

* Ray tracing

- it allows for the fluid simulation of lighting effects.
 - an algorithm emits rays in the form of a 3D graphic, traces the rays' path and then calculates a realistic lighting model.

* Physically Based Rendering (Forward Ray Tracing)

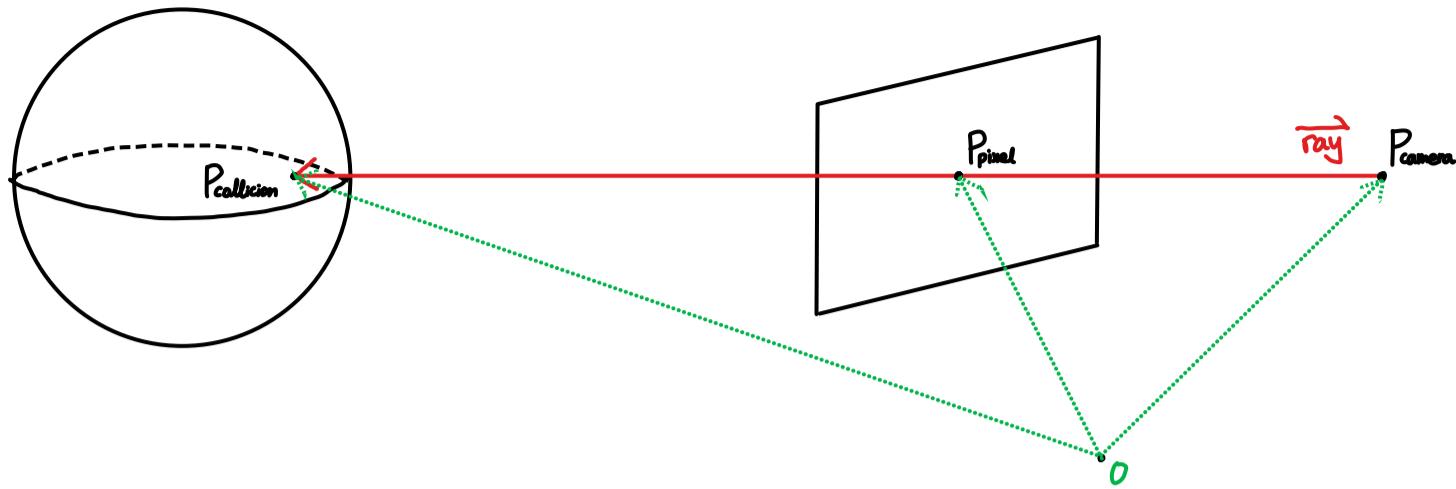
- it follows the light particles (photons) from the light source to the object.
 - it can most accurately determine the coloring of each object, but it is highly inefficient.
 - (many rays from the light source never come through the viewplane and into the eye.)



* Backward Ray Tracing

- an eye ray is created at the eye, it passes through viewplane and on to the world.
 - the first object the eye ray hits is the object that will be visible from that point of the viewplane.
 - after the ray tracer allows that light ray to bounce around,
 - it figures out the exact coloring and shading of that point in the viewplane
 - and displays it on corresponding pixel on the computer monitor screen.
 - it assumes only the right rays that come through the viewplane and on into the eye contribute to the final image of the scene.
 - when an object is transparent or shiny, an error occurs : mix forward and backward ray tracing.

* Ray-Object Intersection



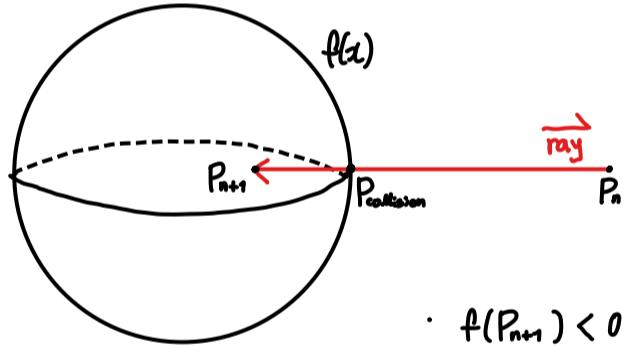
- direction of the pixel ray (\vec{d}_{ray})

$$\vec{d}_{ray} = \frac{\overrightarrow{P_{pixel} - P_{camera}}}{|\overrightarrow{P_{pixel} - P_{camera}}|} \quad \left(= \frac{\overrightarrow{OP_{pixel}} - \overrightarrow{OP_{camera}}}{|\overrightarrow{OP_{pixel}} - \overrightarrow{OP_{camera}}|} \right)$$

- equation of the pixel ray

$$\vec{r}_{pixel} = \overrightarrow{P_{pixel}} + t \cdot \vec{d}_{ray} \quad (= \vec{ray})$$

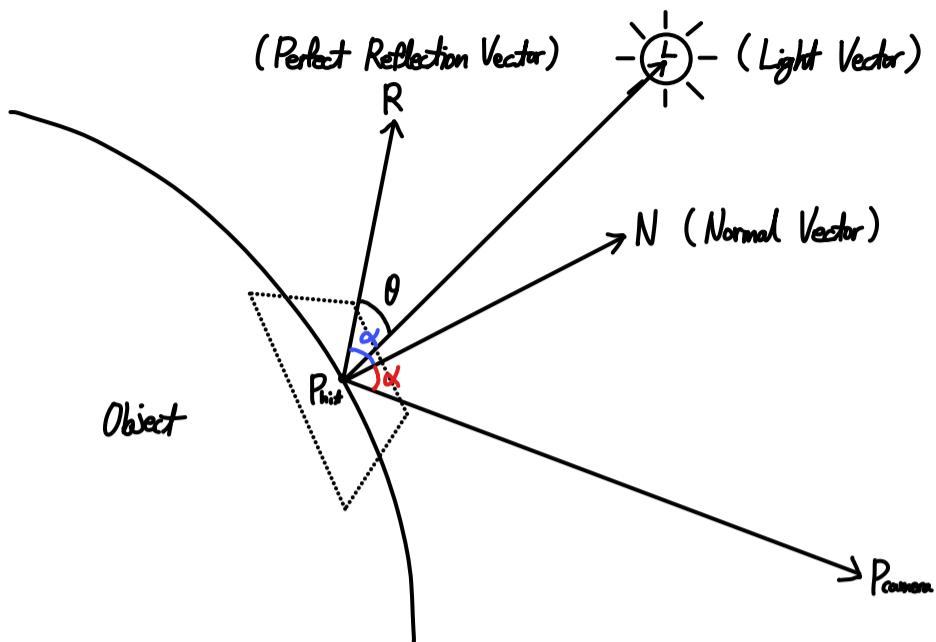
- ray-object intersection



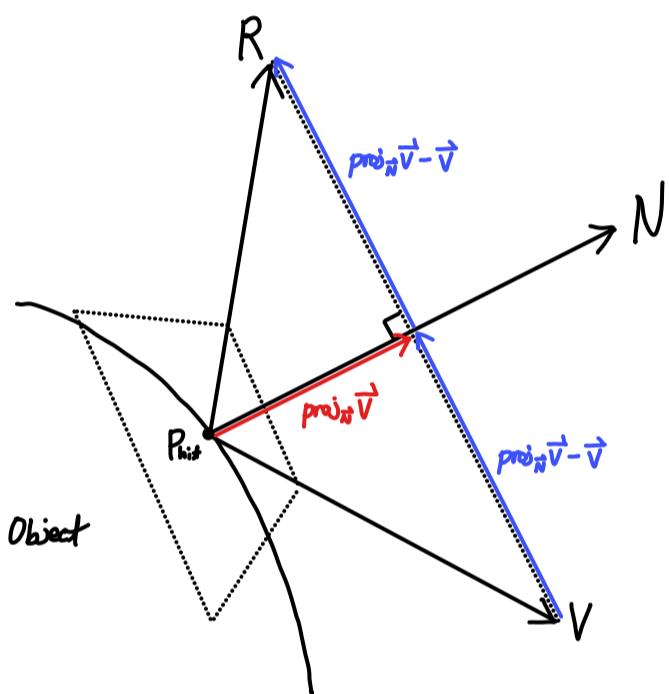
- $f(P_{n+1}) < 0$
- $f(P_{collision}) = 0$
- $f(P_n) > 0$

* Phong Reflection Model

- Perfect Reflection

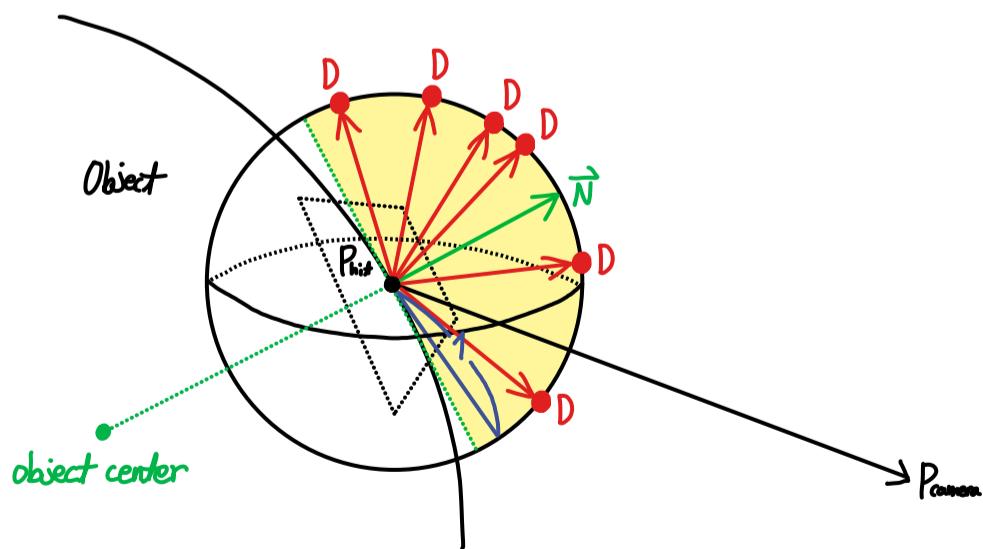


- More light energy will be reflected to $\overrightarrow{P_{\text{hit}} P_{\text{camera}}}$ direction when \vec{L} and \vec{R} are closer to each other.



