# Diversity and Relative Arbitrage in Equity Markets

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# Contents

## List of Figures

# 1 Introduction

This project studies the problem in financial modelling, the integrability conditions when the equivalent martingale measure are removed and/or prohibited. By removing these conditions, we allow for the admission of a wide array of continuous path Itô processes. In particular, we open ourselves to work in a very general semi-martingale setting. As a consequence, issues of relative arbitrage arise under 'weak diversity', a term coined by Fernholz in [F99]. This notion states that on average the equity market is not dominated entirely by a single company. Under this realistic paradigm, the project illustrates with several examples, albeit with certain simplifying assumptions, that relative arbitrage with respect to the market portfolio is a possibility over arbitrary time horizons. Simulations are performed on Australian equities to provide a practical point of view. Along similar lines to [FKK05], the project also considers the issue of pricing general contingent claims under the new setting and its ramifications to European call options. Validity of the put-call parity is also considered in this situation.

Throughout this project, we shall use continuous semi-martingales to model stock prices and will use the following assumptions:

1. Companies do not merge or break up.
2. Dividends are paid continuously, not discretely.
3. There are no transaction costs or taxes.
4. Shares are infinitely divisible.

Since shares are infinitely divisible, without loss of generality, we can assume that each company has a single share outstanding.

# 2 The Model

In this section, we will introduce the model and notation used throughout this project. Most of the material is taken from [F02] and [FKK05].

Stock prices and portfolios are treated as semi-martingales that follow random processes defined on a probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$.

The source of randomness in this model comes from the $m$-dimensional standard independent Brownian motion

$$W = \{W(t) = (W_1(\cdot), \dots, W_m(\cdot)), \mathcal{F}_t, t \in [0, \infty)\}.$$

Here $\{\mathcal{F}_t\}$ is the augmentation under $\mathbf{P}$ of the natural filtration $\{\mathcal{F}_t^W = \sigma(W(s)); 0 \leq s \leq t\}$.

**Definition 2.1** *Let $n$ be a positive integer. A stock price $X_i$ satisfies the stochastic differential equation*

$$dX_i(t) = X_i(t)\left[b_i(t)dt + \sum_{v=1}^{m}\sigma_{iv}(t)dW_v(t)\right], \ i = 1, ..., n, \ t \in [0, \infty) \quad (2.1)$$

*where $m \geq n$. The rates of return $b(\cdot) = (b_1(\cdot), \ldots, b_n(\cdot))'$ and the volatilities $\sigma(\cdot) = \sigma_{iv}(\cdot)_{1\leq i\leq n, 1\leq v\leq m}$ are measurable and adapted.*

In order to ensure that the expression for $X$ makes sense, they further assume the following integrability condition:

$$\sum_{i=1}^{n}\int_0^T\left(|b_i(t)| + \sum_{v=1}^{d}(\sigma_{iv}(t))^2\right)dt < \infty, \quad a.s. \quad (2.2)$$

Under this condition, they can use a wide variety of continuous-path Itô processes with very general distributions ([FK08]) and also consider a very general semi-martingale setting.

The process $\sigma(\cdot)$ represents the sensitivity of $X$ to the $v$th source of uncertainty, $W_v$.

By using Itô's formula, we can represent (2.1) in logarithmic form. As we will see, most of the analysis that follows in this project depends on this logarithmic form. That is, we consider:

$$\begin{aligned} d(\log X_i(t)) &= \frac{1}{X_i(t)}dX_i(t) - \frac{1}{2X_i(t)^2}d\left\langle X_i\right\rangle_t \\ &= b_i(t)dt + \sum_{v=1}^{m}\sigma_{iv}(t)dW_v(t)dt - \frac{1}{2}\sum_{v=1}^{m}\sigma_{iv}^2(t)dt \\ &= \left(b_i(t) - \frac{1}{2}\sum_{v=1}^{m}\sigma_{iv}^2(t)\right)dt + \sum_{v=1}^{m}\sigma_{iv}(t)dW_v(t) \\ &= \gamma_i(t)dt + \sum_{v=1}^{m}\sigma_{iv}(t)dW_v(t), \end{aligned} \quad (2.3)$$

where $\gamma_i(t) := b_i(t) - \frac{1}{2}a_{ii}(t)$ is the individual stock growth rates.

The cross-variation process for $\log X_i$ and $\log X_j$ is denoted by

$$a_{ij}(t)dt = d\left\langle \log X_i, \log X_j\right\rangle(t) = \sum_{v=1}^{m}\sigma_{iv}(t)\sigma_{jv}(t)dt = (\sigma(t)\sigma'(t))_{ij}. \quad (2.4)$$

Equation (2.3) can be integrated directly and we get

$$\log X_i(t) = \log X_i(0) + \int_0^t \gamma_i(s)ds + \int_0^t \sum_{v=1}^{m} \sigma_{iv}(s)dW_v(s),\ t \in [0, \infty).$$

In exponential form, the stock price process becomes

$$X_i(t) = X_i(0) \exp\left( \int_0^t \gamma_i(s)ds + \int_0^t \sum_{v=1}^{n} \sigma_{iv}(s)dW_v(s) \right). \tag{2.5}$$

It follows that $X_i$ is adapted and that $X_i(t) > 0$ for all $t \in [0, \infty)$.

Under this expression, the integrability condition (2.2) is equivalent to:

$$\sum_{i=1}^{n} \int_0^T \left(|b_i(t)| + a_{ii}(t)\right) dt < \infty, \quad a.s. \tag{2.6}$$

**Definition 2.2** *(cf. [F02], 1.1.2 Definition) A market is a family* $\mathcal{M} = \{X_1, \ldots, X_n\}$ *of stocks, each defined as in (2.5).*
*The market* $\mathcal{M}$ *is nondegenerate if there is a number* $\epsilon > 0$ *such that*

$$\xi' \sigma(t)\sigma'(t)\xi \geq \epsilon \|\xi\|^2,\ \xi \in \mathbb{R}^n,\ t \in [0, \infty),\ a.s. \tag{2.7}$$

*The market* $\mathcal{M}$ *has bounded variance if there exists a number* $M > 0$ *such that*

$$\xi' \sigma(t)\sigma'(t)\xi \leq M \|\xi\|^2,\ \xi \in \mathbb{R}^n,\ t \in [0, \infty),\ a.s. \tag{2.8}$$

**Definition 2.3** *A portfolio in the market* $\mathcal{M}$ *is a measurable, adapted vector-valued process* $\pi$, $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))'$, *for* $t \in [0, \infty)$ *such that* $\pi$ *is almost surely bounded on* $[0, \infty)$ *and*

$$\pi_1(t) + \ldots + \pi_n(t) = 1,\ t \in [0, \infty),\ a.s.$$

According to [FKK05], $\pi(\cdot)$ as defined above is regarded as an "extended portfolio": a portfolio that allows for a short and long position.

If we denote $Z^\pi(t)$ to be the value of an investment in $\pi$ at time $t$, then the total change in the portfolio value at time $t$ is

$$\frac{dZ^\pi(t)}{Z^\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dX_i(t)}{X_i(t)} \tag{2.9}$$

with initial capital $Z^\pi(0) = z > 0$.

Using (2.1) and (2.9), we see that

$$\frac{dZ^{\pi}(t)}{Z^{\pi}(t)} = \sum_{i=1}^{n} \pi_i(t) b_i(t) dt + \sum_{v=1}^{m} \left( \sum_{i=1}^{n} \pi_i(t) \sigma_{iv}(t) \right) dW_v(t)$$
$$= b^{\pi}(t) dt + \sum_{v=1}^{m} \sigma_v^{\pi}(t) dW_v(t),$$

where

$$b^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t) b_i(t) \quad \text{and} \quad \sigma_v^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t) \sigma_{iv}(t) \tag{2.10}$$

for $v = 1, \ldots, m$, are the rate-of-return and the volatility coefficients of the portfolio, respectively.

Using (2.3) and by Itô's formula, (2.9) can be expressed in logarithmic form

$$\begin{aligned}
d(\log Z^{\pi}(t)) &= \frac{dZ^{\pi}(t)}{Z^{\pi}(t)} - \frac{1}{2Z^{\pi}(t)^2} d\left\langle Z^{\pi} \right\rangle_t \\
&= \sum_{i=1}^{m} \pi_i(t) b_i(t) + \sum_{v=1}^{m} \sigma_v^{\pi}(t) dW_v(t) - \frac{1}{2} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t) \right) \\
&= \sum_{i=1}^{n} \pi_i(t) \left( \gamma_i(t) + \frac{1}{2} a_{ii}(t) \right) + \sum_{v=1}^{m} \sigma_v^{\pi}(t) dW_v(t) \\
&\quad - \frac{1}{2} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t) \right) \\
&= \gamma^{\pi}(t) + \sum_{v=1}^{m} \sigma_v^{\pi}(t) dW_v(t),
\end{aligned} \tag{2.11}$$

where the growth rate of the portfolio $\pi(\cdot)$ is

$$\gamma^{\pi}(t) := \sum_{i=1}^{n} \pi_i(t) \gamma_i(t) + \gamma_*^{\pi}(t), \tag{2.12}$$

and the excess-growth rate of the portfolio $\pi(\cdot)$ is

$$\gamma_*^{\pi}(t) := \frac{1}{2} \left( \sum_{i=1}^{n} \pi_i(t) a_{ii}(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t) \right) \tag{2.13}$$

Here, the process $a^{\pi\pi}$ is defined as

$$a^{\pi\pi}(t) = \sum_{i=1}^{n} \sum_{k=1}^{n} \pi_i(t) a_{ik}(t) \pi_k(t),$$

and it is called the portfolio variance process. We can therefore write (2.9) using the second line in (2.11) as

$$\frac{dZ^{\pi}(t)}{Z^{\pi}(t)} = d\left(\log Z^{\pi}(t)\right) + \frac{1}{2}a^{\pi\pi}(t). \tag{2.14}$$

Note that $\gamma_*^{\pi}(t)$ can be written as

$$\gamma_*^{\pi}(t) = \frac{1}{2}\left(\sum_{i=1}^{n}\pi_i(t)a_{ii}(t) - a^{\pi\pi}(t)\right),\ t \in [0,\infty),\ a.s. \tag{2.15}$$

The instantaneous logarithmic return of the portfolio and the logarithmic return of the individual stocks can be related by

$$\begin{aligned} d\log Z^{\pi}(t) &= \frac{dZ^{\pi}(t)}{Z^{\pi}(t)} - \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i(t)a_{ij}(t)\pi_j(t)\right)dt \\ &= \sum_{i=1}^{n}\pi_i(t)\frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i(t)a_{ij}(t)\pi_j(t)\right)dt \\ &= \sum_{i=1}^{n}\pi_i(t)d\log X_i(t) + \frac{1}{2}\sum_{i=1}^{n}\pi_i(t)a_{ii}(t)dt - \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i(t)a_{ij}(t)\pi_j(t)\right)dt \\ &= \sum_{i=1}^{n}\pi_i(t)d\log X_i(t) + \gamma_*^{\pi}(t). \end{aligned} \tag{2.16}$$

Therefore $\gamma_*^{\pi}(t)$ also represents the excess growth rate of the portfolio over the weighted average of the instantaneous logarithmic returns of the component stocks.

For the purposes of notation and the analysis to follow, two further important definitions are introduced, which will be used throughout this project.

**Definition 2.4** *The "order-statistics" notation for the weights of a portfolio $\pi(\cdot)$ is ranked at time $t$ in decreasing order, from largest down to the smallest*

$$\max\ \pi_i(t)_{1\le i\le n} =: \pi_{(1)}(t) \le \ldots \le \pi_{(n)}(t) := \min\ \pi_i(t)_{1\le i\le n}. \tag{2.17}$$

**Definition 2.5** *Given any two portfolios $\pi(\cdot)$, $\rho(\cdot)$ with the same initial capital $Z^{\pi}(0) = Z^{\rho}(0) = z > 0$, we shall say that $\pi(\cdot)$ represents an arbitrage opportunity relative to $\rho(\cdot)$ over the time horizon $[0,T]$, with $T > 0$ a given real number, if*

$$\mathbb{P}\left(Z^{\pi}(T) \ge Z^{\rho}(T)\right) = 1 \text{ and } \mathbb{P}\left(Z^{\pi}(T) > Z^{\rho}(T)\right) > 0. \tag{2.18}$$

## 3 The Market Portfolio

**Definition 3.1** *The portfolio $\mu$ is the market portfolio and it has market weights $\mu_1, \ldots, \mu_n$ given as*

$$\mu_i(t) := \frac{X_i(t)}{\sum_{i=1}^{n} X_i(t)}, \ t \in [0, \infty) \tag{3.1}$$

*for $i = 1, \ldots, n$.*

To understand this definition, write the total capitalisation of the market $Z(t)$ and relative capitalisation of the individual components, $\mu_i(t)$ as

$$Z(t) = X_1(t) + \ldots + X_n(t) \quad \text{and} \quad \mu_i(t) := \frac{X_i(t)}{Z(t)}, \ i = 1, \ldots, n. \tag{3.2}$$

Then, the resulting value-process $Z^\mu(\cdot)$ satisfies

$$\begin{aligned} \frac{dZ^\mu(t)}{Z^\mu(t)} &= \sum_{i=1}^{m} \mu_i(t) \frac{dX_i(t)}{X_i(t)} \\ &= \sum_{i=1}^{n} \frac{dX_i(t)}{Z(t)} \\ &= \frac{dZ(t)}{Z(t)} \end{aligned}$$

If we start with initial capital $Z^\mu(0) = Z(0)$, we have $Z^\mu(\cdot) \equiv Z(\cdot)$. Therefore, investing in accordance with the process $\mu(\cdot)$ results in ownership of the entire market.

## 4 The relative return of a stock and arbitrary portfolio with respect to the market portfolio

In this section, we introduce the relative covariance process for an arbitary portfolio $\pi(\cdot)$ and use it to simplify our expression for the excess growth rate and also introduce the "numeraire-invariance" property.

