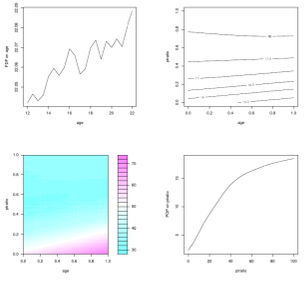
Interpretable Machine Learning

Introduction to Functional Decomposition



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- Basic idea of additive functional decompositions
- Motivation and usefulness of functional decompositions
- Difficulty of obtaining or even defining a functional decomposition
- Several examples

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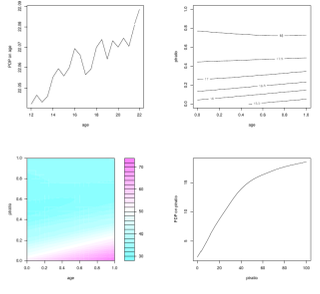
Functional Decompositions Introduction



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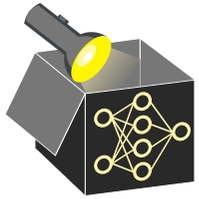

Recap: Interactions

- Interactions between features: Effect of one feature on the prediction output depends on (one or more) other features
- Definition: Features x_j and x_k are considered to interact, if

$$\mathbb{E}\left[\left(\frac{\partial^2 f(\mathbf{x})}{\partial x_j \partial x_k}\right)^2\right] > 0$$

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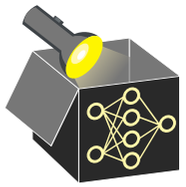

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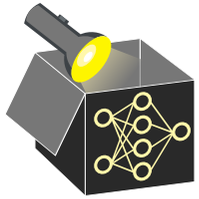

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- Decomposition into only main effects
- Do not contain any interactions

$$\hat{f}(\mathbf{x}) = \theta_0 + g_1(x_1) + g_2(x_2) + \ldots + g_p(x_p)$$

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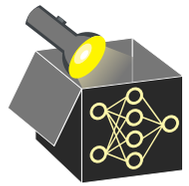
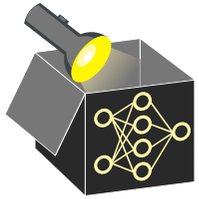


FIRST EXAMPLE: ADDITIVE DECOMPOSITION



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$$\hat{f}(x_1, x_2) = 4 - 2x_1 + 0.3e^{x_2} + |x_1|x_2$$

- Idea: Additive decomposition depending on which features used:

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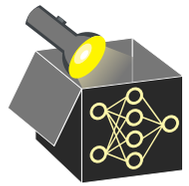

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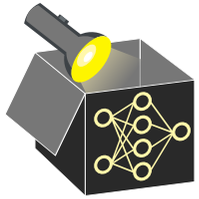

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$$\begin{aligned} g_\emptyset(x_1, x_2) &= 4 && \text{Part depending on no features at all (intercept)} \\ \left.\begin{aligned} g_1(x_1, x_2) &= 2x_1 \\ g_2(x_1, x_2) &= 0.3e^{x_2} \end{aligned}\right\} && \text{Parts depending on a single feature (main effects)} \quad (1) \\ g_{1,2}(x_1, x_2) &= |x_1|x_2 && \text{Part depending on both features (interaction)} \end{aligned}$$

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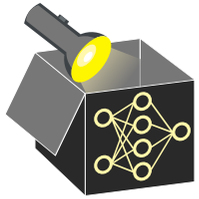

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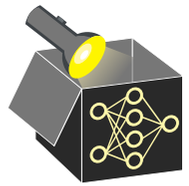

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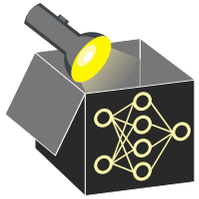
Goal in general: Given a black-box model $\hat{f} : \mathbb{R}^2 \to \mathbb{R}$, find a decomposition

$$\hat{f}(x_1, x_2) = g_\emptyset + g_1(x_1) + g_2(x_2) + g_{1,2}(x_1, x_2) \tag{2}$$



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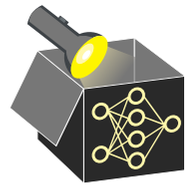

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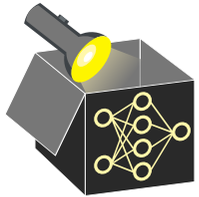

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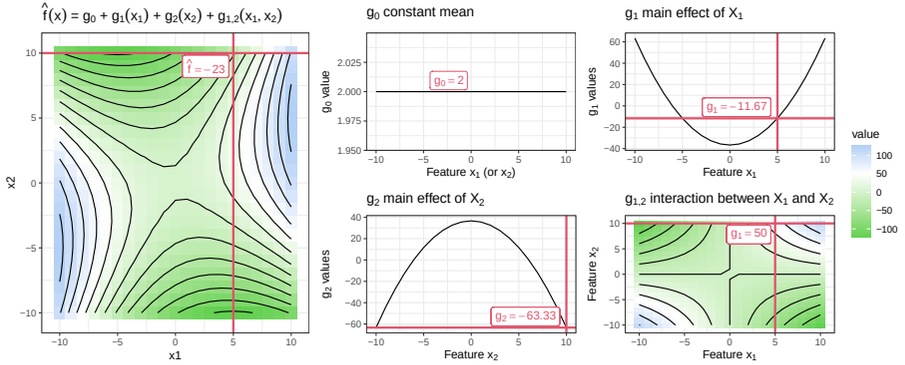

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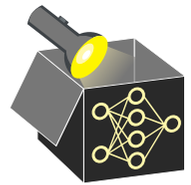

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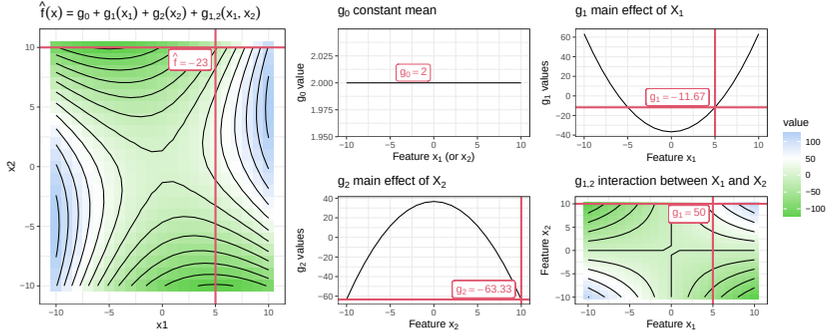

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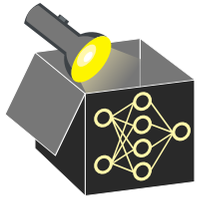

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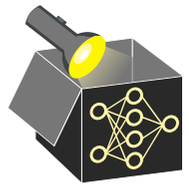

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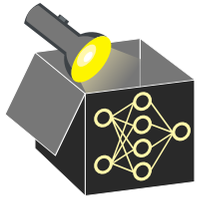

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$\Rightarrow$ 8 components in total, but some empty $\rightsquigarrow$ Certain interactions not present

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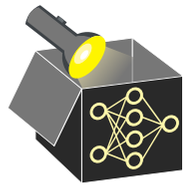

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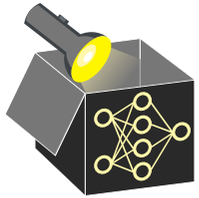

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$\rightsquigarrow$ one component for every possible combination S of indices, allowed to formally only depend on these features / be a function of these features

Problems:

- How to find / compute such a decomposition for arbitrary black-box models $\hat{f}$?
- ... such that the decomposition is useful / has nice properties (w.r.t. the model / w.r.t. the data)?

