部分习题答案

chapter one

3. Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

[**proof**]:For each real number α, there exists an descending sequence $\{r_n\}$ of rational numbers such that $\lim_{n\to\infty} r_n = \alpha$. Moreover,we have

$$(\alpha, +\infty) = \bigcup_{n=1}^{\infty} [r_n, +\infty).$$

Hence,

$$f^{-1}((\alpha, +\infty)) = \bigcup_{n=1}^{\infty} f^{-1}([r_n, +\infty)).$$

Since sets $f^{-1}([r_n, +\infty))$ are measurable for each n, the set $f^{-1}((\alpha, +\infty))$ is also measurable. Then f is measurable.

4. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, +\infty]$, and prove the following assertions:

(a)

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n.$$

(b)

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all n, then

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

Show by an example that strict inequality can hold in (b).

[**proof**]: (a) Since

$$\sup_{k\geq n}(-a_k) = -\inf_{k\geq n} a_k, n = 1, 2, \cdots.$$

Therefore, let $n \to \infty$, it obtains

$$\lim_{n\to\infty} \sup_{k\geq n}(-a_k) = -\lim_{n\to\infty} \inf_{k\geq n} a_k.$$

By the definations of the upper and the lower limits, that is

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n.$$

(b) Since

$$\sup_{k\geq n}(a_k + b_k) \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k, n = 1, 2, \cdots.$$

Hence

$$\lim_{n\to\infty}\sup_{k\geq n}(a_k + b_k) \leq \lim_{n\to\infty}[\sup_{k\geq n} a_k + \sup_{k\geq n} b_k] = \lim_{n\to\infty}\sup_{k\geq n} a_k + \lim_{n\to\infty}\sup_{k\geq n} b_k.$$

By the definations of the upper and the lower limits, that is

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

example: we define

$$a_n = (-1)^n, \quad b_n = (-1)^{n+1}, \quad n = 1, 2, \cdots.$$

Then we have

$$a_n + b_n = 0, \quad n = 1, 2, \cdots.$$

But

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1.$$

(c)Because $a_n \leq b_n$ for all n, then we have

$$\inf_{k\geq n}(a_k) \leq \inf_{k\geq n} b_k, n = 1, 2, \cdots.$$

By the definations of the lower limits, it follows

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

5. (a)Supose $f : X \to [-\infty, +\infty]$ and $g : X \to [-\infty, +\infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}, \{f(x) = g(x)\}$$

are measurable.

(b)Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

[**proof**]: (a)Since for each rational r, the set

$$\{x : f(x) < r < g(x)\} = \{x : f(x) < r\} \cap \{x : r < g(x)\}$$

is measurable. And

$$\{x : f(x) < g(x)\} = \bigcup_{r \in \mathbb{Q}} \{x : f(x) < r < g(x)\}$$

Therefore the set $\{x : f(x) < g(x)\}$ is measurable. Also beacause $|f - g|$ is measurable function and the set

$$\{x : f(x) = g(x)\} = \bigcap_{n=1}^{\infty} \{x : |f(x) - g(x)| < \frac{1}{n}\}$$

is also measurable.

(b) Let $\{f_n(x)\}$ be the sequence of measurable real-valued functions, A be the set of points at which the sequence of measurable real-valued functions $\{f_n(x)\}$ converges (to a finite limit). Hence

$$A = \bigcap_{r=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n,m \geq k}^{\infty} \{x : |f_n(x) - f_m(x)| < \frac{1}{r}\}.$$

Therefore, A is measurable.

7. Suppose that $f_n : X \to [0, +\infty]$ is measurable for $n = 1, 2, 3, \cdots$, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$,for every $x \in X$and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu.$$

and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$"is omitted.

[**proof**]: Define $g_n = f_1 - f_n, n = 1, 2, \cdots$, then g_n is increasing measurable on X for $n = 1, 2, 3, \cdots$. Moreover, $g_n(x) \to f_1(x) - f(x)$ as $n \to \infty$, for every $x \in X$. By the Lebesgue's monotone convergence theorem, we have

$$\lim_{n \to \infty} \int_X g_n d\mu = \int_X (f_1 - f) d\mu.$$

Since $f_1 \in L^1(\mu)$, it shows

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu.$$

[**counterexample**]: Let $X = (-\infty, +\infty)$, and we define

$$f_n = \chi_{E_n}, E_n = [n, +\infty), n = 1, 2, \cdots.$$

8. Suppose that E is measurable subset of measure space (X, μ) with $\mu(E) > 0; \mu(X - E) > 0$, then we can prove that the strict inequality in the Fatou's lemma can hold.

10. Suppose that $\mu(X) < \infty, \{f_n\}$ is a sequence of bounded complex measurable functions on X ,and $f_n \to f$ uniformly on X. Proved that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu,$$

and show that the hypothesis "$\mu(X) < \infty$"cannot be omitted.

[**proof**]: There exist $M_n > 0, n = 1, 2, \cdots$ such that $|f_n(x)| \leq M_n$ for every $x \in X, n = 1, 2, \cdots$. Since $f_n \to f$ uniformly on X, there exists $M > 0$ such that

$$|f_n(x)| \leq M, |f(x)| \leq M \text{ on } X, n = 1, 2, \cdots.$$

By the Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

[**counterexample**]: Let $X = (-\infty, +\infty)$, and we define

$$f_n = \frac{1}{n} \text{ on } X, n = 1, 2, \cdots.$$

12. Suppose $f \in L^1(\mu)$. Proved that to each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

[**proof**]: Since

$$\int_X |f| d\mu = \sup \int_X s d\mu,$$

the supremum being taken over all simple measurable functions s such that $0 \leq s \leq |f|$. Hence, for each $\varepsilon > 0$, there exists a simple measurable function s with $0 \leq s \leq |f|$ such that

$$\int_X |f| d\mu \leq \int_X s d\mu + \frac{\varepsilon}{2}.$$

Suppose that $M > 0$, satisfying $0 \leq s(x) \leq M$ on X, and let $\delta = \frac{\varepsilon}{2M}$, we have

$$\int_E |f| d\mu \leq \int_E s d\mu + \frac{\varepsilon}{2} \leq Mm(E) + \frac{\varepsilon}{2} < \varepsilon$$

when $m(E) < \delta$.

部分习题答案

chapter two

1. Let $\{f_n\}$ be a sequence of real nonnegative functions on $\mathbb{R}^1$, and consider the following four statements:

(a)If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.

(b)If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.

(c)If each $\{f_n\}$ is upper semicontinuous, then $\sum_1^{\infty} f_n$ is upper semicontinuous.

(d)If each $\{f_n\}$ is lower semicontinuous, then $\sum_1^{\infty} f_n$ is lower semicontinuous.

Show that tree of these are true and that one is false. What happens if the word "nonnegative" is omitted? Is the truth of the statements affected if $\mathbb{R}^1$ is replaced by a general topological space?

[**proof**]: (a) Because for every real number α,

$$\{x : f_1(x) + f_2(x) < \alpha\} = \bigcup_{r\in\mathbb{Q}} [\{x : f_1(x) < r\} \cap \{x : f_2(x) < \alpha - r\}]$$

and f_1 and f_2 are upper semicontinuous, therefore $f_1 + f_2$ is upper semicontinuous.

(b) Because for every real number α,

$$\{x : f_1(x) + f_2(x) > \alpha\} = \bigcup_{r\in\mathbb{Q}} [\{x : f_1(x) > r\} \cap \{x : f_2(x) > \alpha - r\}]$$

and f_1 and f_2 are upper semicontinuous, therefore $f_1 + f_2$ is upper semicontinuous.

(c) This conclusion is false. counterexample: for each $n \in \mathbb{N}$, define

$$f_n = \begin{cases} 1; & x = r_n \\ 0; & \text{others} \end{cases}$$

which $\{r_n\}$ are the all rationals on $\mathbb{R}^1$. Then $\{f_n\}$ is a sequence of real nonnegative upper semicontinuous functions on $\mathbb{R}^1$, moreover we have

$$f(x) = \sum_1^\infty f_n(x) = \chi_{\mathbb{Q}}.$$

But $f(x)$ is not upper semicontinuous.

(d) According to the conclusion of (b), it is easy to obtain.

If the word "nonnegative" is omitted, (a) and (b) are still true but (c) and (d) is not. The truth of the statements is not affected if $\mathbb{R}^1$ is replaced by a general topological space.

2. Let f be an arbitrary complex function on $\mathbb{R}^1$, and define

$$\phi(x,\delta) = \sup\{|f(s) - f(t)| : s, t \in (x-\delta, x+\delta)\}$$

$$\phi(x) = \inf\{\phi(x,\delta) : \delta > 0\}$$

Prove that ϕ is upper semicontinuous, that f is continuous at a point x if and only if $\phi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a G_δ.

[**proof**]: It is obvious that

$$\phi(x) = \inf\{\phi(x, \frac{1}{k}) : k \in \mathbb{N}\} \geq 0$$

and

$$\phi(x, \frac{1}{k}) \geq \phi(x, \frac{1}{k+1}), \quad k \in \mathbb{N}$$

For any $\alpha > 0$, we will prove that the set $\{x : \phi(x) < \alpha\}$ is open. Suppose that $\phi(x) < \alpha$, then there exists a $N > 0$ such that

$$\phi(x, \frac{1}{k}) < \alpha, \quad k > N.$$

Take $y \in U(x, \frac{1}{2k})$, since $U(y, \frac{1}{2k}) \subset U(x, \frac{1}{k})$ and then

$$\phi(y) \leq \phi(y, \frac{1}{2k}) \leq \phi(x, \frac{1}{k}) < \alpha$$

So we have $y \in \{x : \phi(x) < \alpha\}$. Therefore,

$$U(x, \frac{1}{2k}) \subset \{x : \phi(x) < \alpha\}$$

Thus, it show that $\{x : \phi(x) < \alpha\}$ is open. Then it follows that ϕ is upper semicontinuous. In the following, we first suppose that $\phi(x) = 0$. for any $\epsilon > 0$, it exists $\delta > 0$ such that

$$\phi(x, \delta) < \epsilon$$

And so for any $y \in (x - \delta, x + \delta)$ we have

$$|f(x) - f(y)| \leq \phi(x, \delta) < \epsilon$$

That is f is continuous at x. Next, provided that f is continuous at x. Then for any $\epsilon > 0$ it exists $k \in \mathbb{N}$ satifying

$$|f(x) - f(y)| \leq \frac{\epsilon}{2}, \quad \forall y \in (x - \frac{1}{k}, x + \frac{1}{k})$$

and so $\phi(x, \frac{1}{k}) \leq \epsilon$. It shows

$$\phi(x) = \inf\{\phi(x, \frac{1}{k}) : k \in \mathbb{N}\} = 0$$

Finally, since

$$\{x : \phi(x) = 0\} = \bigcap_{n=1}^{\infty} \{x : \phi(x) < \frac{1}{n}\}$$

then it shows that the set of points of continuity of an arbitrary complex function is a G_δ.

3. Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) : \; y \in E\}$$

Show that ρ_E is a uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

[**Proof**]: Since for any $x_1, x_2 \in X$ and $y \in E$, we have

$$\rho(x_1, y) \leq \rho(x_1, x_2) + \rho(x_2, y)$$

Hence

$$\rho_E(x_1) \leq \rho(x_1, x_2) + \rho_E(x_2)$$

and so

$$|\rho_E(x_1) - \rho_E(x_2)| \leq \rho(x_1, x_2)$$

Therefore, it shows that ρ_E is a uniformly continuous function on X. In the following, suppose that $K \subset V \subset X$, K is compact subset, and V is open subset. Define

$$f(x) = \frac{\rho_{V^c}(x)}{\rho_{V^c}(x) + \rho_K(x)}$$

then $0 \leq f \leq 1$ and f is continuous on X. Moreover, $K \prec f \prec V$.

4. Examine the proof of the Riesz theorem and prove the following two statements:

(a) If $E_1 \subset V_1$ and $E_2 \subset V_2$, where V_1 and V_2 are disjoint open sets, then

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2),$$

even if E_1 and E_2 are not in $\mathcal{M}$.

(b) If $E \in \mathcal{M}_F$, then $E = N \cup K_1 \cup K_2 \cup \cdots$, where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

[**proof**]: (a) By the proof of the Riesz theorem, there exists a G_δ set G such that

$$E_1 \cup E_2 \subset G \ \text{ and } \ \mu(E_1 \cup E_2) = \mu(G).$$

Set $G_1 = V_1 \cap G,\ G_2 = V_2 \cap G$, then $E_1 \subset G_1,\ E_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$. Therefore

$$\mu(E_1)+\mu(E_2) \leq \mu(G_1)+\mu(G_2) = \mu(G_1 \cup G_2) = \mu(G \cap (V_1 \cup V_2)) \leq \mu(G) = \mu(E_1 \cup E_2).$$

And since

$$\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2),$$

It shows

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2).$$

(b) Since $E \in \mathcal{M}_F$, there exist a sequence of disjoint compact subsets $\{K_n\}$ such that

$$K_n \subset E - \bigcup_{i=0}^{n-1} K_i, m(E - \bigcup_{i=0}^{n-1} K_i) < m(K_n) + \frac{1}{n}, n = 1, 2, \cdots,$$

which we order $K_0 = \emptyset$.

