教材：

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Uri M Ascher、Linda Petzold Computer Methods for ODEs and DAE. SIAM

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一、常微分方程数值解

$y'(t) = f(t, y(t))$

定解条件 $\begin{cases} \text{初值问题 (IVP) (Initial value problem)} \\ \text{边值问题 (BVP) (Boundary ···)} \end{cases}$

解析求解 (pencil and paper, technique)

可分离变量方法.

常数变易法

① 解的存在唯一性

② 解的稳定性

$$\begin{cases} y'(t)=f(t,y(t)) \quad a\le t\le b & (1.1)\\ y(a)=y_0 & (1.2)\end{cases}$$

若 f 满足

① $f(t,y)$ 是实值函数

② $f(t,y)$ 在矩形区域 $\Omega=\{(t,y)\mid t\in[a,b],\, y\in(-\infty,+\infty)\}$ 内连续

③ f 满足全局 Lipschits 条件，即 $\exists L>0$, s.t.

$$|f(t,y)-f(t,z)|<L|y-z|$$

则初值问题 (1.1), (1.2) 在 $[a,b]$ 上有唯一解

且 $y(t)$ 连续可微

(稳定性)

稳定性定义 初值问题 (1.1)(1.2) 称为稳定的，如果存在正常数 K, η, s.t. 对 $\forall \varepsilon<\eta$，当 $|y_0-\tilde{y}_0|<\varepsilon$

$$|f(t,y)-\tilde{f}(t,y)|<\varepsilon \quad t\in[a,b],\, y\in(-\infty,+\infty)$$

有解 $z(t)$ 存在，且满足

$$|y(t)-z(t)|\leq K\varepsilon$$

定理：在解的存在唯一性定理条件下，方程 (1.1) (1.2) 是稳定的

引理（Gronwall - Ballman 不等式）

假设 $u(t)$、$\alpha(t)\in C[a,b]$，非负函数 $\beta(t)\in L^1[a,b]$，若

$$u(t)\leq\alpha(t)+\int_a^t\beta(s)u(s)\,ds\quad t\in[a,b]\quad ①$$

$$\Rightarrow u(t)\leq\alpha(t)+\int_a^t\beta(s)\alpha(s)\exp\left(\int_s^t\beta(\tau)d\tau\right)ds$$

若进一步，假设 $\alpha(t)$ 非减，则有

$$u(t)\leq\alpha(t)\exp\left(\int_a^t\beta(s)\,ds\right)$$

证明： $V(s)=\exp\left\{-\int_a^s\beta(r)dr\right\}\int_a^s\beta(r)u(r)\,dr$ ②

$$\Rightarrow V'(s)=\underbrace{\left(u(s)-\int_a^s\beta(r)u(r)dr\right)}_{\leq\alpha(s)\ (①)}\cdot\beta(s)\exp\left(-\int_a^s\beta(r)dr\right)$$

$$V'(s)\leq\alpha(s)\cdot\beta(s)\cdot\exp\left(-\int_a^s\beta(r)dr\right)$$

两边同时从a到t积分

$$V(t)-V(a) \le \int_a^t \alpha(s)\beta(s)\exp\left(-\int_a^s \beta(r)dr\right)ds$$

$(V(a)=0$ ②$)$

$$\Rightarrow V(t) \le \int_a^t \alpha(s)\beta(s)\exp\left(-\int_a^s \beta(r)dr\right)ds \quad ③$$

② 令 $S=t$ 得:

$$u(t)-\alpha(t) \le \int_a^t \beta(r)u(r)dr = V(t)\cdot\exp\left(\int_a^t \beta(s)ds\right)$$

$$(③\Rightarrow) \quad \le \int_a^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(r)dr\right)ds$$

若 $\alpha(t)$ 在 $[a,b]$ 非减, $\forall S\in[a,t]$,

$$\Rightarrow \alpha(s)\le\alpha(t)$$

$$\Rightarrow \beta(s)\alpha(s)\cdot\exp\left(\int_s^t \beta(r)dr\right) \le \beta(s)\alpha(t)\cdot\exp\left(\int_s^t \beta(r)dr\right)$$

$$\Rightarrow u(t) \le \alpha(t) + \int_a^t \beta(s)\alpha(s)\exp\left(\int_s^t \beta(r)dr\right)ds$$

$$\le \int_a^t \beta(s)\alpha(t)\exp\left(\int_s^t \beta(r)dr\right)ds + \alpha(t)$$

$$=\alpha(t)\left[1+\int_a^t -1\cdot\exp\left(\int_s^t \beta(r)dr\right)d\int_s^t \beta(r)dr\right]$$

$$=\alpha(t)\left[1-\int_a^t d\exp\left(\int_s^t \beta(r)dr\right)\right]$$

$$=\alpha(t)\left[1-\exp\left(\int_s^t \beta(r)dr\right)\Big|_a^t\right]$$

$$=\alpha(t)\cdot\exp\left(\int^t \beta(r)dr\right) \quad t\in[a,b]$$

$$= \alpha(t)\left[1+\int_a^t \beta(s)\exp\left(\int_s^t \beta(r)\,dr\right)ds\right]$$

$$1+\int_a^t \beta(s)\exp\left(\int_s^t \beta(r)\,dr\right)ds$$

$$=1+\int_a^t \exp\left(\int_s^t \beta(r)\,dr\right)\, d\left[-\int_s^t \beta(r)\,dr\right]$$

$$1-\exp\left(\int_s^t \beta(r)\,dr\right)\Big|_a^t$$

$$=1-\left[\exp\int_t^t \beta(r)\,dr-\exp\int_a^t \beta(r)\,dr\right]$$

$$=1-\left[\exp(0)-\exp\int_a^t \beta(r)\,dr\right]$$

$$=\exp\int_a^t \beta(r)\,dr$$