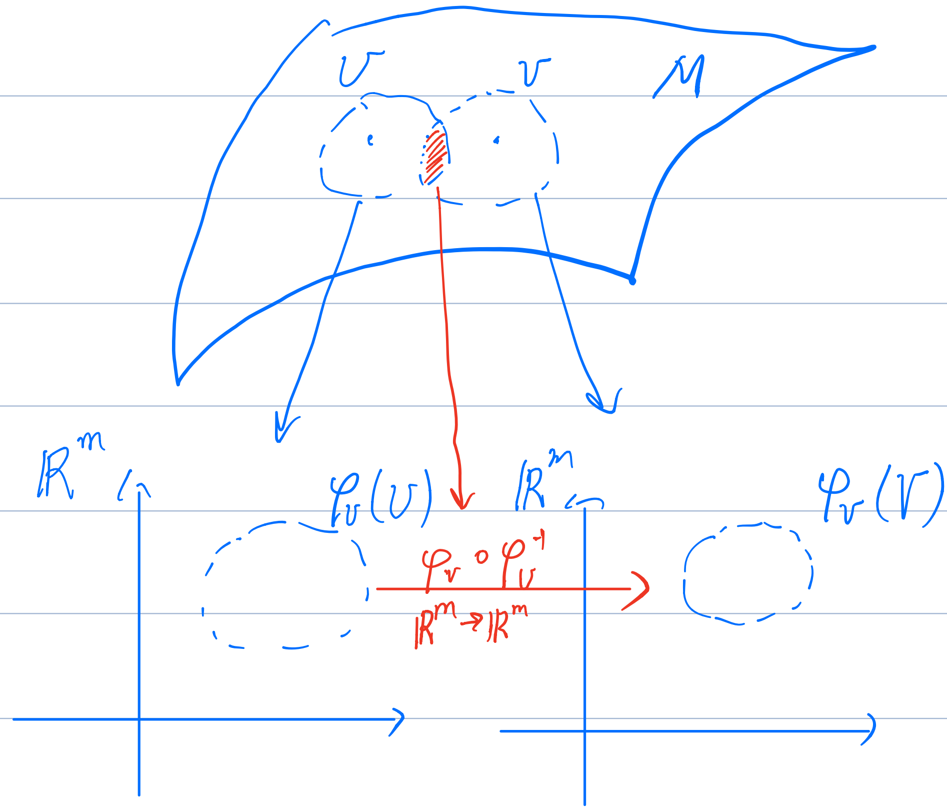


$\forall\, \varphi_V \circ \varphi_U^{-1}$ 为 $C^\infty \Rightarrow$ C^∞ / 光滑流形

Remark: C^r - differential manifold

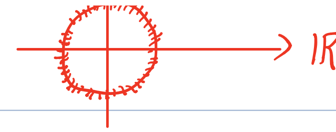
C^0 - topological manifold

Example.

1. $M = \mathbb{R}^m \quad U = \mathbb{R}^m \quad \varphi = id$ (恒同映射)
2. $S^m = \{x \in \mathbb{R}^{m+1} \mid \sum_{i=1}^{m+1} (x^i)^2 = 1\}$

