Honor Mathematics Lecture 2

From Numbers to Sequences

"We define everything as they now defined for a good reason..."

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Overview

Natural Numbers

The natural numbers can be *constructed* from set theory, but that is not our main focus. Instead, we will simply denote the set of natural numbers by $\mathbb{N}$ and define it as:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$$

We will have the following properties we simply take it as *axioms*.

Porperties	Addition	Multiplication
Associativity	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Existence	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
Distributivity	$a \cdot (b + c) = a \cdot b + a \cdot c$	

Mathematical Induction

There are two types of Mathematical Induction.

Mathematical Induction I & II

Let A be a statement that has to do with **natural numbers**. We denote the statement with respect to a specific number n as $A(n)$.

Then, *type-I* induction works by establishing two statements:

1. $A(n_0)$ is true
2. $A(n+1)$ is true whenever $A(n)$ is true for $n_0 \leq n$

And *type-II* induction works by establishing similar two statements:

1. $A(n_0)$ is true
2. $A(k+1)$ is true whenever $A(n)$ is true for all $n_0 \leq n \leq k$

Induction

How, and when to use Induction?

- Examine the complexity of the problem, because using induction is sometimes much more complicated than using a direct method to prove a statement.
- Determine the initial condition (i.e. n_0) for your induction proof.
- Decide the part of the proof that you use induction and which induction you want to use. (There are few more different types of induction.)
- Make a short test of your method on draft paper to see whether it works and is easy to write down.

Rational Numbers

We define that the set of rational numbers is

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} \wedge q \neq 0\}$$

together with the following properties.

Porperties	Addition	Multiplication
Associativity	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
NeutralElement	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
InverseElement	$(-a) + a = a + (-a) = 0$	$a \cdot a^{-1} = a^{-1} \cdot a = 1$
Distributivity	$a \cdot (b + c) = a \cdot b + a \cdot c$	

Trichotomy law

There are still three *axioms* we will define, and together with the nine axioms above, we build our rational numbers $\mathbb{Q}$.

We assume that we know what a strictly positive rational number is, then we know we can find such a set P with the property that :

1. $a = 0$
2. $a \in P$
3. $-a \in P$

which is so-called **trichotomy law**(P10).)
Furthermore, we assume that the set of positive number P is closed under addition and multiplication.(P11, P12)

Real Numbers

We will adapt all axioms we had from rational numbers, except the last three axioms regard to trichotomy law. By replacing P for $\mathbb{R}$, we **almost** define our set of real numbers $\mathbb{R}$

The last thing we need for $\mathbb{R}$ is listed below(P13):

If $A \subset \mathbb{R}$, $A \neq \emptyset$ is bounded above, then there exists a least upper bound for A in $\mathbb{R}$.

(Why? And why we want to define real numbers?)

It is important for us to study whether a set is bounded or not. More importantly, we want to give the **tightest** bounded if possible. Hence, we have

- lower bound
- upper bound
- minimum
- maximum
- supremum(least upper bound)
- infimum(greatest lower bound)

for a set.

Question. Why is P13 sufficient to guarantee the existence of greatest lower bound?

You just need to know how to perform basic complex numbers' computation and some basic properties. Here, we just list some basic computation rules and formulas.

Given $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$,

- $z_1 + z_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1, b_2)$
- $z_1 \cdot z_2 = (a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 - a_2 b_1)$
- $c \cdot z_1 = c(a_1, b_2) = (ca_1, cb_1), c \in \mathbb{R}$
- $\bar{z_1} = (a_1, -b_1)$
- $|z_1| = \sqrt{a_1^2 + b_1^2} = \sqrt{z_1 \bar{z_1}}$
- $|z_1|^2 = z_1 \bar{z_1}$

Sets and Points

We can classify points with respect to a set in following way:

- Interior point
- Exterior point
- Boundary point
- Accumulation Point

(What's the main role for all these definitions?)