Set $K = \bigcup_{n=1}^{\infty} K_n$. Since $m(E) < +\infty$, we obtain that

$$m(E - \bigcup_{i=1}^{n} K_i) < \frac{1}{n}, n = 1, 2, \cdots.$$

Whence we have

$$m(E - K) \le m(E - \bigcup_{i=1}^{n} K_i) < \frac{1}{n}, n = 1, 2, \cdots.$$

Set $N = E - K$, then it shows that $m(N) = 0$ and $E = K \cup N$.

5. Let E be Cantor's familiar "middle thirds" set. Show that $m(E) = 0$. even though E and $\mathbb{R}^1$ have the same cardinality.

[**proof**]: According to the construction of Cantor's set, it shows that the set E is closed set and

$$m(E) = m([0,1]) - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 0.$$

11. Let μ be a regular Borel measure on a compact Hausdorff space X; assume $\mu(X) = 1$. prove that there is a compact set $K \subset X$ such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K.

[**Proof**]: Define the set

$$\Omega = \{K_\alpha : K_\alpha \subset X, \quad K_\alpha \text{ is compact set with } \mu(k_\alpha) = 1\}$$

then $X \in \Omega$. Order

$$K = \bigcap_{K_\alpha \in \Omega} K_\alpha$$

Thus K is compact subset. Assume that $V \subset X$ is an open set with $K \subset V$. Since X is compact, V^c is also compact. Again since

$$V^c \subset \bigcup_{K_\alpha \in \Omega} K_\alpha^c$$

then there exist finite sets $K_{\alpha_1}, \cdots, K_{\alpha_n}$ such that

$$V^c \subset \bigcup_{k=1}^{n} K_{\alpha_k}^c$$

Because $\mu(X) = 1 = \mu(K_\alpha)$,

$$\mu(K_{\alpha_k}^c) = 0, \quad k = 1, 2, \cdots, n$$

and so $\mu(V^c) = 0$. We therefore have $\mu(V) = 1$. Since μ is regular, hence it shows that $\mu(K) = 1$. By the construction of K, it follows that $\mu(H) < 1$ for any proper compact subset of K. If $U \subset X$ is open set with $\mu(U) = 0$, it follows $\mu(U^c) = 1$. U^c is compact subset since X is compact, and so $K \subset U^c$ by the construction of K. Hence, $U \subset K^c$. It thus shows that K^c is the largest open set in X whose measure is 0.

12. Show that every compact subset of $\mathbb{R}^1$ is the support of a Borel measure.

[**Proof**]: Assume that μ is the Lebesgue measure on $\mathbb{R}^1$ and K is a compact subset of $\mathbb{R}^1$. Define

$$\phi(f) = \int_K f d\mu; \quad f \in C_c(\mathbb{R}^1).$$

Then ϕ is a positive linear functional on $C_c(\mathbb{R}^1)$. By Riesz representation theorem, there exists a σ-algebra $\mathfrak{M}$ in $\mathbb{R}^1$ which contains all Borel sets in $\mathbb{R}^1$, and there exists a unique positive measure σ on $\mathfrak{M}$ such that

$$\phi(f) = \int_{\mathbb{R}^1} f d\sigma, \quad f \in C_c(\mathbb{R}^1).$$

By theorem2.14, it shows

$$\sigma(K) = \inf\{\phi(f) : K \prec f\} = \mu(K)$$

$$\sigma(X) = \sup\{\phi(f) : f \prec X\} \leq \mu(K)$$

and so $\sigma(X) = \sigma(K) = \mu(K) < +\infty$. Therefore, it follows for any measurable set $A \subset X$,

$$\sigma(A) = \sigma(A \cap K).$$

That is, K is the support of σ.

14. Let f be a real-valued Lebesgue measurable function on $\mathbb{R}^k$. Prove that there exist Borel functions g and h such that $g(x) = h(x)$ a.e.$[m]$ and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}^k$.

[**Proof**]: Firstly, assume that $1 > f \geq 0$. By the proof of Theorem 2.24 in text, we have

$$f(x) = \sum_{n=1}^{\infty} t_n(x), \quad \forall x \in \mathbb{R}^k$$

and $2^n t_n$ is the characteristic function of a Lebesgue measurable set $T_n \subset \mathbb{R}^k (n \in \mathbb{N})$. According to the construction of Lebesgue measurable set, there exist Borel measurable sets $F_n, E_n (n \in \mathbb{N})$ satisfying

$$F_n \subset T_n \subset E_n; \quad m(E_n - F_n) = 0; \quad n \in \mathbb{N}.$$

Define

$$g = \sum_{n=1}^{\infty} 2^{-n} \chi_{F_n}; \quad h = \sum_{n=1}^{\infty} 2^{-n} \chi_{E_n}$$

then g, h are Borel measurable functions on $\mathbb{R}^k$ and $g(x) \leq f(x) \leq h(x)$. Moreover, $g(x) = h(x)$ a.e.$[m]$.

Secondly, it is easy that the conclusion is correct for $0 \leq f < M$, and hence it is also for bounded real-valued Lebesgue measurable function on $\mathbb{R}^k$.

Finally, if f be a real-valued Lebesgue measurable function on $\mathbb{R}^k$ and if

$$B_n = \{x : |f(x)| \leq n\}, \quad n = 1, 2, \cdots$$

then we have

$$\chi_{B_n}(x) f(x) \to f(x), \quad \text{as } n \to \infty.$$

According to the conclusion above, there exist Borel functions g_n and h_n such that $g_n(x) = h_n(x)$ a.e.$[m]$ and $g_n(x) \leq \chi_{B_n}(x) f(x) \leq h_n(x)$ for every $x \in \mathbb{R}^k$. Take

$$g = \limsup g_n; \quad h = \liminf h_n$$

then $g(x) = h(x)$ a.e.$[m]$ and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}^k$.

15. It is easy to guess the limits of

$$\int_0^n (1 - \frac{x}{n})^n e^{\frac{x}{2}} dx \text{ and } \int_0^n (1 + \frac{x}{n})^n e^{-2x} dx$$

as $n \to \infty$. Prove that your guesses are correct.

[**proof**]: Define

$$f_n(x) = \chi_{[0,n]} (1 - \frac{x}{n})^n e^{\frac{x}{2}}; \quad g_n(x) = \chi_{[0,n]} (1 + \frac{x}{n})^n e^{-2x}; \quad n \in \mathbb{N}$$

Then it is easy that

$$|f_n(x)| \le e^{-\frac{x}{2}}, \text{ and } f_n(x) \to e^{-\frac{x}{2}};$$

$$|g_n(x)| \le e^{-x}, \text{ and } f_n(x) \to e^{-x}.$$

By the Lebesgue's Dominated Convergence Theorem, it shows

$$\lim_{n\to\infty} \int_0^n (1-\frac{x}{n})^n e^{\frac{x}{2}} dx = \lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty e^{-\frac{x}{2}}dx = 2$$

$$\lim_{n\to\infty} \int_0^n (1+\frac{x}{n})^n e^{-2x} dx = \lim_{n\to\infty} \int_0^\infty g_n(x)dx = \int_0^\infty e^{-x}dx = 1$$

17. Define the distance between points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \quad \text{if } x_1 = x_2; \qquad 1 + |y_1 - y_2| \quad \text{if} \quad x_1 \ne x_2.$$

show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let $x_1, x_2, \cdots, x_n$ be those values of x for which $f(x,y) \ne 0$ for at least one y (there are only finitely many such x!), and define

$$\Lambda f = \sum_{j=1}^n \int_{-\infty}^{+\infty} f(x_j, y)dy$$

Let μ be the measure associated with this Λ by Theorem2.14. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

[**proof**]:Order

$$P_i = (x_i, y_i), \quad i = 0, 1, 2, 3, \cdots.$$

$$d(P_1, P_2) = \begin{cases} |y_1 - y_2| & \text{if} \quad x_1 = x_2; \\ 1 + |y_1 - y_2| & \text{if} \quad x_1 \ne x_2 \end{cases}$$

Since

$$d(P_1, P_2) \le \begin{cases} |y_1 - y_3| + |y_2 - y_3| & \text{if} \quad x_1 = x_2; \\ 1 + |y_1 - y_3| + |y_2 - y_3| & \text{if} \quad x_1 \ne x_2 \end{cases}$$

Therefore,

$$d(P_1, P_2) \le d(P_1, P_3) + d(P_2, P_3).$$

That is d is metric. Take $X = (\mathbb{R}^2, d)$. Then

$$P_n \to P \Leftrightarrow \exists\ N > 0, \text{such that if } n > N, x_n = x; \text{and} |y_n - y| \to 0.$$

and for any $0 < \delta < 1$,

$$U(P_0, \delta) = \{P \in X : d(P_0, P) < \delta\} = \{P \in X : x = x_0, \text{and } |y - y_0| < \delta\}$$

$$\overline{U(P_0, \delta)} = \{P \in X : d(P_0, P) \leq \delta\} = \{P \in X : x = x_0, \text{and } |y - y_0| \leq \delta\}$$

Moreover, $\overline{U(P_0, \delta)}$ is compact subset of X. Thus X is locally compact Hausdorff space.

In the following, we order the set for $f \in C_c(X)$

$$\Omega_f = \{x \in \mathbb{R} : \exists\ y \in \mathbb{R} \text{ such that } f(x, y) \neq 0\}$$

Let K be the support of f, K is compact. Then there exists finite $P_1, P_2, \cdots, P_n \in K$ for any $0 < \delta < 1$ such that

$$K \subset \bigcup_{j=1}^{n} \overline{U(P_j, \delta)}$$

and so $\Omega_f = \{x_1, x_2, \cdots, x_n\}$. Since

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{+\infty} f(x_j, y) dy$$

it is easy that Λ is a positive linear functional on $C_c(X)$. There thus exist a measure μ associated with this Λ by Theorem2.14. Let E be the x-axis and $0 < \delta < 1$. Define

$$V = \bigcup_{P_i \in E} U(P_i, \delta); \quad \text{where } P_i = (x_i, 0);$$

it shows V is open and $E \subset V$. For any finite elements $P_1, P_2, \cdots, P_n \in E$, the set

$$F = \bigcup_{j=1}^{n} \overline{U(P_j, \frac{\delta}{2})}$$

is compact with $F \subset V$. By the Urysohn's lemma, there exists $f \in C_c(X)$ satisfying $F \prec f \prec V$. So we have

$$\mu(V) \geq \Lambda f \geq n\delta.$$

Hence, it shows $\mu(V) = \infty$ as $n \to \infty$. Therefore, $\mu(E) = \infty$. Assume that $K \subset E$ is compact, then it is obvious that K is finite set. Take

$$K = \{P_1, P_2, \cdots, P_m\}$$

For any $0 < \epsilon < 1$, define

$$G = \bigcup_{j=1}^{m} U(P_j, \epsilon)$$

then g is open and $K \subset G$. By the Urysohn's lemma, there exists $g \in C_c(X)$ satisfying $K \prec g \prec G$. Since

$$\mu(K) \leq \Lambda g \leq 2\epsilon$$

whence $\mu(K) = 0$.

21. If X is compact and $f : X \to (-\infty, +\infty)$ is upper semicontinuous, prove that f attains its maximum at some point of X.

[**proof**]: for every $t \in f(X)$, define

$$E_t = \{x \in X : f(x) < t\}$$

Assume that f does not attain its maximum at some point of X. Then it shows $X = \bigcup_{t \in f(X)} E_t$. Since f is continuous, E_t is open sets. Again since X is compact, there exists finite $t_i (i = 1, 2, \cdots, n) \in f(X)$ such that $X = \bigcup_{i=1}^{n} E_{t_i}$. Take

$$t_0 = \max\{t_1, t_2, \cdots, t_n\}$$

then we have

$$f(x) < t_0, \quad \forall x \in X.$$

But it is in contradiction with $t_0 \in f(X)$. It therefore is proved that f attains its maximum at some point of X.

部分习题答案

chapter three

1. Prove that the supremum of any collection of convex functions on (a, b) is convex on (a, b) and that pointwise limits of sequences of convex functions are convex. What can you say about upper and lower limits of sequences of convex functions?

