Predicting Drum Vibrations Using the Polar Wave Equation

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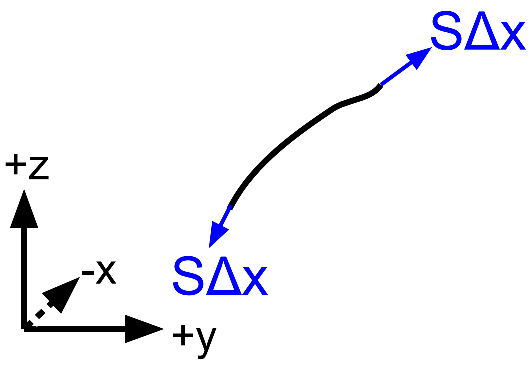
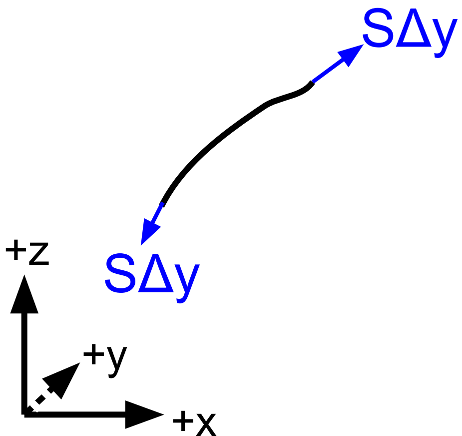
Deriving the (2 + 1)D Wave Equation



$$F_y = S\Delta x\left(\frac{\partial z}{\partial y}(y+\Delta y) - \frac{\partial z}{\partial y}(y)\right)$$

$$= S\Delta x\Delta y\left(\frac{\frac{\partial z}{\partial y}(y+\Delta y) - \frac{\partial z}{\partial y}(y)}{\Delta y}\right)$$

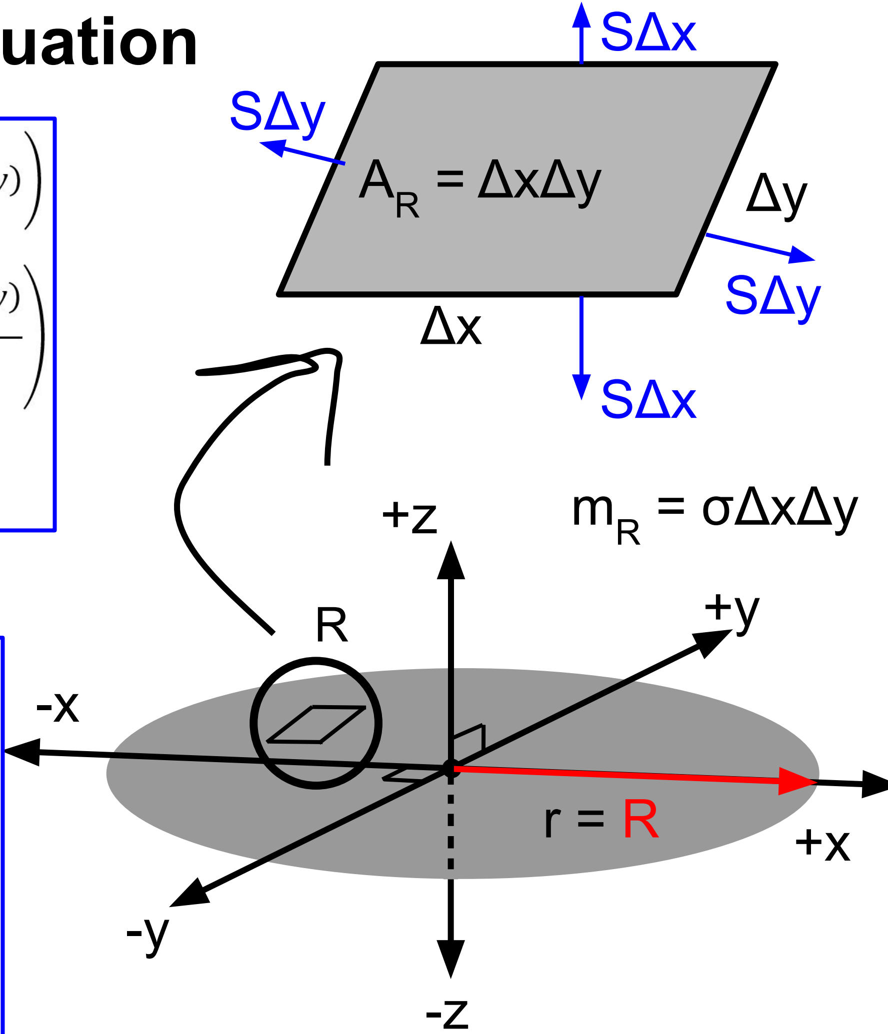
$$= S\Delta x\Delta y\frac{\partial^2 z}{\partial x^2}$$



