With $0 < k < n$ and $k, n \in \mathbb{Z}$, we prove the following lemmas

Lemma 1

$$\prod_{i=k}^{n} i = \frac{n!}{(k-1)!}$$

Proof:

$$\prod_{i=k}^{n} i = \prod_{i=k}^{n} i \frac{(k-1)!}{(k-1)!} = \prod_{i=k}^{n} i \frac{\prod_{i=1}^{k-1} i}{(k-1)!} = \frac{n!}{(k-1)!}$$

Lemma 2

$$\prod_{i=0}^{k} (n-i) = \frac{n!}{(n-k-1)!}$$

Proof:

$$\prod_{i=0}^{k} n-i = (n)(n-1)\cdots(n-k) = \prod_{i=(n-k)}^{n} i$$

And by using lemma 1 we get

$$\prod_{i=(n-k)}^{n} i = \frac{n!}{(n-k-1)!}$$

Problem Statement

We are given a set of n nodes in a valid binary search tree B, with unique keys from a (finite) set of contiguous integers S.
We then pick another key $x \in S$, and begin searching for the key using a linear BST algorithm.[1] How does $p(x \in B)$ change with each node traversed?

Calculating $p(x \in B)_0$

Before beginning the search, ie at time $t = 0$, we calculate $p(x \in B)_0$ as the probability that x is chosen in n tries from a set of $|S|$ items.
The probability of picking an element that is not x on the 1st try is $\frac{1}{|S|}$. On the next try, it will be $\frac{1}{|S|-2}$, accounting for the missing element, and so on until we have the final probability as $\frac{1}{|S|-n+1}$.
We can calculate the probability of picking x by,

$$p(x \in B)_0 = \prod_{i=0}^{n-1} \frac{1}{|S|-i} = \frac{1}{\prod_{i=0}^{n-1} |S|-i} = \frac{(|S|-(n-1)-1)!}{|S|!}$$

[1] Is 'a linear BST' algorithm guaranteed to traverse nodes in a certain order? Can we prove that the linear BST algorithm is unique given certain constraints? (eg "nodes are traversed in same order")

$$= \frac{(|S|-n)!}{|S|!}$$

Finally we can confirm:

$$1 - \frac{|S|!}{(|S|-n-1)!} = \frac{(|S|-n-1)! + |S|!}{(|S|-n-1)!}$$

A different method for calculating the initial probability

There are $\binom{|S|-1}{n-1}$ ways to pick x, and then $n-1$ more elements. There are, in total, $\binom{|S|}{n}$ ways to pick n elements. Thus,

$$p(x \in B)_0 = \frac{\binom{|S|-1}{n-1}}{\binom{|S|}{n}} = \frac{(|S|-1)!}{(n-1)!(|S|-1-(n-1))!} \div \frac{|S|!}{n!(|S|-n)!}$$

$$= \frac{(|S|-1)!}{n!(|S|-1-n+1)!} \times \frac{n!(|S|-n)!}{|S|!} = \frac{(|S|-1)!}{(|S|-n)!} \times \frac{(|S|-n)!}{|S|!}$$

$$= \frac{(|S|-1)!(|S|-n)!}{(|S|-1-n)!|S|!}$$

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The probability of not picking X is thus

$$p(x \notin B)_0 = \prod_{i=0}^{n-1} \left[\frac{|S|-(i+1)}{|S|-i} \right] = \frac{\prod_{i=0}^{n-1} |S|-(i+1)}{\prod_{i=0}^{n-1} |S|-i} = \frac{\prod_{i=0}^{n-1} (|S|-1)-i}{\prod_{i=0}^{n-1} |S|-i}$$

$$= \frac{(|S|-1)!}{((|S|-1)-n-1)!} \frac{(|S|-n-1)!}{|S|!}$$

$$= \frac{(|S|-1)!(|S|-n-1)!}{(|S|-n-2)!|S|!}$$

and through similar reasoning we can determine