线性多步法一般形式

$$\sum_{j=0}^{k} \alpha_j y_{k+j} = h \sum_{j=0}^{n} \beta_j f_{n+j}$$

$$\alpha_k \neq 0 \text{ 且 } \alpha_0^2 + \beta_0^2 = 0$$

局部截断误差.

$$R_n = \sum_{j=0}^{k} \left[\alpha_j y(t_{n+j}) - h\beta_j y'(t_{n+j})\right]$$

统一在 t_n 处 Tayler 展开，得

$$R_n = C_0 y(t_n) + C_1 h y'(t_n) + \cdots + C_p h^p y^{(p)}(t_n) + C_{p+1} h^{p+1} y^{(p)}(t_n)$$

其中
$$\begin{cases} C_0 = \sum_{j=0}^{k} \alpha_j \\ C_1 = \sum_{j=0}^{k} (j\alpha_j - \beta_j) \\ C_p = \frac{1}{p!} \sum_{j=1}^{k} j^{p-1} (j\alpha_j - p\beta_j) \qquad p = 2, 3, \cdots \end{cases}$$

若 $C_0 = C_1 = \cdots = C_p = 0$，$C_{p+1} \neq 0$，则称方称是 p 阶相容的，局部为 $p+1$ 阶，整体(相容)为 p 阶

$C_{p+1} h^{p+1} y^{(p+1)}$ 为误差主项.

① $k=1$，$\alpha_0 y_n + \alpha_1 y_{n+1} = h(\beta_1 f_{n+1} + \beta_1 f_n)$

$$\begin{cases} C_0 = \alpha_0 + \alpha_1 = 0, \\ C_1 = -\beta_0 + \alpha_1 - \beta_1 = 0 \end{cases}$$

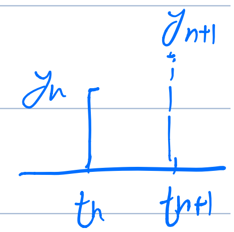
$$C_2 = \frac{1}{2}(\alpha_1 - 2\beta_1) = 0$$

$$C_3 = \frac{1}{3!}(\alpha_1 - 3\beta_1) \neq 0 \Rightarrow \beta_1 \neq \frac{1}{3}\alpha_1$$

不失一般性，令 $\alpha_1 = 1$，

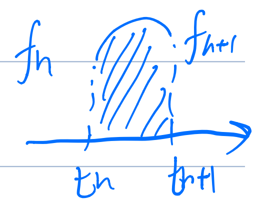
$$C_0 = 0 \Rightarrow \alpha_0 = -1$$

$$C_1 = 0 \Rightarrow \beta_0 + \beta_1 = 1 \quad \beta \in [0, 1]$$



$$y_{n+1} - y_n = h(\beta_0 f_n + (1-\beta_0) f_{n+1})$$

取 $\beta_0 = \theta$ $\quad \theta \in [0, 1]$



$$y_{n+1} = y_n + h[\theta f_n + (1-\theta) f_{n+1}]$$

此称 **线性θ方法** （单步θ-方法）

$$\begin{cases} \theta = 0 & \text{explicit Eular}, \quad y_{n+1} = y_n + h f_n \\ \theta = 1 & y_{n+1} = y_n + h f_{n+1} \quad \text{隐式Eular} \\ \theta = \frac{1}{2} & y_{n+1} = y_n + \frac{h}{2}[f_n + f_{n+1}] \quad \text{(对称的方法)} \end{cases}$$

