Conformal Mapping and the Schwarz-Christoffel Transformation

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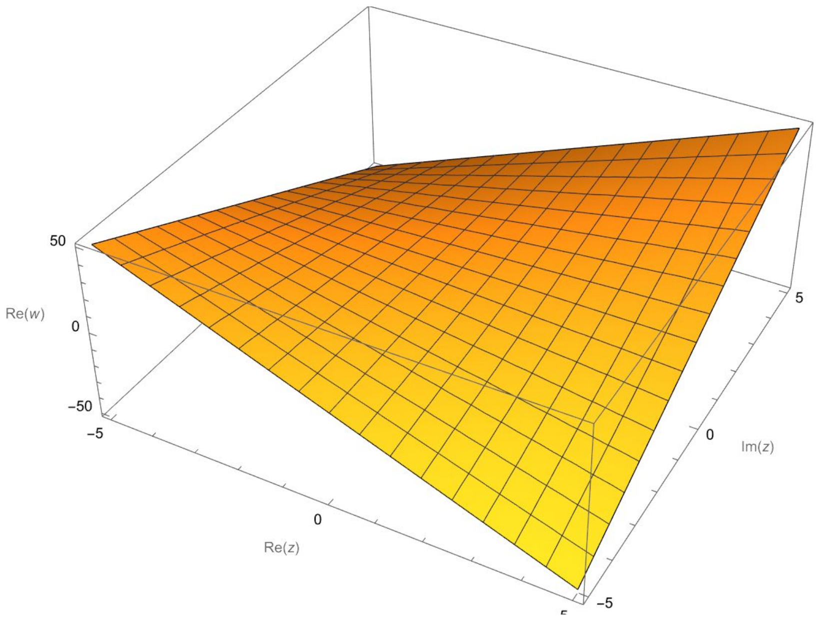
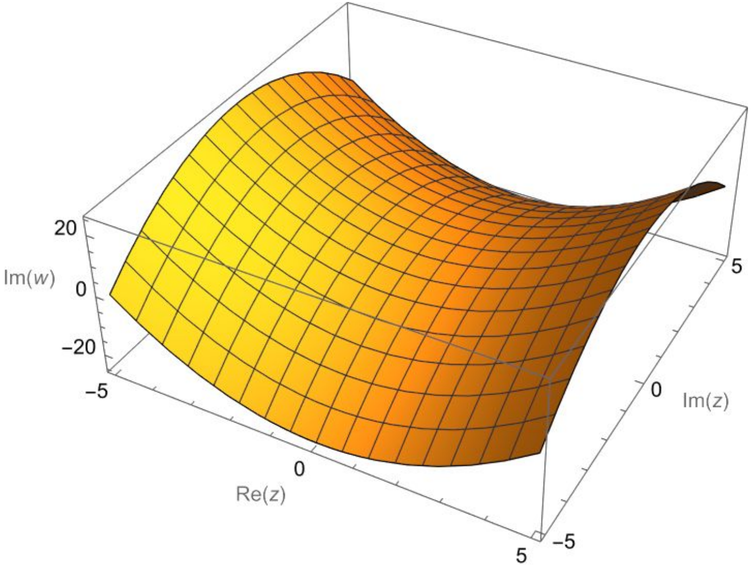
Visualizing Complex Functions



$$w = f(z) = z^2$$

Visualizing Complex Functions



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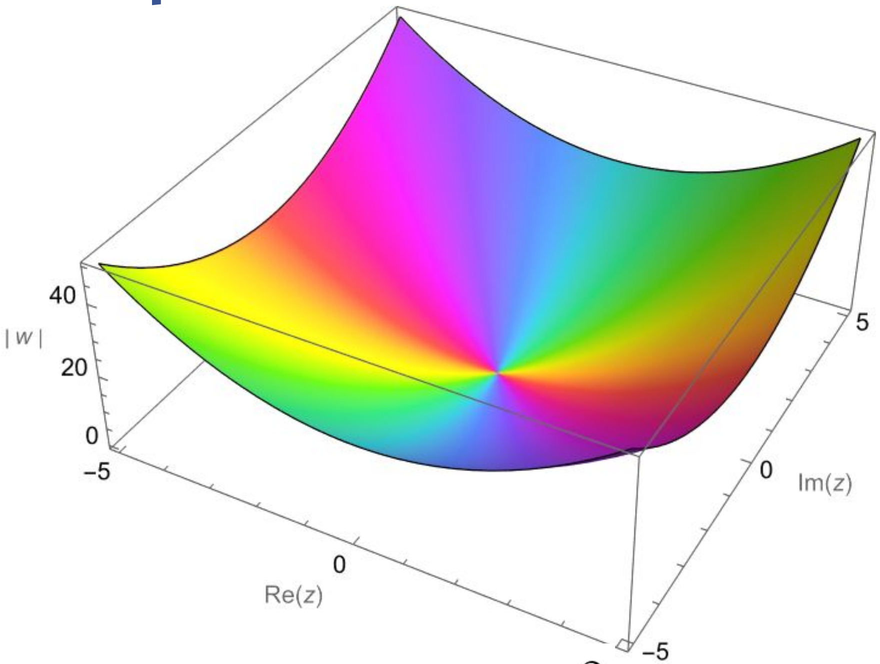
Visualizing Complex Functions



