UM-SJTU Joint Institute

Vv285 / MATH285

1^{st} UM-SJTU JIntegration Bee Problem Set

JIntegration Bee

July 5th, 2021

Answer Sheet

Write Your Name(s) Here:

This is the answer sheet of JIntegration Bee. You need only hand in this sheet at the end of the activity. Only the answers that satisfy the following requirements will be graded.

(a) You should provide only **one answer** for each question.

(b) All the constants in your answer should be in **precise-value form**. For example, you should write π instead of 3.14, and you should write $\log 4$ instead of 1.386.

Part 1

Answer for 1.1:

Answer for 1.2:

Answer for 1.3:

Part 2

Answer for 2.1:

Answer for 2.2:

Answer for 2.3:

Answer for 2.4:

Part 3

Answer for 3.1:

Answer for 3.2:

Answer for 3.3:

Answer for 3.4:

Part 4

Answer for 4.1:

Answer for 4.2:

Answer for 4.3:

Answer for 4.4:

Part 1 : Basic Integration

1.1 An Unusual Integral (2 Marks)

Calculate

$$\int_0^{\ln(x)} \max\{1, y\}\, dy,$$

where $x > e$.

1.2 Even & Odd Functions (3 Marks)

For $a > 0$, calculate

$$\int_{-a}^{a} \frac{\cos(x)}{1 + e(x)^{o(x)}}\, dx,$$

where $e(x)$ is a continuous strictly positive even function, and $o(x)$ is an odd function. **Express your answer in terms of** a.

1.3 A Simple Trigonometric Function (4 Marks)

Calculate

$$\int_0^{\pi} \sin^3(x)\, dx$$

Part 2 : Virtual Reality

In 2035, the UM-SJTU Joint Institute decides to bring in virtual reality equipment in order to keep up with the trend of online teaching systems by using this cutting end technology.

Pingping and Bangbang, both are JI students, really want to try out this high-tech machine, but they don't have permission to do so. Thus, they decide to sneak into the LongBin Building at night. They discover that the front door is guarded by security, so they find a window with a weird shape at the back of the building and try to climb in through it.

2.1 Get Into the Building! (3 Marks)

Since the window is very old, they want to break it first. They decide to weight it and see whether they can do it by themselves.

After looking at it for a moment, Pingping says: "It seems like if we use 2D coordinates to describe it, we can let its vertices be located at

$$(2, 0), (5, 3), (6, 7), (3, 4),$$

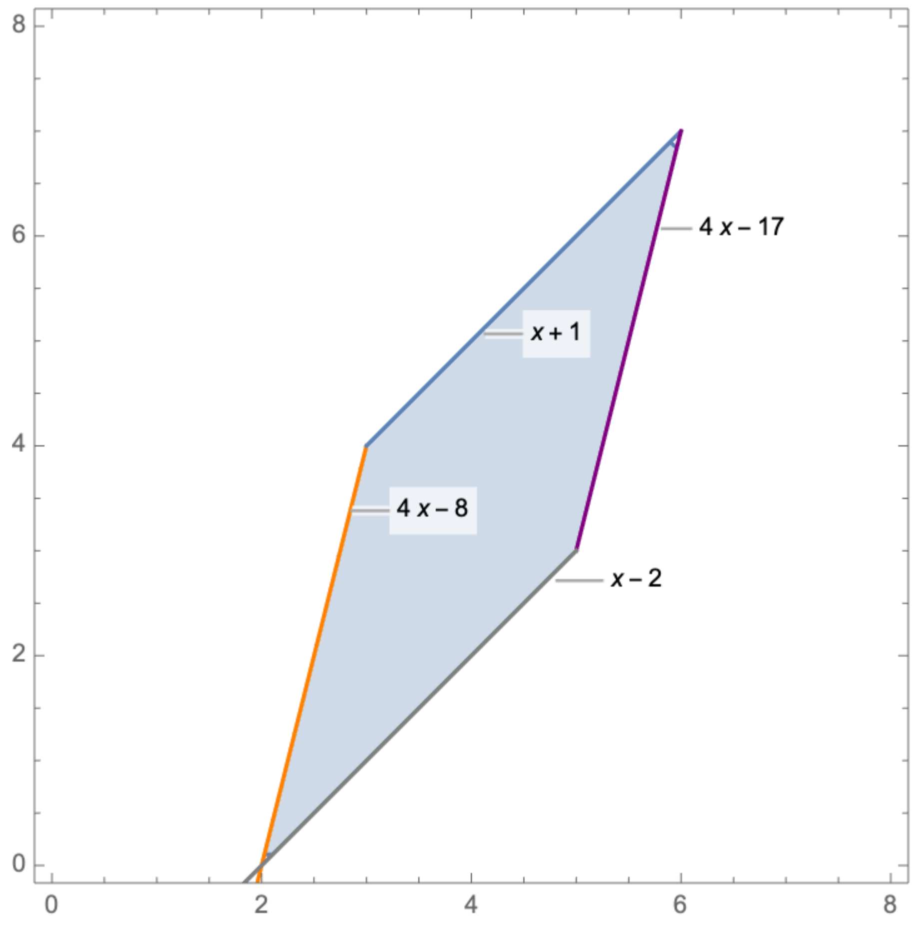
and in this way, it seems to me that the density of this window is described by

