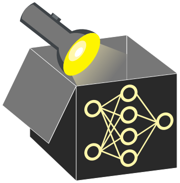
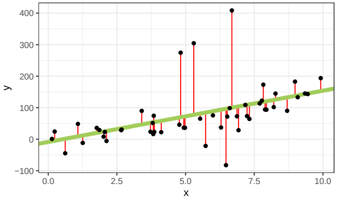


Interpretable Machine Learning
Linear Regression Model

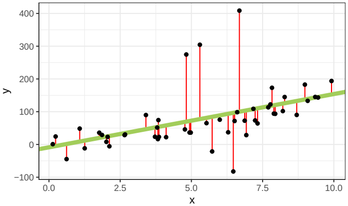


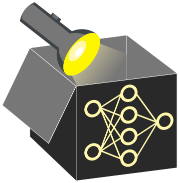
Learning goals

- LM basics and assumptions
- Interpretation of main effects in LM
- What are significant features?

LINEAR REGRESSION

$$y = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_p x_p + \epsilon = \mathbf{x}^\top \boldsymbol{\theta} + \epsilon$$





- y: target / output
- ϵ: remaining error / residual
- θ_j: weight of input feature x_j (intercept θ_0)
  $\rightsquigarrow$ model consists of $p + 1$ weights

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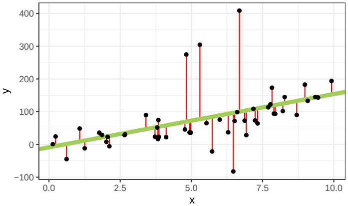

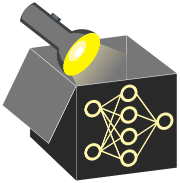

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Properties and assumptions ▶ "Faraway, Ch. 7" 2002

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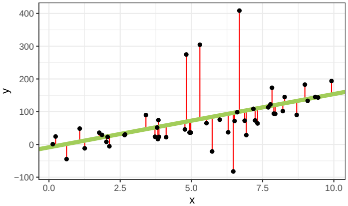

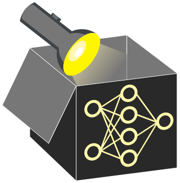

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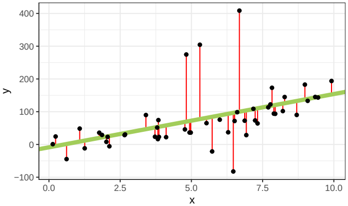

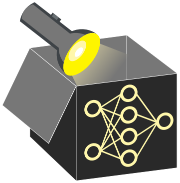

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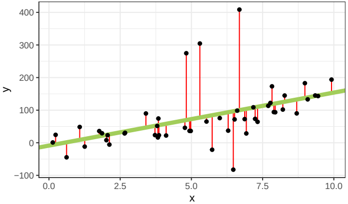

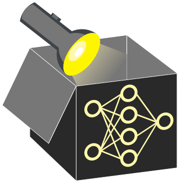

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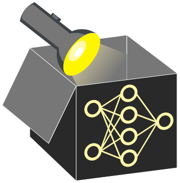

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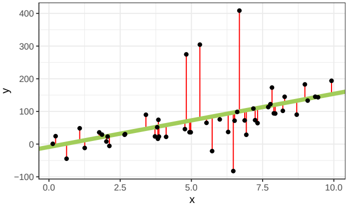

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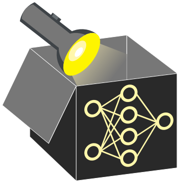
- Features x_j independent from error term ϵ
- No or little multicollinearity (i.e., no strong feature correlations)

LINEAR REGRESSION - INTERPRETATION



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Interpretation of weights (**feature effects**) depend on type of feature:

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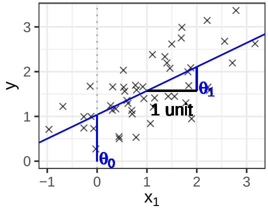

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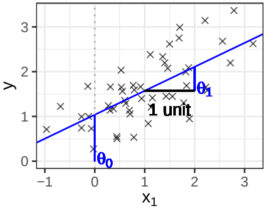

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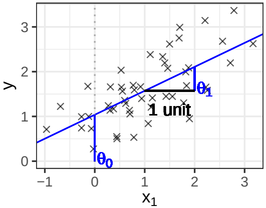
  - Left out cat. is reference ($\hat{=}$ dummy encoding)

  $\rightsquigarrow$ Interpretation: Outcome changes by $\theta_{j,i}$ for category i compared to reference cat., c.p.