[**Proof**]: Suppose that $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta = 1$. Let $\{f_i\}_{i \in I}$ be any collection of convex functions on (a, b). for any $x, y \in (a, b)$, we have

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y), \quad \forall i \in I.$$

It prove that

$$\sup_{i \in I} f_i(\alpha x + \beta y) \leq \alpha \sup_{i \in I} f_i(x) + \beta \sup_{i \in I} f_i(y).$$

So $\sup\limits_{i \in I} f_i$ is convex.
Assume $I = \mathbb{N}$ and f is pointwise limits of sequences of convex functions $\{f_i\}$. according to the properties of limit, we have

$$\lim_{i \to \infty} f_i(\alpha x + \beta y) \leq \alpha \lim_{i \to \infty} f_i(x) + \beta \lim_{i \to \infty} f_i(y)$$

That is $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$ and so f is convex.
By the definition of upper limits and the conclusion above, it is easy that the upper limits of sequences of convex functions is convex. But the lower limits of sequences of convex functions is false.

2. If ϕ is convex on (a, b) and if ψ is convex and nondecreasing on the range of ϕ, prove that $\psi \circ \phi$ is convex on (a, b). For $\phi > 0$, show that the convexity of $\ln \phi$ implies the convexity of ϕ , but not vice versa.

[**Proof**]: Assume that

$$x, y \in (a, b); \alpha, \beta \geq 0 \text{and } \alpha + \beta = 1.$$

Then

$$\psi \circ \phi(\alpha x + \beta y) \leq \psi(\alpha \phi(x) + \beta \phi(y)) \leq \alpha \psi \circ \phi(x) + \beta \psi \circ \phi(y).$$

and so $\psi \circ \phi$ is convex on (a,b). Since e^t is nondecreasing convex on $\mathbb{R}$ and $\phi = e^{\ln\phi}$, it shows from the conclusion above that the convexity of $\ln\phi$ implies the convexity of ϕ for $\phi > 0$.

[**counterexample**]: $\phi(x) = x^2$ is convex on $(0,\infty)$ but $\ln\phi(x) = 2\ln x$ is not convex. Moreover, for $\phi > 0$, it can show that the convexity of $\log_c \phi (c > 1)$ implies the convexity of ϕ.

3. Assume that ϕ is a continuous real function on (a,b) such that

$$\phi(\frac{x+y}{2}) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y)$$

for all x and $y \in (a,b)$. Prove that ϕ is convex.

[**Proof**]: According to the definition of convex function,we only need to prove the case $0 < \lambda < \frac{1}{2}$. Take

$$E = \{\frac{k}{2^n} \mid k = 1,2,\cdots,2^n - 1;\ n \in \mathbb{N}\}.$$

For $n = 1$, it is obvious that

$$\phi(\frac{x+y}{2}) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y)$$

for all x and $y \in (a,b)$.

For $n = 2$, suppose that $\lambda_k = \frac{k}{4}, k = 1$.

$$\begin{aligned}
\phi(\lambda_1 x + (1-\lambda_1)y) &= \phi(\tfrac{1}{2}(\tfrac{x+y}{2} + y)) \\
&\leq \tfrac{1}{2}\phi(\tfrac{x+y}{2}) + \tfrac{1}{2}\phi(y) \\
&\leq \tfrac{1}{2}[\tfrac{1}{2}\phi(x) + \tfrac{1}{2}\phi(y)] + \tfrac{1}{2}\phi(y) \\
&= \tfrac{1}{4}\phi(x) + \tfrac{3}{4}\phi(y) \\
&= \lambda_1\phi(x) + (1-\lambda_1)\phi(y)
\end{aligned}$$

Assume that the case

$$\lambda = \frac{k}{2^{n-1}}, k = 1,2,\cdots,2^{n-1} - 1$$

is correct. For any $\lambda \in E$ with $\lambda = \frac{k}{2^n}; 1 \leq k \leq 2^{n-1}, n \geq 2$, and for all x and $y \in (a,b)$, it shows that

(i)

$$\phi(\lambda x + (1-\lambda)y) = \phi(\frac{x+y}{2}) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y) = \lambda\phi(x) + (1-\lambda)\phi(y)$$

when $k = 2^{n-1}$.

(ii)for $1 \le k < 2^{n-1}$,

$$\lambda x + (1 - \lambda y) = \frac{k}{2^n}x + (1 - \frac{k}{2^n})y = \frac{1}{2}(\frac{k}{2^{n-1}}x + (1 - \frac{k}{2^{n-1}})y + y)$$

Since $\frac{k}{2^{n-1}}x + (1 - \frac{k}{2^{n-1}})y \in (a, b)$ and so

$$\begin{aligned} \phi(\lambda x + (1 - \lambda)y) &\le \tfrac{1}{2}\phi(\tfrac{k}{2^{n-1}}x + (1 - \tfrac{k}{2^{n-1}})y) + \tfrac{1}{2}\phi(y) \\ &\le \tfrac{k}{2^n}\phi(x) + \tfrac{1}{2}(1 - \tfrac{k}{2^{n-1}})\phi(y) + \tfrac{1}{2}\phi(y) \\ &= \lambda\phi(x) + (1 - \lambda)\phi(y) \end{aligned}$$

For any $0 < \lambda < 1$, there exists a sequence $\{\lambda_n\} \subset E$ with $\lim_{n\to\infty} \lambda_n = \lambda$. Thus

$$\begin{aligned} \phi(\lambda x + (1 - \lambda)y) &= \lim_{n\to\infty} \phi(\lambda_n x + (1 - \lambda_n)y) \\ &\le \lim_{n\to\infty} (\lambda_n\phi(x) + (1 - \lambda_n)\phi(y)) \\ &= \lambda\phi(x) + (1 - \lambda)\phi(y) \end{aligned}$$

Therefore, it shows that ϕ is convex on (a, b).

4. Suppose f is a complex measurable function on X, μ is a positive measure on X, and

$$\phi(p) = \int_X |f|^p d\mu = \|f\|_p^p \quad (0 < p < \infty).$$

Let $E = \{p : \phi(p) < \infty\}$, Assume $\|f\|_\infty > 0$.

(a) If $r < p < s,\ r \in E$, and $s \in E$, Prove that $p \in E$.

(b) Prove that $\ln\phi$ is convex in the interior of E and that ϕ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? losed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \le \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that

$$\|f\|_p \to \|f\|_\infty \text{ as } \ p \to \infty.$$

[**Proof**]: (a) If $r < p < s$, there exist $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha + \beta = 1$ and $p = \alpha r + \beta s$. By Hölder's inequality, it follows that

$$\|f\|_p^p = \phi(p) = \int_X |f|^p d\mu = \int_X |f|^{\alpha r + \beta s} d\mu \le \{\int_X |f|^r d\mu\}^\alpha \cdot \{\int_X |f|^s d\mu\}^\beta = \|f\|_r^{\alpha r} \cdot \|f\|_s^{\beta s}.$$

As $r \in E$ and $s \in E$, it prove that $p \in E$.

(b) By (a), it reduces that E is convex. Moreover, E is connected. Since $E \subset (0, \infty)$, the interior of E is an interval and denoted by (a, b). For any s, t in the interior of E and $0 < \alpha < 1$ and $0 < \beta < 1$, it follows from (a) that

$$\phi(\alpha s + \beta t) \leq \phi(s)^{\alpha} \cdot \phi(t)^{\beta}.$$

So it proves

$$\ln \phi(\alpha s + \beta t) \leq \ln[\phi(s)^{\alpha} \cdot \phi(t)^{\beta}] = \alpha \ln \phi(s) + \beta \ln \phi(t).$$

That is, $\ln \phi$ is convex in the interior of E. Moreover, $\ln \phi$ is continuous on (a, b). Thus, ϕ is also continuous on (a, b). Provided that $a \in E$. For any decreasing sequence $\{s_n\}$ of (a, b) with $\lim\limits_{n \to \infty} s_n = a$, it is easy to show that

$$\lim_{n \to \infty} |f(x)|^{s_n} = |f(x)|^{a}, \quad \forall x \in X.$$

Since $|f(x)|^{s_n} \leq \max |f(x)|^{a}, |f(x)|^{s_1}, \quad \forall x \in X$ and $|f(x)|^{a}, |f(x)|^{s_1} \in L^1(\mu)$, by the Lebesgue control convergence theorem, it follows that

$$\lim_{n \to \infty} \phi(s_n) = \phi(a).$$

Similarly, the other cases can be proved. Hence ϕ is continuous on E.

(c) By (a), it shows that E is connected and convex. E can be any interval on $(0, \infty)$. For examples: take $X = (0, \infty)$ and μ is the Lebesgue measure on X.

(i) For $E = (a, b), (0 < a < b)$, take

$$f(x) = \begin{cases} x^{-\frac{1}{b}}, & x \in (0, 1); \\ x^{-\frac{1}{a}}, & x \in [1, \infty) \end{cases}$$

(ii)For $E = [a, b], (0 < a < b)$, take

$$f(x) = \begin{cases} x, & x \in (0, 1); \\ x^{-\frac{1}{a} - 1}, & x \in [1, \infty) \end{cases}$$

(iii)For $E = [a, b), (0 < a < b)$, take

$$f(x) = \begin{cases} x^{-\frac{1}{b}}, & x \in (0, 1); \\ x^{-\frac{1}{a} - 1}, & x \in [1, \infty) \end{cases}$$

(iv)For $E = (a, b], (0 < a < b)$, take

$$f(x) = \begin{cases} x^{-\frac{1}{b+1}}, & x \in (0,1); \\ x^{-\frac{1}{a}}, & x \in [1, \infty) \end{cases}$$

(v)For $E = \emptyset$, take $f \equiv 1$.

(d) If $r < p < s$,by (a), there exist $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha + \beta = 1$ and $p = \alpha r + \beta s$. And it is easy to know

$$\phi(p) = \phi(\alpha r + \beta s) \leq \phi(r)^{\alpha} \cdot \phi(s)^{\beta} \leq \max \phi(r), \phi(s).$$

Thus, $\|f\|_p \leq \max \|f\|_r, \|f\|_s$. Moreover, it shows that $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e)Firstly since $\|f\|_{\infty} > 0$, it shows that $\|f\|_p > 0$ for any $0 < p < \infty$. secondly, assume that $\|f\|_{\infty} = +\infty$. Hence,$\mu(\{x \in x : |f(x)| > 2\}) > 0$. For any $p > r$,

$$0 < 2^p \mu(\{x \in x : |f(x)| > 2\}) \leq \int_X |f|^p d\mu = \|f\|_p^p.$$

Therefore, $\lim_{n \to \infty} \|f\|_p = +\infty$.
Thirdly, assume that $\|f\|_{\infty} = 1$ and $\|f\|_r > 0$. Without generality, assume that $|f(x)| \leq 1, \forall x \in X$. Thus For any $p > r$, it shows that

$$\|f\|_p^p \leq \|f\|_r^r.$$

That is $\|f\|_p \leq \|f\|_r^{\frac{r}{p}}$. And so

$$\lim_{p \to \infty} \|f\|_p = 1 = \|f\|_{\infty}.$$

The general case can reduce to the case $\|f\|_{\infty} = 1$ and $\|f\|_r > 0$. This is completed the proof.

5. Assume, in addition to the hypotheses of Exercise 4, that

$$\mu(X) = 1.$$

(a) Prove that $\|f\|_r \leq \|f\|_s$, if $0 < r < s \leq \infty$.
(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?
(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these

two spaces contain the same functions?
(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p\to 0} \|f\|_p = \exp\{\int_X \log|f| d\mu\}$$

if $\exp\{-\infty\}$ is defined to be 0.

[**Proof**]: (a) When $s = \infty$, it is easy that

$$\int_X |f|^r d\mu \leq \|f\|_\infty^r \mu(X)$$

and so $\|f\|_r \leq \|f\|_\infty$. If $s < \infty$, $\phi(x) = x^{\frac{s}{r}}$ is convex on $(0, \infty)$ and by the Jensen's Inequality, it shows

$$\int_X |f|^s d\mu = \int_X (|f|^r)^{\frac{s}{r}} d\mu \geq (\int_X (|f|^r) d\mu)^{\frac{s}{r}}$$

and whence $\|f\|_s \geq \|f\|_r$.

(b) Assume that $s = \infty$, and $\|f\|_r = \|f\|_\infty < \infty (0 < r < s)$. Then for any $1 < p < \infty$, it shows by (a) that

$$\|f\|_1 = \|f\|_p = \|f\|_q < \infty$$

and so f is constant almost everywhere by the Holder's inequality. Secondly, assume $0 < r < s < \infty$. Take $\alpha = \frac{s}{r} > 1$, then by the Holder's inequality

$$\int_X |f|^r \cdot 1 d\mu \leq (\int_X |f|^s d\mu)^{\frac{r}{s}} = \|f\|_s^r = \|f\|_r^r$$

Therefore, $|f|^r =$ constant $a.e.$ on X. So it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$ if and only if $f =$ constant $a.e.$ on X.