$$F_x = S\Delta y\left(\frac{\partial z}{\partial x}(x+\Delta x) - \frac{\partial z}{\partial x}(x)\right)$$

$$= S\Delta y\Delta x\left(\frac{\frac{\partial z}{\partial x}(x+\Delta x) - \frac{\partial z}{\partial x}(x)}{\Delta x}\right)$$

$$= S\Delta y\Delta x\frac{\partial^2 z}{\partial x^2}$$



Deriving the (2 + 1)D Wave Equation

$$ma = F_{net}$$

$$(\sigma \Delta x \Delta y)\frac{\partial^2 z}{\partial t^2} = F_x + F_y$$

$$(\sigma \Delta x \Delta y)\frac{\partial^2 z}{\partial t^2} = S \Delta y \Delta x \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{S}{\sigma}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z(x, y, t), c = \sqrt{\frac{S}{\sigma}}$$

Assumptions:

- Density is constant across the drum head
- Only forces acting on rectangular patches are tension forces, and tension is constant
- Points in membrane only move in transverse direction up and down, and not parallel to the xy-plane

Physical Representation

- Telegraph Equation
- Can be used to account for friction and other damping forces on the drum
- Does not change spatial solution, only time

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z - \alpha \frac{\partial z}{\partial t}, \quad \alpha > 0$$

Solutions

Undamped

$$z_{m,n}(r,\theta,t) \;=\; J_m\left(\sqrt{\lambda_{m,n}}r\right)\cdot\begin{cases}\cos\left(\sqrt{\mu_m}\theta\right)\\ \sin\left(\sqrt{\mu_m}\theta\right)\end{cases}\cdot\begin{cases}\cos\left(c\sqrt{\lambda_{mn}}t\right)\\ \sin\left(c\sqrt{\lambda_{mn}}t\right)\end{cases}$$

Damped

$$z_{m,n}(r,\theta,t) \;=\; J_m\left(\sqrt{\lambda_{m,n}}r\right)\cdot\begin{cases}\cos\left(\sqrt{\mu_m}\theta\right)\\ \sin\left(\sqrt{\mu_m}\theta\right)\end{cases}\cdot\begin{cases}e^{\frac{-\alpha+\sqrt{\alpha^2-4\lambda c^2}}{2}t}\\ e^{\frac{-\alpha-\sqrt{\alpha^2-4\lambda c^2}}{2}t}\end{cases}$$

.