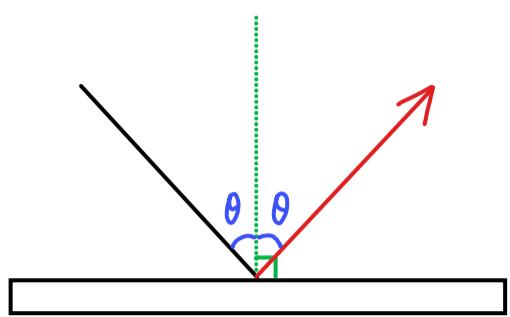
$$\begin{aligned} \vec{R} &= 2(\text{proj}_{\vec{N}} \vec{V} - \vec{V}) \\ &= 2 \left(\frac{\vec{N} \cdot \vec{V}}{\vec{N} \cdot \vec{N}} \cdot \vec{N} - \vec{V} \right) \\ &\quad \boxed{\therefore \text{proj}_{\vec{N}} \vec{V} = t \vec{N}} \end{aligned}$$

- Diffuse Reflection

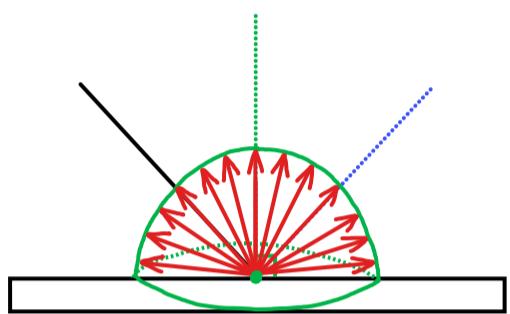


• perfect, diffuse, specular reflection

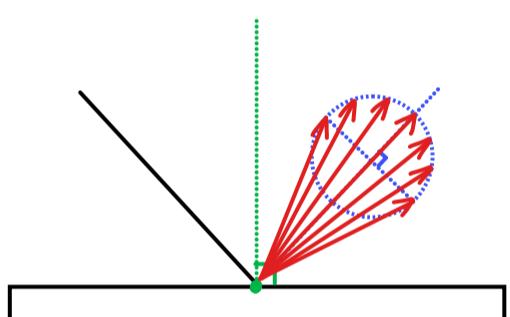
- perfect reflection → metallic surface



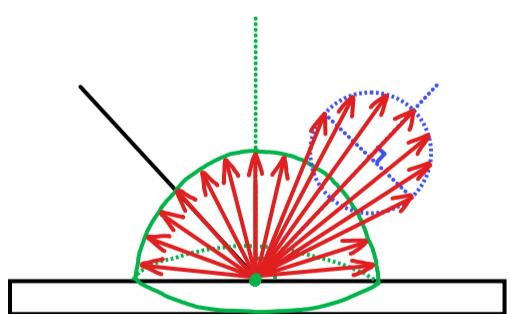
- diffuse reflection



- specular reflection



- diffuse and specular reflection

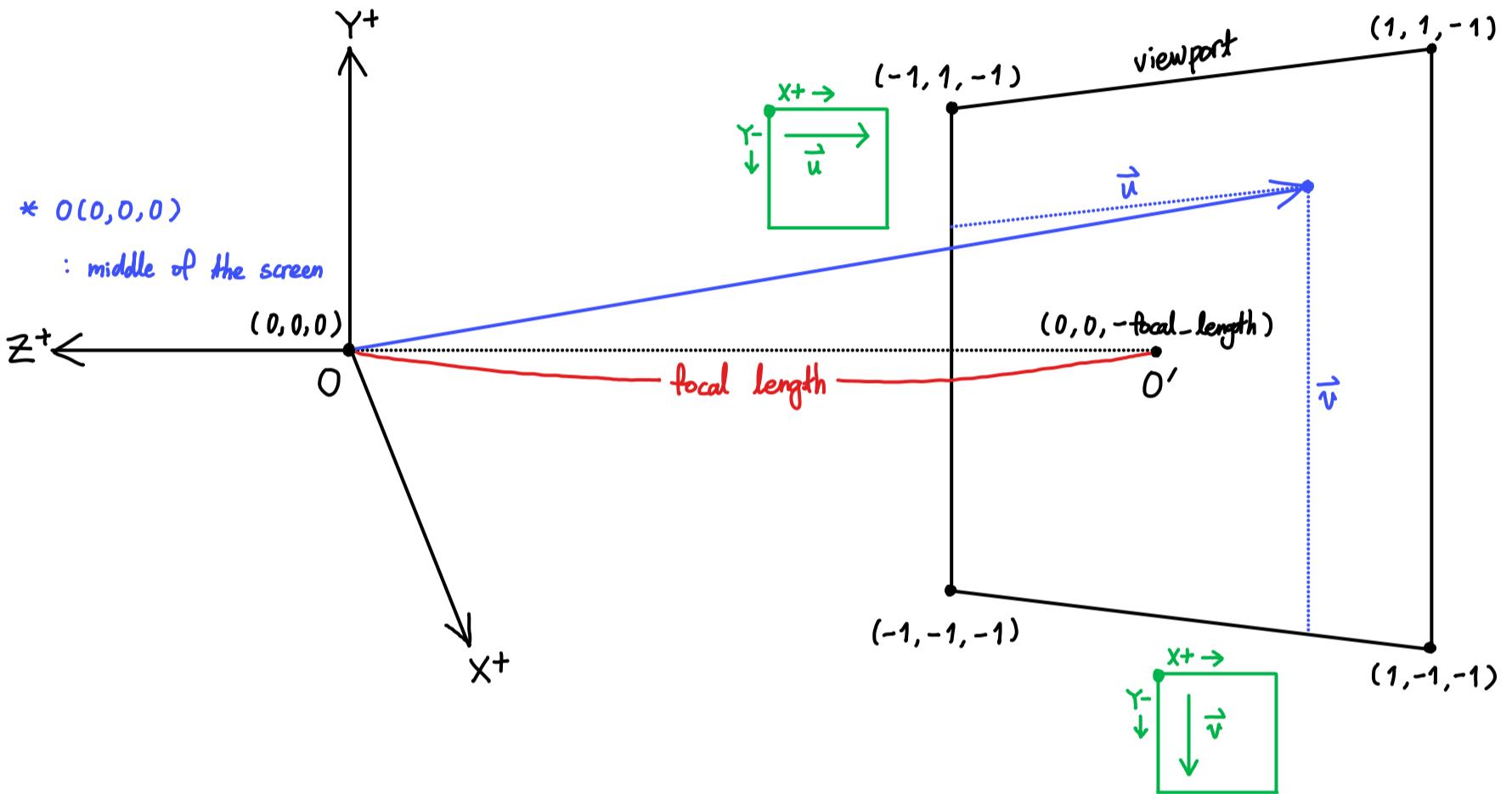


* Ray Tracer

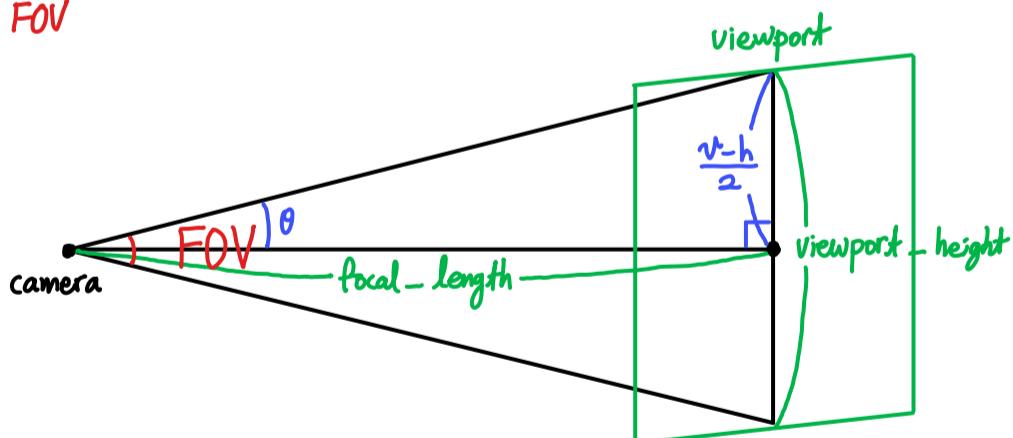
: sends ray through pixel, and calculates color corresponding pixel.

- step 1. calculate distance of camera to object
- step 2. recognize the hit point (on the object)
- step 3. calculate the color
- step 4. apply a light

• setting viewport



• FOV

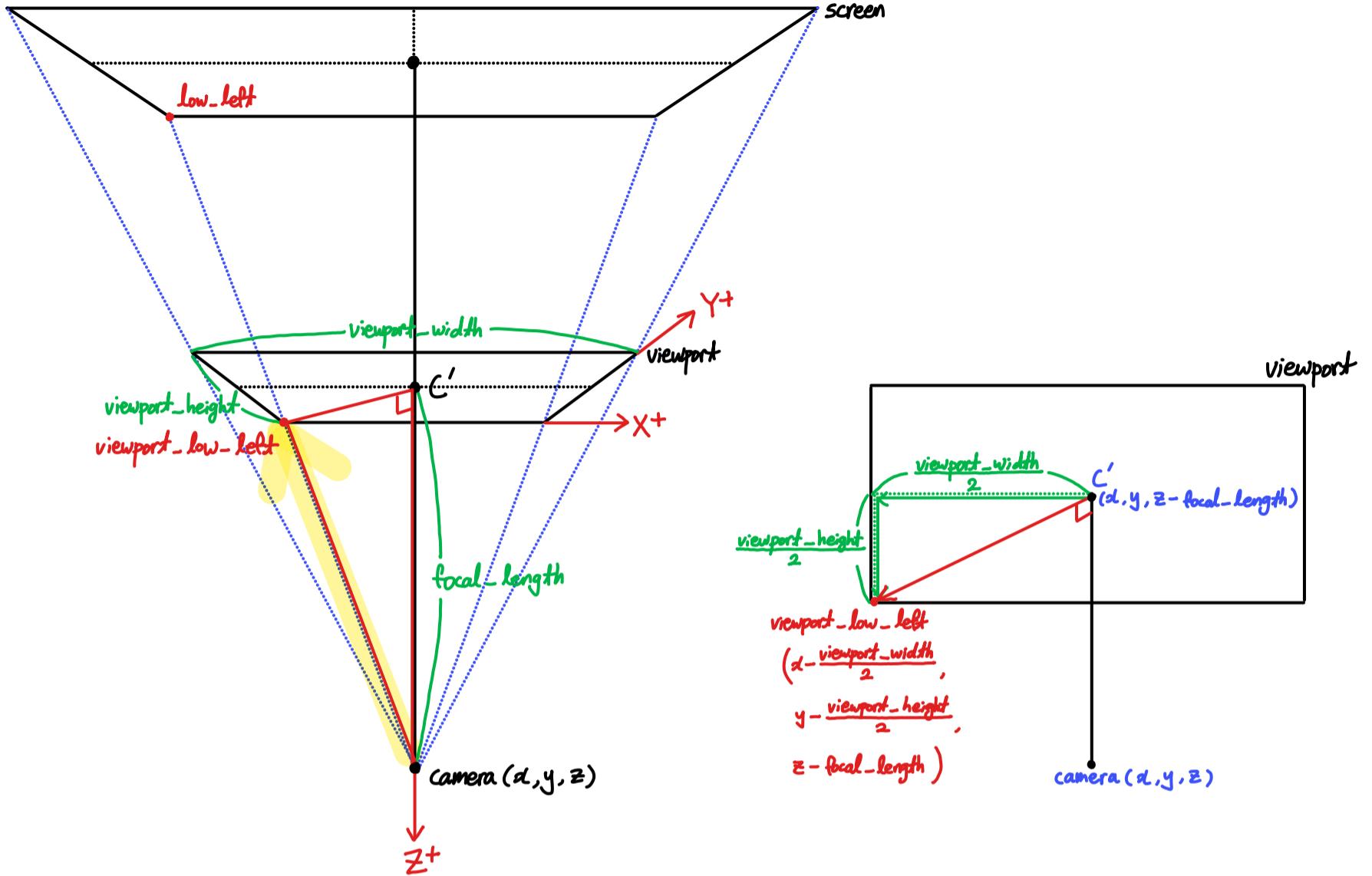


$$\tan \theta = \frac{\frac{\text{viewport_height}}{2}}{\text{focal_length}} \rightarrow \text{viewport_height} = 2 \times \text{focal_length} \times \tan \theta$$

$$\text{also, } \theta (\text{rad}) = \frac{\text{FOV}(\text{°})}{2} = \frac{\pi}{360} \times \text{FOV}$$

$$\therefore \text{viewport_height} = 2 \times \text{focal_length} \times \tan\left(\frac{\pi}{360} \times \text{FOV}\right)$$

• setting ray



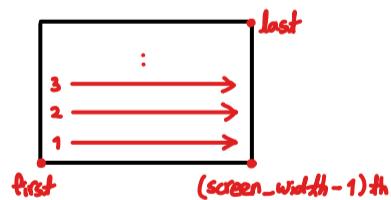
• viewport_low_left : a direction of the first ray.

