Conformal Mapping and the Schwarz-Christoffel Transformation

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Complex Variables and Applications

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Abstract

This paper will explore how functions of complex variables can be viewed as mappings of the complex plane. First, an introduction of conformal mappings will be given, which preserve signed angles and the infinitesimal shapes of figures, but not sizes. Next, various mappings such as the fundamental mapping of a rectangle, elliptic function mappings, and the Schwarz-Christoffel transformation, which is a mapping from the upper half complex plane to the interior of a polygon, will be considered through use of the Riemann Mapping Theorem. Finally, applications of these mappings will be discussed, such as solving Laplace's equation on complicated domains, minimal surfaces, fluid dynamics, and hyperbolic art.

1 Introduction

There are many different ways that a complex function f mapping $z = x+iy$ to $w = u+iv = f(z)$ can be visualized. One can plot $x = Re(z)$ and $y = Im(z)$ against $u = Re(f(z))$, $v = Im(f(z))$, $|f(z)|$, or even $arg(f(z))$. Another way to visualize f is as a *mapping* from the z-plane to the w-plane, as in Figure 1. We can consider a region A in the z-plane and consider how it is *transformed* by f into region B in the w-plane: $f(A) = B$.

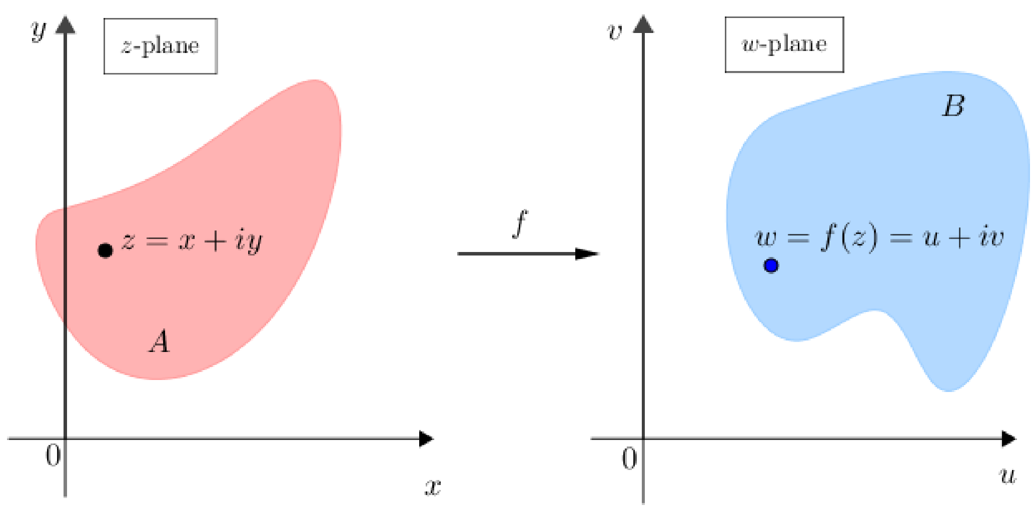

Figure 1: A complex mapping f

Many problems are much easier to solve in certain regions of the complex plane. Solutions to Laplace's equation, for example, can be greatly simplified if the region of interest is the upper half z-plane ($Im(z) > 0$) or the unit disk centered at the origin ($|z| < 1$) [1]. As such, it is useful to consider a transformation $w = f(z)$ that can map a region D in the z-plane to a region D' in the w-plane that makes solving a given problem easier. A certain subset of transformations, known as *conformal maps*, prove to be quite useful for this purpose.

2 Project Description

2.1 Conformal mapping

2.1.1 Definition

*Definition 1: A **conform map** is a function that locally preserves the angles of intersecting curves [1].*

*Theorem 1: Assume that $f(z)$ is analytic and not constant in a domain D of the complex z-plane. For any point $z \in D$ for which $f'(z) \neq 0$, this mapping is **conformal**, meaning it preserves the angle between two different arcs [1].*

Proof: We first prove that, for a curve $C \subset \mathbb{C}$ mapped to C^* under a conformal mapping $f(z)$, the directed tangent to C at a point $z = z_0$ is rotated by the angle $arg(f'(z_0))$. To describe the points $z \in C$, we parameterize C to obtain C: $z(s) = x(s) + iy(s)$, $s \in [a, b]$. The angle of the line tangent to C at $z = z_0 = z(s_0)$ is given by $arg(z'(s_0))$. We then consider $C^* = f(C)$, which is parameterized by C^*: $w(s) = f(z(s))$. The angle of the line tangent to C^* at $z = z_0$ is given by $arg(w'(s_0))$. To find $w'(s)$, we use the chain rule:

$$\begin{aligned} w'(s) = \frac{d}{ds}[w(s)] &= \frac{d}{ds}[f(z(s))] \\ &= \frac{df}{dz} \cdot \frac{dz}{ds} \\ \Rightarrow w'(s_0) &= f'(z_0) z'(s_0). \end{aligned}$$

To find $arg(w'(s_0))$, we let $w'(s_0) = |w'(s_0)|e^{i \cdot arg(w'(s_0))}$, $f'(z_0) = |f'(z_0)|e^{i \cdot arg(f'(z_0))}$, and $z'(s_0) = |z'(s_0)|e^{i \cdot arg(z'(s_0))}$. Substituting, we find that

$$\begin{aligned} |w'(s_0)|e^{i \cdot arg(w'(s_0))} &= |f'(z_0)|e^{i \cdot arg(f'(z_0))} \cdot |z'(s_0)|e^{i \cdot arg(z'(s_0))} \\ &= |f'(z_0)||z'(s_0)|e^{i(arg(f'(z_0)) + arg(z'(s_0)))} \\ \Rightarrow arg(w'(s_0)) &= arg(z'(s_0)) + arg(f'(z_0)). \end{aligned}$$

As stated, the line tangent to C at $z = z_0$ is rotated by $arg(f'(z_0)$ after applying f to obtain C^*.

Now, we consider two curves, C_1: $z_1(s)$ and C_2: $z_2(s)$, intersecting at $z = z_0$ where $z_0 = z_1(s_1) = z_2(s_2)$. The lines tangent to C_1 and C_2 at $z = z_0$ have angles given by $\theta_1 = arg(z_1'(s_1))$ and $\theta_2 = arg(z_2'(s_2))$ respectively. Therefore, the angle between C_1 and C_2 is given by $\theta = \theta_2 - \theta_1$. After applying our conformal map f to C_1 and C_2 to obtain C_1^* and C_2^*, the new angles of the lines tangent to C_1^* and C_2^* are given by $\phi_1 = \theta_1 + arg(f'(z_0))$ and $\phi_2 = \theta_2 + arg(f'(z_0))$. As before, the angle between C_1^* and C_2^* is given by $\phi = \phi_2 - \phi_1$.

