注：空间加 σ-紧，可去掉 (d) 的 $\mu(E)<\infty$.

给定 X, Λ，给出如下构造：

open $V \subset X$,

$\mu(V) := \sup\{\Lambda f \mid f \prec V\}$.

$\mu(E) := \inf\{\mu(V) \mid E \subset V, V \text{ open}\}$.

$\mathfrak{M}_F := \{E \subset X \mid \mu(E)<\infty,\ \mu(E)=\sup\{\mu(K) \mid K \subset E, K \text{ compact}\}\}$.

$\mathfrak{M} := \{E \subset X \mid E \cap K \subset \mathfrak{M}_F, \text{ for } \forall \text{ compact } K\}$

最后再说 $\Lambda f = \int_X f\, d\mu$.

μ 非负已成立（Λ 为正泛函）

只需证可数可加性 $\Rightarrow$ 需要给定 $\mathfrak{M}$

$\Rightarrow$ 证 $\mathfrak{M}$ 为 σ-algebra.

Step 1 $\forall E_1, E_2, \cdots \subset X$.

$$\mu\left(\bigcup_{n=1}^{\infty} E_i\right) \leq \sum_{n=1}^{\infty} \mu(E_i)$$

Proof

- If $\mu(E_i) < \infty$ for some i ✓

- Suppose therefore $\mu(E_i) < \infty$ $\forall i$,

  $\forall \varepsilon > 0$, Since $\mu(E) = \inf\{\mu(V) \mid E \subset V,\ V \text{ open}\}$,

  $\exists$ open set $V_i \supset E_i$, s.t.

  $$\mu(V_i) < \mu(E_i) + 2^{-i}\varepsilon, \quad (i = 1, 2, \cdots)$$

Put $V = \bigcup_{n=1}^{\infty} V_i$ and choose $f \prec V$,

Since f has compact support, we have $f \prec V_1 \cup \cdots \cup V_n$, for some n, Applying induction to

$\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$

we obtain

$$\Lambda f \leq \mu(V_1 \cup \cdots \cup V_n) \leq \mu(V_1) + \cdots \mu(V_n) \leq \sum_{n=1}^{\infty} \mu(E_i) + \varepsilon$$

Since this holds for $\forall f \prec V$

and Since $\bigcup E_i \subset V$, $\mu(V) = \sup\{\Lambda f \mid f \prec V\}$

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} E_i\right) \leq \mu(V) \leq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon.$$

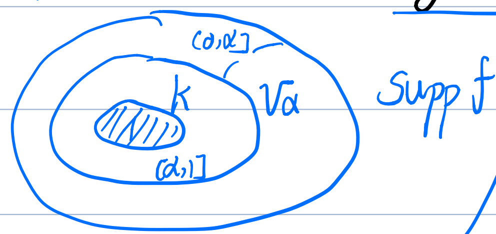
Since ε is arbitrary

Step 11 If K is compact, then $K \in \mathfrak{M}_F$, and

$$\mu(K) = \inf\{\Lambda f : K \prec f\}.$$

Proof: If $K \prec f$, and $0<\alpha<1$, let $V_\alpha = \{x : f(x) > \alpha\}$.

Then $K \subset V_\alpha$ K上取值恒为1. and $\alpha g \leq f$ when $g \prec V_\alpha$



V_α外 $g=0$, ✓

V_α内 $g \leq 1$, $f > \alpha$, $\alpha g \leq \alpha < f$ ✓

$\Rightarrow \mu(K) \leq \mu(V_\alpha) = \sup\{\Lambda g \mid g \prec V_\alpha\} \leq \alpha^{-1} \Lambda f$

Let $\alpha \to 1$.

$\Rightarrow \mu(K) \leq \Lambda f$ ① $\Rightarrow \mu(K) < \infty$.

Since K evidently satisfies

$\mu(E) = \sup\{\mu(K) \mid K \subset E,\ K \text{ compact}\}$, $K \in \mathfrak{M}_F$.