for $\alpha > 2c\sqrt{\lambda}$

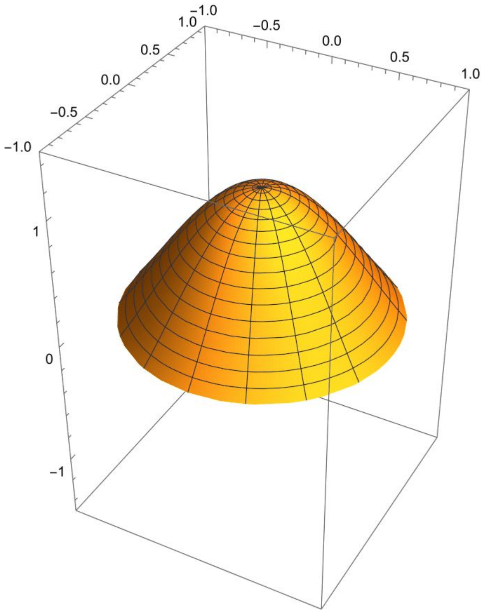
$$z_{m,n}(r,\theta,t) \;=\; J_m\left(\sqrt{\lambda_{m,n}}r\right)\cdot\begin{cases}\cos\left(\sqrt{\mu_m}\theta\right)\\ \sin\left(\sqrt{\mu_m}\theta\right)\end{cases}\cdot\begin{cases}e^{-\alpha t/2}\\ te^{-\alpha t/2}\end{cases}$$

.

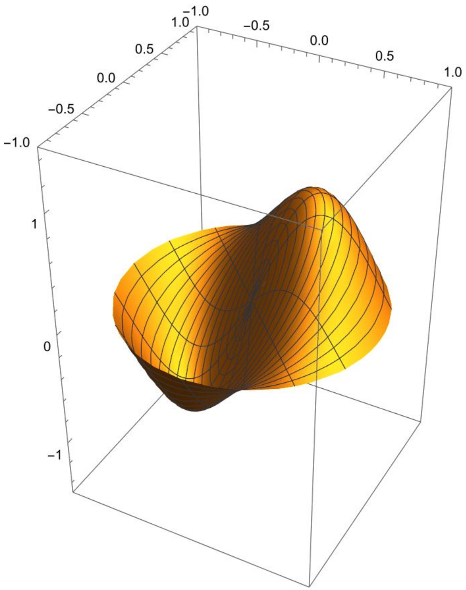
for $\alpha = 2c\sqrt{\lambda}$

$$z_{m,n}(r,\theta,t) = J_m\left(\sqrt{\lambda_{m,n}}r\right)\cdot\begin{cases}\cos\left(\sqrt{\mu_m}\theta\right)\\ \sin\left(\sqrt{\mu_m}\theta\right)\end{cases}\cdot\begin{cases}e^{-\alpha t/2}\cos\left(\alpha^2-4\lambda c^2\right)t\\ e^{-\alpha t/2}\sin\left(\alpha^2-4\lambda c^2\right)t\end{cases}$$

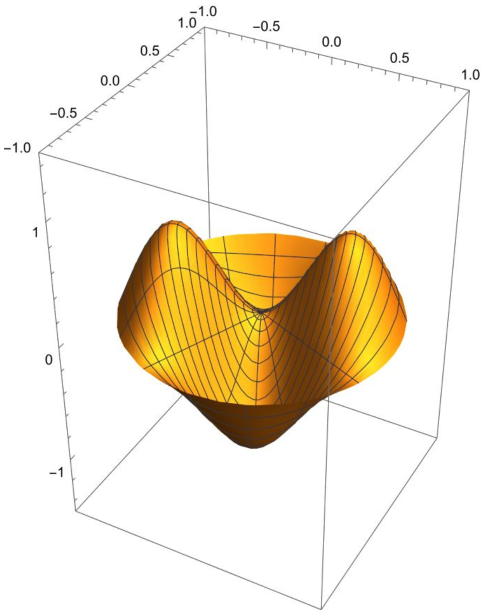
Undamped Modes



Mode (0, 1)