By applying Itô's formula to (2.3) and using the expression for $d \log Z^\pi(t)$, we have

$$d \log\left(\frac{X_i(t)}{Z^\mu(t)}\right) = d\left(\log \mu_i(t)\right) = \left(\gamma_i(t) - \gamma^\mu(t)\right) dt + \sum_{v=1}^{m} \left(\sigma_{iv}(t) - \sigma_v^\mu(t)\right) dW_v(t) \tag{4.1}$$

for $i = 1, \dots, n.$, or equivalently,

$$\frac{d\mu_i(t)}{\mu_i(t)} = \left(\gamma_i(t) - \gamma^\mu(t) + \frac{1}{2}\tau^\mu_{ii}(t)\right) dt + \sum_{v=1}^{m} \left(\sigma_{iv}(t) - \sigma^\mu_v(t)\right) dW_v(t) \quad (4.2)$$

for $i = 1, \dots, n$.

For an arbitrary portfolio $\pi(\cdot)$, and with $e_i$, denoting the $i^{th}$ unit vector in $\mathbb{R}^n$, the relative convariance (matrix-valued) process is given by

$$\begin{aligned}\tau^\pi_{ij}(t) &:= \sum_{v=1}^{d} \left(\sigma_{iv}(t) - \sigma_{\pi v}(t)\right)\left(\sigma_{jv}(t) - \sigma_{\pi v}(t)\right) \\ &= \left(\pi(t) - e_i\right)' a(t) \left(\pi(t) - e_j\right) \\ &= a_{ij}(t) - a^\pi_i(t) - a^\pi_j(t) + a^{\pi\pi}(t) \end{aligned} \quad (4.3)$$

for $1 \leq i, j \leq n$, where

$$a^\pi_i(t) := \sum_{k=1}^{n} \pi_k(t) a_{ik}(t).$$

We also have, by using the definition for nondegeneracy

$$\begin{aligned}\tau^\pi_{ii}(t) &\geq \epsilon \|\pi(t) - e_i\|^2 \\ &\geq \epsilon(1 - \pi_i(t))^2 \\ &\geq \epsilon(1 - \pi_{max}(t))^2 \end{aligned} \quad (4.4)$$

Hence, we can express the excess growth rate (2.13) as

$$\gamma^\pi_*(t) = \frac{1}{2} \sum_{i=1}^{n} \pi_i(t) \tau^\pi_{ii}(t), \quad (4.5)$$

where we have used the elementary property

$$\begin{aligned}\sum_{j=1}^{n} \pi_j(t) \tau^\pi_{ij}(t) &= \sum_{j=1}^{n} \pi_j(t) \left(a_{ij}(t) - a^\pi_i(t) - a^\pi_j(t) + a^{\pi\pi}(t)\right) \\ &= \left(\sum_{j=1}^{n} \pi_j(t) a_{ij}(t) - a^\pi_i(t)\right) - \left(\sum_{j=1}^{n} \pi_j(t) a^\pi_j(t) - a^{\pi\pi}(t)\right) \\ &= 0 \end{aligned} \quad (4.6)$$

for $i = 1, \dots, n$.

Under nondegeneracy, $\gamma_*^\pi(\cdot)$ can be expressed as

$$\begin{aligned}\gamma_*^\pi(t) &\geq \epsilon(1-\pi_{(1)}(t))^2 \\ &\geq \epsilon\delta^2 \\ &\geq \delta \end{aligned}\tag{4.7}$$

For arbitrary portfolios $\pi(\cdot)$, $\rho(\cdot)$, we also have the "numeraire-invariance" property

$$\gamma_*^\pi(t) = \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\tau_{ii}^\rho(t) - \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}^\rho(t)\right)\tag{4.8}$$

(c.f. [F02] *Lemma 1.3.4 and 1.3.6*).

When $\mathcal{M}$ is non-degenerate (2.7), we also have another formulation for the relative variance process $\tau_{ii}^\pi(\cdot)$. These results will prove helpful in determining the existence of relative arbitrage and will be used frequently.

**Lemma 4.1** *For every portfolio $\pi(\cdot)$ that satisfies the non-degeneracy condition, we have the inequality*

$$\tau_{ii}^\pi(t) \geq \epsilon(1-\pi_i(t))^2\tag{4.9}$$

(c.f. [FK08] *Lemma 3.4*)

Using the notations of Definition 2.4, we can reformulate (4.9) as

$$\tau_{ii}^\pi(t) \geq \epsilon(1-\pi_{\max}(t))^2\tag{4.10}$$

Using the notations derived above, we can now represent the relative performance of a portfolio $\pi(\cdot)$ with respect to $\mu(\cdot)$ by

$$\begin{aligned} d\log\left(\frac{Z^\pi(t)}{Z^\mu(t)}\right) &= \gamma_\pi^*(t)dt + \sum_{i=1}^n \pi_i(t)d\log\mu_i(t) \\ &= \left(\gamma_\pi^*(t) - \gamma_\mu^*(t)\right)dt + \sum_{i=1}^n(\pi_i(t)-\mu_i(t))d\log\mu_i(t), \end{aligned}\tag{4.11}$$

where the second equality follows by observing that

$$\begin{aligned}\sum_{i=1}^n \mu_i(t)d\log\mu_i(t) &= \sum_{i=1}^n \mu_i(t)\left(\gamma_i(t)-\gamma_\mu(t)\right)dt \\ &\quad + \sum_{v=1}^m\left(\left(\sum_{i=1}^n \mu_i(t)\sigma_{iv}(t)\right) - \sigma_v^\mu(t)\right)dW_v(t) \\ &= \sum_{i=1}^n \mu_i(t)\left(\gamma_i(t)-\gamma_\mu(t)\right)dt \\ &= -\gamma_\mu^*(t)dt,\end{aligned}$$

and so

$$-\gamma_{\mu}^{*}(t) - \sum_{i=1}^{n} \mu_i(t) d \log \mu_i(t) = 0.$$

Add this quantity to the first line of (4.11) and we get the result.

# 5 Notions of Diversity

The following definition gives a mathematical formulation of "diversity" in the context of equity markets. Plainly put, having a diverse equity market means simply that no single company dominates the market in its entirety.

**Definition 5.1** *(cf. [F02], 2.2.1 Definition) The market $\mathcal{M}$ is diverse if there exists a number $\delta > 0$ such that*

$$\mu_{(1)}(t) \leq 1 - \delta, \ t \in [0, \infty), \ a.s. \tag{5.1}$$

*$\mathcal{M}$ is weakly diverse on $[0, T]$ if there exists a $\delta > 0$ such that*

$$\frac{1}{T} \int_{o}^{T} \mu_{(1)}(t) dt < 1 - \delta, \ a.s. \tag{5.2}$$

For the sake of completeness, we introduce two further forms of diversity.

$\mathcal{M}$ is *uniformly weakly diverse* over $[T_0, \infty]$, if there exists a $\delta \in (0, 1)$ such that (5.2) holds almost surely for every $T \in [T_0, \infty)$ and *asymptotically weakly diverse* if, for some $\delta \in (0, 1)$, we have

$$\limsup_{T \longrightarrow \infty} \frac{1}{T} \int_{0}^{T} \mu_{(1)} dt < 1 - \delta, \ a.s. \tag{5.3}$$

Empirically, in Australia and in any well-developed nation, we observe that equity markets are diverse under all forms as defined above and know for a fact that the largest company doesn't even account for half of the total market capitalisation. As stated in [F02], the above definitions are just a consequence of anti-competition laws that prevent excessive capital concentration into large corporations. We will show however that even this mild, innocuous condition (which is self-evident in real life) when applied to our hypothetical model can have a major effect on portfolio behaviour, in particular, in relation to the existence of relative arbitrage.

In the coming sections, we will demonstrate with examples that in a weakly diverse and nondegenerate market, arbitrage opportunities exist relative to the market portfolio, albeit with a sufficiently long arbitrary time horizon.

However, before we do so, we need to introduce market entropy which is a measure of how diverse the market is and functionally generated portfolios, which is an indispensable tool in determining the existence of relative arbitrage. The entropy-weighed portfolio which is discussed in the following section is also an example of a portfolio that outperforms the market portfolio.

## 6 Market Entropy and its behaviour

[F02] introduces the concept of entropy as a measure of market diversity with a zero entropy occurring when all capital is concentrated in a single stock. Normally, entropy is used in statistical mechanics and information theory as a measure of the uniformity of a probability distribution. For the rest of this section, we assume that the time domain is $[0, T]$.

The entropy function $\mathbf{S}$ is defined as:

$$\mathbf{S}(x) = -\sum_{i=1}^{n} x_i \log x_i \tag{6.1}$$

for all $x$ in

$$\Delta^n = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1,\ 0 < x_i < 1,\ i = 1, 2, \ldots, n \right\} \tag{6.2}$$

This class of portfolios prevents total capital concentration in a single stock.

**Definition 6.1** (c.f. [F02], p.36) *Let $\mu$ be the market portfolio. Then the market entropy process $\mathbf{S}(\mu)$ is*

$$\mathbf{S}(\mu(t)) = -\sum_{i=1}^{n} \mu_i(t) \log \mu_i(t)$$

The properties of $\mathbf{S}(\mu)$ are

(i) it is a continuous semimartingale;

(ii) $0 < \mathbf{S}(\mu(t)) \leq \log n$ for all $t \in [0, T]$ a.s; and

(iii) the market $\mathcal{M}$ is diverse if and only if there is an $\epsilon > 0$ such that

$$\mathbf{S}(\mu(t)) \geq \epsilon,\ t \in [0, T] \text{ a.s.} \tag{6.3}$$

(c.f. [F02], *Proposition 2.3.2*)

Condition (iii) makes sense by our definition of zero entropy.

[F02] defines the entropy weighted portfolio as

**Definition 6.2** (c.f. [F02], p.37) *The entropy-weighted portfolio is defined by*

$$\pi_i(t) = \frac{-\mu_i(t)\log\mu_i(t)}{\mathbf{S}(\mu(t))},\ t \in [0,T] \tag{6.4}$$

*for* $i = 1, 2, \ldots, n$.

Note that

$$\frac{\pi_i(t)}{\mu_i(t)} = \frac{-\log\mu_i(t)}{\mathbf{S}(\mu(t))},\ t \in [0,T]$$

decreases with increasing $\mu_i(t)$. So, with stocks with increasing market weights, $\pi(\cdot)$ will hold less than its counterpart $\mu(\cdot)$. High-priced stocks will be sold and low-price stocks bought under an entropy-weighted portfolio.

Let $Z^\mu$ and $Z^\pi$ be the portfolio value processes of $\mu$ and $\pi$ respectively. Then, a.s., for $t \in [0,T]$,

$$d\log\mathbf{S}(\mu(t)) = d\log\left(\frac{Z^\pi(t)}{Z^\mu(t)}\right) - \frac{\gamma_*^\mu(t)}{\mathbf{S}(\mu(t))}dt \tag{6.5}$$

where $\gamma_*^\mu$ is the excess growth process of $\mu$. (c.f. [F02], *Theorem 2.3.4*)

With the above result, we see that the semi-martingale decomposition of the logarithm of the market entropy satisfies

$$d\log\mathbf{S}(\mu(t)) = \left(\gamma^\pi(t) - \gamma^\mu(t) - \frac{\gamma_*^\mu(t)}{\mathbf{S}(\mu(t))}\right)dt + \sum_{i,v=1}^{n}(\pi_i(t)-\mu_i(t))\sigma_{iv}(t)dW_v(t) \tag{6.6}$$

for $t \in [0,\infty)$ a.s. For a "stable" market in the long term, we need that

$$\lim_{T\to\infty}\frac{1}{T}\log\mathbf{S}(\mu(t)) = 0 \text{ a.s.}$$

That is, a.s.,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\gamma^\pi(t) - \gamma^\mu(t) - \frac{\gamma_*^\mu(t)}{\mathbf{S}(\mu(t))}\right)dt = 0$$

This tells us that on average, we require $\gamma^\pi(t) > \gamma^\mu(t)$ in order to have long term stability which implies that we need the entropy weighted portfolio to outperform the market portfolio.

If the market $\mathcal{M}$ is nondegenerate and diverse, then we have the following results:

$$\begin{aligned} \mathbf{S}(\mu(0)) &\leq \log n \quad \text{by (ii)} \\ \mathbf{S}(\mu(t)) &> \epsilon \quad \text{by (iii)} \\ \gamma_*^{\mu}(t) &> \delta \quad \text{by (4.7).} \end{aligned}$$

Hence,

$$\begin{aligned} \log(Z^{\pi}(T)/Z^{\pi}(0)) &= \log(Z^{\mu}(T)/Z^{\mu}(0)) + \log(\mathbf{S}(\mu(T))/\mathbf{S}(\mu(0)) \\ &\quad + \int_0^T \frac{\gamma_*^{\mu}(t)}{\mathbf{S}(\mu(t))} dt \\ &> \log(Z^{\mu}(T)/Z^{\mu}(0)) + \log\epsilon - \log\log n + \frac{\delta T}{\log n} \\ &> \log(Z^{\mu}(T)/Z^{\mu}(0)) \end{aligned}$$

when $T > \epsilon^{-1}\log n(\log\log n - \log\delta)$.

We have therefore shown that for a sufficiently large number number $T$,

$$\frac{Z^{\pi}(T)}{Z^{\pi}(0)} > \frac{Z^{\mu}(T)}{Z^{\mu}(0)}, \text{ a.s.} \tag{6.7}$$

(c.f. [F02] *Corollary 2.3.5*).