(c)If $0 < r < s$, then by (a) it is easy to prove $L^s(\mu) \subset L^r(\mu)$. Moreover, we will prove the following:

$L^r(\mu) = L^s(\mu) \Leftrightarrow$ there exist finite pairwise disjoint measurable sets at most in

$$\Omega = \{E \subset X : \mu(E) > 0\}.$$

Firstly, assume that there are infinte pairwise disjoint members in Ω. Then it exists a sequence $\{E_n\} \subset X$ of pairwise disjoint measurable sets such that

$$0 < \mu(E_n) < 1, \; n = 1, 2, \cdots.$$

Take $\frac{1}{s} < t < \frac{1}{r}$, and define

$$f = \sum_{n=1}^{\infty} n^{-t} \chi_{E_n}$$

Then

$$\begin{aligned} \int_X |f|^s d\mu &= \int_X \sum_{n=1}^{\infty} n^{-ts} \chi_{E_n} d\mu \\ &= \sum_{n=1}^{\infty} n^{-ts} \mu E_n \\ &\leq \sum_{n=1}^{\infty} n^{-ts} \end{aligned}$$

Since $st > 1$, it has $\sum_{n=1}^{\infty} n^{-t} < +\infty$ and so $f \in L^s(\mu)$. But

$$\begin{aligned} \int_X |f|^r d\mu &= \int_X \sum_{n=1}^{\infty} n^{-tr} \chi_{E_n} d\mu \\ &= \sum_{n=1}^{\infty} n^{-tr} \mu E_n \\ &\geq \sum_{n=1}^{\infty} n^{-tr} \cdot n^{-\alpha} \\ &= \sum_{n=1}^{\infty} n^{-(tr+\alpha)} \end{aligned}$$

which $\alpha = \frac{1-tr}{2}$. Since $0 < \alpha + tr < 1$, it shows that $\sum_{n=1}^{\infty} n^{-(tr+\alpha)} = +\infty$. Hence, $f \notin L^r(\mu)$.

Next, we assume that there exist finite pairwise disjoint measurable sets at most in Ω. Provided that $f \in L^r(\mu)$, then there exists an increasing seqnence $\{h_n\}$ of simple measurable functions such that

$$0 \leq h_n(x) \leq h_{n+1}(x) \leq \cdots \leq |f(x)|, \ forall x \in X$$

and

$$h_n(x) \to |f(x)|, n \to \infty, \forall x \in X.$$

Put

$$t_1(x) = h_1(x); t_n(x) = h_n(x) - h_{n-1}(x), \ n = 2, 3, \cdots$$

then $2^n t_n$ is the characteristic function of a measurable set $T_n \subset X$, and $\{T_n\}$ are pairwise disjoint. Moreover,

$$f(x) = \sum_{n=1}^{\infty} t_n(x), \quad \forall x \in X.$$

According to the assumption, there are finite members of $\{T_n\}$ with nonzero measure at most. Thus f equal to a simple function almost everywhere and so $f \in L^s(\mu)$. Therefore, $L^r(\mu) = L^s(\mu)$.

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, then

$$\|f\|_p \le \|f\|_r < +\infty;\ \|f\|_p \ge e^{\int_X \ln|f|d\mu};\quad 0 < p < r.$$

and $\{\|f\|_p\}$ is increasing relative to p. Therefore, the limit $\lim\limits_{p\to 0} \|f\|_p = c$ exists. If c=0, it is easy that $e^{\int_X \ln|f|d\mu} = 0$. In the following, we assume $c > 0$. Since $\|f\|_r < +\infty$, we may suppose that f is finite everywhere. Let

$$E = [|f| > 1];\quad F = [|f| < 1];\quad G = [|f| = 1]$$

So we have

$$\begin{aligned}\|f\|_p &= (\int_X |f|^p d\mu)^{\frac{1}{p}} \\ &= (\int_E (|f|^p - 1 + 1)d\mu + \int_F (|f|^p - 1 + 1)d\mu + \int_G (|f|^p - 1 + 1)d\mu)^{\frac{1}{p}} \\ &= (1 + \int_E (|f|^p - 1)d\mu + \int_F (|f|^p - 1)d\mu)^{\frac{1}{p}}\end{aligned}$$

Take $\alpha > 1;\quad 0 < \beta < 1$, because

$$\lim_{x\to 1} \frac{x-1}{\ln x} = 1;\quad |f(x)|^p \to 1,\ p \to 0$$

there exists $\delta > 0$ such that

$$|f(x)|^p - 1 < \alpha \ln|f(x)|^p,\ \forall x \in E;\quad |f(x)|^p - 1 < \beta \ln|f(x)|^p,\ \forall x \in F$$

when $p < \delta$. Thus

$$\begin{aligned}\|f\|_p &\le (1 + \int_E \alpha \ln|f|^p d\mu + \int_F \beta \ln|f|^p d\mu)^{\frac{1}{p}} \\ &= (1 + p(\alpha \int_E \ln|f|d\mu + \beta \int_F \ln|f|d\mu)^{\frac{1}{p}}\end{aligned}$$

Then

$$0 < c = \lim_{p\to 0} \|f\|_p \le e^{\alpha \int_E \ln|f|d\mu + \beta \int_F \ln|f|d\mu}$$

Let

$$\alpha \to 1^+;\quad \beta \to 1^-$$

then

$$c = \lim_{p\to 0} \|f\|_p \le e^{\int_E \ln|f|d\mu + \int_F \ln|f|d\mu} = e^{\int_X \ln|f|d\mu}.$$

Hence

$$\lim_{p\to 0} \|f\|_p = e^{\int_X \ln|f|d\mu}.$$

11. Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_\Omega f d\mu \cdot \int_\Omega g d\mu \geq 1.$$

[**Proof**]: Since f and g are positive measurable functions on Ω such that $fg \geq 1$, $f^{\frac{1}{2}}$ and $g^{\frac{1}{2}}$ are positive measurable functions on Ω such that $(fg)^{\frac{1}{2}} \geq 1$. By Hölder's inequality, it follows that

$$1 \leq (\int_\omega (fg)^{\frac{1}{2}} d\mu)^2 \leq \int_\Omega f d\mu \cdot \int_\Omega g d\mu.$$

12. Suppose $\mu(\Omega) = 1$ and $h : \Omega \to [0, \infty]$ is measurable. If

$$A = \int_\Omega h d\mu,$$

prove that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} d\mu \leq 1 + A.$$

[**Proof**]: It is obvious correct when $A = \infty$ or 0 and so we assume $0 < A < +\infty$ and $h : \Omega \to (0, \infty)$. Since $\phi(x) = \sqrt{1 + x^2}$ is convex on $(0, \infty)$, it shows by the Jensen's Inequality that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} d\mu.$$

Because $\sqrt{1 + h^2} \leq 1 + h$,

$$\int_\Omega \sqrt{1 + h^2} d\mu \leq 1 + A.$$

This complete the proof.

14. Suppose $1 < p < \infty$, $f \in L^p = L^p((0, \infty))$, relative to Lebesgue measure and

$$F(x) = \frac{1}{x} \int_0^x f(t) dt \quad (0 < x < \infty).$$

(a) Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p-1} \|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p.

(b) Prove that equality holds only if $f = 0 a.e.$

(c) Prove that the constant $p/(p-1)$ cannot be replaced by a smaller one

.

(d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

[**Proof**]: (a) Firstly, assume that $f \geq 0$ and $f \in C_c((0, \infty))$. Then

$$F(x) = \frac{1}{x} \int_0^x f(t)dt \quad (0 < x < \infty)$$

is differential on $(0, \infty)$ and so $xF'(x) = f(x) - F(x), \forall x \in (0, \infty)$. Therefore

$$\begin{aligned} \int_0^\infty F^p(x)dx &= F^p(x)x \mid_0^\infty - p\int_0^\infty F^{p-1}(x)F'(x)dx \\ &= -p\int_0^\infty F^{p-1}(x)xF'(x)dx \\ &= -p\int_0^\infty F^{p-1}(x)(f(x) - F(x))dx \\ &= -p\int_0^\infty F^{p-1}(x)f(x)dx + p\int_0^\infty F^p(x)dx \end{aligned}$$

Thus

$$(p-1)\int_0^\infty F^p(x)dx = p\int_0^\infty F^{p-1}(x)f(x)dx \leq p(\int_0^\infty F^{(p-1)q}(x)dx)^{\frac{1}{q}}(\int_0^\infty f^p(x)dx)^{\frac{1}{p}}$$

and so

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p.$$

Moreover, it is easy that

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p, \quad \forall f \in C_c((0, \infty)).$$

Given $f \in L^p((0, \infty))$, since $C_c((0, \infty))$ is dense in $L^p((0, \infty))$, there exist a sequence $\{f_n\} \subset C_c((0, \infty))$ such that

$$\|f_n - f\|_p \to 0, \text{ as } n \to \infty.$$

That is, for any $\epsilon > 0$, there exists $N > 0$ such that $\|f_n - f\|_p < \epsilon$ if $n > N$. It shows by the Jensen's Inequality that

$$\begin{aligned} |F_n(x) - F(x)| &\leq \tfrac{1}{x}\int_0^x |f_n(t) - f(t)|dt) \\ &\leq \tfrac{1}{x}(\int_0^x |f_n(t) - f(t)|^p dt)^{\frac{1}{p}} \cdot x^{\frac{1}{q}} \\ &\leq x^{-\frac{1}{p}}\|f_n - f\|_p \end{aligned}$$

and thus

$$F_n(x) \to F(x), \ n \to \infty, \ \forall x \in (0, \infty).$$

Since

$$\|F_n\|_p \le \frac{p}{p-1}\|f_n\|_p,\ n = 1, 2, \cdots$$

it shows that $F_n \in L^p$ and $\{F_n\} \subset L^p$ is Cauchy sequence. By the completion of L^p, it exists $G(x) \in L^p$ satisfying

$$\|F_n - G\|_p \to 0, n \to \infty$$

and so $\|F_n\|_p \to \|G\|_p < +\infty$. By the Fatou's Lemma,

$$\|F\|_p \le \lim_{n\to\infty} \|F_n\|_p = \|G\|_p < +\infty$$

whence $F \in L^p$. Moreover,

$$\|F\|_p \le \|G\|_p = \lim_{n\to\infty} \|F_n\|_p \le \lim_{n\to\infty} \frac{p}{p-1}\|f_n\|_p = \frac{p}{p-1}\|f\|_p.$$

Therefore, the map $f \to F$ is continuous from $L^p(0,\infty)$ to $L^p(0,\infty)$.

(b) Firstly, assume that $f \ge 0$ and $f \in C_c(0,\infty)$. Then, by (a), we have

$$\begin{aligned}
\textstyle\int_0^\infty F^p(x)dx &= \tfrac{p}{p-1}\textstyle\int_0^\infty F^{p-1}(x)f(x)dx \\
&\le \tfrac{p}{p-1}(\textstyle\int_0^\infty F^{(p-1)q}(x)dx)^{\frac{1}{q}}(\textstyle\int_0^\infty f^p(x)dx)^{\frac{1}{p}} \\
&\le \tfrac{p}{p-1}(\textstyle\int_0^\infty F^p(x)dx)^{\frac{1}{q}}(\textstyle\int_0^\infty f^p(x)dx)^{\frac{1}{p}}
\end{aligned}$$

and so

$$\begin{aligned}
\|F\|_p &= \tfrac{p}{p-1}\|f\|_p \\
&\Leftrightarrow \exists\ \alpha,\beta \in \mathbb{R} \text{ and } \alpha\beta \ne 0 \text{ satisfying } \alpha F^p = \beta f^p\ a.e. \\
&\Leftrightarrow \alpha F = \beta f\ a.e. \\
&\Leftrightarrow f = 0\ a.e.
\end{aligned}.$$

Therefore, it shows the conclusion $f = 0$ $a.e.$ if $f \in C_c(0,\infty)$ and $\|F\|_p = \frac{p}{p-1}\|f\|_p$.

(c)

(d)Suppose that $f > 0$ and $f \in L^1$, then $G = \int_0^\infty f d\mu < +\infty$ and so it exists $N > 0$ such that

$$\int_0^x f(t)dt > \frac{G}{2}$$

when $x > N$. Thus

$$\int_0^\infty F(x)dx \ge \int_N^\infty \frac{1}{x}\int_0^x f(t)dt \ge \frac{G}{2}\int_N^\infty \frac{1}{x}dx = +\infty$$

and therefore $F \notin L^1$.