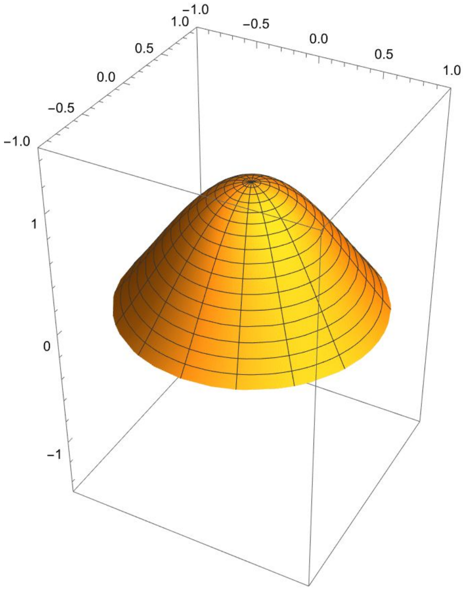


Mode (1, 1)

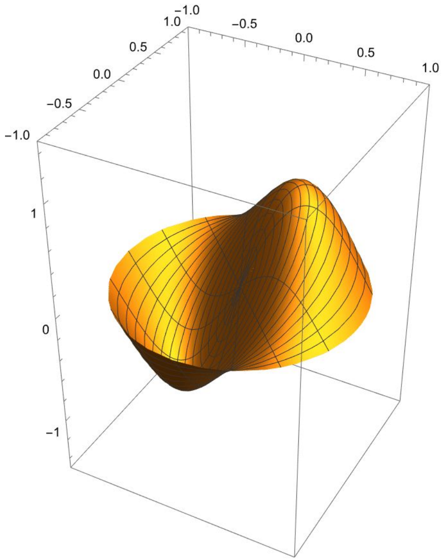


Mode (2, 1)

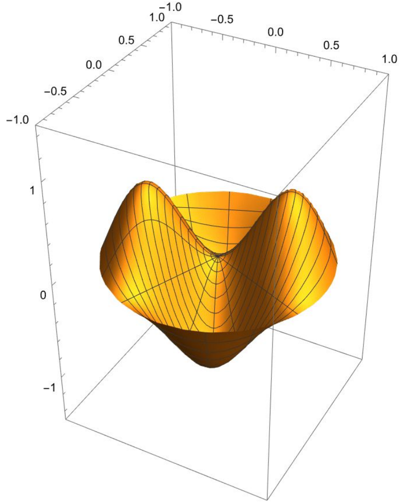
Damped Modes



Mode (0, 1)

