

CS598  
Topics in Graph Algorithms

Pingbang Hu

November 25, 2024

## Abstract

This is an advanced graduate-level graph algorithm course taught by [Chandra Chekuri](#) at University of Illinois Urbana-Champaign. This exploratory seminar-style course will cover selection of topics in graph algorithms with an emphasis on recent developments on fast algorithms for a variety of problems such as shortest paths, flows, cuts, and matchings. Structural results and connections to past ideas and results will also be discussed. More information can be found on the [course website](#).



This course is taken in Fall 2024, and the date on the cover page is the last updated time.

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# Chapter 1

## Introduction

### Lecture 1: Overview

Throughout the course, we consider a graph  $G = (V, E)$  such that  $n := |V|$  and  $m := |E|$ .

27 Aug. 11:00

### 1.1 Minimum Spanning Tree

Finding the minimum cost **spanning tree** (MST) in a connected graph is a basic algorithmic problem that has been long-studied. We introduce the problem formally.

**Definition 1.1.1 (Spanning tree).** A *spanning tree*  $T$  of a connected graph  $G = (V, E)$  is an induced subgraph of  $G$  which spans  $G$ , i.e.,  $V(T) = V$  and  $E(T) \subseteq E$ .

Then, the problem can be formalized as follows.

**Problem 1.1.1 (Minimum spanning tree).** Given a connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , find the min-cost **spanning tree**.

**Remark.** The edge costs need not be positive, but we can make them positive by adding a large number without affecting correctness.

Standard algorithms that are covered in most undergraduate courses are **Kruskal's algorithm**, **Jarník-Prim's (JP) algorithm**, and (sometimes) **Borůvka's algorithm**. There are many algorithms for **MST** and their correctness relies on two simple rules (structural properties), for cut and cycle respectively:

**Lemma 1.1.1 (Cut rule).** If  $e$  is a minimum cost edge in a cut  $\delta(S)$  for some  $S \subseteq V$ , then  $e$  is in some **MST**. In particular, if  $e$  is the unique minimum cost edge in the cut,  $e$  is in every **MST**.

**Definition 1.1.2 (Light).** An edge  $e$  is *light* or *safe* if there exists a cut  $\delta(S)$  such that  $e$  is the cheapest edge crossing the cut. Moreover,  $e$  is *light* w.r.t. a set of edges  $F \subseteq E$  if  $e$  is light in  $(V, F)$ .

**Lemma 1.1.2 (Cycle rule).** If  $e$  is the highest cost edge in a cycle  $C$ , then there exists an **MST** that does not contain  $e$ . In particular, if  $e$  is the unique highest cost edge in  $C$ ,  $e$  cannot be in any **MST**.

**Definition 1.1.3 (Heavy).** An edge  $e$  is *heavy* or *unsafe* if there exists a cycle  $C$  such that  $e$  is the highest cost edge in  $C$ . Moreover,  $e$  is *heavy* w.r.t. a set of edges  $F \subseteq E$  if  $e$  is heavy in  $(V, F)$ .

**Corollary 1.1.1.** Suppose the edge costs are unique and  $G$  is connected. Then the **MST** is unique and consists of the set of all **light** edges.

**Remark.** Without loss of generality, we can assume that the cost are unique by, e.g., perturbation or consistent tie-breaking rule.

### 1.1.1 Standard Algorithms

Let's review the basic algorithms, the data structures they use, and the run-times that they yield.

#### Kruskal's Algorithm

Intuitively speaking, **Kruskal's algorithm** sorts the edges in increasing cost order and greedily inserts edges in this order while maintaining a maximal forest  $F$  at each step. When considering the  $i^{\text{th}}$  edge  $e_i$ , the algorithm needs to decide if  $F + e_i$  is a forest or whether adding  $e$  creates a cycle.

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##### Algorithm 1.1: Kruskal's Algorithm

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**Data:** A connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$

**Result:** A **MST**  $T = (V, F)$

```

1 Sort the edges such that  $c(e_1) \leq c(e_2) \leq \dots \leq c(e_m)$  //  $O(m \log n)^a$ 
2  $F \leftarrow \emptyset$  // Initialize the tree
3 for  $i = 1, \dots, m$  do
4   if  $e_i + F$  has no cycle then
5      $F \leftarrow F + e_i$ 
6 return  $(V, F)$ 
```

---

<sup>a</sup>Since the graph is connected,  $O(m \log m) = O(m \log n)$  as  $n/2 \leq m \leq n^2$ .

**Theorem 1.1.1.** **Kruskal's algorithm** takes  $O(m \log n)$ .

**Proof.** Sorting takes  $O(m \log n)$  time. The standard solution for **line 4** is to use a **union-find** data structure. Union-find data structure with path compression yields a total run time, after sorting, of  $O(m\alpha(m, n))$  where  $\alpha(m, n)$  is **inverse Ackerman function** which is extremely slowly growing. Thus, the bottleneck is sorting, and the run-time is  $O(m \log n)$ . ■

#### Jarnik-Prim's Algorithm

**Jarnik-Prim's algorithm** grows a tree starting at some arbitrary root vertex  $r$  while maintaining a tree  $T$  rooted at  $r$ . In each iteration it adds the cheapest edge leaving  $T$  until  $T$  becomes **spanning**. Thus, the **Jarnik-Prim's algorithm** takes  $n - 1$  iterations.

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##### Algorithm 1.2: Jarnik-Prim's Algorithm

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**Data:** A connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$

**Result:** A **MST**  $T = (V, F)$

```

1  $r \leftarrow \text{uniform}(V)$  // Sample a root
2  $V' \leftarrow \{r\}, F \leftarrow \emptyset$  // Initialize the tree
3 while  $V' \neq V$  do
4    $e \leftarrow \arg \min_{e=(u,v) \in \delta(V'), u \in V'} c(e)$ 
5    $F \leftarrow F + e, V' \leftarrow V' + v$  // Update the tree
6 return  $(V, F)$ 
```

---

**Theorem 1.1.2.** **Jarnik-Prim's algorithm** takes  $O(m + n \log n)$ .

**Proof.** To find the cheapest edge leaving  $T$  (**line 4**), one typically uses a priority queue where we maintain vertices not yet in the tree with a key for  $v$  equal to the cost of the cheapest edge from  $v$  to the current tree. When a new vertex  $v$  is added to  $T$  the algorithm scans the edges in  $\delta(v)$  to update the keys of neighbors of  $v$ . Thus, one sees that there are a total of  $O(m)$  **decrease-key** operations,  $O(n)$  **extract-min** operations, and initially we set up an empty queue. Standard priority queues

implement **decrease-key** and **extract-min** in  $O(\log n)$  time each, so the total time is  $O(m \log n)$ . However, **Fibonacci heaps** and related data structures show that one can implement **decrease-key** in amortized  $O(1)$  time which reduces the total run time to  $O(m + n \log n)$ . ■

**Remark.** The **Jarnik-Prim's algorithm** runs in linear-time for moderately dense graphs!

### Borůvka's Algorithm

**Borůvka's algorithm** seems to be the first **MST** algorithm, which has very nice properties and essentially uses no data structures. The algorithm works in phases. We describe it recursively to simplify the description, while refer to **Algorithm 1.3** for the real implementation. In the first phase the algorithm finds, for each vertex  $v$  the cheapest edge in  $\delta(v)$ . By the **cut rule** this edge is in every **MST**.

**Note.** An edge  $e = uv$  may be the cheapest edge for both  $u$  and  $v$ .

The algorithm collects all these edges, say  $F$ , and adds them to the tree. It then shrinks the connected components induced by  $F$  and recurses on the resulting graph  $H = (V', E')$ . It's easy to see that **Borůvka's algorithm** can be parallelized, unlike the other two algorithms.

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#### Algorithm 1.3: Borůvka's Algorithm

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**Data:** A connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$   
**Result:** A **MST**  $T = (V, F)$

```

1  $F = \emptyset$  // Initialize the tree
2  $\mathcal{S} \leftarrow \{S_v = \{v\}\}$  // Collection of all sets
3 while  $|\mathcal{S}| > 1$  do
4    $\mathcal{S}' \leftarrow \mathcal{S}$  // Make a copy
5   for  $S \in \mathcal{S}$  do
6      $e_S = (u, v) \leftarrow \arg \min_{e \in \delta(S)} c(e)$ 
7      $\mathcal{S}' \leftarrow \mathcal{S}' - \{S_u, S_v\} + S_u \cup S_v^a$  // Merge (i.e., shrink)
8      $F \leftarrow F + e_S$  // Update the tree
9    $\mathcal{S} \leftarrow \mathcal{S}'$  // Update  $\mathcal{S}$ 
10 return  $(V, F)$ 

```

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<sup>a</sup>Here,  $S_u$  and  $S_v$  both refer to  $S := S_u \cup S_v$  later in the algorithm.

**Theorem 1.1.3.** **Borůvka's algorithm** takes  $O(m \log n)$ .

**Proof.** The first phase needs  $O(m)$  from a linear scan of the adjacency lists, and also computing  $H$  (i.e., shrinking) can be done in  $O(m)$  time. The main observation is that  $|V'| \leq |V|/2$  since each vertex  $v$  is in a connected component of size at least 2 as we add an edge leaving  $v$  to  $F$ . Thus, the algorithm terminates in  $O(\log n)$  phases for a total of  $O(m \log n)$  time. ■

### 1.1.2 Faster Algorithms

A natural question is whether there is a linear-time **MST** algorithm. A brief history of this line:

- Very early on, Yao, in 1975, obtained an algorithm that ran in  $O(m \log \log n)$  [Yao75], which leverages the idea developed in 1974 for the linear-time Selection algorithm.
- In 1987, Fredman and Tarjan [FT87] developed the Fibonacci heaps and give an **MST** algorithm which runs in  $O(m \log^* n)$ .<sup>1</sup> This was further improved to  $O(m \log \log^* n)$  [Gab+86].
- Karger, Klein, and Tarjan [KKT95] obtained a linear time randomized algorithm.
- The fastest known deterministic algorithm runs in  $O(m \alpha(m, n))$  [Cha00].

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<sup>1</sup>Formally, it runs in  $O(m \beta(m, n))$ , where  $\beta(m, n)$  is the minimum value of  $i$  such that  $\log^{(i)} n \leq m/n$ , where  $\log^{(i)} n$  is the logarithmic function iterated  $i$  times. Since  $m \leq n^2$ ,  $\beta(m, n) \leq \log^* n$ .

**Note.** Pettie and Ramachandran gave an optimal deterministic algorithm in the comparison model without known what its actual running time is [PR02]!

Perhaps an easier question is the following.

**Problem 1.1.2 (MST verification).** Given a graph  $G$  and a tree  $T$ , decide  $T$  is an **MST** of  $G$  or not.

One can always use an **MST** algorithm to solve the **verification** problem, but not necessarily the other way around. Interestingly, there is indeed a linear-time **MST verification** algorithm based on several non-trivial ideas and data structures and was first developed in the RAM model by Dixon, Rauch, and Tarjan [DRT92] with insights from Komlós [Kom85]. Simplification is done by King [Kin97].

**Note (RAM model).** The RAM model allows bit-wise operation on  $O(\log n)$  bit words in  $O(1)$  time.

**Theorem 1.1.4 (MST verification).** There is a linear-time **MST verification** algorithm in the RAM model. In fact, the algorithm is based on a more general result that we will need: Given a graph  $G = (V, E)$  with edge costs and a **spanning tree**  $T = (V, F)$ , there is an  $O(m)$ -time algorithm that outputs all the  **$F$ -heavy** edge of  $G$ .

**Proof.** The original complicated algorithm has been simplified over the years. See lecture notes of Gupta and Assadi for accessible explanation, also the MST surveys [Eis97; Mar08]. ■

### Fredman-Tarjan's Algorithm

Here we briefly describe Fredman and Tarjan's algorithm [FT87; Mar08] via Fibonacci heaps, which is reasonably simple to describe and analyze modulo a few implementation details that we will gloss over for the sake of brevity. First, we develop a simple  $O(m \log \log n)$  time algorithm by combining **Borůvka's algorithm** and **Jarnik-Prim's algorithm**.

**As previously seen.** **Jarnik-Prim's algorithm** takes  $O(m + n \log n)$  time via Fibonacci heaps where the bottleneck is when  $m = o(n \log n)$ . On the other hand, **Borůvka's algorithm** starts with a graph on  $n$  nodes and after  $i^{\text{th}}$  phases, reduces the number of nodes to  $n/2^i$ ; each phase takes  $O(m)$  times.

**Intuition.** Suppose we run **Borůvka's algorithm** for  $k$  phases and then run **Jarnik-Prim's algorithm** once the number of nodes is reduced. We can see that the total run time is  $O(mk)$  for the  $k$  phases of **Borůvka's algorithm**, and  $O(m + n/2^k \log n/2^k)$  for the **Jarnik-Prim's algorithm** on the reduced graph. Thus, if we choose  $k = \log \log n$ , we obtain a total run-time of  $O(m \log \log n)$ .

Tarjan and Fredman obtained a more sophisticated scheme based on **Jarnik-Prim's algorithm**, where the basic idea is to reduce the number of vertices. The algorithm runs again in phases, and we describe the first phase here.

**Intuition (First phase).** Start growing the tree. If the heap gets too big, we stop.

Consider an integer parameter  $t$  such that  $1 < t \leq n$ . Pick an arbitrary root  $r_1$  and grow a tree  $T_1$  via **Jarnik-Prim's algorithm** with a Fibonacci heap. We stop the tree growth when the heap size exceeds  $t$  for the first time or if we run out of vertices. All the vertices in the tree are marked as visited. Now pick an arbitrary, unmarked vertex as root  $r_2 \in V - T$  and grow a tree  $T_2$ , and we stop growing  $T_2$  if it touches  $T_1$ , in which case it merges with it, or if the heap size exceeds  $t$  or if we run out of vertices. The algorithm proceeds in this fashion by picking new roots and growing them until all nodes are marked.

**Note.** While growing  $T_2$ , the heap may contain previously marked vertices. It is only when the algorithm finds one of the marked vertices as the cheapest neighbor of the current tree that we merge the trees and stop.

It's easy to see that the **first phase of Fredman-Tarjan algorithm** correctly adds a set of **MST** edges  $F$ . After this, we simply shrink these trees and recurse on the smaller graph.

**Algorithm 1.4:** Fredman-Tarjan's Algorithm**Data:** A connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ **Result:** A **MST**  $T = (V, F)$ 

```

1  $V' \leftarrow V, F \leftarrow \emptyset$  // Initialize the tree
2 while  $|V| > 1$  do
3    $T \leftarrow \text{Grow}(G)$  // First phase
4    $F \leftarrow F \cup E(T)$  // Update the tree
5   Shrink  $G$  w.r.t.  $T$ , update  $V$  and  $E$  // Second phase
6 return  $(V', F)$ 
7
8 Grow( $G$ ):
9    $V' \leftarrow \emptyset, F \leftarrow \emptyset, T \leftarrow (V', F)$  // Initialize the forest
10  while  $V' \neq V$  do
11     $r \leftarrow \text{uniform}(V - V')$  // Pick an unmarked vertex
12     $T' \leftarrow (\{r\}, \emptyset)$  // Initialize a tree
13    while  $|N(T')| < t$  or  $V(T') \cap V' \neq \emptyset$  do
14      Run one more step of Jarnik-Prim( $r, T'$ ) // Starting at  $r$ , maintaining  $T'$ 
15       $V' \leftarrow V' \cup V(T')$  // Mark
16       $F \leftarrow F \cup E(T')$  // Update the forest by merging the tree
17  return  $(V, F)$  // Return a forest of  $G$ 

```

**Note.** This can be seen as a parameterized version of **Borůvka's algorithm**.

The difficult part is to determine its runtime. We have the following.

**Theorem 1.1.5.** **Fredman-Tarjan's algorithm** takes  $O(m\beta(m, n))$ .

**Proof.** Firstly, the total time to scan edges and insert vertices into heaps and do **decrease-key** is  $O(m)$  since an edge is only visited twice, once from each end point. Since each heap is not allowed to grow to more than size  $t$ , the total time for all the **extract-min** operations take  $O(n \log t)$ . With the fact that the initialization of each data structure is easy as it starts as an empty one, hence, the **first phase** takes  $O(m + n \log t)$ . We claim that it also reduces the number of vertices to  $2m/t$ .

**Claim.** The number of connected components induced by  $F$  is  $\leq 2m/t$  after the **first phase**.

**Proof.** Let  $C_1, \dots, C_h$  be the connected components of  $F$ . If for every  $C_i$ ,  $\sum_{v \in C_i} \deg(v) \geq t$ ,

$$2m = \sum_{v \in V} \deg(v) = \sum_{i=1}^h \sum_{v \in C_i} \deg(v) \geq ht \Rightarrow h \leq \frac{2m}{t}.$$

To see why the assumption holds, consider the growth of a tree  $T'$  in **line 14**:

- If we stop  $T'$  because heap size  $|N(T')|$  exceeds  $t$ , then each of the vertex in the heap is a witness to a unique edge incident to  $T'$ , hence the property holds.
- If  $T'$  merged with a previous tree, the property holds because the previous tree already had the property and adding vertices only increases the total degree of the component.

The only reason the property may not hold is if **line 17** terminates a tree because all vertices are already included in it, but then that phase finishes the algorithm. ⊗

The question reduces to choosing  $t$ .

**Intuition.** We want linear time in the **first phase**, i.e.,  $n \log t$  to be no more than  $O(m)$ , leading to  $t = 2^{2m/n}$ . If we do this in every iteration, then this leads to  $O(m)$  time per iteration.



We now bound the number of iteration. Consider  $t_1 := 2^{2m/n}$  and  $t_i := 2^{2m/n_i}$ ,<sup>a</sup> where  $n_i$  and  $m_i$  are the number of vertices and edges at the beginning of the  $i^{\text{th}}$  iteration, with  $m_1 = m$  and  $n_1 = n$ . From the previous claim,  $n_{i+1} \leq 2m_i/t_i$ , which gives  $t_{i+1} = 2^{2m/n_{i+1}} \geq 2^{\frac{2m}{2m_i/t_i}} \geq 2^{t_i}$ . Thus,  $t_i$  is a power of twos with  $t_1 = 2^{2m/n}$ , and the [Fredman-Tarjan's algorithm](#) stops if  $t_i \geq n$  since it will [grow](#) a single tree and finish. Thus, the algorithm needs at most  $\beta(m, n)$  iterations, giving the total time  $O(m\beta(m, n))$ . ■

<sup>a</sup>Technically, we need to choose  $t_i := 2^{\lceil 2m/n_i \rceil}$ , but we will be a bit sloppy and ignore the ceilings here.

## Lecture 2: MST and Tree Packing

### Linear-Time Randomized Algorithm

29 Aug. 11:00

Using randomization, it's possible to derive a linear-time algorithm for [MST](#).

**Theorem 1.1.6** ([[KKT95](#)]). [Karger-Klein-Tarjan's algorithm](#) takes  $O(m)$  time that computes the [MST](#) with probability at least  $1 - 1/\text{poly}(m)$ .

[Karger-Klein-Tarjan's algorithm](#) relies on the so-called [sampling lemma](#), which we first discussed.

**Lemma 1.1.3** (Sampling lemma). Given a graph  $G = (V, E)$ , and let  $E' \subseteq E$  be obtained by sampling each edge  $e$  with probability  $p \in (0, 1)$ . Let  $F$  be a minimum spanning forest in  $G' = (V, E')$  (can be disconnected). Then the expected number of [F-light](#) edge in  $G$  is less than  $(n - 1)/p$ .

