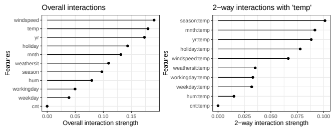
Interpretable Machine Learning

Friedman's H-Statistic

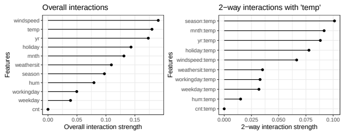


Learning goals

- Friedman's H-statistic with two purposes:
- Measure general k-way interactions between arbitrary features
- Measure a single feature's overall interaction strength

Interpretable Machine Learning

Functional Decompositions
Friedman's H-Statistic



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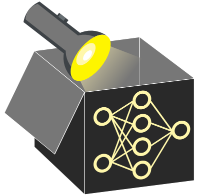
IDEA ▸ Friedman and Popescu (2008)



2-way interaction:

- Two features j and k do not interact, if their 2-way interaction component in functional decomposition $g_{\{j,k\}}$ is 0

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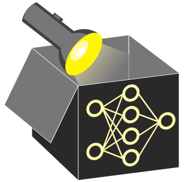

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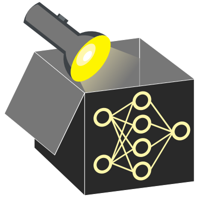

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$$\hat{f}_{\{jk\},PD}(x_j, x_k) = g_{\emptyset} + g_j(x_j) + g_k(x_k) + g_{\{j,k\}}(x_j, x_k)$$

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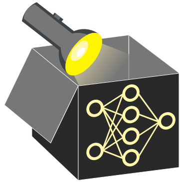

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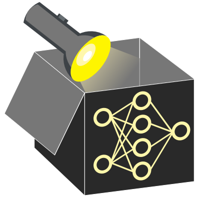

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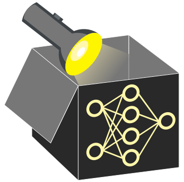

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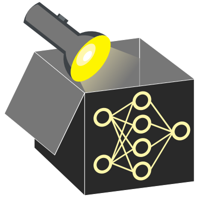

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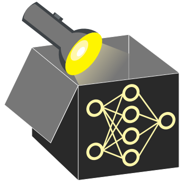

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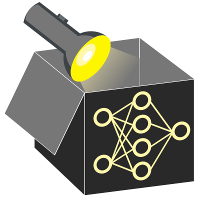

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$\Leftrightarrow$ Every possible decomposition must contain some non-zero term $g_{\{j,k\}}(x_j, x_k)$

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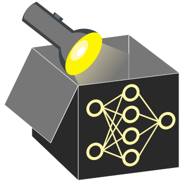

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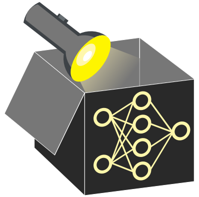

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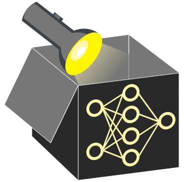

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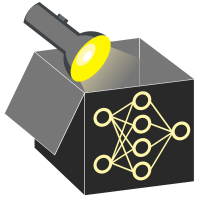

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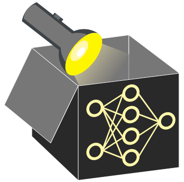

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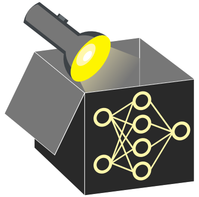

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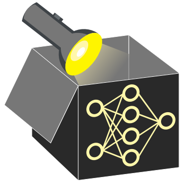

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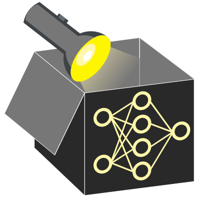

k-way interaction:

- **Analogous** for general k-way interactions between features $S = \{i_1, i_2, \ldots, i_k\}$: No k-way interaction, if

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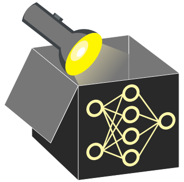

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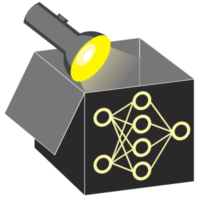

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Overall interaction:

- Question: Does feature j interact with any other feature at all?

$\Rightarrow$ H-statistic analogous to 2-way interactions, but for feature sets $S = \{j\}$ and $-S = \{1, \dots, p\} \setminus \{j\}$ instead of two single features:

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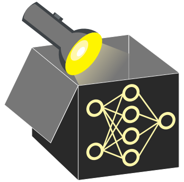

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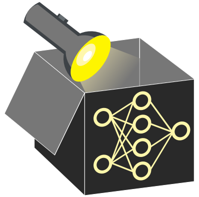

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$$\hat{f}_{S,PD}(x_{i_1}, x_{i_2}, \dots, x_{i_k}) = \sum_{V \subsetneq S} g_V(\mathbf{x}_V) = \sum_{\substack{V \subseteq S \\ |V| < k}} g_V(\mathbf{x}_V)$$

Overall interaction:

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$\Rightarrow$ H-statistic analogous to 2-way interactions, but for feature sets $S = \{j\}$ and $-S = \{1, \dots, p\} \setminus \{j\}$ instead of two single features:

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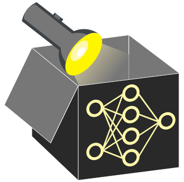
IDEA

POPESCU_2008



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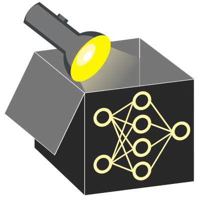

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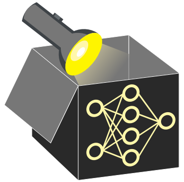

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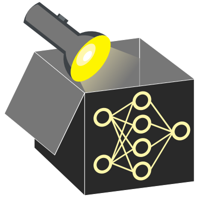

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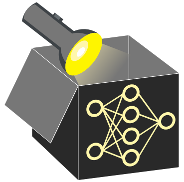

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$$\hat{f}(x_1, x_2) = 4 - 2x_1 + 0.3e^{x_2} + |x_1|x_2 \quad (x_1, x_2) \in [0, 1]^2$$
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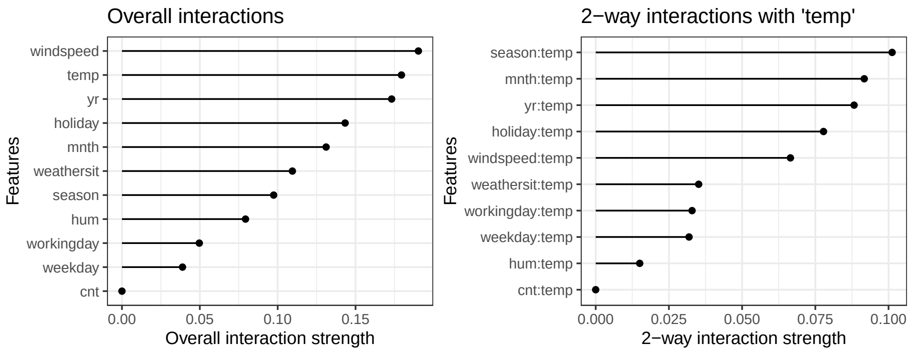
Measure interactions of a random forest for the bike data set



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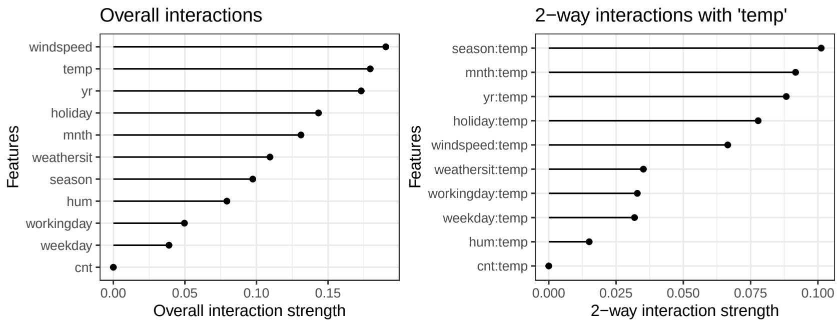

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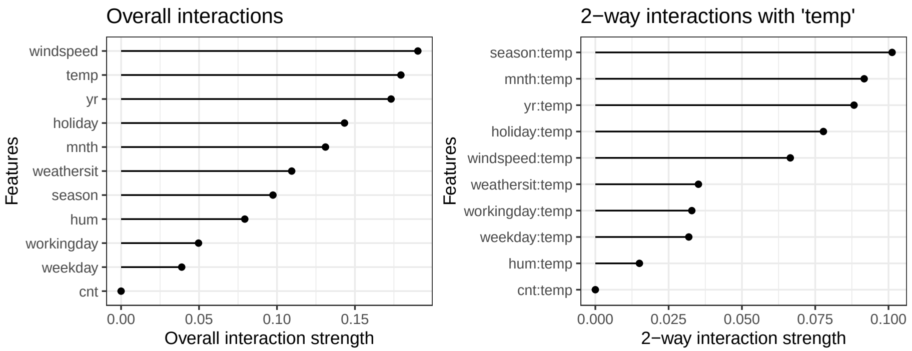

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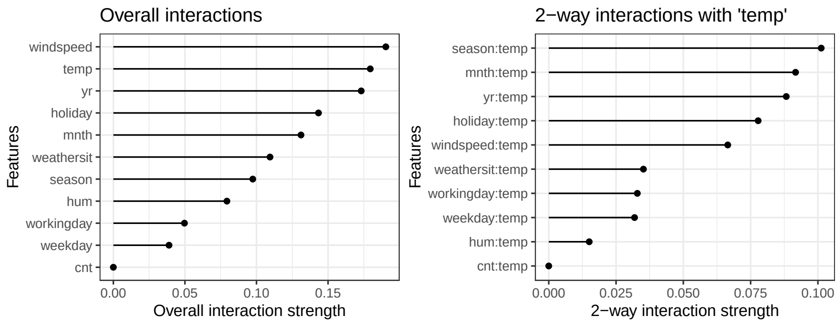

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