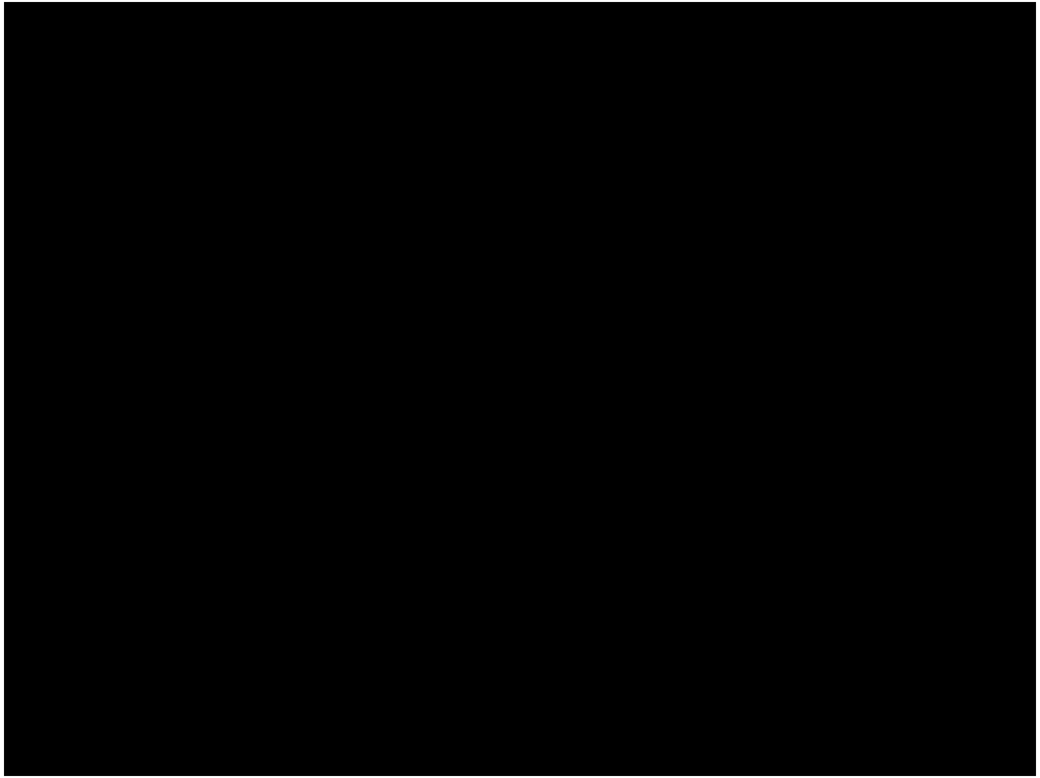


Mode (1, 1)



Mode (2, 1)