We find that

$$\begin{aligned}\phi &= \phi_2 - \phi_1 \\ &= \theta_2 + arg(f'(z_0)) - \theta_1 - arg(f'(z_0)) \\ &= \theta_2 - \theta_1 \\ &= \theta,\end{aligned}$$

which concludes our proof: conformal mappings preserve the angles between arcs.

Besides the angle preservation property, a conformal mapping $w = f(z)$ also has the property that distances near $z = z_0$ are magnified by a factor of $|f'(z_0)|$. By considering the limit definition of $f'(z_0)$, we see that

$$\begin{aligned}f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{z \to z_0} \frac{w - w_0}{z - z_0} \\ |f'(z_0)| &= \lim_{z \to z_0} \frac{|w - w_0|}{|z - z_0|} \\ \Rightarrow |w - w_0| &\approx |f'(z_0)||z - z_0|.\end{aligned}$$

This approximation becomes more exact as the distance from z to z_0 decreases.

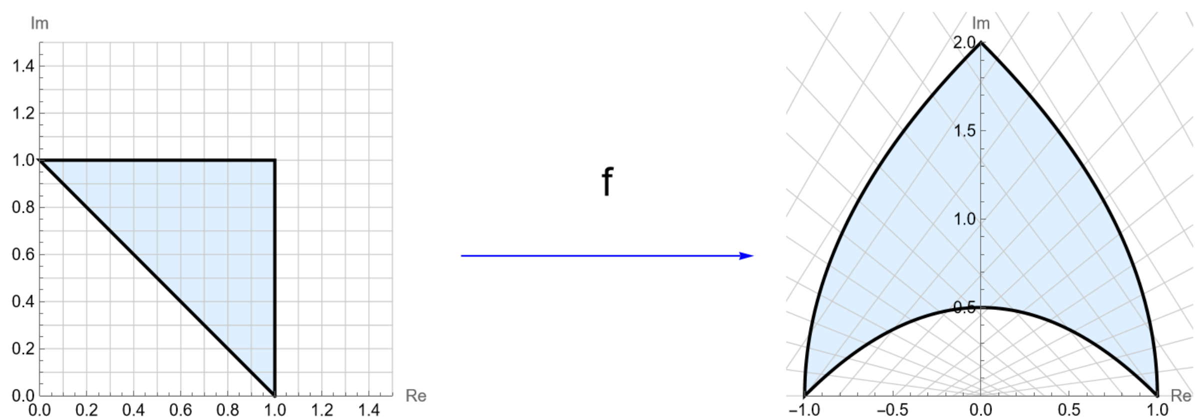

Figure 2: Triangle transformation for $w = f(z) = z^2$.

Figure 2 is a conformal transformation, as it preserves the original angles between the curves from the z-plane to the w-plane. This is because the transformation $f : \mathbb{C} \Rightarrow \mathbb{C}$, where $f(z) = z^2$, is analytic and non-constant within the domain D defined by the interior of the closed contour in the z-plane. Additionally, their first derivatives do not equal zero at the points of intersection. In the case where $f(z)$ has first derivative that equals zero at some point of intersection z_0, $f'(z_0)$ is called a *critical point*.

Theorem 2: Assume that $f(z)$ is analytic and not constant in a domain D of the complex z-plane. Suppose that $f'(z_0) = f''(z_0) = ... = f^{(n-1)}(z_0) = 0$, while $f^{(}n)(z_0) \neq 0$, $z_0 \in D$. Then the mapping $z \Rightarrow f(z)$ magnifies n times the angle between two intersecting differentiable arcs which meet at z_0 [1].

Consider a region D defined by the triangular region bounded by $x = 0$, $y = 0$, and $y = 1$. Figure 3 depict the transformation $w = z^2$.

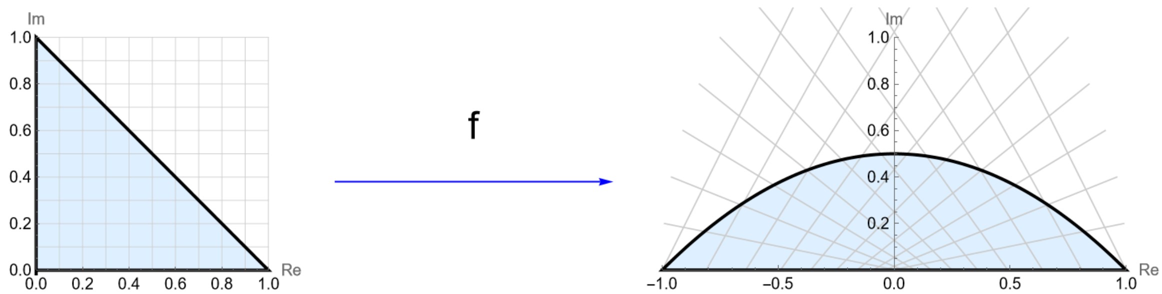

Figure 3: The transformation $w = f(z) = z^2$.

Let $u = x^2 - y^2$, $v = 2xy$. Then, the lines $x = 0$, $y = 0$, and $x + y = 1$ can be mapped to $v = 0$ with $u \leq 0$, $v = 0$ with $u \geq 0$, and $v = \frac{1}{2}(1 - u^2)$ respectfully. The transformation $w = z^2$ is not conformal at $z = 0$. The second derivative of $f(z)$ at $z = 0$ is the first non-zero derivative, so the angle at b should be multiplied by 2 according to Theorem 2. So, the angle b becomes π instead of $\frac{\pi}{2}$.

2.1.2 Other Properties

Theorem 3: If $f(z)$ is analytic and not constant in a domain D of the complex z-plane, then the transformation $w = f(z)$ can be interpreted as a mapping of the domain D onto the domain $D^* = f(D)$ of the complex w-plane [1].

2.2 Transformations

Basic transformation:

Starting with the simple case of transforming a linear function of z, we define:

$$f(z) = az + b \qquad a, b \in \mathbb{C}$$

$$C = z(s) = se^{i\phi}$$

Now we apply the transformation:

$$w = f(z) \implies w(s) = az(s) + b$$

Since a in the z-plane has both a magnitude and angle associated with it ($|a|$ and $arg(a)$) we can write $w(s)$ in polar coordinates as:

$$\begin{aligned} w(s) &= |a|e^{\arg(a)}se^{i\phi} + b \\ &= |a|se^{i\phi + \arg(a)} + b \end{aligned}$$

Thus the original function in the z-plane is rotated by $arg(a)$ in the w-plane.

