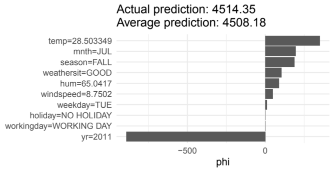
Interpretable Machine Learning

Shapley Values for Local Explanations



Learning goals

- See model predictions as a cooperative game
- Transfer the Shapley value concept from game theory to machine learning

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Shapley

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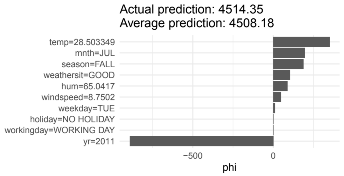

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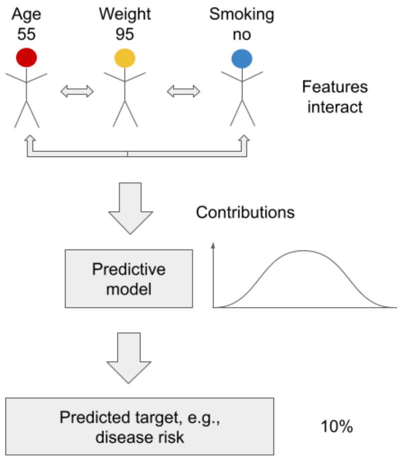
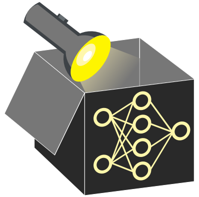
FROM GAME THEORY TO MACHINE LEARNING

- Model prediction depends on feature interactions for a specific observation
- **Goal:** Decompose prediction into **individual feature contributions**
- **Idea:** Treat features as players jointly producing a prediction
- How to fairly assign credit to features? $\Rightarrow$ Shapley values





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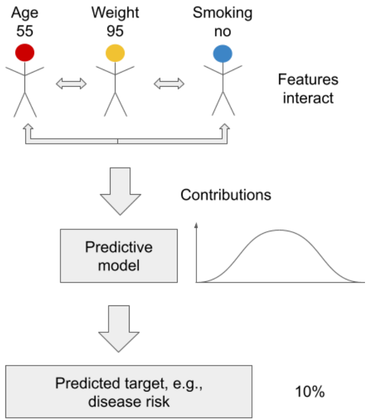

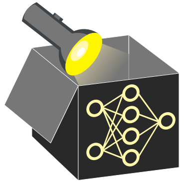

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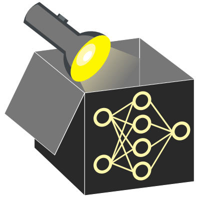

- **Game:** Predict $\hat{f}(x_1, x_2, \ldots, x_p)$ for a single observation $\mathbf{x}$
- **Players:** Features $x_j, j \in \{1, \ldots, p\}$, cooperate to produce a prediction
- **Value function:** Defines payout of coalition $S \subseteq P$ for observation $\mathbf{x}$ by

$$v(S) = \hat{f}_S(\mathbf{x}_S) - \hat{f}_\emptyset, \text{ where}$$

  - $\hat{f}_S : \mathcal{X}_S \mapsto \mathcal{Y}$ is the PD function $\hat{f}_S(\mathbf{x}_S) := \int \hat{f}(\mathbf{x}_S, X_{-S})\, d\mathbb{P}_{X_{-S}}$
    $\rightsquigarrow$ "Removes" features in $-S$ by marginalizing, keeping $\hat{f}$ fixed
  - Mean prediction $\hat{f}_\emptyset := \mathbb{E}_\mathbf{x}(\hat{f}(\mathbf{x}))$ is subtracted to ensure $v(\emptyset) = 0$
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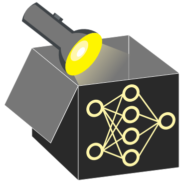

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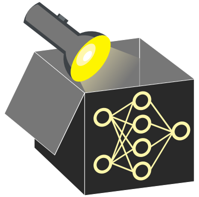

