{j=1}^{6} \frac{\partial\mathcal{L}}{\partial y_j} \cdot x_{i+j-1}$$

This extends straightforwardly to several dimensions.

We now show that

$$\frac{\partial\mathcal{L}}{\partial x_{ij}} = \sum_{k=L_k}^{U_k}\sum_{l=L_l}^{U_l} \frac{\partial\mathcal{L}}{\partial y_{ab}} k_{kl}$$

With

$$\begin{aligned}
a &= i - k + 1 \\
b &= j - l + 1 \\
L_k &= \max(1, i - r(\mathbf{X}) + r(\mathbf{K})) \\
L_l &= \max(1, j - c(\mathbf{X}) + c(\mathbf{K})) \\
U_k &= \min(r(\mathbf{K}), i) \\
U_l &= \min(c(\mathbf{K}), j)
\end{aligned}$$

Going back to the example in one dimension, we now highlight the influence of a fixed element of the input, such as x_3:

We also annotated which element of the kernel is used in every edge.

For brevity, let $y'_i = \partial\mathcal{L}/\partial y_i$. Then, from the image we see that:

$$\frac{\partial \mathcal{L}}{\partial x_3} = y'_3 k_1 + y'_2 k_2 + y'_1 k_3 = \sum_{i=1}^{3} y'_{3-i+1} k_i$$

In particular, note that the indices increse on $\mathbf{K}$ and decrease on $\mathbf{Y}$. Some care needs to be taken at the borders:

$$\frac{\partial \mathcal{L}}{\partial x_1} = k_1 y'_1$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = k_2 y'_1 + k_1 y'_3$$
$$\frac{\partial \mathcal{L}}{\partial x_5} = k_2 y'_4 + k_3 y'_3$$
$$\frac{\partial \mathcal{L}}{\partial x_6} = k_3 y'_4$$

In general, the first and last $c(\mathbf{K}) - 1$ elements will have less summands, i.e. when $i < c(\mathbf{K})$ and $i > c(\mathbf{X}) - c(\mathbf{K}) + 1$. In the former case, we sum the kernel elements from 1 to i, and in the latter case we sum from $i - c(\mathbf{X}) + c(\mathbf{K})$ to 3 (both ends included). As before, this reasoning extends trivially to multiple dimensions.

In [14]:

```
def conv_gradient_wrt_input(dloss_dy : Tensor, kernel) -> Tensor:
    num_rows_kernel, num_cols_kernel = kernel.shape
    num_rows_img = (
        dloss_dy.shape[0] + num_rows_kernel - 1
    )

    num_cols_img = (
        dloss_dy.shape[1] + num_cols_kernel - 1
    )

    img_out = torch.zeros(num_rows_img, num_cols_img)
    for i in range(num_rows_img):
        for j in range(num_cols_img):
            gradient_ij = 0.0

            lk = max(0, i - num_rows_img + num_rows_kernel)
            ll = max(0, j - num_cols_img + num_cols_kernel)
            uk = min(num_rows_kernel, i + 1)
            ul = min(num_cols_kernel, j + 1)

            for k in range(lk, uk):
                for l in range(ll, ul):
                    a = i - k
                    b = j - l

                    gradient_ij += dloss_dy[a, b] * kernel[k, l]

            img_out[i, j] = gradient_ij

    return img_out
```

In [15]:

```
sobel_x = torch.tensor([
        [-1, 0, 1],
        [-2, 0, 2],
        [-1, 0, 1]],
    dtype=torch.float)

def apply_conv2d(img: Tensor, kernel: Tensor) -> Tensor:
    height, width = img.shape
    img_out = F.conv2d(img.view(1, 1, height, width), kernel.view(1, 1, 3, 3))
    return img_out.view(img_out.shape[-2], img_out.shape[-1])

img_sobel_x = apply_conv2d(img, sobel_x)
dloss_dy = 2 * img_sobel_x

img_grad = conv_gradient_wrt_input(dloss_dy, sobel_x)
print(img_grad.shape)

img_grad_normalized = (img_grad - torch.min(img_grad))/(torch.max(img_grad) - torch
plt.imshow(img_grad_normalized, cmap='gray')
plt.axis('off')
plt.show()
```

```
torch.Size([286, 230])
```

We can verify this gradient is correct for a single pixel with finite differences:

In [16]:

```
eps = 1e-6
i = torch.randint(img.shape[0], (1,))
j = torch.randint(img.shape[1], (1,))

# Add epsilon to position i,j and convolve
img[i, j] += eps
conv_pos = apply_conv2d(img, sobel_x)

# Remove epsilon to position i,j and convolve
img[i, j] -= 2 * eps
conv_neg = apply_conv2d(img, sobel_x)

# Undo modification to the image
img[i, j] += eps

# Compute the difference of the losses
# NB: We sum the differences to get a more accurate result
empirical_gradient = torch.sum(conv_pos**2 - conv_neg**2) / (2 * eps)

# Compare empirical and analytical gradients
print('Empirical grad.: {:.5f}\tAnalytical grad.: {:.5f}'
      .format(float(empirical_gradient), float(img_grad[i, j])))
```

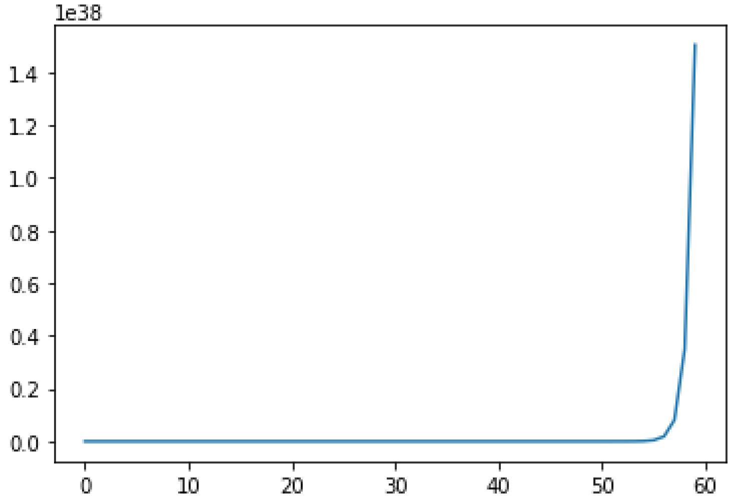
```
Empirical grad.: -0.22957       Analytical grad.: -0.22745
```

If you did everything correctly, these two numbers should have minimal differences.

Now, can you guess what image *maximizes* the loss we just defined? We can find this through gradient *ascent*:

In [17]:

```
img_maximize = torch.rand((9, 9))
losses = []

for _ in range(100):

    conv = apply_convolution(img_maximize, sobel_x)
    loss = torch.sum(conv**2)

    img_grad = conv_gradient_wrt_input(2 * conv, sobel_x)
    img_maximize += 0.01 * img_grad

    losses.append(float(loss))

plt.plot(losses)
plt.show()
```



In [18]:

```
plt.imshow(img_maximize, cmap='gray')
plt.axis('off')
plt.show()
```