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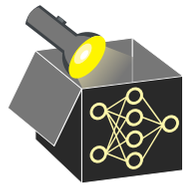
RABITZ_2011 | CHASTAING_2012



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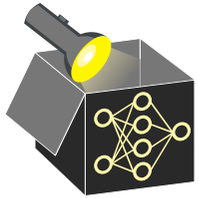

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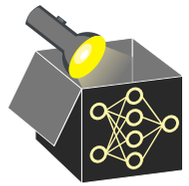
Sort terms according to degree of interaction:

- $g_\emptyset \hat{=}$ Constant mean (intercept)
- $g_j \hat{=}$ first-order or main effect of j-th feature alone on $\hat{f}(\mathbf{x})$
- $g_{j,k} \hat{=}$ second-order interaction effect of features j and k w.r.t. $\hat{f}(\mathbf{x})$
- $g_S(\mathbf{x}_S) \hat{=}$ $|S|$-order effect, depends **only** on features in S

GENERAL FORM OF FUNCTIONAL DECOMPOSITION

RABITZ_2011 | CHASTAING_2012



Definition

Functional decomposition: Additive decomposition of a function $\hat{f} : \mathbb{R}^p \mapsto \mathbb{R}$ into a sum of components of different dimensions w.r.t. inputs $x_1, \ldots, x_p$:

$$\begin{aligned}\hat{f}(\mathbf{x}) &= \sum_{S \subseteq \{1,\ldots,p\}} g_S(\mathbf{x}_S) \\ &= g_\emptyset + g_1(x_1) + g_2(x_2) + \cdots + g_p(x_p) + \\ &\quad g_{1,2}(x_1, x_2) + \cdots + g_{p-1,p}(x_{p-1}, x_p) + \cdots + \\ &\quad g_{1,2,3}(x_1, x_2, x_3) + \cdots + g_{1,2,3,4}(x_1, x_2, x_3, x_4) + \cdots + g_{1,\ldots,p}(x_1, \ldots, x_p)\end{aligned}$$

$\rightsquigarrow$ one component for every possible combination S of indices

Sort terms according to degree of interaction:

- $g_\emptyset \hat{=}$ Constant mean (intercept)
- $g_j \hat{=}$ first-order or main effect of j-th feature alone on $\hat{f}(\mathbf{x})$
- $g_{j,k} \hat{=}$ second-order interaction effect of features j and k w.r.t. $\hat{f}(\mathbf{x})$
- $g_S(\mathbf{x}_S) \hat{=}$ $|S|$-order effect, depends **only** on features in S

PROPERTIES OF THE DEFINITION

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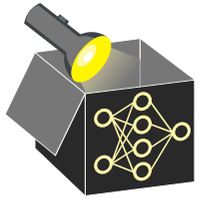

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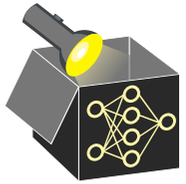

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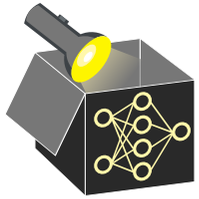

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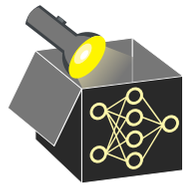

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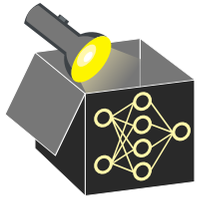

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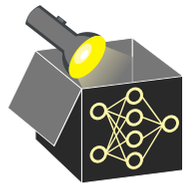

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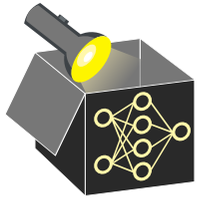

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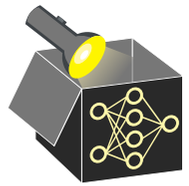

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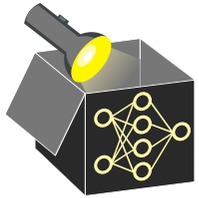

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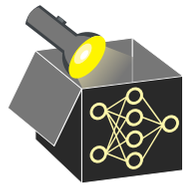

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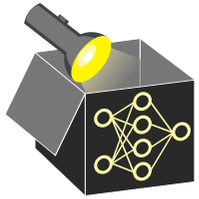
Example

Again consider

$$\hat{f}(x_1, x_2, x_3) = -2x_1 - 2\sin(x_3) + |x_1|x_2 + 0.5x_1x_2x_3 - \sin(x_2x_3) + 1$$



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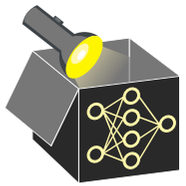

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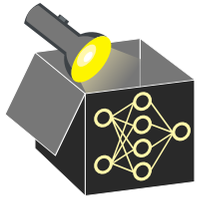

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Two possible decompositions (valid according to definition):

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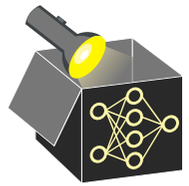

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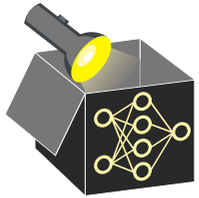

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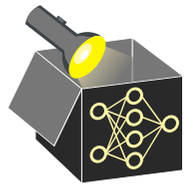

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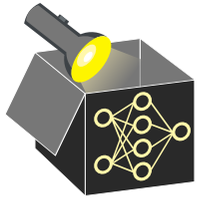

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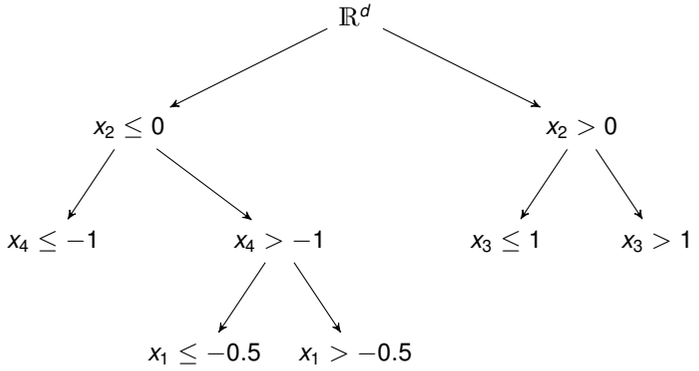

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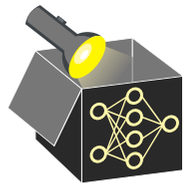

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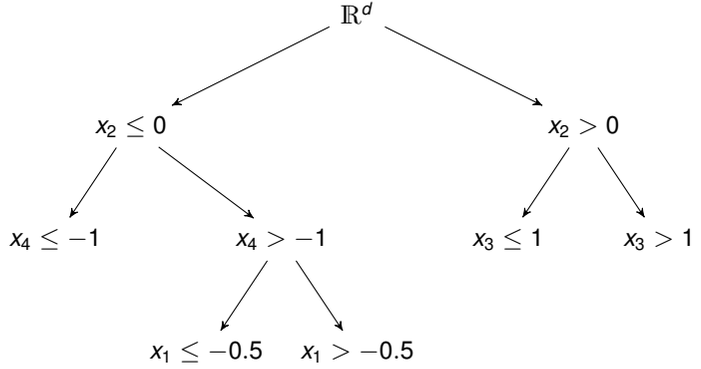

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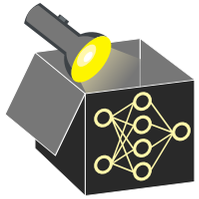

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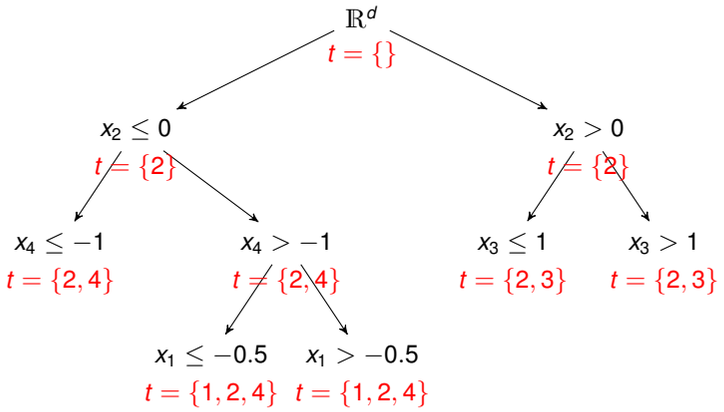

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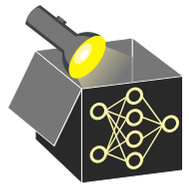

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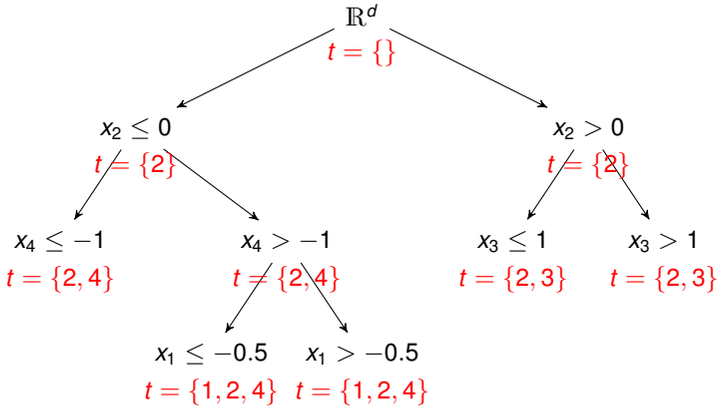