• ray

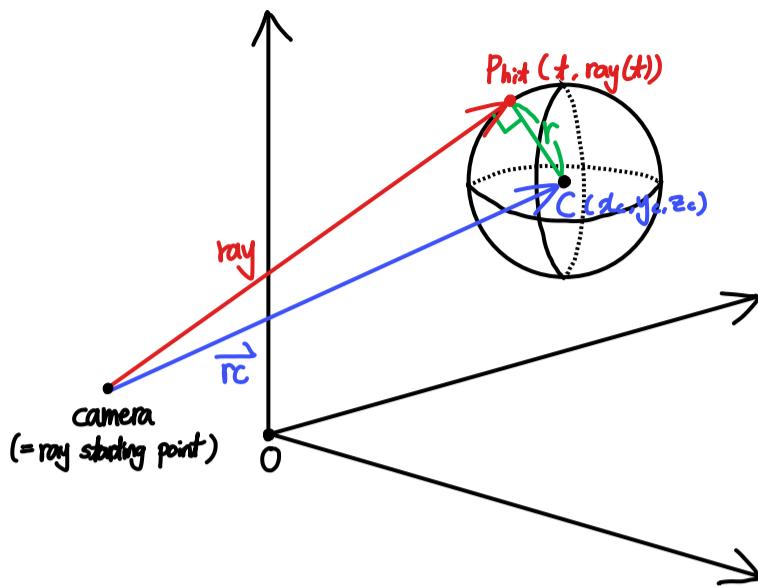
- start point : the position of camera (d, y, z)

- direction : $P(\text{viewport_low_left} + \text{viewport_width} \times \vec{u} + \text{viewport_height} \times \vec{v}) - P_{\text{camera}}(d, y, z)$

$$\left(\vec{u} = \frac{i}{\text{screen_width}-1}, \vec{v} = \frac{j}{\text{screen_height}-1}, 0 \leq i \leq \text{screen_width}-1, 0 \leq j \leq \text{screen_height}-1 \right)$$



• ray - sphere intersection



* equation of sphere

(center: $C(x_c, y_c, z_c)$, radius: r)

- $(x-x_c)^2 + (y-y_c)^2 + (z-z_c)^2 = r^2$
- $|\vec{p} - \vec{oc}| = r$
- $|\vec{p} - \vec{oc}| \cdot |\vec{p} - \vec{oc}| = r^2$

• $|\vec{ray}(t) - \vec{rc}| = r$

$$\rightarrow |\vec{ray}(t) - \vec{rc}| \cdot |\vec{ray}(t) - \vec{rc}| = r^2$$

$$\rightarrow \vec{ray}(t)^2 - 2 \cdot \vec{rc} \cdot \vec{ray}(t) + \vec{rc} \cdot \vec{rc} - r^2 = 0$$

- $\vec{ray}(t)$ is a point on the sphere and a point on the ray at the same time.
on the ray's aspect, we can express it as $\vec{r}_{point} + t \cdot \vec{d}_{ray}$.

• $\vec{ray}(t)^2 - 2 \cdot \vec{rc} \cdot \vec{ray}(t) + \vec{rc} \cdot \vec{rc} - r^2 = 0$

$$\rightarrow (\vec{r}_{point} + t \cdot \vec{d}_{ray})^2 - 2 \cdot \vec{rc} \cdot (\vec{r}_{point} + t \cdot \vec{d}_{ray}) + \vec{rc} \cdot \vec{rc} - r^2 = 0$$

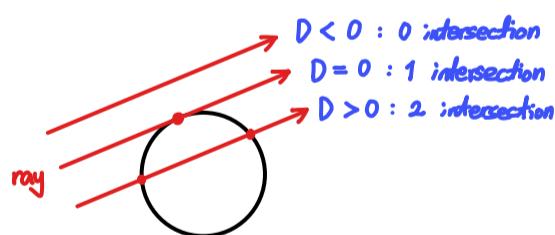
$$\rightarrow \vec{r}_{point}^2 + 2 \cdot \vec{r}_{point} \cdot \vec{d}_{ray} \cdot t + \vec{d}_{ray} \cdot \vec{d}_{ray} \cdot t^2 - 2 \cdot \vec{rc} \cdot \vec{r}_{point} - 2 \cdot \vec{rc} \cdot \vec{d}_{ray} \cdot t + \vec{rc} \cdot \vec{rc} - r^2 = 0$$

$$\rightarrow (\vec{d}_{ray} \cdot \vec{d}_{ray}) t^2 + (2 \cdot \vec{r}_{point} \cdot \vec{d}_{ray} - 2 \cdot \vec{rc} \cdot \vec{d}_{ray}) t + (\vec{r}_{point}^2 - 2 \cdot \vec{rc} \cdot \vec{r}_{point} + \vec{rc} \cdot \vec{rc} - r^2) = 0$$

$$\rightarrow \vec{d}_{ray} \cdot \vec{d}_{ray} \cdot t^2 - 2 \cdot \vec{rc} \cdot \vec{d}_{ray} \cdot t + \vec{rc} \cdot \vec{rc} - r^2 = 0 \quad (\text{quadratic expression for } t)$$

- t is also a point on the sphere, so there should be at least one intersection.

$\rightarrow \text{discriminant} \geq 0$



$$\therefore D = b^2 - 4ac = (-2 \cdot \vec{rc} \cdot \vec{d}_{ray})^2 - 4 \cdot \vec{d}_{ray} \cdot \vec{d}_{ray} \cdot (\vec{rc} \cdot \vec{rc} - r^2) \geq 0,$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \cdot (-2 \cdot \vec{rc} \cdot \vec{d}_{ray}) \pm \sqrt{(-2 \cdot \vec{rc} \cdot \vec{d}_{ray})^2 - 4 \cdot \vec{d}_{ray} \cdot \vec{d}_{ray} \cdot (\vec{rc} \cdot \vec{rc} - r^2)}}{2 \cdot \vec{d}_{ray} \cdot \vec{d}_{ray}}$$

- to reduce calculation, we can use $D/4$.

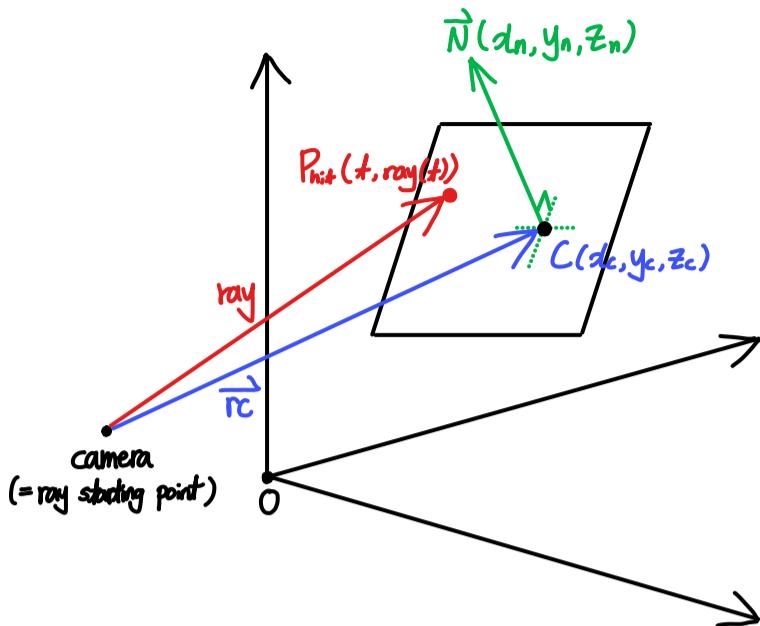
$$D/4 = (b/2)^2 - ac = (-1 \cdot \vec{rc} \cdot \vec{d}_{ray})^2 - \vec{d}_{ray} \cdot \vec{d}_{ray} \cdot (\vec{rc} \cdot \vec{rc} - r^2) \geq 0,$$

$$t = \frac{-(b/2) \pm \sqrt{(b/2)^2 - ac}}{a} = \frac{-1 \cdot (-1 \cdot \vec{rc} \cdot \vec{d}_{ray})^2 \pm \sqrt{(-1 \cdot \vec{rc} \cdot \vec{d}_{ray})^2 - \vec{d}_{ray} \cdot \vec{d}_{ray} \cdot (\vec{rc} \cdot \vec{rc} - r^2)}}{\vec{d}_{ray} \cdot \vec{d}_{ray}}$$

- normal vector at P_{hit}

$$\therefore \vec{N}_{hit} = \frac{\vec{CP}_{hit}}{|\vec{CP}_{hit}|}$$

- ray-plane intersection



* equation of plane

$$ad + by + cz + d = 0$$

(if \vec{N} is (d_n, y_n, z_n) ,

$$d_n \cdot d + y_n \cdot y + z_n \cdot z + d = 0$$

- $ad + by + cz + d = 0$

$$\rightarrow d_n \cdot d + y_n \cdot y + z_n \cdot z + d = 0$$

- we're checking at the hit point $P_{hit}(x_{hit}, y_{hit}, z_{hit})$.

$$\rightarrow d_n \cdot x_{hit} + y_n \cdot y_{hit} + z_n \cdot z_{hit} + d = 0$$

$$\rightarrow \vec{N} \cdot \vec{P}_{hit} + d = 0$$

- the center of the plane $C(x_c, y_c, z_c)$ is also a point on the plane.

$$\rightarrow d_n \cdot x_c + y_n \cdot y_c + z_n \cdot z_c + d = 0$$

$$\rightarrow \vec{N} \cdot \vec{P}_c + d = 0$$

$$\rightarrow d = -\vec{N} \cdot \vec{P}_c$$

- P_{hit} is a point on the plane and a point on the ray at the same time.

on the ray's aspect, we can express it as $r_{point} + t \cdot \vec{d}_{ray}$.