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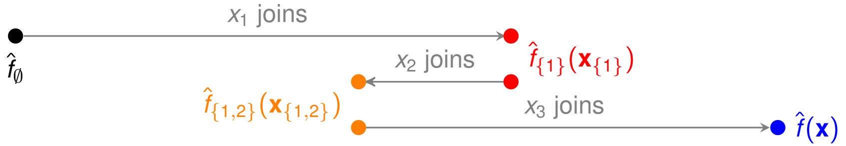
- **Marginal contribution of feature j joining coalition** S ($\hat{f}_\emptyset$ cancels):

$$\Delta(j, S) = v(S \cup \{j\}) - v(S) = \hat{f}_{S \cup \{j\}}(\mathbf{x}_{S \cup \{j\}}) - \hat{f}_S(\mathbf{x}_S)$$

- **Example (3 features):** Feature contributions for joining order $x_1 \to x_2 \to x_3$ toward total payout $v(P) = \hat{f}(\mathbf{x}) - \hat{f}_\emptyset$, each step reflects a marginal contribution



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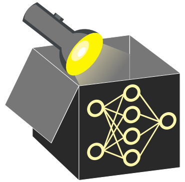

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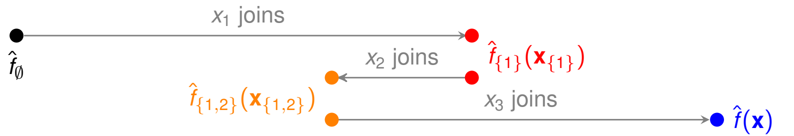
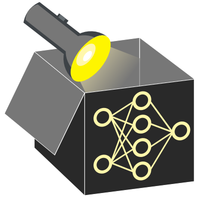


SHAPLEY VALUE - DEFINITION

Shapley (1953) Strumbelj et al. (2014)



Order definition: Shapley value $\phi_j(\mathbf{x})$ quantifies contribution of x_j via

$$\phi_j(\mathbf{x}) = \frac{1}{|P|!} \sum_{\tau \in \Pi} \underbrace{\hat{f}_{S_j^\tau \cup \{j\}}(\mathbf{x}_{S_j^\tau \cup \{j\}}) - \hat{f}_{S_j^\tau}(\mathbf{x}_{S_j^\tau})}_{\Delta(j, S_j^\tau) \text{ marginal contribution of feature } j}$$

- **Interpretation:** $\phi_j(\mathbf{x})$ quantifies how much feature x_j contributes to the difference between $\hat{f}(\mathbf{x})$ and the mean prediction $\hat{f}_\emptyset$
  $\rightsquigarrow$ Marginal contributions and Shapley values can be negative
- **Exact computation of ϕ_j:** Using PD function $\hat{f}_S(\mathbf{x}_S) = \frac{1}{n}\sum_{i=1}^n \hat{f}(\mathbf{x}_S, \mathbf{x}_{-S}^{(i)})$ yields

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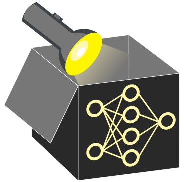
SHAPLEY_1953 STRUMBELJ_2014



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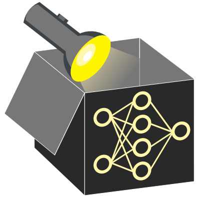

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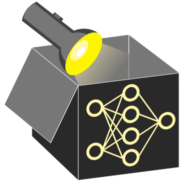

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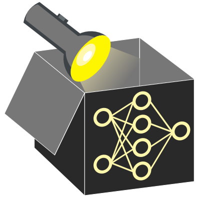

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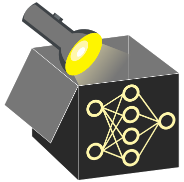

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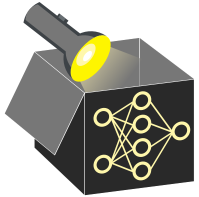

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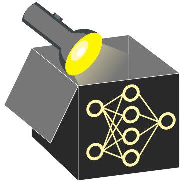

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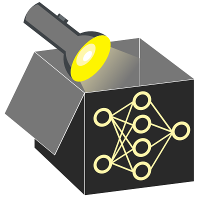
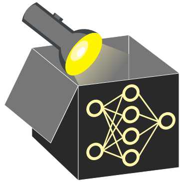
APPROXIMATION ALGORITHM ▸ Strumbelj et al. (2014)



Estimate Shapley value ϕ_j of observation $\mathbf{x}$ for feature j:

- **Input:** $\mathbf{x}$ obs. of interest, j feat. of interest, $\hat{f}$ model, $\mathcal{D}$ data, M iterations