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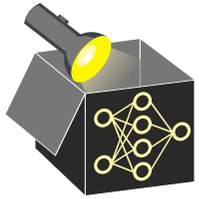

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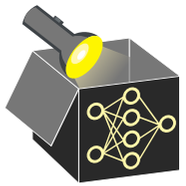

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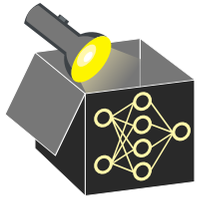

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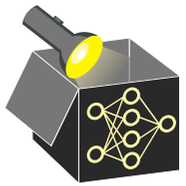

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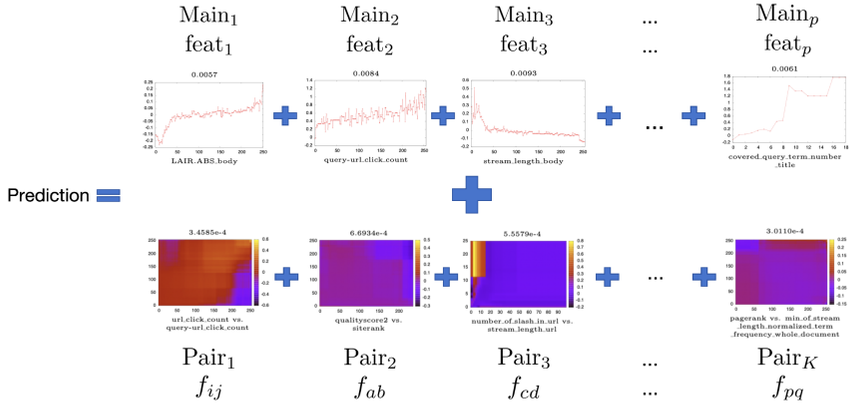
- **EBMs:** Sum of the final one- and two-dimensional components



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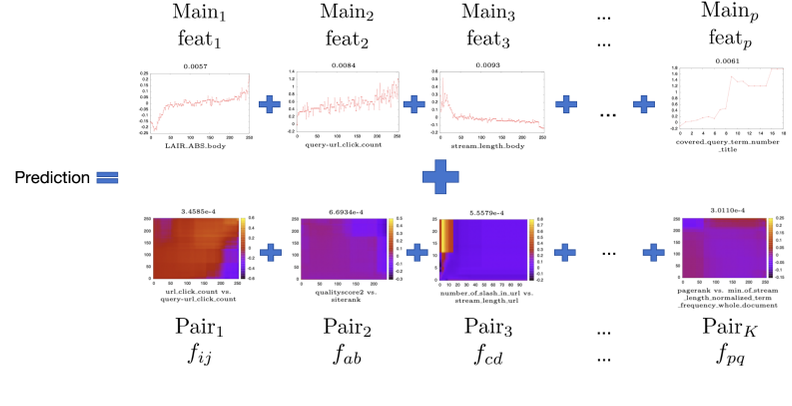

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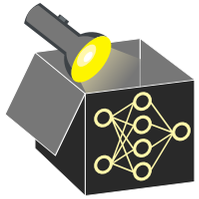

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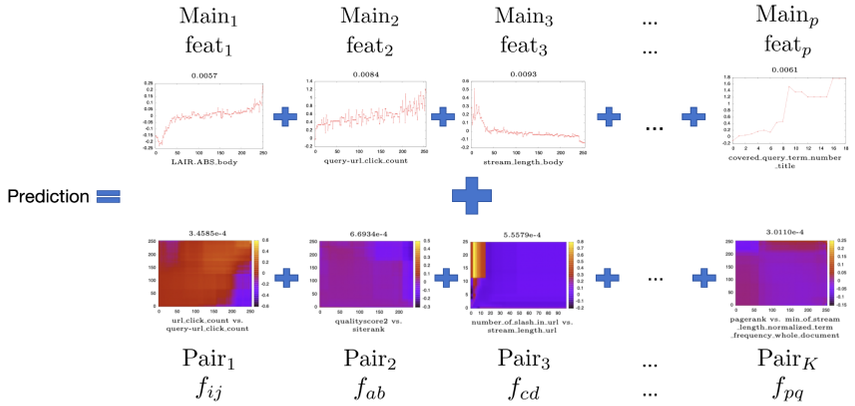

- In general: Model with functional decomposition up to max. order 2 is always “inherently interpretable”
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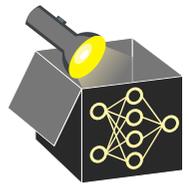

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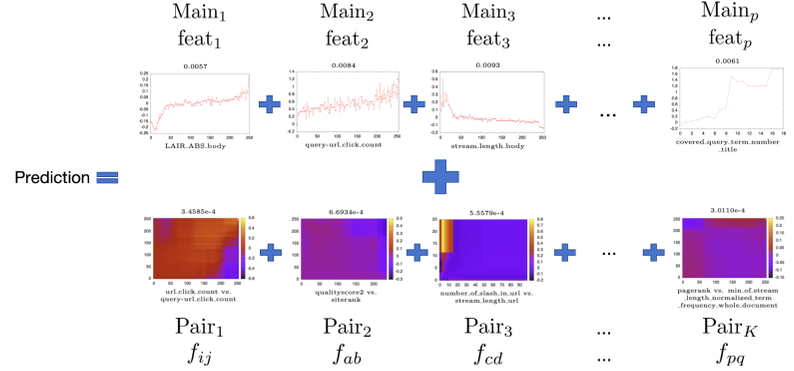

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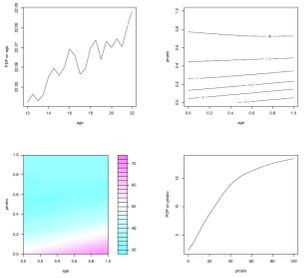
Interpretable Machine Learning

Functional ANOVA

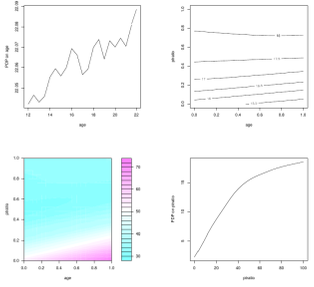


Learning goals

- One method for functional decomposition: Classical functional ANOVA (fANOVA)
- Algorithm for calculating the components in a fANOVA
- Variance decomposition in fANOVA

Interpretable Machine Learning

Functional Decompositions
Functional ANOVA



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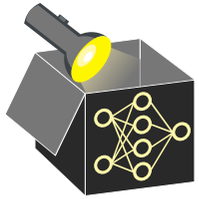
INTRODUCTION AND HISTORY OF FANOVA



- One possible method to obtain functional decomposition
- Since 1940's: Developed under different names in mathematics and sensitivity analysis
- Since 1990's: Developed for probability distributions or statistical data
- Since 2000's: Applied to machine learning, subsequently alternatives developed extending applicability
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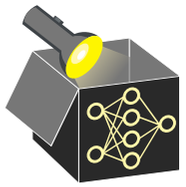

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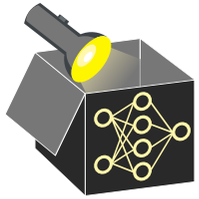
$$\hat{f}(x_1, x_2) = 4 - 2x_1 + 0.3e^{x_2} + |x_1|x_2$$



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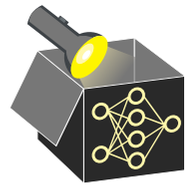

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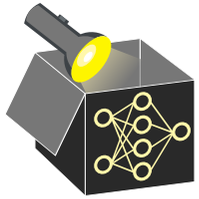

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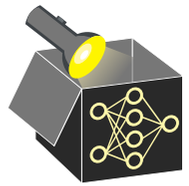

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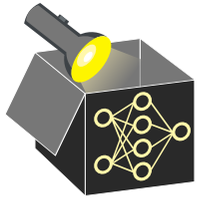

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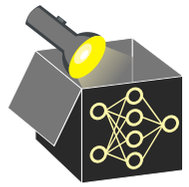