**Proof.** Let  $A$  be the set of [F-light](#) edges. Note that both  $A$  and  $F$  are random sets that are generated by the process of sampling  $E'$ . To analyze  $\mathbb{E}[|A|]$ , we consider [Kruskal's algorithm](#) to obtain  $F$  from  $E'$ , where we generate  $E'$  on the fly:

---

#### Algorithm 1.5: Sampling Process

---

**Data:** A connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , probability  $p \in (0, 1)$

**Result:** A minimum spanning forest  $F$  and the set of [F-light](#) edges  $A$

```

1 Sort the edges such that  $c(e_1) \leq c(e_2) \leq \dots \leq c(e_m)$ 
2  $A \leftarrow \emptyset, F \leftarrow \emptyset, E' \leftarrow \emptyset$ 
3 for  $i = 1, \dots, m$  do
4   if  $\text{Ber}(p) = 1$  then                                     // Toss a biased coin
5      $E' \leftarrow E' + e_i$ 
6     if  $F + e_i$  is a forest then
7        $F \leftarrow F + e_i, A \leftarrow A + e_i$ 
8   else if  $e_i$  is F-light then
9      $A \leftarrow A + e_i$ 
10 return  $F, A$ 
```

---

The following is exactly the same as the above, but easier to analyze:

---

#### Algorithm 1.6: Sampling Process with Tweaks

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**Data:** A connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , probability  $p \in (0, 1)$

**Result:** A minimum spanning forest  $F$  and the set of [F-light](#) edges  $A$

```

1 Sort the edges such that  $c(e_1) \leq c(e_2) \leq \dots \leq c(e_m)$ 
2  $A \leftarrow \emptyset, F \leftarrow \emptyset$ 
3 for  $i = 1, \dots, m$  do
4   if  $e_i$  is F-light then                                     // Sorting implies  $F + e_i$  is a forest  $\Leftrightarrow e_i$  is F-light
5      $A \leftarrow A + e_i$ 
6   if  $\text{Ber}(p) = 1$  then                                     // Toss a biased coin
7      $F \leftarrow F + e_i$ 
8 return  $F, A$ 
```

---

The second algorithm makes the following observation clear.

**Intuition.** An edge  $e_i$  is added to  $A$  implies that it is added to  $F$  with probability  $p$ .

Hence,  $p\mathbb{E}[|A|] = \mathbb{E}[|F|] \leq n - 1$ , hence  $\mathbb{E}[|A|] \leq (n - 1)/p$ . ■

**Remark.** This proof is based on the *principle of deferred decisions* in randomized analysis.

With the [sampling lemma](#), we know that when  $p = 1/2$ , the number of [F-light](#) edges from  $E$  is at most  $2n$ . Hence, we can eliminate most of the edges from  $E \setminus E'$  from consideration given the fact that we can efficiently compute the [F-heavy](#) edges via the [MST verification theorem](#). It's worth noting that to work with the [sampling lemma](#) via the natural recursion that it implies means that we need to work with potentially disconnected graph. That is, we will need to consider disconnected graph. Hence, we make the following generalization.

**Definition 1.1.4 (Spanning forest).** A *spanning forest*  $T$  of a graph  $G = (V, E)$  (potentially disconnected) is an induced subgraph of  $G$  which spans  $G$ , i.e.,  $V(T) = V$  and  $E(T) \subseteq E$ .

**Problem 1.1.3 (Minimum spanning forest).** Given a graph  $G = (V, E)$  (potentially disconnected) with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , find the min-cost [spanning forest](#).

**Note.** [MST](#) and [MSF](#) are closely related and one is reducible to the other in linear time, and the [cut](#) and [cycle rules](#) can be generalized to [MSF](#) easily.

Now, consider the following natural recursive divide and conquer algorithm for computing [MSF](#).

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**Algorithm 1.7:** Natural Recursive Algorithm from [Sampling Lemma](#)

---

**Data:** A graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$

**Result:** A [MSF](#)  $T = (V, F)$

```

1 if  $|V| < n_0$  then                                     //  $n_0$  is some constant
2   return Standard-MST( $G, c$ )                          // Use a standard deterministic algorithm
3
4 Sample each edge i.i.d. from  $\text{Ber}(1/2)$  to obtain  $E_1 \subseteq E$ 
5  $(V, F_1) \leftarrow \text{Karger-Klein-Tarjan}((V, E_1))$       // Recursively compute MSF
6  $E_2 \leftarrow \text{Light-Edge}(G, F_1)$                     // Compute all F1-light edges with Theorem 1.1.4
7  $(V, F_2) \leftarrow \text{Karger-Klein-Tarjan}((V, E_2))$     // Recursively compute MSF
8 return  $(V, F_2)$ 
```

---

The correctness of [Algorithm 1.7](#) is clear from the [cut](#) and [cycle rules](#). The issue is the running time:

**Claim.** [Algorithm 1.7](#) is not efficient enough.

**Proof.** The expected number of edges in  $G_1 := (V, E_1)$  is  $m/2$ , and the expected number of edges in  $G_2 := (V, E_2)$ , via the [sampling lemma](#), is at most  $2n$ . We see that the algorithm does  $O(m + n)$  work outside the two recursive calls ([line 5](#), [line 7](#)). Let  $T(m, n)$  be the expected running time of the algorithm on an  $m$ -edge  $n$ -node graph. Informally, we see the following recurrence:

$$T(m, n) \leq c(m + n) + T(m/2, n) + T(2n, n).$$

If we take the problem size to be  $n + m$ , then [Algorithm 1.7](#) generates two sub-problems of expected size  $m/2 + n$  and  $2n + n$ , with the total size being  $4n + m/2$ . If  $m > 10n$ , say, then the total problem size is shrinking by a constant factor, and we obtain a linear-time algorithm. However, this is generally not the case. ⊛

The problem becomes reducing the graph size, which is the trick of [Karger-Klein-Tarjan's algorithm](#): we run [Borůvka's algorithm](#) for a few iterations as a preprocessing step, reducing the number of vertices:

**Algorithm 1.8:** Karger-Klein-Tarjan's Algorithm [KKT95]

---

**Data:** A connected graph  $G = (V, E)$ <sup>a</sup> with edge capacity  $c: E \rightarrow \mathbb{R}_+$   
**Result:** A **MSF**  $T = (V, F)$

```

1 if  $|V| < n_0$  then                                     //  $n_0$  is some large constant
2   return Standard-MST( $G, c$ )                             // Use a standard deterministic algorithm
3
4  $G' = (V', E'), T' = (V', F') \leftarrow \text{Borůvka}(G, c, 2)$  // Run two iterations with  $|V'| \leq |V|/4$ .
5
6 Sample each edge in  $G'$  i.i.d. from  $\text{Ber}(1/2)$  to obtain  $E_1 \subseteq E'$ 
7  $(V', F_1) \leftarrow \text{Karger-Klein-Tarjan}((V', E_1))$            // Recursively compute MSF
8  $E_2 \leftarrow \text{Light-Edge}(G_1, F_1)$                          // Compute  $F_2$ -light edges with Theorem 1.1.4
9  $(V', F_2) \leftarrow \text{Karger-Klein-Tarjan}((V', E_2))$          // Recursively compute MSF
10 return  $(V, F' \cup F_2)$ 
```

---

<sup>a</sup>Assume no connected component of  $G$  is small.

Now, we provide the proof sketch of **Theorem 1.1.6**, which can be made precise with expectation.

**Proof Sketch of Theorem 1.1.6.** The correctness is easy to see as before. As for the running time, we see that **Borůvka's algorithm** takes  $O(m)$  time for each phase, so the total time for the preprocessing (line 4) is  $O(m)$ . Then, the recurrence for  $T(m, n)$  is

$$T(m, n) \leq c(m + n) + T(m/2, n/4) + T(2n/4 + n/4),$$

i.e., the resulting sub-problem is of size  $n/4 + m/2 + n/4 + n/2 = n + m/2$ , which is good enough assuming  $m \geq n - 1$ .<sup>a</sup> By a simple inductive proof, we can show that  $T(m, n) = O(n + m)$ . ■

<sup>a</sup>Since we eliminate small components including singletons.

**Remark.** A more refined analysis of the **sampling lemma** can be used to show that the running time is linear with high probability as well.

Many properties of forests and spanning trees can be understood in the more general context of *matroids*. In many cases this perspective is insightful and also useful. The **sampling lemma** applies in this more general context and has various applications [Kar95; Kar98]. Obtaining a deterministic  $O(m)$  time algorithm is a major open problem. Obtaining a simpler linear-time **MST verification** algorithm, even randomized, is also a very interesting open problem.

## 1.2 Tree Packing

We turn to another interesting problem, **tree packing**.

**Problem 1.2.1 (Tree packing).** Given a multigraph  $G = (V, E)$ , find all the edge-disjoint **spanning trees** in  $G$ . In particular, find the maximum number,  $\tau(G)$ , of edge-disjoint **spanning trees** of  $G$

### 1.2.1 Bound on the Tree Packing Number

There is a beautiful theorem that provides a min-max formula for this. We first introduce some notation.

**Notation.** Let  $\mathcal{P}$  be the collection of partitions of  $V$ , and  $E_P$  is the edge between connected components induced by a partition  $P \in \mathcal{P}$ , i.e.,  $e \in E_P$  if its endpoints are in different parts of  $P$ .

It's easy to see that any **spanning tree** must contain at least  $|P| - 1$  edges from  $E_P$ . Thus, if  $G$  has  $k$  edge-disjoint **spanning trees**, then

$$k \leq \frac{|E_P|}{|P| - 1}.$$

More generally, we have the following.

**Theorem 1.2.1.** The maximum number of edge-disjoint **spanning trees** in a graph  $G$  is given by

$$\tau(G) = \left\lfloor \min_{P \in \mathcal{P}} \frac{|E_P|}{|P| - 1} \right\rfloor.$$

**Remark.** Theorem 1.2.1 is a special case of a theorem on matroid base packing where it is perhaps more natural to see [Sch+03].

A weaker version of the theorem is regarding fractional packing. In fractional packing, we allow one to use a fraction amount of a tree. The total amount to which an edge can be used is at most 1 (or  $c(e)$  in the capacitated case). Clearly, an integer packing is also a fractional packing. The advantage of fractional packings is that one can write a linear program for it, and they often have some nice properties. Let  $\tau_{\text{frac}}(G)$  be the fraction **tree packing** number. Clearly, we have  $\tau_{\text{frac}}(G) \geq \tau(G)$ .

**Corollary 1.2.1.** Given a graph  $G$ , we have

$$\tau_{\text{frac}}(G) = \min_{P \in \mathcal{P}} \frac{|E_P|}{|P| - 1}.$$

**Proof.** Assuming Theorem 1.2.1, then with  $c := |P^*| - 1$  for  $P^* = \arg \min_{P \in \mathcal{P}} |E_P| / (|P| - 1)$ ,

$$\tau(G_c) - \min_{P \in \mathcal{P}} \frac{c|E_P|}{|P| - 1} = [|E_{P^*}|] - |E_{P^*}| = 0,$$

where  $G_c$  is with edge capacity scaled up by  $c$ . This implies that  $\tau_{\text{frac}}(G_c) = \tau(G_c)$ . As this holds for every  $c$  (with different graphs), this can only happen if  $\tau_{\text{frac}}(G) = \min_{P \in \mathcal{P}} |E_P| / (|P| - 1)$ . ■

The second important corollary that is frequently used is about the min-cut. We see that while the min-cut size  $\lambda(G)$  of  $G$  is upper-bounding  $\tau(G)$ , i.e.,  $\tau(G) \leq \lambda(G)$ , this is not tight at all.

**Corollary 1.2.2.** Let  $G$  be a capacitated graph and let  $\lambda(G)$  be the **global min-cut** size. Then

$$\tau_{\text{frac}}(G) \geq \frac{\lambda(G)}{2} \frac{n}{n-1}.$$

**Proof.** Let  $P^*$  be the optimum partition that induces  $\tau_{\text{frac}}(G)$ . Then,  $\tau(G) = |E_{P^*}| / (|P^*| - 1)$ . Since for every connected component induced by  $P^*$ , at least  $\lambda(G)$  edges are going out, hence

$$\tau_{\text{frac}}(G) = \frac{|E_{P^*}|}{|P^*| - 1} \geq \frac{\lambda(G)/2 \cdot |P^*|}{|P^*| - 1} \geq \frac{\lambda(G)}{2} \frac{n}{n-1},$$

where we use the fact that  $|P^*| \leq n$  and  $i/(i-1)$  is decreasing. ■

We first see a tight example.

**Example (Cycle).** Consider the  $n$ -node cycle  $C_n$ . Clearly,  $\tau(C_n) = 1$ , and  $\tau_{\text{frac}}(C_n) \leq n/(n-1)$  since each tree has  $n-1$  edges and there are  $n$  edges in the graph. Indeed, we have  $\tau_{\text{frac}}(C_n) = n/(n-1)$ . Finally, we see that  $\lambda(G) = 2$ .

**Proof.** Consider  $n$  trees in  $C_n$  by deleting each of the  $n$  edges and assign a fraction value of  $1/(n-1)$  for each of them. The corresponding tight partition consists of the  $n$  singleton vertices. ⊛

**Note.** Theorem 1.2.1 and its corollaries naturally extend to the capacitated case. For integer packing, we can assume  $c_e$  is an integer for each edge  $e$ , and the formula is changed to

$$\tau(G) = \left\lfloor \min_{P \in \mathcal{P}} \frac{c(E_P)}{|P| - 1} \right\rfloor.$$

Corollary 1.2.1 can also be proved in the same way when the edge capacity is rational.

Typically, one uses the connection between [tree packing](#) and min-cut to argue about the existence of many disjoint [trees](#), since the global minimum cut is easier to understand than  $\tau(G)$ . However, we will see that one can use [tree packing](#) to compute  $\lambda(G)$  exactly which may seem surprising at first due to the approximate relationship [Corollary 1.2.2](#).

### 1.2.2 Proof of [Corollary 1.2.1](#)

Now, we give a different proof for [Corollary 1.2.1](#) via LP duality without relying on [Theorem 1.2.1](#).<sup>2</sup>

**Proof of [Corollary 1.2.1](#) [CQ17].** Consider  $\mathcal{T}_G := \{T \mid T \text{ is a spanning tree of } G\}$ . Then, consider the following primal and the dual linear program:

$$\begin{aligned}
 \max \quad & \sum_{T \in \mathcal{T}_G} y_T & \min \quad & \sum_{e \in E} c(e)x_e \\
 \sum_{T \ni e} y_T & \leq c(e) \quad \forall e \in E; & \sum_{e \in T} x_e & \geq 1 \quad \forall T \in \mathcal{T}_G; \\
 \text{(P)} \quad y_T & \geq 0 \quad \forall T \in \mathcal{T}_G; & \text{(D)} \quad x_e & \geq 0 \quad \forall e \in E.
 \end{aligned} \tag{1.1}$$

Let  $y^*$  and  $x^*$  be the optimal solution to the primal and the dual. Then from the strong duality,

$$\sum_{T \in \mathcal{T}_G} y_T^* = \tau_{\text{frac}}(G) = \sum_{e \in E} c(e)x_e^*.$$

We see that if there exists  $e$  such that  $x_e^* = 0$ , then we can just contract all these edges, so without loss of generality,  $x_e^* > 0$  for all  $e \in E$ .

**Intuition.** If  $x_e^* = 0$ , we can effectively increase  $c(e)$  to  $\infty$  without affecting the value of the dual solution, i.e.,  $e$  is not a bottleneck in the primal [tree packing](#), hence safe to contract.

**Claim.** If  $x_e^* > 0$  for all  $e \in E$ , then  $\tau_{\text{frac}}(G)$  is achieved via the singleton partition  $P$ . In particular,

$$\tau_{\text{frac}}(G) = \frac{\sum_{e \in E} c(e)}{n-1}.$$

**Proof.** From complementary slackness, we know that  $\sum_{T \ni e} y_T^* = c(e)$  for all  $e \in E$ . Hence,

$$(n-1) \sum_{T \in \mathcal{T}_G} y_T^* = \sum_{T \in \mathcal{T}_G} \sum_{e \in T} y_T^* = \sum_{e \in E} \sum_{T \ni e} y_T^* = \sum_{e \in E} c(e),$$

implying that  $\sum_{T \in \mathcal{T}_G} y_T^* = \sum_{e \in E} c(e)/(n-1)$ . ⊛

Finally, we recall that  $\tau_{\text{frac}}(G) \leq \min_P |E_P|/(|P|-1)$ , hence, the above claim gives us the desired conclusion via induction: this is true if  $x_e^* > 0$  for all  $e \in E$ ; otherwise, we contract edges with  $x_e^* = 0$  and reduce to this case. ■

**Remark.** In the above proof, the dual can be interpreted as a relaxation for the min-cut problem. In fact, if  $x_e \in \{0, 1\}$ , then this is exact.

### 1.2.3 Finding an Optimum Tree Packing and Approximating Tree Packing

If the [linear program](#) in the [proof](#) of [Corollary 1.2.1](#) can be solved efficient to get  $\tau_{\text{frac}}(G)$ , then it will also yield an algorithm for the value of the integer packing  $\tau(G)$  since it's just the floor of which. The problem is that while the primal has an exponentially many variables, the dual has an exponentially many constraints. We recall the following fact.

<sup>2</sup>Indeed, this is a hard theorem to prove, so we will not touch on this.

**As previously seen.** The Ellipsoid method needs a *separation oracle*. For example, applying it to the dual, we need to answer the following question efficiently:

- Given  $x \in \mathbb{R}^E$ , is it the case that  $\sum_{e \in T} x_e \geq 1$  for all  $T \in \mathcal{T}_G$ ?
- If not, find a tree  $T$  such that  $\sum_{e \in T} x_e < 1$ .

We see that this corresponds to solving **MST**, hence, the dual admits an efficient solution via the Ellipsoid method. One can convert an exact algorithm for the dual to an exact algorithm for the primal.

**Remark.** There are combinatorial algorithms for solving **tree packing** (both integer version and fraction versions) in strongly polynomial time [Sch+03].

On the other hand, we're also interested in whether we can find a faster algorithm for **tree packing** if one allows approximation. With an adaption of the *multiplicative weights update* method and data structures for **MST** maintenance, there is a near-linear time algorithm.

**Theorem 1.2.2** ([CQ17]). There is a deterministic algorithm to compute a  $(1 - \epsilon)$ -approximate fractional **tree packing** in  $\tilde{O}(m/\epsilon^2)$  time.

## Lecture 3: Global Min-Cut with Tree Packing

### 1.3 Min-Cuts and Steiner Min-cuts via Tree Packing

3 Sep. 11:00

Consider the following famous problems about min-cuts.

**Problem 1.3.1** (*s-t min-cut*). Given a graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , the *s-t min-cut* problem aims to find  $\min_{S \subseteq V: s \in S, t \in V \setminus S} c(\delta(S))$ .

**Problem 1.3.2** (*Global min-cut*). Given a graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , the *global min-cut* problem aims to find  $\min_{\emptyset \neq S \subseteq V} c(\delta(S))$ .

In what follows, we will simply use *min-cut* to refer to **Problem 1.3.2**. For convenience, we also introduce its well-known dual, the **max-flow** problem. We first define the **flow** formally.

**Definition 1.3.1** (*Flow*). Given a directed graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , a *flow*  $f: E \rightarrow \mathbb{R}_+$  from  $s$  to  $t$  for some pair of  $s, t \in V$  satisfies

- $0 \leq f(e) \leq c(e)$  for all  $e \in E$ ;
- $\sum_{e \in \delta^-(v)} f(e) = \sum_{e \in \delta^+(v)} f(e)$  for all  $v \in V \setminus \{s, t\}$ .

