Exercise Sheet: Boosting with Decision Stumps

Exercise: Two Iterations of Componentwise Boosting

You are given a boosting model using decision stumps (depth-1 trees) as base learners. The goal is to approximate a target function y over a single feature $x_1 \in [0, 2]$. The boosting algorithm proceeds as follows with step size $\nu = 1$.

Iteration 1

- Base learner: decision stump splitting at $x_1 > 0.3$

- Predictions: $f_1(x_1) = 1 \cdot \mathbb{I}(x_1 \leq 0.3) + 2 \cdot \mathbb{I}(x_1 > 0.3)$

- Initial model: $\hat{f}^{[1]}(x_1) = f_1(x_1)$

Iteration 2

- Residuals: $r^{[1]} = y - \hat{f}^{[1]}$

- Fit a new stump to residuals: split at $x_1 > 1.2$

- Predictions: $f_2(x_1) = 0 \cdot \mathbb{I}(x_1 \leq 1.2) + (-1) \cdot \mathbb{I}(x_1 > 1.2)$

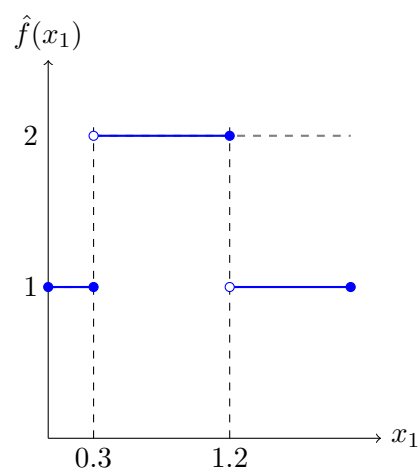- Updated model: $\hat{f}^{[2]}(x_1) = \hat{f}^{[1]}(x_1) + f_2(x_1)$

Tasks

1. Derive the predictor in the second iteration $\hat{f}^{[2]}(x_1)$ and sketch $\hat{f}^{[2]}(x_1)$.

2. Compute $\hat{f}^{[2]}(x_1)$ for $x_1 = 0.2, 0.8, 1.5$.

Solution

Final Predictor

$$\hat{f}^{[2]}(x_1) = 1 \cdot \mathbb{I}(x_1 \leq 0.3) + 2 \cdot \mathbb{I}(0.3 < x_1 \leq 1.2) + 1 \cdot \mathbb{I}(x_1 > 1.2)$$

TikZ sketch:

Step-by-step evaluations:

- $x_1 = 0.2$: $\hat{f}^{[1]} = 1$, $f_2 = 0$, $\hat{f}^{[2]} = 1$

- $x_1 = 0.8$: $\hat{f}^{[1]} = 2$, $f_2 = 0$, $\hat{f}^{[2]} = 2$

- $x_1 = 1.5$: $\hat{f}^{[1]} = 2$, $f_2 = -1$, $\hat{f}^{[2]} = 1$