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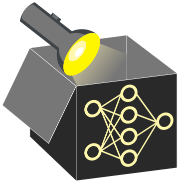

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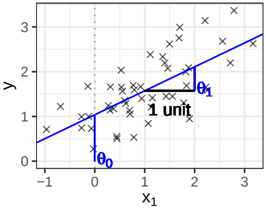
- **Intercept** θ_0: Expected outcome if all feature values are set to 0



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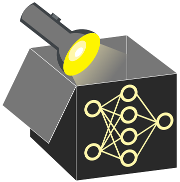

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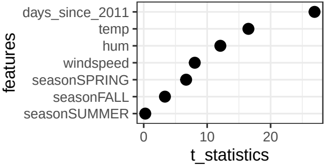
Feature importance:

- Absolute **t-statistic** value: $\hat{\theta}_j$ scaled with standard error ($SE(\hat{\theta}_j) \hat{=}$ reliability of estimate)

$$|t_{\hat{\theta}_j}| = \left| \frac{\hat{\theta}_j}{SE(\hat{\theta}_j)} \right|$$



- High t-values $\Rightarrow$ important (significant) feat.

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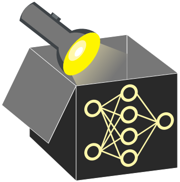

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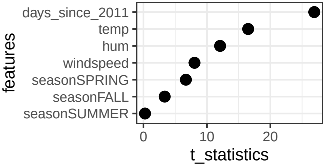
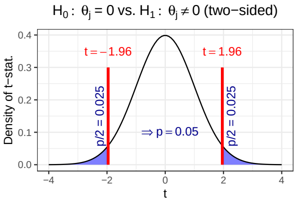
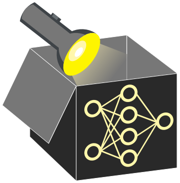


- High t-values $\Rightarrow$ important (significant) feat.
- **p-value**: probability of obtaining a more extreme test statistic assuming H_0 is correct (here: $\theta_j = 0$, i.e., feat. j not significant) $\rightsquigarrow$ High $|t| \Rightarrow$ small p-val. (speak against H_0)



EXAMPLE: LIN. REGRESSION - MAIN EFFECTS



Bike data: predict no. of rented bikes using 4 numeric, 1 cat. feat. (season)

$$
\begin{aligned}
\hat{y} = & \hat{\theta}_0 + \hat{\theta}_1 \mathbb{1}_{x_{season}=SPRING} + \\
& \hat{\theta}_2 \mathbb{1}_{x_{season}=SUMMER} + \\
& \hat{\theta}_3 \mathbb{1}_{x_{season}=FALL} + \hat{\theta}_4 x_{temp} + \\
& \hat{\theta}_5 x_{hum} + \hat{\theta}_6 x_{windspeed} + \\
& \hat{\theta}_7 x_{days\_since\_2011}
\end{aligned}
$$

	Weights	SE	t-stat.	p-val.
(Intercept)	3229.3	220.6	14.6	0.00
seasonSPRING	862.0	129.0	6.7	0.00
seasonSUMMER	41.6	170.2	0.2	0.81
seasonFALL	390.1	116.6	3.3	0.00
temp	120.5	7.3	16.5	0.00
hum	-31.1	2.6	-12.1	0.00
windspeed	-56.9	7.1	-8.0	0.00
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- **Intercept**: If all feature values are 0 (and season is `WINTER` $\hat{=}$ reference cat.), the expected number of bike rentals is $\hat{\theta}_0 = 3229.3$

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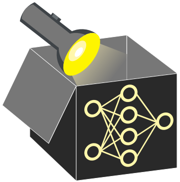

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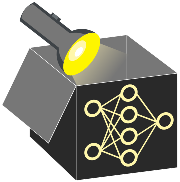

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- **Numerical**: Rentals increase by $\hat{\theta}_4 = 120.5$ if `temp` increases by 1 °C, c.p.