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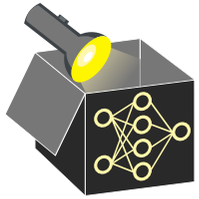

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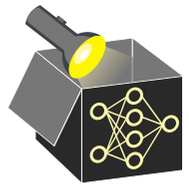

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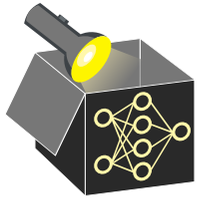

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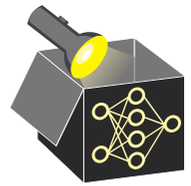

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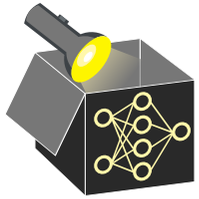
FORMAL DEFINITION AND COMPUTATION

Hooker (2004)



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Recursive computation using PD-functions
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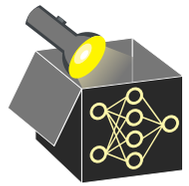
FORMAL DEFINITION AND COMPUTATION

HOOKER_2004



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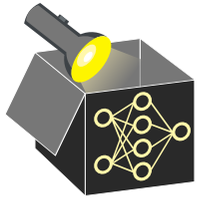

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FORMAL DEFINITION AND COMPUTATION

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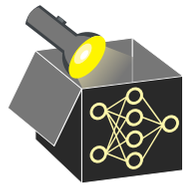

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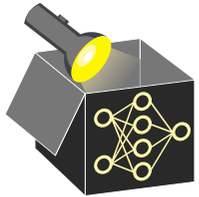
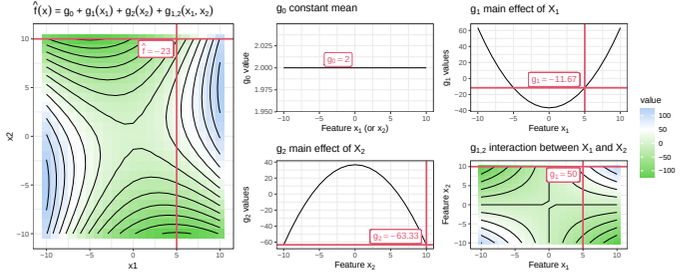
STANDARD FANOVA – EXAMPLE



Example: $\hat{f}(\mathbf{x}) = 2 + x_1^2 - x_2^2 + x_1 \cdot x_2$ (e.g., for $x_1 = 5$ and $x_2 = 10$ we have $\hat{f}(\mathbf{x}) = -23$)

- Computation of components using feature values $x_1 = x_2 = (-10, -9, \ldots, 10)^\top$ gives:

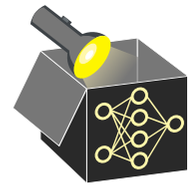


For $x_1 = 5$ and $x_2 = 10$:

- $g_\emptyset = 2$
- $g_1(x_1) = -9.67$
- $g_2(x_2) = -65.33$
- $g_{1,2}(x_1, x_2) = 50$

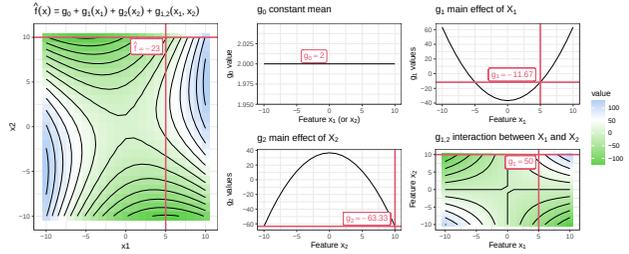
$\Rightarrow \hat{f}(\mathbf{x}) = -23$

STANDARD FANOVA – EXAMPLE



Example: $\hat{f}(\mathbf{x}) = 2 + x_1^2 - x_2^2 + x_1 \cdot x_2$ (e.g., for $x_1 = 5$ and $x_2 = 10$ we have $\hat{f}(\mathbf{x}) = -23$)

- Computation of components using feature values $x_1 = x_2 = (-10, -9, \ldots, 10)^\top$ gives:



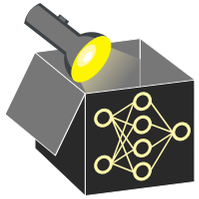
For $x_1 = 5$ and $x_2 = 10$:

- $g_\emptyset = 2$
- $g_1(x_1) = -9.67$
- $g_2(x_2) = -65.33$
- $g_{1,2}(x_1, x_2) = 50$

$\Rightarrow \hat{f}(\mathbf{x}) = -23$

STANDARD FANOVA - EXAMPLE

In-class task



STANDARD FANOVA - EXAMPLE

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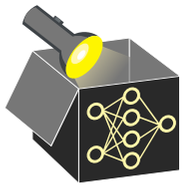
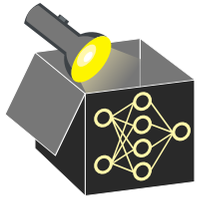


STANDARD FANOVA - EXAMPLE REVISITED



Example

$\hat{f}(x_1, x_2) = 4 - 2x_1 + 0.3e^{x_2} + |x_1|x_2 \qquad (x_1, x_2) \in [0,1]^2 \qquad$ uniformly distributed

- **Intercept:**

$$\begin{aligned} g_\emptyset &= \mathbb{E}\left[\hat{f}(x_1, x_2)\right] = \int_0^1 \int_0^1 4 - 2x_1 + 0.3e^{x_2} + |x_1|x_2 \, dx_1 \, dx_2 \\ &= 4 - \left(\int_0^1 2x_1 \, dx_1\right) + \left(\int_0^1 0.3e^{x_2} \, dx_2\right) + \left(\int_0^1 |x_1| \, dx_1\right)\left(\int_0^1 x_2 \, dx_2\right) \\ &= 4 - 1 + 0.3(e-1) + 0.5^2 = 2.95 + 0.3e. \end{aligned}$$

STANDARD FANOVA - EXAMPLE REVISITED

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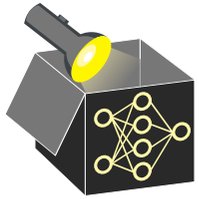

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- **First-order components:**

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\begin{aligned}
g_1(x_1) = \hat{f}_{1;PD}(x_1) - g_\emptyset &= \left(\int_0^1 4 - 2x_1 + 0.3e^{x_2} + |x_1|x_2 \, dx_2\right) - g_\emptyset \\
&= 4 - 2x_1 + 0.3(e-1) + |x_1| \cdot \frac{1}{2} - (2.95 + 0.3e) \\
&= -2x_1 + 0.5|x_1| + 0.75 \\
g_2(x_2) = \hat{f}_{2;PD}(x_2) - g_\emptyset &= \left(\int_0^1 4 - 2x_1 + 0.3e^{x_2} + |x_1|x_2 \, dx_1\right) - g_\emptyset \\
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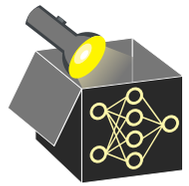
\end{aligned}
$$

STANDARD FANOVA - EXAMPLE REVISITED



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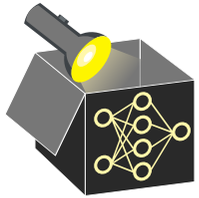

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- **Second-order component:**

$$\begin{aligned} g_{12}(x_1, x_2) &= \hat{f}_{\{1,2\};PD}(x_1, x_2) - g_\emptyset - g_1(x_1) - g_2(x_2) \\ &= 4 - 2x_1 + 0.3e^{x_2} + |x_1|x_2 - (2.95 + 0.3e) \\ &\quad - (-2x_1 + 0.5|x_1| + 0.75) - (0.3e^{x_2} + 0.5x_2 - 0.3e + 0.05) \\ &= |x_1|x_2 - 0.5|x_1| - 0.5x_2 + 0.25 \end{aligned}$$

$\Rightarrow$ All components shifted to have mean 0

$\Rightarrow$ Parts of $|x_1|x_2$, which intuitively seems to be the "interaction term", is attributed to the main effects (correctly, depends on distribution!)