Transforming a 2-D region:

Define region D in the z-plane as the area bounded by $y = 0, y = 1, y = \sqrt{3}x$, $y = \sqrt{3}x - 2$

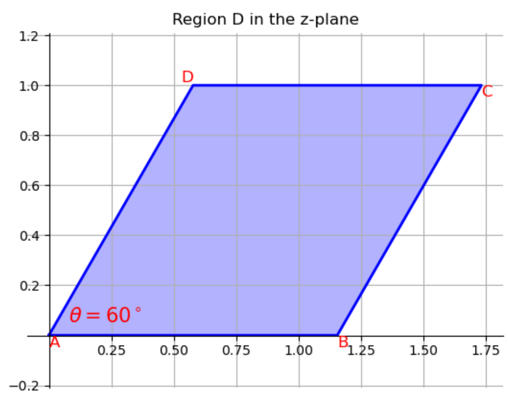

Figure 4: Transformation of 2-D region D.

To map this region D to a region in the w-plane, D', that is rotated 30 degrees we apply the following transformation:

$$\begin{aligned} w &= (\sqrt{3} + i)z \\ &= (\sqrt{3} + i)e^{\frac{i\pi}{6}}z \end{aligned}$$

$$z = x + iy \qquad \text{for } 0 < x \leq \sqrt{3} \text{ and } 0 < y \leq 1$$

$$\begin{aligned} w(z) = u + iv &= (\sqrt{3} + i)(x + iy) \\ &= (\sqrt{3} + i)x + (\sqrt{3}i - 1)y \\ &= (\sqrt{3}x - y) + (x + \sqrt{3}y)i \\ &\implies u = \sqrt{3}x - y \text{ and } v = x + \sqrt{3}y \end{aligned}$$

Using this result we can transform the points $(0,0), (\frac{2\sqrt{3}}{\sqrt{3}}, 0), (\frac{\sqrt{3}}{3}, 1)$, and $(\sqrt{3}, 1)$ (A, B, C, and D in the above figure) in the z-plane to points in the w-plane:

$$\begin{aligned} \text{z-plane} &\implies \text{w-plane} \\ (0,0) &\implies (0,0) \\ (\frac{2\sqrt{3}}{\sqrt{3}}, 0) &\implies (2, \frac{2\sqrt{3}}{3}) \\ (\frac{\sqrt{3}}{\sqrt{3}}, 1) &\implies (0, \frac{4\sqrt{3}}{3}) \\ (\sqrt{3}, 1) &\implies (2, 2\sqrt{3}) \end{aligned}$$

And we are left with the resulting figure:

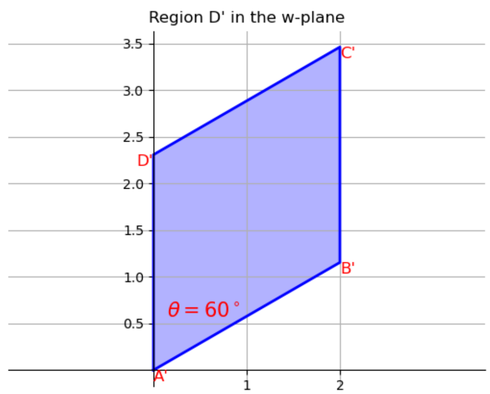

Figure 5: Transformation of 2-D region D'.

Transforming with a non-linear function:

Let region D in the z-plane be bounded by the unit circle

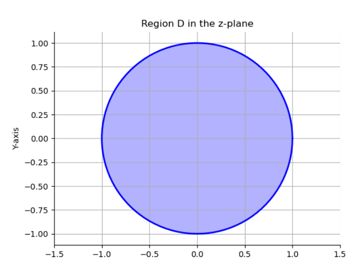

Figure 6: Circle in z-plane

We can apply the following non-linear transformation to map this circle to a line in the w-plane:

$$
\begin{aligned}
w &= z + \frac{r^2}{z} \\
&= z + \frac{1}{z}
\end{aligned}
$$

$$
z = x + iy \qquad \text{for } -1 < x \leq 1 \text{ and } -1 < y \leq 1
$$

$$
\begin{aligned}
w(z) = u + iv &= (x + iy) + \frac{1}{(x + iy)} \\
&= (x + iy) + \frac{(x - iy)}{x^2 + y^2} \\
&= x + \frac{x}{x^2 + y^2} + i(y - \frac{y}{x^2 + y^2}) \\
&\implies u = x + \frac{x}{x^2 + y^2} \text{ and } v = y - \frac{y}{x^2 + y^2} \\
&\implies u = x + x \text{ and } v = y - y \\
&\implies u = 2x \text{ and } v = 0
\end{aligned}
$$

Since $-1 < x \leq 1$ we can see that the circle in the z-plane has been transformed into a line in the w-plane extending from -2 to 2 on the real axis [4].

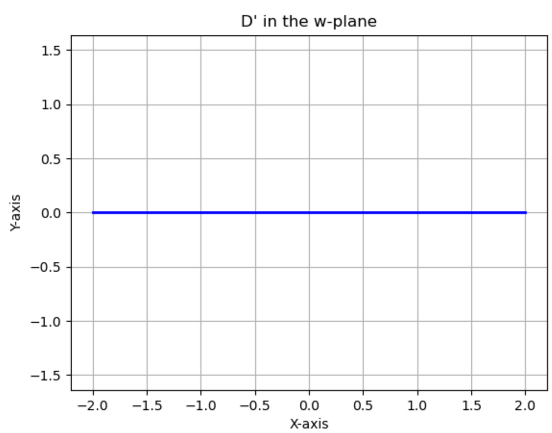

Figure 7: Circle mapped to line in w-plane

2.3 Schwarz-Christoffel Transformation

Definition 2: An analytic function $f(z)$ is ***univalent*** *in a domain D if it takes no value more than once in D [1].*

Theorem 4 (Riemann Mapping): Let D be a simply connected domain in the z-plane, which is neither the z-plane or the extended z-plane. Then there exists a univalent function f(z), such that w = f(z) maps D onto the disc $|w| < 1$ [1].

The Schwarz-Christoffel transformation (SCT) is a specific case of the Riemann Mapping Theorem that maps a polygon on a halfplane or circular disk. Because the SCT formula consists of factors that are binomials to some power, it is worthwhile to note the bilinear transformation:

$$w = i(\frac{1-z}{1+z}) \tag{1}$$

The bilinear transformation maps the interior of the unit circle centered at the origin of the z-plane to the upper half of the w-plane. Therefore, the Riemann Mapping Theorem asserts that if there exists a univalent function $f(z)$ that maps the unit disk D onto the disk $|w| < 1$, then there exists a univalent function $f(z)$ that maps D onto the upper half of the w-plane.

The Riemann Mapping Theorem is unfortunately non-constructive. But if the simply connected domain D in the z-plane is the interior of a polygon, then the Schwarz-Christoffel transformation supplies a constructive formula for mapping D to the upper half plane of w.

To obtain the unit disk, we can use the bilinear transformation

to map the upper half of the z-plane onto the disk . Before delving into the theorem for the Schwarz-Christoffel transformation, we will go through a simple example to provide insight into SCT.