**Notation** (Value). Given a **flow**, the *value* of  $f$  is defined as  $|f| := \sum_{e \in \delta^+(s)} f(e) - \sum_{e \in \delta^-(s)} f(e)$ .

Then the problem of **s-t max-flow** can be formulated as follows.

**Problem 1.3.3** (*s-t max-flow*). Given a directed graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ . The *s-t max-flow* problem aims to find  $\max_{f \in \mathcal{F}_{s,t}} |f|$  where  $\mathcal{F}_{s,t}$  is the set of all **s-t flows**.

A naive way to solve the **min-cut** problem is to first fix one end  $s \in V$ , and compute the **s-t min-cut** for all  $t \in V - s$ . Fairly recent work shows how one can do it with only polynomial-logarithmically many **max-flow** computations (**Theorem 1.3.7**).

Over the years, several very different algorithmic approaches have been developed for these problems. One of the surprising ones is based on MA-orderings [NI92], which is a combinatorial  $O(mn + n^2 \log n)$  time algorithm that does not rely on **flow** at all.<sup>3</sup> Another approach is to combine several **flow** com-

<sup>3</sup>This approach generalizes to symmetric submodular functions.

putations together via the push-relabel method [HO94], which also works for directed graphs. Karger developed elegant and powerful random contraction based algorithms for [global min-cuts](#) [Kar95], leading to many results. Two notable consequences are the following.

**Theorem 1.3.1** ([KS96]). There is a randomized algorithm that runs in  $O(n^2 \log n)$  time and outputs the [min-cut](#) with high probability.<sup>a</sup>

<sup>a</sup>This is a Monte-Carlo algorithm, so we cannot guarantee that the min-cut found is the correct one.

The following is a consequence of Karger's contraction algorithm [Kar95].

**Theorem 1.3.2** (Approximate min-cut [Kar00]). The number of [α-approximate min-cuts](#) in a graph is at most  $O(n^{2\alpha})$ .

**Definition 1.3.2** (Approximate min-cut). For  $\alpha \geq 1$  an  $\alpha$ -approximate min-cut is a cut  $(S, V \setminus S)$  such that  $c(\delta(S)) \leq \alpha \lambda(G)$ .

Karger then developed another approach via [tree packing](#) to obtain a randomized near-linear time algorithm for [min-cut](#). He also was able to refine the bound on [approximate min-cuts](#) via this approach.

**Theorem 1.3.3** ([Kar00]). There is a randomized algorithm that runs in time  $O(m \log^3 n)$  and outputs the [min-cut](#) with high probability.

While the random contraction based algorithm is taught quite frequently due to its elegance and simplicity, the [tree packing](#) approach is more technical. More recently, the [tree packing](#) approach has led to several new results, which we now discuss.

### 1.3.1 Tree Packing-Based Algorithm for Min-Cut

Recall [Corollary 1.2.2](#), which gives  $\tau_{\text{frac}}(G) \in [\frac{\lambda(G)}{2} \frac{n}{n-1}, \lambda(G)]$ . Intuitively, even if we can compute  $\tau_{\text{frac}}(G)$  exactly, we have a 2-approximation to  $\lambda(G)$ . However, this already leads a crucial observation:

**Intuition.** On average, each tree can't cross the [min-cut](#) more than twice.

To formalize the above intuition, consider the following definition.

**Definition 1.3.3** (Respecting). Let  $T = (V, E_T)$  be a [spanning tree](#) and  $(S, V \setminus S)$  be a cut. The for an integer  $h \geq 1$ , we say  $T$  is  $h$ -respecting w.r.t.  $S$  if  $|E_T \cap \delta(S)| \leq h$ .

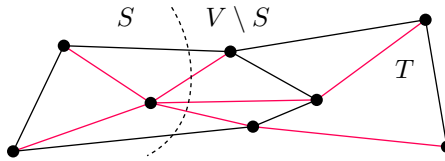


Figure 1.1: The [spanning tree](#)  $T$  is shown in red edges.  $T$  is [3-respecting](#) the cut  $(S, V \setminus S)$ .

We can now formalize the intuition in [Lemma 1.3.1](#).

**Lemma 1.3.1.** Suppose  $\{y_T\}_{T \in \mathcal{T}_G}$  is a  $(1 - \epsilon)$ -approximate [tree packing](#) of  $G$ , and  $\delta(S)$  is a [min-cut](#) of  $G$ . Let  $\ell_T := |E_T \cap \delta(S)|$  be the number of edges of  $T$  that cross the cut  $S$ . Furthermore, let  $p_T = y_T / \sum_{T \in \mathcal{T}_G} y_T$  and  $q := \sum_{T: \ell(T) \leq 2} p_T$ . Then,

$$q \geq \frac{1}{2} \left( 3 - \frac{2}{1 - \epsilon} \left( 1 - \frac{1}{n} \right) \right).$$



In particular, if  $\epsilon = 0$ , then  $1 \geq 1/2 + 1/n$ , and if  $\epsilon < 1/5$ , then  $q > 1/4$ .

**Proof.** From the assumption,  $\sum_{T \in \mathcal{T}_G} y_T \geq (1 - \epsilon)\tau_{\text{frac}}(G)$ . With [Corollary 1.2.2](#), we have

$$\sum_{T \in \mathcal{T}_G} y_T \geq (1 - \epsilon) \frac{n}{n-1} \frac{\lambda(G)}{2}.$$

Let  $S \subseteq V$  be a [min-cut](#), we have  $1 = \sum_{T \in \mathcal{T}_G} p(T) = \sum_{T: \ell(T) \leq 2} p_T + \sum_{T: \ell(T) \geq 3} p_T$ . Observe that

- each tree  $T$  with  $\ell(T) \geq 3$  uses up at least 3 edges from  $\delta(S)$ ; while
- each tree  $T$  with  $\ell(T) \leq 2$  uses up at least 1 edge from  $\delta(S)$ .

Since the total capacity of  $\delta(S)$  is  $\lambda(G)$ , and the [tree packing](#) solution is valid, we have

$$\sum_{T: \ell(T) \leq 2} y_T + 3 \sum_{T: \ell(T) \geq 3} y_T \leq \lambda(G) \Rightarrow q + 3(1 - q) \leq \frac{\lambda(G)}{\sum_{T \in \mathcal{T}_G} y_T} \leq \frac{2}{1 - \epsilon} \left(1 - \frac{1}{n}\right),$$

where the last inequality follows from the very first inequality we have derived. ■

**Remark.** [Lemma 1.3.1](#) states that if the [tree packing](#) is sufficiently good, then a constant fraction of the trees in the [packing](#) will cross the [min-cut](#) at most twice.

Now, we're ready to see Karger's algorithm for [min-cut](#) [\[Kar00\]](#). However, the original algorithm was more involved since at that time, there was no near-linear time approximation algorithm for [tree packing](#), so he used a form of sparsification and then applied an approximation [tree packing](#) algorithm on the sparsified graph which is quite a feat. In our case, recall that following.

**As previously seen.** [Theorem 1.2.2](#) states that we can compute a  $(1 - \epsilon)$ -approximate [tree packing](#) of  $G$ , given by  $\{y_T\}_{T \in \mathcal{T}_G}$ , in  $O(m \log^3 n / \epsilon^2)$  time.

By black-boxing this near-linear time [tree packing](#) algorithm, consider the following.

---

**Algorithm 1.9:** [Tree Packing-Based Min-Cut Algorithm](#) [\[Kar00; CQ17\]](#)

---

**Data:** A connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ ,  $\epsilon_0 \in (0, 1/5)$

**Result:** A cut  $S$

- 1  $\{y_T\}_{T \in \mathcal{T}_G} \leftarrow \text{Approximate-Tree-Packing}(G, c, \epsilon_0)$  //  $O(m \log^3 n)$
  - 2 Sample a tree  $T$  with probability  $p_T = y_T / \sum_{T \in \mathcal{T}_G} y_T$
  - 3 Find the cheapest cut  $(S, V \setminus S)$  in  $G$  such that  $T$  is [2-respecting](#) w.r.t.  $S$
  - 4 **return**  $S$
- 

Firstly, we see that [Algorithm 1.9](#) admits the following.

**Lemma 1.3.2.** [Algorithm 1.9](#) outputs the [min-cut](#) of  $G$  with probability at least  $1/4$ .

**Proof.** It's immediate from [Lemma 1.3.1](#). ■

To boost the success probability, we can simply repeat the last two steps ([line 2](#), [line 3](#))  $\Theta(\log n)$  times, which results in a success probability to at least  $1 - 1/n^c$  for any constant  $c$ . To analyze the running time, a key ingredient is [line 3](#). Karger showed that one can implement [line 3](#) via a clever dynamic programming coupled with [link-cut tree](#) data structure:

**Theorem 1.3.4** ([\[Kar00\]](#)). Given a graph  $G = (V, E)$  and a [spanning tree](#)  $T = (V, E_T)$ . There is a deterministic algorithm that computes a minimum cut  $(S, V \setminus S)$  such that  $T$  is [2-respecting](#) w.r.t.  $S$  in  $O(m \log^2 n)$  time.

We can now prove [Theorem 1.3.3](#).

**Proof of Theorem 1.3.3.** Since [line 1](#) takes  $O(m \log^3 n)$  for  $\epsilon_0$  being a constant, and observe that once the approximated [tree packing](#)  $\{y_T\}_{T \in \mathcal{T}}$  is computed, we can reuse them and apply the repe-



tition for [line 2](#) and [line 3](#) to boost the probability of success. With  $\Theta(\log n)$  repetitions, we obtain an  $O(m \log^3 n)$  time algorithm as desired with the running time guaranteed by [Theorem 1.3.4](#). ■

### 1.3.2 Bounding the Number of Approximate Min-Cuts

As hinted in [Theorem 1.3.2](#), we're now interested in how many distinct [min-cuts](#) can an undirected graph have. The following theorem was shown a long time ago:

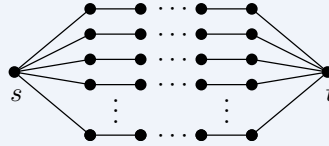
**Theorem 1.3.5** ([DKL76]). The number of distinct [min-cuts](#) in an undirected graph is at most  $\binom{n}{2}$ .

**Example** (Cycle). The worst case example is an  $n$ -cycle  $C_n$ .

**Remark.** All the [min-cuts](#) of a graph can be represented in a nice and compact data structure called the cactus (cactus representation), which was also shown in [DKL76].

In contrast, for  $s$ - $t$  [min-cuts](#), it can be exponentially many in  $n$ .

**Example.** Consider the following multi-highway-like graph, which has exponentially many  $s$ - $t$  [min-cuts](#) since if we choose one of the road section in each line of the road, it'll be a  $s$ - $t$  [min-cut](#).



Recall earlier in [Theorem 1.3.2](#), we stated that Karger used [tree packing](#) to prove that the number of  $\alpha$ -approximation [min-cuts](#) is at most  $O_\alpha(n^{\lfloor 2\alpha \rfloor})$ . We now prove it formally.

**Proof of Theorem 1.3.2** [CQX20]. Consider an optimum fraction [tree packing](#) solution  $\{(T, y_T^*)\}_{T \in \mathcal{T}_G}$ . In the [proof](#) of [Corollary 1.2.1](#), where we define the fractional [tree packing](#) linear program, we know that there are only  $m$  non-trivial constraints, hence there are only  $m$  many  $T$ 's such that  $y_T^* > 0$ .

**As previously seen.** A solution  $x^*$  to a linear program which has  $n$  non-trivial constraints means that the support size of  $x$  is at most  $n$ , i.e.,  $x_i > 0$  for at most  $n$  many  $i$ 's.

Consider an  $\alpha$ -approximate [min-cut](#)  $S \subseteq V$ , and let  $h = \lceil 2\alpha \rceil$ . Now, let  $q_{h,\alpha}$  be the fraction of [tree packing](#) that  $h$ -respects  $S \subseteq V$ , i.e.,  $q_{h,\alpha} := \sum_{T: \ell(T) \leq h} p_T$ . Using a similar analysis as the one in [Lemma 1.3.1](#), we can argue that

$$q_{h,\alpha} \geq \frac{1}{h} (1 - (2\alpha - \lfloor 2\alpha \rfloor)) \left(1 - \frac{1}{n}\right).$$

The main intuition is the following:

**Intuition.** Say at least one tree in the [packing](#)  $h$ -respects the cut (which is the case). Then, the total number of  $\alpha$ -approximate [min-cuts](#) is at most  $m \cdot n^h \leq m \cdot n^{\lfloor 2\alpha \rfloor}$ .

But we can do better by noticing that  $q_{h,\alpha} > 0$  is a fixed constant for any fixed  $\alpha$ . Suppose  $N$  is the number of  $\alpha$ -approximate [min-cuts](#). For any fixed  $\alpha$ -approximate [min-cut](#),  $q_{h,\alpha}$  fraction of the [tree packing](#) is  $h$ -respecting w.r.t. the cut. Consider the following question:

**Problem.** Fix a single tree  $T$ , how many distinct cuts are there such that  $T$   $h$ -respects w.r.t.?

**Answer.** We can remove at most  $h$  edges from  $T$  to create at most  $h + 1$  components and combine these components into two sides of a cut, hence, each tree  $T$  correspond to at most  $2^{h+1} \binom{n-1}{h} \leq 2^{h+1} n^h$  cuts. ⊛

Thus, the number of  $\alpha$ -approximate [min-cuts](#) is at most  $2^{h+1} n^h / q_{h,\alpha}$ . ■

## Lecture 4: Steiner Min-Cut with Isolating Cuts

### 1.3.3 Steiner Min-Cut

5 Sep. 11:00

Consider the following problem that generalizes the [s-t min-cut](#) and [global min-cut](#).

**Problem 1.3.4 (Steiner min-cut).** Given a graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$  and a set  $T \subseteq V$  of terminals, the *Steiner min-cut* problem aims to find the min-cut  $(S, V \setminus S)$  which separates some pair of terminals, i.e.,  $S \cap T \neq \emptyset$  and  $(V \setminus S) \cap T \neq \emptyset$ .

**Remark.** Steiner min-cut generalizes both [s-t min-cut](#) and [global min-cut](#).

**Proof.** [s-t min-cut](#) corresponds to  $T = \{s, t\}$ , while [global min-cut](#) corresponds to  $T = V$ . ⊛

A simple algorithm for the [Steiner min-cut](#) is the same as the [global min-cut](#) by solving [s-t min-cut](#): for  $T = \{t_1, \dots, t_k\}$ , fix a terminal, say  $t_1$ , then compute [t<sub>1</sub>-t<sub>i</sub> min-cut](#) for all  $i \geq 2$ . This requires  $|T| - 1$  [max-flow](#) computations. In fact, this is the best known algorithm even for the [global min-cut](#) till [NI92].

Quite recently, a simple yet striking approach that computes the [Steiner min-cut](#) with high probability using only  $O(\log^3 n)$  [s-t cut](#) computations is developed [LP20], which is based on [isolating cut](#).

### Submodular Function

The main interest here, i.e., solving [isolating cut](#), will be essentially based on properties of [symmetric submodular functions](#). Although we can prove various properties by appealing to first principles, it's useful to see the proofs via [submodularity](#). Here, we give some necessarily background.

**Definition.** Given a finite ground set  $V$ , consider a real-valued set function  $f: 2^V \rightarrow \mathbb{R}$ .

**Definition 1.3.4 (Modular).** The function  $f$  is *modular* if for all  $A, B \subseteq V$ ,

$$f(A) + f(B) = f(A \cap B) + f(A \cup B).$$

**Definition 1.3.5 (Submodular).** The function  $f$  is *submodular* if for all  $A, B \subseteq V$ ,

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B).$$

**Definition 1.3.6 (Supermodular).** The function  $f$  is *supermodular* if for all  $A, B \subseteq V$ ,

$$f(A \cap B) + f(A \cup B) \geq f(A) + f(B).$$

**Definition 1.3.7 (Posi-modular).** The function  $f$  is *posi-modular* if for all  $A, B \subseteq V$ ,

$$f(A - B) + f(B - A) \geq f(A) + f(B).$$

We note that perhaps a more common definition of [submodularity](#) is *diminishing marginal utility*, i.e., if  $f(A + v) - f(A) \geq f(B + v) - f(B)$  for all  $A \subseteq B$ . Here, we see some examples.

**Example (Modular function as weight function).**  $f$  is [modular](#) if and only if there exists some  $w: V \rightarrow \mathbb{R}$  such that  $f(A) = \sum_{v \in A} w(v) + c$  for some shift  $c$ .

**Example.** If  $f$  and  $g$  are [submodular](#), then so is  $\alpha f + \beta g$  for some  $\alpha, \beta \geq 0$ .

One of the reason that [submodularity](#) is important for graphs is because of the following.

**Example (Cut).** Given a graph  $G = (V, E)$ , the cut size function  $|\delta_G(\cdot)|: 2^V \rightarrow \mathbb{R}_+$  is [submodular](#).

**Proof.** We simply note that for any  $A, B \subseteq V$ ,

$$|\delta_G(A)| + |\delta_G(B)| = |\delta_G(A \cap B)| + |\delta_G(A \cup B)| + 2|E(A \setminus B, B \setminus A)| \geq |\delta_G(A \cap B)| + |\delta_G(A \cup B)|,$$

where  $E(X, Y)$  is the set of edges crossing  $X$  and  $Y$  for some  $X, Y \subseteq V$ . ⊗

The above argument extends naturally to non-negative capacitated graphs and directed graphs.

**Example.** For a directed graph,  $|\delta^+(\cdot)|$  is **submodular** (so does  $|\delta^-(\cdot)|$  by symmetry).

We're also interested in the following property.

**Definition 1.3.8 (Symmetric).** A set function is *symmetric* if  $f(A) = f(V \setminus A)$  for all  $A \subseteq V$ .

Clearly,  $|\delta_G(\cdot)|$  is **symmetric**. However, for directed graph, this is not necessarily the case. Finally, we see that **symmetric submodular** function satisfies another important property.

**Example.** A **symmetric submodular** function is automatically **posi-modular**.

Now, we discuss uncrossing, a common and powerful technique that is frequently used in working with **submodular functions**. We illustrate this in the context of **min-cuts**.

**Lemma 1.3.3.** Let  $G = (V, E)$  be a graph and  $(A, V \setminus A)$ ,  $(B, V \setminus B)$  be two **s-t min-cuts**. Then  $(A \cap B, V \setminus (A \cap B))$  and  $(A \cup B, V \setminus (A \cup B))$  are also **s-t min-cuts**.

**Proof.** From **submodularity**, we have  $|\delta(A)| + |\delta(B)| \geq |\delta(A \cap B)| + |\delta(A \cup B)|$ . However, as both  $A \cup B$  and  $A \cap B$  are themselves **s-t cuts**, all terms need to be equal. ■

**Corollary 1.3.1.** For any graph  $G = (V, E)$ , there is a unique (inclusion-wise) minimal **s-t min-cut**.<sup>a</sup>

<sup>a</sup>While maybe not that useful, from the same logic, there is a unique maximal **s-t min-cut**.

**Proof.** If there are two **s-t min-cuts**  $A, B$  that are both minimal and distinct, then  $A \setminus B \neq \emptyset$  and  $B \setminus A \neq \emptyset$  since otherwise one will be contained in the other, contradicting the minimality. From **Lemma 1.3.3**,  $A \cap B$  is also a **s-t min-cut** and  $A \cap B$  is a strict subset of  $A$  and  $B$ , again contradicting minimality of  $A$  and  $B$ . ■

The above proof applies to directed graph as well since we only used **submodularity**.