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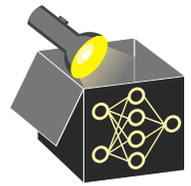
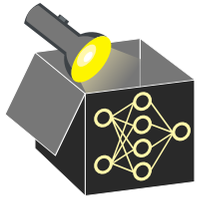


ESTIMATE FANOVA IN PRACTICE



Main part: Calculate all PD-functions $\rightarrow 2^p$ **many PD-functions**
Estimation of a single PD-function: **Sampling**
(so-called **Monte-Carlo integration**)

- Same idea as for PDPs: Fix **grid values** for features x_S
  Here: Same grid for all features over the whole algorithm
- Estimate integral by sampling: for grid value $\mathbf{x}_S^*$:

$$\hat{f}_{S,PD}(\mathbf{x}_S^*) = \mathbb{E}_{\mathbf{X}_{-S}}\left[\hat{f}(\mathbf{x}_S^*, \mathbf{X}_{-S})\right] \approx \frac{1}{n}\sum_{i=1}^{n}\hat{f}(\mathbf{x}_S^*, \mathbf{x}_{-S}^{(i)})$$

- Or: for each grid value $\mathbf{x}_S^*$, sample only $n_s < n$ many random samples (e.g. sampling uniformly)

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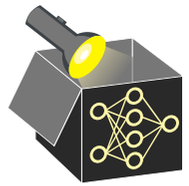

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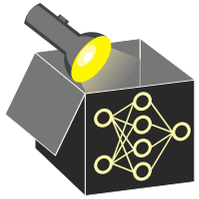
VARIANCE DECOMPOSITION - WHY "FUNCTIONAL ANOVA"?

- Decomposition of $\hat{f}(\mathbf{x})$ allows for "functional analysis of variance" (fANOVA)



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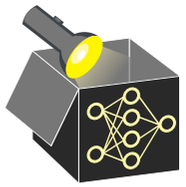

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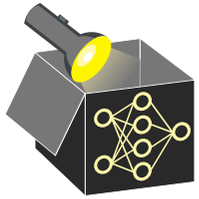

- Decomposition of $\hat{f}(\mathbf{x})$ allows for "functional analysis of variance" (fANOVA)
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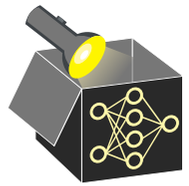

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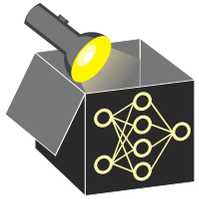

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- In other words: Single components uncorrelated (see later)

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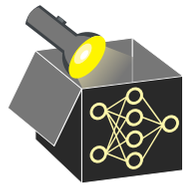

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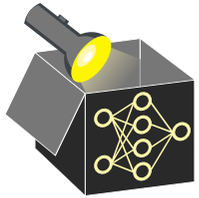
VARIANCE DECOMPOSITION - WHY "FUNCTIONAL ANOVA"?



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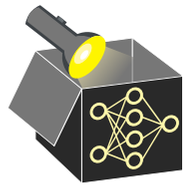
$\rightarrow$ **Sobol index**: Fraction of variance explained by some component $g_V(\mathbf{x}_V)$:

$$S_V = \frac{\text{Var}\left[g_V(\mathbf{x}_V)\right]}{\text{Var}\left[\hat{f}(\mathbf{x})\right]}$$

$\rightsquigarrow$ Usable as importance measure of component $g_V(\mathbf{x}_V)$

VARIANCE DECOMPOSITION - WHY "FUNCTIONAL ANOVA"?



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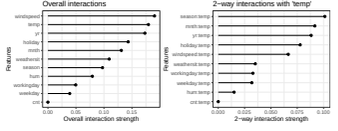
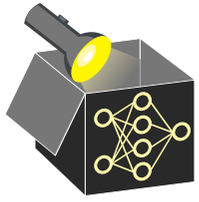
Interpretable Machine Learning

Friedman's H-Statistic



Learning goals

- Friedman's H-statistic with two purposes:
- Measure general *k*-way interactions between arbitrary features
- Measure a single feature's overall interaction strength



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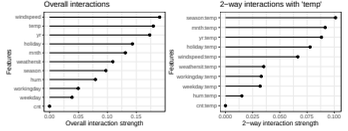
Functional Decompositions
Friedman's H-Statistic



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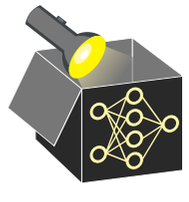

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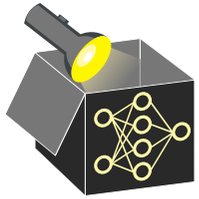
Friedman and Popescu (2008)

2-way interaction:

- Two features j and k do not interact, if their 2-way interaction component in functional decomposition $g_{\{j,k\}}$ is 0



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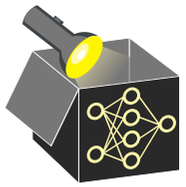

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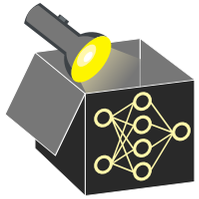

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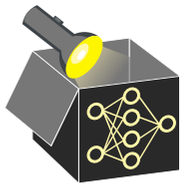

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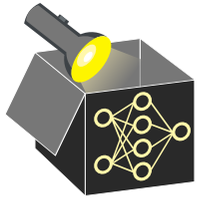

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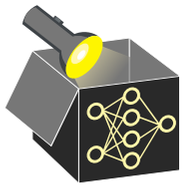

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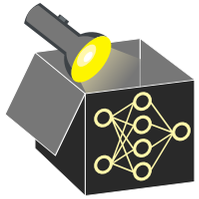

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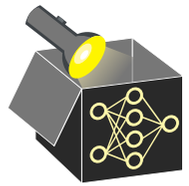

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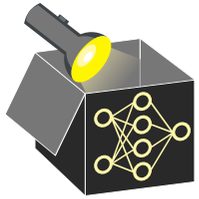

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- This means: There are interactions
  $\Leftrightarrow$ Every possible decomposition must contain some non-zero term $g_{\{j,k\}}(x_j, x_k)$

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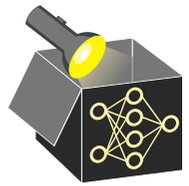

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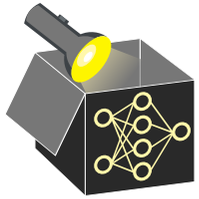

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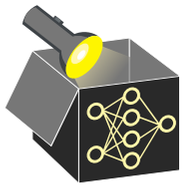

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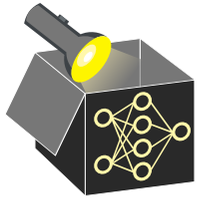

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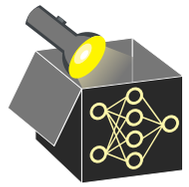

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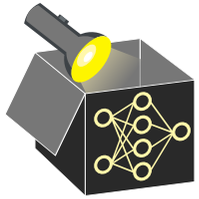

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$$\hat{f}(x_1, x_2, x_3) = -2x_1 - 2\sin(x_3) + |x_1|x_2 - \sin(x_2 x_3) + 1$$

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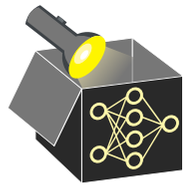

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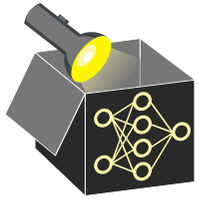

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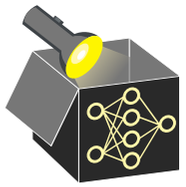

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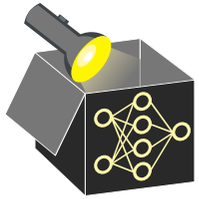
$$\begin{aligned}\hat{f}^c_{\{ijk\},PD}(x_i, x_j, x_k) =& \hat{f}^c_{\{ij\},PD}(x_i, x_j) + \hat{f}^c_{\{ik\},PD}(x_i, x_k) + \hat{f}^c_{\{jk\},PD}(x_j, x_k) \\ & - \hat{f}^c_{i,PD}(x_i) - \hat{f}^c_{j,PD}(x_j) - \hat{f}^c_{k,PD}(x_k)\end{aligned}$$

IDEA

Friedman and Popescu (2008)



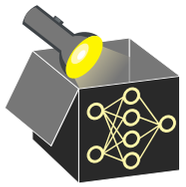
k-**way interaction:**

- **Analogous** for general *k*-way interactions between features $S = \{i_1, i_2, \ldots, i_k\}$: No *k*-way interaction, if

$$\hat{f}_{S,PD}(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = \sum_{V \subsetneq S} g_V(\mathbf{x}_V) = \sum_{\substack{V \subseteq S \\ |V|<k}} g_V(\mathbf{x}_V)$$