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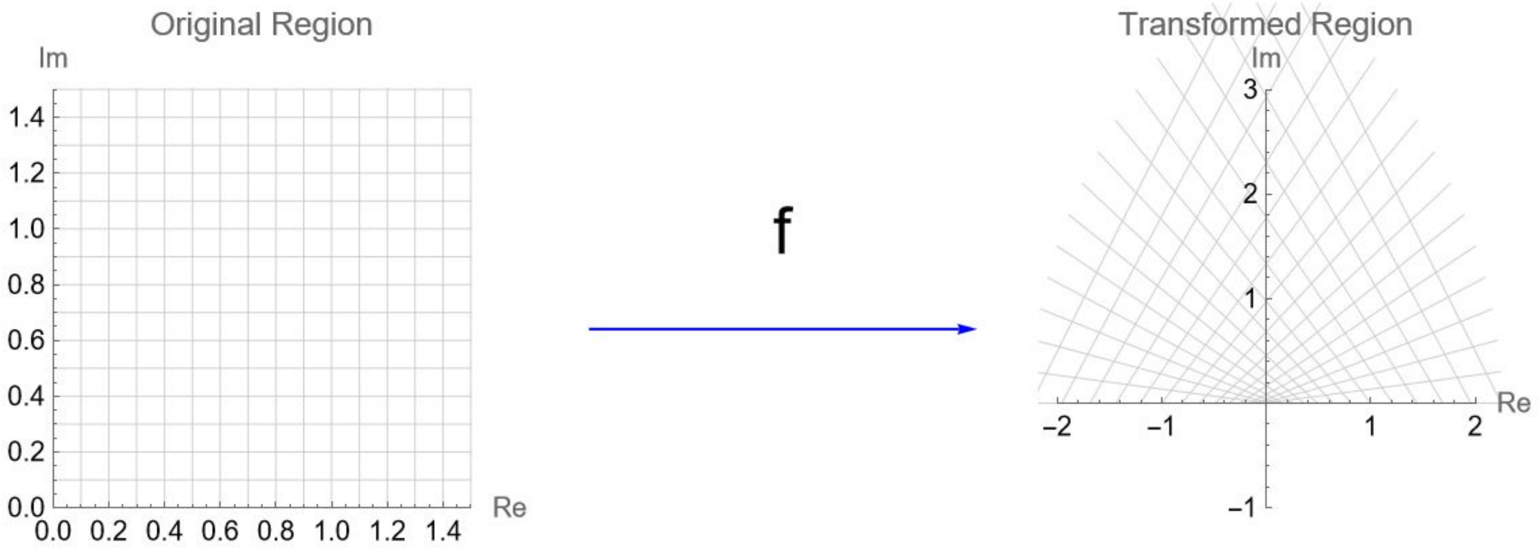
Visualizing Complex Functions As Mappings

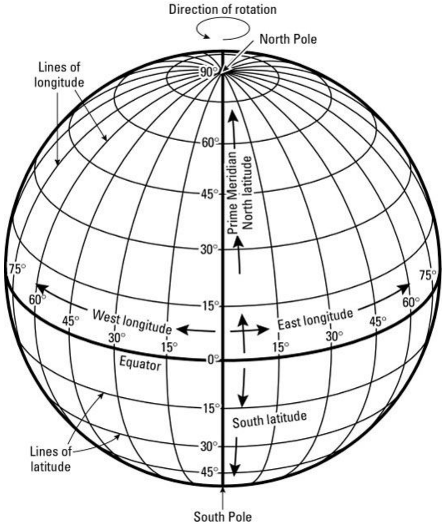


$$w = f(z) = z^2$$

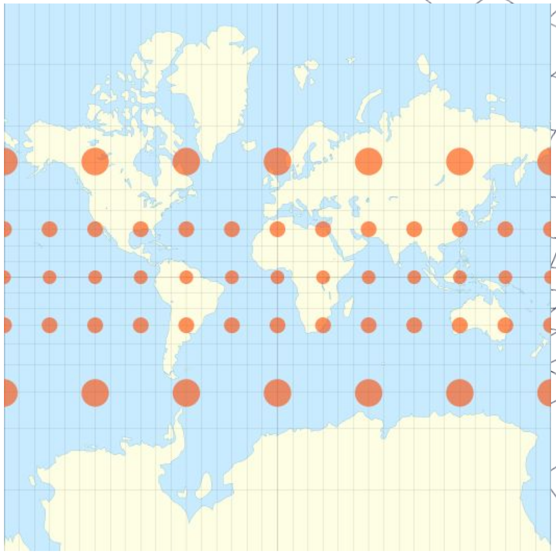
Conformal mapping

Conformal map: a function that locally preserves all angles between intersecting curves

Theorem: Assume a function f(z) is analytic and not constant in a domain *D* in the complex z-plane. f(z) is ***conformal*** for any point z_0 where $f'(z_0) \neq 0$.