$$d(x, y) = 6x - 3y.$$

."

Bangbang: "..., I don't even want to ask how you came up with this..., but anyway, assume you're right..."

After some careful drawing, Pingping has the following figure:



Calculate the weight of the window for him, i.e., find

$$\iint_R (6x - 3y)\, dA.$$

where R describes the region of the window. You may want to use the following transformation:

$$x = \frac{1}{3}(v - u), \quad y = \frac{1}{3}(4v - u).$$

2.2 Breathe! (4 Marks)

After some struggles, Pingping and Bangbang finally get into the building. When they see the machine, they are shocked by how complicated it is. It has far too many switches! Pingping is too excited about it, so he decides to just lie in the machine and closes it.

After 5 minutes, Pingping feels the effects of hypoxia (lack of oxygen). Meanwhile, Bangbang is reading the instructions for the machine. He finds out that when one wants to use it, one needs to turn on the air circulation! Bangbang is pretty certain that Pingping did not do this (which is true). But Bangbang is not able to open the machine from the outside. In this emergency situation, Bangbang decides to call the security for help...

What is Pingping doing during this time? He finds out that the shape of the machine can be described as

$$x = 2y^2 + 2z^2 - 5$$

with a cap described by

$$x = 1.$$

Furthermore, it seems to him that the current oxygen density is given by the equation

$$d(x, y, z) = yz.$$

Disregarding how Pingping comes up with all of this, find the total amount of oxygen left in the machine for him! Calculate

$$\iiint_M yz\, dV$$

where M is the region of the machine described above.

Wait, what? You want me to provide you the graph of M just like in the last question? Hey, it's time to grow up!

(P.S. Don't always trust Pingping's intuition, he may be wrong some of the time!)

2.3 FIVE Dimension (3 Marks)

After Pingping realizes that his intuition is wrong, he feels sad. He opens the machine and climbs out of it. But he does not see Bangbang. The only thing he sees is the instruction book for this virtual machine, which is left on the floor opening to the page that explains how to turn on the air circulation function for this machine.

Pingping is excited again! He immediately decides to go back into the virtual machine, turn on the air circulation function and start to explore the magic of this machine. After some time, when Pingping finally finds out how to enter the virtual reality, Bangbang and the security arrive. The security guard is so afraid to be fired becasue of his inattention that he decides to open the machine from the outside by force. But at that time, the system of the machine detects this and decides to turn on its protection mechanism.

Just then, Pingping starts to see a strange shape that he never saw before. He is fascinated by it. It's so complex, so beautiful, and so symmetric. Long time passes by and Pingping murmurs: "It's a five dimensional ball... I know it... It must be... It is a five dimensional sphere with radius R..."

He decides to calculate the volume of this five dimensional ball, can you help him? Namely, please calculate

$$|B_5| = \int \cdots \int_{B_5} 1 \, dB_5.$$

What? again? You want me to provide you a graph for that ball? Get out of here!

(p.s. Maybe you will need the result from **1.3**, but who knows...)

2.4 Challenge from Bangbang (4 Marks)

After solving the volume of a five dimensional ball, the whole virtual reality world blacks out. Just then, Pingping is forced to leave the virtual environment and re-enter reality. He notices that there is a warning from the machine telling him that someone is trying to breakout the machine from outside.

Pingping thinks that this is because Bangbang is jealous that he is not the first one to try this machine, so he yells out: "How can you cut down the power and do this to me? I'm enjoying the magic of this machine! I even saw a five-dimensional ball!"

Bangbang: "Stop spouting nonsense! If you are able to identify a five dimensional ball, you must be able to solve this! I bet you can't!"