$\uparrow \mathbb{R}^m$

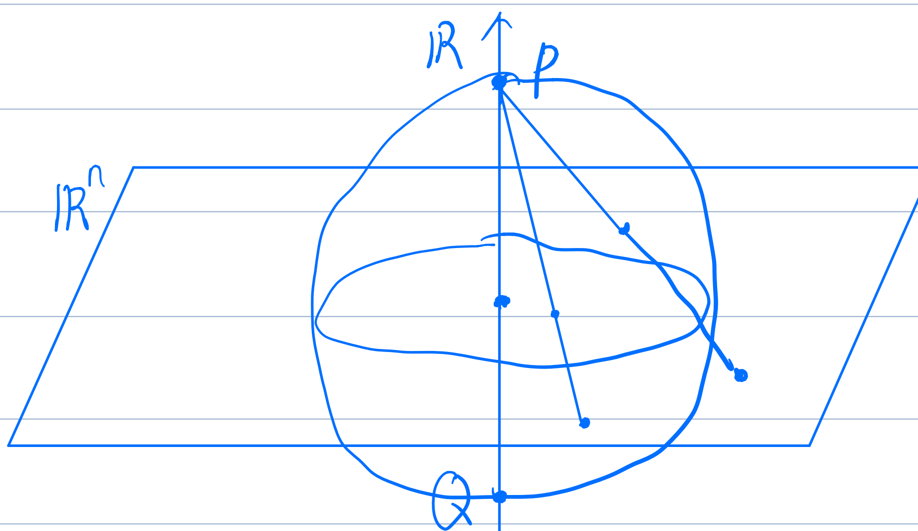
m维单位球面



用m维，而不用$m+1$维研究

即不在原空间研究几何性质，

而采用各点邻域 $\to \mathbb{R}^m$



北级投影 $\{S^n-P\} \xrightarrow{\varphi_U} \mathbb{R}^n$

南级…… $\{S^n-Q\} \xrightarrow{\varphi_V} \mathbb{R}^n$

注：要挖去一点（撕破）才能将

n维球面同胚到 $\mathbb{R}^n$

可验证/求解 φ_U、φ_V 表达式

验证 $\varphi_V \circ \varphi_U^{-1}$ 是否为 C^∞

$\Rightarrow S^n$ 为 C^∞ - Manifold.

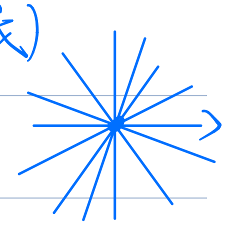
3. $\mathbb{P}^m$ Projective Space 射影空间

$\sim$ 等价类 $\quad \forall x, y \in \mathbb{R}^{m+1} - \{0\}$ (挖零点)

(先回到 $\mathbb{R}^{m+1}$) $\quad x \sim y \Leftrightarrow x = \alpha y$ for some

$\alpha \in \mathbb{R}, \alpha \neq 0$ 即同一直线上的点

(或直线)



等价关系 ① 自反: $x \sim x$

② 对称: $x \sim y \Rightarrow y \sim x$

③ 传递: $x \sim y, y \sim z \Rightarrow x \sim z$

quotient space

$\mathbb{R}^m = (\mathbb{R}^{m+1} - \{0\}) / \sim$ (关于该等价关系的商空间)

$= \{[x] \mid x \in \mathbb{R}^{m+1} - \{0\}\}$

$[x] := \{y \in \mathbb{R}^{m+1} - \{0\} \mid x \sim y\}$ (关于该等价关系的

等价类 equivalence class

$\begin{cases} U_i = \{[x^1, \cdots, x^{m+1}] \mid x^i \neq 0\} \\ \subset \mathbb{P}^m \end{cases}$

$$\varphi_i : U_i \xrightarrow{\cong} \varphi_i(U_i) \subset \mathbb{R}^m$$

$$[x] \longmapsto \varphi_i([x])$$

$$= \left({}_i\xi_1, \cdots, {}_i\xi_{i-1}, {}_i\xi_{i+1}, \cdots, {}_i\xi_{m+1} \right)$$

where ${}_i\xi_h = \dfrac{x_h}{x_i} \quad (h \neq i)$

$(x^1, \cdots, x^i, \cdots, x^{m+1})$

$\to \left(\dfrac{x^1}{x^i}, \cdots, 1, \cdots, \dfrac{x^{m+1}}{x^i}\right)$ 失去意义

Obviously, $\{U_i \mid 1 \leq i \leq m+1\}$ is a covering of $\mathbb{P}^m$ and on $U_i \cap U_j \ (i \neq j)$ we have