Mercator Projection



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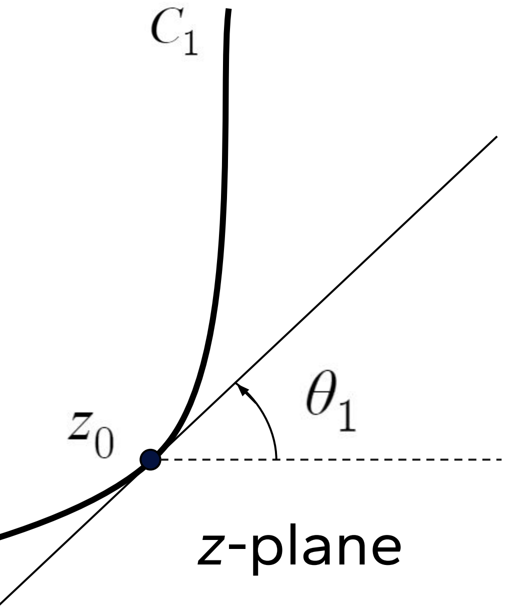
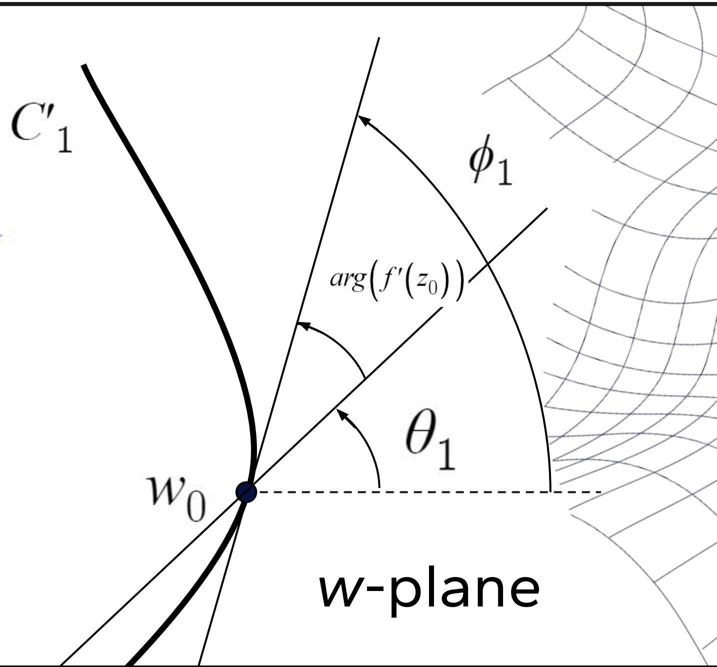


$$w = f(z)$$

Lemma: The tangent line to a curve C at z_0 is rotated by $arg\left(f'(z_0)\right)$ under the transformation $w = f(z)$



Lemma: The tangent line to a curve C at z_0 is rotated by $arg\left(f'(z_0)\right)$ under the transformation $w = f(z)$

Conformal mapping

Proof: $$C:\; z(s) = x(s) + iy(s)$$

At $z = z_0 = z(s_0)$: $$\theta_1 = arg\left(z'\left(s_0\right)\right)$$

At $w = w_0 = f\left(z\left(s_0\right)\right)$: $$w'(s) = \frac{d}{ds}[w(s)] = \frac{d}{ds}[f(z(s))]$$

$$= \frac{df}{dz} \cdot \frac{dz}{ds}$$

$$\Rightarrow w'(s_0) = f'(z_0) z'(s_0).$$

Lemma: The tangent line to a curve C at z_0 is rotated by $arg\left(f'(z_0)\right)$ under the transformation $w = f(z)$

Conformal mapping

Proof:

At $w = w_0 = f\left(z(s_0)\right)$: $w'(s_0) = f'(z_0)z'(s_0)$

$$|w'(s_0)|e^{i \cdot arg(w'(s_0))} = |f'(z_0)|e^{i \cdot arg(f'(z_0))} \cdot |z'(s_0)|e^{i \cdot arg(z'(s_0))}$$

$$\Rightarrow arg(w'(s_0)) = arg(z'(s_0)) + arg(f'(z_0))$$

Conformal mapping

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Theorem: Assume a function f(z) is analytic and not constant in a domain *D* in the complex z-plane. f(z) is ***conformal*** for any point z_0 where $f'(z_0) \neq 0$.

$$C_1: \; z_1(s) \quad \text{intersecting at } z = z_0 \quad C_2: \; z_2(s)$$

Tangent line angles:

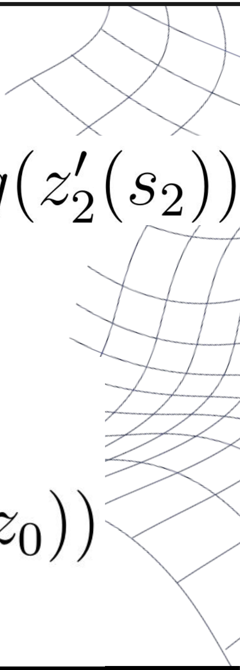
$$\theta_1 = arg(z_1'(s_1)) \qquad \theta_2 = arg(z_2'(s_2))$$

Intersection angle (pre-transform):

$$\theta = \theta_2 - \theta_1$$

Intersection angle (post-transform):

$$\begin{aligned} \phi &= \phi_2 - \phi_1 \\ &= \theta_2 + arg(f'(z_0)) - \theta_1 - arg(f'(z_0)) \\ &= \theta_2 - \theta_1 = \theta \end{aligned}$$



Conformal mapping

Distances are usually not preserved in conformal mappings!

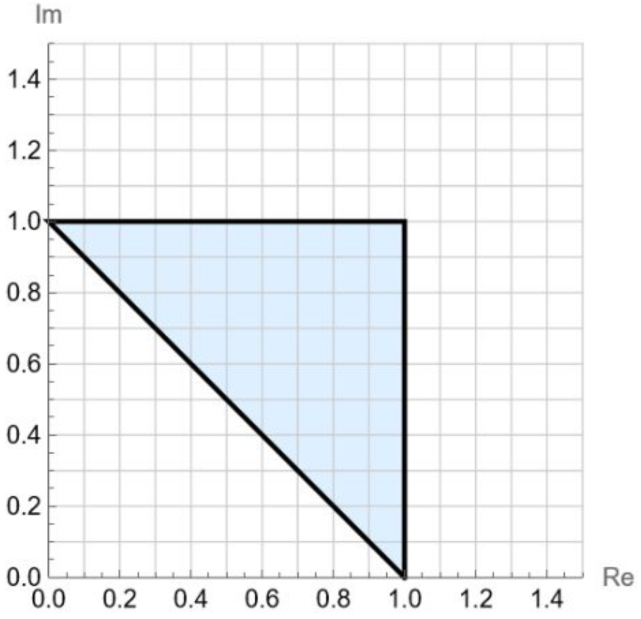
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{w - w_0}{z - z_0}$$

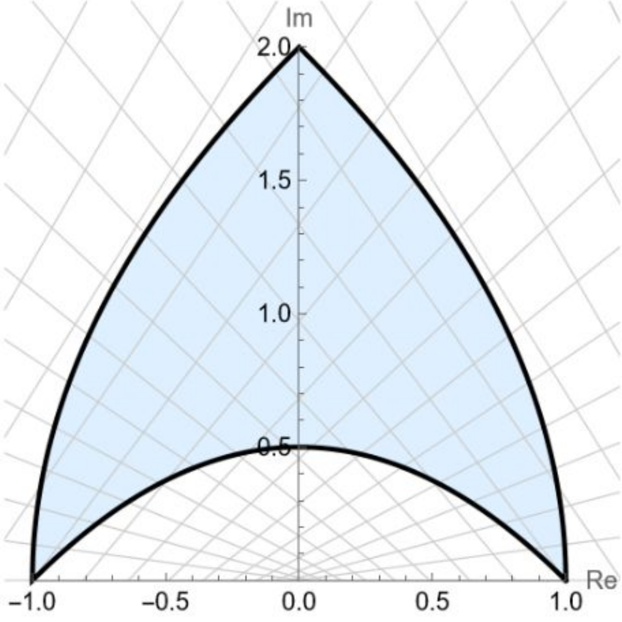
$$|f'(z_0)| = \lim_{z \to z_0} \frac{|w - w_0|}{|z - z_0|}$$

$$\Rightarrow |w - w_0| \approx |f'(z_0)||z - z_0|.$$

Conformal mapping

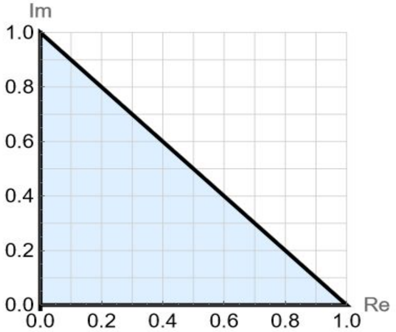


$$w = f(z) = z^2$$

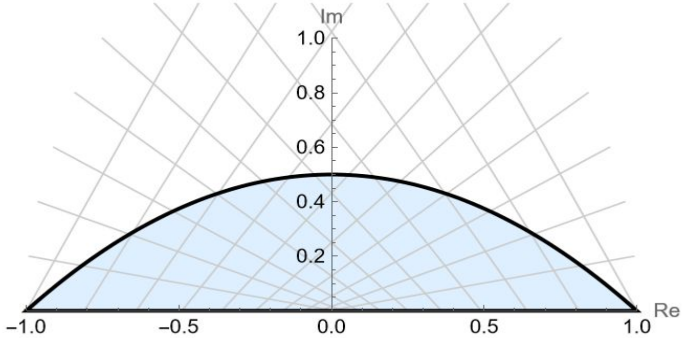


Conformal mapping

Theorem: Suppose that $f'(z_0) = f''(z_0) = \ldots = f^{(n-1)}(z_0) = 0$, but $f^{(n)}(z_0) \neq 0$, then the mapping $z \rightarrow w = f(z)$ magnifies n times the angle between the two intersecting arcs which meet at z_0.