IDEA

POPESCU_2008



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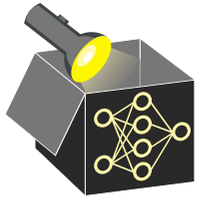

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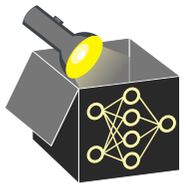
POPESCU_2008



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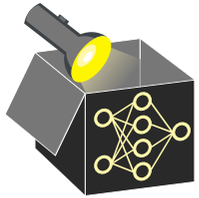

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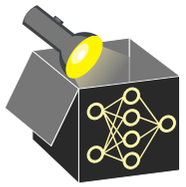
IDEA

POPESCU_2008



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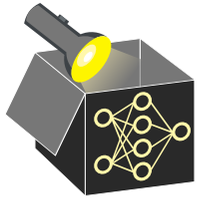

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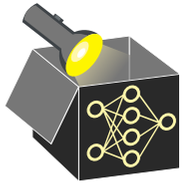

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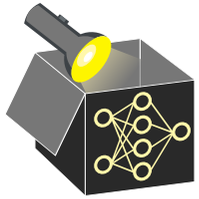

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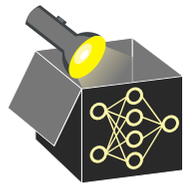

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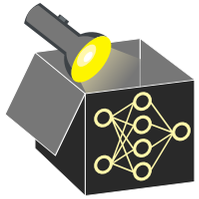

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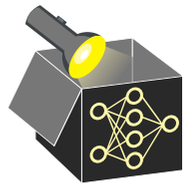

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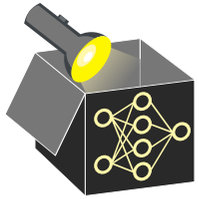

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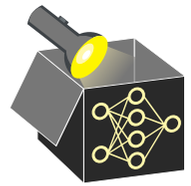

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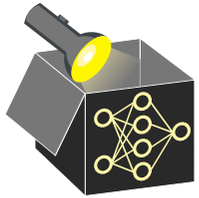

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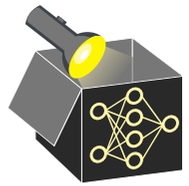

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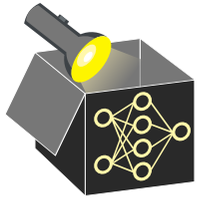

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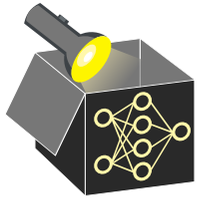

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$$= \frac{\mathbb{E}\left[\left(|x_1|x_2 - 0.5|x_1| - 0.5x_2 + 0.25\right)^2\right]}{\mathbb{E}\left[\left(1.05 - 2x_1 + 0.3e^{x_2} + |x_1|x_2 - 0.3e\right)^2\right]} > 0$$

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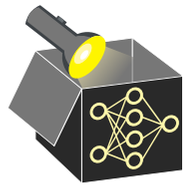

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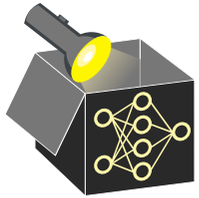
$$= \frac{\mathbb{E}\left[\left(|x_1|x_2 - 0.5|x_1| - 0.5x_2 + 0.25\right)^2\right]}{\mathbb{E}\left[\left(1.05 - 2x_1 + 0.3e^{x_2} + |x_1|x_2 - 0.3e\right)^2\right]} > 0$$

3-WAY INTERACTION STRENGTH



- Same idea as for 2-way, but different formula (see before):

$$\hat{f}^c_{\{ijk\},PD}(x_i, x_j, x_k) = \hat{f}^c_{\{ij\},PD}(x_i, x_j) + \hat{f}^c_{\{ik\},PD}(x_i, x_k) + \hat{f}^c_{\{jk\},PD}(x_j, x_k) - \hat{f}^c_{i,PD}(x_i) - \hat{f}^c_{j,PD}(x_j) - \hat{f}^c_{k,PD}(x_k)$$

$\Rightarrow$ H-statistic for a 3-way interaction between features i, j and k:

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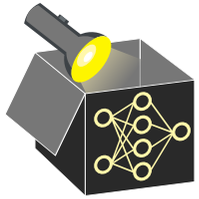

- Measure overall strength of interactions between feature j and all other features

$\Rightarrow$ **H-statistic** analogous to 2-way interaction:

$$H_j^2 = \frac{\text{Var}\left[\hat{f}^c(\mathbf{X}) - \hat{f}^c_{j,PD}(X_j) - \hat{f}^c_{-j,PD}(\mathbf{X}_{-j})\right]}{\text{Var}\left[\hat{f}^c(\mathbf{X})\right]}$$

$$= \frac{\sum_{i=1}^n \left(\hat{f}^c(\mathbf{x}^{(i)}) - \hat{f}^c_{j,PD}(x_j^{(i)}) - \hat{f}^c_{-j,PD}(\mathbf{x}^{(i)}_{-j})\right)^2}{\sum_{i=1}^n \left(\hat{f}^c(\mathbf{x}^{(i)})\right)^2}$$

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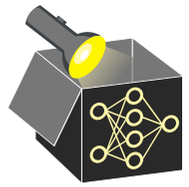

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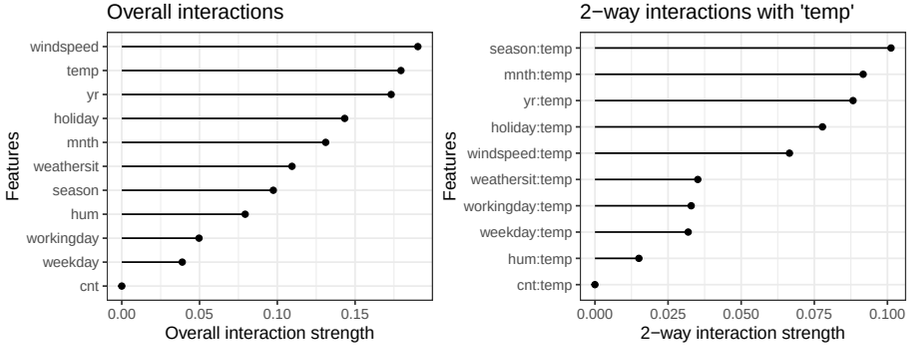
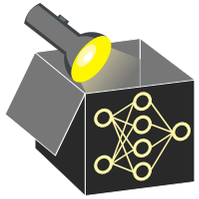
H-STATISTIC: EXAMPLE

Measure interactions of a random forest for the bike data set





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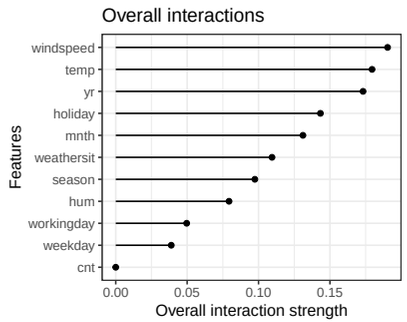

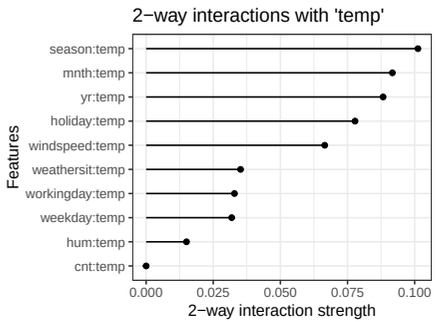

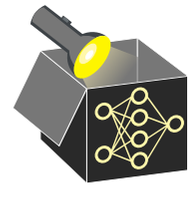

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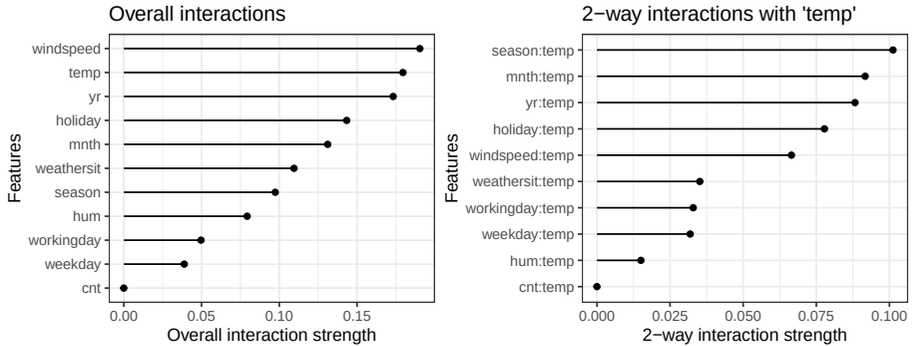

