§3 Regularity Properties of Borel Measures

注：为去掉(4)的测度有限，
需添加条件……

Definition 1. A measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is called a Borel measure on X.

A Borel set $E \subset X$ is outer regular or inner regular, respectively, if E has property (3) or (4). of the Riesz Representation Theorem.

If every Borel set in X is both outer and inner regular, μ is called regular.

Definition 2. A set E in a topological space is called σ-compact if E is a countable union of compact sets. 或可数闭集的并

A set E in a measure space (with measure μ) is said to have σ-finite measure if E is a countable union of sets E_i with $\mu(E_i) < \infty$

Remark. In the situation described in Representation Th.

- Every σ-compact set has σ-finite measure.

  每个紧集都测度有限

- It's easy to say that. if $E \in \mathfrak{M}$ and E has σ-finite measure, then E is inner regular.

  ☆ 回去想下为什么

Theorem L Suppose X is a locally compact, σ-compact Hausdorff space.

If $\mathfrak{M}$ and μ are as described in the statement of Riesz -- Th. Then $\mathfrak{M}$ and μ have the following properties:

(a) If $E \in \mathfrak{M}$ and $\varepsilon > 0$. $\exists$ a closed set F and an open set V. s.t. $F \subset E \subset V$, and $\mu(V - F) < \varepsilon$.

(b). μ is a regular Borel measure on X.

(c). If $E \in \mathfrak{M}$, $\exists$ F_σ A and G_δ B, s.t. $A \subset E \subset B$ and $\mu(B - A) = 0$.

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(c) $\Rightarrow$ $\forall E \in \mathfrak{M}$, $E = $ (an F_σ set) $\cup$ (measure 0 set).

注：用好的表示/逼近

例：多项式 $\xrightarrow{\text{泰勒}}$ 一般

Proof: (1). Let $X = \bigcup_{n=1}^{\infty} K_n$, where each K_n is compact. If $E \in \mathfrak{M}$ and $\varepsilon > 0$, Then $\mu(K_n \cap E) < \infty$.　σ-compact $\Rightarrow$ σ-finite measure.

and $\exists$ open sets $V_n \supset K_n \cap E$ s.t.

$$\mu(V_n - (K_n \cap E)) < \frac{\varepsilon}{2^{n+1}}$$ (Riesz (3))

Denote $V = \bigcup V_n$

then $V - E \subset \bigcup (V_n - (K_n \cap E))$

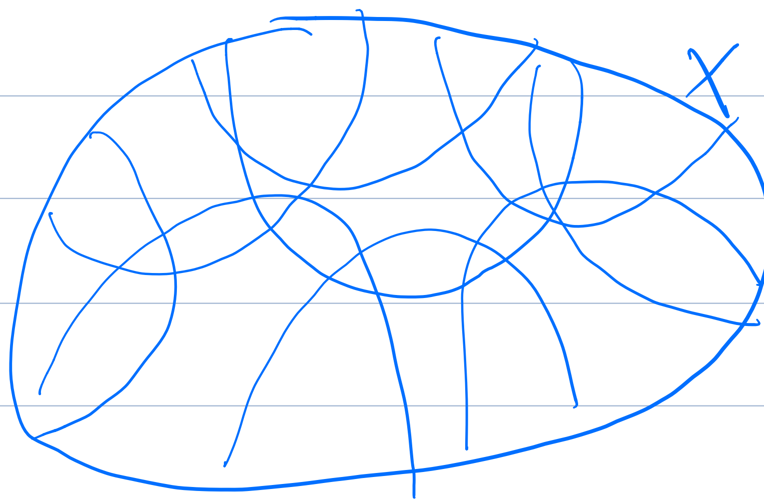
s.t. $\mu(V - E) < \frac{\varepsilon}{2}$

Taking E^c in place of E:

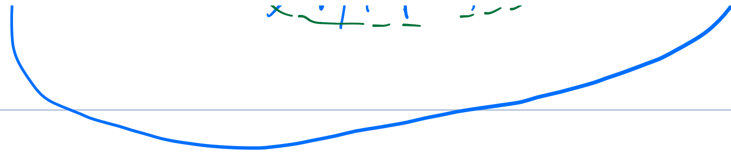
$\exists$ an open set $W \supset E^c$ s.t. $\mu(W - E^c) < \frac{\varepsilon}{2}$.

$F = W^c$

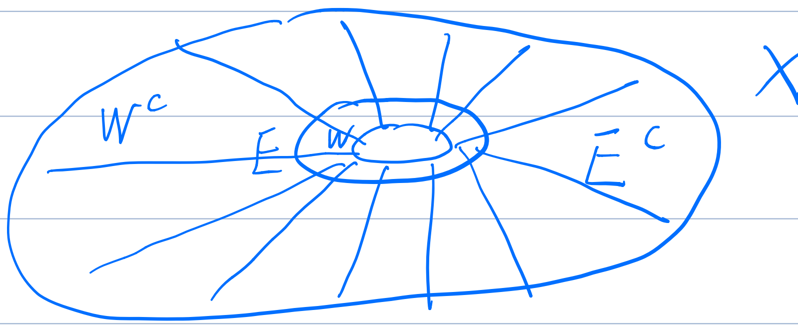
$X = \bigcup K_n$



$E \cap K_n$



$W \subset E$



(2). Every closed set $F \subset X$ is σ-compact

紧集闭子集为紧集 $\rightarrow$ $F \cap K_n$ is σ-compact

可数个可数还是可数 $\rightarrow$ $F = F \cap K_n$ is σ-compact

Hence (1) implies that $\forall E \in \mathfrak{M}$ is inner regular.

(3). If we apply (1) with $\varepsilon = \frac{1}{j}$ $(j = 1, 2, \cdots)$ we obtain closed sets $F_j \subset E \subset V_j$, and Put $A = \bigcup F_j$ and $B = \bigcap V_j$. Then $A \subset E \subset B$, A is an F_σ, B is a G_δ and $\mu(B - A) = 0$. since $B - A \subset V_j - F_j$,

$j = 1, 2, \cdots,$ $\square$

$\mu(B-A) < \mu(V_j - F_j) < \frac{1}{j} \quad (j \to \infty)$

注：以后可以直接用这几个性质.

Theorem 2. Let X be a locally compact Hausdorff space in which every open set is σ-compact. Let λ be any positive Borel measure on X s.t. $\lambda(K) < \infty$ for $\forall$ compact K. Then λ is regular.

Th1 中给出了 Riesz Th 给出的 μ 和 m.
此处无该条件.

Remark. R^k satisfies the present hypothesis