**Remark (Graphic matroid).** A second aspect of **submodularity** in graphs comes via *matroids*. We will not discuss it here but the rank function of a matroid is a special class of **submodular functions**; and in a formal sense, matroid rank functions are building blocks for all **submodular functions**. Given an undirected graph  $G = (V, E)$  there is a fundamental matroid associated with the edge set of  $G$  called the *graphic matroid*.<sup>a</sup> Several properties of trees and forests can be better understood in the context of the graphic matroid including the **Tutte-Nash-Williams theorem**.

<sup>a</sup>There are other matroids that are also defined from graphs including the dual graphic matroid for instance.

## Isolating Cuts via Poly-Logarithmic Many Max-Flow Computations

We can now formally introduce the **isolating cut** problem.

**Problem 1.3.5 (Isolating cut).** Given a graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$  and a set  $T \subseteq V$  of terminals. The  $t_i$ -*isolating cut* problem aims to find a cut  $(S_i, V \setminus S_i)$  such that  $t_i \in S_i$  and  $t_j \notin S_i$  (i.e.,  $t_j \in V \setminus S_i$ ) for all  $j \neq i$ , i.e., the cut isolates  $t_i$  from the rest of the terminals.

The minimum capacity  $t_i$ -*isolating cut* can be found by a single **max-flow** computation: by shrinking the terminals in  $T - t_i$  into a single vertex  $s$  and computing the **s- $t_i$  min-cut**. Thus, naively, computing all **isolating cuts** require  $k$  **max-flow** computations. The upshot is that this can be done in only  $O(\log k)$

**max-flow.** Before we describe the algorithm, we first note that from **submodularity**, we also have a similar structural result, just like **Corollary 1.3.1**.

**Lemma 1.3.4.** There is a unique minimal  $t_i$ -isolating min-cut  $(S_i^*, V \setminus S_i^*)$  such that if  $(S_i, V \setminus S_i)$  is any  $t_i$ -isolating min-cut, then  $S_i^* \subseteq S_i$ .

We now describe the algorithm for computing the **isolating cuts**. Basically, we consider  $h$  bi-partitions  $(A_1, B_1), \dots, (A_h, B_h)$  with  $h = \lceil \log k \rceil$ , and compute a cut separating each bi-partition. Then, we take the intersection among the resulting cut sets, which will be **isolating cuts** as we will see. Finally, with the structural property **Lemma 1.3.4**, we can then find the minimum **isolating cuts** from them.

**Algorithm 1.10: Isolating Cut [LP20]** (also developed independently in [AKT21])

**Data:** A connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , terminal  $T = \{t_i\}_{i=1}^k$

**Result:** A set of **isolating cuts**  $\{(S_i^*, V \setminus S_i^*)\}_{i=1}^k$  isolating  $t_i$ 's

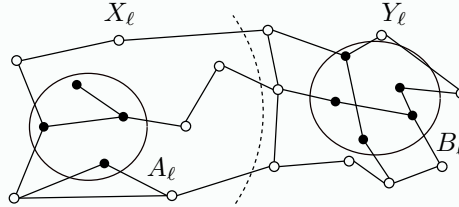
```

1 for  $\ell = 1, \dots, \lceil \log k \rceil$  do //  $\lceil \log k \rceil$  bi-partitions  $(A_\ell, B_\ell)$  of  $T$ 
2    $A_\ell \leftarrow \{t_i \mid \text{binary representation of } i \text{ has 1 in } \ell^{\text{th}} \text{ bit}\}$ 
3    $B_\ell \leftarrow T \setminus A_\ell$ 
4    $(X_\ell, Y_\ell) \leftarrow \text{s-t-min-cut}(G, c, A_\ell, B_\ell)^a$  //  $Y_\ell = V \setminus X_\ell$ 
5 for  $i = 1, \dots, k$  do
6    $S_i \leftarrow \bigcap_{\ell: t_i \in A_\ell} X_\ell \cap \bigcap_{\ell: t_i \in B_\ell} Y_\ell$ 
7    $H_i = (V_i, E_i) \leftarrow \text{shrinking } V \setminus S_i \text{ to a single vertex } s_i$ 
8    $(S_i^*, V \setminus S_i^*) \leftarrow \text{s-t-min-cut}(H_i, c, t_i, s_i)$ 
9 return  $\{(S_i^*, V \setminus S_i^*)\}_{i=1}^k$ 

```

<sup>a</sup>This can be done via shrinking  $A_\ell$  and  $B_\ell$  into two separate nodes, and compute the **s-t min-cut**.

**Intuition.** The following illustrates **line 4**, where terminals are black vertices.  $(A_\ell, B_\ell)$  is a bi-partition of  $T$ , while  $(X_\ell, Y_\ell)$  is a **min-cut** that separates  $(A_\ell, B_\ell)$ .



Additionally, **line 6** is created by considering the intersections of all  $X_\ell$  (or  $Y_\ell$ ) that includes  $t_i$ .

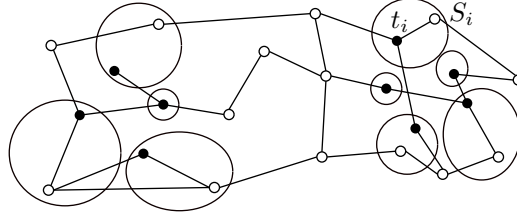
To see the correctness of the **Algorithm 1.10**, we want to say that **s-t min-cut** in  $H_i$  is exactly the minimum cost  $t_i$ -isolating cut in  $G$ . This is due to **Lemma 1.3.5**.

**Lemma 1.3.5.** For each  $i$ ,  $(S_i, V \setminus S_i)$  is a  $t_i$ -isolating cut. Furthermore,  $S_i$ 's are pairwise disjoint.

**Proof.** Firstly,  $t_i \in A_\ell \subseteq X_\ell$  or  $t_i \in B_\ell \subseteq Y_\ell$ , implying  $t_i \in S_i$ . Consider  $t_j$  with  $j \neq i$ . As  $i \neq j$ , there is some index  $\ell$  in the binary representation of  $i$  and  $j$  differ in the bit position. Suppose  $i$  has 1 in the  $\ell^{\text{th}}$  position and  $j$  has 0, then  $t_i \in A_\ell$  and  $t_j \in B_\ell$ , implying  $t_i \in X_\ell$  and  $t_j \notin X_\ell$  as  $t_j \in B_\ell \subseteq Y_\ell$  and  $Y_\ell \cap X_\ell = \emptyset$ . This means  $t_j \notin S_i$ .

We now prove that  $S_i \cap S_j = \emptyset$  for all  $i \neq j$ . Firstly, there exists some  $\ell$  such that  $t_i \in A_\ell$  and  $t_j \in B_\ell$  (or  $t_i \in B_\ell$  and  $t_j \in A_\ell$ ). Suppose  $v \in X_\ell$ , then  $v$  can't be in  $S_j \subseteq Y_\ell$  and if  $v \in Y_\ell$ , then  $v$  can't be in  $S_i \subseteq X_\ell$ , hence  $v$  can't be in both  $S_i$  and  $S_j$ . ■

**Lemma 1.3.5** gives the following picture, where each  $t_i$  lives in exactly one  $S_i$ .



Hence, for each  $i$ , we have a  $t_i$ -isolating cut  $(S_i, V \setminus S_i)$ . Now, [Lemma 1.3.4](#) states that there is a  $t_i$ -isolating min-cut  $(S_i^*, V \setminus S_i^*)$  where  $S_i^*$  is a subset of any  $t_i$ -isolating min-cut, it doesn't say it will be a subset of  $S_i$  in particular, as  $(S_i, V \setminus S_i)$  is only a  $t_i$ -isolating cut. However, we do not lose anything:

**Lemma 1.3.6.** The minimal  $t_i$ -isolating min-cut  $(S_i^*, V \setminus S_i^*)$  is in  $(S_i, V \setminus S_i)$ , i.e.,  $S_i^* \subseteq S_i$ .

**Proof.** It suffices to prove that if  $t_i \in A_\ell$  then  $S_i^* \subseteq X_\ell$ . Assume not, then  $S_i^* \cap (V \setminus X_\ell) \neq \emptyset$ . But since  $S_i^* \cap X_\ell$  is a  $t_i$ -isolating cut, while  $S_i^*$  is the minimal  $t_i$ -isolating min-cut,  $|\delta(S_i^* \cap X_\ell)| > |\delta(S_i^*)|$ .

Moreover, it's trivial to see that  $S_i^* \cup X_\ell$  is a  $A_\ell$ - $B_\ell$  cut (not necessarily minimum, just a cut), hence we also have  $|\delta(S_i^* \cup X_\ell)| \geq |\delta(X_\ell)|$ . From [submodularity](#) of  $|\delta(\cdot)|$ , we have

$$|\delta(S_i^*)| + |\delta(X_\ell)| \geq |\delta(S_i^* \cap X_\ell)| + |\delta(S_i^* \cup X_\ell)|,$$

which is a contradiction. ■

With all the lemmas, it's now easy to see that [Algorithm 1.10](#) is at least correct. Firstly, from [Lemma 1.3.6](#), we know that the optimal  $t_i$ -isolating min-cut  $S_i^*$  is a subset of  $S_i$  (here,  $S_i^*$  is not necessary the one found by [Algorithm 1.10](#), we're trying to argue this). As  $S_i$ 's are disjoint from [Lemma 1.3.5](#), each terminal  $t_i$  lives in exactly one  $S_i$ , hence computing  $s_i$ - $t_i$  min-cut will indeed recover  $S_i^*$ .

**Intuition.** When we contract  $V \setminus S_i$ , we do not lose the optimal isolating cut  $S_i^*$ .

**Theorem 1.3.6** ([LP20]). [Algorithm 1.10](#) is a deterministic algorithm that given  $G = (V, E)$  and a terminal set  $T \subseteq V$  with  $|T| = k$ , computes all the isolating cuts using  $O(\log k)$  max-flow computations on graphs with  $|V|$  vertices and  $|E|$  edges each.

**Proof.** We analyze the runtime. It's easy to see that [line 1](#) requires  $O(\log k)$  max-flow computations on  $G$ . It's also easy to show that computing  $S_i$ 's in [line 6](#) can be done in  $O((m+n) \log k)$  time given  $(X_\ell, Y_\ell)$  for  $\ell \in [\log k]$ . However, [line 7](#) and [line 8](#) seem to require  $k$  max-flow computations.

**Claim.** In total, [line 8](#) only requires  $O(\log k)$  max-flow computations.

**Proof.** Let us understand the size of  $H_i$ . It has  $n_i + 1$  vertices where  $n_i = |S_i|$ , and it has  $m_i$  edges where  $m_i = |E(S_i)| + |\delta(S_i)|$ . Thus, the running time of max-flow on  $H_i$  is  $T(n_i + 1, m_i)$  where  $T(a, b)$  is the running time of max-flow on graph with  $a$  nodes and  $b$  edges. We observe that  $\sum_i (n_i + 1) \leq 2n$  since  $S_i$ 's are disjoint, while  $\sum_i m_i \leq 2m$ : consider any edge  $uv \in E$ . If  $uv \in E(S_i)$  for some  $i$ , then it does not contribute to any other  $H_j$ . If  $uv \in \delta(S_i)$  for some  $i$ , then it can be in  $\delta(S_j)$  for only one more index  $j \neq i$ .

Thus, the total time to compute all  $k$  max-flows is  $\sum_i T(n_i, m_i) \leq T(2n, 2m)$  under reasonable assumption, specifically,  $T(a, b)$  is super-additive.<sup>a</sup> ⊛

<sup>a</sup>Formally, we first create a single  $H$  that includes each  $H_i$  as a copy in it, and we can run a single max-flow on  $H$  to recover all the max-flow values in each  $H_i$ .  $H$  will have  $O(n)$  vertices and  $O(m)$  edges.

With the correctness of [Algorithm 1.10](#), the theorem is proved. ■

We see that this could have been discovered many years ago in terms of its simplicity. [Algorithm 1.10](#) has been very influential in the last few years for a number of problems.

**Note.** Another perspective of the bi-partitions is that they are a way to *derandomize* a natural randomized algorithm that picks some  $O(\log k)$  bi-partitions of  $T$  at random and computes the cuts between them. With high probability, every  $t_i, t_j$  with  $i \neq j$  will be separated in at least one of the random bi-partitions.

**Remark.** The core idea of **isolating cuts** relies only on **submodularity** and symmetry, thus, this applies in much more generality and to several other problems. This is explicitly discussed in [CQ21], though the ideas are implicit in [LP20].

### Randomized Algorithm for Steiner Min-Cut via Isolating Cuts

**Isolating cut** naturally lead to a simple randomized algorithm for **Steiner min-cut**. The basic idea is quite simple. Consider an optimum **Steiner min-cut**  $(S, V \setminus S)$  and let  $T_1 := S \cap T$  and  $T_2 := (V \setminus S) \cap T$ , with  $k_1 = |T_1|$  and  $k_2 = |T_2|$ . We may assume that  $1 \leq k_1 \leq k_2$ .

**Note.**  $(S, V \setminus S)$  is a  $t_i$ - $t_j$  **min-cut** for any  $t_i \in T_1, t_j \in T_2$  since otherwise, a lower-cost cut exists.

The basic intuition is the following.

**Intuition.** If we can sample exactly one terminal in one side of the **Steiner min-cut**, then we can simply use the **isolating cut** to recover the **Steiner min-cut**.

Say we know  $k_1$ . We can sample each terminal in  $T$  independently with probability  $1/k_1$  to obtain  $T' \subseteq T$  such that with constant probability,  $|T' \cap T_1| = 1$  and  $|T' \cap T_2| \geq 1$  (recall  $k_1 \leq k_2$ ). Suppose  $T'$  satisfies these properties and let  $T' \cap T_1 = \{t_i\}$ . Then,  $(S, V \setminus S)$  is a minimum cost  $t_i$ -**isolating cut** w.r.t.  $T'$ . Hence, by computing  $t_i$ -**isolating cuts** for all  $t_i \in T'$  and choosing the cheapest one identifies the **Steiner min-cut** for  $T$ . The problem is that we don't know  $k_1$ , and trying all possible values for  $k_1$  (from 1 to  $k/2$ ) will be too expensive.

**Intuition.** The above sampling procedure is robust: if we sample with probability, say,  $1/2k_1$ , everything still happens with constant probability. Hence, we only need to try  $k_i = 2^i$ , i.e.,  $O(\log k)$  different sampling probabilities.

We formally describe this algorithm as follows.

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#### Algorithm 1.11: Steiner Min-Cut

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**Data:** A connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , terminal  $T = \{t_i\}_{i=1}^k$

**Result:** A possible **Steiner min-cut**  $(S^*, V \setminus S^*)$

---

```

1  $S^* \leftarrow \emptyset$  // Initialize Steiner min-cut
2 for  $i = 0, \dots, \lceil \log k \rceil$  do
3    $T' \leftarrow \text{Sample}(T, 1/2^i)$  // Sample each terminal in  $T$  with probability  $1/2^i$ 
4    $\{(S_i^*, V \setminus S_i^*)\}_{i=1}^{|T'|} \leftarrow \text{Isolating-Cut}(G, c, T')$ 
5    $S^* \leftarrow \text{Min-Cost}(\{(S_i^*, V \setminus S_i^*)\}_{i=1}^{|T'|} \cup \{(S^*, V \setminus S^*)\})$  // Update minimum cost cut
6 return  $(S^*, V \setminus S^*)$ 

```

---

We now formally prove the robustness we have mentioned.

**Lemma 1.3.7.** Algorithm 1.11 finds the **Steiner min-cut** for  $T$  with a constant probability.

**Proof.** We see that for  $k_1 = 1$ , Algorithm 1.11 is correct (deterministically) since  $i$  can only be 0 and  $T' = T$ . Hence, let  $k_1 > 1$ . Consider the case that  $1/2^{i+1} < 1/k_1 \leq 1/2^i$ , where  $i$  will be tried at some point during  $i = 0$  to  $\lceil \log k \rceil$  since  $1 \leq k_1 \leq k/2$ . Let  $\ell = 2^i$ , i.e.,  $\ell \leq k_1 \leq 2\ell$ .

Now, let  $\mathcal{E}_1$  be the event that  $|T_1 \cap T'| = 1$ , i.e., exactly one terminal from  $T_1$  is chosen. Then

$$\Pr(\mathcal{E}_1) = k_1 \cdot \frac{1}{\ell} \cdot \left(1 - \frac{1}{\ell}\right)^{k_1-1} \geq \left(1 - \frac{1}{\ell}\right)^{2\ell} \geq \frac{1}{e^2}.$$

On the other hand, let  $\mathcal{E}_2$  be the event that  $T_2 \cap T' \neq \emptyset$ . We see that

$$\Pr(\mathcal{E}_2) \geq 1 - \left(1 - \frac{1}{\ell}\right)^{k_2} \geq \left(1 - \frac{1}{\ell}\right)^\ell \geq 1 - \frac{1}{e}.$$

Since  $T_1$  and  $T_2$  are disjoint,  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are independent, we have

$$\Pr(\mathcal{E}_1 \cap \mathcal{E}_2) \geq \left(1 - \frac{1}{e}\right) \cdot \frac{1}{e^2},$$

which is a constant. ■

**Theorem 1.3.7.** There is a randomized algorithm that given  $G = (V, E)$  and terminal set  $T \subseteq V$  of size  $k$ , outputs the **Steiner min-cut** with high probability using  $O(\log^2 k \log n)$  **max-flow** computations on graphs with  $|V|$  vertices and  $|E|$  edges.

**Proof.** From **Lemma 1.3.7**, **Algorithm 1.11** succeeds with a constant probability. We further boost the overall success probability by rerunning **Algorithm 1.11**  $\Theta(\log n)$  times. With **Theorem 1.3.6**, this requires  $O(\log^2 k \log n)$  **max-flow** computations. ■

**Remark (Deterministic algorithm).** Li and Panigraphy [LP20] also developed deterministic **min-cut** and **Steiner min-cut** algorithms using additional ideas based on **expander decomposition**.

## Lecture 5: Metric Embedding and Multi-Cut Problem

### 1.3.4 Max-Flow Min-Cut Theorem

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Finally, we conclude this section by proving the well-known **max-flow min-cut theorem**. Consider the following linear program relaxation of the **s-t min-cut** where  $s, t$  are two distinct vertices:

$$\begin{aligned} \min \quad & \sum_{e \in E} c(e)x_e & \max \quad & \sum_{P \in \mathcal{P}_{s,t}} y_P \\ & \sum_{e \in P} x_e \geq 1 \quad P \in \mathcal{P}_{s,t}; & & \sum_{P \ni e} y_P \leq c(e) \quad \forall e \in E; \\ \text{(P)} \quad & x_e \geq 0 \quad \forall e \in E; & \text{(D)} \quad & y_P \geq 0 \quad \forall P \in \mathcal{P}_{s,t}, \end{aligned} \tag{1.2}$$

where  $\mathcal{P}_{s,t}$  is the set of all  $s$ - $t$  paths. The integer version is with constraints  $x_e \in \{0, 1\}$ .

**Remark.** An **s-t cut** is often also defined as  $\delta^+(S)$  for some  $S \subseteq V$  where  $s \in S$  and  $t \in V \setminus S$ .

**Proof.** Suppose  $E'$  is an **s-t cut** and  $S$  is the set of nodes reachable from  $s$  in  $G - E'$ . Then,  $\delta(S) \subseteq E'$  and  $\delta(S)$  is an **s-t cut**. Hence, it suffices to focus on such limited type of cuts.<sup>a</sup> ⊛

<sup>a</sup>In some more general settings, it is useful to keep these notions separate.