$w = f(z) = z^2$



$$f'(z) = 2z \qquad f''(z) = 2$$

$$f'(0) = 0 \qquad f''(0) = 2$$

$$\Rightarrow n = 2$$

Transformations

A function f(z) is ***univalent*** in a domain *D* if it takes no value more than once.

Riemann Mapping Theorem: Let *D* be a simply connected domain in the z-plane. Then there exists a *univalent* function f(z) such that w = f(z) maps *D* onto the disk |w| < 1.

Linear Mapping: maps a curve in the z-plane to a curve in the w-plane that has been rotated by the argument of a in $w = az + b$

Bilinear Transform: maps the interior of the unit circle centered at the origin in the z-plane to the upper half of the w-plane. $w = i(\frac{1-z}{1+z})$

Nonlinear Transforms: more complex mappings between the z-plane and w-plane, useful for applications. Ex: $w = z + \frac{R^2}{z}$ can be used to map circles into lines and vice versa

Example

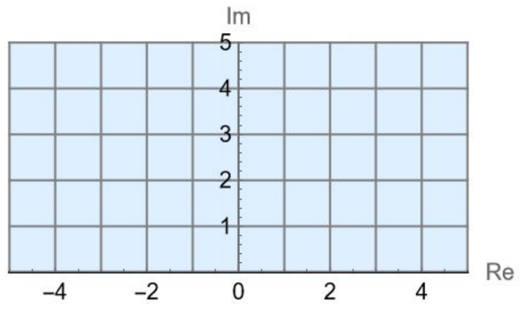
$w = f(z) = z^{\alpha}$, $0 < \alpha < 2$, maps the interior of an open triangle of angle $\pi\alpha$, with vertex at the origin of the w-plane to the upper half z-plane.

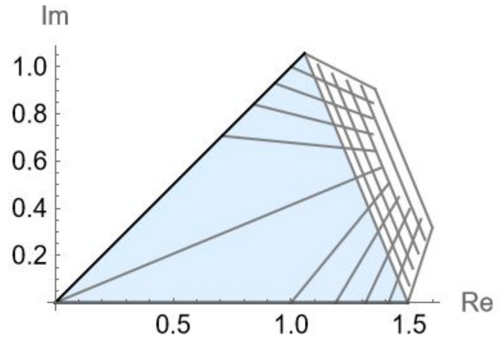
Let $z = re^{i\theta}$ and $w = \rho e^{i\varphi}$
$\theta = 0$ and $\theta = \pi$ in z-plane are mapped to $\varphi = 0$ and $\varphi = \alpha\pi$ in the w-plane

$f(z) = z^{\alpha}$ is not conformal at $z = 0$ when $\alpha < 1$.

Take the inverse of f(z) and get $g(w) = w^{1/\alpha}$. Must restrict $0 < \alpha < 2$ so that the conform map in the z-plane does not overlap itself.

$f = g(w) = w^{1/\alpha}$ maps the z-plane to the w-plane.





Schwarz-Christoffel Transformation

Schwarz-Christoffel transformation (SCT): a conformal map of the upper half plane or the complex unit disk onto the interior of a simple polygon

Schwarz-Christoffel theorem: Let Γ be the piecewise linear boundary of a polygon in the w-plane, and let the interior angles at successive vertices be $\alpha_1\pi,\ldots,\alpha_n\pi$. The transformation defined by the equation

$$\frac{dw}{dz} = \gamma(z - a_1)^{\alpha_1 - 1}(z - a_2)^{\alpha_2 - 1}\ldots(z - a_n)^{\alpha_n - 1}$$

where γ is a complex number and $a_1,\ldots,a_n$ are real numbers, maps Γ onto the real axis of the z-plane and the interior of the polygon to the upper half z-plane.

SCT of an open polygon

Find an analytic function w = f(z) that maps the upper half z-plane to an open polygon.