Consider the interior of an open triangle of angle $\pi\alpha$, with vertex at the origin of the w-plane, is mapped tot he upper half z-plane by $w = f(z) = z^{\alpha}$, for $0 < \alpha < 2$.

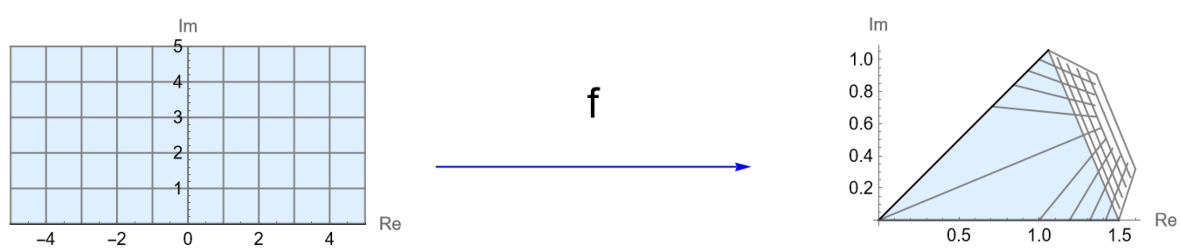

Figure 8: The transformation $w = f(z) = z^{\alpha}$, $\alpha = 1/4$.

Figure 8 a transformation of $f(z) = z^{\alpha}$ with $0 < \alpha < 2$ from the upper half z-plane to the w-plane. If $z = re^{i\theta}$ and $w = pe^{i\varphi}$, then we see that the rays $\theta = 0$ and $\theta = \pi$ of the z-plane are mapped to the rays of $\varphi = 0$ and $\varphi = \pi\alpha$ of the w-plane. Notice that $f(z) = z^\alpha$ is not conformal at $z = 0$ because $f(z)$ is not analytics there when $\alpha \neq 1$.

To get the mapping from the w-plane to the z-plane, we take the inverse of $f(z)$ and get $g(w) = w^{1/\alpha}$. We must restrict $0 < \alpha < 2$ so that the conform map in the z-plane does not overlap itself.

Theorem (Schwarz-Christoffel): Let Γ be the piecewise linear boundary of a polygon in the w-plane, and let the interior angles at successive vertices be $\alpha_1\pi, ..., \alpha_n\pi$. The transformation defined by the equation

$$\frac{dw}{dz} = (\gamma(z - a_1)^{\alpha_1 - 1}(z - a_2)^{\alpha_2 - 1}...(z - a_n)^{\alpha_n - 1}), \tag{2}$$

where γ is a complex number and $a_1, ..., a_n$ are real numbers, maps Γ onto the real axis of the z-plane and the interior of the polygon to the upper half z-plane. The vertices of the polygon $A_1, A_2, ..., A_n$, are mapped to the points $a_1, ..., a_n$ on the real axis. The map is an analytic one-to-one conformal transformation between the upper half z-plane and the interior of the polygon [1].

We can use the above theorem to map the w-plane to the z-plane with $z = g(w)$ or the z-plane to the w-plane with $w = f(z)$.

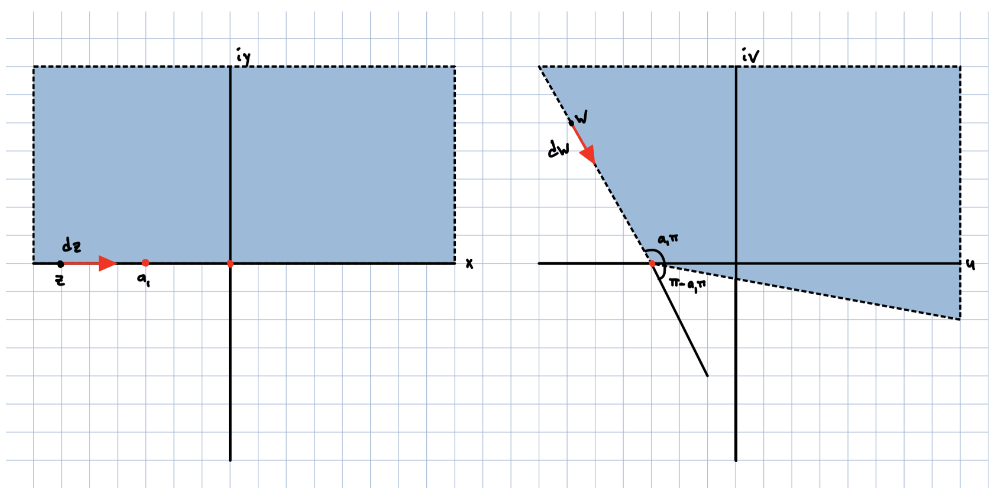

Figure 9: Transformation of upper half z-plane to open polygon with one angle.

Consider Figure 9 which is a transformation of the upper half z-plane, $Im(z) > 0$, to an open polygon with one angle in the z-plane. In this example, we must consider both the map and its inverse. We need to find an analytic function $f(z)$ in the upper half z-plane such that $w = f(z)$ maps the real axis of the z-plane onto the boundary of the polygon. Consider the derivative of the mapping $\frac{dw}{dz} = f'(z)$, also written as $dw = f'(z)dz$. Consider a point w on the polygon to the left of the vertex A_1 and its corresponding point z to the left of a_1 in the z-plane. Think of dw and dz as vectors on these contours. As dz crosses a_1, arg(dw) changes by $\pi - \alpha_1\pi$. As said previously, the goal is to find a function

$w = f(z)$ that causes dw to change by $\pi - \alpha_1\pi$ as dz crosses a_1. Thus, let $dz = re^{i\theta}$ and $dw = pe^{i\phi}$. Then, using the above formula $\frac{dw}{dz} = f'(z)$, we get that $f'(z) = \frac{p}{r}e^{i(\phi-\theta)}$. Moreover, $arg(f'(z)) = arg(dw) - arg(dz)$, so it follows $arg(f'(z)) = \phi - 0 = \phi = arg(dw)$ since $arg(dw) = \theta = 0$ always. So, as dz crosses a_1, we want $arg(f'(z)) = arg(dw)$ to change by $\pi - \alpha_1\pi$. Using the formula from the previous theorem, let $f'(z) = (z - a_1)^{\alpha_1 - 1}$, so $arg(dw)$ changes by $\pi - \alpha_1\pi$ at a_1. To see this, we can let $z - a_1 = re^{i\theta}$ and set $f'(z) = (z - a_1)^{\alpha_1-1} = (re^{i\theta})^{\alpha_1-1} = r^{\alpha_1-1}e^{i\theta(\alpha_1-1)}$. From this, we get that $arg(f'(z)) = \theta(\alpha_1 - 1)$. As we traverse the point $z = a_1$, $arg(z - a_1) = \pi$ if z is real and on the left of a_1 and $arg(z - a_1) = 0$ if z is real and on the right of a_1. Therefore, $arg(f'(z))$ changes by $\pi(1 - \alpha_1)$, which is what we were looking for. We can continue this process similarly to get an open polygon with more angles.