18. Let μ be a positive measure on X. A sequence $\{f_n\}$ of complex measurable functions on X is said to converge in measure to the measurable function f if to every $\varepsilon > 0$ there corresponds an N such that

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$$

for all $n > N$. (This notion is of importance in probability theory.) Assume $\mu(X) < \infty$ and prove the following statements:

(a) If $f_n(x) \to f(x) a.e.$, then $f_n \to f$ in measure.

(b) If $f_n \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure; here $1 \leq p \leq \infty$.

(c) If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence which converges to $f a.e.$

Investigate the converses of (a) and (b). What happen to (a), (b),and (c) if $\mu(X) = \infty$, for instance, if μ is Lebesgue measure on $\mathbb{R}^1$?

[**Proof**]: (a) Set $E = \{x \in X : \lim\limits_{n\to\infty} f_n(x) \neq f(x)\}$, then $\mu(E) = 0$. For every $\varepsilon > 0$, we define

$$E_k = \{x \in X : |f_n(x) - f(x)| > \varepsilon\},\ k = 1, 2, \cdots;$$

$$F_n = \bigcup_{k \geq n}^{\infty} E_k,\ n = 1, 2, \cdots; F = \bigcap_{n=1}^{\infty} F_n.$$

It is easy that $F \subset E$ and so $\mu(F) = 0$. Since $\{F_n\}$ is a sequence of decreasing measurable sets and $\mu(X) < +\infty$, it shows that

$$\lim_{n\to\infty} \mu(F_n) = \mu(F) = 0.$$

As $E_n \subset F_n,\ n = 1, 2, \cdots$, it is trival that

$$\lim_{n\to\infty} \mu(E_n) = 0.$$

It is then proved that $f_n \to f$ in measure.

(b) For every $\varepsilon > 0$, suppose that

$$E_k = \{x \in X : |f_n(x) - f(x)| > \varepsilon\},\ k = 1, 2, \cdots.$$

Then it follows that for $1 \leq p < \infty$,

$$\varepsilon^p \mu(E_n) \leq \int_X |f_n - f|^p d\mu = \|f_n - f\|_p^p.$$

Since $\|f_n - f\|_p \to 0$, it shows that

$$\lim_{n\to\infty} \mu(E_n) = 0.$$

It is then proved that $f_n \to f$ in measure. If $p = +\infty$ and $\|f_n - f\|_p \to 0$, there exists a national $N > 0$ satisfying

$$\|f_n - f\|_p < \varepsilon, \text{ whenever } n > N.$$

Hence it shows

$$\mu(E_n) = 0, \text{ whenever } n > N.$$

It is then proved that $f_n \to f$ in measure.

(c)Suppose that $f_n \to f$ in measure. For every national k,there exists national n_k such that

$$\mu(E_k) < \frac{1}{2^k}, \ k = 1, 2, \cdots.$$

where $E_k = \{x \in X : |f_{n_k}(x) - f(x)| > \frac{1}{2^k}\}$.
Define

$$F = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n}^{\infty} E_k$$

then we have

$$\mu(F) \leq \mu(\bigcup_{k\geq n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} \mu(E_k) = \frac{1}{2^{n-1}}, \ n = 1, 2, \cdots.$$

It is obvious that $\mu(F) = 0$. For any $x \in X - F$, it follows that there exists national N such that

$$|f_{n_k}(x) - f(x)| \leq \frac{1}{2^k}, \text{ for each national } k > N.$$

Therefore, we obtain

$$\lim_{k\to\infty} f_{n_k}(x) = f(x).$$

It then shows that $\{f_n\}$ has a subsequence which converges to f *a.e.*

20. Suppose ϕ is a real function on $\mathbb{R}$ such that

$$\phi(\int_0^1 f(x)dx) \leq \int_0^1 \phi(f)dx$$

for every real bounded measurable f. Prove that ϕ is then convex.

[**Proof**]: Assume that

$$x, y \in \mathbb{R}; \quad 0 < \lambda < 1.$$

Then

$$\lambda x + (1-\lambda)y = \int_0^1 (x\chi_{[0,\lambda]}(t) + y\chi_{[\lambda,1]})(t)dt.$$

Thus

$$\begin{aligned} \phi(\lambda x + (1-\lambda)y) &= \phi(\textstyle\int_0^1 (x\chi_{[0,\lambda]}(t) + y\chi_{[\lambda,1]})(t)dt) \\ &\leq \textstyle\int_0^1 \phi \circ (x\chi_{[0,\lambda]}(t) + y\chi_{[\lambda,1]})(t)dt \\ &= \lambda\phi(x) + (1-\lambda)\phi(y). \end{aligned}$$

Therefore, ϕ is convex.

部分习题答案

chapter four

In this set of exercises, H always denotes a Hilbert space.

1. If M is a closed subspace of H, prove that $M = (M^{\perp})^{\perp}$. Is there a similar true statement for subspaces M which are not necessarily closed?

[**Proof**]: It is easy $M \subset (M^{\perp})^{\perp}$. on the other hand, for any $z \in (M^{\perp})^{\perp}$, it exist $x \in M$ and $y \in M^{\perp}$ with $z = x + y$ when M is a closed subspace of H. Thus,

$$0 = (z, y) = (x, y) + (y, y) = (y, y)$$

and so $y = 0$. Therefore, $z = x \in M$. It thus follows that $M \supset (M^{\perp})^{\perp}$ and whence $M = (M^{\perp})^{\perp}$.

Next, assume that M is not closed subspace. Since $M \subset (M^{\perp})^{\perp}$ and $(M^{\perp})^{\perp}$ is closed, it is obvious that $\overline{M} \subset (M^{\perp})^{\perp}$. since $M \subset \overline{M}$, it shows $\overline{M}^{\perp} \subset M^{\perp}$. Moreover, $(M^{\perp})^{\perp} \subset (\overline{M}^{\perp})^{\perp}$. Again since $\overline{M} = (\overline{M}^{\perp})^{\perp}$, it is proved that $\overline{M} = (M^{\perp})^{\perp}$.

2. Let $\{x_n : n = 1, 2, 3, \cdots\}$ be a linearly independent set of vectors in H. Show that the following construction yields an orthonormal set $\{u_n\}$ such that $\{x_1, x_2, \cdots, x_N\}$ and $\{u_1, u_2, \cdots, u_N\}$ have the same span for all N.

[**Proof**]: Let $\{x_n : n = 1, 2, 3, \cdots\}$ be a linearly independent set of vectors in H. Put

$$u_1 = \frac{x_1}{\|x_1\|};\ v_n = x_n - \sum_{k=1}^{n-1}(x_n, u_k)u_k;\ u_n = \frac{v_n}{\|v_n\|};\ n = 2, 3, \cdots$$

Then it is easy that

$$(v_2, u_1) = 0; (u_2, u_1) = 0$$

and so $(v_n, u_k) = 0, k = 1, 2, \cdots, n-1$. Therefore, $\{u_n\}$ is an orthonormal set. Moreover,

$$span\{x_n\} = span\{u_n\}.$$

3. Show that $L^p(T)$ is separable if $1 \leq p < \infty$, but that $L^{\infty}(T)$ is not separable.

[**Proof**]: Firstly, assume that $1 \leq p < \infty$, then $C(T)$ is dense in $L^p(T)$. Let $\mathcal{P}$ be the set of all trigonometric polynomials and $\tilde{\mathcal{P}}$ be the set of all trigonometric polynomials with rational coefficients, it is obvious that $\mathcal{P}$ is dense in $C(T)$and $\tilde{\mathcal{P}}$ is also. Since

$$\|f-g\|_p \leq \|f-g\|_\infty, \quad \forall f, g \in C(T)$$

and so it shows that $\tilde{\mathcal{P}}$ is countable dense subset in $L^p(T)$. This proves that $L^p(T)$ is separable.

Next, we prove that $L^\infty(T)$ is not separable. Assume that $L^\infty(T)$ is separable and let $\{u_n\}$ is the countable dense subset of $L^\infty(T)$. Put

$$f_t(s) = \chi_{[0,t]}(s); 0 < t < 2\pi$$

and extend it to the real axis $\mathbb{R}$ with period 2π. Thus,

$$f_t \in L^\infty(T) \text{ and } \|f_t\|_\infty = 1.$$

Take $0 < \epsilon < \frac{1}{2}$, according to the assumption, it exists some u_k satisfying

$$U(u_k, \epsilon) \cap \{u_n\} \text{ is infinite set.}$$

Put $f_t, f_{t'} \in U(u_k, \epsilon); 0 < t < t' \leq 2\pi$, then

$$1 = \|f_t - f_{t'}\|_\infty \leq \|f_t - u_k\|_\infty + \|f_{t'} - u_k\|_\infty < 2\epsilon < 1$$

It is contradiction. Therefore, $L^\infty(T)$ is not separable.

4. Show that H is separable if and only if H contains a maximal orthonormal system which is at most countable.

[**Proof**]: First, there exists a countable dense subset M of H if H is separable. Let N is the maximal linearly independent subset of M. By the conclusion of Exercise 2, it shows the existence of a maximal orthonormal system of H which is at most separable.

Second, suppose that $\{u_n\}$ is a countabe maximal orthonormal system of H. Put $M = \overline{span\{u_n\}}$. If $M \neq H$, ther exists an element $v \in H$ and $v \notin M$. Furthermore,

$$\overline{v} = \sum_{n=1}^{\infty} (v, u_n) u_n \in M; \quad v - \overline{v} \neq 0.$$

Let $u_0 = \frac{v-\overline{v}}{\|v-\overline{v}\|}$, then

$$\|u_0\| = 1; \ (u_0, u_n) = 0, \ n = 1, 2, \cdots.$$

Therefore, $\{u_0, u_1, u_2, \cdots\} \subset H$ is a orthonormal set but this contradict the maximality of $\{u_1, u_2, \cdots\} \subset H$. It thus follows that $M = H$. So H is separable.

5. If $M = \{x : Lx = 0\}$, where L is a continuous linear functional on H, prove that $M^\perp$ is a vector space of dimension 1(unless $M = H$).

[**Proof**]: If $L = 0$, then $M = H$ and $M^\perp = \{0\}$. in the following, assume that $L \neq 0$. Then it exists unique element $y \neq 0$ of H such that

$$Lx = (x, y); \quad \forall x \in H.$$

Thus, $y \perp M$ and so $M = span\{y\}^\perp$. That is, $M^\perp = span\{y\}$ since M is a closed subspace by the continuity of L. This complete the proof.

7. Suppose that $\{a_n\}$ is a sequence of positive numbers such that $\sum a_n b_n < \infty$ whenever $b_n \geq 0$ and $\sum b_n^2 < \infty$. Prove that $\sum a_n^2 < \infty$.

[**Proof**]: Assume that $\sum\limits_{n=1}^{\infty} a_n^2 = +\infty$. For any $k \in \mathbb{N}$, it then exists $n_k \in \mathbb{N}$ such that

$$\sum_{n=1}^{n_k} a_n^2 > k; \quad 1 < n_1 < n_2 < n_3 < \cdots$$

Hence

$$c_k^2 = \sum_{n_k+1}^{n_{k+1}} a_n^2 > 1; \quad k = 1, 2, \cdots.$$

Put

$$b_n = \begin{cases} \frac{a_n}{kc_k}; & n_k \leq n < n_{k+1}, \quad k = 1, 2, \cdots \\ 0; & 1 \leq n < n_1 \end{cases}$$

Then

$$\sum_{n=1}^{\infty} b_n^2 = \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} \frac{a_n^2}{k^2 c_k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty$$

but

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} \frac{a_n^2}{kc_k} = \sum_{k=1}^{\infty} \frac{c_k}{k} = +\infty$$

Therefore, the assumption is error and $\sum a_n^2 < \infty$.

9. If $A \subset [0, 2\pi]$ and A is measurable, prove that

$$\lim_{n\to\infty}\int_A \cos nx dx = \lim_{n\to\infty}\int_A \sin nx dx = 0.$$

[**Proof**]: If $A \subset [0, 2\pi]$ and A is measurable, then $\chi_A \in L^2([0, 2\pi])$. Since $L^2([0, 2\pi])$ is Hilbert space and $\{e^{int} : n \in \mathbb{Z}\}$ is the maximal orthonormal set of $L^2([0, 2\pi])$, and so we have

$$\sum_{n\in\mathbb{Z}} |(\chi_A, e^{int})|^2 = \|\chi_A\|_2 < +\infty$$

Therefore, it shows that

$$|(\chi_A, e^{int})| \to 0, \quad |n| \to \infty$$

Furthermore, it follows that

$$\lim_{n\to\infty}\int_A \cos nx dx = \lim_{n\to\infty}\int_A \sin nx dx = 0.$$

16. If $x_0 \in H$ and M is a closed linear subspace of H, prove that

$$\min\{\|x - x_0\| : x \in M\} = \max\{|(x_0, y)| : y \in M^\perp, \|y\| = 1\}.$$

[**Proof**]: Suppose that P and Q are the projections on M, $M^\perp$ respectively. For $x_0 \in H$, it shows that $x_0 = Px_0 + Qx_0$ and

$$\|Qx_0\| = \min\{\|x - x_0\| : x \in M\}.$$

On the other hand, for any $y \in M^\perp$ with $\|y\| = 1$, then

$$(x_0, y) = (Px_0 + Qx_0, y) = (Qx_0, y)$$

and whence

$$|(x_0, y)| = |(Qx_0, y)| \leq \|Qx_0\| \cdot \|y\| = \|Qx_0\|.$$

Since $Qx_0 \in M^\perp$, we may put $y_0 = \frac{Qx_0}{\|Qx_0\|}$ when $Qx_0 \neq 0$. Then $(x_0, y_0) = \|Qx_0\|$. Thus

$$\max\{|(x_0, y)| : y \in M^\perp, \|y\| = 1\} = \|Qx_0\|.$$

This complet the proof.

