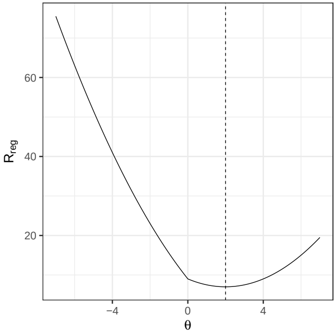
Introduction to Machine Learning

Regularization
Soft-thresholding and lasso (Deep-Dive)



Learning goals

- Understand the relationship between soft-thresholding and L1 regularization

SOFT-THRESHOLDING AND L1 REGULARIZATION

In the lecture, we wanted to solve

$$\min_{\boldsymbol{\theta}} \tilde{\mathcal{R}}_{\text{reg}}(\boldsymbol{\theta}) = \min_{\boldsymbol{\theta}} \mathcal{R}_{\text{emp}}(\hat{\theta}) + \sum_j \left[\frac{1}{2} H_{j,j}(\theta_j - \hat{\theta}_j)^2\right] + \sum_j \lambda |\theta_j|$$

with $H_{j,j} \geq 0, \lambda > 0$. Note that we can separate the dimensions, i.e.,

$$\tilde{\mathcal{R}}_{\text{reg}}(\boldsymbol{\theta}) = \sum_j z_j(\theta_j) \text{ with } z_j(\theta_j) = \frac{1}{2} H_{j,j}(\theta_j - \hat{\theta}_j)^2 + \lambda |\theta_j|.$$

Hence, we can minimize each z_j separately to find the global minimum.

If $H_{j,j} = 0$, then z_j is clearly minimized by $\hat{\theta}_{\text{lasso},j} = 0$. Otherwise, z_j is strictly convex since $\frac{1}{2} H_{j,j}(\theta_j - \hat{\theta}_j)^2$ is strictly convex and the sum of a strictly convex function and a convex function is strictly convex.

SOFT-THRESHOLDING AND L1 REGULARIZATION / 2

For strictly convex functions, there exists only one unique minimum and for convex functions a stationary point (if it exists) is a minimum.

We now separately investigate z_j for $\theta_j > 0$ and $\theta_j < 0$.

NB: on these halflines z_j is differentiable (with possible stationary point) since

- for $\theta_j > 0$: $\frac{d}{d\theta_j}|\theta_j| = \frac{d}{d\theta_j}\theta_j = 1$,
- for $\theta_j < 0$: $\frac{d}{d\theta_j}|\theta_j| = \frac{d}{d\theta_j}\left(-\theta_j\right) = -1$.

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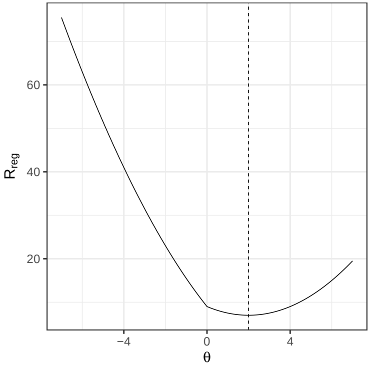
1) $\theta_j > 0$:



$$\frac{d}{d\theta_j} z_j(\theta_j) = H_{j,j}\theta_j - H_{j,j}\hat{\theta}_j + \lambda \overset{!}{=} 0$$

$$\Rightarrow \hat{\theta}_{\text{lasso},j} = \hat{\theta}_j - \frac{\lambda}{H_{j,j}} > 0$$

This minimum is only valid if $\hat{\theta}_{\text{lasso},j} > 0$ and so it must hold that

$$\hat{\theta}_j > \frac{\lambda}{H_{j,j}}.$$

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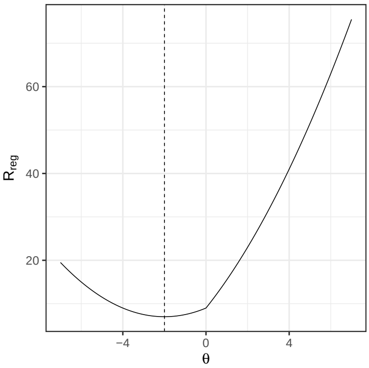
2) $\theta_j < 0$:



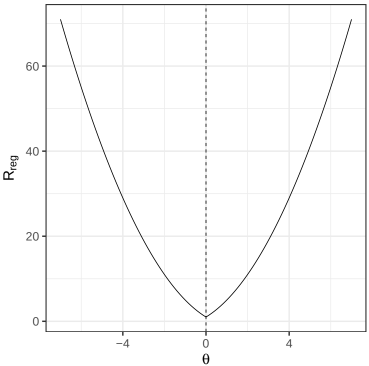
$$\frac{d}{d\theta_j} z_j(\theta_j) = H_{j,j}\theta_j - H_{j,j}\hat{\theta}_j - \lambda \overset{!}{=} 0$$

$$\Rightarrow \hat{\theta}_{\text{lasso},j} = \hat{\theta}_j + \frac{\lambda}{H_{j,j}} < 0$$

This minimum is only valid if $\hat{\theta}_{\text{lasso},j} < 0$ and so it must hold that

$$\hat{\theta}_j < -\frac{\lambda}{H_{j,j}}.$$

SOFT-THRESHOLDING AND L1 REGULARIZATION / 5



$\Rightarrow$ If $\hat{\theta}_j \in [-\frac{\lambda}{H_{j,j}}, \frac{\lambda}{H_{j,j}}]$ then z_j has no stationary point with

$$\hat{\theta}_{\text{lasso},j} < 0 \text{ or } \hat{\theta}_{\text{lasso},j} > 0.$$

However, a unique minimum must exist since z_j is strictly convex for $H_{j,j} > 0$. This means the only possible minimizer of z_j is $\hat{\theta}_{\text{lasso},j} = 0$.

$$\Rightarrow \hat{\theta}_{\text{lasso},j} = \begin{cases} \hat{\theta}_j + \frac{\lambda}{H_{j,j}} & \text{, if } \hat{\theta}_j < -\frac{\lambda}{H_{j,j}} \text{ and } H_{j,j} > 0 \\ 0 & \text{, if } \hat{\theta}_j \in [-\frac{\lambda}{H_{j,j}}, \frac{\lambda}{H_{j,j}}] \text{ or } H_{j,j} = 0 \\ \hat{\theta}_j - \frac{\lambda}{H_{j,j}} & \text{, if } \hat{\theta}_j > \frac{\lambda}{H_{j,j}} \text{ and } H_{j,j} > 0 \end{cases}$$