$$\rightarrow \vec{N} \cdot \vec{P}_{hit} + d = 0$$

$$\rightarrow \vec{N} \cdot \vec{P}_{hit} - \vec{N} \cdot \vec{P}_c = 0$$

$$\rightarrow \vec{N} \cdot (r_{point} + t \cdot \vec{d}_{ray}) - \vec{N} \cdot \vec{P}_c = 0$$

$$\rightarrow \vec{N} \cdot \vec{d}_{ray} \cdot t + (\vec{N} \cdot r_{point} - \vec{N} \cdot \vec{P}_c) = 0$$

$$\rightarrow \vec{N} \cdot \vec{d}_{ray} \cdot t + \vec{N} \cdot (r_{point} - \vec{P}_c) = 0$$

$$\rightarrow \vec{N} \cdot \vec{d}_{ray} \cdot t - \vec{N} \cdot \vec{r}_c = 0$$

$$\therefore t = \frac{\vec{N} \cdot \vec{r}_c}{\vec{N} \cdot \vec{d}_{ray}} \quad (\vec{N} \cdot \vec{d}_{ray} \neq 0)$$

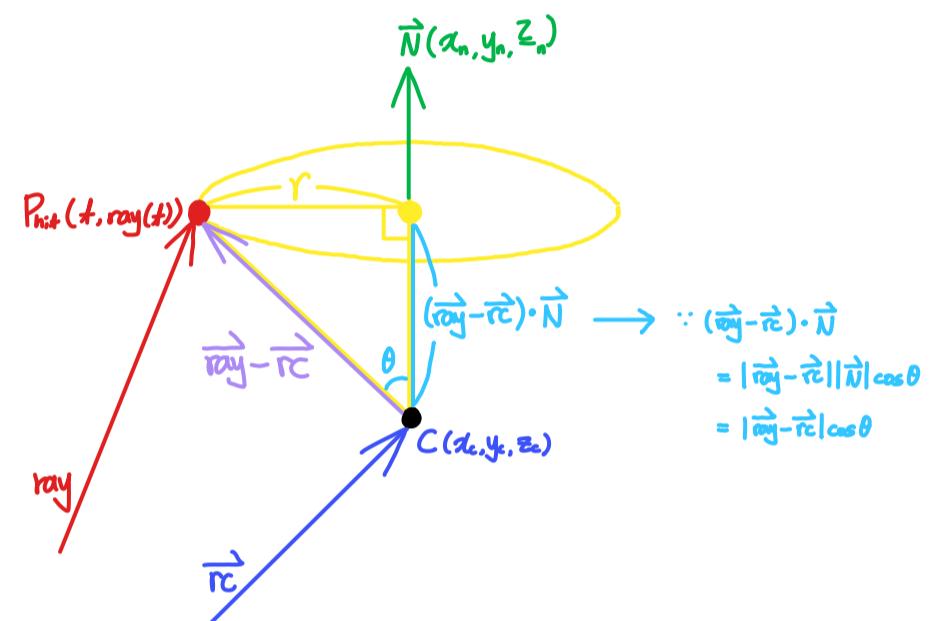
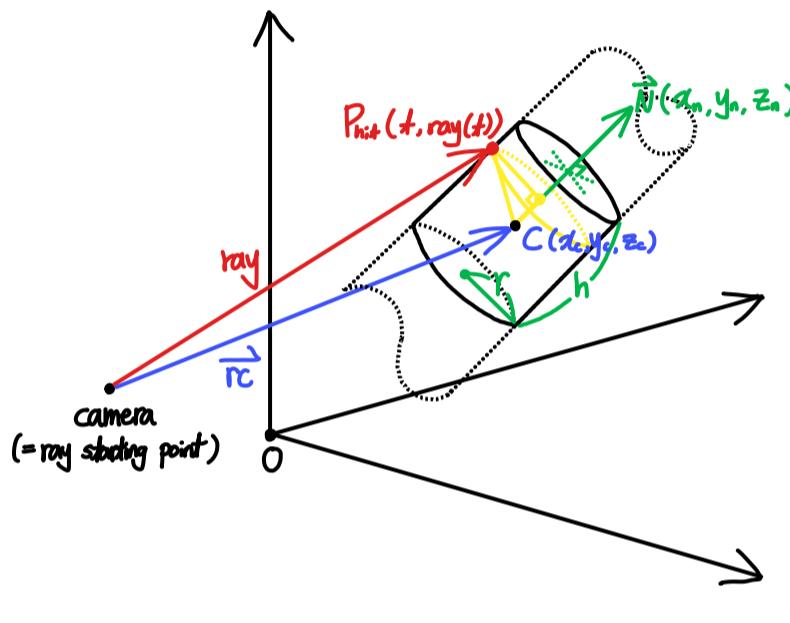
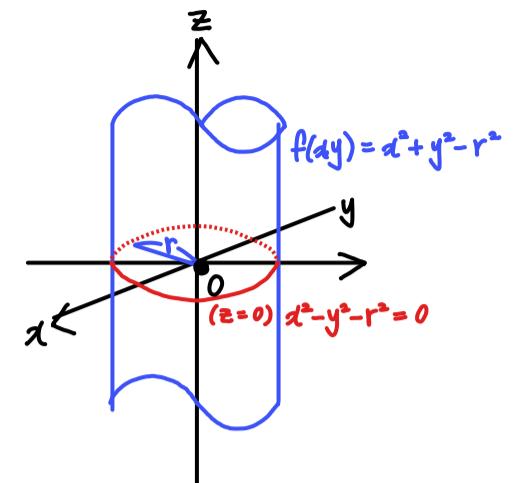
- normal vector at P_{hit}

$$\therefore \vec{N}_{hit} = \vec{N}$$

• ray - cylinder intersection

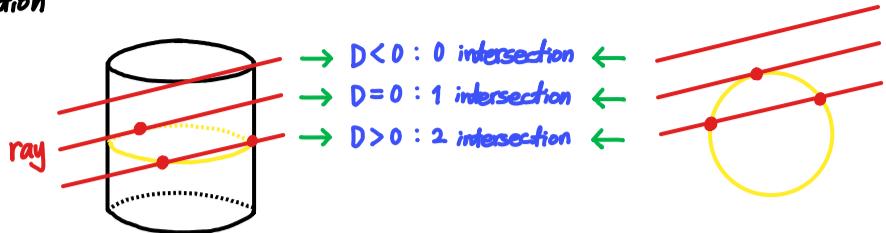
- step 1 : check hit point of the body of the cylinder.

- a basic form of cylinder is a cylinder that has infinite height without top and bottom circles.



- $|\vec{ray}(t) - \vec{rc}|^2 = \{(\vec{ray}(t) - \vec{rc}) \cdot \vec{N}\}^2 + r^2$
 $\rightarrow |\vec{ray}(t) - \vec{rc}| \cdot |\vec{ray}(t) - \vec{rc}| - \{(\vec{ray}(t) - \vec{rc}) \cdot \vec{N}\} \cdot \{(\vec{ray}(t) - \vec{rc}) \cdot \vec{N}\} - r^2 = 0$
 $\rightarrow \vec{ray}(t)^2 - 2 \cdot \vec{rc} \cdot \vec{ray}(t) + \vec{rc} \cdot \vec{rc} - (\vec{ray}(t) \cdot \vec{N} - \vec{rc} \cdot \vec{N})^2 - r^2 = 0$
 $\rightarrow \vec{ray}(t)^2 - 2 \cdot \vec{rc} \cdot \vec{ray}(t) + \vec{rc} \cdot \vec{rc} - (\vec{ray}(t) \cdot \vec{N})^2 + 2 \cdot (\vec{ray}(t) \cdot \vec{N})(\vec{rc} \cdot \vec{N}) + (\vec{rc} \cdot \vec{N})^2 - r^2 = 0$
- $\vec{ray}(t)$ is a point on the sphere and a point on the ray at the same time.
on the ray's aspect, we can express it as $r_{point} + t \cdot \vec{d}_{ray}$.
- $\vec{ray}(t)^2 - 2 \cdot \vec{rc} \cdot \vec{ray}(t) + \vec{rc} \cdot \vec{rc} - (\vec{ray}(t) \cdot \vec{N})^2 + 2 \cdot (\vec{ray}(t) \cdot \vec{N})(\vec{rc} \cdot \vec{N}) - (\vec{rc} \cdot \vec{N})^2 - r^2 = 0$
 $\rightarrow (r_{point} + t \cdot \vec{d}_{ray})^2 - 2 \cdot \vec{rc} \cdot (r_{point} + t \cdot \vec{d}_{ray}) + \vec{rc} \cdot \vec{rc} - \{(r_{point} + t \cdot \vec{d}_{ray}) \cdot \vec{N}\}^2 + 2 \cdot \{(r_{point} + t \cdot \vec{d}_{ray}) \cdot \vec{N}\}(\vec{rc} \cdot \vec{N}) - (\vec{rc} \cdot \vec{N})^2 - r^2 = 0$
 $\rightarrow r_{point}^2 + 2 \cdot r_{point} \cdot \vec{d}_{ray} \cdot t + \vec{d}_{ray} \cdot \vec{d}_{ray} \cdot t^2 - 2 \cdot \vec{rc} \cdot r_{point} - 2 \cdot \vec{rc} \cdot \vec{d}_{ray} \cdot t + \vec{rc} \cdot \vec{rc}$
 $- (r_{point} \cdot \vec{N} + t \cdot \vec{d}_{ray} \cdot \vec{N})^2 + 2(r_{point} \cdot \vec{N} + t \cdot \vec{d}_{ray} \cdot \vec{N})(\vec{rc} \cdot \vec{N}) - (\vec{rc} \cdot \vec{N})^2 - r^2 = 0$
 $\rightarrow r_{point}^2 + 2 \cdot r_{point} \cdot \vec{d}_{ray} \cdot t + \vec{d}_{ray} \cdot \vec{d}_{ray} \cdot t^2 - 2 \cdot \vec{rc} \cdot r_{point} - 2 \cdot \vec{rc} \cdot \vec{d}_{ray} \cdot t + \vec{rc} \cdot \vec{rc}$
 $- (r_{point} \cdot \vec{N})^2 - 2 \cdot (r_{point} \cdot \vec{N})(\vec{d}_{ray} \cdot \vec{N}) + - (\vec{d}_{ray} \cdot \vec{N})^2 \cdot t^2 + 2 \cdot r_{point} \cdot \vec{N} \cdot (\vec{rc} \cdot \vec{N}) + 2 \cdot (\vec{d}_{ray} \cdot \vec{N})(\vec{rc} \cdot \vec{N}) \cdot t - (\vec{rc} \cdot \vec{N})^2 - r^2 = 0$
 $\rightarrow \{\vec{d}_{ray} \cdot \vec{d}_{ray} - (\vec{d}_{ray} \cdot \vec{N})^2\}t^2 + \{2 \cdot r_{point} \cdot \vec{d}_{ray} - 2 \cdot \vec{rc} \cdot \vec{d}_{ray} - 2 \cdot (r_{point} \cdot \vec{N})(\vec{d}_{ray} \cdot \vec{N}) + 2 \cdot (\vec{d}_{ray} \cdot \vec{N})(\vec{rc} \cdot \vec{N})\}t +$
 $+ \{r_{point}^2 - 2 \cdot \vec{rc} \cdot r_{point} + \vec{rc} \cdot \vec{rc} - (r_{point} \cdot \vec{N})^2 + 2 \cdot r_{point} \cdot \vec{N} \cdot (\vec{rc} \cdot \vec{N}) - (\vec{rc} \cdot \vec{N})^2 - r^2\} = 0$
 $\rightarrow \{\vec{d}_{ray} \cdot \vec{d}_{ray} - (\vec{d}_{ray} \cdot \vec{N})^2\}t^2 - 2 \{\vec{rc} \cdot \vec{d}_{ray} - (\vec{d}_{ray} \cdot \vec{N})(\vec{rc} \cdot \vec{N})\}t + \{\vec{rc} \cdot \vec{rc} - (\vec{rc} \cdot \vec{N})^2 - r^2\} = 0$
(quadratic expression for t)

- t is also a point on the body of the cylinder,
so there should be at least one intersection
 \rightarrow discriminant ≥ 0



$$\therefore D = b^2 - 4ac = [-2 \cdot \{\vec{rc} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{rc} \cdot \vec{N})\}]^2 - 4 \cdot \{\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})^2\} \cdot \{\vec{rc} \cdot \vec{rc} - (\vec{rc} \cdot \vec{N})^2 - r^2\} \geq 0,$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \cdot -2 \{\vec{rc} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{rc} \cdot \vec{N})\} \pm \sqrt{[-2 \{\vec{rc} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{rc} \cdot \vec{N})\}]^2 - 4 \cdot \{\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})^2\} \cdot \{\vec{rc} \cdot \vec{rc} - (\vec{rc} \cdot \vec{N})^2 - r^2\}}}{2 \cdot \{\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})^2\}}$$

- to reduce calculation, we can use D/4.