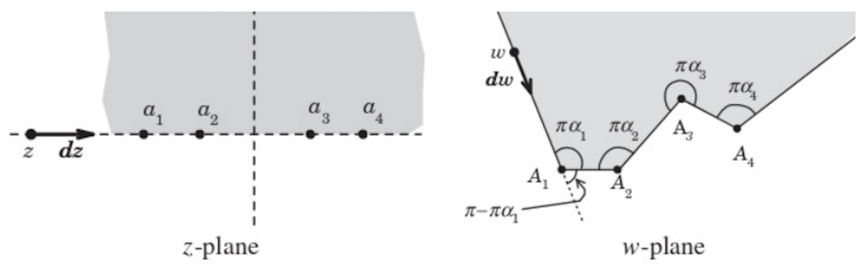

Figure 10: Transformation of upper half z-plane to open polygon with four angles [1].

Consider Figure 10 and let $f'(z) = (z-a_1)^{\alpha_1-1}(z-a_2)^{\alpha_2-1}(z-a_3)^{\alpha_3-1}(z-a_4)^{\alpha_4-1}$ with

$$z - a_1 = r_1e^{i\theta_1}, z - a_2 = r_2e^{i\theta_2}$$
$$z - a_3 = r_3e^{i\theta_3}, z - a_4 = r_4e^{i\theta_4}$$

So,

$$\begin{aligned} f'(z) &= (z-a_1)^{\alpha_1-1}(z-a_2)^{\alpha_2-1}(z-a_3)^{\alpha_3-1}(z-a_4)^{\alpha_4-1} \\ &= (r_1e^{i\theta_1})^{\alpha_1-1}(r_2e^{i\theta_2})^{\alpha_2-1}(r_3e^{i\theta_3})^{\alpha_3-1}(r_4e^{i\theta_4})^{\alpha_4-1} \\ &= r_1^{\alpha_1-1}r_2^{\alpha_2-1}r_3^{\alpha_3-1}r_4^{\alpha_4-1}e^{i(\theta_1(\alpha_1-1)+\theta_2(\alpha_2-1)+\theta_3(\alpha_3-1)+\theta_4(\alpha_4-1))} \end{aligned}$$

Thus, $arg(f'(z)) = (\theta_1(\alpha_1 - 1) + \theta_2(\alpha_2 - 1) + \theta_3(\alpha_3 - 1) + \theta_4(\alpha_4 - 1))$. This means that when we cross a_1 in the z-plane, θ_1 changes by $-\pi$ while θ_2, θ_3 and θ_4 do not change. This gives us that $arg(f'(z))$ changes by $\pi(1 - a_1)$ like the last example. When we cross a_2 in the z-plane, θ_2 changes by $-\pi$ while θ_1, θ_3 and θ_4 do not change, so we get that $arg(f'(z))$ changes by $\pi(1 - a_2)$. This pattern follows for a_3 and a_4.

Although this example broke down the transformation for an open polygon with exactly four angles, this process can be extended to any open polygon with n angles. This is how we get the equation $\frac{dw}{dz} = (\gamma(z-a_1)^{\alpha_1-1}(z-a_2)^{\alpha_2-1}...(z-a_n)^{\alpha_n-1})$ from the Schwarz-Christoffel theorem.

2.4 Jacobian Elliptic Function

Theorem 5 (Schwarz reflection principle): Suppose that $f(z)$ is analytic in a domain D that lies in the upper half of the complex z-plane. Let $\tilde{D}$ denote the domain obtained by D by reflect with respect to the real axis. Then corresponding to every point $z \in \tilde{D}$, the function $\tilde{f}(z) = \overline{f(\tilde{z})}$ is analytic in $\tilde{D}$ [1].

We have shown that the SCT is great for mapping a conformal map of the upper half of the z-plane onto the interior of a simple polygon, but as we increase the number of angles, SCT become increasingly complicated quickly. We can use the Schwarz reflection principle (Theorem 5) to define a family of functions called **Jacobian elliptic functions**.

The incomplete elliptic integral of the first kind is defined as

$$F(z,k) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \tag{3}$$

where the parameter k is the modulus of the elliptic integral. When $z = 1$, we get the complete elliptic integral usually defined as $K(k) = F(1,k)$.

From this, we define the associated elliptic integral K' as

$$\begin{aligned} K'(k) &= \int_1^{1/k} \frac{d\xi}{\sqrt{(\xi^2-1)(1-k^2\xi^2)}} \\ &= \int_1^{1/k} \frac{d\xi}{\sqrt{-(1-\xi^2)(1-k^2\xi^2)}} \\ &= \int_1^{1/k} \frac{d\xi}{i\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \\ iK'(k) &= \int_1^{1/k} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} \end{aligned}$$

where $\xi = (1-k'^2\xi'^2)^{-1/2}$ and $k' = \sqrt{1-k^2}$.

Thus,

$$F(\frac{1}{k},k) = K(k) + iK'(k). \tag{4}$$

2.5 Derivation

We will use SCT to derive the Jacobian elliptic functions [3]. Consider a rectangle with corners ± 1 and $\pm 1 + is$. As seen in Figure 11, vertex $A_1(-1+is)$ corresponds with $a_1(-1/k)$, $A_2(-1)$ corresponds with $a_2(-1)$, $A_3(1)$ corresponds with $a_3(1)$, $A_4(1+is)$ corresponds with $a_4(1/k)$, and $z = 0$ corresponds with $w = 0$. Suppose we want to map this rectangle to the upper half plane. We need to determine the transformation $w = f(z)$ and the constant s as a function of k. Let $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{1}{k}$, $a_1 = -\frac{1}{k}$, $a_2 = -1$, $a_3 = 1$, and $a_4 = \frac{1}{k}$.

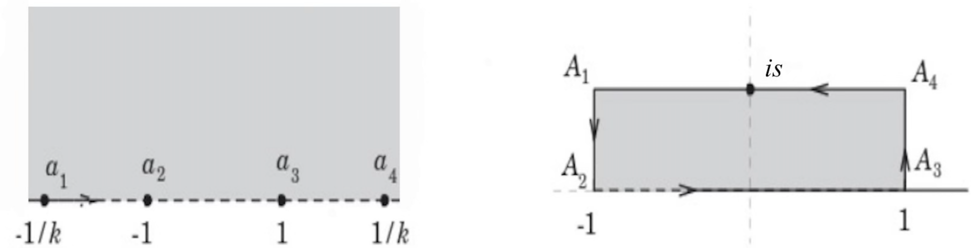

Figure 11: SCT of a rectangle to the upper half z-plane [1].