单步唯一个高阶方法 哈密尔顿，保结构

可能会考：

例：用Taylor展开法求2步3阶方法

$$\alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h[\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n]$$

$$C_0 = C_1 = C_2 = C_3 = 0 \quad C_4 \neq 0$$

$$\Rightarrow \begin{cases} C_0 = \alpha_0 + \alpha_1 + \alpha_2 = 0 \\ C_1 = -\beta_0 + (\alpha_1 - \beta_1) + (2\alpha_2 - \beta_2) \\ \quad = \alpha_1 + 2\alpha_2 - (\beta_0 + \beta_1 + \beta_2) = 0 \\ C_2 = \frac{1}{2}\left((\alpha_1 - 2\beta_1) + 2(2\alpha_2 - 2\beta_2)\right) = 0 \\ C_3 = \frac{1}{3!}\left(\alpha_1 - 3\beta_1 + 4(2\alpha_2 - 3\beta_2)\right) = 0 \end{cases} \qquad C_1 = \sum_{j=0}^{k} (j\alpha_j - \beta_j)$$

不失一般性，令 $\alpha_2 = 1$

$$\Rightarrow \begin{cases} \alpha_1 = -1 - \alpha_0 \\ \beta_0 = -\frac{1}{12}(1 + 5\alpha_0) \\ \beta_1 = \frac{2}{3}(1 - \alpha_0) \\ \beta_2 = \frac{1}{12}(5 + \alpha_0) \end{cases}$$

$$\Rightarrow \begin{cases} C_4 = -\frac{1}{24}(1 + \alpha_0) \\ C_5 = -\frac{1}{360}(17 + 13\alpha_0) \end{cases}$$

① 取 $\alpha_0 = -1$，$C_4 = 0$，$C_5 \neq 0$

$$\begin{cases} \alpha_1 = 0 \end{cases}$$

$$\begin{cases} \beta_0 = \frac{1}{3} \\ \beta_1 = \frac{4}{3} \\ \beta_2 = \frac{1}{3} \end{cases} \Rightarrow y_{n+2} = y_n + \frac{h}{3}[f_{n+2} + 4f_{n+1} + f_n]$$

Miles, 2步4阶

② 取 $\alpha_0 = 0$,

$\alpha_1 = -1$, $\beta_0 = -\frac{1}{12}$, $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{5}{12}$

$$\Rightarrow y_{n+2} = y_{n+1} + \frac{h}{12}[5f_{n+2} + 8f_{n+1} - f_n]$$

Adams 2步 3阶

作业: ① 分析 3步 Adams 方法的相容阶

$$y_{n+3} = y_{n+2} + \frac{h}{12}[23f_{n+2} - 16f_{n+1} + 5f_n]$$

②. 用待定系数法求3步4阶方法类, 并确定 3步4阶显式方法.

③. 满足条件 $\beta_j = 0$, $j = 0, 1, 2, \cdots, k-1$ 的 k 步 k 阶方法称为 Gear 方法, 试对 $k = 2, 3$, 求 Gear 方法的表达式.

线性多步法的收敛性.

收敛 = 稳定 + 相容.

$\hookrightarrow \boxed{y_n - y(t_n)} \to 0$,

即整体误差 $\to 0$.

k阶方法称为 p阶收敛，若以该方法求解任意满足 Lipschtiz 条件的初值问题，只需 f 充分可微

启动值为p阶

且 $\boxed{\max_{0 \le j \le k} |y(t_j) - y_j| = O(h^p)}$ $h \to 0$

都有 $\varepsilon_n = O(h^p)$ $h \to 0$

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$$

$$R_n = \sum_{j=0}^{k} (\alpha_j y(t_{n+j}) - h\beta_j y'(t_{n+j}))$$

$$y'(t) = f(t, y(t))$$

$$\Downarrow$$

$$\frac{y(t_{n+1}) - y(t_n)}{h} \approx f(t_k, y(t_n))$$

$$\Downarrow$$

$$\frac{\sum_{j=1}^{k} \alpha_j y(t_{n+j})}{h} \approx \sum_{j=1}^{k} \beta_j f(t_{n+j}, y(t_{n+j}))$$

$$\Downarrow$$

$$\sum_{j=1}^{k} \alpha_j \, y(t_{n+j}) = \sum_{j=1}^{k} h \, \beta_j \, f(t_{n+j}, y(t_{n+j})) + R_n .$$