# 7 Portfolio Generating Functions

Suppose that $\mathbf{S}$ is a positive continuous function on $\Delta^n$ and $\pi$ be a portfolio. [F02] says that $\mathbf{S}$ generates $\pi$ if there exists a measurable process of bounded variation $\Theta$ such that

$$\log\left(\frac{Z^{\pi}(t)}{Z^{\mu}(t)}\right) = \log\mathbf{S}(\mu(t)) + \Theta(t) \text{ a.s.} \tag{7.1}$$

Here $\Theta(t)$ is the drift process corresponding to $\mathbf{S}$. [F02] states that if $\mathbf{S}$ generates $\pi$, then $\mathbf{S}$ is the generating function of $\pi$ so that $\pi$ is functionally generated.

Observe that $\Theta$ is continuous and adapted since both $\log\left(\frac{Z^{\pi}}{Z^{\mu}}\right)$ and $\log(\mathbf{S}(\mu)$ are continuous and adapted. Further, $\Theta$ is a function of bounded variation, hence $\log\mathbf{S}(\mu)$ is a continuous semimartingale. We can, therefore, write (7.1) in differential form

$$d\log\left(\frac{Z^{\pi}(t)}{Z^{\mu}(t)}\right) = d\log\mathbf{S}(\mu(t)) + d\Theta(t),\ t\in[0,T],\ \text{a.s.} \tag{7.2}$$

Note that $\Theta$ will dominate the long-term behaviour of the relative return, especially if $\log \mathbf{S}$ is bounded on $\Delta^n$. (c.f. [F02], p.44)

[F02] also shows a connection between the growth rate of a functionally generated portfolio $\pi$, the market portfolio $\mu$ and the drift process $\Theta$.

If we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{S}(\mu(t)) = 0 \tag{7.3}$$

Then

$$\lim_{T\to\infty} \frac{1}{T} \left( \int_0^T \gamma^{\pi}(t)dt - \int_0^T \gamma^{\mu}(t)dt - \Theta(t) \right) = 0, \text{ a.s.} \tag{7.4}$$

Note that (7.3) will be satisfied especially if $\mathbf{S}$ is bounded on $\Delta^n$, which is the case in most financial applications.

According to [F02], the total return process of a functionally generated portfolio can be written as

$$\log \left( \frac{\hat{\mathrm{Z}}^{\pi}(t)}{\hat{\mathrm{Z}}^{\mu}(t)} \right) = \log \mathbf{S}(\mu(t)) + \int_0^t \left(\delta^{\pi}(s) - \delta^{\mu}(s)\right) ds + \Theta(t) \tag{7.5}$$

where $\delta$ represents the dividend rate.

[F02] assumes that the generating function $\mathbf{S}$ is in the class of $\mathbf{C}^2$ (i.e. real-valued functions defined on the open subset of $\mathbb{R}^n$ which are twice continuously differentially in all $n$ variables.)

Using [F02]'s notation, $D_i$ represents the partial derivative with respect to the $i$th variable and $D_{ij}$ represents the second partial derivative with respect to the $i$th and $j$th variables. Now, if $\mathbf{S}$ is a positive $\mathbf{C}^2$ functioned defined on a neighbourhood $\mathbf{U}$ of $\Delta^n$ such that for all $i$, $x_i D_i \log \mathbf{S}(x)$ is bounded on $\Delta^n$, then $\mathbf{S}$ generates the following weighted portfolio $\pi$

$$\pi_i(t) = \left( D_i \log \mathbf{S}(\mu(t)) + \mathbf{1} - \sum_{\mathbf{j=1}}^{\mathbf{n}} \mu_{\mathbf{j}}(\mathbf{t}) \mathbf{D}_{\mathbf{j}} \log \mathbf{S}(\mu(\mathbf{t})) \right) \mu_i(t), \tag{7.6}$$

for $t \in [0, T]$ and $i = 1, 2, \ldots, n$. Further, the drift process $\Theta$ are a.s for $t \in [0, T]$,

$$d\Theta(t) = \frac{-1}{2\mathbf{S}(\mu(t))} \sum_{i,j=1}^{n} D_{ij} \mathbf{S}(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t) dt \tag{7.7}$$

where $\tau_{ij}(t)$ is the relative covariance process.

Some examples of generating functions and their portfolios are

1 Let $\mathbf{S}(x) = 1$. This generates the market portfolio $\mu$ with $\Theta(t) = 0$.

2 Let $\mathbf{S}(x) = w_1 x_1 + \ldots + w_n x_n$, where $w_i$ is non-negative constants for which one of them is positive. This generating function generates the buy and hold portfolio. The holder holds $w_i$ shares of the $i$th stock. In this case $\Theta(t) = 0$.

3 Let $\mathbf{S}(x) = (x_1, \ldots, x_n)^{\frac{1}{n}}$. This generates an equal weighted portfolio where $d\Theta(t) = \gamma_*^\pi(t)dt$. An example of this is the value line index.

4 (A single stock, with leverage) The function $\mathbf{S}(x) = x_1^2$ generates the portfolio $\pi$ with weights $\pi_1(t) = 2 - \mu_1(t)$, and $\pi_i(t) = -\mu)i(t)$, for $i = 2, \ldots, n$. In this case $d\Theta(t) = -\tau_{11}(t)dt$, so the drift term is decreasing. This shows that while investing in a single share may seem reasonable, leveraging it by shorting the market may not be wise, at least over the long term.

A more complicated case is the *weighted average capitalisation of the market.* This portfolio is commonly used as a measure of the concentration of capital in the market. Suppose the value of the weighted average is

$$\sum_{i=1}^{n} \mu_i(t) X_i(t), \ t \in [0, T]$$

and the weighted average for the weights of capitalisation is

$$\sum_{i=1}^{n} \mu_i^2(t), \ t \in [0, T]$$

which are proportional to the weighted-average capitalisation. In this case,

$$\mathbf{S}(x) = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$$

generates the portfolio $\pi$ with

$$\pi_i(t) = \frac{\mu_i^2(t)}{\mu_1^2(t) + \ldots + \mu_n^2(t)}, \ t \in [0, T]$$

for $i = 1, 2, \ldots, n$. That is, $\pi$ is overweighted in the larger stocks and underweighted in the smaller stocks, when compared to the market portfolio. The drift process is

$$d\Theta(t) = -\gamma_*^\pi(t)dt, \ t \in [0, T].$$

Note that

$$\log \mathbf{S}(\mu(t)) = \frac{1}{2} \log \left( \sum_{i=1}^{n} \mu_i^2(t) \right), \ t \in [0, T].$$

So, if the weighted average of the capitalisation weights is the same as the beginning and end of a time interval, i.e. $\log \mathbf{S}(\mu(t))$ is a constant, then $\pi$ will have a lower return than the market in the same time interval since the drift process is a decreasing process. (c.f. [F02] *Example 3.1.9*).

How do we know which portfolios are functionally generated? The next result gives us an answer.

**Proposition 7.1** *(*c.f. [F02]*, Proposition 3.1.11) Let $f_1, \ldots, f_n$ be continuously differentiable real-valued functions defined on a neighbourhood of $\Delta^n$ with $\sum_{i=1}^n f_i(x) = 1$ for all $x \in \Delta^n$. Then the portfolio $\pi$ defined by $\pi_i(t) = f_i(\mu(t))$ for $t \in [0, T]$ and $i = 1, \ldots, n$ is functionally generated if and only if there exists a continuously differentiable real-valued function $\mathbf{F}$ defined on a neighbourhood of $\Delta^n$ such that*

$$\sum_{i=1}^{n} \left( \frac{f_i(x)}{x_i} + F(x) \right) dx_i \tag{7.8}$$

*is an exact differential.*

Recall that a differential is exact if it is of the form $\sum_i D_i G(x) dx_i$ for some differentiable function $G$. (see [S65]).

It turns out the values of $\mathbf{S}$ on $\Delta^n$ uniquely determines the portfolio. That is, if $f$ is a continuously differentiable real-valued function defined on a neighbourhood $U$ of $\Delta^n$, then $f$ is a constant on $\Delta^n$ if and only if for all $x \in \Delta^n$, $D_i f(x) = D_j f(x)$ for all $i$ and (j) (c.f. [F02] *Lemma 3.1.13 p.52*). Further, two $\mathbf{C}^2$ functions $\mathbf{S}_1$ and $\mathbf{S}_2$ generate the same portfolio for all realisation of the market if and only if

$$D_i \log \mathbf{S}_1(x) + 1 - \sum_{j=1}^{n} x_j D_j \log \mathbf{S}_1(x) = D_i \log \mathbf{S}_2(x) + 1 - \sum_{j=1}^{n} x_j D_j \log \mathbf{S}_2(x) \tag{7.9}$$

for $i = 1, \ldots, n$ and for all $x \in \Delta^n$. In other words, (7.9) holds if and only if $\log \left( \frac{\mathbf{S}_1}{\mathbf{S}_2} \right)$ is a constant on $\Delta^n$.

# 8 Weakly diverse markets contains arbitrage opportunities

In this section, we consider detailed examples to illustrate the concept that if the model $\mathcal{M}$ consisting of stocks that pay no dividends (as defined in Section 1) is weakly diverse and non-degenerate, then arbitrage opportunities relative to the market portfolio do exist, for a sufficient large real number $T > 0$. We have already seen an example in Section 6: the entropy-weighted portfolio. We also deal with the case when dividends are included in the analysis. Not surprisingly, when we impose certain dividend payment structures, the relative arbitrage opportunities disappear.

## 8.1 Admissible, market-dominating portfolio

To prevent the possibility of double strategies and to ensure realistic constructions of the portfolio process, we introduce the following definition:

**Definition 8.1** *(c.f. [F02], 3.3.1 Definition) A portfolio is admissible if:*

*(i) for* $i = 1, \ldots, n,\ \pi_i(t) \geq 0,\ t \in [0, T]$*;*

*(ii) there exists a constant* $c > 0$ *such that*

$$\frac{\hat{\mathrm{Z}}^{\pi}(t)}{\hat{\mathrm{Z}}^{\pi}(0)} \geq c\frac{\hat{\mathrm{Z}}^{\mu}(t)}{\hat{\mathrm{Z}}^{\mu}(0)},\ t \in [0, T],\ \text{a.s.};$$

*(iii) there exists a constant* $M$ *such that for* $i = 1, \ldots, n$*,*

$$\frac{\pi_i(t)}{\mu_i(t)} \leq M,\ t \in [0, T],\ \text{a.s.}$$

Note that in this definition

(i) does not allow for short-selling.

(ii) limits negative performance relative to the market as numeraire.

(iii) restricts arbitrarily high overweighting of any particular stock relative to the market weighting.

So, the arbitrage opportunities in this situation are made up of these admissible portfolios.

Suppose $\eta$ and $\xi$ are two portfolios. [F02] says that $\eta$ dominates $\xi$ in $[0, T]$ if

$$\frac{\hat{\mathrm{Z}}_{\eta}(T)}{\hat{\mathrm{Z}}_{\eta}(0)} \geq \frac{\hat{\mathrm{Z}}_{\xi}(T)}{\hat{\mathrm{Z}}_{\xi}(0)}\ \text{a.s.}$$

and

$$\frac{\hat{Z}_\eta(T)}{\hat{Z}_\eta(0)} > \frac{\hat{Z}_\xi(T)}{\hat{Z}_\xi(0)} \text{ a.s}$$

If we have the latter, $\eta$ strictly dominates $\xi$ in $[0, T]$.

Consider a market $\mathcal{M}$ which is non-degenerate, weakly diverse in $[0, T]$ and consists of non-dividend paying stocks with a portfolio process generated by

$$\mathbf{S}(\mu(t)) = 1 - \frac{1}{2}\sum_{i=1}^{n} \mu_i^2(t) \tag{8.1}$$

The weights of the portfolio $\pi$ are, using that $D_i \log \mathbf{S}(\mu(t)) = -\frac{\mu_i(t)}{\mathbf{S}(\mu(t))}$

$$\begin{aligned} \pi_i(t) &= \left( \frac{-\mu_i(t)}{\mathbf{S}(\mu(t))} + 1 + \frac{\sum_{j=1}^{n} \mu_j^2(t)}{\mathbf{S}(\mu(t))} \right) \mu_i(t) \\ &= \left( \frac{-\mu_i(t)}{\mathbf{S}(\mu(t))} + 1 + \frac{2(1 - \mathbf{S}(\mu(t)))}{\mathbf{S}(\mu(t))} \right) \mu_i(t) \\ &= \left( \frac{2 - \mu_i(t)}{\mathbf{S}(\mu(t))} - 1 \right) \mu_i(t), \end{aligned} \tag{8.2}$$

for $i = 1, \ldots, n$, and the drift process (using that $D_{ij}\mathbf{S}(\mu(t)) = -1$ when $j = i$ and 0 when $j \neq i$)

$$d\Theta(t) = \frac{1}{2\mathbf{S}(\mu(t))} \sum_{i=1}^{n} \mu_i^2(t) \tau_{ii}(t) dt. \tag{8.3}$$

We want to show that $\pi$ is admissible in the sense above and strictly dominates the market portfolio for time $T$ sufficiently large (to be specified).