Then, from Schwarz-Christoffel Theorem, we get

$$\frac{dw}{dz} = (\gamma(z-1)^{-1/2}(z+1)^{-1/2}(z-1/k)^{-1/2}(z+1/k)^{-1/2}). \tag{5}$$

After integration manipulation and changing γ to $\tilde{\gamma}$, we get

$$\begin{aligned} w(z,k) &= \gamma \int_0^z \frac{d\zeta}{\sqrt{(\zeta^2-1)(\zeta^2-\frac{1}{k^2}}} \\ &= \tilde{\gamma} \int_0^z \frac{d\zeta}{\sqrt{(\zeta^2-1)(k^2\zeta^2-1)}} \\ &= \tilde{\gamma} F(z,k) \end{aligned}$$

Using the complete elliptic integral $K(k) = F(1,k) = \int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$, the association of $z = 1$ and $w = 1$ implies that $\tilde{\gamma} = 1/K(k)$.

It can be easily verified that this value for $\tilde{\gamma}$ will map A_2 to a_2. From here, we get the function $f(z)$ and the constant s that maps the upper half plane to the interior of a rectangle:

$$w = \frac{F(z,k)}{K(k)}, s = \frac{K'(k)}{K(k)} \tag{6}$$

The inverse of Equation 6 gives one of the Jacobian elliptic functions:

$$w = \frac{F(z,k)}{K(k)}, z = sn(wK,k) \tag{7}$$

The remaining corners, A_1 and A_4, are fixed by A_2 and A_3 since the SCT ensures that our transformation will be a rectangle. All that is left is to find a function of k and s that describes the mapping of the upper half z-plane to the rectangle in the w-plane. We can plug $w(\frac{1}{k}, k) = 1 + is$ into Equation 6.

$$w(\frac{1}{k}, k) = \frac{1}{K(k)}F(\frac{1}{k}, k)$$
$$1 + is = \frac{1}{K(k)}(K(k) + iK'(k))$$
$$1 + is = 1 + i\frac{K'(k)}{K(k)}$$

$$s = \frac{K'(k)}{K(k)}$$

In conclusion, the function that describes the mapping of the upper half z-plane to the rectangle in the w-plane is

$$w = \frac{F(z,k)}{K(k)}, s = \frac{K'(k)}{K(k)}$$

This function is also univalent by the Schwarz-Christoffel Theorem, so it has an analytic inverse function that maps the rectangle to the upper half plane.

We will introduce a function called the **elliptic sine**. The elliptic sine is derived from the inverse of the incomplete elliptic integral of the first kind. Let $w = F(z, k)$ be the "normalized" function.

$$w = F(z, k), s = K'(k)$$

Then, the elliptic sine is

$$sn(w, k) = F^{-1}(w, k) = z$$

The elliptic sign is one of the Jacobian elliptic functions and can help with visualizing the elliptic integral on the complex plane. The fundamental properties of the elliptic functions are their "double periodicity" and single valuedness.

2.5.1 Double Periodicity

We will show the double periodicity

$$sn(w + n\omega_1 + im\omega_2, k) = sn(w, k) \tag{8}$$

where m and n are integers, $\omega_1 = 4K(k)$, and $\omega_2 = 2K'(k)$.

We can prove this using the Schwarz reflection principle. Suppose that $f(z) = u(x, y) + iv(x, y)$ in a domain D that lies in the upper half z-plane and $\tilde{D}$ is the domain obtained by reflecting D about the real axis to the lower half plane (Schwarz reflection principle).

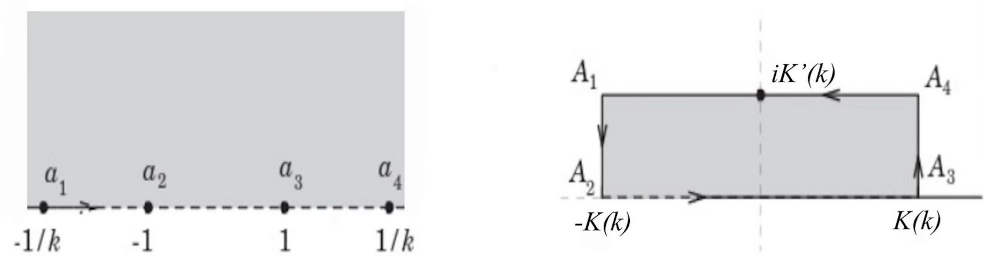

Figure 12: Rectangle scaled by factor of $K(k)$.

Then, the complex conjugate $\overline{f(\tilde{z})} = u(x, -y) - iv(x, -y)$. Given that the Cauchy-Riemann conditions are satisfied for $\tilde{D}$. Therefore, $\overline{f(\tilde{z})}$ is analytic in $\tilde{D}$.

Using the rectangle from the previous example, we will scale it be a factor of $K(k)$ to get Figure 12. We will call the rectangular region R. The function $z = sn(w, k)$ can be analytically continued by the Schwarz reflection principle.

Starting with any point w in R, notice that we can obtain the same point w by either symmetrically reflecting twice about a horizontal side of the rectangle, or twice about a vertical side of the rectangle. This results in the double periodicity relationship.

Suppose we reflect R over the edge connected by A_1 and A_2, giving us a rectangle $\tilde{R}$. By the Schwarz reflection principle, $\overline{w(\tilde{z}, k)}$ is analytic in the lower half plane that maps to $\tilde{R}$. Reflecting R twice over the edge connected by A_1 and A_2, we reflect the lower half plane the maps to $\tilde{R}$ back to the upper half plane. This takes us back where we started and gives the same function $w(w, k)$. This shows that

$$sn(0 + 2(2K(k)) + i(0), k) = sn(0, k)$$

and the periodicity of $4K(k)$ comes from us having to reflect twice around either a horizontal side of the rectangle or a vertical side of the rectangle to get the original mapping of the upper half plane.

Thus, reflecting about a horizontal side of rectangle gives us the periodicity $2K'(k)$ from Equation 8. With this, we get the double periodicity

$$sn(w + 4nK(k) + 2imK'(k), k) = sn(w, k)$$

where m and n are integers.

2.5.2 Single Valuedness

We see that $sn(w, k)$ is single valued as well. Consider a "period rectangle" that consists of any four rectangles meeting at a corner, such as R, R_1, R_3, and R_4 in Figure 13. Two of the rectangle map to the upper half plane while two rectangles map to the lower half plane. So, the "period rectangle" covers the z-plane twice. Thus, there are two values of w that corresponds to a value z for $z = sn(w, k)$.