It is well-known that **s-t min-cut** can be computed efficiently via **s-t max-flow**, establishing the **max-flow min-cut theorem**. This fundamental theorem in combinatorial optimization has many applications, and is typically established via the augmenting path algorithm. Here, we give another proof.

**Theorem 1.3.8 (Max-flow min-cut).** Let  $G = (V, E)$  be a directed graph with rational edge capacities  $c: E \rightarrow \mathbb{Q}_+$  and let  $s, t \in V$  be two distinct vertices. The **s-t max-flow** value in  $G$  is equal to the **s-t min-cut** value, and both can be computed in strongly polynomial time. Furthermore, if  $c$  is integer valued, then there exists an integer-valued **max-flow** as well.

**Proof.** To start, we observe that the primal **linear program** assigns lengths to edges such that the  $s$ - $t$  shortest path according to which is at least 1. This is a fractional relaxation of the cut. We claim that it's possible to round the fractional solution of the primal to the exact **s-t min-cut** without any



loss. Consider the following rounding algorithms for the primal [linear program](#).

---

**Algorithm 1.12:**  $\theta$ -Rounding Algorithm

---

**Data:** A directed graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ ,  $s, t \in V$

**Result:** A  $s$ - $t$  min-cut  $F$

---

```

1  $\{x_e\}_{e \in E} \leftarrow \text{LP-Solve}(\text{Min-Cut-LP}(G, c, s, t))$  // Solve the primal
2  $\theta \leftarrow \text{Uniform}((0, 1))$ 
3 for  $v \in V$  do
4    $d_x(s, v) \leftarrow \text{Shortest-Path-Dist}(s, v, G, x)$ 
5 return  $F = \delta^+(B_x(s, \theta))$  //  $B_x(s, \theta) = \{v \in V \mid d_x(s, v) \leq \theta\}$ 
```

---

Firstly, [Algorithm 1.12](#) will output a valid  $s$ - $t$  cut since  $d_x(s, t) \geq 1$  by feasibility of the linear program solution  $x$  and hence  $t \notin B_x(s, \theta)$  for any  $\theta < 1$ .

**Claim.** For any  $e \in E$ ,  $\Pr(e \text{ is cut by Algorithm 1.12}) \leq x_e$ .

**Proof.** The edge  $e = (u, v)$  is cut if and only if  $d_x(s, u) \leq \theta < d_x(s, v)$ . Hence, the edge is not cut if  $d_x(s, v) \leq d_x(s, u)$ . If  $d_x(s, v) > d_x(s, u)$ , we have  $d_x(s, v) - d_x(s, u) \leq x_{(u, v)}$ . Since  $\theta$  is chosen uniformly at random from  $(0, 1)$ , the probability that  $\theta$  lies in the interval  $[d_x(s, u), d_x(s, v)]$  is at most  $x_{(u, v)}$ .  $\otimes$

With this claim, from linearity of expectation, we see that  $\mathbb{E}[c(\delta^+(B_x(s, \theta)))] \leq \sum_{e \in E} c(e)x_e$ . As  $B_x(s, \theta)$  will always be a valid  $s$ - $t$  cut, this implies that there is an integral cut whose cost is at most that of the linear program relaxation, implying that the linear program relaxation yields an optimum solution.

Finally, observe that the dual is the *path version* of the  $s$ - $t$  max-flow. Hence, from strong duality, the optimal value of  $s$ - $t$  min-cut is the same as the  $s$ - $t$  max-flow. Moreover, we note that the primal is strongly polynomial-time solvable if we have a separation oracle. In this case, given  $\{x_e\}_{e \in E}$ , we need to answer either this is a feasible solution, or outputs some path  $p$  such that  $\sum_{e \in p} x_e > 1$ , which is exactly the shortest  $s$ - $t$  path algorithm and can be solved efficiently.  $\blacksquare$

**Intuition (Line embedding).** The rounding can be thought as putting every vertex on a line from  $s$  to  $t$ , sorting by their distances given by  $x_e$ 's. Then, observe that on that line, any two vertices  $u, v \in V$  has distance at most  $x_{(u, v)}$ , and picking  $\theta$  corresponds to picking a threshold on the line.

The above intuition not only helps the analysis, but also gives a way to derandomize [Algorithm 1.12](#). Basically, we can try all possible  $\theta$ 's, and if we adapt this line embedding viewpoint, the only interesting  $\theta$ 's are given by the  $n$  values  $d_x(s, v)$ .



## Chapter 2

# Metric Methods

In this chapter, we will see a series of techniques that are based on the structure of the metric spaces underlying the graphs.

### 2.1 Multi-cut via Metric Decomposition

Recall that [s-t min-cut problem](#) we just saw. Now, consider a more general problem called [multi-min-cut](#).

**Problem 2.1.1 (Multi-min-cut).** Given a graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$  and  $k$  pairs of vertices  $\{(s_i, t_i)\}_{i=1}^k$ , the *multi-min-cut problem* aims to find a minimum capacity cut that separates all pairs.

[Multi-min-cut](#) is NP-hard even on trees. In general, it is a NP-complete problem. Hence, we ask for an approximation algorithm instead. An  $O(k)$ -approximation algorithm is trivial by simply outputting the union of all [s<sub>i</sub>-t<sub>i</sub> min-cuts](#). The goal is an  $O(\log k)$ -approximation algorithm, which also proves the multi-commodity flow-cut gap.

**Note.** It turns out that  $O(\log k)$  is tight in general graphs. For planar graphs, one can get an  $O(1)$ -approximation and flow-cut gap. These results are only for undirected graphs since the situation is more complicated in directed graphs, and we will discuss that later.

Again, we can write the following linear program relaxation for the [multi-min-cut problem](#):

$$\begin{aligned}
 \min \quad & \sum_{e \in E} c(e)x_e & \max \quad & \sum_{i=1}^k \sum_{P \in \mathcal{P}_{s_i, t_i}} y_P \\
 \sum_{e \in P} x_e \geq 1 \quad & P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}; & \sum_{i=1}^k \sum_{P \ni e} y_P \leq c(e) & \quad \forall e \in E; \\
 \text{(P)} \quad x_e \geq 0 & \quad \forall e \in E; & \text{(D)} \quad y_P \geq 0 & \quad \forall P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}.
 \end{aligned} \tag{2.1}$$

The primal assigns distance labels  $x_e$  to edges so that, on each path  $P$  between  $s_i$  and  $t_i$ , the distance labels of these edges on  $P$  sum up to at least one, just like the [s-t min-cut](#).

**Remark.** The primal (with exponentially many constraints) is efficiently solvable.

**Proof.** With the ellipsoid method, we just need a separation oracle. Consider setting the length of each edge to  $x_e$  and for each pair  $(s_i, t_i)$ , compute the length of the shortest path between  $s_i$  and  $t_i$  and check whether it is at least 1. This only takes  $k$  many times compared to the previous separation oracle for the [s-t min-cut](#) linear program relaxation, hence it's still polynomial time.  $\otimes$

On the other hand, the dual variable can be interpreted as the amount of [flow](#) between  $s_i$  and  $t_i$  that is routed along the path  $P$ . This is called the *maximum throughput multi-commodity flow* problem,

where we don't care about individual demands, but only the overall **flow**. The dual tries to assign an amount of **flow**  $y_P$  to each path  $P$  so that the total **flow** on each edge is at most the capacity of the edge.

**Note.** The **flow conservation constraints** are automatically satisfied and the endpoints of the path  $P$  determine which kind of commodity is routed along the path.

### 2.1.1 Approximation via Randomized Decomposition

The first algorithm [GVY93] (see [Vaz01; WS11]) that achieved an  $O(\log k)$ -approximation for **multi-min-cut** is based on the region growing technique [LR99]. Here, we present the randomized rounding algorithm due to its future application for metric embedding [CKR05], which in particular also shows the integrality gap of the primal **linear program** to be  $O(\log k)$ . The goal is to find a procedure to cut (i.e., decompose) the graph into components that satisfies the requirement of being a **multi-cut**, i.e.,

- each component has diameter at most 1 when the edge capacity is induced by the primal solution;
- the probability that edge  $e$  is cut is at most  $\alpha x_e$ ,

then we will get an  $\alpha$ -approximation algorithm. To do this, we consider the following algorithm.

---

**Algorithm 2.1:** Random Partition [CKR05]

---

**Data:** A connected graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ ,  $\{(s_i, t_i) \mid s_i, t_i \in V\}_{i=1}^k$

**Result:** A **multi-min-cut**  $F$

```

1  $\{x_e\}_{e \in E} \leftarrow \text{LP-Solve}(\text{Multi-Min-Cut-LP}(G, c, \{(s_i, t_i)\}_{i=1}^k))$  // Solve the primal
2  $\theta \leftarrow \text{Uniform}([0, 1/2])$ 
3  $\sigma \leftarrow \text{Uniform}(S([k]))$  // Random permutation from permutation group  $S([k])$ 
4 for  $i = 1, \dots, k$  do
5    $V_{\sigma(i)} \leftarrow B_x(s_{\sigma(i)}, \theta) \setminus \bigcup_{j < i} V_{\sigma(j)}$ 
6  $F \leftarrow \bigcup_{i=1}^k \delta(V_i)$ 
7 return  $F$ 

```

---

**Intuition.** It essentially reduces to *low-diameter decomposition* in a metric space.

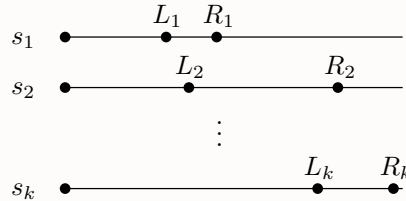
**Lemma 2.1.1.** Algorithm 2.1 outputs a feasible **multi-cut** for the given instance.

**Proof.** Suppose not, then there exists a pair  $(s_i, t_i)$  are still connected in  $G - F$ . This can only happen if  $s_i$  is “grabbed” by some terminals  $s_j$  which is proceeded before  $s_i$ , i.e., there exists some  $V_j \subseteq B_x(s_j, \theta)$  that contains both  $s_i$  and  $t_i$ . However, if  $s_j$  grabs both  $s_i$  and  $t_i$ , it means the distance between  $s_i$  and  $t_i$  is at most  $2\theta < 1$ , a contradiction to the feasibility of  $\{x_e\}_{e \in E}$ . ■

**Lemma 2.1.2.** For any  $e \in E$ ,  $\Pr(e = uv \text{ is cut by Algorithm 2.1}) \leq 2H_k x_e \leq O(\log k)x_e$ .<sup>a</sup>

<sup>a</sup> $H_k$  is the  $k^{\text{th}}$  harmonic number.

**Proof.** Let  $L_i := \min(d_x(s_i, u), d_x(s_i, v))$  and  $R_i := \max(d_x(s_i, u), d_x(s_i, v))$ . Without loss of generality, we can renumber  $s_i$ 's such that  $L_1 \leq L_2 \leq \dots \leq L_k$ .

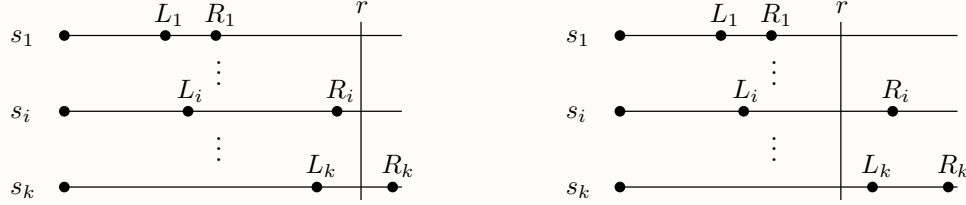


Let  $A_i$  be the event that  $s_i$  cut  $e = uv$  first, i.e.,  $A_i$  is the event that  $|V_i \cap \{u, v\}| = 1$  and  $|V_j \cap \{u, v\}| = 0$  for all  $j$  such that  $\sigma(j) < \sigma(i)$ , where  $|V_i \cap \{u, v\}| = 1$  simply says that  $s_i$  cuts the edge  $e$ . If  $A_i$  happens, then for all  $j$  that come before  $i$  in  $\sigma$ , neither  $u$  nor  $v$  can be in  $V_j$  since:

- if only one of  $u$  and  $v$  is in  $V_j$ , then  $s_j$  cuts  $e$ ;
- if both  $u$  and  $v$  are in  $V_j$ ,  $s_i$  can't cut  $e$  as the cut set only grabs the leftover vertices (line 5).

Let  $A$  be the event that  $e$  is cut, which is the union of the disjoint events  $A_i$ 's, hence  $\Pr(A) = \sum_{i=1}^k \Pr(A_i)$ . Now, for any fixed  $r \in [0, 1/2)$ , we see that

- $r \notin [L_i, R_i]$ : This is easy to understand as if  $r \notin [L_i, R_i]$ ,  $s_i$  is impossible to cut  $e$ ;
- $r \in [L_i, R_i]$ : Consider some  $j < i$  and suppose  $j$  comes before  $i$  in the permutation (i.e.,  $\sigma(j) < \sigma(i)$ ). Since  $j < i$ ,  $L_j \leq L_i \leq r$ . Hence, at least one of  $u$  and  $v$  is inside the ball of radius  $r$  centered at  $s_j$ . Consequently,  $s_i$  can't be the first to cut  $e$ , resulting in the fact that  $s_i$  is the first to cut the edge  $e$  if  $\sigma(i) < \sigma(j)$  for all  $j < i$ .



Since  $\sigma$  is a random permutation,  $i$  appears before  $j$  for all  $j < i$  with probability  $1/i$ . Hence,

$$\begin{cases} \Pr(A_i \mid \theta = r) = 0, & \text{if } \theta \notin [L_i, R_i]; \\ \Pr(A_i \mid \theta = r) \leq 1/i, & \text{if } \theta \in [L_i, R_i]. \end{cases}$$

As  $\theta$  is independent of  $\sigma$ , we have

$$\Pr(A_i) \leq \frac{1}{i} \cdot \mathbb{P}(\theta \in [L_i, R_i]) = \frac{2}{i}(R_i - L_i) \leq \frac{2x_e}{i}$$

from the triangle inequality  $R_i \leq L_i + x_e$ . This finally leads to

$$\Pr(A) = \sum_{i=1}^k \Pr(A_i) \leq \sum_{i=1}^k \frac{2x_e}{i} \leq 2H_k x_e,$$

and we conclude the theorem by noting that  $H_k = O(\log k)$ . ■

**Theorem 2.1.1.** Algorithm 2.1 is an  $O(\log k)$ -approximation (in expectation) algorithm for the multi-min-cut problem. Furthermore, the integrality gap of the multi-min-cut linear program is  $O(\log k)$ .

**Proof.** Let  $F$  be the set of edges outputted by Algorithm 2.1. For each edge  $e$ , let  $\xi_e = \mathbb{1}_{e \in F}$  in an indicator random variable. Hence, we have  $\mathbb{E}[\xi_e] = \mathbb{P}(\xi_e = 1) \leq 2H_k x_e$  from Lemma 2.1.2. This leads to

$$\mathbb{E}[c(F)] := \mathbb{E} \left[ \sum_{e \in F} c(e) \right] = \mathbb{E} \left[ \sum_{e \in E} c(e) \xi_e \right] = \sum_{e \in E} c(e) \Pr(\xi_e) \leq 2H_k \sum_{e \in E} c(e) x_e = 2H_k \text{OPT}_{\text{LP}}$$

where  $\text{OPT}_{\text{LP}}$  is the optimal value of the linear program. Since  $\text{OPT}_{\text{LP}} \leq \text{OPT}$  where  $\text{OPT}$  is the optimum value of the multi-min-cut problem, we have

$$\mathbb{E}[c(F)] \leq 2H_k \text{OPT}_{\text{LP}} \leq 2H_k \text{OPT} = O(\log k) \text{OPT}.$$

This also implies that there exists a set of edges  $F$  such that the total capacity of edges in  $F$  is at most  $2H_k \text{OPT}_{\text{LP}}$ , i.e.,  $\text{OPT}_{\text{LP}} \leq \text{OPT} \leq 2H_k \text{OPT}_{\text{LP}}$ , which proves the integrality gap result. ■

The expected cost analysis can be used to obtain a randomized algorithm via repetition that outputs an  $O(\log k)$ -approximation with high probability. The algorithm can also be derandomized, but it's not straight forward.

**Remark (Flow-cut gap).** Recall that when  $k = 1$ , we have the [max-flow min-cut theorem](#). The integrality gap of the standard linear program for [multi-min-cut](#) is the same as the relative gap between [flow](#) and cut when  $k$  is arbitrary. The upper bound on the integrality gap gives an upper bound on the gap.

## Lecture 6: Low-Diameter Decomposition and Tree Embeddings

### 2.1.2 Low-Diameter Decomposition

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Given a graph  $G = (V, E)$  and edge length  $\ell: E \rightarrow \mathbb{R}_+$ , which define a metric space  $(V, d)$  where  $d(u, v)$  is the shortest path distances between  $u$  and  $v$  in  $G$  as before. As we have seen, a useful notion is to decompose or partition the graph into subgraphs (or clusters) of small diameter. More precisely, given a graph  $G = (V, E)$ , we would like to partition  $V$  into clusters with vertex sets  $\{V_i\}_{i=1}^h$  such that each  $V_i$  has diameter at most some given parameter  $\delta$ .

**Example (Singleton).** It's trivial to consider the singleton partition, where  $V_i = \{v\}$  for all  $v \in V$ .

However, the goal in partitioning is to ensure that two vertices  $u, v \in V$  that are close to each other, say  $d(u, v) < \delta$ , should ideally not be split apart into different clusters. However, as the graph (or metric space) is connected, it's impossible to do this deterministically.

**Example (Line graph).** Considered a line graph  $L_n$ . The most natural randomized algorithm is to shift the line by  $\theta \in [0, \delta)$ , and then we separate the line graph by  $\delta$ -length clusters. In this way, the probability that any pair  $u, v \in V$  is cut is at most  $d(u, v)/\delta$ .

This is the best we can hope for, i.e.,  $u, v$  are separated only with probability proportional to  $d(u, v)/\delta$ .

**Definition 2.1.1 (Low-diameter decomposition).** Let  $G = (V, E)$  be a graph with edge lengths  $\ell: E \rightarrow \mathbb{R}_+$ , which induces a distance metric  $d: V \times V \rightarrow \mathbb{R}_+$ . Let  $\Delta$  be the diameter of the metric space  $(V, d)$ . For a given  $\delta \in [0, \Delta]$ , a *low-diameter decomposition* with cutting probability parameter  $\alpha$  is a probability distribution  $\mathcal{D}$  over the set  $\mathcal{P}$  of all partitions of  $V$  such that

- for any partition  $P = (V_1, \dots, V_h) \in \mathcal{P}$  in  $\text{supp}(\mathcal{D})$ ,  $\text{diam}((V_i, d)) \leq \delta$ ;
- for all  $u, v \in V$ ,  $\Pr(u, v \text{ are separated}) \leq \alpha d(u, v)/\delta$ .

Given a partition  $P \in \mathcal{P}$ , we write  $E_P$  be the set of edges (pairs) that are separated by  $P$ . Hence, in [Definition 2.1.1](#), the second point is equivalent to  $\Pr((u, v) \in E_P) \leq \alpha d(u, v)/\delta$ .

**Definition 2.1.2 (Low-diameter decomposition scheme).** A *low-diameter decomposition scheme* with parameter  $\alpha$  is a family of algorithms that given any  $\delta \in [0, \Delta]$ , generates a [low-diameter decomposition](#) with cutting (separation) probability parameter at most  $\alpha$ .