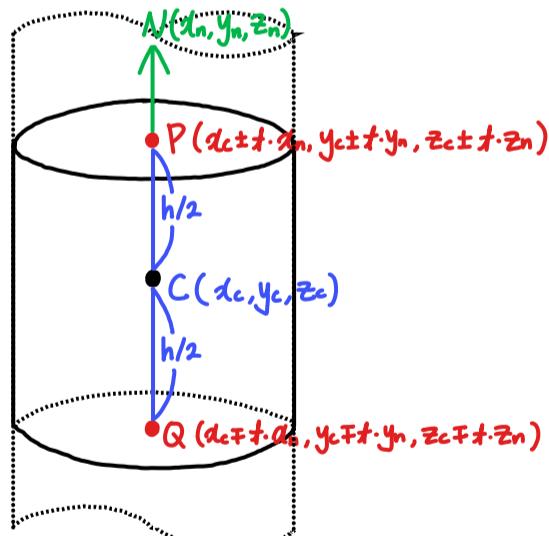
$$D/4 = (b/2)^2 - ac = [-1 \cdot \{\vec{rc} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{rc} \cdot \vec{N})\}]^2 - \{\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})^2\} \cdot \{\vec{rc} \cdot \vec{rc} - (\vec{rc} \cdot \vec{N})^2 - r^2\} \geq 0,$$

$$t = \frac{-(b/2)^2 \pm \sqrt{(b/2)^2 - ac}}{a}$$

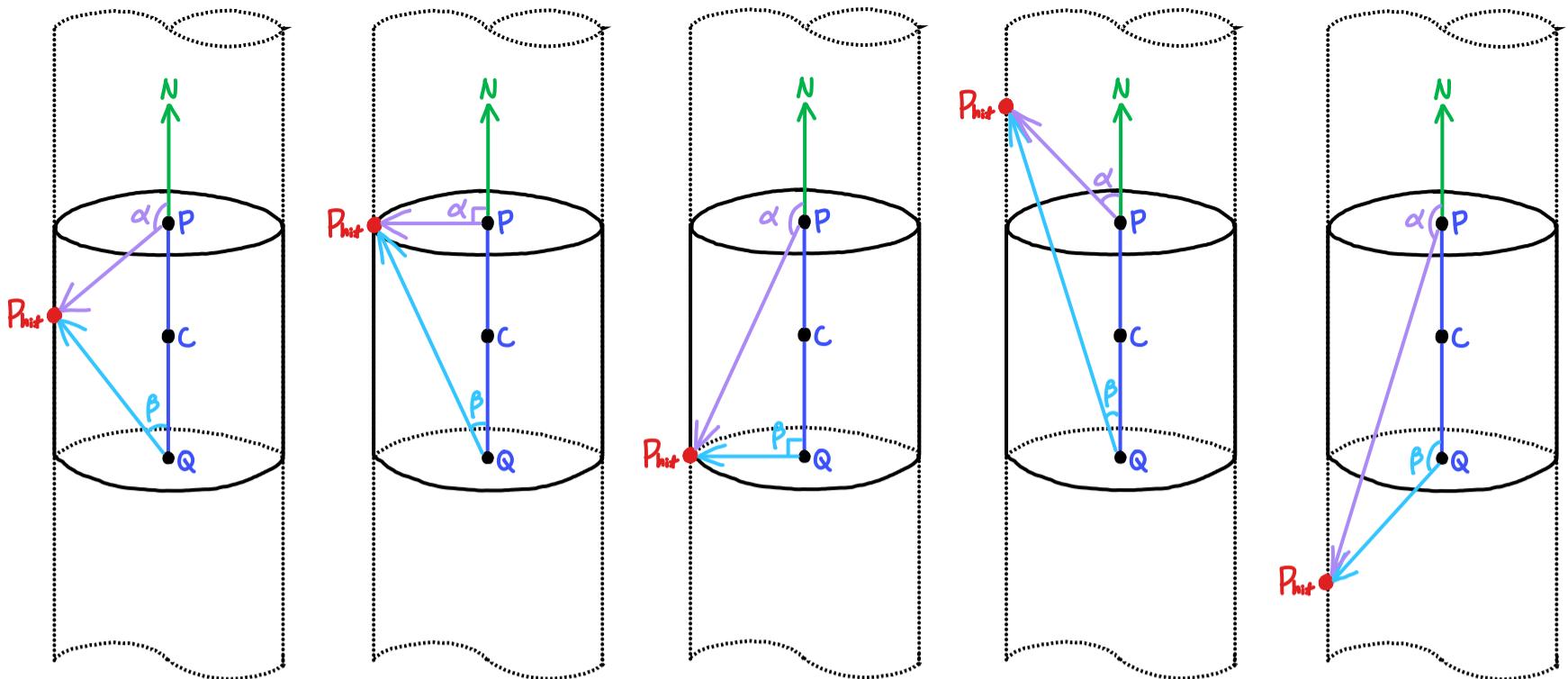
$$= \frac{-1 \cdot -1 \{\vec{rc} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{rc} \cdot \vec{N})\} \pm \sqrt{[-1 \cdot \{\vec{rc} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{rc} \cdot \vec{N})\}]^2 - \{\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})^2\} \cdot \{\vec{rc} \cdot \vec{rc} - (\vec{rc} \cdot \vec{N})^2 - r^2\}}}{\{\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}} - (\vec{d}_{\text{ray}} \cdot \vec{N})^2\}}$$

- Step 2 : check height of the cylinder.

- to check the height, we need positions of the center of top and bottom circles.



- P, C, Q are on the same line, start point is C and direction is $\vec{N} (C + t \cdot \vec{N})$.
- P and Q are point symmetric
 \rightarrow if the position of P is $(x_c + t \cdot d_n, y_c + t \cdot y_n, z_c + t \cdot z_n)$, Q is $(x_c - t \cdot d_n, y_c - t \cdot y_n, z_c - t \cdot z_n)$.
- also, $\overline{CP} = \overline{CQ} = h/2$
 $\rightarrow \sqrt{(x_c + t \cdot d_n - x_c)^2 + (y_c + t \cdot y_n - y_c)^2 + (z_c + t \cdot z_n - z_c)^2} = \pm h/2$
 $\rightarrow \sqrt{d_n^2 \cdot t^2 + y_n^2 \cdot t^2 + z_n^2 \cdot t^2} = \pm h/2$
 $\rightarrow \sqrt{(d_n^2 + y_n^2 + z_n^2) t^2} = \pm h/2$
 $\therefore t = \pm h/2 \cdot \sqrt{1 / (d_n^2 + y_n^2 + z_n^2)}$



• we can check if P_{hit} is on cylinder or not by comparing angles between \vec{N} and $\overrightarrow{PP_{hit}}$, and \vec{N} and $\overrightarrow{QP_{hit}}$.

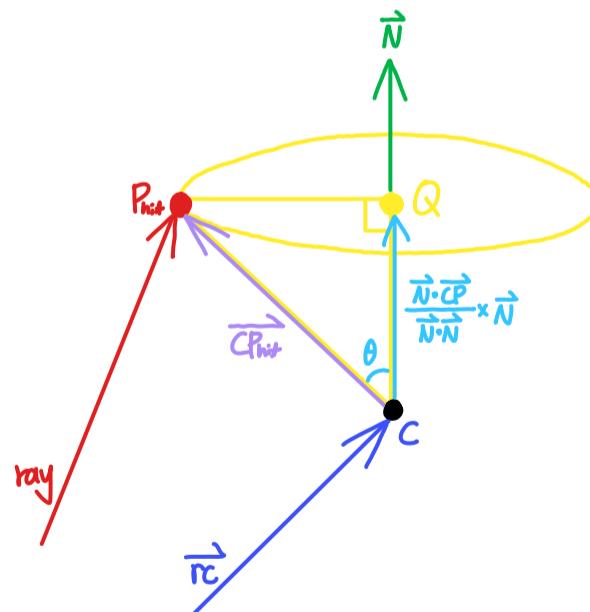
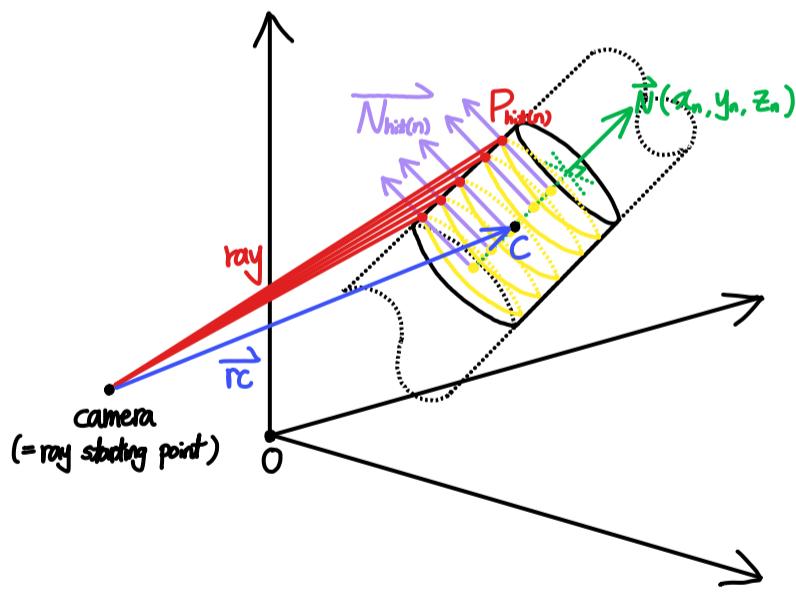
$$\rightarrow \pi/2 \leq \alpha < \pi, 0 < \beta \leq \pi/2$$

(α : angle between \vec{N} and $\overrightarrow{PP_{hit}}$, β : angle between \vec{N} and $\overrightarrow{QP_{hit}}$)