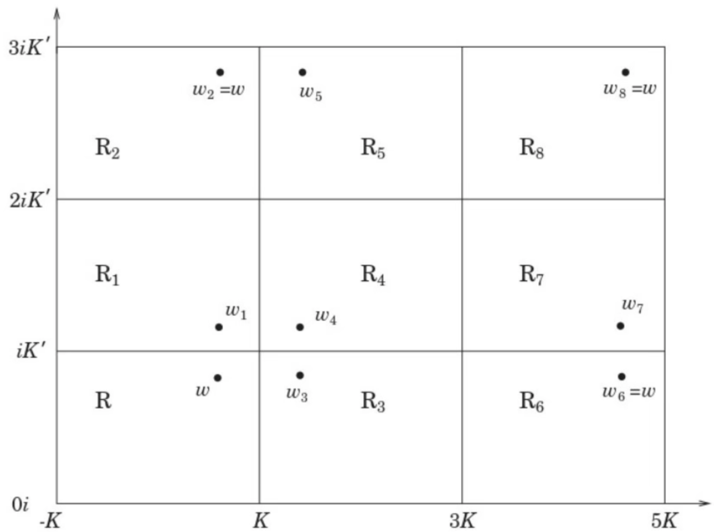

Figure 13: Double periodicity in the w-plane [1].

The zeros of $sn(w, k)$ are $w = 2nK + 2miK'$ where m and n are integers. From the definition of $F(z, k)$, we get that $F(0, k) = 0$. So, if we reflect R to R_1 say, the location of this zero is now $w = 2K'$. Thus, there are two zeroes located in each "period rectangle". Continuing this, we can find $z = -1$ located at $w = -K$, $z = -\frac{1}{k}$ located at $w = -K + iK'$, and $z = \infty$ located at $w = iK'$. This demonstrates the single valuedness of $sn(w, k)$.

2.6 Double periodicity of other polygons

In previous examples, the double periodicity came of the Jacobian elliptic functions came from applying the Schwarz reflection principle to a rectangle in the w-plane. We got the double periodicity from repeatedly reflecting the rectangle to cover the entire w-plane. Similarly, we can do this with other polygons. We will show how an equilateral triangle can similarly be reflected repeatedly to cover the entire w-plane. It is important to note that the only polygons we can do this with are equilateral triangles and hexagons.

In the previous Schwarz-Christoffel tranformations, we have used the upper half of the z-plane. While this is convenient, it is impossible to know where to place the remaining vertices when we have polygons with three or more vertices unless they are symmetric polygons. But SCT can also map polygons from a unit disk, which will increase the symmetry and fix the previous issue.

We will derive the SCT for a unit disk. By the Schwarz-Christoffel theorem, the bilinear transformation maps to the upper half z-plane and a unit disk in the ζ-plane. The chain rule gives

$$\frac{dw}{d\zeta} = \frac{dz}{d\zeta}\frac{dw}{dz}$$

Like Equation 2 where we mapped the interior of an open polygon to the upper half z-plane, the Schwarz-Christoffel formula for mapping the exterior of a closed polygon to the upper half z-plane is given by

$$\frac{dw}{d\zeta} = \gamma(\zeta - \zeta_1)^{1-\alpha_1}(\zeta - \zeta_2)^{1-\alpha_2}...(\zeta - \zeta_n)^{1-\alpha_n}) \tag{9}$$

where the points $\zeta_k = e^{2\pi ik/n}$, $k = 1, 2, ..., n$ lie on the unit circle.

From this, if we integrate $\frac{dw}{d\zeta}$ from 0 to ζ, we get [2]

$$w(\zeta) = A + \gamma \int_0^\zeta \frac{d\zeta'}{(\zeta_k - 1)^{2/n}}$$

This maps a unit disk to a polygon with n sides, interior angles α_k, and vertices $f(\zeta_k)$.

2.6.1 SCT of an equilateral triangle and covering the entire w-plane

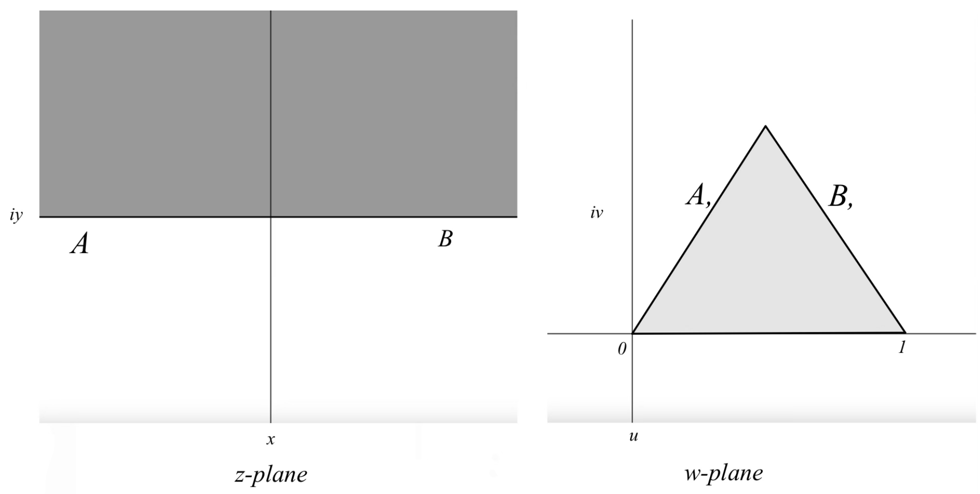

Figure 14: SCT of upper half plane to the interior of an equilateral triangle.

We will use SCT to construct a mapping from the upper half z-plane to the interior of an equilateral triangle shown in Figure 14.

The region bounded by the equilateral triangle having two vertices at 0 and 1 will have the third vertex at $\frac{1+i\sqrt{3}}{2}$. So, we have a region with interior angles $\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{3}$.

By the Schwarz-Cristoffel theorem and vertices $w_1 = 0$ and $w_2 = 1$, we have

$$\begin{aligned} w'(\zeta) &= \gamma(\zeta - 0)^{-2/3}(\zeta - 1)^{-2/3} \\ &= \gamma\zeta^{-2/3}(\zeta - 1)^{-2/3} \end{aligned}$$

In the given region $z \in \mathbb{C}, Im(z) > 0$, the function $w'(\zeta)$ is analytic. Thus,

$$w(\zeta) = \gamma \int_0^\zeta \frac{d\epsilon}{\epsilon^{2/3}(\epsilon - 1)^{2/3}} + A$$

where γ and A are complex constants.

Now,

$$w(0) = \gamma \int_0^0 \frac{d\epsilon}{\epsilon^{2/3}(\epsilon - 1)^{2/3}} + A = 0$$

implies that $A = f(0)$, which then leads to $A = 0$.
Then,

$$w(1) = \gamma \int_0^1 \frac{d\epsilon}{\epsilon^{2/3}(\epsilon - 1)^{2/3}} = 1.$$

Thus,

$$\gamma = \frac{1}{r}, r = \int_0^1 \frac{d\epsilon}{\epsilon^{2/3}(\epsilon - 1)^{2/3}}.$$

So,

$$w(\zeta) = \frac{1}{r} \int_0^1 \frac{d\epsilon}{\epsilon^{2/3}(\epsilon - 1)^{2/3}}.$$

As said earlier, we can use the equilateral triangle to cover the entire w-plane without any gaps. We can do this by repeatedly reflecting the triangle over its sides. When we do this, the points of 6 triangles meet at each vertex. So, when we reflect the triangle 6 times around itself, we end up back where we started. Thus, the inverse of the function that maps the equilateral triangle is single-valued.