Theorem 2.1 Let $E \subseteq \mathbb{R}$ be a proper subset of $\mathbb{R}$ with at least two points. If $\forall a \notin E$, either $(a, +\infty) \cap E$ or $(-\infty, a) \cap E$ is an empty set, then E is an interval.

Proof:

First we note that if E is an interval, then for any $x, y \in E$ with $x < y$, the set $[x, y]$ is in E.

Then we prove Theorem 2.1 by contraposition. Suppose E is not an interval, then there is some $x, y \in E$ with $x < y$, such that $[x, y]$ is not in E. This means there exists some point $a \in [x, y]$ such that $a \notin E$. Both the set $(a, +\infty) \cap E$ and $(\infty, a) \cap E$ are non-empty, since the first one contain y, and the second one contains x.

Bound

- A bound is defined in the total set of a set A, i.e., if the total set of A is $\mathbb{Q}$, then the bound is in $\mathbb{Q}$; if it's $\mathbb{R}$, then the bound is in $\mathbb{R}$.
- Usually we *assume* the total set is $\mathbb{R}$
- A bound may not be an element in the A
- Therefore, if A doesn't have a maximum (or minimum), then the least upper bound (or greatest lower bound) of A is not in A.

Bound

Example:

1. The set $A = (-\infty, a)$ is bounded above in $\mathbb{R}$ with $supA = a$. It isn't in A.
2. The set $B = [b, +\infty)$ is bounded below in $\mathbb{R}$ with $infB = b$. It's in B since b is the minimum of B.
3. The set $C = [c, d) \cup (e, f)$ is bounded above and below in $\mathbb{R}$, so it's bounded with $supC = f$, $infC = c$.
4. The set $D = \{x \in \mathbb{Q}^+ : x = \frac{1}{n}, n \in \mathbb{N}^*\}$ is bounded above in $\mathbb{Q}^+$, but not bounded below in $\mathbb{Q}^+$.

Exercises-I

1. When learning the axioms of rational number, one student found that the operation of subsets of a non-empty set X is somewhat similar to that of rational number:

If we regard $\cup$ as $+$, $\cap$ as $\cdot$, then the equation

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

is just the distributivity law. Help him check whether P1 – P9 also hold for such operations.

2.

1. Prove that for $a_i \in \mathbb{Q}, i \in \mathbb{N}^*$, and n is a natural number,

$$|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$$

2. Prove that $|a-c| \leq |a-b| + |c-b|$

Exercises-I

3. We define $\mathbb{R}$ as a set containing rational numbers and the limits of rational numbers. Let $E \subseteq \mathbb{R}$ a non-empty subset of that is bounded above in $\mathbb{R}$. Now we don't assume P13 is an axiom. We want to prove P13 with P1 – P12 and other facts that we know:

- Let's call the set of form $(a, +\infty)$ or $[a, +\infty)$ *up interval*. Prove that the set of E's upper bound, denoted by U, is an *up interval*.
- (**Theorem 2.2**) E has a least upper bound.

Exercise-I*

4. Let A be bounded set in $\mathbb{R}$ (which means that the total set is $\mathbb{R}$), for any $\epsilon > 0$, there is an element x in A such that $|x - supA| < \epsilon$.

Function

What is a function?

There are some crucial properties should be mentioned when you want to describe a function, they are:

- Domain:
  $dom\ f := \{x \in X : (x, y)\ satisfies\ the\ requirement\ of\ f\}$
- Co-domain(target set):
  $Y = \{all\ "y"s\ such\ that\ (x, y)\ satisfies\ the\ requirement\ of\ f\}$
- Range:
  $ran\ f = \{y\ is\ in\ the\ target\ set:\ \ y = f(x)\}$
- How does it *map*?

(In general, which set is bigger? Co-domain or Range?)