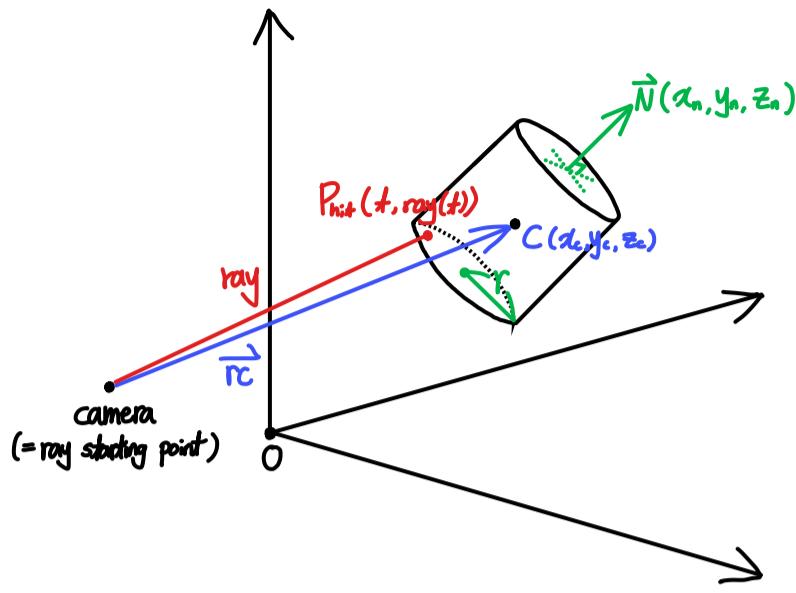
$$\rightarrow \vec{N} \cdot \overrightarrow{PP_{hit}} \leq 0, \vec{N} \cdot \overrightarrow{QP_{hit}} \geq 0$$

• normal vector at P_{hit}

$$\vec{N}_{hit} = \frac{\overrightarrow{QP_{hit}}}{|\overrightarrow{QP_{hit}}|} = \frac{\overrightarrow{P_{hit}} - (C + \frac{\vec{N} \cdot \vec{CP}}{\vec{N} \cdot \vec{N}} \times \vec{N})}{|\overrightarrow{P_{hit}} - (C + \frac{\vec{N} \cdot \vec{CP}}{\vec{N} \cdot \vec{N}} \times \vec{N})|}$$

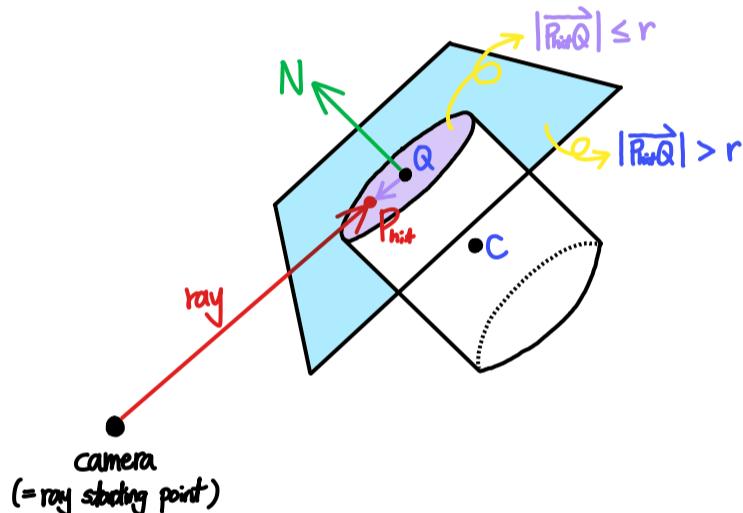


- step 3 : check hit point of the top and the bottom circles of the cylinder.



* top circle of the cylinder

- think as ray-plane intersection, and then limit the range of $\overrightarrow{P_{hit}Q}$.



$$1) Q(x_c + h/2 \cdot \sqrt{1/x_n^2 + y_n^2 + z_n^2} \cdot x_n, y_c + h/2 \cdot \sqrt{1/x_n^2 + y_n^2 + z_n^2} \cdot y_n, z_c + h/2 \cdot \sqrt{1/x_n^2 + y_n^2 + z_n^2} \cdot z_n),$$

$$\text{plane hit point } t = \frac{\vec{N} \cdot \vec{rQ}}{\vec{N} \cdot \vec{rcy}} \quad (\vec{N} \cdot \vec{rcy} \neq 0)$$

$$2) |\overrightarrow{P_{hit}Q}| \leq r$$

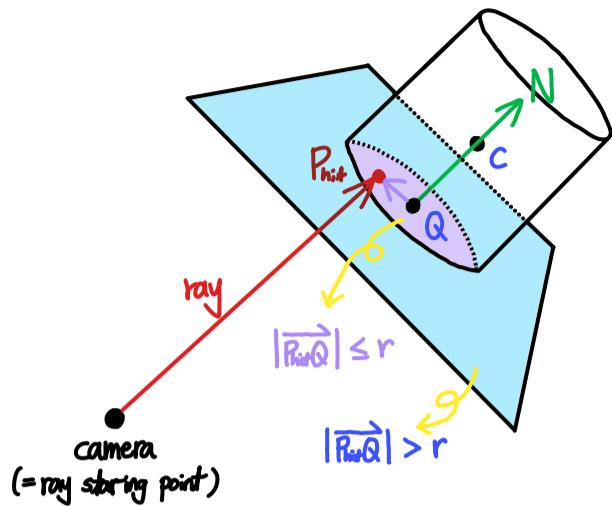
$\rightarrow 1) \cap 2)$ is a set of hit point on the top circle of the cylinder.

- normal vector at P_{hit}

$$: \vec{N}_{hit} = \vec{N}$$

* bottom circle of the cylinder

- think as ray-plane intersection, and then limit the range of $\vec{P}_{hit}Q$.



$$1) Q(x_c - h/2 \cdot \sqrt{1/x_n^2 + y_n^2 + z_n^2} \cdot x_n, y_c - h/2 \cdot \sqrt{1/x_n^2 + y_n^2 + z_n^2} \cdot y_n, z_c - h/2 \cdot \sqrt{1/x_n^2 + y_n^2 + z_n^2} \cdot z_n),$$

$$\text{plane hit point } t = \frac{\vec{N} \cdot \vec{r}_Q}{\vec{N} \cdot \vec{d}_{ray}} \quad (\vec{N} \cdot \vec{d}_{ray} \neq 0)$$

$$2) |\vec{P}_{hit}Q| \leq r$$

$\rightarrow 1) \cap 2)$ is a set of hit point on the top circle of the cylinder.

- normal vector at P_{hit}

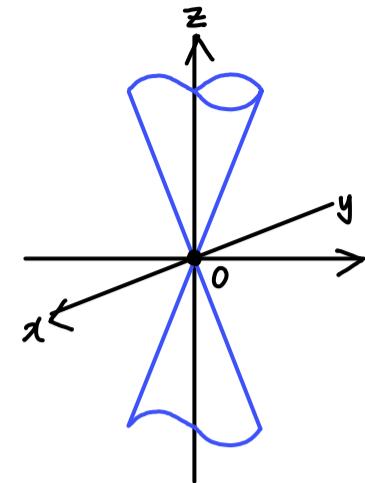
$$: \vec{N}_{hit} = \vec{N}$$

- if a hit point of the top circle t_{top} is less than a hit point of the bottom circle t_{bottom} ,
the top circle of the cylinder will be drawn. (we'll ignore the bottom circle)
- if a hit point of the bottom circle t_{bottom} is less than a hit point of the top circle t_{top} ,
the bottom circle of the cylinder will be drawn. (we'll ignore the top circle)

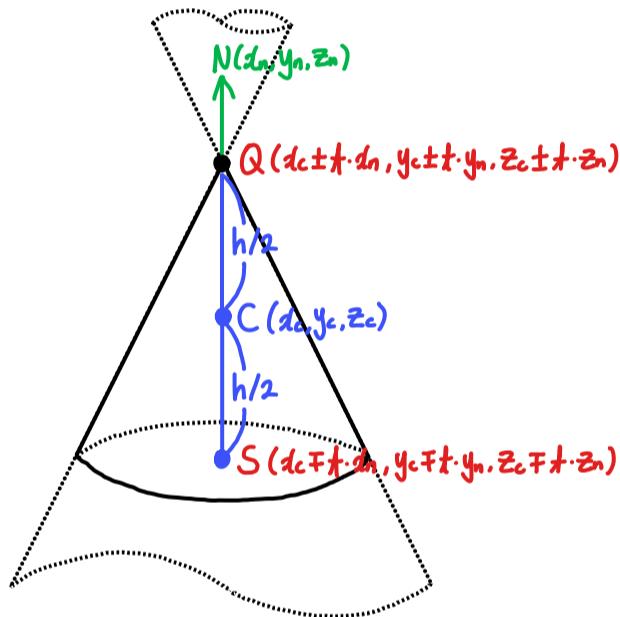
• ray-cone intersection

- step 1 : check hit point of the body of the cone.

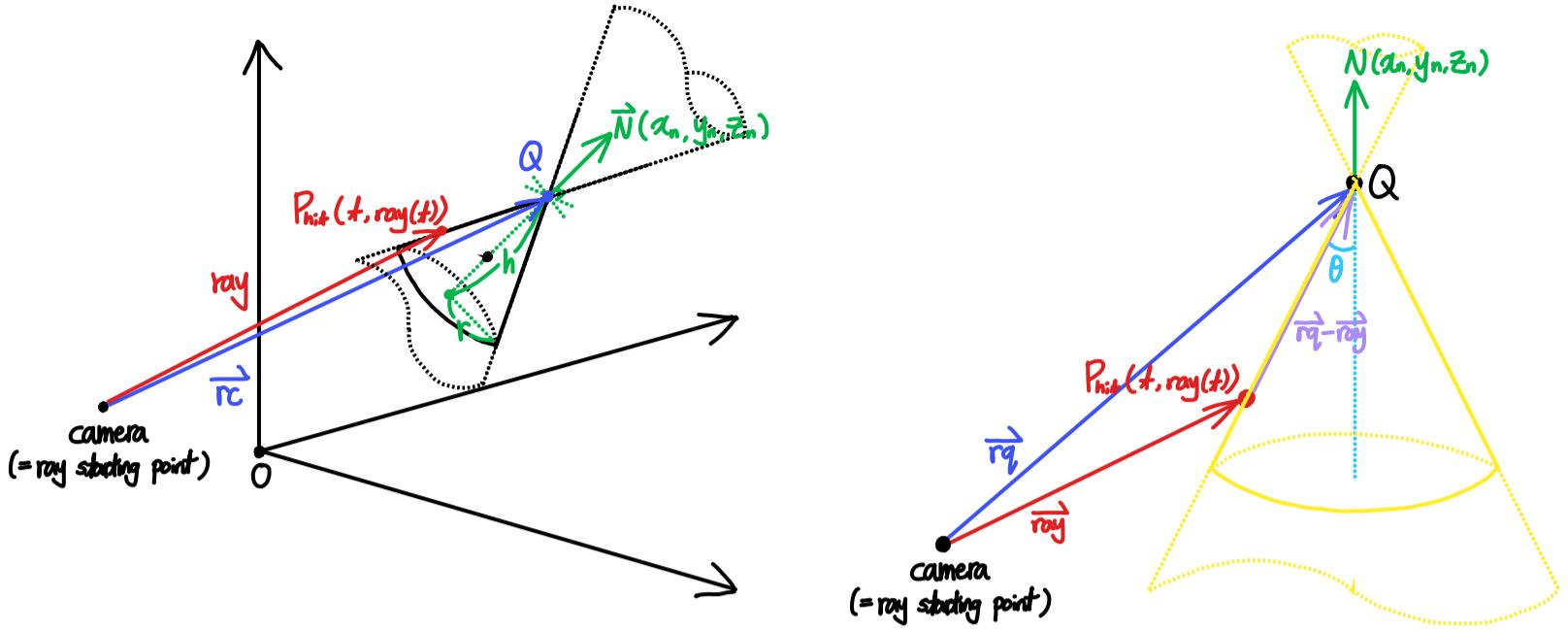
- a basic form of cylinder is a cone that has infinite height without top and bottom circles.



