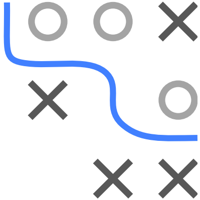
BATCH LEARNING



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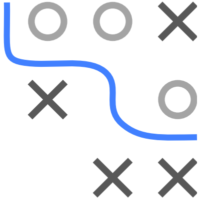

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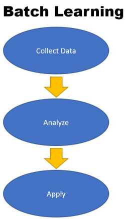
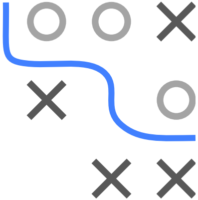
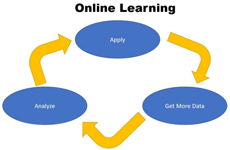


- The learning task on the available data beforehand is called the *training phase* and the prediction on the unseen data is called the *testing phase.* Both phases are **separated**.

ONLINE LEARNING



- However, many real-world problems are *dynamic* with the following aspects:
  - *Sequential order* — data is generated only bit by bit;
  - *On-the-fly decisions* — decisions or predictions have to be made during the data generating process;
  - *Unforeseeable consequences* — decisions can have a drastic influence on the data generating process;
  - *Constraints* — there is a specific time limit or computational limit for the decision.



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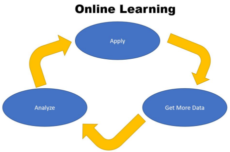
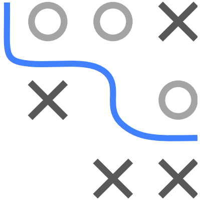


- These dynamic aspects outline the framework where **online learning** is settled.
- Characteristically: In the online learning scenario the training phase and the testing phase are **interleaved**.

ONLINE LEARNING: EXAMPLES



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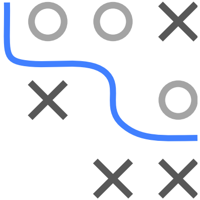

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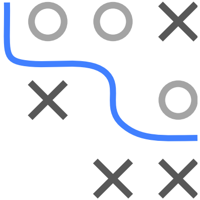

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- *Autonomous driving systems* — Steer the automotive, while constantly monitoring the environment and react appropriately to any changes.
- ...

ONLINE LEARNING: ILLUSTRATION

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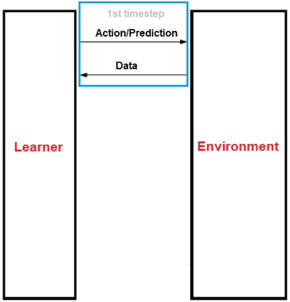

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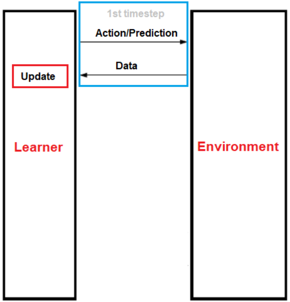

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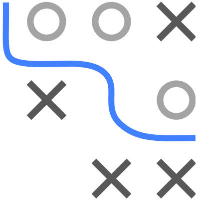

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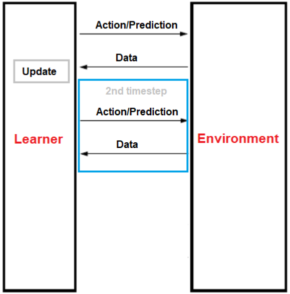

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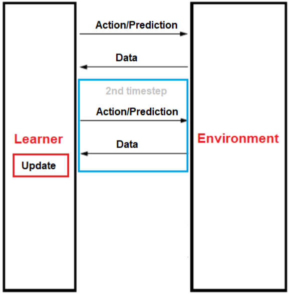

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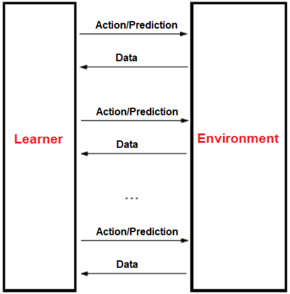
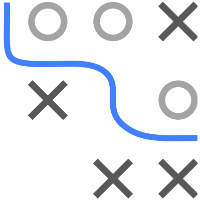


⇒ The learner and the environment are alternately performing their actions.

THE BASIC ONLINE LEARNING PROTOCOL



Formally, an online learning problem consists of:

- a learner (forecaster, agent resp. decision maker), an environment (user resp. adversary, system resp. nature),
- time steps $1, 2, \ldots, T$ (may be infinite),
- available actions $\mathcal{A}$ for the learner (may be infinite),
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Typically $\mathcal{A} = \mathcal{Z} = \mathcal{Y}$, so that

- the learner's chosen action $a_t = \hat{y}_t$ corresponds to a prediction,
- the generated data point $z_t = y_t$ is the revealed outcome.