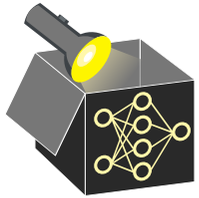

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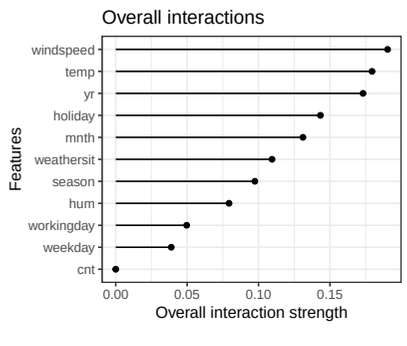

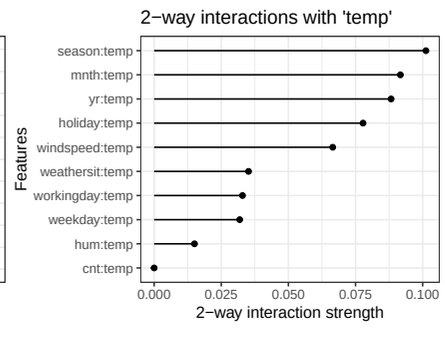

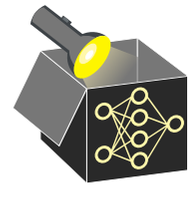

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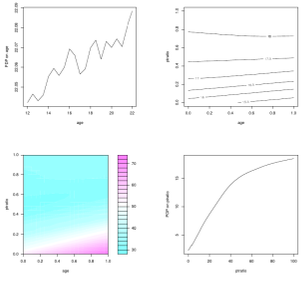
Interpretable Machine Learning

Theory of Standard fANOVA

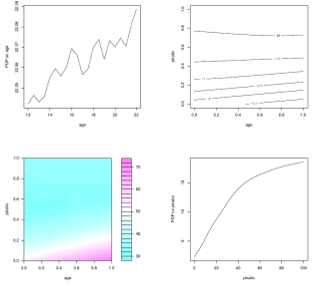


Learning goals

- Properties of classical fANOVA, reason for its popularity
- Equivalent definition of classical fANOVA
- Understand the role constraints play for any functional decomposition

Interpretable Machine Learning

Functional Decompositions
Theory of Standard fANOVA



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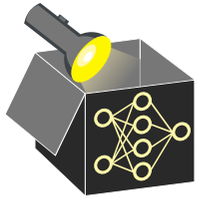

- Remember: Functional decomposition in general not unique
- **Standard fANOVA** only one possible approach
- Example:

$$\begin{aligned}\hat{f}(x_1, x_2) &= 4 - 2x_1 + 0.3e^{x_2} + |x_1|x_2 \\ &= \underbrace{2.95 + 0.3e.}_{g_\emptyset} + \underbrace{-2x_1 + 0.5|x_1| + 0.75}_{g_1(x_1)} \\ &\quad + \underbrace{0.3e^{x_2} + 0.5x_2 - 0.3e + 0.05}_{g_2(x_2)} + \underbrace{|x_1|x_2 - 0.5|x_1| - 0.5x_2 + 0.25}_{g_{1,2}(x_1,x_2)}\end{aligned}$$

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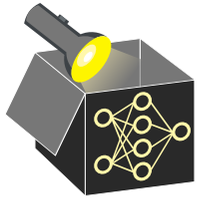
CONSTRAINTS FOR STANDARD FANOVA ALGORITHM



Theorem

Features independent $\implies$ *The components defined by standard fANOVA fulfill the so-called vanishing conditions:*

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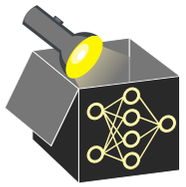

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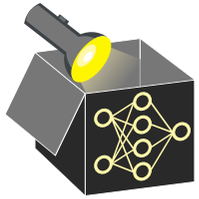

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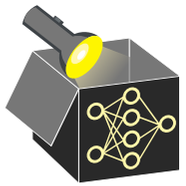

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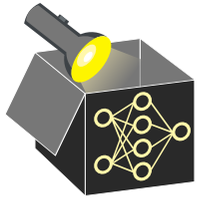

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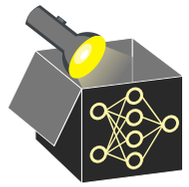

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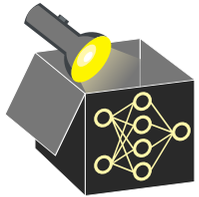
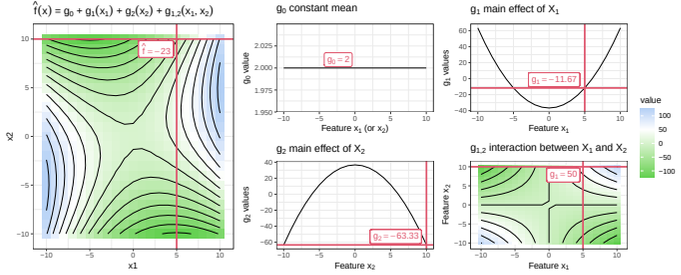
EXAMPLES REVISITED



Example: $\hat{f}(\mathbf{x}) = 2 + x_1^2 - x_2^2 + x_1 \cdot x_2$ (e.g., for $x_1 = 5$ and $x_2 = 10$ we have $\hat{f}(\mathbf{x}) = -23$)

- Computation of components using feature values $x_1 = x_2 = (-10, -9, \ldots, 10)^\top$ gives:



For $x_1 = 5$ and $x_2 = 10$:

- $g_\emptyset = 2$
- $g_1(x_1) = -9.67$
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- $g_{1,2}(x_1, x_2) = 50$

$\Rightarrow \hat{f}(\mathbf{x}) = -23$

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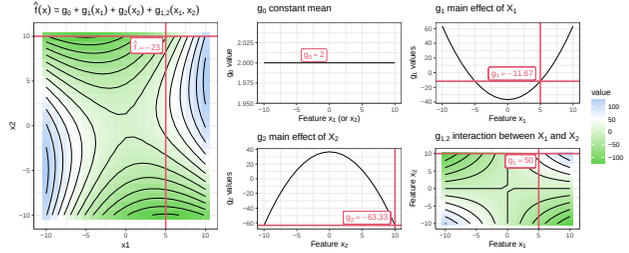

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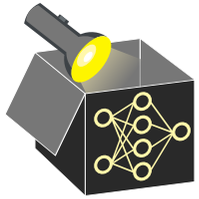

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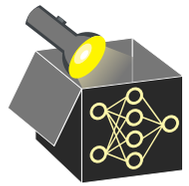

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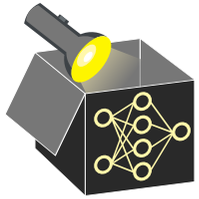

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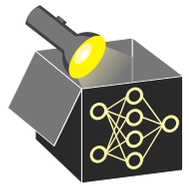

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Interpretable Machine Learning – 4 / 5

CONSTRAINTS: EQUIVALENT CHARACTERIZATION

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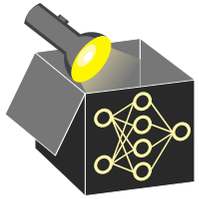

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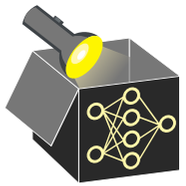

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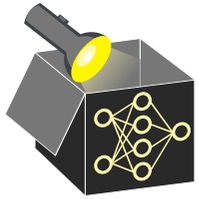

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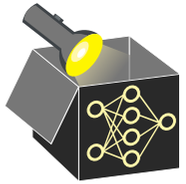

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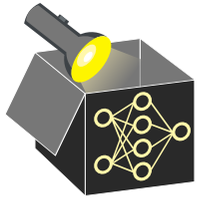