APPROXIMATION ALGORITHM ▸ STRUMBELJ_2014



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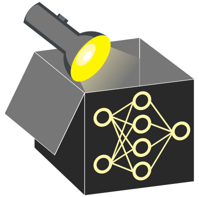

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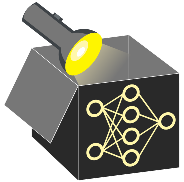

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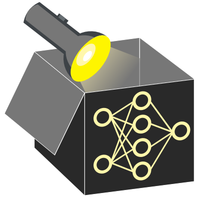

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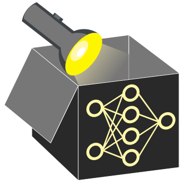

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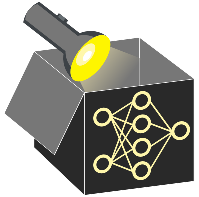

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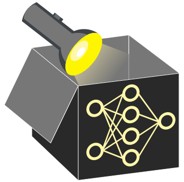

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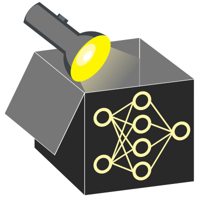

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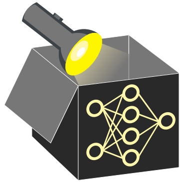
APPROXIMATION ALGORITHM ▸ STRUMBELJ_2014



Estimate Shapley value ϕ_j of observation $\mathbf{x}$ for feature j:

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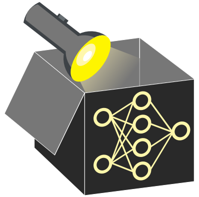
APPROXIMATION ALGORITHM ▸ Strumbelj et al. (2014)



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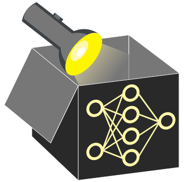

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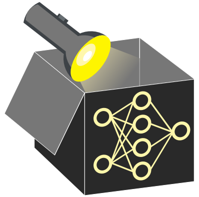

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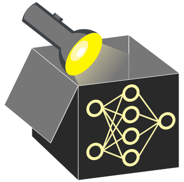

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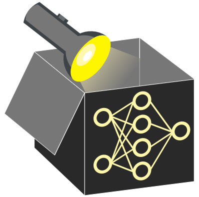

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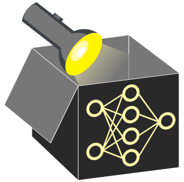
APPROXIMATION ALGORITHM ▸ STRUMBELJ_2014



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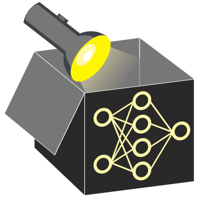
APPROXIMATION ALGORITHM ▸ Strumbelj et al. (2014)



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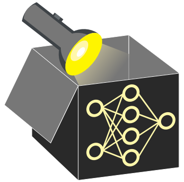
   5. Compute marginal contribution $\Delta(j, S_m) = \hat{f}(\mathbf{x}_{+j}^{(m)}) - \hat{f}(\mathbf{x}_{-j}^{(m)})$

APPROXIMATION ALGORITHM ▸ STRUMBELJ_2014



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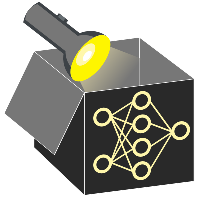

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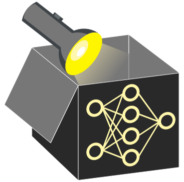
APPROXIMATION ALGORITHM ▸ STRUMBELJ_2014



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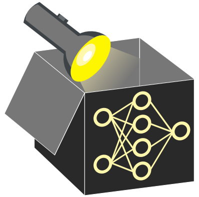

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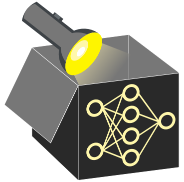
APPROXIMATION ALGORITHM ▸ STRUMBELJ_2014



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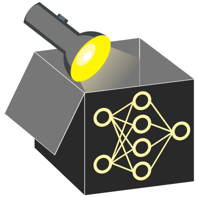
SHAPLEY VALUE APPROXIMATION - ILLUSTRATION



Definition

x: obs. of interest

x with feature values in $\mathbf{x}_{S_m}$ (other are replaced)

