

# FTL FOR ONLINE LINEAR OPTIMIZATION

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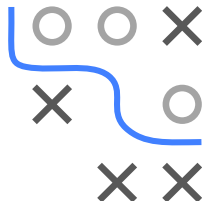
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- No matter how we choose the first action  $a_1^{\text{FTL}}$ , it will hold that FTL has a cumulative loss greater than (or equal)  $T - 3/2$ , while the best action in hindsight has a cumulative loss of  $-1/2$ .

- Thus, FTL's cumulative regret is at least  $T - 1$ , which is linearly growing in  $T$ .



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- Indeed, note that

$$\begin{aligned} a_{t+1}^{\text{FTL}} &= \arg \min_{a \in \mathcal{A}} \sum_{s=1}^t (a, z_s) = \arg \min_{a \in [-1, 1]} a \sum_{s=1}^t z_s \\ &= \begin{cases} -1, & \text{if } \sum_{s=1}^t z_s > 0, \\ 1, & \text{if } \sum_{s=1}^t z_s < 0, \\ \text{arbitrary}, & \text{if } \sum_{s=1}^t z_s = 0. \end{cases} \end{aligned}$$

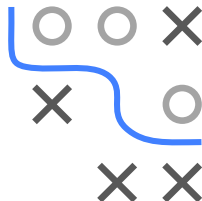


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$t$	$a_t^{\text{FTL}}$	$z_t$	$(a_t^{\text{FTL}}, z_t)$	$\sum_{s=1}^t (a_s^{\text{FTL}}, z_s)$	$\sum_{s=1}^t z_s$
1	1	$-1/2$	$-1/2$	$-1/2$	$-1/2$

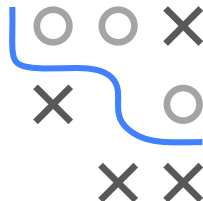


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2	1	1	1	$1 - 1/2$	$1/2$

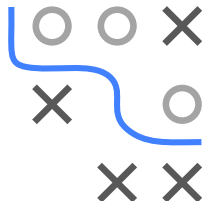


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$T$	$(-1)^T$	$(-1)^T$	1	$T - 1 - 1/2$	$(-1/2)^T$





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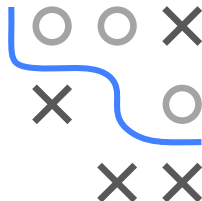
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$T$	$(-1)^T$	$(-1)^T$	1	$T - 1 - 1/2$	$(-1/2)^T$

- The best action has cumulative loss

$$\inf_{a \in \mathcal{A}} \sum_{s=1}^T (a, z_s) = \inf_{a \in [-1, 1]} a \underbrace{\sum_{s=1}^T z_s}_{=(-1/2)^T} = -1/2.$$



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- Thus, we see: FTL can fail for **online linear optimization problems**, although it is well suited for **online quadratic optimization problems**!
- The reason is that the action selection of FTL is not stable enough (caused by the loss function), which is fine for **the latter problem**, but problematic for **the former**.
- One has to note that the online linear optimization problem example above, where FTL fails, is in fact an adversarial learning setting: The environmental data is generated in such a way that the FTL learner is fooled in each time step.