From (8.1) and using that $\sum_{i=1}^{n} \mu_i^2(t) < 1$, we see that

$$\frac{1}{2} < \mathbf{S}(\mu(t)) < 1 \text{ for all } t \in [0, T], \text{ a.s.} \tag{8.4}$$

Now, by inserting suitable inequalities for $\mathbf{S}(\mu(t))$ into (8.2), we get

$$\pi_i(t) > (2 - \mu_i(t) - 1)\mu_i(t) = (1 - \mu_i(t))\mu_i(t) > 0$$

and

$$\pi_i(t) < (4 - 2\mu_i(t) - 1)\mu_i(t) = (3 - 2\mu_i(t))\mu_i(t) < 3\mu_i(t).$$

We thus get

$$0 < \pi_i(t) < 3\mu_i(t), \; t \in [0, T], \; \text{a.s.}$$

Conditions $(i)$ and $(ii)$ of Definition 8.1 are therefore satisfied. Since $\tau_{ii}(t) \geq 0$ for $i = 1, \ldots, n$, the drift process (8.3) is non-decreasing and we get from (7.2),

$$\log\left(\frac{Z^\pi(T)}{Z^\pi(0)}\right) - \log\left(\frac{Z^\mu(T)}{Z^\mu(0)}\right) \geq \log\left(\frac{\mathbf{S}(\mu(T))}{\mathbf{S}(\mu(0))}\right), \; \text{a.s.}$$

From (8.4), it can easily be shown that $\frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))} \geq \frac{1}{2}$, so we get

$$\frac{Z^\pi(T)}{Z^\pi(0)} \geq \frac{1}{2} \frac{Z^\mu(t)}{Z^\mu(0)}, \; \text{a.s.}$$

Thus, the second condition of Definition 8.1 is satisfied and so $\pi$ is admissible. Next, we need to check to see if $\pi$ strictly dominates $\mu$. Since we know that $\mathcal{M}$ is non-degenerate, we make use of (4.4) and (8.4) to get

$$\begin{aligned}
\Theta(T) &= \frac{1}{2} \int_0^T \frac{1}{\mathbf{S}(\mu(t))} \sum_{i=1}^n \mu_i^2(t) \tau_{ii}(t) dt \\
&\geq \frac{\epsilon}{2} \int_0^T \sum_{i=1}^n \mu_i^2(t)(1 - \mu_{\max}(t))^2 dt \\
&\geq \frac{\epsilon}{2n} \int_0^T (1 - \mu_{\max}(t))^2 dt,
\end{aligned}$$

where the last inequality $\sum_{i=1}^n \mu_i^2 \geq \frac{1}{n}$ is derived by an application of the power mean inequality.

Now, we make use of the important assumption that $\mathcal{M}$ is weakly diverse (5.2).

$$\frac{1}{T} \int_0^T (1 - \mu_{\max}(t)) dt > \delta, \; \text{a.s.}$$

By Schwarz's inequality, we get

$$\frac{1}{T} \int_0^T (1 - \mu_{\max}(t))^2 dt > \delta^2, \; \text{a.s.}$$

Hence,

$$\Theta(T) \geq \frac{\epsilon \delta^2 T}{2n}, \; \text{a.s.}$$

We therefore have

$$\begin{aligned}
\log\left(\frac{Z^\pi(T)}{Z^\pi(0)}\right) - \log\left(\frac{Z^\mu(T)}{Z^\mu(0)}\right) &= \log\left(\frac{\mathbf{S}(\mu(T))}{\mathbf{S}(\mu(0))}\right) + \Theta(T) \\
&\geq \log(\mathbf{S}(\mu(T)) - \log(\mathbf{S}(\mu(0)) + \frac{\epsilon\delta^2 T}{2n} \\
&\geq \log\left(\frac{1}{2}\right) - \log 1 + \frac{\epsilon\delta^2 T}{2n} \text{ (by 8.4)} \\
&= -\log 2 + \frac{\epsilon\delta^2 T}{2n} \\
&\geq 0.
\end{aligned}$$

This tells us that if

$$T > \frac{2n\log 2}{\epsilon\delta^2}$$

then $\pi$ strictly dominates the market portfolio in $[0, T]$. Since $\epsilon$ and $\delta$ are usually small quantities, this sort of arbitrage would be effective for portfolios that adopt a long-time horizon. (c.f. [F02] *Example 3.3.3*).

It is commented in [F02] that it is hard, if not impossible to test the validity of the no-arbitrage hypothesis empirically. In the theoretical literature of financial mathematics, we basically assume the existence of an equivalent martingale measure that assures that there is no arbitrage in the market. This example however illustrates that it is possible to have an arbitrage opportunity in a fairly well behaved market (without dividends). In particular, where the only condition imposed is that the market is weakly diverse which by common sense is apparent in real equity markets.

We now divert our attention to the case when dividends are included in the analysis. With dividends, (7.2) is equivalent to

$$d\log\left(\frac{\hat{\mathrm{Z}}^\pi(t)}{\hat{\mathrm{Z}}^\mu(t)}\right) = d\log \mathbf{S}(\mu(t)) + (\delta^\pi(t) - \delta^\mu(t))\,dt + d\Theta(t),\ \ t \in [0, T],\ \text{ a.s.} \tag{8.5}$$

When we integrate (8.5) and make appropriate adjustments, we get

$$\log\left(\frac{\hat{\mathrm{Z}}^\mu(t)}{\hat{\mathrm{Z}}^\mu(0)}\right) = \log\left(\frac{\hat{\mathrm{Z}}^\pi(t)}{\hat{\mathrm{Z}}^\pi(0)}\right) - \log\left(\frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))}\right) + \int_0^t (\delta^\mu(s) - \delta^\pi(s))ds - \int_0^t d\Theta(s).$$

Since we want to avoid $\pi$ strictly dominating the market portfolio, we want that

$$\mathbb{P}\left\{\log\left(\frac{\hat{\mathrm{Z}}^\mu(t)}{\hat{\mathrm{Z}}^\mu(0)}\right) \geq \log\left(\frac{\hat{\mathrm{Z}}^\pi(t)}{\hat{\mathrm{Z}}^\pi(0)}\right)\right\} > 0,$$

which is equivalent to

$$\mathbb{P}\left\{\int_0^t (\delta^\mu(s) - \delta^\pi(s))ds - \int_0^t d\Theta(s) \geq \log\left(\frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))}\right)\right\} > 0.$$

Now, we know from the previous example that

$$\log\left(\frac{\mathbf{S}(\mu(t))}{\mathbf{S}(\mu(0))}\right) \geq \log\left(\frac{1}{2}\right).$$

It follows that for all $t > 0$,

$$\mathbb{P}\left\{t^{-1}\left(\int_0^t (\delta^\mu(s) - \delta^\pi(s))ds - \int_0^t d\Theta(s)\right) \geq t^{-1}\log\left(\frac{1}{2}\right)\right\} > 0.$$

If we let $T \geq \epsilon^{-1}\log 2$, then for $t > T$,

$$\mathbb{P}\left\{t^{-1}\left(\int_0^t (\delta^\mu(s) - \delta^\pi(s))ds - \int_0^t d\Theta(s)\right) \geq -\epsilon\right\} > 0$$

We know from the no-dividend example that if the market is diverse, then the drift process is non-decreasing and bounded away from zero when $t > T$. Therefore, in order for the above inequality to hold in probability, it must be possible for the market portfolio to have a higher average dividend rate than $\pi$. What is clear is that the average dividend rate for the two portfolios cannot be the same.

Under the entropy weighted portfolio, the market portfolio has a greater concentration in the larger stocks than the entropy weighted portfolio. This means that it must be possible in certain cases that the dividend rates of the larger stocks must exceed those of the smaller stocks.

## 8.2 Diversity-Weighted Portfolio

Let $\mathbf{D}_p(\mu) = (\sum_{i=1}^n \mu_i^p)^{1/p}$ where $0 < p < 1$, gives

$$\pi_i(t) = \frac{(\mu_i(t))^p}{(\mathbf{D}_p(\mu(t)))^p}$$

and relative returns

$$\log\left(\frac{Z^\pi(T)}{Z^\mu(T)}\right) = \log\left(\frac{\mathbf{D}_p(\mu(T))}{\mathbf{D}_p(\mu(0))}\right) + (1-p)\int_0^T \gamma_*^\pi(s)ds.$$

Since integral term is increasing, relative arbitrage results if the other term in the return equation can be bounded from below.

If one assumes the market is weakly diverse and nondegenerate, then we have

$$1 = \sum_{i=1}^{n} \mu_i(t) \leq \sum_{i=1}^{n} (\mu_i(t))^p = (\mathbf{D}_p(\mu(t)))^p \leq \sum_{i=1}^{n} \frac{1}{n^p} = \frac{n}{n^p}.$$

So, we get

$$\log \left( \frac{\mathbf{D}_p(\mu(T))}{\mathbf{D}_p(\mu(0))} \right) \geq -\frac{1-p}{p} \cdot \log n. \tag{8.6}$$

Now, using that the largest weight of the portfolio which does not exceed the maximum weight of the market portfolio, we see that

$$\int_0^T \gamma_*^\pi(t)dt \geq \frac{\epsilon}{2} \cdot \int_0^T \left(1 - \mu_{(1)}(t)\right) dt > \frac{\epsilon}{2}\delta T.$$

This leads to

$$\log \left( \frac{Z^\pi(T)}{Z^\mu(T)} \right) > (1-p) \left[ \frac{\epsilon T}{2}\delta - \frac{1}{p} \log n \right], \tag{8.7}$$

which implies the relative arbitrage for $T > 2 \log n / p\epsilon\delta$.

# 9 Mirror portfolios and relative arbitrage for arbitrary times

We first start off this section by relaxing condition (i) of "admissibility" and work with "extended portfolios". By being able to short-sell, we can always find a portfolio (in a weakly diverse market) that consistently underperforms the market portfolio.

[FKK05] first starts off by defining the *p-mirror-image of* $\pi(\cdot)$ *with respect to* $\mu(\cdot)$ by

$$\tilde{\pi}^{(p)}(\cdot) := p\pi(\cdot) + (1-p)\mu(\cdot), \tag{9.1}$$

where $p \neq 0$ is any fixed real number. Note that $\tilde{\pi}^{(p)}$ is an extended portfolio which allows for short-selling if $p > 1$. If $p = -1$, $\tilde{\pi}^{(-1)}(\cdot) := -\pi(\cdot) + 2\mu(\cdot)$ is called the "mirror image" of $\pi(\cdot)$ with respect to $\mu(\cdot)$.

We notice that

$$\left(\tilde{\pi}^{(p)}(\cdot)\right)^{\tilde{}\,(q)} = qp\pi(\cdot) + q(1-p)\mu(\cdot) + (1-q)\mu(\cdot)$$
$$= pq\pi(\cdot) + (1-pq)\mu(\cdot)$$
$$= \tilde{\pi}^{(pq)}(\cdot) \tag{9.2}$$

and

$$\left(\tilde{\pi}^{(p)}(\cdot)\right)^{\tilde{}\,(\frac{1}{p})} = \pi(\cdot) \tag{9.3}$$

Recalling the notation in (4.3) of $\tau^{\pi}(\cdot)$, [FKK05] define the relative covariance of $\pi(\cdot)$ with respect to $\mu(\cdot)$ as

$$\tau^{\mu}_{\pi\pi}(t) := (\pi(t) - \mu(t))' a(t)(\pi(t) - \mu(t))$$
$$\geq \epsilon \|\pi(t) - \mu(t)\|^2 \tag{9.4}$$

and make the following observations

$$\tau^{\mu}(\cdot)\mu(\cdot) \equiv 0 \text{ (as shown in 4.6)}, \tag{9.5}$$
$$\tau^{\mu}_{\pi\pi}(\cdot) = \pi'(\cdot)\tau^{\mu}(\cdot)\pi(\cdot) = \tau^{\pi}_{\mu\mu}(\cdot), \tag{9.6}$$
$$\tau^{\mu}_{\tilde{\pi}^{(p)}\tilde{\pi}^{(p)}}(\cdot) = p(\pi(\cdot) - \mu(\cdot))' a(t) p(\pi(\cdot) - \mu(\cdot)) = p^2 \tau^{\mu}_{\pi\pi}(\cdot). \tag{9.7}$$

Recalling the expression for the relative performance of $\pi(\cdot)$ with respect to $\mu(\cdot)$ given in (4.11), we can now give an expression for $\tilde{\pi}^{(p)}(\cdot)$, but to make our calculations easier, we derive certain terms

$$d\log\left(\frac{Z^{\tilde{\pi}^{(p)}}(t)}{Z^{\mu}(t)}\right) = \left(\gamma_*^{\tilde{\pi}^{(p)}}(t) - \gamma_*^{\mu}(t)\right) dt + p\sum_{i=1}^{n} (\pi_i(t) - \mu_i(t))\, d\log \mu_i(t),$$

$$p \cdot d\log\left(\frac{Z^{\pi}(t)}{Z^{\mu}(t)}\right) = p\left(\gamma_*^{\pi}(t) - \gamma_*^{\mu}(t)\right) dt + p\sum_{i=1}^{n} (\pi_i(t) - \mu_i(t))\, d\log \mu_i(t).$$

Subtracting the second term from the first, we get

$$d\log\left(\frac{Z^{\tilde{\pi}^{(p)}}(t)}{Z^{\mu}(t)}\right) = p \cdot d\log\left(\frac{Z^{\pi}(t)}{Z^{\mu}(t)}\right) + (p-1)\gamma_*^{\mu}(t)dt + \left(\gamma_*^{\tilde{\pi}^{(p)}}(t) - p\gamma_*^{\pi}(t)\right) dt. \tag{9.8}$$

The last term in the above equation can be re-expressed using (9.5), (4.5) and (4.9)

$$2\left(\gamma_*^{\tilde{\pi}^{(p)}}(t) - p\gamma_*^{\pi}(t)\right) = \sum_{i=1}^{n} \tilde{\pi}_i^{(p)}(t)\tau^{\mu}_{ii}(t) - \tau^{\mu}_{\tilde{\pi}^{(p)}\tilde{\pi}^{(p)}} - p\left(\sum_{i=1}^{n} \pi_i(t)\tau^{\mu}_{ii}(t) - \tau^{\mu}_{\pi\pi}(t)\right)$$
$$= (1-p) \cdot \sum_{i=1}^{n} \mu_i(t)\tau^{\mu}_{ii}(t) - p^2\tau^{\mu}_{\pi\pi}(t) + p\tau^{\mu}_{\pi\pi}(t)$$
$$= (1-p) \cdot \left[2\gamma_*^{\mu}(t) + p\tau^{\mu}_{\pi\pi}(t)\right].$$

Substituting this into (9.8), we get

$$\log\left(\frac{Z^{\tilde{\pi}^{(p)}}(T)}{Z^{\mu}(T)}\right) = p \cdot \log\left(\frac{Z^{\pi}(T)}{Z^{\mu}(T)}\right) + \frac{p(1-p)}{2}\int_0^T \tau^{\mu}_{\pi\pi}(t)dt, \qquad (9.9)$$

where the last term is non-negative by (9.4).