$\frac{dw}{dz} = f'(z)$, also $dw = f'(z)dz$

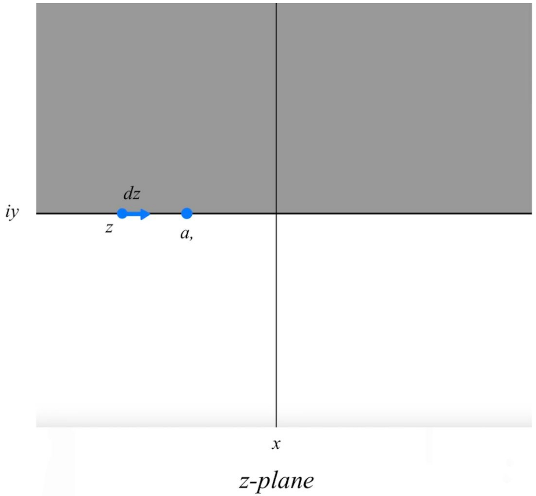
Choose point w in w-plane left of A_1

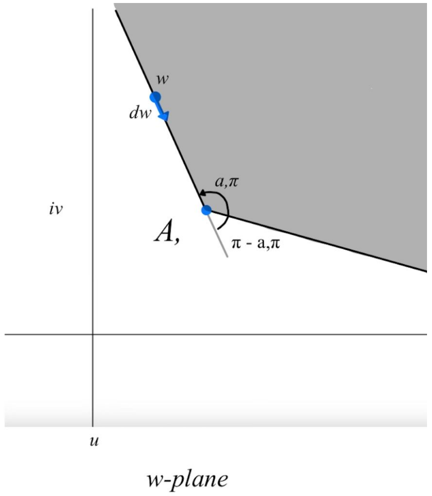
As dz crosses a_1, $arg(dw) = \pi(1 - \alpha_1)$

Let $dz = re^{i\theta}$ and $dw = pe^{i\phi}$

$$f'(z) = \frac{p}{r}e^{i(\phi-\theta)}$$

$$arg(f'(z)) = arg(dw) - arg(dz)$$
$$= \phi$$
$$= arg(dw)$$
$$arg(dz) = 0$$





SCT of an open polygon

From the previous slide, $arg(f'(z)) = arg(dw)$

$f'(z) = (z - a_1)^{\alpha_1 - 1}$, so $arg(dw)$ changes by $\pi(1 - \alpha_1)$

To see this, let $z - a_1 = re^{i\theta}$

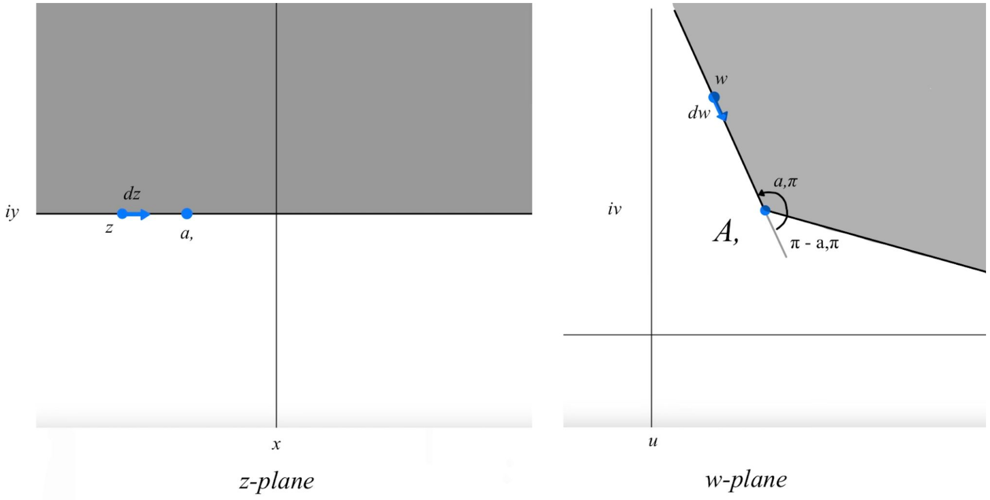
$$f'(z) = (z - a_1)^{\alpha_1 - 1} = (re^{i\theta})^{\alpha_1 - 1}$$

$$= r^{\alpha_1 - 1} e^{i\theta(\alpha_1 - 1)}$$

$$arg(f'(z)) = \theta(\alpha_1 - 1)$$

Open polygon with *n* angles:

$$\frac{dw}{dz} = \gamma(z - a_1)^{\alpha_1 - 1}(z - a_2)^{\alpha_2 - 1} \dots (z - a_n)^{\alpha_n - 1}$$



Jacobian elliptic functions

Schwarz reflection principle: Suppose that f(z) is analytic in a domain *D* that lies in the upper half z-plane. Let *D'* denote the domain obtained by *D* by reflecting with respect to the real axis. Then corresponding to every point z in $\tilde{D}$ the function $\tilde{f}(z) = \overline{f(\tilde{z})}$ is analytic in $\tilde{D}$

The incomplete elliptic integral of the first kind is: $F(z,k) = \int_0^z \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}$

The complete elliptic integral is *K(k) = F(1,k)*

$$K'(k) = \int_1^{1/k} \frac{d\xi}{\sqrt{(\xi^2-1)(1-k^2\xi^2)}} = \int_1^{1/k} \frac{d\xi}{\sqrt{-(1-\xi^2)(1-k^2\xi^2)}}$$

$$iK'(k) = \int_1^{1/k} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}$$

$$F(\tfrac{1}{k}, k) = K(k) + iK'(k)$$

$$\xi = (1 - k'^2\xi'^2)^{-1/2} \qquad k' = \sqrt{1-k^2}$$

Derivation

Find w = f(z) and the constant *s* as a function of *k*.