As soon as Bangbang challenges Pingping, Pingping opens the machine and jumps out. He has never been **insulted** like this before! No one can judge his ability of doing integration! He yells at Bangbang: "Show me what you have!" Bangbang slowly enunciates the integral, which is

$$\int_{\mathbb{R}} \frac{\cos(x)}{x^2+1} \, dx.$$

After Bangbang is done, Pingping sneers at him. "Ha... Haha... Hahaha... Are you kidding me? Only one variable? You are insulting me again! I'll do this in 2 minutes!"

Please help Pingping to find out the answer of this integral!

(Probably not solving it in 2 minutes is also OK, but Pingping will be upset...)

Part 3 : Things May Not Behave the Way You Expected

Suzumiya Haruhi is Watahashi Yasumi's math tutor. She is teaching her how to solve integral problems. She does not like the ordinary approach and the too normal daily life. She looks forward to making the world more interesting and to spreading cheer and excitement. Some problems may seem simple, but indeed are tricky. Some problems seem huge and complicated, but are easy to solve. Some problems have the most ugly parameters, but the solution is quite concise. These are quite interesting problems in Haruhi's mind. Haruhi offered Yasumi three integral exercises. You are Yasumi – please solve these problems!

In your answer, leave expressions involving arctan as they are. Do not calculate them out except to let $\arctan 0 = 0$. Do not leave any square root in the denominator. Do not write log and arctan more than once in your answer.

3.1 A Simple Integral (3 Marks)

$$K = \int_1^{\sqrt{2}} \frac{1}{1+x^4}\,dx.$$

3.2 A Complicated Double Integral (3 Marks)

$$M = \iint_{x^2+y^2\leqslant 1} \left|\frac{x+y}{\sqrt{2}} - x^2 - y^2\right| \mathrm{d}x\ \mathrm{d}y.$$

3.3 An Ugly Line with a Beautiful Line Integral Result (2 Marks)

Given the curve $\mathit{\Gamma}$

$$\begin{cases} (x-y)^2 = a(x+y) \\ x^2 - y^2 = \frac{9}{8}z^2. \end{cases}$$

calculate the length ℓ on $\mathit{\Gamma}$ from $A\colon\ (0,0,0)$ to $B\colon\ (1, y_0, z_0)$.

3.4 Last Exercise for Yasumi (4 Marks)

Yasumi successfully solved the three questions in **Part 3** and it's time to go visit the SOS Brigade to have some fun in after-class club activities. In the club activity room, Haruhi is quite proud that Yasumi's ability to solve integrals has improved a lot. Now she asks Yasumi to solve one problem on the blackboard. ou are Yasumi – please solve this problem!

$$P = \iiint_{\mathit{\Omega}} \left[\frac{1}{yz}\frac{\partial F}{\partial x} + \frac{1}{xz}\frac{\partial F}{\partial y} + \frac{1}{xy}\frac{\partial F}{\partial x}\right] \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z.$$

where $\mathit{\Omega} = \{1 \leqslant yz \leqslant 2,\ 1 \leqslant xz \leqslant 2,\ 1 \leqslant xy \leqslant 2\}$ and $F(x,y,z) = 2x + My + \lfloor \ell \rfloor z$, where M and ℓ were introduced in 3.2 and 3.3, respectively.

Based on characters in *The Surprise of Haruhi Suzumiya*, Nagaru Tanigawa

Part 4 :

AppleDog and DT, both are robot players, would like to create and run a small video website in their spare time. To achieve this, they need to define a system for their website that assigns ranks for existing videos and videos under review. By investigating the ranking system of *Piggy Piggy* (a famous video website), they list some criteria for their system.

(a) The score of a video decreases as time (x_1) passes by;

(b) The score of a video increases as the heat (x_2) of the topic of the video increases;

(c) The score of a video increases as the number of views (x_3) of it increases;

(d) The score of a video increases as the total number of likes and collections (x_4) it obtains increases;

(e) x_2, x_3, x_4 are all related to time x_1.

After some thought, DT derives a function

$$f : \mathbb{R}^4 \to \mathbb{R}, \quad f(x_1, x_2, x_3, x_4) = 2x_2(\alpha + x_1^2)^{-1} + x_4 e^{x_4}(1 + x_3)$$

where x_i's are defined as above, and $\alpha > 0$ is a parameter depending on the type of a video. The score of an *existing video* is the line integral of f along its data curve

$$\gamma(t) = (t, x_2(t), x_3(t), x_4(t)),$$

and the predicted score of a video *under review* is the integral of f over the domain E bounded by

$$0 \le x_1 \le 7; \quad 0 \le x_2 \le \sqrt{(\beta + x_1^2)^{-1}};$$
$$0 \le x_3 \le 3; \quad 0 \le x_4 \le \log(1 + x_3),$$

where $\beta > 0$ is a parameter depending on the topic of a video, and "log" is the inverse function of e^x.