部分习题答案

chapter five

2. Prove that the unit ball(open or closed) is convex in every normed linear space.

[**Proof**]: Let X be a normed linear space and S be the open nuit ball of X. Assume that $0 \le \alpha, \beta \le 1$ and $\alpha + \beta = 1$. For any $x, y \in S$,

$$\|\alpha x + \beta y\| \le \|\alpha x\| + \|\beta y\| = \alpha\|x\| + \beta\|y\| < \alpha + \beta = 1$$

and so $\alpha x + \beta y \in S$. It shows that S is convex.

4. Let C be the space of all continuous functions on $[0,1]$, with the supremum norm. Let M consist of all $f \in C$ for which

$$\int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt = 1.$$

Prove that M is closed convex subset of C which contains no element of minimal norm.

[**Proof**]: By use of the Lebesgue dominated convergence theorem, it is easy that M is closed convex set of C. Moreover, for any $f \in M$,

$$1 = \int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt \le \int_0^1 |f(t)|dt = \|f\|_1 \le \|f\|_\infty.$$

If $f \in C$ and $\|f\|_\infty = 1$, it holds that

$$\int_0^1 |f(t)|dt = 1; \quad \int_0^1 (1 - |f(t)|)dt = 0; \quad 1 - |f(t)| \ge 0,\ \forall t \in [0,1]$$

and so $|f(t)| = 1,\ \ \forall t \in [0,1]$. But in this case, $f \notin M$. Therefore, M is closed convex subset of C which contains no element of minimal norm.

5. Let M be the set of all $f \in L^1([0,1])$, relative to Lebesgue measure, such that

$$\int_0^1 f(t)dt = 1.$$

Show that M is a closed convex subset of $L^1([0,1])$ which contains infinitely many elements of minimal norm.

[**Proof**]: Assume that $0 \leq \alpha, \beta \leq 1$ and $\alpha + \beta = 1$. For any $f, g \in M \subset L^1([0,1])$,

$$\int_0^1 [\alpha f + \beta g](t)dt = \alpha \int_0^1 f(t)dt + \beta \int_0^1 g(t)dt = \alpha + \beta = 1$$

whence $\alpha f + \beta g \in M$. So M is convex. Next, given

$$\{f_n\} \subset M; \quad \text{and } \|f_n - f\|_1 \to 0, \ n \to \infty.$$

Then

$$\int_0^1 f_n(t)dt - \int_0^1 f(t)dt = \int_0^1 \leq \int_0^1 |f_n(t) - f(t)|dt = \|f_n - f\|_1 \to 0$$

and so

$$\lim_{n\to\infty} \int_0^1 f_n(t)dt = \int_0^1 f(t).$$

It follows that $f \in M$ since $\{f_n\} \subset M$ and so M is closed convex subset. Moreover,

$$\|f\|_1 \geq 1, \quad \forall f \in M.$$

Since

$$nt^{n-1} \in M, \ \forall n \in \mathbb{N}; \quad \text{and} \quad \|nt^{n-1}\|_1 = 1$$

it shows that M contains infinitely many elements of minimal norm.

6. Let f be a bounded linear functional on a subspace M of a Hilbert space H. Prove that f has a unique norm-preserving extension to a bounded linear functional on H, and that this extension vanishes on $M^\perp$.

[**Proof**]: If $f = 0$, we only need to take the bounded linear functional $F = 0$ on H. In the following, we assume $f \neq 0$. Since f is a bounded linear functional on a subspace M, it is easy to extend f to a bounded linear functional $\tilde{f}$ on $\overline{M}$ and $\|f\| = \|\tilde{f}\|$. In fact, for any $x \in \overline{M}$, there exists a sequence $\{x_n\} \subset M$ such that $\|x_n - x\| \to 0$ and then define $\tilde{f}(x) = \lim_{n\to\infty} f(x_n)$. So we can define a linear functional of H such that

$$F(x) = \begin{cases} \tilde{f}(x); & x \in \overline{M} \\ 0; & x \in M^\perp \end{cases}$$

and then it is proved that F is the norm-preserving linear extension on H of f and this extension vanishes on $M^\perp$. Moreover, for any norm-preserving linear extension g on H of f, $g\mid_{\overline{M}}=\tilde{f}$. Since $M^{\perp}=\overline{M}^{\perp}$ and $H=\overline{M}\oplus M^{\perp}$, it shows that it is unique that the norm-preserving extension of f to a bounded linear functional on H which vanishes on $M^{\perp}$.

8. Let X be a normed linear space, and let X^* be its dual space, as defined in Sec. 5.21, with the norm

$$\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}.$$

(a)Prove that X^* is a Banach space.
(b) Prove that the mapping $f \to f(x)$ is, for each $x \in X$, a bounded linear functional on X^*, of norm $\|x\|$.
(c) Prove that $\{\|x_n\|\}$ is bounded if $\{x_n\}$ is a sequence in X such that $\{f(x_n)\}$ is bounded for every $f \in X^*$.

[**Proof**]: (a) It is obvious that X^* is a normed linear space. Given $\{f_n\} \subset X^*$ is a cauchy sequence, then for any $\epsilon > 0$, there is a $N > 0$ such that

$$\|f_n - f_m\| < \epsilon; \quad \forall n, m > N$$

and so

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| \cdot \|x\| < \|x\| \cdot \epsilon; \quad \forall n, m > N, \ \forall x \in X.$$

Thus $\{f_n(x)\} \subset \mathbb{C}$ is also a cauchy sequence and so we can define

$$f(x) = \lim_{n\to\infty} f_n(x); \quad \forall x \in X.$$

It is easy that f is a linear functional on X. Since $\{\|f_n\|\}$ is a cauchy sequence, ther is $M > 0$ satisfying

$$\|f_n\| \leq M, \ n = 1, 2, \cdots.$$

Hence, for ϵ, N as above, it shows for $n > N$ the following

$$|f(x)| \leq |f_n(x)| + \|x\|\epsilon \leq \|x\|(M + \epsilon); \quad \forall x \in X.$$

It forces

$$|f(x)| \leq M\|x\|; \quad \forall x \in X$$

and so $\|f\| \leq M$. That is, $f \in X^*$. Therefore, X^* is a Banach space.

(b) For each $x \in X$, define

$$\lambda(f) = f(x); \quad \forall f \in X^*$$

Then for any $f, g \in X^*$ and $\alpha, \beta \in \mathbb{C}$,

$$\Lambda(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha\Lambda(f) + \beta\Lambda(g)$$

and we see that Λ is a linear functional on X^*. Since

$$|\Lambda(f)| = |f(x)| \leq \|f\| \cdot \|x\|; \quad \forall f \in X^*$$

whence $\|\Lambda\| \leq \|x\|$. From the Hahn-Banach Theorem, there is $g \in X^*$ such that $g(x) = \|x\|$ and $\|g\| = 1$. It implies that $\|\Lambda\| = \|x\|$.

(c) Suppose that $\{x_n\} \subset X$ such that $\{f(x_n)\}$ is bounded for every $f \in X^*$. By (a),(b) and the Banach-Steinhaus Theorem, it is easy to prove that $\{\|x_n\|\}$ is bounded.

9. Let c_0, l^1, l^∞ be the Banach spaces consisting of all complex sequences $x = \{\xi_i\},\ i = 1, 2, \cdots$, define as follows:

$$x \in l^1 \text{if and only if} \|x\|_1 = \sum |\xi_i| < \infty.$$

$$x \in L^\infty \text{if and only if} \|x\|_\infty = \sup |\xi_i| < \infty.$$

c_0 is the subspace of l^∞ consisting of all $x \in l^\infty$ for which $\xi_i \to 0$ as $i \to \infty$. Prove the following four statements.

(a) If $y = \{\eta_i\} \in l^1$ and $\Lambda x = \sum \xi_i \eta_i$ for every $x \in c_0$, then Λ is a bounded linear functional on c_0, and $\|\Lambda\| = \|y\|_1$. Moreover, every $\Lambda \in (c_0)^*$ is obtained in this way. In brief, $(c_0)^* = l^1$.

(b) In the same sense, $(l^1)^* = l^\infty$.

(c) Every $y \in l^1$ induces a bounded linear functional on l^∞, as in (a). However, this does not give all of $(l^\infty)^*$, since $(l^\infty)^*$ contains nontrivial functionals that vanish on all of c_0.

(d) c_0 and l^1 are separable but l^∞ is not.

[**Proof**]: (a) For each $y = \{\eta_i\} \in l^1$, we define

$$\Lambda x = \sum_{i=1}^{\infty} \xi_i \eta_i; \quad \forall x = \{\xi_i\} \in c_0.$$

Then the above series converges absolutely and so Λ is linear mapping. Since

$$|\Lambda x| \leq \|x\|_\infty \cdot \|y\|_1$$

it follows that $\|\Lambda\| \leq \|y\|_1$. Thus, $\Lambda \in (c_0)^*$. On the other hand, put

$$e_i = \{\overbrace{0, 0, \cdots, 1}^{i}, 0, 0, \cdots, \}; \quad i = 1, 2, \cdots.$$

It is obvious that $e_i \in c_0$ and $\|e_i\|_\infty = 1,\ i = 1, 2, \cdots$. For any $\Lambda \in (c_0)^*$,take

$$\eta_i = \Lambda e_i, \quad i = 1, 2, \cdots.$$

Then

$$|\eta_i| = |\Lambda e_i| \leq \|\Lambda\| \cdot \|e_i\|_\infty = \|\Lambda\|.$$

Let $y = \{\eta_i\}$ and then $y \in l^\infty$ with $\|y\|_\infty \leq \|\Lambda\|$. Assume $x_n = \{\xi_i^n\}$, where

$$\xi_i^n = \begin{cases} \frac{\overline{\eta_i}}{|\eta_i|}; & \eta_i \neq 0,\ i = 1, 2, \cdots, n; \\ 0; & \eta_i = 0, \text{or } i > n. \end{cases}$$

then $x_n \in c_0$ with $\|x_n\|_\infty = 1, n = 1, 2, \cdots$. Moreover,

$$\Lambda x_n = \sum_{i=1}^{n} |\eta_i| \leq \|\Lambda\| \cdot \|x_n\|_\infty = \|\Lambda\|$$

whence

$$\sum_{i=1}^{\infty} |\eta_i| \leq \|\Lambda\|$$

Thus $y \in l^1$ and $\|y\|_1 \leq \|\Lambda\|$. Since for any $x = \{\xi_i\} \in c_0$, $x = \sum\limits_{i=1}^{\infty} \xi_i e_i$ and so $\Lambda x = \sum\limits_{i=1}^{\infty} \xi_i \eta_i$. From the proof above, it implies that $\|\Lambda\| \leq \|y\|_1$ and so $\|\Lambda\| = \|y\|_1$. Therefore, $(c_0)^* = l^1$.

(b) For each $y = \{\eta_i\} \in l^\infty$, define

$$\Lambda x = \sum_{i=1}^{\infty} \xi_i \eta_i; \quad \forall x = \{\xi_i\} \in l^1.$$

Then the above series converges absolutely and so Λ is linear mapping. Since

$$|\Lambda x| \leq \|y\|_\infty \cdot \|x\|_1$$

it shows that $\|\Lambda\| \leq \|y\|_\infty$. Thus, $\Lambda \in (l^1)^*$. On the other hand, suppose that $\Lambda \in (l^1)^*$. Put

$$\eta_i = \Lambda e_i, \quad i = 1, 2, \cdots.$$

Then

$$|\eta_i| = |\Lambda e_i| \leq \|\Lambda\| \cdot \|e_i\|_1 = \|\Lambda\|.$$

Let $y = \{\eta_i\}$ and then $y \in l^\infty$ with $\|y\|_\infty \leq \|\Lambda\|$. Since for any $x = \{\xi_i\} \in l^1$, $x = \sum_{i=1}^{\infty} \xi_i e_i$ and so $\Lambda x = \sum_{i=1}^{\infty} \xi_i \eta_i$. It implies that $\|\Lambda\| \leq \|y\|_\infty$ and so $\|\Lambda\| = \|y\|_\infty$. Therefore, $(l^1)^* = l^\infty$.