**Lemma 9.1** (c.f. [FKK05], Lemma 8.1) *Suppose that the extended portfolio* $\pi(\cdot)$ *is such that the conditions*

$$\mathbb{P}\left(\frac{Z^{\pi}(T)}{Z^{\mu}(T)} \geq \beta\right) = 1 \quad \text{or} \quad \mathbb{P}\left(\frac{Z^{\pi}(T)}{Z^{\mu}(T)} \leq \frac{1}{\beta}\right) = 1 \qquad (9.10)$$

*and*

$$\mathbb{P}\left(\int_0^T \tau^{\mu}_{\pi\pi}(t)dt \geq \eta\right) = 1 \qquad (9.11)$$

*hold, for some* $0 < \beta < 1$ *and* $\eta > 0$. *Then there exists an extended portfolio* $\widehat{\pi}(\cdot)$ *such that*

$$\mathbb{P}\left(Z^{\widehat{\pi}}(T) < Z^{\mu}(T)\right) = 1. \qquad (9.12)$$

We see that in order for (9.11) to hold, we require that $\|\pi - \mu\|_{\mathbf{L}^2([0,T])}$ is bounded away from zero, recalling the condition in (9.4).

We shall now construct a portfolio that will be used as a basis for creating all-long portfolios that underperform or outperform the market portfolio over any given time horizon $T \in (0, \infty)$. [FK08] refer to this portfolio as the "seed portfolio".

With $\pi = e_1 = (1, 0, \ldots, 0)'$, we define the extended portfolio

$$\widehat{\pi}(t) := \tilde{\pi}^{(p)}(t) = pe_1 + (1-p)\mu(t), \quad 0 \leq t < \infty \qquad (9.13)$$

where $p > 1$ will need to be specified. We see that this portfolio takes a long position in the first stock and a short position in the market (much like the leveraged portfolio):

$$\begin{aligned} \widehat{\pi}_1(t) &= p + (1-p)\mu_1(t), \\ \widehat{\pi}_i(t) &= (1-p)\mu_i(t), \qquad i = 1, 2, \ldots, n. \end{aligned}$$

We therefore get, assuming that $Z^{\mu}(0) = Z^{\widehat{\pi}}(0) = 1$:

$$\begin{aligned}
\log\left(\frac{Z^{\widehat{\pi}}(T)}{Z^{\mu}(T)}\right) &= \log\left(\frac{Z^{\widehat{\pi}}(0)}{Z^{\mu}(0)}\right) + p\left(\log\frac{Z^{\pi}(T)}{Z^{\mu}(T)} - \log\frac{Z^{\pi}(0)}{Z^{\mu}(0)}\right) \\
&\quad - \frac{p(1-p)}{2}\int_0^T \tau_{11}^{\mu}(t)dt \\
&= p\left(\log\frac{X_1(T)}{Z^{\mu}(T)} - \log\frac{X_1(0)}{Z^{\mu}(0)}\right) - \frac{p(1-p)}{2}\int_0^T \tau_{11}^{\mu}(t)dt \\
&= p\left(\log\frac{\mu_1(T)}{\mu_1(0)}\right) - \frac{p(1-p)}{2}\int_0^T \tau_{11}^{\mu}(t)dt. \qquad (9.14)
\end{aligned}$$

If we take $\beta := \mu_1(0)$, we have that $\frac{\mu_1(T)}{\mu_1(0)} \leq \frac{1}{\beta}$, and if the market is weakly diverse on $[0,T]$ we obtain from (4.10) and the Cauchy-Schwarz inequality:

$$\int_0^T \tau_{11}^{\mu}(t)dt \geq \epsilon\int_0^T (1-\mu_{(1)})^2 dt > \epsilon\delta^2 T =: \eta. \qquad (9.15)$$

By Lemma 9.1, the market portfolio then represents an arbitrary opportunity with respect to the "seed portfolio" provided that for any given $T \in (0,\infty)$. So we take $p > p(T) := 1 + \frac{2}{\epsilon\delta^2 T}\cdot\log\left(\frac{1}{\mu_1(0)}\right)$. Notice that $\lim_{T\downarrow 0} p(T) = \infty$ and also the following which is a direct consequence of (9.14):

$$Z^{\widehat{\pi}}(t) \leq \left(\frac{\mu_1(t)}{\mu_1(0)}\right)^p \cdot Z^{\mu}(t), \quad 0 \leq t < \infty. \qquad (9.16)$$

We now provide examples of all-long portfolios which underperform and outperform the market portfolio over any given time horizon.

Consider an investment strategy $\rho(\cdot)$ that places a dollar in $\widehat{\pi}(\cdot)$ of (9.13) and $\frac{p-1}{(\mu_1(0))^p}$ dollars in the market portfolio $\mu(\cdot)$ at $t = 0$ and makes no changes afterward where $p$ is determined as above. The value of this strategy is:

$$Z^{\rho}(t) = Z^{\widehat{\pi}}(t) + \frac{p-1}{(\mu_1(0))^p}\cdot Z^{\mu}(t) > 0, \quad 0 \leq t < \infty,$$

with weights:

$$\rho_i(t) = \frac{1}{Z^{\rho}(t)}\left[\widehat{\pi}_i(t)\cdot Z^{\widehat{\pi}}(t) + \frac{p-1}{(\mu_1(0))^p}\cdot\mu_i(t)Z^{\mu}(t)\right], \quad i = 1,\ldots,n.$$

Since $\widehat{\pi}_1(t)$ and $\mu_1(t)$ are positive, we have $\rho_1(t) > 0$. To check that $\rho(\cdot)$ is an all-long portfolio, we need to consider the dollar amount invested in each stock $(i = 2,\ldots,n)$ at time $t$.

$$\begin{aligned}
(1-p)\mu_i(t)Z^{\widehat{\pi}}(t) + \frac{p-1}{(\mu_1(0))^p}\mu_i(t)Z^{\mu}(t) &= \frac{(p-1)\mu_i(t)}{(\mu_1(0))^p}Z^{\mu}(t)\left[1 - \frac{(\mu_1(0))^p Z^{\widehat{\pi}}(t)}{Z^{\mu}(t)}\right] \\
&\geq \frac{(p-1)\mu_i(t)}{(\mu_1(0))^p}Z^{\mu}(t)\left[1 - (\mu_1(t))^p\right],
\end{aligned}$$

using

$$-\frac{Z^{\widehat{\pi}}}{Z^{\mu}} \geq -\left(\frac{\mu_1(t)}{\mu_1(0)}\right)^p \tag{9.17}$$

from (9.16). Given that the market portfolio and $\rho(\cdot)$ starts with the same initial capital $z := Z^{\rho}(0) = 1 + \frac{p-1}{(\mu_1(0))^p}$, we know from the results from the behaviour of the "seed portfolio" ($\widehat{\pi}(\cdot)$) that at $t = T$, $\rho(\cdot)$ will underperform the market portfolio because $\rho(\cdot)$ holds a combination of $\mu(\cdot)$ and $\widehat{\pi}(\cdot)$.

Now, we consider an example of a long-only portfolio that always outperforms the market portfolio. Let $\eta(\cdot)$ be a trading strategy that invests $\frac{p}{(\mu_1(0))^p}$ dollars in the market portfolio and $-1$ dollars in $\widehat{\pi}(\cdot)$ at $t = 0$, and makes no changes afterwards. Before we go any further, we can already see by the fact that we are shorting a portfolio that underperforms the market portfolio, that this trading strategy is going to outperform the market.

The value of this strategy is:

$$\begin{aligned}
Z^{\eta}(t) &= \frac{p}{(\mu_1(0))^p} \cdot Z^{\mu}(t) - Z^{\widehat{\pi}}(t) \\
&= \frac{Z^{\mu}(t)}{(\mu_1(0))^p} \left[p - \frac{Z^{\widehat{\pi}}(t)(\mu_1(0))^p}{Z^{\mu}(t)}\right] \\
&\geq \frac{Z^{\mu}(t)}{(\mu_1(0))^p} \left[p - (\mu_1(t))^p\right] \\
&> 0
\end{aligned}$$

by (9.17), and the weights are:

$$\eta_i(t) = \frac{1}{Z^{\eta}(t)} \left[\frac{p}{(\mu_1(0))^p} \cdot \mu_i(t) Z^{\mu}(t) - \widehat{\pi}_i(t) \cdot Z^{\widehat{\pi}}(t)\right], \quad i = 1, \ldots, n.$$

For $i = 2, \ldots, n$, we have $\widehat{\pi}_i(t) = (1-p)\mu_i(t) < 0$. Since, we are shorting $\widehat{\pi}(\cdot)$, $\eta_2(\cdot), \ldots, \eta_n(\cdot)$ are strictly positive. Therefore, we need to check that $\eta_1(t) \geq 0$. The amount invested in the first stock is:

$$\frac{p\mu_1(t)}{(\mu_1(0))^p} \cdot Z^{\mu}(t) - [p - (p-1)\mu_1(t)] \cdot Z^{\widehat{\pi}}(t) \tag{9.18}$$

which dominates by (9.16):

$$\frac{p\mu_1(t)}{(\mu_1(0))^p} \cdot Z^{\mu}(t) - [p - (p-1)\mu_1(t)] \cdot \left(\frac{\mu_1(t)}{\mu_1(0)}\right)^p Z^{\mu}(t)$$

or, equivalently,

$$\frac{Z^{\mu}(t)\mu_1(t)}{(\mu_1(0))^p} \cdot \left[(p-1)(\mu_1(t))^p + p\left(1 - (\mu_1(t))^{p-1}\right)\right] > 0,$$

thus showing that $\eta(\cdot)$ is an all-long portfolio. Similar to the reasoning for the previous example, $\eta(\cdot)$ outperforms at $t = T$ the market portfolio with the same initial capital $Z^{\eta}(0) = \frac{p}{(\mu_1(0))^p} - 1$ because the portfolio is long in $\mu(\cdot)$ and short in the extended portfolio $\widehat{\pi}(\cdot)$, which, as shown earlier, underperforms the market at $t = T$.

# 10 Application of the Girsanov Theorem

Thus far, we have only been in the setting where the only integrability conditions that hold are (2.2). In this section, we see how our model is affected when we assume the Novikov condition, or alternatively, assume that the coefficients in (2.1) are a.s. bounded.

We start off this section by assuming that there exists a $\mathbb{F}$-progressively measurable process $\theta$ where

$$\theta : [0, \infty) \times \Omega \to \mathbb{R}^m,$$

with

$$\sigma(t)\theta(t) = b(t), \quad \forall\ 0 \leq t \leq T, \tag{10.1}$$

that satisfies the Novikov condition:

$$\mathbb{E}_{\mathbb{P}}\left[exp\left\{\frac{1}{2}\int_0^T \|\theta\|^2 dt\right\}\right] < \infty, \quad \forall\ 0 \leq t \leq T. \tag{10.2}$$

By imposing this condition we guarantee that the following supermartingale $L_t$ is a martingale (c.f. [M00] p.g. 182):

$$L_t = \exp\left(-\int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right\}, \quad 0 \leq t < \infty, \tag{10.3}$$

and that the the following process is a Brownian motion under the measure $\mathbb{Q}$:

$$\widehat{W}(t) := W(t) + \int_0^t \theta(s)ds, \quad 0 \leq t < \infty \tag{10.4}$$

where $\mathbb{Q}$ is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = L_t.$$

If we assume that the above holds true, then we are in a setting where all our stock price processes $\{X_1(\cdot), \ldots, X_n(\cdot)\}$ are martingales:

$$\begin{aligned} dX_i(t) &= X_i(t)\left[b_i(t)dt + \sum_{v=1}^{m} \sigma_{iv}(t)dW_v(t)\right] \\ &= X_i(t)\sum_{v=1}^{m} \sigma_{iv}(t)d\widehat{W}_v(t). \end{aligned}$$

By (2.9), we also get that our value process $Z^\pi(\cdot)$ is a martingale:

$$
\begin{aligned}
dZ^\pi(t) &= Z^\pi(t)\left[\sum_{i=1}^{n}\pi_i(t)\frac{dX_i(t)}{X_i(t)}\right] \\
&= Z^\pi(t)\sum_{i=1}^{n}\sum_{v=1}^{m}\pi_i(t)\sigma_{iv}(t)d\widehat{W}_v(t) \\
&= Z^\pi(t)\pi'(t)\sigma(t)d\widehat{W}(t),
\end{aligned}
$$

with $Z^\pi(0) = 1$.

Therefore, the difference $Z^\pi(t) - Z^\rho(t),\ 0 \le t \le T$ is also martingale under $\mathbb{Q}_T$ for any portfolio $\rho(\cdot)$ with $Z^\rho(0) = 1$:

$$\mathbb{E}_{\mathbb{Q}}\left[Z^\pi(T) - Z^\rho(T)\right] = 0.$$

Recalling our definition for relative arbitrage in Definition 2.5, if $Z^\pi(T) - Z^\rho(T) \ge 0$ hold a.s., then this gives $Z^\pi(T) - Z^\rho(T) = 0$, ruling out the possibility of the second requirement:

$$\mathbb{P}\left[Z^\pi(T) - Z^\rho(T) > 0\right] > 0$$

From the above analysis, it is apparent that if we assume a condition of the Novikov type, then we cannot possibly have a weakly diverse market, let alone a diverse market.

# 11 Hedging in Weakly Diverse Markets

In this section, we will show that the familiar methods for option pricing and hedging in financial modelling remain intact even in a diverse market, where we assume that an equivalent martingale measure does not exist.

Before we show this result, we need to introduce the notion of a trading strategy. A trading strategy allows the investor to invest in (or borrow from) the money market with interest rate $r : [0,\infty)\times\Omega \to [0,\infty)$; a progressively measurable and locally integrable process, where a dollar invested now grows to $B(T) = \exp\left\{\int_0^T r(u)du\right\}$ at $t = T$.