- check a center of the bottom circle and a point of the top.



- Q, C, S are on the same line , start point is C and direction is $\vec{N} (C + t \cdot \vec{N})$.
- Q and S are point symmetric
→ if the position of Q is $(d_c + t \cdot d_n, y_c + t \cdot y_n, z_c + t \cdot z_n)$, S is $(d_c - t \cdot d_n, y_c - t \cdot y_n, z_c - t \cdot z_n)$.
- also, $\overline{CQ} = \overline{CS} = h/2$
 $\rightarrow \sqrt{(d_c + t \cdot d_n - d_c)^2 + (y_c + t \cdot y_n - y_c)^2 + (z_c + t \cdot z_n - z_c)^2} = \pm h/2$
 $\rightarrow \sqrt{d_n^2 \cdot t^2 + y_n^2 \cdot t^2 + z_n^2 \cdot t^2} = \pm h/2$
 $\rightarrow \sqrt{(d_n^2 + y_n^2 + z_n^2) t^2} = \pm h/2$
 $\therefore t = \pm h/2 \cdot \sqrt{1 / (d_n^2 + y_n^2 + z_n^2)}$



- $$(\vec{r}_q - \text{ray}(t)) \cdot \vec{N} = |\vec{r}_q - \text{ray}(t)| |\vec{N}| \cos \theta$$

→ since $|\vec{N}| = 1$, $(\vec{r}_q - \text{ray}(t)) \cdot \vec{N} = |\vec{r}_q - \text{ray}(t)| \cos \theta$

$$\rightarrow (\vec{r}_q \cdot \vec{N} - \text{ray}(t) \cdot \vec{N})^2 = (\vec{r}_q - \text{ray}(t))^2 \cos^2 \theta$$

$$\rightarrow \{(\vec{r}_q \cdot \vec{N})^2 - 2 \cdot (\vec{r}_q \cdot \vec{N}) \cdot \text{ray}(t) + \text{ray}(t)^2 \cdot (\vec{N} \cdot \vec{N})\} - \{ \vec{r}_q \cdot \vec{r}_q - 2 \cdot \vec{r}_q \cdot \text{ray}(t) + \text{ray}(t)^2 \} \cdot \cos^2 \theta = 0$$
- $\text{ray}(t)$ is a point on the sphere and a point on the ray at the same time.
on the ray's aspect, we can express it as $r_{\text{point}} + t \cdot \vec{d}_{\text{ray}}$.
- $$\{(\vec{r}_q \cdot \vec{N})^2 - 2 \cdot (\vec{r}_q \cdot \vec{N}) \cdot \text{ray}(t) + \text{ray}(t)^2 \cdot (\vec{N} \cdot \vec{N})\} - \{ \vec{r}_q \cdot \vec{r}_q - 2 \cdot \vec{r}_q \cdot \text{ray}(t) + \text{ray}(t)^2 \} \cdot \cos^2 \theta = 0$$

$$\rightarrow \{(\vec{r}_q \cdot \vec{N})(\vec{r}_q \cdot \vec{N}) - 2 \cdot (\vec{r}_q \cdot \vec{N}) \cdot \vec{N} \cdot (r_p + t \cdot \vec{d}_{\text{ray}}) + (r_p \cdot \vec{N} + t \cdot \vec{d}_{\text{ray}} \cdot \vec{N})^2$$

$$- (\vec{r}_q \cdot \vec{r}_q) \cos^2 \theta + 2 \cdot \vec{r}_q \cdot (r_p + t \cdot \vec{d}_{\text{ray}}) \cos^2 \theta - (r_p + t \cdot \vec{d}_{\text{ray}})^2 \cos^2 \theta = 0$$

$$\rightarrow (\vec{r}_q \cdot \vec{N})(\vec{r}_q \cdot \vec{N}) - 2 \cdot (\vec{r}_q \cdot \vec{N}) \cdot \vec{N} \cdot r_p - 2 \cdot (\vec{r}_q \cdot \vec{N}) \cdot (\vec{N} \cdot \vec{d}_{\text{ray}}) t + \cancel{r_p^2 \cdot (\vec{N} \cdot \vec{N})} + \cancel{2 \cdot r_p \cdot \vec{N} \cdot (\vec{d}_{\text{ray}} \cdot \vec{N}) t} + (\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{d}_{\text{ray}} \cdot \vec{N}) t^2$$

$$- (\vec{r}_q \cdot \vec{r}_q) \cos^2 \theta + \cancel{2 \cdot \vec{r}_q \cdot r_p \cos^2 \theta} + 2 \cdot (\vec{r}_q \cdot \vec{d}_{\text{ray}}) \cos^2 \theta t - \cancel{r_p^2 \cos^2 \theta} - \cancel{2 \cdot r_p \cdot \vec{d}_{\text{ray}} \cos^2 \theta t} - (\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}}) \cos^2 \theta t^2 = 0$$

$$\rightarrow \{(\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{d}_{\text{ray}} \cdot \vec{N}) - (\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}}) \cos^2 \theta\} t^2 - 2 \{(\vec{r}_q \cdot \vec{N})(\vec{N} \cdot \vec{d}_{\text{ray}}) - (\vec{r}_q \cdot \vec{d}_{\text{ray}}) \cos^2 \theta\} t + \{(\vec{r}_q \cdot \vec{N})(\vec{r}_q \cdot \vec{N}) - (\vec{r}_q \cdot \vec{r}_q) \cos^2 \theta\} = 0$$

(quadratic expression for t)

- t is also a point on the body of the cylinder,
so there should be at least one intersection
 $\rightarrow \text{discriminant} \geq 0$



$$\therefore D = b^2 - 4ac = [-2 \{ (\vec{r}_q \cdot \vec{N})(\vec{N} \cdot \vec{d}_{\text{ray}}) - (\vec{r}_q \cdot \vec{d}_{\text{ray}}) \cos^2 \theta \}]^2 - 4 \cdot \{ (\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{d}_{\text{ray}} \cdot \vec{N}) - (\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}}) \cos^2 \theta \} \cdot \{ (\vec{r}_q \cdot \vec{N})(\vec{r}_q \cdot \vec{N}) - (\vec{r}_q \cdot \vec{r}_q) \cos^2 \theta \} \geq 0,$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 - 2 \{ (\vec{r}_q \cdot \vec{N})(\vec{N} \cdot \vec{d}_{\text{ray}}) - (\vec{r}_q \cdot \vec{d}_{\text{ray}}) \cos^2 \theta \} \pm \sqrt{[-2 \{ (\vec{r}_q \cdot \vec{N})(\vec{N} \cdot \vec{d}_{\text{ray}}) - (\vec{r}_q \cdot \vec{d}_{\text{ray}}) \cos^2 \theta \}]^2 - 4 \cdot \{ (\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{d}_{\text{ray}} \cdot \vec{N}) - (\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}}) \cos^2 \theta \} \cdot \{ (\vec{r}_q \cdot \vec{N})(\vec{r}_q \cdot \vec{N}) - (\vec{r}_q \cdot \vec{r}_q) \cos^2 \theta \} }}{2 \cdot \{ (\vec{d}_{\text{ray}} \cdot \vec{N})(\vec{d}_{\text{ray}} \cdot \vec{N}) - (\vec{d}_{\text{ray}} \cdot \vec{d}_{\text{ray}}) \cos^2 \theta \}}$$

- to reduce calculation, we can use D/4.

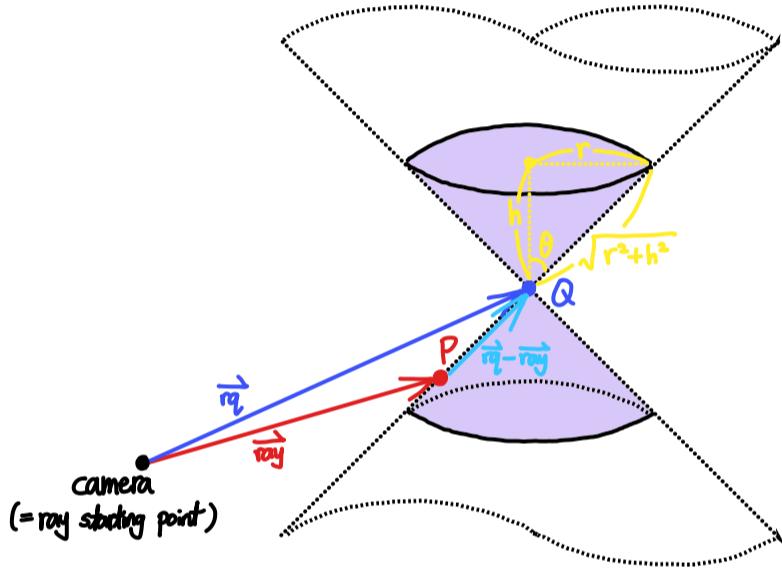
$$D/4 = (b/2)^2 - ac = [-1 \cdot \{(\vec{r}_q \cdot \vec{N})(\vec{N} \cdot \vec{d}_{ray}) - (\vec{r}_q \cdot \vec{d}_{ray}) \cdot \cos^2 \theta\}]^2 - 1 \cdot \{(\vec{d}_{ray} \cdot \vec{N})(\vec{d}_{ray} \cdot \vec{N}) - (\vec{d}_{ray} \cdot \vec{d}_{ray}) \cdot \cos^2 \theta\} \cdot \{(\vec{r}_q \cdot \vec{N})(\vec{r}_q \cdot \vec{N}) - (\vec{r}_q \cdot \vec{r}_q) \cdot \cos^2 \theta\} \geq 0,$$

$$t = \frac{-(b/2)^2 \pm \sqrt{(b/2)^2 - ac}}{a}$$

$$= \frac{-1 \cdot -1 \cdot \{(\vec{r}_q \cdot \vec{N})(\vec{N} \cdot \vec{d}_{ray}) - (\vec{r}_q \cdot \vec{d}_{ray}) \cdot \cos^2 \theta\} \pm \sqrt{[-1 \cdot \{(\vec{r}_q \cdot \vec{N})(\vec{N} \cdot \vec{d}_{ray}) - (\vec{r}_q \cdot \vec{d}_{ray}) \cdot \cos^2 \theta\}]^2 - \{(\vec{d}_{ray} \cdot \vec{N})(\vec{d}_{ray} \cdot \vec{N}) - (\vec{d}_{ray} \cdot \vec{d}_{ray}) \cdot \cos^2 \theta\} \cdot \{(\vec{r}_q \cdot \vec{N})(\vec{r}_q \cdot \vec{N}) - (\vec{r}_q \cdot \vec{r}_q) \cdot \cos^2 \theta\}}}{\{(\vec{d}_{ray} \cdot \vec{N})(\vec{d}_{ray} \cdot \vec{N}) - (\vec{d}_{ray} \cdot \vec{d}_{ray}) \cdot \cos^2 \theta\}}$$