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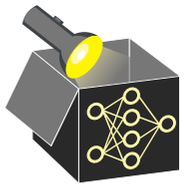

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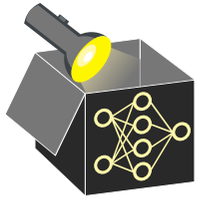

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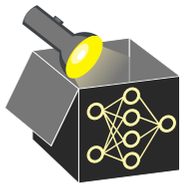

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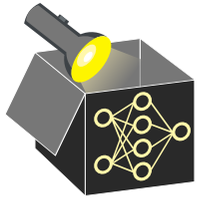

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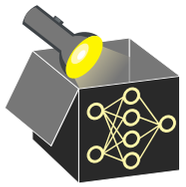

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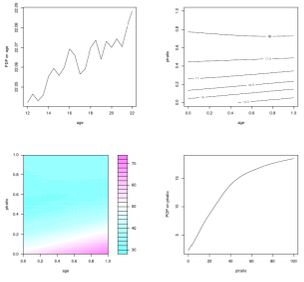
Interpretable Machine Learning

Functional Decompositions: Further Methods



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- Limitations of classical fANOVA
- Alternatives: Generalized fANOVA and ALE
- Advantages and relevance of functional decompositions

Interpretable Machine Learning

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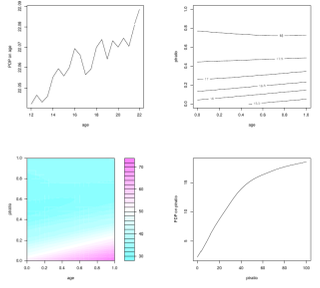

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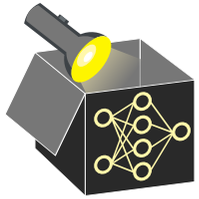
- Standard fANOVA builds on PD-functions
- *Remember:* Problems of PDPs for **correlated** / **dependent features**
- Here: Dependent features $\implies$ Standard fANOVA does NOT fulfill vanishing conditions



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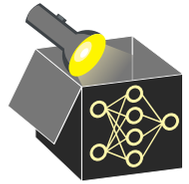

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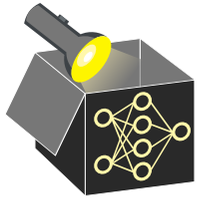

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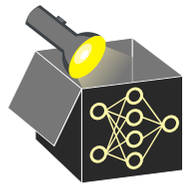

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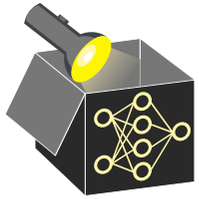

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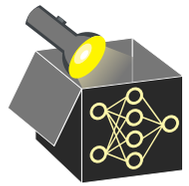

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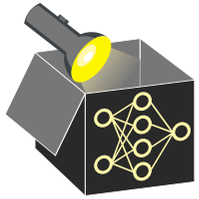

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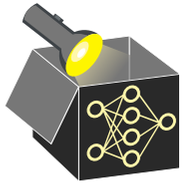

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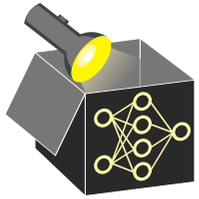

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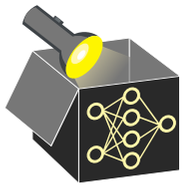

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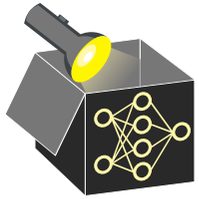

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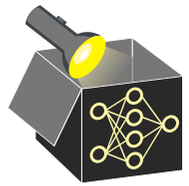

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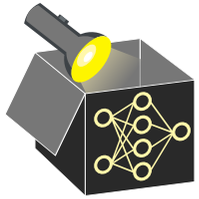

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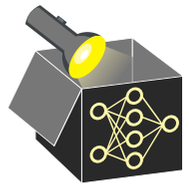

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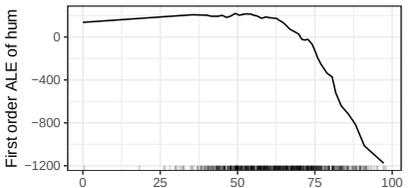

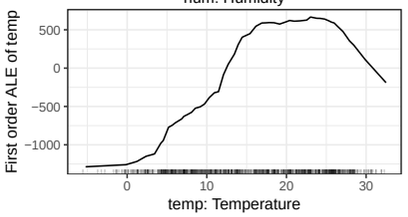

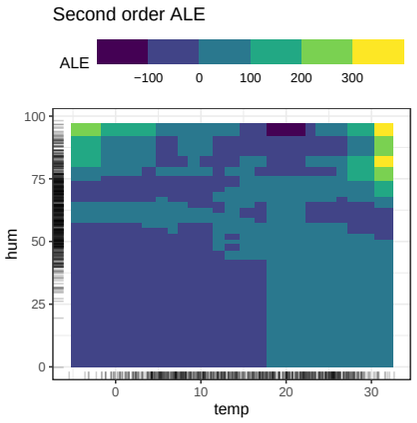

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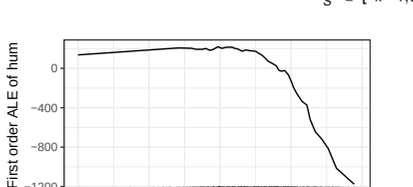

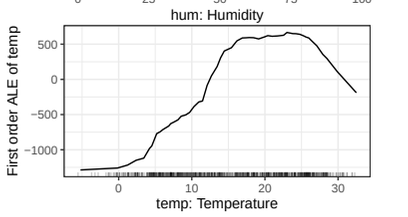

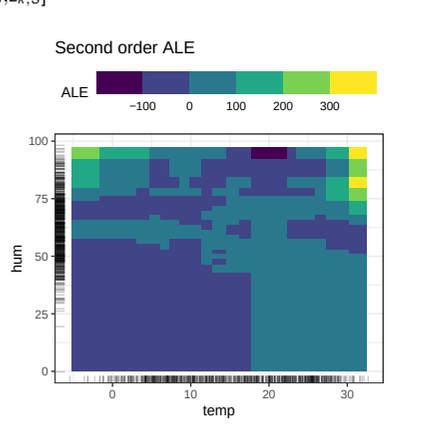

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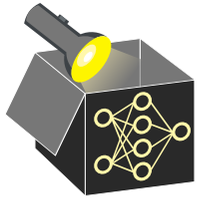

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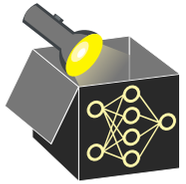

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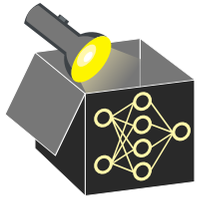

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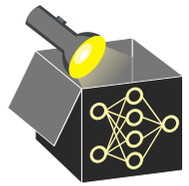

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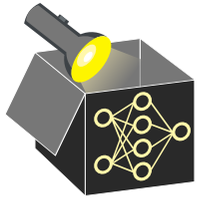

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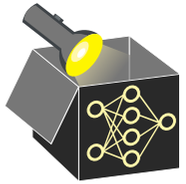

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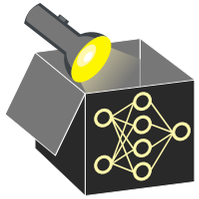

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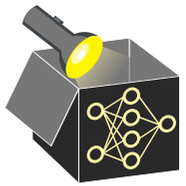

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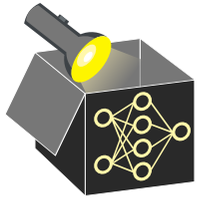

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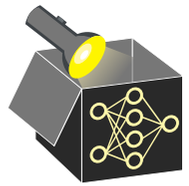

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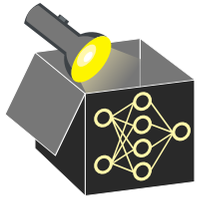

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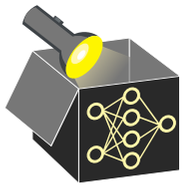

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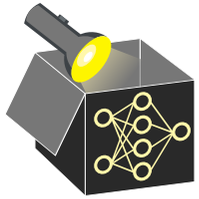

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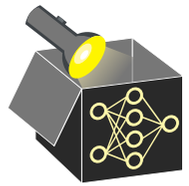

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