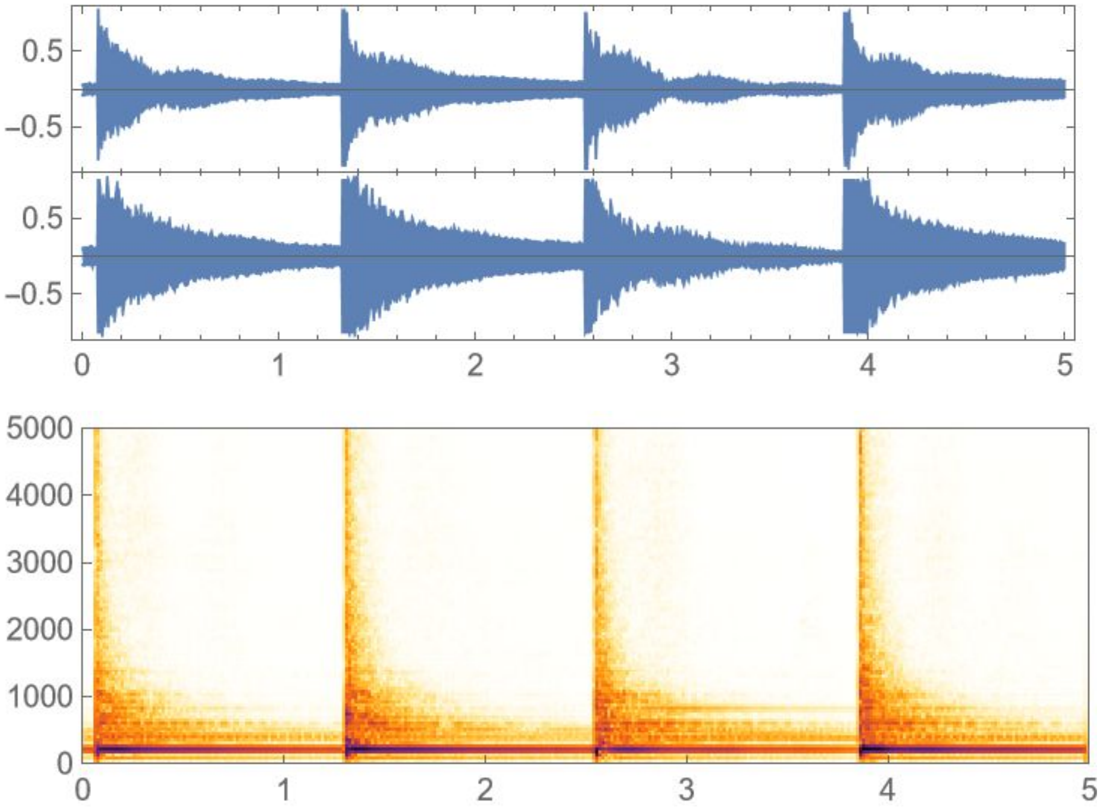
Comparison to Real World



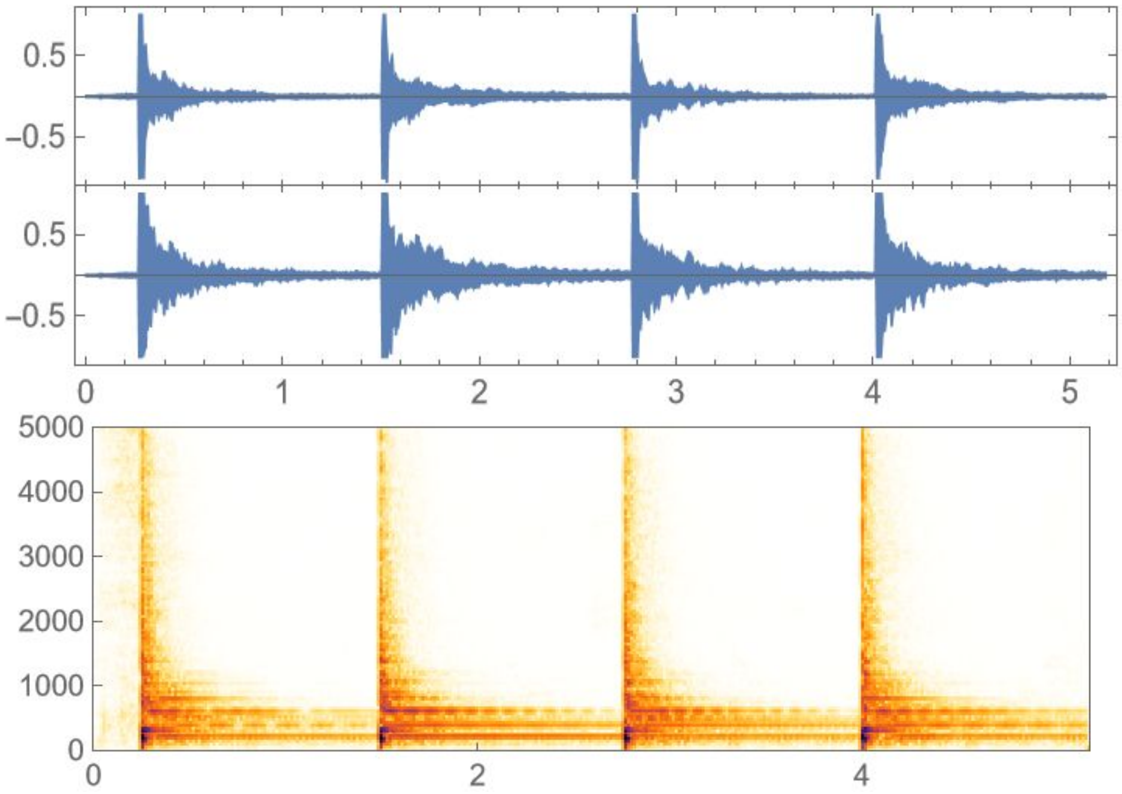
Experimental results

- Used 6 inch radius floor tom
- Polyethylene Terephthalate drum head with density around 1.38 g/cm^3
- Tensioned with roughly 16 N/cm of force

Spectrogram and Waveform - Center hit



Spectrogram and Waveform - Off-center hit



Conclusion

- Striking the drum in the center results in lower frequencies being more pronounced
- When struck off center, frequencies are more varied, resulting in a fuller sound
- The frequency of individual drum modes is not dependent on where or how hard the drum is struck
  - Location and velocity of the drum strike still result in a different sound
  - This is due to changes in amplitude and a difference of overlapping frequencies
- The tension and material of the drum head determine the resulting frequency

Further goals

- Construct a more accurate model for initial conditions, primarily for strikes that are off center on the drum head
- Take measurements for assumed constants in experimental tests
  - Measure drum tension rather than just use an assumed average