(c) For each $y = \{\eta_i\} \in l^\infty$, define

$$\Lambda x = \sum_{i=1}^{\infty} \xi_i \eta_i; \quad \forall x = \{\xi_i\} \in l^\infty.$$

Then the above series converges absolutely and so Λ is linear mapping. Since

$$|\Lambda x| \leq \|x\|_\infty \cdot \|y\|_1$$

it shows that $\|\Lambda\| \leq \|y\|_1$. Thus, $\Lambda \in (l^\infty)^*$. Since $c_0 \subset l^\infty$ is closed subspace, by the Hahn-Banach theorem, there is $f \in (l^\infty)^*$ such that $f\mid_{c_0} = 0$. By (a), $(c_0)^* = l^1$ and so $f \notin l^1$. Therefore, $l^1 \neq (l^\infty)^*$.

(d) Since

$$\overline{span\{e_i\}}^{c_0} = c_0; \quad \overline{span\{e_i\}}^{l^1} = l^1$$

it is easy to prove that c_0, l^1 are separable.

10. If $\sum \alpha_i \xi_i$ converges for every sequence (ξ_i) such that $\xi_i \to 0$ as $i \to \infty$, prove that $\sum |\alpha_i| < \infty$.

[**Proof**]: Put

$$x = \{\alpha_i\}; \quad x_n = \{\alpha_1, \alpha_2, \cdots, \alpha_n, 0, 0, \cdots\}; \quad \xi = \{\xi_n\} \in c_0.$$

and define

$$\rho(\xi) = \sum_{i=1}^{\infty} \alpha_i \xi_i; \quad \rho_n(\xi) = \sum_{i=1}^{n} \alpha_i \xi_i.$$

Then ρ is linear and $\rho_n (n \in \mathbb{N})$ is bounded linear functionals on c_0. Moreover,

$$\|\rho_n\| = \sum_{i=1}^{n} |\alpha_i|, \quad n \in \mathbb{N}.$$

Since

$$\lim_{n\to\infty} \rho_n(\xi) = \rho(\xi); \quad \forall \xi \in c_0$$

exists, by the Banach-Steinhaus Theorem, there is $M > 0$ satisfying

$$\|\rho_n\| \leq M; \quad n \in \mathbb{N}$$

That is,

$$\sum_{i=1}^{n} |\alpha_i| \leq M, \quad n \in \mathbb{N}.$$

It thus forces that $\sum_{i=1}^{\infty} |\alpha_i| < +\infty$.

11. For $0 < \alpha \leq 1$, let Lipα denote the space of all complex functions f on $[a, b]$ for which

$$M_f = \sup_{s\neq t} \frac{|f(s) - f(t)|}{|s - t|^{\alpha}} < \infty.$$

Prove that Lipα is a Banach space, if $\|f\| = |f(a)| + M_f$; also, if

$$\|f\| = M_f + \sup_{x\in[a,b]} |f(x)|.$$

[**Proof**]: It is easy to prove that Lipα is a normed linear space with the corresponding norm. Let $\{f_n\}$ be a cauchy sequence of Lipα.

(a) Assume

$$\|f\| = |f(a)| + M_f.$$

There thus exists $G > 0$ such that $M_{f_n} \leq G$. For any $\epsilon > 0$, there is $N > 0$ such that

$$|f_n(a) - f_m(a)| < \epsilon; M_{f_n - f_m} < \epsilon$$

when $n, m > N$. For every $s \in (a, b]$,

$$\begin{aligned} \tfrac{|f_n(s)-f_m(s)|}{|s-a|^{\alpha}} &\leq \tfrac{|f_n(a)-f_m(a)|}{|s-a|^{\alpha}} + \tfrac{|f_n(s)-f_m(s)-f_n(a)+f_m(a)|}{|s-a|^{\alpha}} \\ &\leq M_{f_n-f_m} + \tfrac{|f_n(a)-f_m(a)|}{|s-a|^{\alpha}} \end{aligned}$$

It implies that $\{f_n(s)\}$ is cauchy sequence for each $s \in [a, b]$. Therefore, we can define

$$\lim_{n\to\infty} f_n(s) = f(s); \quad \forall s \in [a, b].$$

Since for any $s, t \in [a, b]$ and $s \neq t$, for each $n \in \mathbb{N}$,

$$\begin{aligned} \tfrac{|f(s)-f(t)|}{|s-t|^{\alpha}} &\leq \tfrac{|f_n(s)-f_n(t)|}{|s-t|^{\alpha}} + \tfrac{|f_n(s)-f(s)|}{|s-t|^{\alpha}} + \tfrac{|f_n(t)-f(t)|}{|s-t|^{\alpha}} \\ &\leq \tfrac{|f_n(s)-f(s)|}{|s-t|^{\alpha}} + \tfrac{|f_n(t)-f(t)|}{|s-t|^{\alpha}} + G \end{aligned}$$

Let $n \to \infty$, then

$$\sup_{s \neq t} \frac{|f(s) - f(t)|}{|s-t|^\alpha} \leq G < +\infty.$$

That is, $f \in \text{Lip}\alpha$ and so $\text{Lip}\alpha$ is a Banach space.

(b) Assume

$$\|f\| = M_f + \sup_{x \in [a,b]} |f(x)|.$$

There thus exists $G > 0$ such that $M_{f_n} \leq G$. Put $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$ and so $\{\|f_n\|_\infty\}$ is cauchy sequence. Then it exists a complex function f on $[a, b]$ such that

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| \to 0; \quad n \to 0.$$

Moreover, for any $s, t \in [a, b]$ and $s \neq t$, for each $n \in \mathbb{N}$,

$$\begin{array}{rl} \frac{|f(s)-f(t)|}{|s-t|^\alpha} & \leq \frac{|f_n(s)-f_n(t)|}{|s-t|^\alpha} + \frac{|f_n(s)-f(s)|}{|s-t|^\alpha} + \frac{|f_n(t)-f(t)|}{|s-t|^\alpha} \\ & \leq \frac{|f_n(s)-f(s)|}{|s-t|^\alpha} + \frac{|f_n(t)-f(t)|}{|s-t|^\alpha} + G \end{array}$$

Let $n \to \infty$, then

$$\sup_{s \neq t} \frac{|f(s) - f(t)|}{|s-t|^\alpha} \leq G < +\infty.$$

That is, $f \in \text{Lip}\alpha$ and so $\text{Lip}\alpha$ is a Banach space.

16. Suppose X and Y are Banach spaces, and suppose Λ is a linear mapping of X into Y, with the following property: For every sequence $\{x_n\}$ in X for which $x = \lim x_n$ and $y = \lim \Lambda x_n$ exist, it is true that $y = \Lambda x$. Prove that Λ is continuous.

[**Proof**]: Provided that X, Y are Banach spaces. Define

$$X \oplus Y = \{(x, y) \mid x \in X,\ y \in Y\}$$

The addition and scalar multiplication on $X \oplus Y$ are difined in the obvious manner as follows:

$$(x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+y_2);\ \ \alpha(x, y) = (\alpha x, \alpha y);\ \ \forall x, x_i \in X,\ y, y_i \in Y, i = 1, 2.$$

If put

$$\|(x, y)\| = \|x\| + \|y\|; \quad \forall x \in X, \quad y \in Y$$

it then implies that $X \oplus Y$ is normed linear space. Given $\{(x_n, y_n)\} \subset X \oplus Y$ is cauchy sequence, and so $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ are also cauchy sequences. Therefore, it exist $x \in X$ and $y \in Y$ such that

$$\|x_n - x\| \to 0; \quad \|y_n - y\| \to 0; \text{ if } \ n \to \infty.$$

Hence,

$$\|(x_n, y_n) - (x, y)\| = \|(x_n - x, y_n - y)\| = \|x_n - x\| + \|y_n - y\| \to 0; \ n \to \infty.$$

This implies that $X \oplus Y$ is Banach space. Put

$$G = \{(x, \Lambda x) \mid x \in X\}$$

it is easy that G is a subspace of $X \oplus Y$. Assume that $\{(x_n, \Lambda x_n)\} \subset G$ is cauchy sequence, and then $\{x_n\} \subset x$ and $\{\Lambda x_n\} \subset Y$ are aslo cauchy sequences. So there exist $x \in X$ and $y \in Y$ such that

$$\|x_n - x\| \to 0; \quad \|\Lambda x_n - y\| \to 0; \quad n \to \infty.$$

According to the property of Λ, it holds that $\Lambda x = y$ and $(x, \Lambda x) \in G$. It thus shows that G is closed and so G is a Banach space. Take $T(x, \Lambda x) = x$ for each $x \in x$, then T is linear mapping and

$$\|T(x, \Lambda x)\| = \|x\| \leq \|x\| + \|\Lambda x\| = \|(x, \Lambda x)\|; \quad \forall x \in X$$

whence $\|T\| \leq 1$. Moreover, T is injective and surjective bounded linear mapping. By the open mapping theorem, it show that T^{-1} is continuous and so there is $M > 0$ such that $\|T^{-1}\| \leq M$. Thus

$$\|x\| + \|\Lambda x\| = \|(x, \Lambda x)\| = \|T^{-1}x\| \leq M\|x\|; \quad \forall x \in X$$

and so

$$\|\Lambda x\| \leq M\|x\|; \quad \forall x \in X.$$

It follows that Λ is continuous.

17. If μ is a positive measure, each $f \in L^\infty(\mu)$ defines a pultiplication operator M_f on $L^2(\mu)$ into $L^2(\mu)$, such that $M_f g = fg$. Prove that $\|M_f\| \leq \|f\|_\infty$. For which measure μ is it true that $\|M_f\| = \|f\|_\infty$ for all $f \in L^\infty(\mu)$? For which $f \in L^\infty(\mu)$ does M_f map $L^2(\mu)$ onto $L^2(\mu)$?

[**Proof**]: For each $f \in L^\infty(\mu)$, define

$$M_f g = fg; \quad \forall g \in L^2(\mu)$$

Then it is easy that M_f is a linear operator from $L^2(\mu)$ into $L^2(\mu)$. Furthermore,

$$\|M_f g\|_2 = \|fg\|_2 \le \|f\|_\infty \cdot \|g\|_2; \quad \forall g \in L^2(\mu)$$

Thus it shows that $\|M_f\| \le \|f\|_\infty$.

If $f \in L^\infty(\mu)$ with $\frac{1}{f} \in L^\infty(\mu)$, then M_f maps $L^2(\mu)$ onto $L^2(\mu)$.

18. Suppose $\{\Lambda_n\}$ is a sequence of bounded linear transformations from a normed linear space X to a Banach space Y, suppose $\|\Lambda_n\| \le M < \infty$ for all n, and suppose there is a dense set $E \subset X$ such that $\{\Lambda_n x\}$ converges for each $x \in E$. Prove that $\{\Lambda_n x\}$ converges for each $x \in X$.

[**Proof**]: For every $x \in X$, there is a sequence $\{x_k\} \subset E$ such that $\|x_k - x\| \to 0 (k \to \infty)$ since $E \subset X$ is a dense subset. Moreover, $\{\Lambda_n x_k\} \subset Y$ converges for each $k \in \mathbb{N}$. It thus exists a sequence $\{y_k\} \subset Y$ such that

$$\Lambda_n x_k \to y_k, \quad n \to \infty; \quad k = 1, 2, \cdots.$$

For any $\epsilon > 0$, there exists $N > 0$ such that

$$\|x_k - x_{k'}\| < \frac{\epsilon}{M}; \quad \forall k, k' > N.$$

and so

$$\|y_k - y_{k'}\| = \lim_{n\to\infty} \|\Lambda_n(x_k - x_{k'})\| \le \lim_{n\to\infty} \|\Lambda_n\| \cdot \|x_k - x_{k'}\| \le M \cdot \frac{\epsilon}{M} = \epsilon$$

It hence implies that $\{y_k\} \subset Y$ is a cauchy sequence. Since Y is a Banach space, there is $y \in Y$ with $\|y_k - y\| \to 0$. Therefore,

$$\begin{aligned} \|\Lambda_n x - y\| &= \|\Lambda_n x - \Lambda_n x_k + \Lambda_n x_k - y_k + y_k - y\| \\ &\le \|\Lambda_n\| \cdot \|x_k - x\| + \|\Lambda_n x_k - y_k\| + \|y_k - y\| \\ &\le M \cdot \|x_k - x\| + \|\Lambda_n x_k - y_k\| + \|y_k - y\| \end{aligned}$$

This implies that $\lim_{n\to\infty} \Lambda_n x = y$.