Let $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$

From SC theorem, we get

$$\frac{dw}{dz} = \gamma(z - 1)^{-1/2}(z + 1)^{-1/2}(z - 1/k)^{-1/2}(z + 1/k)^{-1/2}$$

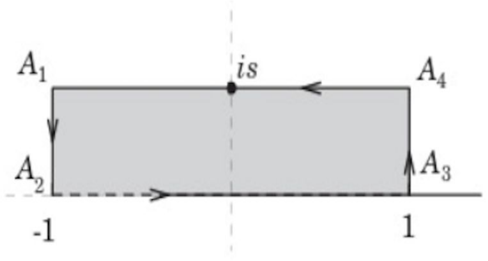
Integrate and change γ to λ:

$$w(z,k) = \gamma\int_0^z \frac{d\xi}{\sqrt{(\zeta^2-1)(\zeta^2-1/k^2)}} = \lambda\int_0^z \frac{d\xi}{\sqrt{(\zeta^2-1)(k^2\zeta^2-1)}} = \lambda F(z,k)$$

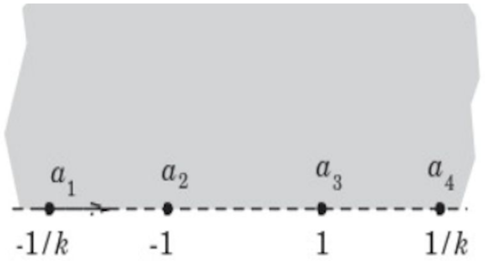
The complete elliptic integral $K(k) = F(1,k) = \int_0^1 \frac{d\xi}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$ implies that λ = 1/K(k).

From this, we get: $w = \frac{F(z,k)}{K(k)},\ s = \frac{K'(k)}{K(k)}$

The inverse gives a **Jacobian elliptic function**: $w = \frac{F(z,k)}{K(k)},\ z = sn(wK,k)$



w-plane



z-plane $(0 < k < 1)$

Properties of elliptic functions

If we “normalize” the previous function to be w = F(z,k), then s=K’(k).

The **elliptic sine** is $sn(w,k) = F^{-1}(w,k) = z$

- **Double periodicity**: $sn(w + n\omega_1 + im\omega_2, k) = sn(w,k)$
  - Can be proved with the Schwarz reflection principle.
  - Rectangle example: $sn(w + 4nK(k) + 2imK'(k), k) = sn(w,k)$
- **Single valuedness**
  - The symmetric relationship implies z = sn(w,k) is single valued.
  - Any point in z in the UHP is uniquely determined and corresponds to an even number of reflections.

Double periodicity of other polygons

SCT for mapping unit disk to a polygon with *n* sides

The **SCT formula** for mapping from a disk is identical to mapping from the UHP!

$$\frac{dw}{d\zeta} = \gamma(\zeta - \zeta_1)^{\alpha_1 - 1} \dots (\zeta - \zeta_k)^{\alpha_k - 1}$$

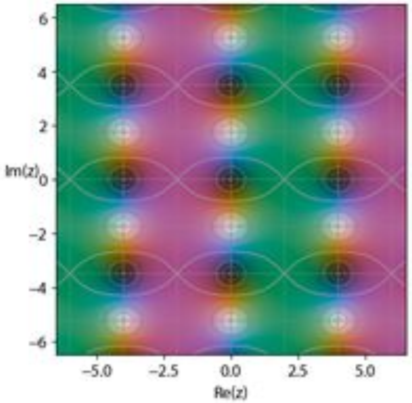
where the points $\zeta_k = e^{2\pi i k/n}$, k=1,...,n lie on the unit circle.

Every angles has the same magnitude and the sum of the interior angles is π(n-2).

Thus, $\alpha_k = 1 - \frac{2}{n}$

Plugging that in, we get $\frac{dw}{d\zeta} = \gamma\left(\prod_{k=1}^{n}\left(\zeta - e^{2pik/n}\right)\right)^{-\frac{2}{n}}$, $\quad w(\zeta) = A + \gamma\int_0^{\zeta}\frac{d\zeta'}{(\zeta_k - 1)^{2/n}}$

which maps a unit disk to a polygon with *n* sides, interior angles α_k, and vertices f(ζ_k).