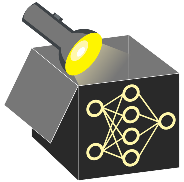
$$\phi_j(\mathbf{x}) = \frac{1}{M}\sum_{m=1}^{M}\left[\hat{f}(\mathbf{x}_{+j}{}^{(m)}) - \hat{f}(\mathbf{x}_{-j}{}^{(m)})\right]$$

x with feature values in $\mathbf{x}_{S_m \cup \{j\}}$

	Temperature	Humidity	Windspeed	Year
x	10.66	56	11	2012
x_{+j}	10.66	56	$random : z^{(m)}_{windspeed}$	2012
x_{-j}	10.66	56	$random : z^{(m)}_{windspeed}$	$random : z^{(m)}_{year}$

j

SHAPLEY VALUE APPROX. - ILLUSTRATION



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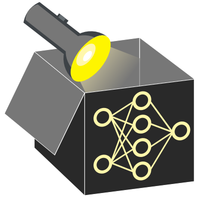


Definition

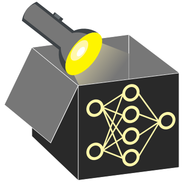
Contribution of feature j to coalition S_m

$$\phi_j(\mathbf{x}) = \frac{1}{M}\sum_{m=1}^{M} \underbrace{\left[\hat{f}(\mathbf{x}_{+j}{}^{(m)}) - \hat{f}(\mathbf{x}_{-j}{}^{(m)})\right]}_{:= \Delta(j, S_m)}$$

- $\Delta(j, S_m) = \hat{f}(\mathbf{x}_{+j}^{(m)}) - \hat{f}(\mathbf{x}_{-j}^{(m)})$ is marginal contribution of feature j to coalition S_m
- Here: Feature *year* contributes +700 bike rentals if it joins coalition $S_m = \{temp, hum\}$

	Temperature	Humidity	Windspeed	Year	Count	
x	10.66	56	11	2012		
x_{+j}	10.66	56	$random : z^{(m)}_{windspeed}$	2012	5600	700
x_{-j}	10.66	56	$random : z^{(m)}_{windspeed}$	$random : z^{(m)}_{year}$	4900	
				j	$\hat{f}$	$\Delta(j, S_m)$ marginal contribution

SHAPLEY VALUE APPROX. - ILLUSTRATION



Definition

Contribution of feature j to coalition S_m

$$\phi_j(\mathbf{x}) = \frac{1}{M}\sum_{m=1}^{M} \underbrace{\left[\hat{f}(\mathbf{x}_{+j}{}^{(m)}) - \hat{f}(\mathbf{x}_{-j}{}^{(m)})\right]}_{:= \Delta(j, S_m)}$$

- $\Delta(j, S_m) = \hat{f}(\mathbf{x}_{+j}^{(m)}) - \hat{f}(\mathbf{x}_{-j}^{(m)})$ is marginal contribution of feature j to coalition S_m
- Here: Feature *year* contributes +700 bike rentals if it joins coalition $S_m = \{temp, hum\}$

	Temperature	Humidity	Windspeed	Year	Count	
x	10.66	56	11	2012		
x_{+j}	10.66	56	$random : z^{(m)}_{windspeed}$	2012	5600	700
x_{-j}	10.66	56	$random : z^{(m)}_{windspeed}$	$random : z^{(m)}_{year}$	4900	
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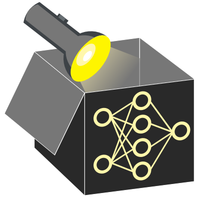
SHAPLEY VALUE APPROXIMATION - ILLUSTRATION



Definition

average the contributions of feature j

$$\phi_j(\mathbf{x}) = \frac{1}{M}\sum_{m=1}^{M}\left[\hat{f}(\mathbf{x}_{+j}^{(m)}) - \hat{f}(\mathbf{x}_{-j}^{(m)})\right]$$

- Compute marginal contribution of feature j towards the prediction across all randomly drawn feature coalitions $S_1, \dots, S_m$
- Average all M marginal contributions of feature j
- Shapley value ϕ_j is the payout of feature j, i.e., how much feature *year* contributed to the overall prediction in bicycle counts of a specific observation $\mathbf{x}$

$$\begin{array}{ccccccc} m = & 1 & 2 & & M & & \text{Shapley value} \\ & \underbrace{700 \quad 503 \quad \cdots \quad 300}_{\Delta(j, S_m)} & & & & & \underbrace{501}_{\phi_j} \end{array}$$