A trading strategy is a $\mathbb{F}-$ progressively measurable, $\mathbb{R}^n-$ valued processes $\phi(t) = (\phi_1(t),\ldots,\phi_n(t))'$ that satisfy the integrability condition:

$$\sum_{i=1}^{n}\int_0^T \left(|\phi_i(t)||b_i(t) - r(t)| + \phi_i^2(t)a_{ii}(t)\right)dt < \infty, \quad a.s.$$

for every $T \in (0, \infty)$. The quantity $\phi_i(t)$ is the dollar amount invested in the $i^{th}$ stock and $Z^{z,\phi}(t) - \sum_{i=1}^{n} \phi_i(t)$ the amount held in the money-market. Here $z > 0$ represents the initial capital. We have

$$\begin{aligned}
dZ^{z,\phi}(t) &= \left( Z^{z,\phi}(t) - \sum_{i=1}^{n} \phi_i(t) \right) r(t)dt + \sum_{i=1}^{n} \phi_i(t) \frac{dX_i(t)}{X_i(t)} \\
&= \left( Z^{z,\phi}(t) - \sum_{i=1}^{n} \phi_i(t) \right) r(t)dt \\
&\quad + \sum_{i=1}^{n} \phi_i(t) \left( b_i(t)dt + \sum_{v=1}^{m} \sigma_{iv}(t)dW_v(t) \right) \\
dZ^{z,\phi}(t) - r(t)Z^{z,\phi}(t)dt &= -\sum_{i=1}^{n} \phi_i(t)r(t)dt + \sum_{i=1}^{n} \phi_i(t) \left( b_i(t)dt + \sum_{v=1}^{m} \sigma_{iv}(t)dW_v(t) \right) \\
B(t) \cdot d\left( \frac{Z^{z,\phi}}{B(t)} \right) &= \sum_{i=1}^{n} \phi_i(t) \left[ b_i(t) - r(t) \right] dt + \sum_{i=1}^{n} \sum_{v=1}^{m} \phi_i(t)\sigma_{iv}(t)dW_v(t) \\
d\left( \frac{Z^{z,\phi}}{B(t)} \right) &= \frac{1}{B(t)} \left\{ \phi'(t) \left( \left[ b(t) - r(t)\mathbf{I} \right] dt + \sigma(t)dW(t) \right) \right\}
\end{aligned} \tag{11.1}$$

or, in integral form:

$$\frac{Z^{z,\phi}}{B(t)} = z + \int_0^t \frac{\phi'(s)}{B(s)} \left( \left[ b(s) - r(s)\mathbf{I} \right] ds + \sigma(s)dW(s) \right), \quad 0 \leq t < \infty. \tag{11.2}$$

Here, $\mathbf{I}$ is the n-dimensional column vector of 1's.

Since $\phi_i(\cdot)$ for all $i$ and $Z^{z,\phi}(\cdot) - \phi'(\cdot)\mathbf{I}$ are allowed to take negative values, this opens the door to doubling strategies (c.f. [E02] p.g. 113). To prevent this from occurring on a given time horizon $[0, T]$, we only focus on trading strategies $\phi(\cdot)$ that satisfies:

$$\mathbb{P}\left( Z^{z,\phi}(t) \geq 0, \quad \forall\, 0 \leq t \leq T \right) = 1. \tag{11.3}$$

A portfolio generated by such a trading strategy will be self-financing (c.f. [E02] p.g. 113). We will denote these class of trading strategies $\phi(\cdot)$ by $\Phi_T(z)$.

Using similar arguments as in the last section, it can easily be established that if $\mathcal{M}$ is weakly diverse on some finite time period $[0, T]$, then the process

$$L(t) := \exp\left\{ -\int_0^t \kappa'(s)dW(s) - \frac{1}{2}\int_0^t \|\kappa(s)\|^2 ds \right\}, \quad 0 \leq t < \infty, \tag{11.4}$$

where $\sigma(t)\kappa(t) = b(t) - r(t)\mathbf{I}$, is a local martingale and a supermartingale but not a martingale. If the process is a martingale, we get that the discounted price processes $\frac{X_i(\cdot)}{B(\cdot)}$ and the discounted value processes $\frac{Z^{z,\phi}(\cdot)}{B(\cdot)}$ are martingales on $[0, T]$, eliminating the existence of relative arbitrage between arbitrary portfolios $\pi(\cdot)$ and $\rho(\cdot)$. Thus, in a weakly diverse market the process $L(\cdot)$ is a strict local martingale and we have $\mathbb{E}[L(t)] < 1$.

Since $L(\cdot)$ is a local martingale, there exists an increasing sequence $\{S_k\}_{k\in\mathbb{N}}$ of stopping times with $\lim_{k\to\infty} = \infty a.s.$ such that $L(\cdot \wedge S_k)$ is a martingale for every $k \in \mathbb{N}$. [FKK05] take $S_k$ to be

$$S_k = \inf\left\{t \geq 0 \mid \int_0^t \|\kappa(s)\|^2 ds \geq k\right\}.$$

Therefore, if we replace $T$ by $T \wedge S_k$ in Definition 2.5, there is no possibility for relative arbitrage on the time interval $[0, T \wedge S_k]$ for any $k \in \mathbb{N}$. But, if we take the limit as $k \to \infty$, relative arbitrage of the type in Definition 2.5 appears if the market $\mathcal{M}$ is weakly diverse on $[0, T]$. This seems reasonable since as $k \to \infty$, under our formulation for $S_k$, the market price of risk $\kappa(\cdot)$ is not square integrable which negates the possible application of the Girsanov Theorem.

As stated in [FKK05] and [FK08], the failure of the process $L(\cdot)$ to be a martingale does not prevent the possibility of hedging contingent claims in a weakly diverse market $\mathcal{M}$. They proceed first by considering deflated stock price and wealth processes:

$$\tilde{X}_i(t) := \frac{L(t)X_i(t)}{B(t)},\ i = 1, \ldots, n \quad \text{and} \quad \tilde{Z}^{\phi}(t) := \frac{L(t)Z^{z,\phi}}{B(t)}, \tag{11.5}$$

for $0 \leq t < \infty$. By an application of the product formula and using the differential equation $dL(t) = -L(t)\kappa'(t)dW(t)$, we obtain that each of the processes:

$$\begin{aligned}
d\tilde{X}_i(t) &= d\left(\frac{L(t)X_i(t)}{B(t)}\right) \\
&= \frac{X_i(t)}{B(t)}dL(t) + L(t)d\left(\frac{X_i(t)}{B(t)}\right) + d\left\langle L, \frac{X_i}{B}\right\rangle_t \\
&= -\tilde{X}_i(t)\kappa'(t)dW(t) + \tilde{X}_i(t)\sum_{v=1}^m \sigma_{iv}(t)\left[dW_v(t) + \kappa_v(t)dt\right] \\
&\quad - \tilde{X}_i(t)\sum_{v=1}^m \sigma_{iv}(t)\kappa_v(t)dt \\
&= \tilde{X}_i(t) \cdot \sum_{v=1}^m \left(\sigma_{iv}(t) - \kappa_v(t)\right) dW_v(t)
\end{aligned} \tag{11.6}$$

$$
\begin{aligned}
d\tilde{Z}^{\phi}(t) &= d\left(\frac{L(t)Z^{z,\phi}(t)}{B(t)}\right) \\
&= \frac{Z^{z,\phi}(t)}{B(t)}dL(t) + L(t)d\left(\frac{Z^{z,\phi}(t)}{B(t)}\right) + d\left\langle L, \frac{Z^{z,\phi}}{B}\right\rangle_t \\
&= -\tilde{Z}^{\phi}(t)\kappa'(t)dW(t) + \frac{L(t)\kappa'(t)\sigma(t)}{B(t)}\left[dW(t) + \kappa(t)dt\right] \\
&\quad - \frac{L(t)\kappa'(t)\sigma(t)}{B(t)}\kappa(t)dt \\
&= \left(\frac{L(t)\kappa'(t)\sigma(t)}{B(t)} - \tilde{Z}^{\phi}(t)\kappa'(t)\right)dW(t) \qquad (11.7)
\end{aligned}
$$

where $\tilde{Z}^{z,\phi}(0) = z$, are non-negative local martingales (and supermartingales) under $\mathbb{P}$.

It is shown in [FKK05] and [FK08] that the processes $\tilde{X}_i(\cdot),\ i = 1, \dots, n$ are strict local martingales and in particular that,

$$
\mathbb{E}_{\mathbb{P}}\left[\frac{L(T)X_i(T)}{B(T)}\right] < X_i(0) \qquad (11.8)
$$

holds for all $T \in (0, \infty)$. Similar conclusion is reached for $\tilde{Z}^{\phi}(\cdot)$.

We now consider an $\mathcal{F}(T)$-measurable random variable $Y : \Omega \to [0, \infty)$ that satisfies

$$
0 < y_0 := \mathbb{E}\left[YL(T)/B(T)\right] < \infty. \qquad (11.9)
$$

If we view $Y$ as a contingent claim that an investor has to hedge at $t = T$, [FKK05] characterises the smallest amount of capital required to hedge this claim by

$$
h := \inf\left\{z > 0 \mid \text{there exists } \phi(\cdot) \in \Phi_T(z) \text{ such that } Z^{z,\phi}(T) \geq Y \text{ holds a.s.}\right\}. \qquad (11.10)
$$

Using that $Z^{z,\phi}(\cdot)$ is a supermartingale, we have that there exists some $\phi(\cdot) \in \Phi_T(z)$ such that

$$
\mathbb{E}\left[YL(T)/B(T)\right] \leq \mathbb{E}\left[Z^{z,\phi}(T)L(T)/B(T)\right] \leq z, \qquad (11.11)
$$

and so $y_0 = \mathbb{E}\left[YL(T)/B(T)\right] \leq h$.

To get the inverse inequality, [FKK05] start by assuming that $m = n$, that is, we have exactly as many sources of randomness as we have stocks in the market, that the (now) square matrix $\sigma(t, \omega)$ is invertible for every $(t, \omega) \in [0, T] \times \Omega$ and that the filtration $\mathbb{F} = \mathcal{F}(t)_{0 \leq t \leq T}$ is generated by the Brownian motion $W(\cdot)$ itself. By the martingale representation property of this Brownian motion, we can represent the price of our claim at time $t$ by a non-negative martingale

$$M(t) := \mathbb{E}\left[YL(T)/B(T)|\mathcal{F}(t)\right] = y_0 + \int_0^t \psi'(s)dW(s), \quad 0 \le t \le T \quad (11.12)$$

for some progressively measurable and a.s. square-integrable process $\psi : [0,T] \times \Omega \to \mathbb{R}^n$. Setting

$$\widehat{Z}(\cdot) := M(\cdot)B(\cdot)/L(\cdot) \quad \text{and} \quad \widehat{\phi}(\cdot) := (B(\cdot)/L(\cdot)) \left(\sigma^{-1}(\cdot)\right)' (\psi(\cdot)+M(\cdot)\kappa(\cdot)),$$

then comparing (11.7) and (11.12), we observe that $\widehat{Z}(0) = y_0$, $\widehat{Z}(T) = Y$ and $\widehat{Z}(\cdot) \equiv Z^{y_0,\widehat{\phi}}(\cdot) \ge 0$, almost surely.
Therefore, we have that $\widehat{\phi} \in \Phi_T(y_0)$ and satisfies $Z^{y_0,\widehat{\phi}}(T) = Y$ a.s. This shows that $y_0$ belongs to the right-hand side of (11.10) and so $y_0 \ge h$.

[FKK05] arrives at the hedging price

$$h = \mathbb{E}\left[YL(T)/B(T)\right]. \quad (11.13)$$

We conclude this section by the observation that a weakly diverse market $\mathcal{M}$ which admits no equivalent martingale measure can nevertheless be complete in the sense that any contingent claims under this model have replicating strategies.

# 12 Applications of hedging in weakly diverse markets

We now consider an application of hedging in weakly diverse markets by investing the behaviour of the price of a European call option as $T \to \infty$, that is, as time to maturity approaches infinity. Intuition would require that the price should approach 0 since the call is never exercised, and this proves true in [FKK05]'s weakly diverse market.

Consider the call pay-off $Y = (X_1(T) - K)^+$ of the first stock where $K$ is the strike price. We assume that the interest rate process is bounded away from 0.

$$\mathcal{P}\left[r(t) \ge r,\ \forall t \ge 0\right] = 1 \quad (12.1)$$

for some $r > 0$.

Using that $L(\cdot)X_1(\cdot)/B(\cdot)$ is a supermartingale and a strict local martingale and (12.1), we get the following:

$$\begin{aligned}
X_1(0) &> \mathbb{E}[L(T)X_1(T)/B(T)] \geq \mathbb{E}[L(T)(X_1(T)-K)^+/B(T)] = h(T) \\
&\geq \mathbb{E}[L(T)X_1(T)/B(T)] - K\cdot\mathbb{E}\left[L(T)\cdot e^{-\int_0^T r(t)dt}\right] \\
&\geq \mathbb{E}[L(T)X_1(T)/B(T)] - Ke^{-rT}\mathbb{E}[L(T)] > \mathbb{E}[L(T)X_1(T)/B(T)] - Ke^{-rT}.
\end{aligned}$$

Hence

$$0 \leq h(\infty) := \lim_{T\to\infty} h(T) \leq \lim_{T\to\infty} \mathbb{E}\left(\frac{L(T)X_1(T)}{B(T)}\right) < X_1(0).$$

This shows that the option is strictly less than the price of the underlying stock at time $t = 0$.