**Notation (Strong v.s. weak diameter guarantee).** A [low-diameter decomposition](#) is said to have the *strong diameter* guarantee if the diameter of each cluster  $V_i$  in the induced graphs  $G[V_i]$  is at most  $\delta$ . Note that [Definition 2.1.2](#) does not require that because it is based on the metric closure  $(V, d)$  of the given graph. The standard definition is called the *weak diameter* guarantee. Some applications require the strong diameter guarantee. We will, by default, work with weak diameter guarantee and mention strong diameter guarantee when needed.

**Note (Padded decomposition).** [Definition 2.1.1](#) is often strengthened to require more. Given a point  $u$ , let  $B_d(u, r) = \{v \in V \mid d(u, v) \leq r\}$ . In *padded decomposition*, we require that for each  $u$ , and for each  $r \leq \delta$ , the probability of  $B_d(u, r)$  being contained in the same part is at least  $e^{-\beta r}$ .

**Remark (Sparse cover).** A *sparse cover* consists of several clusters  $\{V_i\}_{i=1}^h$ . Each cluster should have weak/strong diameter at most  $\delta$ . For each  $u, v$  with distance at most  $\delta$ , there must be some cluster  $V_i$  that contains both  $u$  and  $v$ , and no vertex  $u$  must be in more than some number  $s$  of clusters. The techniques underlying sparse covers and [low-diameter decomposition](#) are related though we will mostly work only with the latter.

The main question here for the [low-diameter decomposition](#) is the smallest  $\alpha$  that one can obtain. It turns out that for general metric spaces,  $\alpha = O(\log n)$  is a tight bound for both strong and weak diameter guarantee [Bar96]. For planar graph metrics,  $\alpha = O(1)$  is achievable, where weak diameter guarantee was shown in [KPR93], and the strong diameter guarantee was more difficult and was shown later. See [Fil24] for some recent work and pointers to literature on these topics.

Now, we present the algorithm for weak diameter guarantee. Let  $(V, d)$  be a metric space with  $|V| = n$ . Borrowing ideas from [Algorithm 2.1](#), we see that implicitly this is a [metric partitioning scheme](#). We simply modify the algorithm to ensure that the weak diameter of each cluster is at most a given parameter  $\delta$ .

---

**Algorithm 2.2:** Random Partition [CKR05]

---

**Data:** A metric space  $(V, d)$ ,  $\delta$   
**Result:**  $E_P$  for the partition  $P$

```

1  $\theta \leftarrow \text{Uniform}([0, \delta/2])$ 
2  $\sigma \leftarrow \text{Uniform}(S(V))$  // Random permutation from permutation group  $S(V)$ 
3 for  $i = 1, \dots, n$  do
4    $V_{\sigma(i)} \leftarrow B_d(v_{\sigma(i)}, \theta) \setminus \bigcup_{j < i} V_{\sigma(j)}$ 
5 return  $\bigcup_{i=1}^n \delta(V_i)$ 
```

---

From the same analysis as in [Lemma 2.1.2](#), claim the following.

**Claim.** [Algorithm 2.2](#) correctly outputs a partition of  $V$  into clusters, each of which has weak diameter at most  $\delta$ .

Furthermore, the probability guarantee can also be stated.

**Theorem 2.1.2.** The probability that  $u$  and  $v$  are in different clusters outputted by [Algorithm 2.2](#) is at most  $2H_n d(u, v)/\delta$ , i.e.,  $\alpha = H_k = O(\log n)$ .

**Proof.** We see that  $\Pr(A_j) \leq 2d(u, v)/\delta \cdot 1/j$ , hence  $\Pr(A) \leq 2H_n d(u, v)/\delta$ . ■

Finally, we consider a not so apparent modification of [Algorithm 2.2](#): we sample  $\theta$  from  $[\delta/4, \delta/2]$  instead of  $[0, \delta/2]$ .

---

**Algorithm 2.3:** Refined Random Partition [CKR05]

---

**Data:** A metric space  $(V, d)$ ,  $\delta$   
**Result:**  $E_P$  for the partition  $P$

```

1  $\theta \leftarrow \text{Uniform}([\delta/4, \delta/2])$ 
2  $\sigma \leftarrow \text{Uniform}(S(V))$  // Random permutation from permutation group  $S(V)$ 
3 for  $i = 1, \dots, n$  do
4    $V_{\sigma(i)} \leftarrow B_d(v_{\sigma(i)}, \theta) \setminus \bigcup_{j < i} V_{\sigma(j)}$ 
5 return  $\bigcup_{i=1}^n \delta(V_i)$ 
```

---

**Intuition.** Intuitively, this will preserve closer points.

It's clear that the guarantee about the diameter remains the same.

**Claim.** [Algorithm 2.3](#) correctly outputs a partition of  $V$  into clusters, each of which has weak diameter at most  $\delta$ .

The main difference is in the probability guarantee which is refinement of the previous bound.

**Theorem 2.1.3.** The probability that  $u$  and  $v$  are in different clusters outputted by Algorithm 2.3 is at most  $\frac{4d(u,v)}{\delta} \log \frac{|B(u,\delta/2)|}{|B(u,\delta/8)|}$ , i.e.,  $\alpha = \alpha(\delta) = 4 \log \frac{|B(u,\delta/2)|}{|B(u,\delta/8)|}$ .

**Proof.** We sketch the proof based on the proof of Lemma 2.1.2. Assuming the exact same notation, and we fix  $u, v \in V$  and think of  $V$  as  $v_1, \dots, v_n$ . If  $d(u, v) \geq \delta/8$ , then the edge is going to get cut with constant probability, and the bound is not giving anything interesting, so we are primarily interested in the case when  $d(u, v) < \delta/8$ .

We consider the event  $A_i$  which is that  $v_i$  is the first vertex to separate the pair  $u, v$ . We can argue as before that  $\Pr(A_i) \leq 1/i \cdot \Pr(\theta \in [L_i, R_i])$ , and this is at most  $4d(u, v)/\delta i$  since we're choosing the radius from  $[\delta/4, \delta/2]$ . The new twist is that since we choose  $\theta \in [\delta/4, \delta/2]$  and  $d(u, v) < \delta/8$ , no vertex  $v_j \in B(u, \delta/8)$  can separate  $u, v$  because  $L_j \leq d(v_j, u) < \delta/8$  and  $R_j \leq L_j + d(u, v_j) \leq \delta/8 + \delta/8 = \delta/4$ . Any such vertex will capture both  $u, v$  if they are not already separated. Similarly, any vertex  $v_j \notin B(u, \delta)$  can cut the pair because  $L_j \geq \delta - d(u, v) \geq \delta - \delta/8 \geq \delta/2$ . Therefore, if  $A$  is the event of  $u, v$  being cut, then

$$\Pr(A) \leq \sum_{j \in B(u, \delta) \setminus B(u, \delta/8)} \Pr(A_j) \leq \frac{4d(u, v)}{\delta} \sum_{|B(u, \delta/8)| < j \leq |B(u, \delta)|} \frac{1}{j} \leq \frac{4d(u, v)}{\delta} \log \frac{|B(u, \delta)|}{|B(u, \delta/8)|},$$

proving the result. ■

## 2.2 Dominating Tree Metrics Embedding

Using tree representations of graphs is a powerful tool in algorithm design. Here, we are interested in representing the distances in an undirected graph via distances in a [spanning tree](#). Let  $G = (V, E)$  be a graph with edge length  $\ell: E \rightarrow \mathbb{R}_+$ , which induces a metric space  $(V, d)$  via shortest path distances. The main question is the following:

**Problem 2.2.1 (Tree embedding).** Given a graph  $G = (V, E)$  with edge length  $\ell: E \rightarrow \mathbb{R}_+$ , the *tree embedding* problem aims to find a [spanning tree](#)  $T = (V, E_T)$  of  $G$  such that for any  $u, v \in V$ ,  $d_T(u, v) \leq \alpha d_G(u, v)$  where  $\alpha$  is called the *distortion* or *stretch*.<sup>a</sup>

<sup>a</sup>Clearly,  $d_T(u, v) \geq d_G(u, v)$ .

**Example (Cycle).** Consider a cycle  $C_n$ . We see that for a fixed edge  $(u, v)$ , there exists one spanning tree  $T$  such that  $d_T(u, v) = n - 1$ .

Motivated by applications of [spanning tree](#) based metric approximations, we observe that if we are allowed to pick a probability distribution over [spanning trees](#), then the expected distance for any pair of vertices can be much better than the above worst-case example.

**Example (Cycle).** Again, consider a cycle  $C_n$ . If we allow randomization (picking trees randomly),

$$\mathbb{E}[d_T(u, v)] = \frac{n-1}{n} \cdot 1 + \frac{1}{n} \cdot (n-1) \leq 2.$$

It's showed that [Alo+95] for any weighted graph  $G = (V, E)$ , there is a distribution  $\mathcal{D}$  over [spanning trees](#) of  $G$  such that for any  $u, v \in V$ ,

$$\mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] \leq \exp\left(\sqrt{\log n \log \log n}\right) \cdot d_G(u, v) < n^{o(1)} d_G(u, v).$$

**Intuition.** This is a probabilistic approximation of a graph metric by [spanning tree](#) metrics.

This can also be viewed as a metric *embedding* result. In keeping with metric embedding terminology, we're interested in the worst-case guarantee of how much the expected distance for any pair increases, i.e., minimizing  $\alpha$ . In the above example  $\alpha \leq \exp(\sqrt{\log n \log \log n})$ .

**Note (Lower bound).** A lower bound of  $\alpha = \Omega(\log n)$  is required for probabilistic [tree](#) approximation, and it's conjectured that this is tight [\[Alo+95\]](#).

### 2.2.1 Dominating Tree Metric

It turned out that this is quite difficult to obtain even a poly-logarithmic bound [\[Elk+05\]](#), and currently the best known bound is  $O(\log n \log \log n)$  [\[ABN08\]](#). To make the problem easier, [\[Bar96\]](#) proposed to forget about the graph topology and just focus on  $(V, d)$ , i.e., consider the *metric embeddings* instead of [spanning tree embeddings](#). More generally, we work with the metric completion  $(V, d)$  and view it as a complete graph on  $V$ , where any [spanning tree](#) of the complete graph is now allowed. Moreover, we allow additional vertices. This is formalized as follows.

**Definition 2.2.1 (Dominating tree metric).** A tree  $T = (V_T, E_T)$  with edge length  $\ell_T: E_T \rightarrow \mathbb{R}_+$  is a *dominating tree metric* for a finite metric space  $(V, d)$  if  $V \subseteq V_T$  and for all  $u, v \in V$ ,  $d_T(u, v) \geq d_G(u, v)$ .

Then, we're interested in approximating metrics probabilistically by [dominating tree metrics](#):

**Definition 2.2.2 (Probabilistic approximation).** A *probabilistic approximation* of a metric space  $(V, d)$  by [dominating tree metrics](#) is a probability distribution  $\mathcal{D}$  over a collection of trees  $\{T_i\}_{i=1}^h$  if each  $T_i$  is a [dominating tree metric](#) for  $(V, d)$ .

Furthermore, we say that the [probabilistic approximation](#)  $\mathcal{D}$  has *stretch*  $\alpha$  if for all  $u, v \in V$ ,

$$\mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] \leq \alpha d(u, v).$$

### 2.2.2 Tree Embedding with Low-Diameter Decomposition

One can use [low-diameter decomposition](#) to efficiently sample from a distribution that has stretch  $\alpha = O(\log^2 n)$  [\[Bar96\]](#), and this was subsequently improved to  $O(\log n \log \log n)$  [\[Bar98\]](#), and finally improved to the optimal  $O(\log n)$  [\[FRT03\]](#) using [Algorithm 2.3](#).

**Intuition.** Recursively decompose  $(V, d)$  using [low-diameter decomposition](#).

Specifically, let  $(V, d)$  be a metric space with diameter  $\Delta$ . We use [low-diameter decomposition](#) with parameter  $\delta = \Delta/2$  to randomly partition  $V$  into clusters  $\{V_i\}_{i=1}^h$  of diameter at most  $\Delta/2$ , then we recursively find a tree for each of the  $V_i$ , with root  $r_i$ . We create a new dummy root  $r$  and connect each  $r_i$  to  $r$  with an edge of length  $\Delta$ .

---

#### Algorithm 2.4: Tree Embedding

---

**Data:** A metric space  $(V, d)$ , diameter  $D$

**Result:** A rooted tree  $(T, r)$

```

1 if  $|V| = 1$  then
2    $T \leftarrow (V, \emptyset)$ 
3   return  $(T, v)$                                      //  $V = \{v\}$ 
4
5 Create a tree  $T$  with root  $r$ 
6  $\{V_i\}_{i=1}^h \leftarrow \text{Low-Diameter-Decomposition}((V, d), D/2)$ 
7 for  $j = 1, \dots, h$  do
8    $(T_j, r_j) \leftarrow \text{Tree-Embedding}((V_j, d), D/2)$ 
9   Connect  $T_j$  to  $T$  by adding edge  $(r, r_j)$  of length  $D$ 
10 return  $(T, r)$ 

```

---





Figure 2.1: Illustration of Algorithm 2.4.

**Notation.** We say  $\delta = \Delta/2^i$  at level  $i$  to make the analysis cleaner.

We will now assume that the minimum distance is at least 1 by scaling, and let  $\Delta$  be the diameter of the metric space with this assumption.

**Remark.** If the minimum distance is at least 1, Algorithm 2.4 yields a stretch of  $O(\log n \log \Delta)$ .

**Theorem 2.2.1.** Let  $\Delta$  be the diameter of  $(V, d)$ . Algorithm 2.4 outputs a random dominating tree metric  $T = (V_T, E_T)$  with length  $\ell_T$  such that for each  $u, v \in V$ ,  $\mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] \leq O(\alpha \log \Delta) d(u, v)$  where  $\alpha$  is the cutting probability of the low-diameter decomposition algorithm used.

**Proof.** We prove this by induction. It's clear that the base case is trivial. If we start with  $D = \Delta$ , then at depth  $i$  of the recursion, the parameter is  $\Delta/2^{i-1}$ , and it is the upper-bound on the diameter of the metric space in that recursive call. We note the following claims.

**Claim.** The length of the root to leaf path of a tree created at level  $i$  of the recursion is at most  $\sum_{j \geq i} \Delta/2^{j-1} \leq 2\Delta/2^{i-1}$ .

Suppose  $u$  and  $v$  are first separated at level  $i$  of the recursion. Then,  $d_T(u, v) \leq 4\Delta/2^{i-1}$  from the above claim. We see that if  $u$  and  $v$  are separated in the first level of the recursion due to the low-diameter decomposition algorithm, its probability is at most  $\alpha d(u, v)/(\Delta/2) \leq 2\alpha d(u, v)/\Delta$ , in which case their distance in the tree is at most  $4\Delta$ . Otherwise, they are in the same cluster, and we can apply induction. Note that  $u$  and  $v$  are definitely separated by level  $t$  where  $t$  is the smallest integer such that  $\Delta/2^{t+1} < d(u, v)$ . Hence, the depth of the recursion is at most  $1 + \lceil \log \Delta \rceil \leq 2 \log \Delta$ . It's easy to unroll the induction and use the preceding claim to obtain

$$\mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] \leq \sum_{i=0}^{t+1} 2\alpha \frac{d(u, v)}{(\Delta/2^{i-1})} \cdot \left( 4 \frac{\Delta}{2^{i-1}} \right) \leq O(\alpha \log \Delta) d(u, v) \quad (2.2)$$

since the depth of the recursion is  $O(\log \Delta)$ . ■

If we use Algorithm 2.2 as the low-diameter decomposition algorithm, then we have  $\alpha = O(\log n)$ . From Theorem 2.2.1, we see that the tree may require depth  $\log \Delta$  to provide a good approximation, and in general  $\log \Delta$  can be as large as  $n$ , so we get an  $O(n \log n)$  approximation.

**Example.** Consider the metric induced by a path with  $n$  edges and edge lengths are  $2^i$  for all  $i = 1, \dots, n$ . In such cases, the dependence of the stretch on  $\log \Delta$  is undesirable.

One can alter Algorithm 2.4 to make the stretch bound  $O(\log^2 n)$ : in applying the low-diameter decomposition algorithm with parameter  $\delta$ , we ensure that any pair  $u, v$  such that  $d(u, v) \leq \delta/n^2$  is not cut during the procedure. We can do this by contracting all such pairs without changing the diameter of the resulting metric space too much. This will ensure that in the tree construction process, a pair  $u, v$  participates in only  $O(\log n)$  levels and hence the expected stretch can be bounded by  $O(\alpha \log n)$ .



**Remark (Hierarchically well-separated tree).** The trees constructed by Algorithm 2.4 have an additional strong property: the edge lengths at each level are the same and the length from the root to the leaf go down by a factor of 2 at each level. A tree metric with such a property is called *hierarchically well-separated tree metric* and this additional property can be exploited in algorithms and comes for free in the construction.

Finally, we note that if we choose Algorithm 2.3 as the *low-diameter decomposition* algorithm specifically, the expected stretch is actually  $O(\log n)$ , which is optimal [FRT03].

**Theorem 2.2.2.** Let  $\Delta$  be the diameter of  $(V, d)$ . When Algorithm 2.3 is used as the *low-diameter decomposition* algorithm in Algorithm 2.4, Algorithm 2.4 outputs a random *dominating tree metric*  $T = (V_T, E_T)$  with length  $\ell_T$  such that for each  $u, v \in V$ ,  $\mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] \leq O(\log n)d(u, v)$ .

**Proof.** Firstly, we recall that from Theorem 2.1.3, the probability that  $u$  and  $v$  are in different clusters outputted by Algorithm 2.3 is at most  $\frac{4d(u, v)}{\delta} \log \frac{|B(u, \delta/2)|}{|B(u, \delta/8)|}$ . Thus, the guarantee  $\alpha(\delta)$  from the *low-diameter decomposition* algorithm is no longer uniform but depends on the diameter. Plugging this to Equation 2.2, we have

$$\begin{aligned} \mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] &\leq \sum_{i=0}^{t+1} 2 \log \frac{|B(u, \Delta/2^i)|}{|B(u, \Delta/2^{i+3})|} \frac{d(u, v)}{\Delta/2^{i-1}} \cdot \left(4 \frac{\Delta}{2^{i-1}}\right) \\ &\leq 8d(u, v) (\log |B(u, \Delta/2)| + \log |B(u, \Delta/4)| + \log |B(u, \Delta/8)|) \leq O(\log n)d(u, v), \end{aligned}$$

proving the desired result.  $\blacksquare$

**Remark (Lower bound).**  $O(\log n)$  bound for *low-diameter decomposition* algorithm and *tree embeddings* are near optimal (modulo precise constant factors).

**Proof.** We can use the existence of low-girth graphs which are closely tied to *expanders* in a direct fashion. Another way is via indirection. Previously, in the lecture note, we saw that the integrality gap of the *linear program relaxation* for *multi-min-cut* is  $\Omega(\log k)$  (the upper-bound is proved in Theorem 2.1.1 via a *low-diameter decomposition* algorithm). In fact, if  $\alpha$  is the factor for the *low-diameter decomposition* algorithm, then we get an  $O(\alpha)$ -approximation for *multi-min-cut* via the *linear program*. Thus, one see that  $\alpha = \Omega(\log n)$  for general metrics (in the integrality gap example for *multi-min-cut*  $k = \Omega(n^2)$ ). One can use similar approaches to prove for *tree embedding*.  $\circledast$

**Note (Efficient algorithms).** While we focus on the quality of *low-diameter decomposition* algorithms and *tree embedding* but not so much on the running times, it is easy to see that the algorithms themselves can be implemented in polynomial time. The main computation is about the shortest paths. If one computes all pairs shortest paths (APSP), then the algorithms are pretty simple. However, APSP is slow, which takes  $O(mn)$  times. It is possible to compute the *low-diameter decomposition* algorithm and metric *tree embeddings* in close to linear time on a weighted graph with  $m$  edges. This involves computing approximate shortest paths and several tricks, and sometimes we give up on the quality of the approximation by logarithmic factors.