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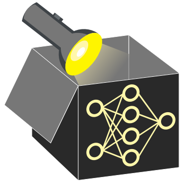

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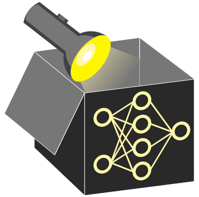
REVISITED: AXIOMS FOR FAIR ATTRIBUTIONS



We adapt the classic Shapley axioms to the setting of model predictions:

- **Efficiency**: Sum of Shapley values adds up to the centered prediction:

$$\sum_{j=1}^{p} \phi_j(\mathbf{x}) = \hat{f}(\mathbf{x}) - \mathbb{E}_{\mathbf{x}}[\hat{f}(\mathbf{x})]$$

  $\rightsquigarrow$ All predictive contribution is fully distributed among features
- **Symmetry**: Identical contributors receive equal value:

$$\hat{f}_{S\cup\{j\}}(\mathbf{x}_{S\cup\{j\}}) = \hat{f}_{S\cup\{k\}}(\mathbf{x}_{S\cup\{k\}})\ \forall S \subseteq P \setminus \{j,k\} \Rightarrow \phi_j = \phi_k$$

  $\rightsquigarrow$ Interaction effects are shared equitably
- **Dummy (Null Player)**: Irrelevant features receive zero attribution:

$$\hat{f}_{S\cup\{j\}}(\mathbf{x}_{S\cup\{j\}}) = \hat{f}_S(\mathbf{x}_S)\ \forall S \subseteq P \Rightarrow \phi_j = 0$$

  $\rightsquigarrow$ Shapley value is zero for unused features (e.g., trees or LASSO)
- **Additivity**: Attributions are additive across models:

$$\phi_j(v_1 + v_2) = \phi_j(v_1) + \phi_j(v_2)$$

  $\rightsquigarrow$ Enables combining Shapley values for model ensembles

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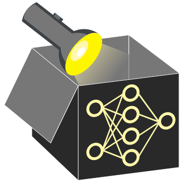

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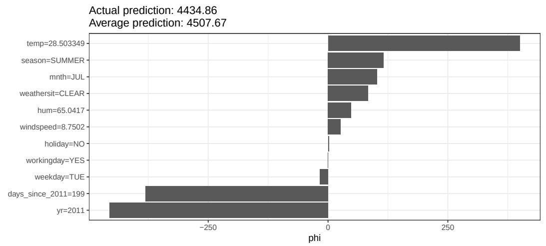
BIKE SHARING DATASET



- Shapley decomposition for a single prediction in bike sharing dataset
- Model prediction: $\hat{f}(\mathbf{x}^{(200)}) = \mathbf{4434.86}$ vs. dataset average: $\mathbb{E}_{\mathbf{x}}[\hat{f}(\mathbf{x})] = \mathbf{4507.67}$
- Total feature attribution: $\sum_j \phi_j =$ **-72.81**
  $\rightsquigarrow$ Explain downward shift from mean prediction
- Temperature (with value 28.5°C) is the strongest positive contributor: $+400$
- Features `yr = 2011` and `days_since_2011 = 199` strongly reduce prediction
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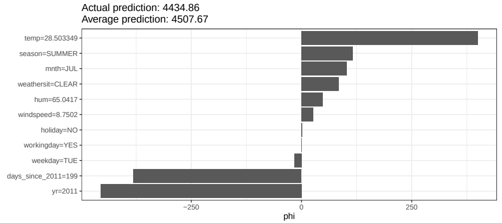

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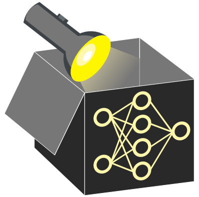
ADVANTAGES AND DISADVANTAGES



Advantages:

- **Strong theoretical foundation** from cooperative game theory
- **Fair attribution:** Prediction is additively distributed across features
  $\rightsquigarrow$ Easy to interpret for users
- **Contrastive explanations:** Quantify each feature's role in deviating from the average prediction

Disadvantages:

- **Computational cost:** Exact computation scales factorially with feature count
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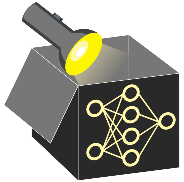

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