For every fixed $p \in (0,1)$ and $T \geq \frac{2\log n}{p\epsilon\delta}$, we get that

$$\begin{aligned}
\mathbb{E}\left(\frac{L(T)}{B(T)}X_1(T)\right) &\leq \mathbb{E}\left(\frac{L(T)}{B(T)}Z^{\mu}(T)\right) \\
&\leq \mathbb{E}\left(\frac{L(T)}{B(T)}Z^{\pi^{(p)}}(T)\right)\cdot n^{\frac{1-p}{p}}e^{-\epsilon\delta(1-p)T/2} \\
&< Z^{\pi^{(p)}}(0)n^{\frac{1-p}{p}}e^{-\epsilon\delta(1-p)T/2},
\end{aligned}$$

where $Z^{\pi^{(p)}}(\cdot)$ is the value process of a diversity-weighted portfolio. Letting $T \to \infty$, this leads to $h(\infty) = 0$.

Suppose now that there exists an equivalent martingale measure on any finite time-horizon, then we have that $L(\cdot)$ and $L(\cdot)X_1(\cdot)/B(\cdot)$ are martingales under $\mathbb{P}$ ($L(\cdot)$ is the Radon-Nikodym derivative). Then, we have that $\mathbb{E}_{\mathbb{P}}\left[L(T)X_1(T)/B(T)\right] = X_1(0)$ holds for all $T \in (0,\infty)$, and that

$$\lim_{T\to\infty} h(T) = \lim_{T\to\infty} \mathbb{E}_{\mathbb{P}}\left[L(T)(X_1(T)-q)^+/B(T)\right] = X_1(0).$$

using the inequality

$$X_1(0) \geq h(T) > \mathbb{E}_{\mathbb{P}}\left[L(T)X_1(T)/B(T)\right] - qe^{-rT}.$$

Hence, as the time-horizon increases without limit, the hedging price of the call-option approaches the current stock price.

Quite surprisingly, one is led to more realistic values for European calls under weak diversity for exceedingly long time-horizons than under a market which assumes no-arbitrage.

[FKK05] however shows that under the setting of weakly diverse markets, the put-call parity may not hold.
Set
$$Y_1 := (\Xi_1(T) - \Xi_2(T))^+ \quad \text{and} \quad Y_2 := (\Xi_2(T) - \Xi_1(T))^+ .$$
We say two assets are in put-call parity if

$$\begin{aligned} h_1 - h_2 &= \mathbb{E}[L(T)(\Xi_1(T) - \Xi_2(T))] \\ &= \Xi_1(0) - \Xi_2(0). \end{aligned}$$

Suppose that $\Xi_1(\cdot) \equiv Z^\mu(\cdot)$, $\Xi_2(\cdot) \equiv Z^{\widehat{\pi}}(\cdot)$ and $Z^\mu(0) = Z^{\widehat{\pi}}(0)$ where $Z^{\widehat{\pi}}(\cdot)$ is the portfolio considered in Section 9 that underperforms the market portfolio. Using these constructs we have that the put-call parity is violated:

$$\begin{aligned} h_1 - h_2 &= \mathbb{E}[L(T)(Z^\mu(T) - Z^{\widehat{\pi}}(T))] \\ &> 0 \\ &= Z^\mu(0) - Z^{\widehat{\pi}}(0). \\ &= \Xi_1(0) - \Xi_2(0). \end{aligned}$$

# 13 Performance of Functionally Generated Portfolios

In this section, we analyse the simulated behaviour of several functionally generated portfolios in the Australian stock market. The portfolios we consider are the entropy-weighted portfolio, the $\mathbf{D}_p$-weighted portfolio, the admissible, market-dominating portfolio of Section 8.1, and the leveraged portfolio of Section 7.

The data that we used for all the simulations in this section are daily trading prices of 31 Australian stocks trading on the stock exchange over the time period 2004 to 2014. For the purposes of analysis, the historical daily stocks prices have been plotted in Figure 1 (top). Note the significant drop in prices at the start of trading day 1200 and smaller dives occurring subsequently. These were caused by recessional fears of the US market after the global financial crisis and the European markets after their credit crisis.

Over the period, the entropy-weighted, the $\mathbf{D}_p$-weighted portfolios and the admissible, market-dominating portfolio that we have simulated show higher returns than the capitalisation-weighted portfolios of the same 31 stocks. This tells us that in the absence of significant change in market structure, in particular stability in the market diversity, we would expect that the same type of out-performance in the future. The existence of such portfolios, as stated in [F02], suggests that the no-arbitrage hypothesis is questionable in

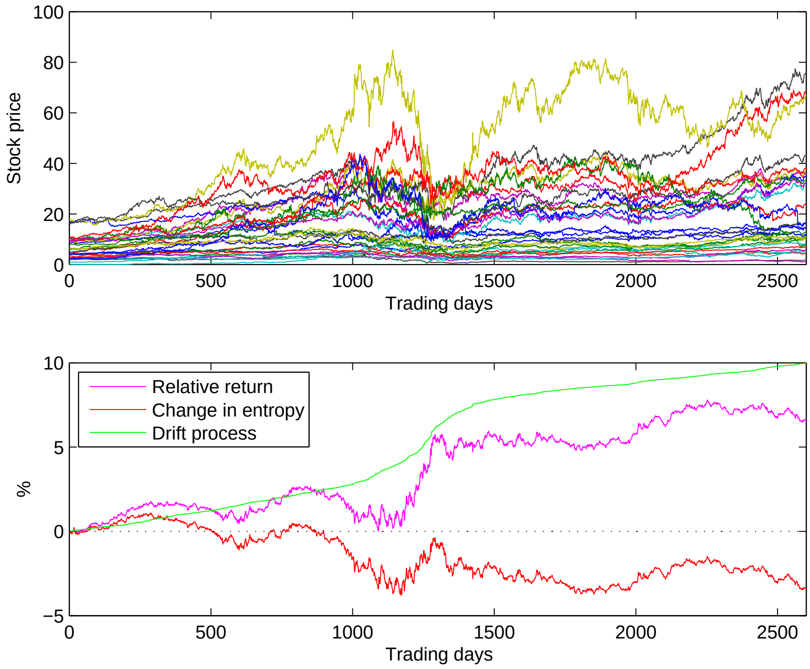


Figure 1: Performance of the entropy-weighted portfolio relative to the capitalisation-weighted portfolio. The first plot shows the historical daily trading stock prices for all 31 stocks from 2004 to 2014.

real markets, at least over a long time period.

Recall by Section 4, that relative return always refers to relative logarithmic return. In the examples that follow, no transaction costs are subtracted from the returns of any of the portfolios considered. We wish to note that since we re-balance these portfolios daily, transaction costs will, in reality, play a significant role in the performance of our selected portfolios.

## 13.1 Entropy-weighted portfolio

The results of the simulation are presented in Figure 1, which show the historical daily stock prices for the 31 stocks over ten years (top), and the relative logarithmic return of the entropy-weighted portfolio (bottom, magenta). The two plots were included so that comparisons could be made between the return of the entropy-weighted portfolio and the performance and structure of the market (represented by the 31 stocks). We have also plotted on the bottom graph the cumulative values of the components of the relative logarithmic return of the entropy-weighted portfolio: the drift process and the cumulative change in entropy.

Figure 1 shows that the entropy-weighted portfolio generated about 6.5% higher logarithmic return than the market over the 10-year period, which is an average annual relative return of 0.65%. The cumulative change in entropy was about $-3.5\%$, or $-0.35\%$ a year. Since entropy is a measure of diversity, it seems that the diversity of our market is deteriorating. However, this is expected considering the nature of our market which consists only of 31 stocks. We would expect that the diversity of a large equity market such as the ASX to remain fairly stable over the long term.

The drift process contributed about 10%, or 1% a year, to the relative return of the entropy-weighted porfolio over the period studied. Since the drift process satisfies $d\Theta(t) = \gamma_*^\mu / \mathbf{S}(\mu(t))dt$, the slope of the curve depends on the relative variance of the stocks in the market. These relative variances appear to have been fairly stable most of the time, except for the jump in the middle.

Financial theory implies that a portfolio that has a higher expected return relative to the market must have a beta ($\beta$) greater than 1. But the calculated beta for the entropy-weighted portfolio is $\beta = 0.9764$ sampling over daily-intervals. This means that the entropy-weighted portfolio is not behaving as it should be. The other portfolios we consider also display this inconsistency with financial theory.

## 13.2 Diversity-weighted portfolio

The results of the simulation are presented in Figure 2.

For this part, we will consider $\mathbf{D}_p$ with $p = 0.5$. This portfolio generated about 12% higher logarithmic return than the capitalisation-weighted portfolio over the 10-year period (1.2% average annual relative return). Over the 10-years, change in diversity was about $-5\%$, or $-0.5\%$ a year and the drift accounted for about 17%, or 1.7% a year.

Note that as $p \to 1$, $\pi$ approaches the market portfolio and as $p \to 0$, $\pi$ approaches the equal-weighted portfolio (constant weights $1/n$ - "buy low, sell high" portfolio). We see from Figure 2 that although the drift process for $p = 0.01$ is steeper than any other portfolio, it is highly sensitive to changes in diversity. This sensitivity stems from the fact that these portfolios lose comparatively more money than other portfolios if high performing stocks perform even higher and poor performing stocks perform even worse. If, on the contrary, prices move as expected, then these portfolios generate "supe-

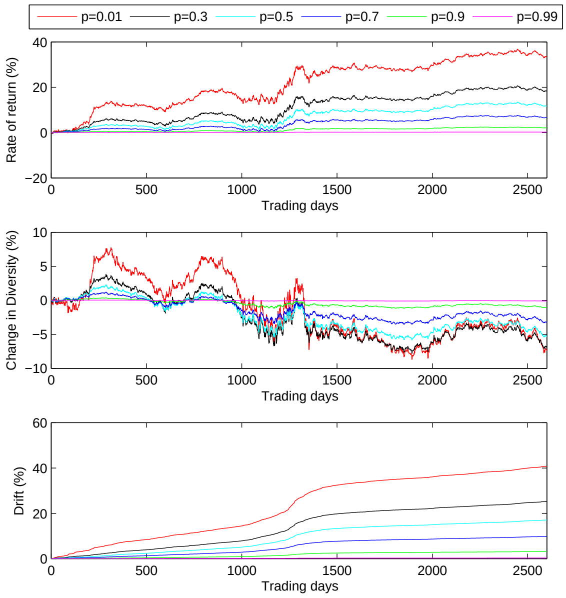


Figure 2: Performance of the diversity-weighted portfolio for different values of $p$: relative return (top), cumulative change in diversity (mid) and drift process (bot)

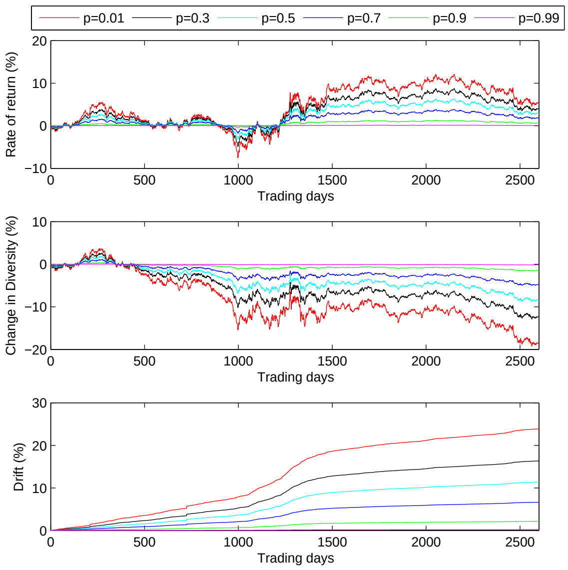


Figure 3: Performance of the diversity-weighted portfolio for different values of $p$ with decreasing diversity

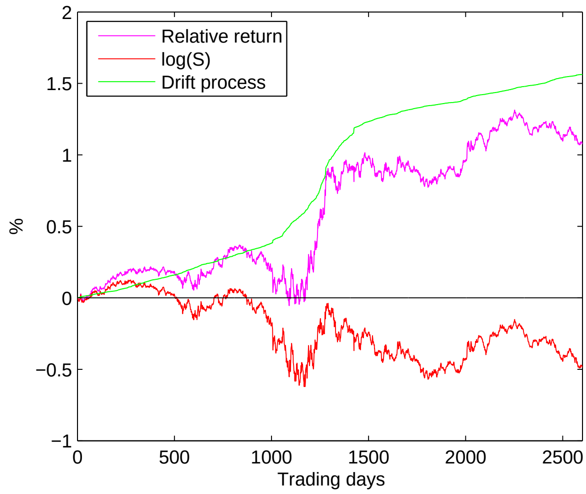


Figure 4: Performance of the admissible, market-dominating portfolio

rior" returns [FK08]. In a market that is becoming less diverse, investing in this portfolio may pose a problem, as shown in Figure 3, at least in the short term. In Figure 3, we have used 10 stocks in the Australian stock market rather than 31 in order to illustrate the effect of decreasing diversity on an equal-weighted portfolio. We conclude that the equal-weighted portfolio is highly sensitive to changes in diversity and it can drop below zero in the short-term if we have low diversity. However, in the long run, generally it will be a sound investment.

### 13.3 Admissible, market-dominating portfolio

The results of the simulation are presented in Figure 4.

Note the similarity with the entropy-weighted portfolio. However, we see that the relative returns are significantly lower. Recall the conditions of "admissibility" in Section 8. These conditions/constraints restrict the portfolio from being able to perform optimally and generate better returns.

### 13.4 Leveraged Portfolios

The results of the simulation are presented in Figure 5 and Figure 6, where a different choice of stock to take long is chosen. Take note that the drift process is negative ($d\Theta(t) = -\tau_{11}(t)dt$).

Without the benefit of hindsight, it may be unwise to leverage a single stock by shorting the market, at least over the long term.

### 13.5 Mirrored Portfolios

An example of a mirrored portfolio is presented in Figure 7.

Recalling our results in Section 9, by choosing $p$ large enough, we can ensure arbitrage on any positive time horizon.

## 14 Simulations - Extreme Cases

In this section, we use simulated data to drive home key concepts shown in [F99] and [FKK05].