## Lecture 7: Linear Programming for Sparsest Cut

### 2.3 Sparsest Cut

17 Sep. 11:00

Consider building a network of  $n$  vertices. The best network might be the complete graph, which can do everything and is robust. However, the problem is that the degree is too high ( $n - 1$ ).

**Example.** For degree equal to 2, the best we can hope for is a cycle.

The magic happens whenever the degree goes up to 3.

**Intuition.** If we can down-weight edges in a complete graph  $K_n$  by  $3/(n-1)$ , then any cut  $S$  has

$$\delta(S) = |S| \cdot |V \setminus S| \cdot \frac{3}{n-1} \approx c|S|$$

for  $|S| \ll |V \setminus S|$  and some  $c \geq 0$ .

This notion can be formalized as the so-called [expander](#) [HLW06]. We postpone the formal introduction of [expander](#), and first focus on a closely related problem, the [sparsest cut problem](#), specifically, the [non-uniform](#) version. It turns out that solving this helps us answer various questions for [expanders](#).

### 2.3.1 Uniform and Non-Uniform Sparsest Cut Problem

To introduce the problem, we first define the [sparsity](#) for a cut.

**Definition 2.3.1 (Sparsity).** Given a graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$  and a demand graph  $H = (V, F)$  with demand capacity  $D: F \rightarrow \mathbb{R}_+$ , for any cut  $S \subseteq V$ , its *sparsity* is defined as

$$\frac{c(\delta(S))}{\sum_{i: |S \cap \{s_i, t_i\}|=1} D_i}.$$

That is, the [sparsity](#) of a cut is the ratio of the capacity of the cut and the total demand of the pairs separated by  $S$ . Now, we can introduce the [non-uniform sparsest cut problem](#).

**Problem 2.3.1 (Non-uniform sparsest cut).** Given a supply graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$  and  $k$  pairs of vertices  $\{(s_i, t_i)\}_{i=1}^k$  along with non-negative demand values  $D_1, \dots, D_k$ .<sup>a</sup> The *non-uniform sparsest cut problem* aims to find a cut  $S$  with minimum [sparsity](#).

<sup>a</sup>If  $G$  is undirected, then the demand pairs are unordered, i.e., we do not distinguish  $(s_i, t_i)$  from  $(t_i, s_i)$ .

Here, demands forms a demand graph  $H = (V, F)$  with edges  $(s_i, t_i)$  and demand capacity  $D: F \rightarrow \mathbb{R}_+$  such that  $D((s_i, t_i)) = D_i$ . With this representation, the [sparsity](#) of cut  $S$  is simply  $c(\delta_G(S))/D(\delta_H(S))$  where  $\delta_G(S)$  (respectively,  $\delta_H(S)$ ) represents the supply (respectively, demand) edges crossing  $S$ .

**Intuition.** We're trying to find the best “bang per buck” cut, i.e., how much capacity do we need to remove per amount of demand separated to satisfy the demand?

**Remark.** We can define a cut as removing a set of edges, leading to more than two components. In the case of [sparsest cut](#) in undirected graphs, it suffices to restrict attention to cuts of the form  $\delta(S)$  for some  $S \subseteq V$ . This is not necessarily true for directed graphs or even in undirected graphs with node-weights or in hypergraphs.

**Problem 2.3.2 ((Uniform) sparsest cut).** The *sparsest cut* problem is the same as [Problem 2.3.1](#) with all  $D(u, v) = 1$  for each unordered pair of vertices  $(u, v)$ .<sup>a</sup>

<sup>a</sup>That is,  $\{(s_i, t_i)\}_{i=1}^k$  is the set of all unordered pairs of vertices.

We see that in the [uniform sparsest cut problem](#), the [sparsity](#) of a cut  $S$  is given by  $c(\delta_G(S))/|S||V \setminus S|$ . In this case, the demand graph  $H$  is a complete graph with unit demand values on each edge.

Finally, to further motivate the problem, we see that the [uniform sparsest cut](#) helps us directly and indirectly solve the [graph bisection](#), a central problem in graph algorithm:

**Problem 2.3.3 (Graph bisection).** Given a graph  $G = (V, E)$ , the *graph bisection problem* aims to find a partition of  $G$  into  $(S, V \setminus S)$  such that  $|S| = |V \setminus S|$  while minimizing  $|\delta(S)|$ .

### 2.3.2 Linear Program Relaxation and Maximum Concurrent Flow

It's not clear how to start solving the [uniform sparsest cut problem](#) as writing a linear program relaxation for which is not obvious, compared to [multi-min-cut](#) and other cut problems where we have explicit terminal pairs that we wish to separate. Hence, to write an integer program for which, we let  $y_i \in \{0, 1\}$  being the indicator variable of whether we want to separate  $(s_i, t_i)$ . Moreover, let  $x_e \in \{0, 1\}$  for all  $e \in E$  to be the cut indicator variables.

**Intuition.** If we decide to separate  $(s_i, t_i)$ , then for every path between  $s_i$  and  $t_i$  we should cut at least one edge on the path.

Hence, a natural integer program of [non-uniform sparsest cut](#) and its relaxation is given by:

$$\begin{aligned}
 \min \quad & \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k D_i y_i} \\
 \sum_{e \in P} x_e & \geq y_i \quad \forall P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}; \\
 x_e & \in \{0, 1\} \quad \forall e \in E; \\
 y_i & \in \{0, 1\} \quad \forall i = 1, \dots, k;
 \end{aligned}
 \rightarrow
 \begin{aligned}
 \min \quad & \sum_{e \in E} c(e)x_e \\
 \sum_{i=1}^k D_i y_i & = 1 \\
 \sum_{e \in P} x_e & \geq y_i \quad \forall P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}; \\
 x_e & \geq 0 \quad \forall e \in E; \\
 (P) \quad y_i & \geq 0 \quad \forall i = 1, \dots, k,
 \end{aligned}
 \tag{2.3}$$

where we use the standard trick, i.e., linearization to normalize the denominator to be 1, to make the ratio of the integer program into a linear program. With the dual variable  $z_P$  for each path such that it indicates the amount of “[flow](#)” sent on the path  $P$ , the dual of the above is

$$\begin{aligned}
 \max \quad & \lambda \\
 \sum_{P \in \mathcal{P}_{s_i, t_i}} z_P & \geq \lambda D_i \quad \forall i = 1, \dots, k; \\
 \sum_{i=1}^k \sum_{\substack{P \in \mathcal{P}_{s_i, t_i} \\ P \ni e}} z_P & \leq c(e) \quad \forall e \in E; \\
 z_P & \geq 0 \quad \forall P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}; \\
 (D) \quad \lambda & \geq 0,
 \end{aligned}
 \tag{2.4}$$

which is a multi-commodity [flow](#). In particular, it solves the *maximum concurrent multi-commodity flow* problem for the given instance, i.e., it finds the largest value of  $\lambda$  such that there is a feasible multi-commodity [flow](#) for the given pairs in which the [flow](#) routed for pair  $(s_i, t_i)$  is at least  $\lambda D_i$ .

**Notation** (Concurrent flow). It is called *concurrent flow* since we need to route all demand pairs to the same factor which is in contrast to the dual of [multi-min-cut](#), which corresponds to the maximum throughput multi-commodity [flow](#).<sup>a</sup>

<sup>a</sup>Recall that in this case, some pairs may have zero [flow](#) while others have a lot of [flow](#).

We note that this dual can be solved efficiently via ellipsoid method since the separation oracle is just the shortest path problem. One can also write a compact linear program via distance variables as

$$\begin{aligned}
 \min \quad & \sum_{uv \in E} c(uv)d(uv) \\
 \sum_{i=1}^k D_i d(s_i, t_i) & = 1 \\
 d & \text{ is a metric on } V.
 \end{aligned}$$

In this context, we can understand the *flow-cut gap* of [non-uniform sparsest cut](#) by the following equivalent way of thinking about the problem:

**Intuition (Cut-condition).** Given a multi-commodity flow instance on  $G$ . A necessary condition to route all the demand pairs is that if  $G$  satisfies the *cut-condition*, i.e., for every  $S \subseteq V$ , the capacity  $c(\delta(S))$  is at least the demand separated by  $S$ . However, the converse is not necessarily true, i.e., the cut-condition is not sufficient.

The cut-condition is sufficient when  $k = 1$  but is not true in general even for  $k = 3$  in undirected graphs. The question is the maximum value of  $\lambda$  such that we can route  $\lambda D_i$  for each pair  $i$ . The worst-case integrality gap of the *proceeding linear program* is precisely the flow-cut gap.

### 2.3.3 Rounding Linear Program via $\ell_1$ Embeddings

It's known that there is an  $O(\log n)$ -approximation algorithm and the flow-cut gap for *uniform sparsest cut*, together with a lower bound of  $\Omega(\log n)$  on the flow-cut gap for *uniform sparsest cut* via *expanders* [LR99]. This leads to an  $O(\log^2 n)$ -approximation for *non-uniform sparsest cut*, and it was an open problem to obtain a tight conjectured bound of  $O(\log n)$ . It turns out to be possible via the optimal rounding algorithm for the *linear program relaxation*. This goes via metric embedding theory [LLR95; AR98], hence we need some basics in metric embeddings to point out the connection and rounding.

**Note.** Even though the metric embedding machinery is powerful, it can seem like magic. The more basic ideas for *uniform sparsest cut* based on region growing is useful to know [WS11].

**Remark (Rounding via multi-min-cut).** There are also close connections between *sparsest cut* and *multi-min-cut*. In particular, suppose there is an  $\alpha(k, n)$ -approximation for *non-uniform sparsest cut*, then we have an  $O(\alpha(k, n) \ln k)$ -approximation for *multi-min-cut*. The converse is also true.<sup>a</sup>

<sup>a</sup>See the *note* for a reference.

Before we start, consider the following simple setting when  $G$  is a tree  $T = (V, E)$ .

**Example (Tree).** Given a tree  $G = T = (V, E)$ , for each edge  $e \in T$ , we can associate a cut  $S_e$  which is one side of the two components in  $T - e$ . The capacity of the cut  $\delta(S_e)$  is  $c(e)$ . Let  $D(e) = \sum_{i: |S_e \cup \{s_i, t_i\}|=1} D_i$  be the demand separated by  $e$ . The *sparsity* of the cut  $S_e$  is simply  $c(e)/D_e$ , hence finding the *sparsest cut* in this case is easy. Interestingly, the *linear program relaxation* give an optimum solution on a tree.

**Proof.** Let  $(x, y)$  be a feasible solution to the *dual* of the *linear program relaxation* with objective value  $\lambda$ . We want to prove that if  $G$  is a tree  $T$ , then there is an edge  $e \in T$  such that  $c_e/D_e \leq \lambda$ . We note that by considering the compact linear program with distance variables, we have

$$\lambda = \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k D_i d_x(s_i, t_i)}$$

where  $d_x(s_i, t_i)$  is the shortest path distance between  $s_i$  and  $t_i$  induced by  $x$ . Since there is a unique path  $P_{s_i, t_i}$  from  $s_i$  to  $t_i$  in  $T$  such that  $d_x(s_i, t_i) = \sum_{e \in P_{s_i, t_i}} x_e$ , we have

$$\lambda = \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k D_i d_x(s_i, t_i)} = \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k D_i \sum_{e \in P_{s_i, t_i}} x_e} = \frac{\sum_{e \in E} c(e)x_e}{\sum_{e \in E} x_e \sum_{i: e \in P_{s_i, t_i}} D_i} = \frac{\sum_{e \in E} c(e)x_e}{\sum_{e \in E} D(e)x_e}.$$

Finally, the result follows from  $\sum_i a_i / \sum_i b_i \geq \min_i a_i / b_i$  for positive  $a_i$  and  $b_i$ 's.  $\circledast$

**Example (Ring).** For a ring graph, the same technique works where we need to remove two edges.

The reason why the above proof works for trees is because of a more general phenomenon: the shortest path distances are  $\ell_1$  metrics, or equivalently, *cut metrics*.

### Cut, Line, and $\ell_1$ Metrics, and Metric Embedding

To explain the above phenomenon, consider a finite metric space  $(V, d)$  with following metrics:

**Definition 2.3.2 (Cut metric).** Let  $(V, d)$  be a finite metric space. The metric  $d$  is a *cut metric* if there is a set  $S \subseteq V$  such that  $d = d_S$ , where  $d_S$  associated with the cut  $S$  is defined as

$$d_S(u, v) = \begin{cases} 1, & \text{if } |S \cap \{u, v\}| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The *cut-cone* consists of non-negative combination of *cut metrics*:

**Definition 2.3.3 (Cut cone).** Let  $(V, d)$  be a finite metric space. The metric  $d$  is in the *cut cone* if there exist non-negative scalars  $y_S$  where  $S \subseteq V$  such that for all  $u, v \in V$ ,

$$d(u, v) = \sum_{S \subseteq V} y_S d_S(u, v).$$

Beside the *cut metric*, another useful metric is the *line metric* and the well-known  $\ell_1$  metric:

**Definition 2.3.4 (Line metric).** Let  $(V, d)$  be a finite metric space. The metric  $d$  is a *line metric* if there is a mapping  $f: V \rightarrow \mathbb{R}$  such that for all  $u, v \in V$ ,

$$d(u, v) = |f(u) - f(v)|.$$

**Definition 2.3.5 ( $\ell_1$  metric).** Let  $(V, d)$  be a finite metric space. The metric  $d$  is an  $\ell_1$  metric if there is some integer  $d$  and a mapping  $f: V \rightarrow \mathbb{R}^d$  such that for all  $u, v \in V$ ,

$$d(u, v) = \|f(u) - f(v)\|_1$$

It might not be too surprising that the following holds.

**Lemma 2.3.1.** A metric  $d$  of a metric space  $(V, d)$  is an  $\ell_1$  metric if and only if it is a non-negative combination of *line metrics* (in the cone of the *line metrics*).

**Proof.** If  $d$  is an  $\ell_1$  metric then each dimension corresponds to a *line metric*. Conversely, any non-negative combination of *line metrics* can be made into an  $\ell_1$  metric where each *line metric* becomes a separate dimension (scalar multiplication for a *line metric* is also a *line metric*). ■

A more interesting observation is that any *cut metric*  $d_S$  is a simple *line metric*: map all vertices in  $S$  to 0 and all vertices in  $V \setminus S$  to 1. This leads to the following.

**Lemma 2.3.2.** A metric  $d$  is an  $\ell_1$  metric if and only if  $d$  is in the *cut cone*.

**Proof.** If  $d$  is in the *cut cone*, then it is a non-negative combination of the *cut metrics*, hence it is a non-negative combination of *line metrics* by the above observation, hence an  $\ell_1$  metric.

For the converse, it suffices to argue that any *line metric* is in the *cut cone*. Let  $V = \{v_i\}_{i=1}^n$  and let  $d$  be a *line metric* on  $V$ . Without loss of generality, assume that the coordinates  $x_i$  of the points for each  $v_i$  corresponding to the *line metric*  $d$  are  $x_1 \leq x_2 \leq \dots \leq x_n$  on the real line. For  $1 \leq i < n$ , let  $S_i = \{v_1, v_2, \dots, v_i\}$ . It is not hard to verify that  $\sum_{i=1}^{n-1} |x_{i+1} - x_i| d_{S_i} = d$ . ■

Now we have introduced all the necessary metrics we will use. Consider a finite metric space  $(V, d)$ .

**Claim.** Any finite metric space can be viewed as one that is derived from the shortest path metric induced on a graph with some non-negative edge lengths.

If  $G = (V, E)$  is a simple graph and  $\ell: E \rightarrow \mathbb{R}_+$  are some edge-lengths, the metric induced on  $V$  depends both on the *topology* of  $G$  and the lengths as well, i.e., finite metrics can encode graph structure, hence it can be diverse. When trying to round we may want to work with simpler metric spaces.

**Intuition (Embedding).** Embed a given metric space  $(V, d)$  into a simpler host metric space  $(V', d')$  via an embedding  $f: V \rightarrow V'$ .

**Note.** Even though we may be interested in finite metric spaces, the host metric space can be continuous or infinite such as the  $\mathbb{R}^h$  for dimension  $h$ .

As embedding typically distorts the distances, thus, we want to find embeddings with small [distortion](#).

**Definition 2.3.6 (Distortion).** Let  $(V, d)$  and  $(V', d')$  be two metric spaces and let  $f: V \rightarrow V'$  be an embedding. The *distortion* of  $f$  is given by<sup>a</sup>

$$\max_{\substack{u, v \in V \\ u \neq v}} \left( \frac{d'(f(u), f(v))}{d(u, v)}, \frac{d(u, v)}{d'(f(u), f(v))} \right).$$

<sup>a</sup>Additive version are also explored, although they are very restrictive due to lack of scale invariance.

Additionally, we're interested in the following kind of embeddings:

**Definition.** Let  $(V, d)$  and  $(V', d')$  be two metric spaces and let  $f: V \rightarrow V'$  be an embedding.

**Definition 2.3.7 (Isometric embedding).** The embedding  $f$  is *isometric* if for all  $u, v \in V$ ,

$$d(u, v) = d'(f(u), f(v)).$$

**Definition 2.3.8 (Contraction).** The embedding  $f$  is a *contraction* if for all  $u, v \in V$ ,

$$d(u, v) \geq d'(f(u), f(v)).$$

**Definition 2.3.9 (Non-contracting).** The embedding  $f$  is *non-contracting* if for all  $u, v \in V$ ,

$$d(u, v) \leq d'(f(u), f(v)).$$

Of particular importance are embeddings of finite metric spaces into  $\mathbb{R}^h$ , where the distance in the host space is measured under a norm such as  $\ell_p$  norm. The dimension  $h$  is also important in various applications but in some settings like with [non-uniform sparsest cut](#), it is not. We assume the following:

**Theorem 2.3.1 (Bourgain [Bou85; LLR95]).** Any  $n$ -point finite metric space can be embedded into  $\ell_2$  (hence also  $\ell_1$ ) with [distortion](#)  $O(\log n)$ . Moreover, the embedding is a [contraction](#) and can be constructed in randomized polynomial time and embeds points into  $\mathbb{R}^h$  where  $h = O(\log^2 n)$ .

In fact, one can obtain a refined version of [Theorem 2.3.1](#) that is useful for [non-uniform sparsest cut](#).

**Theorem 2.3.2 (Bourgain [LLR95]).** Let  $(V, d)$  be an  $n$ -point finite metric space and let  $S \subseteq V$  with  $|S| = k$ . Then there is a randomized polynomial time algorithm to compute an embedding  $f: V \rightarrow \mathbb{R}^{O(\log^2 n)}$  such that the embedding is a [contraction](#)<sup>a</sup> and for every  $u, v \in S$ ,  $\|f(u) - f(v)\|_1 \geq cd(u, v)/\log k$  for some universal constant  $c$ .

<sup>a</sup>I.e.,  $\|f(u) - f(v)\|_1 \leq d(u, v)$  for all  $u, v \in V$

By utilizing [Theorem 2.3.2](#) with the previous insight on tree, we can provide a general guarantee.