Function

There are few ways to express a function:

Ways to express function	*Comment*
$f : X \to Y, x \longmapsto f(x)$	Pay attention to the forms of arrows.
$f : X \to Y, f(x) = \dots$	An intuitive way to express a function.
$f = \{(x, y) : P(x, y)\}$	When hard to show the explicit form of f.

Function

Common Misunderstanding:

Function is a *graph*

Rather, one can view a function as a machine that sends each element in its domain to its target set. This is the passive view of function. The function is like an "bow "that shoots an element in its domain to its target set. This is the active point view of a function.

However, often it's intuitive (and useful) to express a function using graphs, especially when f is an $\mathbb{R}$ to $\mathbb{R}$ function.

Common Misunderstanding:

The *co-domain is range* of f

The target set just contains $ranf$, it may be larger than $ranf$. For example, we can define a function

$$g : \mathbb{R} \longmapsto \mathbb{R}, g(x) = x^2$$

The range of f is $[0, +\infty)$, but the target set can simply be $\mathbb{R}$.

Comment. On the contrary, the domain of f contains exactly all the elements that have assignment with an element in f's target set.

Sequence

A sequence is defined as:

$$a_{(\cdot)} : \mathbb{N} \to \mathbb{N}, \ \ n \longmapsto a_n$$

And we start from looking at some common misunderstanding of sequences.

Sequence

Common misunderstanding:

A sequence may *not* contain infinitely many terms.

When we say "sequence" , we usually assume that it is infinite. If it's finite, i.e., it contains only $(a_0, a_1, a_2, a_3, \dots)$, we usually say it is a "*tuple*". This means the "Ω"on slides 114 is an *infinite set*. Similarly, a subsequence of a sequence is also infinite.

Sequence

Common misunderstanding:

A sequence is either convergent or divergent to *infinity*.

Of course, if a sequence is not convergent, we say it's "divergent". However, it doesn't mean it diverge to infinity.

A classical example is

$$a_n := (-1)^n$$

Important Results & Theorems & Comments.

- A convergent sequence is bounded. (Slides 122)
- A convergent sequence has precisely one limit. (Slides 124)
- (**Squeeze Theorem**)
  Let $(a_n), (b_n)$ and (c_n) be real sequences with $a_n < c_n < b_n$ for sufficiently large $n \in \mathbb{N}$. Suppose that $\lim\ a_n = \lim\ b_n =: a$. Then (c_n) converges and $\lim\ c_n = a$. (Slide 127)
  Comment. It is extremely useful for examining the convergence of a sequence that is bounded.
- Let (a_n) be a convergent sequence with limit a. Then any subsequence of (a_n) is convergent with the same limit. (Slide 139)
- Every real sequence has a monotonic subsequence. (Slide 140)

Sequence

- If a sequence has an accumulation point x, then there is a subsequence that converge to this point x.
- (**Bolzano–Weierstraß**)
  Every bounded real sequence has an accumulation point.
  (Slide 144)
  Comment. There are at least two proofs, which we will discuss later.
- Every monotonic and bounded (real) sequence is convergent.
  (Slide 135)
  Comment. This result holds for sequence in any space with an ordering (otherwise it's strange to even define "monotonic").

Bolzano–Weierstraß

Bolzano–Weierstraß Every bounded real sequence has an accumulation point.

1. Proof–1: On Horst's Slides.
2. Proof–2: Since(a_n) is bounded, assume $-M \leq a_n \leq M$ for all n. Divide the interval $[-M, M]$ into 2 sections:$[-M, 0], [0, M]$. One of the interval, denoted by $I^{(1)}$, must contain infinitely many "a_n"s(otherwise $(a_n)$is finite). Choose an $a_{(n,1)}$ in $I^{(1)}$. We bisect $I^{(1)}$ into two intervals, one of which, denoted by $I^{(2)}$ must contain infinitely many "a_n "s. Choose an $a_{(n,2)}$ in $I^{(2)}$ that is different from $a_{(n,1)}$. By repeatedly doing this procedure, we find a subsequence $(a_{n,k})_{k\in\mathbb{N}}$ that converges.

