Step 5. If $E \in \mathfrak{M}_F$ and $\varepsilon > 0$ $\exists$ a compact K and an open V s.t. $K \subset E \subset V$. $\mu(V-K) < \varepsilon$

Proof: Our definition shows that $\exists$ $K \subset E$ and $V \supset E$ s.t.

$$\mu(V) - \frac{\varepsilon}{2} < \mu(E) < \mu(K) + \frac{\varepsilon}{2}.$$

Since $V-K$ is open, $V-K \in \mathfrak{M}_F$. by Step 3. Step 4 $\Rightarrow$ $\mu(K) + \mu(V-K) = \mu(V) < \mu(K) + \varepsilon$.

Step 6. If $A, B \in \mathfrak{M}_F$, then $A-B$. $A \cup B$, $A \cap B \in \mathfrak{M}_F$.

Proof: $\forall \varepsilon > 0$, $\exists K_i, V_i$ $(i=1,2)$

s.t. $K_1 \subset A \subset V_1$, $K_2 \subset B \subset V_2$.

and $\mu(V_i - K_i) < \varepsilon$ $i=1,2$.

Since $A-B \subset V_1 - K_2 \subset (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2)$

Step 1 shows that

$\mu(A-B) \le \varepsilon + \mu(K_1 - V_2) + \varepsilon.$

$K_1 - V_2 \subset A-B$, compact $\Rightarrow$ $A-B$ satisfies (3)

差集成立 $\Rightarrow A-B \in \mathfrak{M}_F$. $A, B \in \mathfrak{M}_F$, $A \cap B = \phi \Rightarrow A \cup B \in \mathfrak{M}_F$

Since $A \cup B = (A-B) \cup B$ step 4 $\Rightarrow$ $A \cup B \in \mathfrak{M}_F$

$A \cap B = A - \underbrace{(A-B)}_{A-B \in \mathfrak{M}_F} \Longrightarrow A \cap B \in \mathfrak{M}_F$

Step 7. $\mathfrak{M}$ is σ-algebra in X which contains all Borel sets.

Proof. $\forall K$ compact in X.

(1). $X \in \mathfrak{M}$, Obviously.

(2). If $A \in \mathfrak{M}$, then

$$A^c \cap K = \underbrace{K}_{\in \mathfrak{M}_F} - \underbrace{(A \cap K)}_{\in \mathfrak{M}_F} \in \mathfrak{M}_F$$

$\Rightarrow A \in \mathfrak{M}$

(3). Suppose $A = \bigcup_{i=1}^{\infty} A_i$, where $A_i \in \mathfrak{M}$.

Put $B_1 = A_1 \cap K$.

$$B_n = (A_n - K) - (B_1 \cup \cdots \cup B_{n-1}) \quad (n = 2, 3, \cdots)$$

Then $\{B_n\}$ is a disjoint sequence of members of $\mathfrak{M}_F$, by step 6, and $A \cap K = \bigcup_{n=1}^{\infty} B_n \in \mathfrak{M}_F \Rightarrow A \in \mathfrak{M}$

Finally, if C is closed, then $C \cap K$ is compact, hence $C \cap K \in \mathfrak{M}_F \Rightarrow C \in \mathfrak{M}$

闭 $\in \mathfrak{M}$, 补 $\in \mathfrak{M} \Rightarrow$ 开 $\in \mathfrak{M}$

$\Rightarrow \mathfrak{M}$ 包含全体 Borel

Step 8. $\mathfrak{M}_F$ consists of precisely those sets $E \in \mathfrak{M}$ for which $\mu(E) < \infty$.

This implies assertion (d) of the theorem

Proof: ($\Rightarrow$). If $E \in \mathfrak{M}_F$, Step 2 and 4 imply that $E \cap K \in \mathfrak{M}_F$ for $\forall$ compact K, hence $E \in \mathfrak{M}$.

($\Leftarrow$). Suppose $E \in \mathfrak{M}$ and $\mu(E) < \infty$, $\forall \varepsilon > 0$, $\exists$ an open set $V \supset E$ with $\mu(V) < \infty$, By step 3, step 5, $\exists$ a compact $K \subset V$ with $\mu(V - K) < \varepsilon$.

Since $E \cap K \in \mathfrak{M}_F$, $\exists$ a compact set

$H \subset E \cap K$ with $\mu(E \cap K) < \mu(H) + \varepsilon$

Since $E \subset (E \cap K) \cup (V - K)$

$$\mu(E) \leq \mu(E \cap K) + \mu(V - K) < \mu(H) + 2\varepsilon$$

Step 9 μ is a measure on $\mathfrak{M}$.

The countable additivity of μ on $\mathfrak{M}$ follows immediately from Steps 4 and 8

Step 10. $\forall f \in C_c(X)$, $\Lambda f = \int_X f \, d\mu$.

Proof: Clearly, it is enough to prove this for real f. Also, it is enough to prove the inequality

$$\Lambda f \leq \int_X f \, d\mu$$

(for every $f \in C_c(X)$. For once it is established, the linearity of Λ shows that

$$-\Lambda f = \Lambda(-f) \leq \int_X (-f) \, d\mu = -\int_X f \, d\mu$$

)

Let K be the support of a real $f \in C_c(X)$, let $[a,b]$ be an interval which contains the range of f.

Choose $\varepsilon > 0$, and choose y_i, $i = 0, 1, 2, \cdots, n$ s.t.

$$y_0 < a < y_1 < \cdots < y_n = b$$

Put.

$$E_i = \{x \mid y_{i-1} < f(x) \le y_i\} \cap K \quad (i = 1, 2, \cdots n).$$

Since f is continous, f is Borel measurable, and the sets E_i are therefore disjoint Borel sets whose union is K.

$\exists$ open sets $V_i \supset E_i$, s.t.

$$\mu(V_i) < \mu(E_i) + \frac{\varepsilon}{n} \ (i = 1, \cdots, n)$$

and $f(x) < y_i + \varepsilon$ 连续性，自变量改变很小，因变量改变也很小

单位分解

$\exists$ $h_i \prec V_i$ s.t. $\boxed{\sum h_i = 1}$ on K. Hence $f = \sum h_i f$, and step 2 shows that

$$\mu(K) \leq \Lambda(\Sigma h) = \Sigma \Lambda h_i$$

Since $h_i f \leq (y_i + \varepsilon) h_i$ and $y_i - \varepsilon < f(x)$ on E_i,

$$\Lambda f = \sum_{i=1}^{n} \Lambda(h_i f) \leq \sum_{i=1}^{n} (y_i + \varepsilon) \Lambda h_i$$

$$= \sum_{i=1}^{n} (|a| + y_i + \varepsilon) \Lambda h_i - |a| \sum_{i=1}^{n} \Lambda h_i$$

$$\leq \sum_{i=1}^{n} (|a| + y_i + \varepsilon) [\mu(E_i) + \varepsilon/n] - |a| \mu(K)$$

$$= \sum_{i=1}^{n} (y_i - \varepsilon) \mu(E_i) + 2\varepsilon \mu(K) + \frac{\varepsilon}{n} \sum_{i=1}^{n} (|a| + y_i + \varepsilon)$$

$$\leq \int_X f \, d\mu + \varepsilon [2\mu(K) + |a| + b + \varepsilon].$$