We can also find the periodicity of the function. Referring back to Figure 14, we can see that the triangle can be reflected either over the real axis or along one of its other sides. The angle between the two sides and the real axis is $\frac{\pi}{3}$. Using the Pythagorean theorem, we find that our reflected triangle will be translated a distance of $\sqrt{3}$ from the original triangle. Therefore, the inverse function is has double periodicity

$$\zeta(w + n\sqrt{3} + m\sqrt{3}e^{i\pi/3}) = \zeta(w)$$

where m and n are integers.

It is important to note that the above does not return every reflected triangle. There is also a factor of rotation. The angle of rotation is $e^{2\pi i/3}$, which results in

$$\zeta(e^{l\frac{2\pi i}{3}} w + n\sqrt{3} + m\sqrt{3}e^{i\pi/3}) = \zeta(w)$$

where l, m, and n are integers.

2.7 Application

Conformal mapping can be applied to potential flow problems by allowing us to transform the flow past a complicated shape to a flow past a more simple shape. This approach can convert a more involved problem into one with a solution that is easier to find [4].

For example, say we are attempting to solve for potential flow around a cylinder, given by the equations:

$$\Phi = \phi(x, y) + i\psi(x, y)$$
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$
$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

We can map this cylinder in the z-plane to a plate in the w-plane using the transform $w = z + \frac{R^2}{z}$, where R is the radius of the cylinder. The process for this transformation is outlined in section 2.2 above.

The equation for potential flow around a plate is given by:

$$\Phi' = Aw \qquad \text{where A is a constant}$$

thus:

$$\Phi = A(z + \frac{R^2}{z})$$

Using this mapping we can go back to find expressions for the velocity potential and stream function.

$$\begin{aligned}
\Phi &= Az + \frac{AR^2}{z} \\
&= Are^{i\theta} + \frac{AR^2}{re^{i\theta}} \\
&= Ar(\cos(\theta) + i\sin(\theta)) + \frac{AR^2}{r}(\cos(\theta) - i\sin(\theta)) \\
&= (Ar + \frac{AR^2}{r})\cos(\theta) + i(Ar - \frac{AR^2}{r})\sin(\theta) \\
&\implies \phi = (Ar + \frac{AR^2}{r})\cos(\theta) \text{ and } \psi = (Ar - \frac{AR^2}{r})\sin(\theta)
\end{aligned}$$

We then find the velocity field components:

$$v_r = \frac{\partial \phi}{\partial r}$$
$$= (A + \frac{AR^2}{r^2}) \cos(\theta)$$

$$v_\theta = \frac{\partial \phi}{r \partial \theta}$$
$$= -(A + \frac{AR^2}{r^2}) \sin(\theta)$$

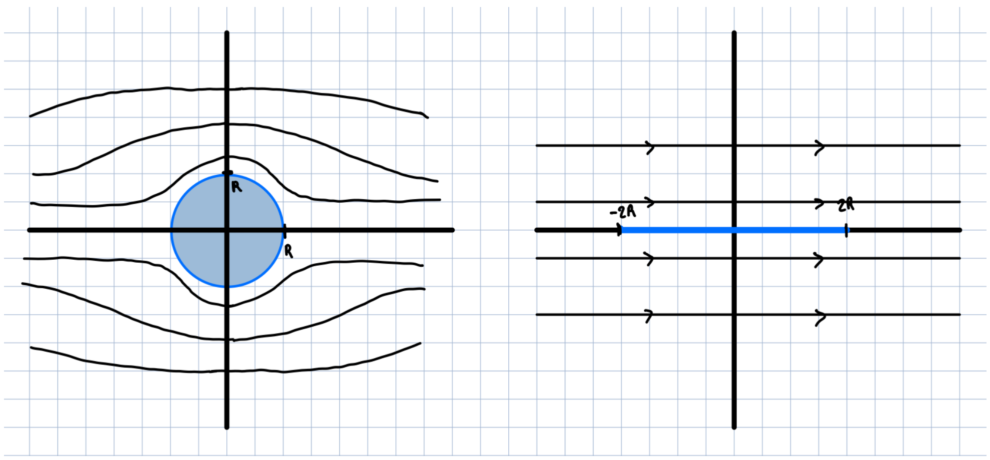

Figure 15: Potential flow over a cyliner (left) and over a flat plate (right)

3 Conclusion

Conformal maps provide both extreme utility in mathematical applications as well as elegance in their underlying mathematical theory. The foundational property of conformal mappings, angle preservation, allows for solutions of problems in complicated domains to be solved in much simpler ones. In this paper, we examined their use in modeling fluid dynamics using Laplace's equation, but the techniques presented can be easily extended to problems in other fields, such as aerodynamics and electrostatics.

While the Riemann Mapping theorem demonstrates the existence of conformal maps from any simply connected domain to the unit disk, the main conformal map discussed in this paper, the Schwarz-Christoffel transformation, provides a concrete method for finding these mappings for polygonal domains. Furthermore, the use of Jacobian elliptic functions allows one to simplify the Schwarz-Christoffel transformation even further, with their properties of double periodicity and single valuedness making them interesting in their own right.

One possible future extension to our discussion on conformal mapping would be a discussion on the numerical methods used for producing such maps. Defining a conformal map using the Schwarz-Christoffel transformation requires integrating over a contour, for which an exact solution may not exist; instead, we must resort to the use of numerical methods. Given that most numerical integration methods have inherent error in them, it would be interesting to find error bounds and orders for different polygons that can be mapped by the Schwarz-Christoffel transformation.

Another topic our group would like to research more given time is applying conformal mapping to solve for potential flow on more complex shapes rather than just a cylinder. A similar method to the one we used can be utilized to map air foil geometries to simpler shape. This application could be used to find ideal shapes to maximize lift in aircraft dynamics.

References

[1] Mark J. Ablowitz and Anthanassios S. Fokas. *Introduction to Complex Variables and Applications*. Cambridge University Press, 1st edition, 2021.

[2] Steve Cheng. Schwarz-christoffel transformation (circular version). 2013.

[3] W. H. Hendriks. The Schwarz-christoffel transformation and elliptic functions, 2009.

[4] Nitin Kapania, Katherine Terracciano, and Shannon Taylor. Modeling the fluid flow around airfoils using conformal mapping. *SIAM Undergraduate Research Online*, 1(2):70–99, Jan 2008.