double periodicity of Jacobian elliptic function ***sn***

SCT of equilateral triangle

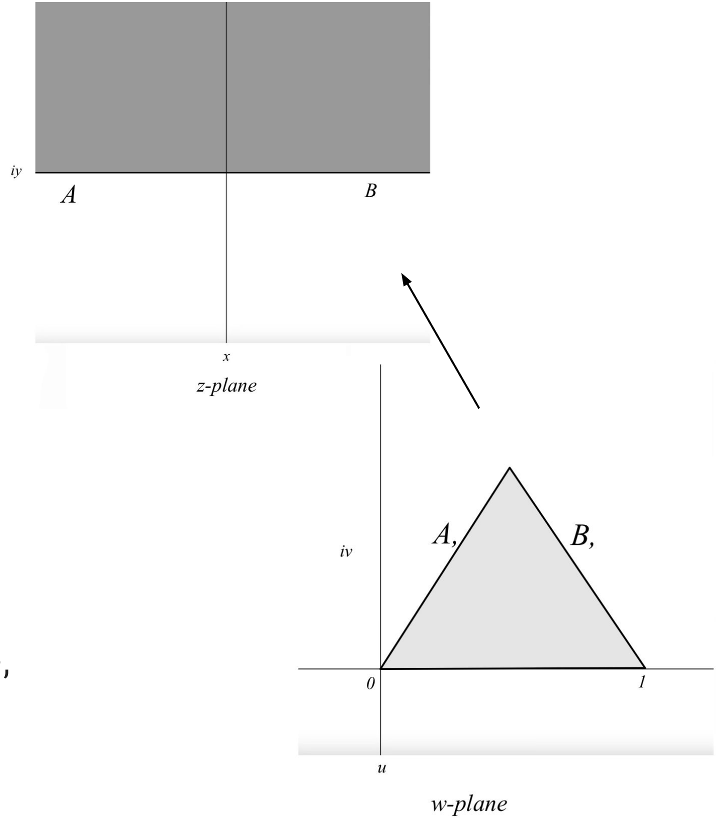
With vertices $\zeta_0 = 0$ and $\zeta_1 = 1$ and interior angles $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$,

$$w'(\zeta) = \gamma\,\zeta^{-2/3}(\zeta - 1)^{-2/3}$$

$w'(\zeta)$ is analytic in the UHP, so

$$w(\zeta) = A + \gamma\int_0^{\zeta} \frac{d\zeta'}{\zeta'^{2/3}(\zeta'-1)^{2/3}}$$

$w(\zeta)$ maps the equilateral triangle to the upper half z-plane, with A and γ complex constants.



SCT of equilateral triangle

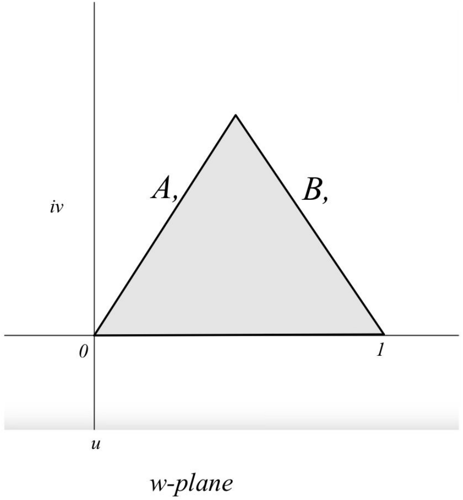
Single valuedness

- When we reflect the triangle to cover the w-plane, an even number of 6 triangles meet at each vertex.
- When we reflect the triangle around a vertex back onto itself, we return to the interior of the unit circle in the plane.

Double periodicity

- The triangle can be reflected over the real axis or along one of its other sides. The angle between the two sides and the real axis is π/3.
- Using pythagorean theorem, the reflected triangle will be translated a distance of √3.
- The inverse function has double periodicity $\zeta(w + n\sqrt{3} + m\sqrt{3}e^{\frac{i\pi}{3}}) = \zeta(w)$

The angle of rotation is $e^{2\pi i/3}$, which results in $\zeta(e^{l\frac{2\pi i}{3}} w + n\sqrt{3} + m\sqrt{3}e^{\frac{i\pi}{3}}) = \zeta(w)$



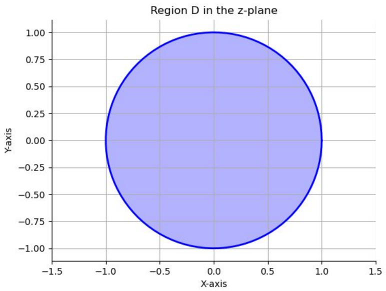
Applications

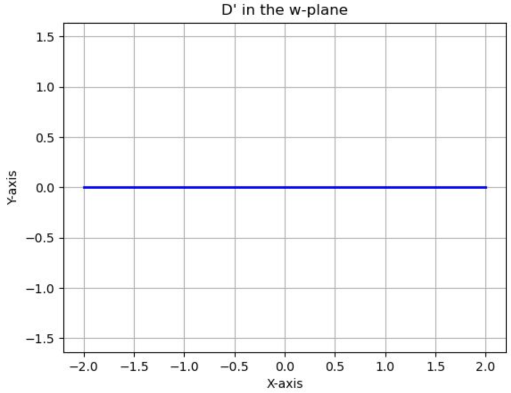
Conformal mappings can be applied to potential flow problems

- Use conform mapping function to transform complicated geometries into simpler shapes

Example:

- Find potential flow around a cylinder
- Mapping function: $$w = z + \frac{R^2}{z}$$
- Now we can find the potential flow around a plate
- This is just uniform flow: Uw





Questions?