We adopt codes generated for a Merton Normal Jump-Diffusion in [D08] to generate a geometric Brownian motion with a compound Poisson process.

In Figures 8 and 9, we consider the case when stocks exhibit almost identical behaviour.

Suppose that all stocks in the market have the same growth rate process. Then it can be shown that

$$\mathbb{P}\left\{\lim_{t\to\infty} t^{-1}\int_0^t \gamma_*^\pi(s)ds = 0\right\} = 1$$

(c.f. [F99] *Proposition 3.2*).

Recall our drift term for the entropy portfolio in (6.5). The drift process would approach 0 as we took the limit as $t \to \infty$. Although in Figures 8 an 9, we can see that the drift process is positive (due to the perturbations), and it is extremely close to 0. To pull the drift process away from zero, it is shown in [FKK05] that we require the largest stock to have “strongly negative” rate of growth and that all other stocks have ”sufficiently high” rates of growth.

[F99] states that this change of growth can be done through dividend payments. In order to achieve this, he shows that the average dividend rate must be at least equal the average excess growth rate of the market over the long term and that the dividend rates of the larger stocks be greater than those of the smaller stocks. This makes intuitive sense because large

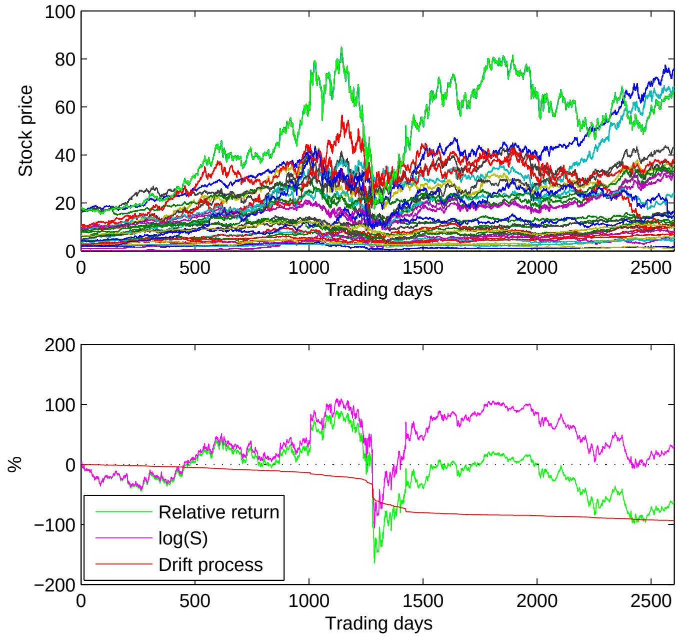


Figure 5: Performance of a leveraged portfolio. Here the stock process coloured fluro-green represents the stock we take long.

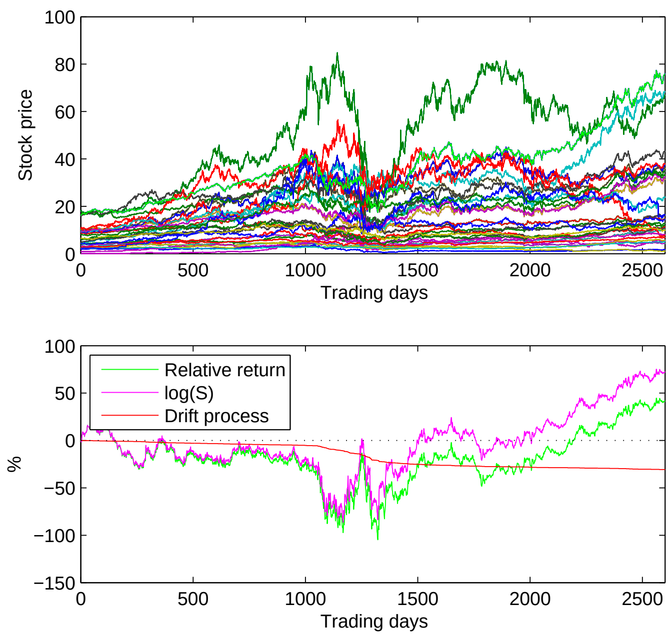


Figure 6: Performance of a leveraged portfolio. Here the stock process coloured fluro-green represents the stock we take long.

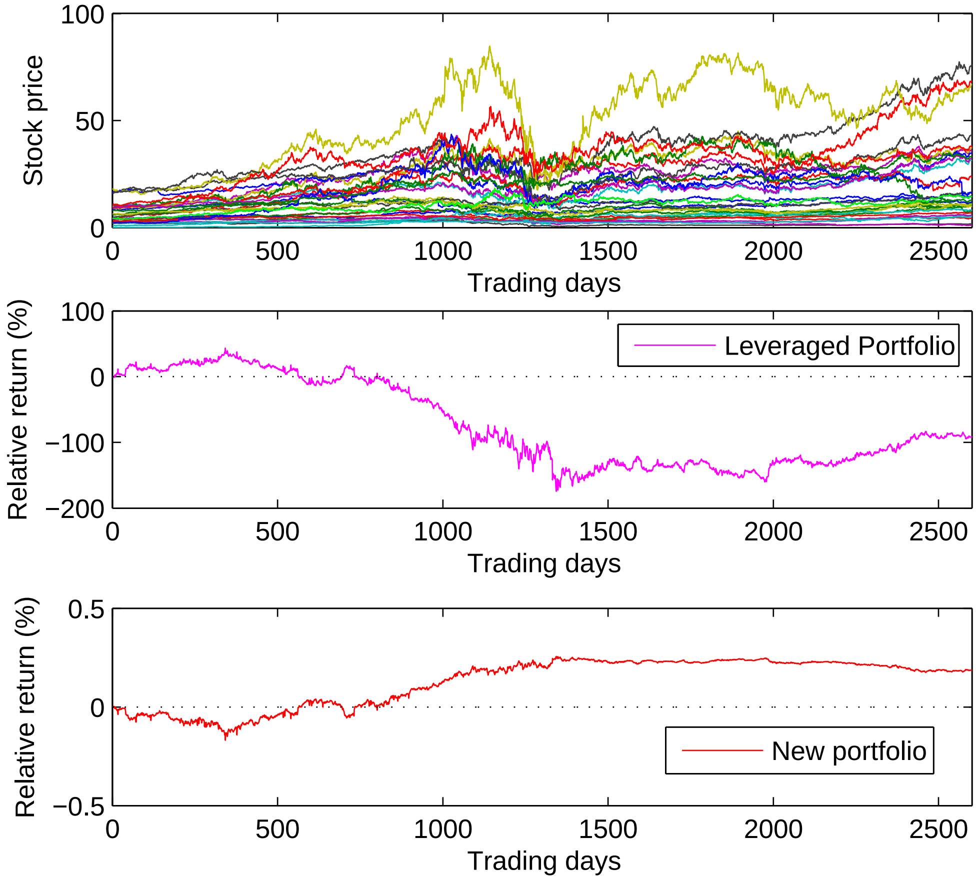


Figure 7: Mirror portfolio with $p = 1.5$

amounts of capital are drawn from the larger companies (in the form of dividends) leading to a drop in their prices and investors re-invest most of this capital to smaller companies increasing their prices. The growth rate changes for all stocks and we get $\gamma_*^\pi(t) > 0$ and a positive drift process.

Now, we look at the extreme case, where a single stock starts to dominate a market in Figure 10. As should be expected, any possibility of arbitrage relative to the market disappears. Again, similar to the reasoning above, it has been shown in Fernholz (1999) that in order to maintain diversity in a market such as this, dividend payments must be made available to the investors so that they can channel capital from larger companies to smaller companies.

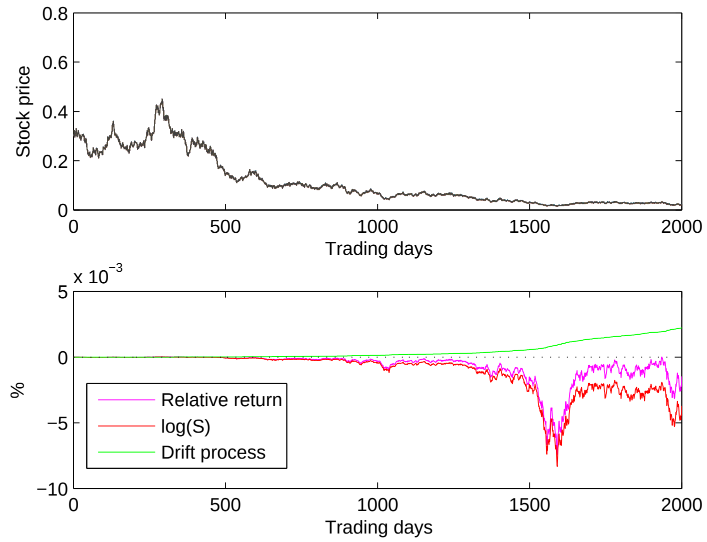


Figure 8: A single stock process was generated and sixty two additional stock processes were created by adding/subtracting a small perturbation vector (elements 0.001) to the generated process.

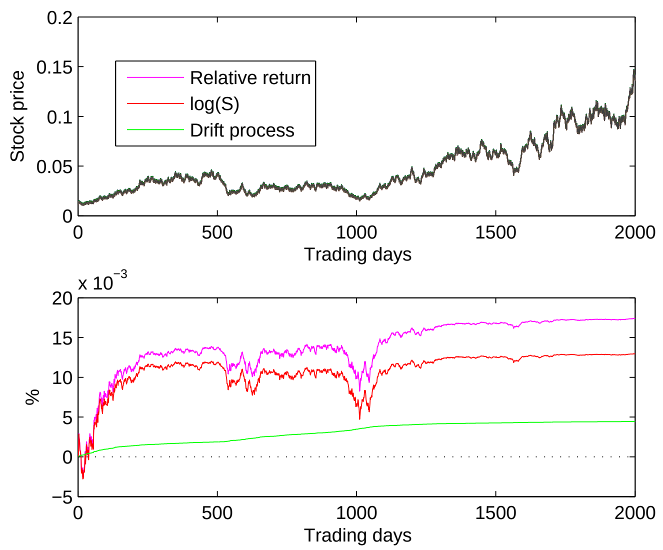


Figure 9: A single stock process was generated and sixty two additional stock processes were created by adding/subtracting a small perturbation vector (elements 0.001) to the generated process.

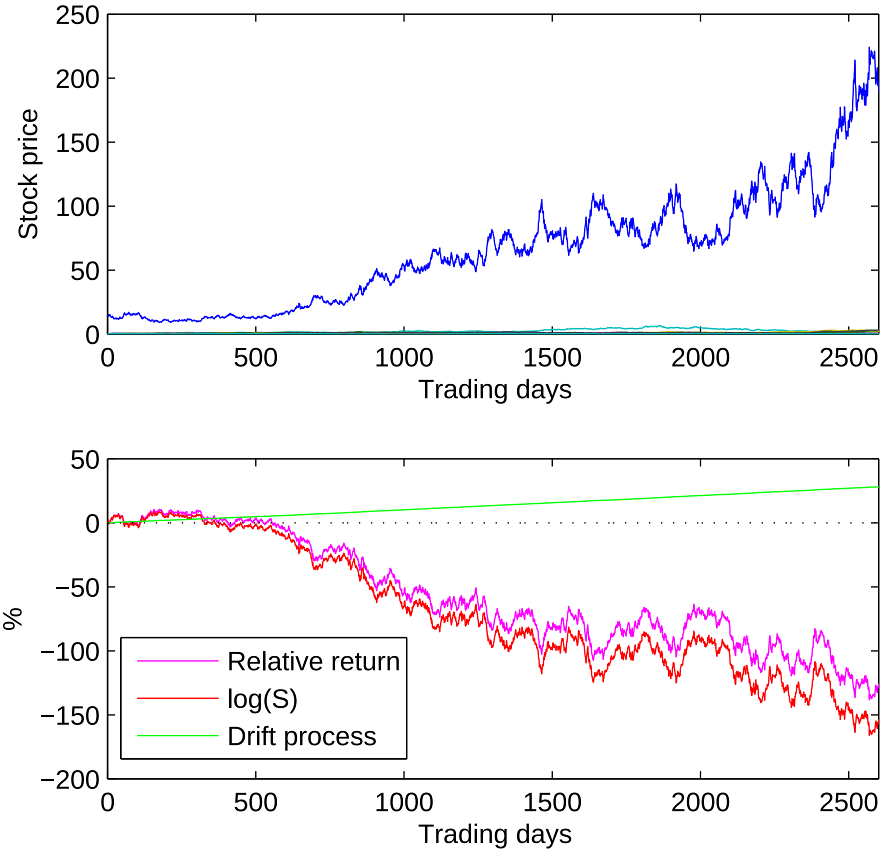


Figure 10: Eighty penny stocks and one dominating stock was generated.

## 15 Conclusion

In this project, we review fundamental concepts introduced in [F02] and [FKK05]. We apply their results on Australian equity market. In particular, the presence of relative arbitrage in a weakly diverse and nondegenerate market in absence of an equivalent martingale measure which is a key finding in [FKK05] is a major focus of this project and in our simulations. Several functionally generated portfolios are examined and we discuss their numerical traits. Although the analysis covered only thirty one listed stocks in the ASX, they were enough to demonstrate the key results, as shown in [FKK05]. Adding additional stocks may have made the analysis more realistic.. Hedging issues are also considered within the context of this model with some surprising results. A major drawback of this analysis undertaken in this project is the generality of the market model. We were restricted to working in the general semi-martingale setting. Therefore, a possible area of further research is to work in the context of volatility-stabilized market models as discussed in [BF08]. They are able to answer affirmatively that there exists relative arbitrage opportunities over arbitrarily short time horizons in the context of certain volatility-stabilized market models. However, they were only able to prove it for two stocks. It is still an open question for more than three stocks. Another possible direction is to introduce time-delays in this market model.

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