AppleDog is curious about DT's model, so she tries with some examples. **Your task is to help Appledog calculate the results**.

4.1 Score of Existing Video (2 Marks)

Suppose in the first 7 days the data of a video (i.e., heat, number of views, etc.) follows the curve $\mathcal{C}$ parametrized by

$$\gamma : [0, 7] \to \mathbb{R}^4, \quad \gamma(t) = (t, \sqrt{\alpha}, t, 0)$$

Determine the score of this video, i.e., evaluate the integral below (**express your answer in terms of α and β**)

$$\int_{\mathcal{C}} f.$$

Then determine the value of α when the score of this video is $2\sqrt{2}$. This α will be used in **4.4**.

4.2 Area of the Domain E (3 Marks)

AppleDog first wants to see the area of E, where E is defined above. Help her calculate it (**express your answer in α**).

4.3 Predicted Score of Video Under Review (4 Marks)

Help AppleDog calculate the predicted score of a video under review, i.e., evaluate the following integral, where f is defined as above (**express your answer in** α **and** β)

$$\int_0^7 \int_0^{\sqrt{(\beta+x_1^2)^{-1}}} \int_0^3 \int_0^{\log(1+x_3)} f(x_1, x_2, x_3, x_4)\, dx_4 dx_3 dx_2 dx_1.$$

4.4 Challenge From AppleDog (3 Marks)

Knowing that DT has been attending robot competitions since the age of 16, AppleDog believes that he must be smart and good at math. Thus, she gives him an integral as a reward for his contribution to the video ranking system:

Let α be the value calculated in **4.1**. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and

$$\int_0^1 f(x)dx = \alpha.$$

Evaluate the following integral (your answer **should not** contain α).

$$\int_0^1 \int_0^x \int_0^y f(x)f(y)f(z)\, dzdydx.$$

The story is based on the TV series "DT/AppleDog's Time", 2021

Answers

Answer for 1.1:

$$\frac{\ln^2(x)}{2} + \frac{1}{2}$$

Answer for 1.2:

$$\sin(a)$$

Answer for 1.3:

$$\frac{4}{3}$$

Answer for 2.1:

$$\frac{243}{2}$$

Answer for 2.2:

$$0$$

Answer for 2.3:

$$\frac{8\pi^2}{15}R^5$$

Answer for 2.4:

$$\frac{\pi}{e}$$

Answer for 3.1:

$$-\frac{\sqrt{2}}{8}\ln\frac{3+2\sqrt{2}}{5} + \frac{\sqrt{2}}{4}\arctan\frac{1}{2}$$

Answer for 3.2:

$$\frac{9\pi}{16}$$

Answer for 3.3:

$$\sqrt{2}$$

Answer for 3.4:

$$\left(1 + \frac{9\pi}{16}\right)\ln^2 2 + 2\ln 2$$

Answer for 4.1:

$$2\sqrt{2}\arctan\bigl(7/\sqrt{\alpha}\bigr), \quad \alpha = 28^2/\pi^2 \text{ or } \alpha = 784/\pi^2$$

Answer for 4.2:

$$(4\log 4 - 3)\operatorname{arcsinh}(\frac{7}{\sqrt{\beta}}) \quad \text{or}$$
$$(4\log 4 - 3)\log\left(\frac{\sqrt{49+\beta}+7}{\sqrt{\beta}}\right) \quad \text{or}$$
$$(4\log 4 - 3)\log\left(\sqrt{(\frac{7}{\sqrt{\beta}})^2 + 1} + \frac{7}{\sqrt{\beta}}\right)$$

Answer for 4.3:

$$\frac{4\log 4 - 3}{\beta - \alpha}\left[\frac{\arctan(7/\sqrt{\alpha})}{\sqrt{\alpha}} - \frac{\arctan\bigl(7/\sqrt{\beta}\bigr)}{\sqrt{\beta}}\right] + \left(\frac{64}{3}\log 4 - \frac{41}{2}\right)\operatorname{arcsinh}\left(\frac{7}{\sqrt{\beta}}\right)$$

Answer for 4.4:

$$\frac{392}{3\pi^2} \quad \text{or} \quad \frac{28^2}{6\pi^2} \quad \text{or} \quad \frac{784}{6\pi^2}$$