THE EXTENDED ONLINE LEARNING PROTOCOL

- In some applications, the environmental data consists of two parts: $z_t = (z_t^{(1)}, z_t^{(2)})$, where the first part of the data, $z_t^{(1)}$, is revealed to the learner **before** the action is made. After the learner carries out its action, the remaining part of the environmental data is revealed, that is, $z_t^{(2)}$.

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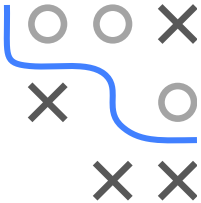
- We call this setting the *extended online learning protocol.*
- Typically $\mathcal{A} = \mathcal{Y}$ and $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, so that
  - the first part $z_t^{(1)} = \mathbf{x}_t$ is some feature information,
  - the learner's chosen action $a_t = \hat{y}_t$ corresponds to a prediction (dep. on $\mathbf{x}_t$),
  - the second part $z_t^{(2)} = y_t$ is the corresponding outcome.

DATA GENERATION IN ONLINE LEARNING



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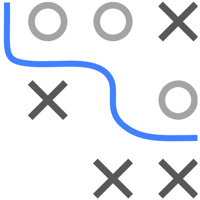

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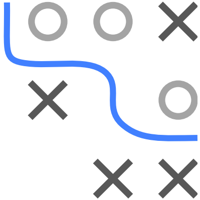
- In particular, the environmental data are not necessarily generated by a probability distribution!
- This also covers the area of *adversarial learning*: the data can even be generated by an adversary trying to fool the learner.
- However, the online learning setting can of course also be considered in a statistical setting.

ONLINE LEARNING: REQUIREMENTS



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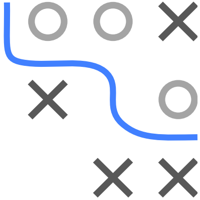

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- Roughly speaking, one seeks to construct an online learning algorithm which is adaptive to the environment and allows incremental as well as preferably cheap updates over time.
- Although consideration of time and memory constraints is important for practical purposes, we will only implicitly consider these constraints in this lecture.
- We will mainly focus our theoretical analysis on the performance of the learner in terms of its (cumulative) loss, which, however, will usually ignore computational aspects of the learner.

MEASURE OF QUALITY IN ONLINE LEARNING

- In order to measure the quality of an online learner one can compute the difference between the cumulative loss of the learner and the cumulative loss by taking some competing action $a \in \mathcal{A}$:

$$R_T(a) = \sum_{t=1}^{T} L(a_t, z_t) - \sum_{t=1}^{T} L(a, z_t).$$

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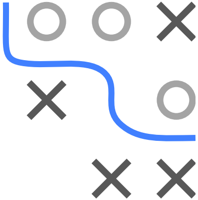

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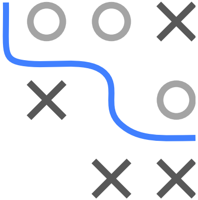

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- We refer to R_T as the *(cumulative) regret* of the online learner. It is easy to see that $R_T = \sup_{a \in \mathcal{A}} R_T(a)$.

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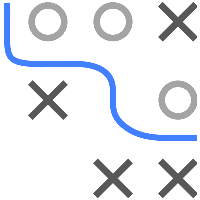

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- Note that the cumulative regret can be in principle negative as the action sequence could be such that $L(a_s, z_s) < L(a^*, z_s)$ holds for specific time steps s, where $a^* \in \arg\min_{a \in \mathcal{A}} \sum_{s=1}^{T} L(a, z_s)$ is one of the best actions in hindsight (may be unique).

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- Formally, the following should hold

$$R_T = o(T).$$

Interpretation: The average regret per time step (or per example) goes to zero:

$$\frac{1}{T}\left(\sum_{t=1}^{T} L(a_t, z_t) - \inf_{a \in \mathcal{A}} \sum_{t=1}^{T} L(a, z_t)\right) = \frac{R_T}{T} = o(1).$$

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- The rationale behind this measure of quality is that the best fixed action in hindsight is already reasonably good over all the time steps: it performs almost as well as a batch learner that observes the entire sequence and picks the best action in hindsight.
- However, this is too optimistic and may not hold in changing environments, where data are evolving and the optimal action is drifting over the time.

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- The rationale behind this measure of quality is that the best fixed action in hindsight is already reasonably good over all the time steps: it performs almost as well as a batch learner that observes the entire sequence and picks the best action in hindsight.
- However, this is too optimistic and may not hold in changing environments, where data are evolving and the optimal action is drifting over the time.
- To address this limitation, recent works have also considered the *dynamic regret*:

$$R_T^D(a_1^*, a_2^*, \ldots, a_T^*) = \sum\nolimits_{t=1}^{T} L(a_t, z_t) - \sum\nolimits_{t=1}^{T} L(a_t^*, z_t).$$

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- We will cover only the static regret in this lecture.