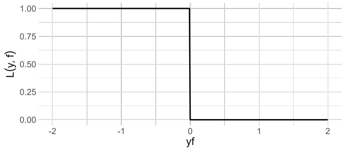
Introduction to Machine Learning

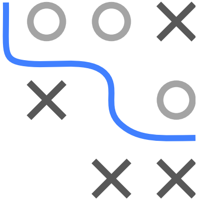
Advanced Risk Minimization
0-1-Loss



Learning goals

- Derive the risk minimizer of the 0-1-loss
- Derive the optimal constant model for the 0-1-loss

0-1-LOSS



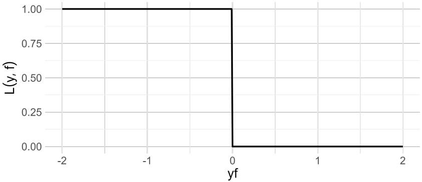
- Let us first consider a discrete classifier $h : \mathcal{X} \to \mathcal{Y}$.
- The most natural choice for $L(y, h)$ is the 0-1-loss

$$L(y, h) = \mathbb{1}_{\{y \neq h\}} = \begin{cases} 1 & \text{if } y \neq h \\ 0 & \text{if } y = h \end{cases}$$

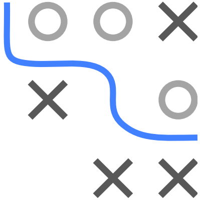
- For the binary case ($g = 2$) we can express the 0-1-loss for a scoring classifier f based on the margin $\nu := yf$

$$L(y, f) = \mathbb{1}_{\{\nu < 0\}} = \mathbb{1}_{\{yf < 0\}}.$$

- Analytic properties: Not continuous, even for linear f the optimization problem is NP-hard and close to intractable.



0-1-LOSS: RISK MINIMIZER



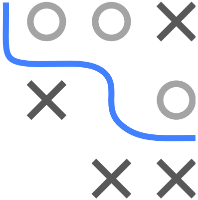
By the law of total expection we can in general rewrite the risk as (this all works for the multiclass case with 0-1)

$$
\begin{aligned}
\mathcal{R}(f) &= \mathbb{E}_{xy}\left[L\left(y, f(\mathbf{x})\right)\right] = \mathbb{E}_x\left[\mathbb{E}_{y|x}[L\left(y, f(\mathbf{x})\right)]\right] \\
&= \mathbb{E}_x\left[\sum_{k \in \mathcal{Y}} L(k, f(\mathbf{x}))\mathbb{P}(y = k \mid \mathbf{x})\right],
\end{aligned}
$$

with $\mathbb{P}(y = k|\mathbf{x})$ the posterior probability for class k. For the binary case we denote $\eta(\mathbf{x}) := \mathbb{P}(y = 1 \mid \mathbf{x})$ and the expression becomes

$$
\mathcal{R}(f) = \mathbb{E}_x\left[L(1, \pi(\mathbf{x})) \cdot \eta(\mathbf{x}) + L(0, \pi(\mathbf{x})) \cdot (1 - \eta(\mathbf{x}))\right].
$$

0-1-LOSS: RISK MINIMIZER / 2



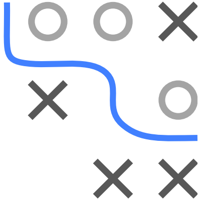
We compute the point-wise optimizer of the above term for the 0-1-loss (defined on a discrete classifier $h(\mathbf{x})$):

$$
\begin{aligned}
h^*(\mathbf{x}) &= \underset{l \in \mathcal{Y}}{\arg\min} \sum_{k \in \mathcal{Y}} L(k, l) \cdot \mathbb{P}(y = k \mid \mathbf{x} = \mathbf{x}) \\
&= \underset{l \in \mathcal{Y}}{\arg\min} \sum_{k \neq l} \mathbb{P}(y = k \mid \mathbf{x} = \mathbf{x}) \\
&= \underset{l \in \mathcal{Y}}{\arg\min}\, 1 - \mathbb{P}(y = l \mid \mathbf{x} = \mathbf{x}) \\
&= \underset{l \in \mathcal{Y}}{\arg\max}\, \mathbb{P}(y = l \mid \mathbf{x} = \mathbf{x}),
\end{aligned}
$$

which corresponds to predicting the most probable class.

Note that sometimes $h^*(\mathbf{x}) = \arg\max_{l \in \mathcal{Y}} \mathbb{P}(y = l \mid \mathbf{x} = \mathbf{x})$ is referred to as the **Bayes optimal classifier** (without closer specification of the the loss function used).

0-1-LOSS: RISK MINIMIZER / 3



The Bayes risk for the 0-1-loss (also: Bayes error rate) is

$$\mathcal{R}^* \quad = \quad 1 - \mathbb{E}_x \left[\max_{l \in \mathcal{Y}} \mathbb{P}(y = l \mid \mathbf{x}) \right].$$

In the binary case ($g = 2$) we can write risk minimizer and Bayes risk as follows:

$$h^*(\mathbf{x}) \quad = \quad \begin{cases} 1 & \eta(\mathbf{x}) \geq \frac{1}{2} \\ 0 & \eta(\mathbf{x}) < \frac{1}{2} \end{cases}$$

$$\mathcal{R}^* \quad = \quad \mathbb{E}_x \left[\min(\eta(\mathbf{x}), 1 - \eta(\mathbf{x})) \right] = 1 - \mathbb{E}_x \left[\max(\eta(\mathbf{x}), 1 - \eta(\mathbf{x})) \right].$$

0-1-LOSS: RISK MINIMIZER / 4

Example: Assume that $\mathbb{P}(y = 1) = \frac{1}{2}$ and

$$\mathbb{P}(x \mid y) = \begin{cases} \phi_{\mu_1,\sigma^2}(x) & \text{for } y = 0 \\ \phi_{\mu_2,\sigma^2}(x) & \text{for } y = 1 \end{cases}$$

The decision boundary of the Bayes optimal classifier is shown in orange and the Bayes error rate is highlighted as red area.