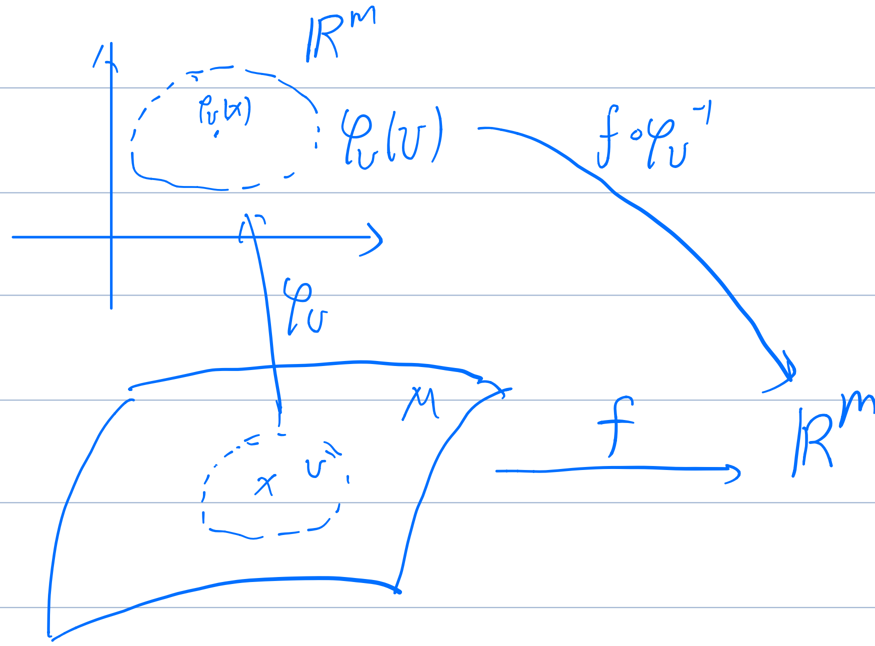
$$\begin{cases} {}_j\xi_h = \dfrac{{}_i\xi_h}{{}_i\xi_j} \\ {}_j\xi_i = \dfrac{1}{{}_i\xi_j} \end{cases} \quad \left({}_j\xi_h = \frac{x^h}{x_j} = \frac{x_h / x_i}{x_j / x_i} = \frac{{}_i\xi_h}{{}_i\xi_j} \right) \quad h \neq i, j$$

$$\begin{tikzcd} \mathbb{R}^m \arrow[dr, "f \circ \varphi_U^{-1}"] & \\ M \arrow[u, "\varphi_U"] \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

Definition 3. A function $f \quad M \xrightarrow{f} \mathbb{R}$ is called C^∞ at $x \in M$, if $f \circ \varphi_U^{-1}$ is C^∞ at $\varphi_U(x) \in \mathbb{R}^m$

Moreover, f is called C^∞ if it is C^∞

at every points of M.



Definition 4. An atlas for a differentiable manifold is called oriented (可定向) if all chart transitions has positive functional determinant.

S^n 可定向，莫比乌斯带不可定向. 雅可比行列式

A differentiable manifold is called orientable if it posesses an oriented atlas.

注：Manifold 是 $(M, (U_\alpha, \varphi_\alpha))$

类似拓扑/可测空间.

给定结构 $(U_\alpha, \varphi_\alpha)$ 后，叫 Manifold.

Definition 5. A mapping $h: M \to M'$ between differentiable manifolds M and M' with charts $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ is called differentable if all $\psi_\beta \circ h \circ \varphi_\alpha^{-1}$ are differentiable (C^∞) are defined

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\psi_\beta \circ h \circ \varphi_\alpha^{-1}} & \mathbb{R}^m \\ \uparrow \varphi_\alpha & & \uparrow \psi_\beta \\ M & \xrightarrow{h} & M' \end{array}$$

Such a map is called a differmorphism (微分同胚) if bijective (双射) and differentiable in both diveritions.

$$M \underset{h^{-1}}{\overset{h}{\rightleftarrows}} M'$$

$h: C^r \Rightarrow C^r$ – 微分同胚

§ 2. Tangent Space

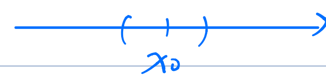
M a differentiable manifold of dim m

$p \in M$, a fixed point.

C_p^∞: the set of all C^∞-functions defined in a neighborhood of p.

仅在这里有定义就够了.

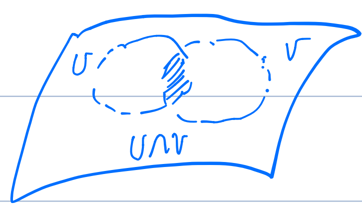
$\lim\limits_{x\to x_0} f(x)$



只考虑邻域内

$f(x)$在x_0处连续

In the space of C_p^∞, we can define $f+g$, $f\cdot g$ on $U \cap V$. ($f \in U$, $g \in V$)



i.e. $f+g,\ f\cdot g \in C_p^\infty$

封闭的.