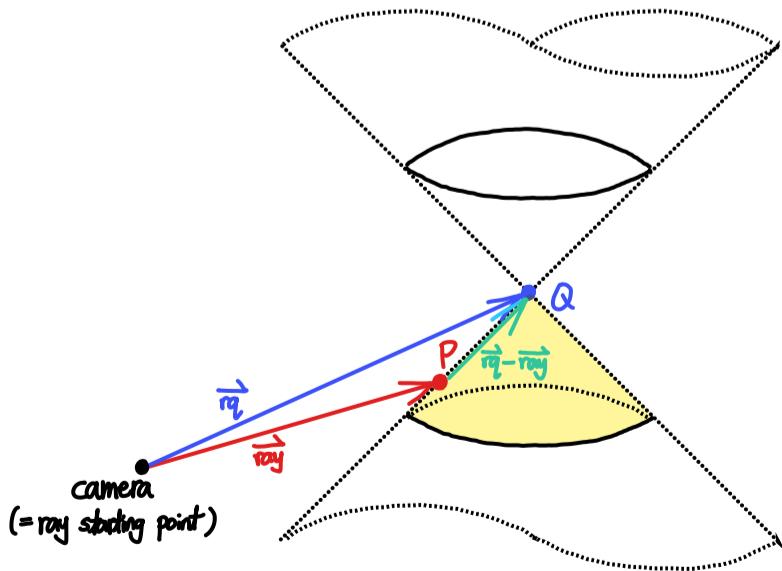
- Step 2 : check height of the cone.

- limit the range of height



$$* |\vec{PQ}| = |\vec{r}_q - \text{ray}(t)| \leq \sqrt{r^2 + h^2}$$

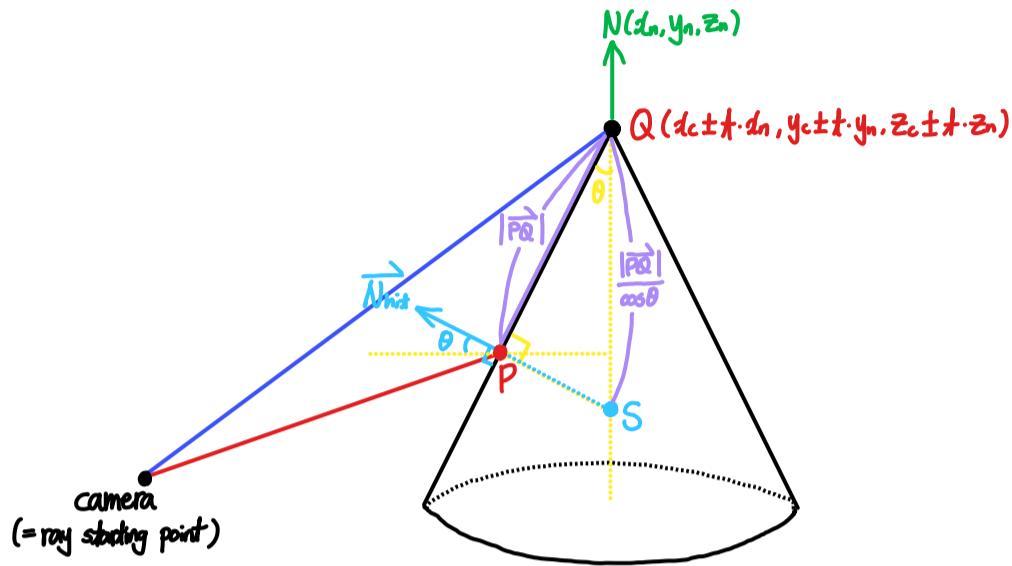
- take the bottom side



$$* \vec{r}_q \cdot \vec{PQ} = \vec{r}_q \cdot (\vec{r}_q - \text{ray}) \geq 0$$

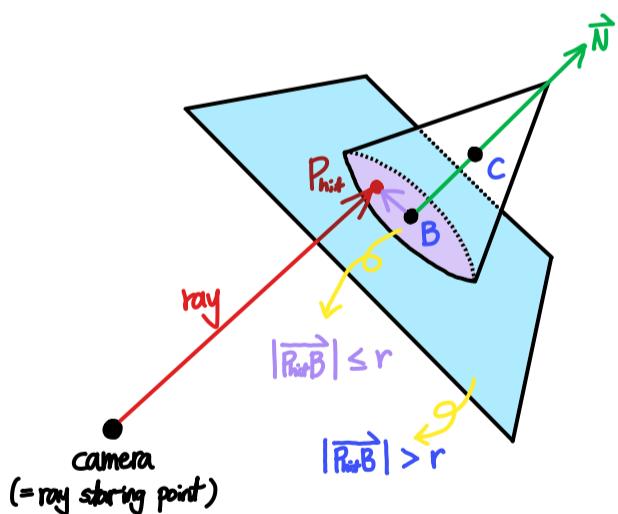
- normal vector at P_{hit}

$$\vec{N}_{hit} = \frac{\overrightarrow{SP_{hit}}}{|\overrightarrow{SP_{hit}}|} = \frac{\overrightarrow{Q} + \frac{\overrightarrow{PQ}}{\cos\theta} \times \vec{N}}{\left| \overrightarrow{Q} + \frac{\overrightarrow{PQ}}{\cos\theta} \times \vec{N} \right|}$$



- step 3 : check hit point of the top circle of the cone.

- think as ray-plane intersection, and then limit the range of $\overrightarrow{P_{hit}B}$.



$$1) B(d_c - h/2 \cdot \sqrt{(1/x_n^2 + y_n^2 + z_n^2)} \cdot x_n, y_c - h/2 \cdot \sqrt{(1/x_n^2 + y_n^2 + z_n^2)} \cdot y_n, z_c - h/2 \cdot \sqrt{(1/x_n^2 + y_n^2 + z_n^2)} \cdot z_n),$$

$$\text{plane hit point } t = \frac{\vec{N} \cdot \overrightarrow{PB}}{\vec{N} \cdot \overrightarrow{d_{ray}}} \quad (\vec{N} \cdot \overrightarrow{d_{ray}} \neq 0)$$

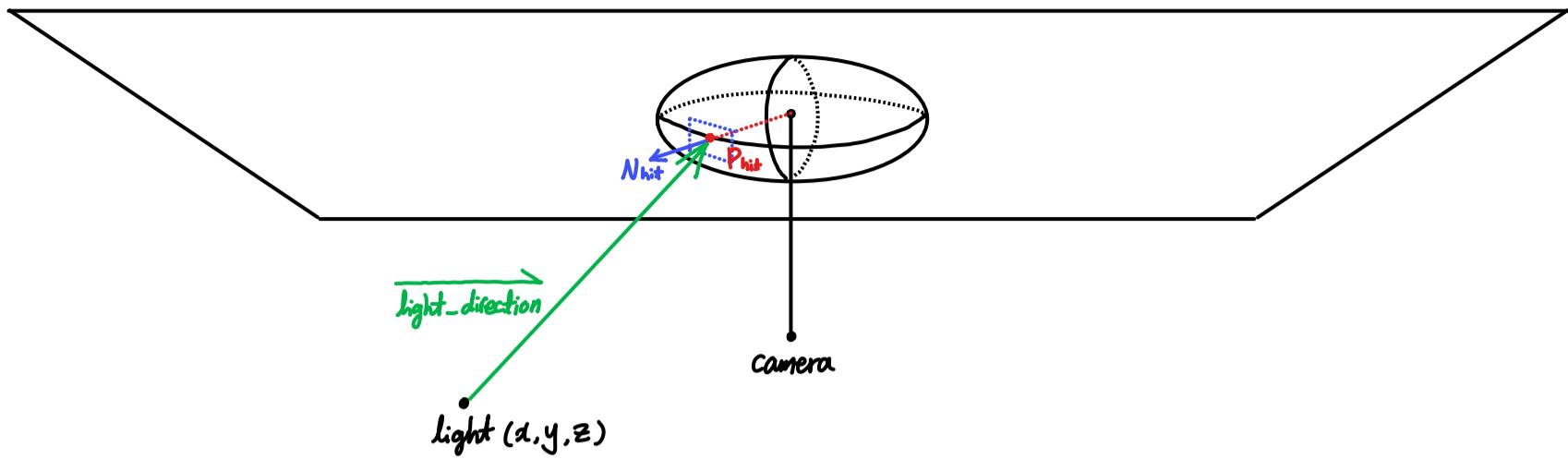
$$2) |\overrightarrow{P_{hit}B}| \leq r$$

$\rightarrow 1) \cap 2)$ is a set of hit point on the top circle of the cylinder.

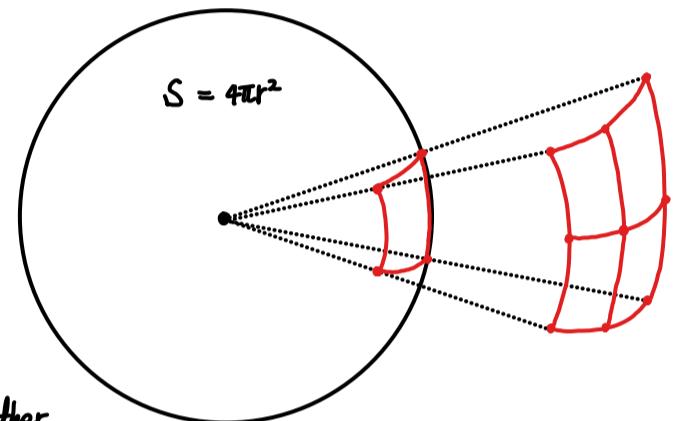
- normal vector at P_{hit}

$$\vec{N}_{hit} = \vec{N}$$

• apply light



- $\overrightarrow{\text{light_direction}} = P_{hit} - P_{\text{light}}(x, y, z)$
- $\text{distance} = |\overrightarrow{\text{light_direction}}|$
- by inverse square law of light, $\text{light} \propto \frac{1}{\text{distance}^2}$.
- also, we should consider the angle of the $\overrightarrow{\text{light_direction}}$ and the normal vector of each hit point (= light-intensity)
- more light will reflect when $\overrightarrow{\text{light_direction}}$ and $\overrightarrow{N_{hit}}$ are closer to each other



$$\therefore \text{light_color} = \frac{\text{light_rgb} \times \text{light_brightness}}{4 \times \pi \times \text{distance} \times \text{distance}} \times \text{light_intensity}$$

- light should apply when a line that starts from P_{hit} and direction is $-\overrightarrow{\text{light_direction}}$ can reach the light without hitting other objects.
- (when it can't reach, light-color is 0)