Limit

Now we take a step back, see some simple results for limit.

$$\text{Suppose } (a_n) \to a \text{ in } \mathbb{R} \text{ and } (b_n) \to b \text{ in } \mathbb{R}$$

1. $\lim(a_n + b_n) = a + b$
2. $\lim(a_n \cdot b_n) = a \cdot b$
3. $\lim \frac{a_n}{b_n} = \frac{a}{b}, b \neq 0$

We will prove 3 now.

$$\lim \frac{a_n}{b_n} = \frac{a}{b}, b \neq 0$$

Proof:

We want to prove that $|\frac{a_n}{b_n} - \frac{a}{b}| \to 0$ as $n \to \infty$. Since we don't want b_n to be zero, we fix some $M \in \mathbb{N}$ such that $|b_n - b| < \frac{1}{2}|b|$ for all $n > M$. Now, when $n > M$, we have:

$$\begin{aligned}
|\frac{a_n}{b_n} - \frac{a}{b}| &= |\frac{a_n b - b_n a}{b_n b}| = |\frac{a_n b - ab + ab - b_n a}{b_n b}| \\
&\leq \frac{|a_n - a||b|}{|b_n b|} + \frac{|a||b - b_n|}{|b_n b|} < \frac{2|a_n - a||b|}{b^2} + \frac{2|a||b - b_n|}{b^2}
\end{aligned}$$

Given $\epsilon > 0$, choose $N > M$ such that

$$\forall n > N, |a_n - a| < \frac{|b|\epsilon}{4}|b_n - b| < \frac{b^2}{|a|} \cdot \frac{\epsilon}{4}$$

then we have:

$$\forall n > N, |\frac{a_n}{b_n} - \frac{a}{b}| < \frac{|b|\epsilon}{4} \cdot \frac{2|b|}{b^2} + \frac{2|a|}{b^2} \cdot \frac{b^2}{|a|} \cdot \frac{\epsilon}{4} < \epsilon$$

Limit

Here we list some common results for limit:

- $\lim_{n\to\infty}(\frac{1}{n^{\alpha}}) = 0, \alpha \in (0, +\infty)$
- $\lim_{n\to\infty} \sqrt[n]{a} = 1, a > 0$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$

5. A sequence is defined as

$$(S_n)_{n\in\mathbb{N}}, S_1 = \sqrt{2}, S_2 = \sqrt{2\sqrt{2}}, S_3 = \sqrt{2\sqrt{2\sqrt{2}}}$$

Please calculate the limit of (S_n) as $n \to \infty$, if it exists.

Exercise-II

6. Let $(a_n), (b_n)$ be two real sequences. Furthermore, assume that $a_n < b_n$ for all n, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, $\lim(a_n - b_n) = 0$. Prove that there is an unique $m \in [a_n, b_n]$ for all n, such that

$$\lim a_n = \lim b_n = m$$

Exercise-II

7. Let (x_n) be a bounded real sequence. Then define

$$a_n := \sup_{m\geq n}(x_m), \quad b_n := \inf_{m\geq n}(x_m)$$

1. Prove that (a_n) is decreasing, while (b_n) is increasing.
2. Since both $(a_n), (b_n)$ are monotonic and bounded, they are convergent. We denote $\underline{\lim} x_n = \lim b_n$; $\overline{\lim} x_n = \lim a_n$. Show that:

$$\underline{\lim} y_n + \underline{\lim} z_n \leq \underline{\lim}(y_n + z_n) \leq \overline{\lim} y_n + \underline{\lim} z_n \leq \overline{\lim} y_n + \overline{\lim} z_n$$

Exercise-II*

8. Let (a_n) be a sequence such that

$$a_n = \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

Calculate the limit of (a_n).

9. Prove that $\lim \sqrt[n]{n} = 1$.

Have Fun and Learn Well!