As announced in the lecture, the exercises for this week are about the topics that were not covered anymore in the lecture, namely correlation, dependency, and interactions. Therefore, we recommend watching the respective videos (Chapters 1.4 and 1.5 on the website) first before working on the exercises, although it is also possible to directly do the exercises without having watched the videos.

Exercise 1: Correlation and dependency

You received a dataset with 9 observations and two features:

	1	2	3	4	5	6	7	8	9	$\sum_{i=1}^{n}$
y	-7.79	-5.37	-4.08	-1.97	0.02	2.05	1.93	2.16	2.13	-10.92
x1	-1.00	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75	1.00	0
x2	0.95	0.57	0.29	-0.03	0.02	0.08	0.23	0.54	0.98	3.63

The last column corresponds to the sum of values of each row.

a) Compute the Pearson correlation of x_1 and x_2. The formula is:

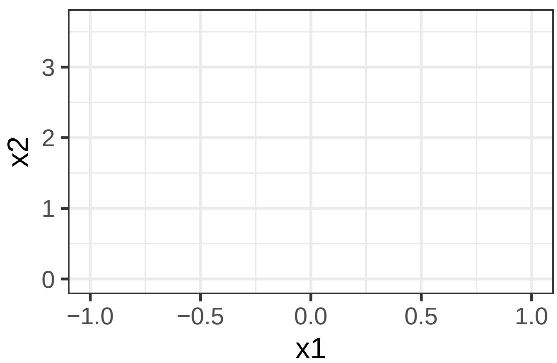
$$\rho(x_1, x_2) = \frac{\sum_{i=1}^{n}(x_1^{(i)} - \overline{x}_1)(x_2^{(i)} - \overline{x}_2)}{\sqrt{\sum_{i=1}^{n}(x_1^{(i)} - \overline{x}_1)^2}\sqrt{\sum_{i=1}^{n}(x_2^{(i)} - \overline{x}_2)^2}}$$

To speed things up, the individual differences to the means ($x_1^{(i)} - \overline{x_1}$, $x_2^{(i)} - \overline{x_2}$), are given in the following table.

	1	2	3	4	5	6	7	8	9
$x_1^{(i)} - \overline{x_1}$	-1.00	-0.75	-0.50	-0.25	0.00	0.25	0.50	0.75	1.00
$x_2^{(i)} - \overline{x_2}$	0.55	0.17	-0.11	-0.43	-0.38	-0.32	-0.17	0.14	0.58

Interpret the results. Based on $\rho(x_1, x_2)$, are x_1 and x_2 correlated?

b) Add points (x_1, x_2) to the following figure:



Based on your drawing, do you consider the Pearson correlation coefficient a reliable measure to detect dependencies for the above use case?

c) We now want to consider the mutual information (MI) instead of the Pearson correlation coefficient. The formula for the MI is:

$$MI(x_1; x_2) = \mathbb{E}_{p(x_1,x_2)}\left[\log\left(\frac{p(x_1,x_2)}{p(x_1)p(x_2)}\right)\right] = \sum_{x_1}\sum_{x_2} p(x_1,x_2)\log\left(\frac{p(x_1,x_2)}{p(x_1)p(x_2)}\right)$$

Here $p(x_1, x_2)$ denotes the joint probability distribution and $p(x_1)$ and $p(x_2)$ respectively the marginal ones. What problem do you encounter when using this formula to calculate the mutual information between x_1 and x_2?

Exercise 2* (Bonus Exercise): Mutual information

In exercise 1c), find a way to work around the problem encountered and calculate the MI between x_1 and x_2. Interpret the result and compare it with your previous findings.

Exercise 3: Correlation and fraction of explained variance

Show that the following relationship between the Pearson correlation coefficient and the coefficient of determination holds:

$$R^2 = \rho^2.$$

This means that for two numerical one-dimensional variables, the degree of (linear) correlation between them is exactly equal to how well a simple linear model fits their relationship.

Recap:

$$\rho = \frac{\sum_{i=1}^n (x^{(i)} - \bar{x}) \cdot (y^{(i)} - \bar{y})}{\sqrt{\sum_{i=1}^n (x^{(i)} - \bar{x})^2}\sqrt{\sum_{i=1}^n (y^{(i)} - \bar{y})^2}},$$
$$R^2 = 1 - \frac{SSE_{LM}}{SSE_c},$$

where

$$SSE_{LM} = \sum_{i=1}^n (y^{(i)} - \hat{f}_{LM}(x^{(i)}))^2,$$
$$SSE_c = \sum_{i=1}^n (y^{(i)} - \bar{y})^2$$

are the sum of squares due to regression and the total sum of squares, respectively.

Exercise 4: Interactions

Consider the following function:

$$f(\mathbf{x}) = f(x_1, x_2) = 2x_1 + 3x_2 - x_1|x_2|$$

Mathematically check whether interactions are present.

Here, we consider the following definition of interactions: A twice differentiable function (or model) f depending on two features x_i and x_j (among others) exhibits an interaction between x_i and x_j if and only if

$$\mathbb{E}\left[\left(\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}\right)^2\right] > 0.$$

Exercise 5* (Bonus Exercise): Interactions

a) Given the function

$$g(x_1, x_2) = 0.01e^{x_1^2} + \sin(x_2)\sqrt{x_1 \mathbf{1}_{(0,\infty)}(x_1)} - 1.5x_2^3,$$

where $\mathbf{1}$ denotes an indicator function

$$\mathbf{1}_S(x) = \begin{cases} 1 \text{ iff } x \in S, \\ 0 \text{ iff } x \notin S, \end{cases} \quad \text{and i.p.} \quad \mathbf{1}_{(a,b)}(x) = \begin{cases} 1 \text{ iff } a < x < b, \\ 0 \text{ else,} \end{cases}$$

find two distributions on $\mathbb{R}^2$ such that the function g contains interactions w.r.t. one of them, but not the other.

b) Again, consider the function f from exercise 4, does it exhibit interactions w.r.t. each of the two distributions you found in a)? Can you find any distribution that would change your result?