Honor Mathematics Lecture 1

Logic

"Without logic, mathematics will fall apart… "

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October 29, 2021

Pay attention to following...

For studying Honor Mathematics series well...

- **Forget EVERYTHING you have learned in school before.**
- Do think more about the question in "()".
  e.g. "(How to prove?)"
- You are welcome to ask questions in an adequate manner.
- The class is designed to be interactive. Don't be so shy!
- Focus more on the idea of a proof. Don't just "recite" everything.

Overview

1. Statement
2. Logical Operation
3. Truth Table
4. Relations between Statements
5. Logical Quantifiers
6. Sets
7. Set Operations
8. Ordered Pairs
9. Russel Antinomy
10. Exercises

Statement

Statement, also called as proposition, is anything we can regard as being either *true* or *false*.
Some important things about statements:

- True statement
- False statement
- *Vacuous Truth*
- Statement variable
- Statement structure
  e.g. quantifier, predicate, specific value

Logical Operation

Well, statements are important, but it will not be in this case without logical operation to combine different statements together.

Logical Operation	
$\neg$	Negation
$\wedge$	Conjunction
$\vee$	Disjunction

For example, we see that we can have $3 > 2$ is a true statement, and $x^3 > 0$ is not a statement since we can not decide whether it is true or not. In the above example, the variable x is so-called *statement variable*.

If you want to go deeper, Let us see another example.

Logical Operation

Consider the statement

$$\text{"For any number } x,\ x^3 > 0\text{"}$$

The first part of the statement is a *quantifier*("for any number x"), while the second part is called a *statement frame* or *predicate*("$x^3 > 0$").

A statement frame becomes a statement(which can then be either true or false) when the variable takes on a specific value.

Logical Operation

Back to logical operation. It is clear that we can have something like

$$A \vee B$$

where A and B are two statements. We called these kinds of expression as *compound statement*.

A compound statement that is always true is called a *tautology*. On the other hand, if the compound statement is always false is called a *contradiction*.

Truth Table

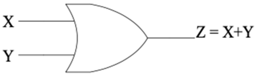
How to use Truth Table?

- Understand what the problem is about.
- Set up a Truth Table.
- Always cover all possible cases.

Just to show you how powerful the concept of Truth Table is, here is an example.

Application of Truth Table

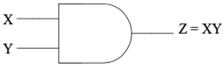
Have you ever consider how a computer add two number?
A: It is implemented by logic gates, and then circuits.
Introduce some logic gates.



2-input OR gate

X	Y	Z = X + Y
0	0	0
0	1	1
1	0	1
1	1	1

Truth Table



2-input AND gate

X	Y	Z = XY
0	0	0
0	1	0
1	0	0
1	1	1

Truth Table

Then, we can consider following:

Given two numbers as input, how to add them together?

As we all know, circuits work with binary number, e.g. 01011...
So, the question becomes

How to add two one digit binary numbers?

The idea is quite simple, that is let the circuit *mimic* the action when we perform addition. In other words, we want this circuit to cover all possible cases when we do the basic addition, namely

$$00 + 00 = 00 \quad 00 + 01 = 01 \quad 01 + 00 = 01 \quad 01 + 01 = 10$$

Adder

We consider how each result's digit will react to the changing of input by constructing a Truth Table as follows:

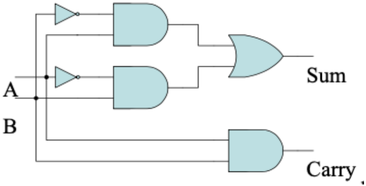
A	B	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Then if you done your assignment 1.4, you should know you can construct a *logic equation* from this Truth Table for **Sum** and **Carry**.

Logic gates for Adder

$$Sum = A'B + AB' \quad Carry = AB$$

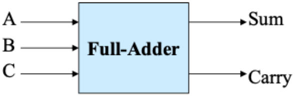
Then we can get following circuit



Which is so-called *Half Adder*.

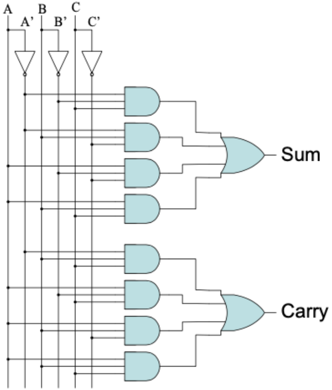
More about Adder . . .

You can design a more complicated circuit by using same method, like:



A	B	C	Sum	Carry
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

$$\begin{aligned}\text{Sum} &= A'B'C + A'BC' + AB'C' + ABC \\ &= \Sigma\, m(1, 2, 4, 7) \\ \text{Carry} &= A'BC + AB'C + ABC' + ABC \\ &= \Sigma\, m(3, 5, 6, 7)\end{aligned}$$



Which is so-called *Full Adder*. You will learn these interesting concept in VE270.

Relations between Statements

- Implication
$$A \Rightarrow B$$
- Equivalence
$$A \Leftrightarrow B$$
- Contraposition
$$(A \Rightarrow B) \Leftrightarrow (\neg A \Leftarrow \neg B)$$

Proof of the contraposition (**de Morgan rules**):

A	B	$\neg A$	$\neg B$	$\neg B \Rightarrow \neg A$	$A \Rightarrow B$	$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

And this is the concept of *tautology*.

Relations between Statements

Something very important related to tautology is called *vacuously true*. Consider this.