$\forall \varepsilon > 0$, $\exists V \supset K$ with $\mu(V) < \mu(K) + \varepsilon$,

By Urysohn's Lemma, $K \prec f \prec V$ for some f, Thus $\quad \Lambda f \le \mu(V) < \mu(K) + \varepsilon \quad$ ②

① ② $\Rightarrow \quad \mu(K) = \inf\{\Lambda f \mid K \prec f\}$.

$\hookrightarrow \Lambda f < \mu(K) + \varepsilon \le \Lambda f + \varepsilon \qquad \forall \varepsilon$

$\Rightarrow \quad \Lambda f - \varepsilon < \mu(K) \le \Lambda f \qquad \forall \varepsilon$

$\Rightarrow \quad \mu(K) = \inf\{\Lambda f \mid K \prec f\}$.

Step III Every open set satisfies (3).

Hence $\mathfrak{M}_F$ contains every open set V with $\mu(V) < \infty$.

Proof: (1) $\mu(V) \ge \mu(K) \quad K \subset V, \quad K$ compact.

(2). $\forall \varepsilon > 0, \quad \alpha := \mu(V) - \varepsilon < \mu(V) \left(:= \sup\{\Lambda f \mid f \prec V\} \right)$

If W is any open set which contains the support K of f, then $f \prec W$.

hence $\Lambda f \le \mu(W) \Rightarrow \Lambda f \le \mu(K)$

i.e. $\quad \alpha < \Lambda f \le \mu(K)$

$\mu(V) - \varepsilon < \mu(K) \qquad \square$.

看书先跳着看，看每一步干什么，再看细节

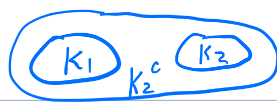
Step 4 Suppose $E = \bigcup_{i=1}^{\infty} E_i$, where $E_1, E_2, \cdots$, are pairwise disjoint members of $\mathfrak{M}_F$.

Then $$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \qquad (4)$$

If, in addition, $\mu(E) < \infty$, then also $E \in \mathfrak{M}_F$.

Proof: We first show that $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$ if K_1, K_2 are disjoint compact sets.

$\forall \varepsilon > 0$, By Urysohn's Lemma, $(K_1 \subset K_2^c)$ $f \in C_c(X)$, s.t. $f(x) = 1$ on K_1, $f(x) = 0$ on K_2 and $0 \le f \le 1$.



By step 2, $\exists g$ s.t.

$$\begin{cases} K_1 \cup K_2 \prec g \\ \Lambda g < \mu(K_1 \cup K_2) + \varepsilon. \end{cases}$$

Note that $K_1 \prec fg$, $K_2 \prec (1-f)g$

Since Λ is linear, $\mu(K_1) + \mu(K_2) \le \Lambda(fg) + \Lambda(g - fg)$

$= \Lambda g < \mu(K_1 \cup K_2) + \varepsilon$

Combining with Step 1, $\mu(K_1) = \mu(K_1) + \mu(K_2)$

If $\mu(E) = \infty$, (4) follows from Step 1.
Assume therefore that $\mu(E) < \infty$.

$\forall \varepsilon > 0$ Since $E_i \in \mathfrak{M}_F$, $\exists$ compact $H_i \subset E_i$, s.t.

$$\mu(H_i) > \mu(E_i) - 2^{-i}\varepsilon \qquad i = 1, 2, 3, \cdots;$$

$$\mu(E) \geq \mu\left(\bigcup_{i=1}^{n} H_i\right) = \sum_{i=1}^{n} \mu(H_i) > \sum_{i=1}^{n} \mu(E) - \varepsilon. \qquad (*)$$

If $\mu(E) < \infty$, $\forall \varepsilon > 0$ (4) shows that

$$\mu(E) \leq \sum_{i=1}^{N} \mu(E_i) + \varepsilon \quad \text{for some } N.$$

By $(*)$ $\mu(E) \leq \mu\left(\bigcup_{i=1}^{N} H_i\right) + 2\varepsilon$. this shows that E satisfies (3).

Hence $E \in \mathfrak{M}_F$.