部分习题答案

chapter six

2. Prove that the example given at the end of Sec. 6.10 has stated properties.

[**Proof**]:

3. Prove that the vector space $M(X)$ of all complex regular Borel measures on a locally compact hausdorff space X is a Banach space if $\|\mu\| = |\mu|(X)$.

[**Proof**]: Firstly, assume $\mu, \lambda \in M(X)$ and $\alpha \in \mathbb{C}$. Then $\alpha\mu \in M(X)$ and $|\mu|, |\lambda|$ are regular Borel measures. For any measurable set E in X and $\epsilon > 0$, it exist compact sets $K_1, K_2 \subset E$ and open sets $G_1, G_2 \supset E$ satisfying:

$$|\mu|(E) < |\mu|(K_1) + \frac{\epsilon}{2}; \quad |\lambda(E)| < |\lambda|(K_2) + \frac{\epsilon}{2}$$
$$|\mu|(G_1) < |\mu|(E) + \frac{\epsilon}{2}; \quad |\lambda|(G_2) < |\lambda|(E) + \frac{\epsilon}{2}$$

Take

$$K = K_1 \cup K_2; \quad g = G_1 \cap G_2$$

it implies that $K$is compact and G is open. Moreover,

$$(|\mu| + |\lambda|)(E) < (|\mu| + |\lambda|)(K) + \epsilon; \quad (|\mu| + |\lambda|)(G) < (|\mu| + |\lambda|)(E) + \epsilon$$

and so $|\mu|+|\lambda|$ is regular Norel measure. Since $|\mu+\lambda|$ and $|\mu|+|\lambda|$ are bounded positive measures and

$$|\mu + \lambda| \leq |\mu| + |\lambda|$$

it shows that $|\mu + \lambda|$ is regular Borel measure. In fact, we have the following conclusion:

Suppose that μ is bounded positive measure, then
μ is regular $\Leftrightarrow$ for any measurable set E and $\epsilon > 0$, there are compact subset $K \subset E$ and open set $G \supset E$ such that

$$\mu(E - K) < \epsilon; \quad \mu(G - E) < \epsilon.$$

Therefore, it shows that $M(X)$ is a linear space. Define

$$\|\mu\| = |\mu|(X), \quad \forall \mu \in M(X)$$

it is easy to show that $\|\mu\|$ is norm and $M(X)$ is normed linear space. By Theorem 6.19, it is obtained that $M(X) \cong (C_0(X))^*$. Since $(C_0(X))^*$ is a Banach space and so $M(X)$ is also.

4. Suppose $1 \le p \le \infty$, and q is the exponent conjugate to p. Suppose μ is a positive σ-finite measure and g is a measurable function such that $fg \in L^1(\mu)$ for every $f \in L^p(\mu)$. Prove that then $g \in L^q(\mu)$.

[**Proof**]: Since μ is a positive σ-finite measure, there is a sequence of disjoint measurable sets $\{X_n\}$ of X such that

$$X = \bigcup_{n=1}^{\infty} X_n; \quad \mu(X_n) < +\infty, \quad n = 1, 2, \cdots.$$

It thus shows that

$$\chi_{X_n} \in L^p(\mu), \quad n = 1, 2, \cdots.$$

Therefore,

$$g \in L^1(X_n, \mu), \quad n = 1, 2, \cdots.$$

It implies that g is finite almost everywhere on X and so we may assume that g is finite on X. Let

$$E_n = \{x \in X \mid |g(x)| \le n\}, \quad n = 1, 2, \cdots.$$

Then $\{E_n\}$ is a sequence of increasing measurable sets and $X = \bigcup_{n=1}^{\infty} E_n$. Put $g_n = \chi_{E_n} \cdot g$ and then $\{g_n\}$ are measurable on X. Moreover,

(a) $g_n(x) \to g(x)$; $\forall x \in X$.

(b) $|g_n(x)| \le n$; $\forall x \in X$, $n = 1, 2, \cdots$.

(c) $g_n \in L^q(\mu)$, $n = 1, 2, \cdots$.

Define

$$T_n(f) = \int_X f g_n d\mu; \quad \forall f \in L^p(\mu)$$

it is easy to prove that $\{T_n\}$ is bounded linear functionals on $L^p(\mu)$ with

$$\|T_n\| = \|g_n\|_q; \quad n = 1, 2, \cdots.$$

Since

$$|T_n(f)| \le \int_X |f g_n| d\mu = \int_{E_n} |fg| d\mu \le \int_X |fg| d\mu \le \|fg\|_1; \quad n = 1, 2, \cdots$$

By use of the Banach-Steinhaus Theorem, there is $M > 0$ such that

$$\|T_n\| \leq M; \quad n = 1, 2, \cdots.$$

It hence implies that

$$\|g_n\|_q \leq M; \quad n = 1, 2, \cdots.$$

If $q = \infty$, it shows from $X = \bigcup_{n=1}^{\infty} E_n$ that there is $n > 0$ for any $x \in X$ with $x \in E_n$ and so

$$|g(x)| = |g_n(x)| \leq \|g_n\|_\infty \leq M.$$

it shows that $\|g\|_\infty \leq M$ and $g \in L^\infty(\mu)$.
In the case $1 \leq q < \infty$, because

$$|g_n(x)| \leq |g_{n+1}(x)|, \quad n = 1, 2, \cdots; \quad |g_n(x)| \to |g(x)|, \quad \forall x \in X$$

By the Lebesgue's monotone convergence Theorem, it shows

$$\int_x |g_n|^q d\mu \to \int_X |g|^q d\mu, \quad n \to \infty$$

and so

$$(\int_X |g|^q d\mu)^{\frac{1}{q}} \leq M.$$

Therefore, $\|g\|_q \leq M$ and $g \in L^q(\mu)$.

6. Sppose $1 < p < \infty$ and prove that $L^q(\mu)$ is the dual space of $L^p(\mu)$ even if μ is not σ-finite.

[**Proof**]: Sppose $1 < p < \infty$ and μ is not σ-finite.
(a) Since for any $f \in L^p(\mu)$ and $g \in L^q(\mu)$, it is easy that $f \cdot g \in L^1(\mu)$. Then we can define

$$T_g(f) = \int_x f \cdot g d\mu, \quad \forall f \in L^q(\mu)$$

and $T_g \in (L^p(\mu))^*$ with $\|T_g\| \leq \|g\|_q$. Assume

$$E_n = \{x \in X : |g| \geq \frac{1}{n}\}, \quad n = 1, 2, \cdots.$$

It is obvious that $\{E_n\}$ is an increasing sequence of measurable sets and

$$\bigcup_{n=1}^{\infty} E_n = E = \{x \in X : |g| > 0\}.$$

Put

$$f_n = \chi_{E_n} |g|^{q-1} \alpha$$

where $\alpha g = |g|$; $|\alpha| = 1$. Then $f_n \in L^p(\mu)$ and

$$\|f_n\|_p = (\int_{E_n} |g|^q d\mu)^{\frac{1}{p}}; \quad n = 1, 2, \cdots.$$

Moreover, $\{|f_n|\}$ converges increasingly to $|g|^{q-1}$. Therefore,

$$T_g(f_n) = \int_X f_n \cdot g d\mu = \int_{E_n} |g|^q d\mu; \quad n = 1, 2, \cdots.$$

and so

$$\int_{E_n} |g|^q d\mu \leq \|T_g\| \cdot \|f_n\|_p = \|T_g\| \cdot (\int_{E_n} |g|^q d\mu)^{\frac{1}{p}}$$

It shows that

$$\int_{E_n} |g|^q d\mu \leq \|T_g\|^q; \quad n = 1, 2, \cdots.$$

By the Lebesgue's monotone convergence Theorem, it shows that

$$\int_{E_n} |g|^q d\mu = \int_X |f_n| \cdot |g| d\mu \to \int_X |g|^q d\mu; \quad n \to \infty.$$

Hence, $\int_X |g|^q d\mu \leq \|T_g\|^q$ and so $\|g\|_q = \|T_g\|$.

(b) Assume $\Phi \in (L^p(\mu))^*$ and $\|\Phi\| \neq 0$. At first, for each $f \in L^p(\mu)$, define

$$D_f = \{x \in X : f(x) \neq 0\}; \quad E_n = \{x \in X : |f(x)| \geq \frac{1}{n}\}.$$

Since $D_f = \bigcup_{n=1}^{\infty} E_n$, and

$$\mu(E_n)(\frac{1}{n})^p \leq \int_{E_n} |f|^p d\mu \leq \int_X |f|^p d\mu < +\infty$$

it thus follows that $\mu(E_n) < +\infty$, $n = 1, 2, \cdots$; and so D_f is σ-finite measurable set.

Secondly, given $E \subset X$ is σ-finite measurable set, and put

$$M = \{f \in L^p(\mu) : D_f \subset E\}$$

then $M \subset L^p(\mu)$ is a Banach subspace. In fact, provided that $\{f_n\} \subset M$ is a cauchy sequence, there is $f \in L^p(\mu)$ such that

$$\|f_n - f\|_p \to 0, \quad n \to \infty.$$

So it exists a subsequence of $\{f_n\} \subset M$ which converges to f almost everywhere. It implies that $f \in M$. Therefore, $\Phi \mid_M$ is a bounded linear functional on M. Since it also that $M \subset L^p(E, \mu)$, by the Hahn-Banach Theorem, it can extend $\Phi \mid_M$ to a bounded linear functional $\Phi \mid_E$ of $L^p(E, \mu)$ and so there is unique $h_E \in L^q(E, \mu)$ such that

$$\Phi(f) = \int_E f \cdot h_E d\mu, \quad \forall f \in M; \quad \|h_E\|_q = \|\Phi \mid_E \| = \|\Phi \mid_M \| \leq \|\Phi\|.$$

Take $g_E = h_E$ on E and $g_E = 0$ when $x \in X - E$. It is easy that

$$g_E \in L^q(\mu); \quad D_{g_E} \subset E; \ \|g_E\|_q = \|h_E\|_q \leq \|\Phi\|$$

and

$$\Phi(f) = \int_E f \cdot h_E d\mu = \int_X f \cdot g_E d\mu; \quad \forall f \in M.$$

Order

$$\alpha = \sup\{\|g_E\|_q : E \subset X \text{ is } \sigma - \text{finite measurable subset}\}$$

it is obvious that $0 \leq \alpha \leq \|\Phi\|$. Because there is a sequence $\{f_n\} \subset L^p(\mu)$ satisfying:

$$\|f_n\|_p = 1, \ n = 1, 2, \cdots; \quad |\Phi(f_n)| \to \|\Phi\|, \ n \to \infty.$$

Take $F = \bigcup_{n=1}^{\infty} D_{f_n}$, then F is a σ-finite measurable set. Moreover,

$$|\Phi(f_n)| \leq \|\Phi \mid_F \| = \|g_F\|_q \leq \alpha \leq \|\Phi\|, \quad n = 1, 2, \cdots.$$

Therefore, it holds that $\|\Phi\| = \|g_F\|_q = \alpha$. Write $g = g_F$, then $g \in L^q(\mu)$ and $\|\Phi\| = \|g\|_q$.

Thirdly, assume that $A \subset X$ is σ-finite measurable set and write $B = A \cup F$, B is also σ-finite measurable set. Then

$$\|\Phi\| = \alpha = \|g_F\|_q \leq \|g_B\|_q \leq \alpha \leq \|\Phi\|.$$

That is,

$$\|\Phi\| = \|g_F\|_q = \|g_B\|_q.$$

Since $L^p(F, \mu) \subset L^p(B, \mu)$, it show sthat $h_B = h_F$ on F almost everywhere and so $g_B = g_F$ a.e. on F. Again since

$$\begin{aligned} \|\Phi\|^q &= \int_X |g_B|^q d\mu \\ &= \int_B |g_B|^q d\mu \\ &= \int_{A-F} |g_B|^q d\mu + \int_F |g_B|^q d\mu \\ &= \int_{A-F} |g_B|^q d\mu + \int_F |g_F|^q d\mu \\ &\geq \int_F |g_F|^q d\mu \\ &= \|\Phi\|^q \end{aligned}$$

Hence, it holds that $\int_{A-F} |g_B|^q d\mu = 0$ and so $g_B = 0$ a.e. on $A - F$. It implies that $g = g_F = g_B$ a.e. on X.

finally, from the conclusion above, it shows that

$$\Phi(f) = \int_X g_{D_f \cup F} f d\mu = \int_X g f d\mu.$$

Therefore, $(L^p(\mu))^* = L^q(\mu)$.