Example

Let M be the set of real numbers x such that $x = x + 1$. Then the statement

$$\underset{x \in M}{\forall} x > x$$

is true.

Why?

It's essentially similar to say that "All pink elephants can fly." is a true statement, because it is impossible to find a pink elephant that can't fly.

Exercise

Let P, Q be two sets such that $P \subseteq Q$. Then what is the relation between these two statements?

Statement A: $x \in P$. Statement B: $x \in Q$.

Logical Quantifiers

Logical Quantifiers		
Sign	Type	Interpretation
$\forall$	universal	for any; for all
$\exists$	existential	there exist; there is some
$\forall \ldots \forall \ldots$	nesting quantifier	for all ... for all ...
$\exists \ldots \exists \ldots$	nesting quantifier	there exists ... (such that) there exist ...
$\vdots$	$\vdots$	$\vdots$

We see that in order to use logical quantifiers properly, we need a *domain* of our predicates. This is where set is needed.

Sets

1. What is a set?
2. Common set types
   - Empty set: $\varnothing := \{x \colon x \neq x\}$
   - Total set
   - Subset
   - Proper subset
   - Power set

Simple question:

Why is $\varnothing$ a subset for any set X?

Example

let $A := \{4, 5, 6\}$ be a set.

- The total set can be $\mathbb{N}$
- $B := \{4, 5, 5, 6, 6\} = A$
- $C = \{1, 5\} \subseteq A$
- $P := \{\emptyset, \{4\}, \{5\}, \{6\}, \{4, 5\}, \{5, 6\}, \{4, 6\}, \{4, 5, 6\}\}$
  is the power set of A. (What is the cardinality of A ?)

Naive Set Theorey: Sets via Predicates

We see that we want to be able to talk about the *collection of objects*. However, it's hard to strictly define what an "object" or a "collection" is. The problem with what we are using, namely naive set theory is that any attempt to make a formal definition will lead to a contradiction.
However, we are not going into the detail for this, we just stick with what we need, and naive set theory is enough. We indicate that an object, called an *element* x is part of a collection, called a *set* X by writing $x \in X$. We characterize this relation by relating this with a predicate $P(x)$ such that

$$x \in X \iff P(x).$$

We write such a set X in the form of

$$X = \{x \colon P(x)\}.$$

Set Operations

Define

$$A := \{1, 2\} \quad B := \{2, 3\} \quad M := \{1, 2, 3, 4, 5\}$$

Set Operations		
$A \cup B$	Union	$\{1, 2, 3\}$
$A \cap B$	Intersection	$\{2\}$
$A \setminus B$	Difference	$\{1\}$
A^c	Complement	$\{3, 4, 5\}$

Simple question:

What is M^c ? Also, what is $\varnothing^c$?

Ordered Pairs

- What is an ordered pair?
- What is the difference between ordered pair and set?
- Concept of *Cartesian product*.

Question: How can we show the order relation by what we have defined, namely only use set?

There exist several paradoxes in native set theory, including:

1. Russel Antinomy
2. Cantor's paradox
3. Burali-Forti paradox

The above paradoxes illustrate the fundamental flaw of our naive theory, namely it's not **well-defined**.

However, these problems can be solved if we replaced naive set theory by a *modern axiomatic set theory*, but the detail about it is beyond our scope. However, we'll show Russel Antinomy mathematically rather than tell that barber story.

If you're interesting, you can see ZF-set theory, which can be considered the first success attempt for formalizing set theory.

Russel Antinomy

Theorem

Russel Anatinomy. The predicate $P(x)\colon x \notin x$ does not define a set

$$A = \{x \colon P(x)\}.$$

Proof.

If $A = \{x \colon x \notin x\}$ were a set, then we should be able to tell if for any set y such that it is in A or not.
But we see that if we consider $y = A$, then

- Assume $A \in A$. Then $P(A)$, which means $A \notin A$. $\lightning$
- Assume $A \notin A$. Then $\not P(A)$, which means $A \in A$. $\lightning$

We see that we can't decide $A \in A$ or $A \notin A$, hence A can not be a set. ■

Exercises

1. Let A, B, C be three statements. Use truth table to prove that

$$(A \vee B) \wedge C \equiv (A \wedge C) \vee (B \wedge C)$$

2. Let A, B, C be three sets. Prove that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

From Exercise 1 & 2, we can see that sets and statements are similar.

Exercises

3. Check whether the following sentences are true statement, false statement, or not a statement.

- $\forall x, y \in \mathbb{R}, x^2 + y^3 \geq 0$
- Let $f(a) = a^4$, then $f(0) > 0$
- For any $a \in \mathbb{R}, a^4 > 0$
- An African Elephant is very big.
- Let A, B be two statements, then $(A \vee B) \Leftrightarrow \neg(\neg A \wedge \neg B)$

Simple question

Rewrite above sentences in quantifiers form if they are statement.

Exercises

4. Use quantifiers to rewrite the following definition of covergence:

Let $(a_n)_{n \in \mathbb{N}}$ be a real sequence. If for some fixed $c \in \mathbb{R}$, for any $a > 0$, there is an $N \in \mathbb{N}$, such that for all $n > N$, $|a_n - c| < a$, then we say (a_n) converges.

Reference

Reference.

- Exercises from 2019–Vv186 TA-Zhang Leyang.
- Figure for circuits from Ve270 T2-Logic Gate Slides.

Have Fun and Learn Well!