**As previously seen.** The integrality gap of the [linear program](#) is 1 on trees since the shortest path metric on trees is in the [cut cone](#), i.e.,  $\ell_1$ -embeddable.

One can prove that if the shortest path metric on a graph  $G$  embeds into  $\ell_1$  with [distortion](#)  $\alpha$ , then the integrality gap of the [linear program](#) is at most  $\alpha$ . This will imply an  $O(\log n)$ -integrality gap via [Theorem 2.3.2](#) since any  $n$ -point finite metric space embeds into  $\ell_1$  with [distortion](#)  $O(\log n)$ .

## Lecture 8: Randomized Rounding for Sparsest Cut and Expanders

### Randomized Rounding Algorithm with Metric Embeddings

19 Sep. 11:00

Now, we see how to utilize [Theorem 2.3.2](#) to design a randomized rounding algorithm for the [non-uniform sparsest cut problem](#). The main theorem is the following.

**Theorem 2.3.3.** Let  $G = (V, E)$  be a graph. Suppose any finite metric induced by edge lengths on  $E$  can be embedded into  $\ell_1$  with [distortion](#)  $\alpha$ , then the integrality gap of the [linear program](#) for the [non-uniform sparsest cut](#) is at most  $\alpha$  for any instance on  $G$ .

**Proof.** Let  $(x, y)$  be a feasible fraction solution of the [linear program relaxation](#), and let  $d$  be the metric induced by edge lengths given by  $x$ . Let  $\lambda$  be the value of the solution, i.e.,

$$\lambda = \frac{\sum_{uv \in E} c(uv)d(u, v)}{\sum_{i=1}^k D_i d(s_i, t_i)}.$$

Since  $d$  can be embedded into  $\ell_1$  with [distortion](#) at most  $\alpha$ , and any  $\ell_1$  metric is in the [cut cone](#) from [Lemma 2.3.2](#), it implies that there are scalars  $z_S$ ,  $S \subseteq V$ , such that for all  $u, v \in V$ ,

$$\frac{1}{\alpha} \sum_{S \subseteq V} z_S d_S(u, v) \leq d(u, v) \leq \sum_{S \subseteq V} z_S d_S(u, v).$$

Without loss of generality, we assume that the embedding is a [contraction](#). Then, we have

$$\begin{aligned} \lambda &= \frac{\sum_{uv \in E} c(uv)d(u, v)}{\sum_{i=1}^k D_i d(s_i, t_i)} \geq \frac{1}{\alpha} \frac{\sum_{uv \in E} c(uv) \sum_{S \subseteq V} z_S d_S(u, v)}{\sum_{i=1}^k D_i \sum_{S \subseteq V} d_S(s_i, t_i)} \\ &= \frac{1}{\alpha} \frac{\sum_{S \subseteq V} z_S c(\delta_G(S))}{\sum_{S \subseteq V} z_S D(\delta_H(S))} \geq \frac{1}{\alpha} \min_{S \subseteq V} \frac{c(\delta_G(S))}{D(\delta_H(S))}. \end{aligned}$$

Hence, there is a cut whose [sparsity](#) is at most  $\alpha\lambda$ . ■

[Theorem 2.3.3](#) shows that one of the cuts with  $z_S > 0$  has [sparsity](#) at most  $\alpha\lambda$ . Now, suppose we have an  $\ell_1$  embedding into  $h$ -dimensions, i.e.,  $\mathbb{R}^h$ . Observe the following.

**Intuition.** First, each dimension in  $\mathbb{R}^h$  corresponds to a [line](#) embedding, and each [line](#) embedding is in the [cut cone](#) with only  $n - 1$  cuts used to express it (recall [Lemma 2.3.2](#)). Thus, given an  $\ell_1$  embedding into  $\mathbb{R}^h$  with [distortion](#)  $\alpha$ , we only need to try  $d(n - 1)$  cuts and one of them will be guaranteed to have [sparsity](#) at most  $\alpha\lambda$ .

[Algorithm 2.5](#) exploits this intuition.

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#### Algorithm 2.5: Non-Uniform Sparsest Cut via Embedding

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**Data:** A supply graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , demand graph  $H = (V, F)$  with demand capacity  $D: F \rightarrow \mathbb{R}_+$

**Result:** The [sparsest cut](#)  $S \subseteq V$

```

1  $(\{x_e\}_{e \in E}, \{y_i\}_{i=1}^k) \leftarrow \text{LP-Solve}(\text{Non-Uniform-Sparsest-Cut-LP}(G, c, H, D))$ 
2  $f \leftarrow \text{Bourgain-Embedding}((V, d_x))$  //  $f: V \rightarrow \mathbb{R}^h, h = O(\log^2 n)$ 
3 for  $\ell = 1, \dots, h$  do
4   Sort  $\ell^{\text{th}}$  coordinate of  $\{f(v)\}_{v \in V}$  as  $x_1^{(\ell)} \leq x_2^{(\ell)} \leq \dots \leq x_n^{(\ell)}$  //  $f(v)_\ell = x_i^{(\ell)}$  for some  $i$ 
5   for  $j = 1, \dots, n - 1$  do
6      $S_j^{(\ell)} \leftarrow \{v \in V \mid f(v)_\ell \leq x_j^{(\ell)}\}$ 
7  $S \leftarrow \arg \min_{\ell \in [h], j \in [n-1]} c(\delta_G(S_j^{(\ell)})) / D(\delta_H(S_j^{(\ell)}))$ 
8 return  $S$ 
```

---

The guarantee of [Algorithm 2.5](#) can be derived from [Theorem 2.3.3](#).



**Theorem 2.3.4.** Algorithm 2.5 outputs cuts of sparsity at most  $\alpha\lambda^*$ ,<sup>a</sup> in particular, it's a randomized  $O(\log k)$ -approximation algorithm for non-uniform sparsest cut.

<sup>a</sup> $\lambda^*$  is the optimal solution of the dual linear program.

**Proof.** From Theorem 2.3.2,  $f$  is a contraction and with distortion  $\alpha = O(\log k)$ . From a similar argument as in Theorem 2.3.3, we see that for some  $z_S$ ,  $S \subseteq V$ , we have

$$\begin{aligned}\lambda^* &= \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k D_i d_x(s_i, t_i)} \\ &\geq \frac{\sum_{uv \in E} c(uv) \|f(u) - f(v)\|_1}{\alpha \sum_{i=1}^k D_i \|f(s_i) - f(t_i)\|_1} \\ &= \frac{1}{\alpha} \frac{\sum_{uv \in E} c(uv) \sum_{\ell=1}^h |f(u)_\ell - f(v)_\ell|}{\sum_{i=1}^k D_i \sum_{\ell=1}^h |f(s_i)_\ell - f(t_i)_\ell|} \\ &= \frac{1}{\alpha} \frac{\sum_{\ell=1}^h \sum_{j=1}^{n-1} |x_{j+1}^{(\ell)} - x_j^{(\ell)}| c(\delta_G(S_j^{(\ell)}))}{\sum_{\ell=1}^h \sum_{j=1}^{n-1} |x_{j+1}^{(\ell)} - x_j^{(\ell)}| D(\delta_H(S_j^{(\ell)}))} = \frac{1}{\alpha} \frac{\sum_{S \subseteq V} z_S c(\delta_G(S))}{\sum_{S \subseteq V} z_S D(\delta_H(S))} \geq \frac{1}{\alpha} \min_{S \subseteq V} \frac{c(\delta_G(S))}{D(\delta_H(S))},\end{aligned}$$

where the second last equality follows from the fact that  $s_i, t_i \in V$  as well. ■

### 2.3.4 Line, $\ell_1$ , and Tree Embeddings, and State-of-the-Art

Theorem 2.3.2 shows that any finite metric space on  $n$  points embeds into  $\ell_1$  with distortion  $O(\log n)$ . Here, we hint on the underlying algorithm of the construction: from Lemma 2.3.1,  $\ell_1$  embeddings are a non-negative combination of line embeddings. A particular type of line embedding is the following.

**Definition 2.3.10** (Fréchet embedding). Let  $(V, d)$  be a metric space and let  $S \subseteq V$ . The *Fréchet embedding* is a contraction  $f: V \rightarrow \mathbb{R}$  such that  $f(v) = d(S, v)$ .

Many results in embeddings into  $\ell_p$  spaces are based on using Fréchet embeddings in various clever and often highly non-trivial ways. In particular, Theorem 2.3.2 is based on picking many random sets and combining the resulting Fréchet embeddings.

Now, we note that Theorem 2.3.2 can also be derived via probabilistic tree embeddings because every tree metric embeds into  $\ell_1$  isometrically. For general metrics, tree embeddings provide a more constrained space while yielding the same worst-case distortion. However, one can ask if  $\ell_1$  embeddings provide better distortion for concrete graph classes. This is indeed the case.

**Example (Ring).** Consider a ring graph (a cycle with capacities). One can prove that tree embeddings require a distortion 2, while the ring metric can be isometrically embedded into  $\ell_1$ . Thus, the flow-cut gap on ring is 1 which is not obvious.

Rather than looking at distortion, one can ask about the flow-cut gap obtained via different embeddings for a particular graph class. First, recall the followings for the non-uniform sparsest cut.

**As previously seen** (Theorem 2.3.4). The flow-cut gap in general undirected graphs is  $O(\log k)$ .

Additionally, for general graph, a lower bound is also known.

**Remark (Lower bound).** Expanders give a lower bound on the flow-cut gap to be  $\Omega(\log k)$  even for uniform sparsest cut [LR99].

With these general bounds in mind, we now consider the flow-cut gap for planar graphs in particular.

**Example (Planar graph).** There is a famous conjecture that the flow-cut gap in planar graphs is  $O(1)$  [Gup+04]. Interestingly, for tree embeddings, there is a lower bound of  $\Omega(\log n)$  even on the special case of planar graphs called series parallel graphs. Hence, tree embeddings are not powerful



enough to prove the conjecture.

The best flow-cut gap so far is  $O(\sqrt{\log n})$  via  $\ell_1$  embeddings, thus separating the general graph case from the planar graph case. For series parallel graphs, we know that the flow-cut gap is a tight bound of 2 and establishing this tight bound took a fair amount of work.

For **uniform sparsest cut**, the flow-cut gap in planar graphs is  $O(1)$  [KPR93]. One can show a tight connection between embeddability into  $\ell_1$  and flow-cut gap [Gup+04].

On the other hand, we can ask for a better approximation guarantee. First, note the following.

**Note.** Approximating the **non-uniform sparsest cut problem** is not the same as establishing the flow-cut gap as the flow-cut gap relies on the **linear program relaxation**.

**Problem.** Can we obtain a better approximation than  $O(\log k)$  for **non-uniform sparsest cut**?

**Answer.** Yes! By using semi-definite programming based relaxation, one can obtain an  $O(\sqrt{\log n})$ -approximation for **uniform sparsest cut** [ARV09] and also the **product instances**. Based on this, an  $O(\sqrt{\log n} \log \log n)$ -approximation for **non-uniform sparsest cut** is achieved [ALN05; ALN07].<sup>a</sup> \*

<sup>a</sup>There was a conjecture that the SDP based relaxation would yield an  $O(1)$ -approximation, but it was shown that the integrality gap is essentially close to  $\Omega(\sqrt{\log n})$ .

### 2.3.5 Node Capacities<sup>1</sup>

A slight generalization of the **uniform sparsest cut problem** is obtained by considering demands induced by weights on vertices, which we refer to **product instance**.

**Problem 2.3.4 (Product instance of sparsest cut).** Given a graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$  and vertex weight  $\pi: V \rightarrow \mathbb{R}_+$ . The *product instance of sparsest cut problem* is the **non-uniform sparsest cut problem** with demand  $D(u, v)$  for edge  $uv \in E$  setting to be  $\pi(u)\pi(v)$ .

**Notation.** In this case, the dual **flow** instances are called *product multi-commodity flow*.

We see that the **product instances** indeed generalizes **uniform sparsest cut**: If  $\pi(u) = 1$  for all  $u$ , then this reduces to the **uniform sparsest cut problem**.

**Remark (Subset sparsity).** If  $\pi(u) \in \{0, 1\}$  for all  $u$ , then we are focusing our attention on the **sparsity** w.r.t. the set  $V' = \{v \in V \mid \pi(v) = 1\}$ . Since vertices with  $\pi(u) = 0$  play no role.

## 2.4 Expander and Well-Linked Set

We now introduce **expanders** [HLW06], which relate to the **non-uniform sparsest cut** in an intricate way.

**Definition 2.4.1 (Expander).** An *expander* with parameter  $\alpha$  is a graph  $G = (V, E)$  such that for all  $S \subseteq V$  with  $|S| \leq |V|/2$  such that  $|\delta(S)| \geq \alpha|S|$ .

**Definition 2.4.2 (Expansion).** The *expansion* of a graph  $G = (V, E)$  is  $\min_{S: |S| \leq |V|/2} |\delta(S)|/|S|$ .

Another related notion called **conductance** also has a nice connection to **uniform sparsest cut**.

**Definition 2.4.3 (Conductance).** Given a graph  $G = (V, E)$  and a cut  $S \subseteq V$ , the *conductance*  $\phi(G)$  of  $G$  is defined as  $|\delta(S)|/\text{vol}(S)$  where  $\text{vol}(S) = \sum_{v \in S} \deg(v)$ .

It's clear that  $G$  is an  $\alpha$ -**expander** if the **expansion** of  $G$  is at least  $\alpha$ . Initially, **expansion** arose from the **graph bisection problem** as we have seen before. However, as we will soon see, **expander** itself is quite

<sup>1</sup>We refer to the **note** for further information on further generalizations to node capacitated graph and directed graph.

interesting. Firstly, we note that **expanders** do exist, and they are quite common.

**Lemma 2.4.1.** There exists **expanders** with degree 3 with **expansion**  $\alpha = \Omega(1)$ . More specifically, a random 3-regular graph is an **expander** with high probability.

Hence, with **Lemma 2.4.1**, a natural way to generate an **expander** is to first sample a random regular graph, then check its **expansion**. However, computing the **expansion** is **coNP-hard**.

### 2.4.1 Expansion and Conductance via Sparsest Cut

The first connection of **expander** to the **uniform sparsest cut** is that the latter can be used to find the **expansion** of a graph. In particular, we see that when  $|S| \leq |V|/2$ , we have

$$\frac{1}{|V|} \frac{|\delta(S)|}{|S|} \leq \frac{|\delta(S)|}{|S||V \setminus S|} \leq \frac{2}{|V|} \frac{|\delta(S)|}{|S|}.$$

**Remark.** The **expansion** and the **uniform sparsest cut's sparsity** are within a factor of 2 of each other. Hence, while determining the **expansion** exactly is **coNP-hard**, we can use **uniform sparsest cut** to certify the **expansion** of a graph within a factor of 2.

Sometimes it is useful to consider **expansion** with vertex weights  $w: V \rightarrow \mathbb{R}_+$  as well. In this case, the expansion is defined as  $\min_{S: w(S) \leq w(V)/2} |\delta(S)|/w(S)$ .

**Note.** It's dual corresponds to the *product multi-commodity flow instances* where  $\pi(v) = w(v)$ .

As for **conductance**, it's easy to see that one can capture it by **expansion** via setting weights on vertices with  $w(v) = \deg(v)$ , which further reduces to the **product instance of sparsest cut**.

**Claim.** For regular graphs, **expansion** and **conductance** are the same.

### 2.4.2 Spectral Relaxation for Conductance

In several applications it is important to obtain constant-degree **expanders** with constant **expansion**.

**Intuition.** The  $O(\sqrt{\log k})$ -approximation **algorithms** we saw are not useful in this regime.

It turns out that there is a very different method based on spectral graph theory that helps in this regime. For an undirected graph on  $n$  vertices, consider the Laplacian  $\mathcal{L}_G := D - A$ , where  $D$  is the diagonal degree matrix and  $A$  is the adjacent matrix. In particular, we have

$$(\mathcal{L}_G)_{ij} := \begin{cases} \deg(v_i), & \text{if } i = j; \\ -1, & \text{if } i \neq j \text{ and } A_{ij} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Since  $\mathcal{L}_G$  is symmetric, all its eigenvalues are real. Moreover, this matrix is also positive semi-definite, hence all its eigenvalues are actually non-negative. Let  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  be its eigenvalues, then a well-known and famous result in spectral graph theory is the following.

**Theorem 2.4.1 (Cheeger's inequality).** Given a graph  $G$  with **conductance**  $\phi(G)$ ,

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}.$$

**Remark.**  $\lambda_2$  provides a constant factor approximation for the **conductance** when it is a constant!

Since the **expansion** and **conductance** are related by the maximum degree, when the degree is a small constant, one can use  $\lambda_2$  to certify **expansion**. Due to its importance for certifying **expansion/conductance**, some use  $\lambda_2$  as the definition of **expansion** since it is computable and also helps in construction of **expanders**.

### 2.4.3 Expander Decomposition

Even if a graph is not an [expander](#) at first, it is possible to decompose a graph into smaller subgraphs such that each of them has good [expansion/conductance](#). More explicitly, the goal is to remove as few edges as possible such that the graph decomposes into [expanders](#). This is useful since [expander](#) leads to good algorithms. The question is the trade-off between the number of edges that we remove and the [expansion](#) that we can guarantee for the pieces.

**Notation.** Technically the process works with [conductance](#), but we use the terminology of [expander decomposition](#) for historical reasons.

Since one is often interested in finding fast algorithms for [expander decomposition](#), it is common to explore trade-offs between the quality of the [conductance](#) of the pieces and the number of edges.

**Definition 2.4.4** (Expander decomposition). A  $(\phi, \epsilon)$ -*expander decomposition* for some  $\epsilon \in (0, 1)$  is a partition of the (connected) graph  $G = (V, E)$  into vertex induced subgraphs  $\{G_i = G[V_i]\}_{i=1}^h$  such that each  $G_i$  has [conductance](#) at least  $\phi$ , and the number of inter-cluster edges is at most  $\epsilon m$ , i.e.,

$$\frac{1}{2} \sum_{i=1}^h |\delta(V_i)| \leq \epsilon m.$$

## Lecture 9: Expander Decomposition and Well-Linked Sets

We first see an algorithm that compute the [expander decomposition](#).

24 Sep. 11:00

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### Algorithm 2.6: Expander Decomposition

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**Data:** A connected graph  $G = (V, E)$ , base graph edge set size  $M$ ,<sup>a</sup> parameter  $\epsilon \in (0, 1)$

**Result:** An  $(\Omega(\frac{\epsilon}{\alpha \log m}), \epsilon)$ -[expander decomposition](#)  $\{G_i\}_{i=1}^h$

```

1 if  $m \leq 10 \log M / \epsilon$  then                                     // Base case
2   return  $G$ 
3
4  $(S, V \setminus S) \leftarrow \alpha$ -Uniform-Spsest-Cut( $G$ )b           //  $\text{vol}(S) \leq \text{vol}(V \setminus S)$ 
5
6 if  $|\delta(S)| / \text{vol}(S) > c / 10 \log M$  then                       // Check sparsity
7   return  $G$ 
8 else
9    $\{G_i^{(1)}\}_{i=1}^{h_1} \leftarrow \text{Expander-Decomposition}(G[S], M, \epsilon)$ 
10   $\{G_i^{(2)}\}_{i=1}^{h_2} \leftarrow \text{Expander-Decomposition}(G[V \setminus S], M, \epsilon)$ 
11  return  $\{G_i^{(1)}\}_{i=1}^{h_1} \cup \{G_i^{(2)}\}_{i=1}^{h_2}$ 

```

---

<sup>a</sup>In the initial call, it's  $m$ .

<sup>b</sup>Recall that this is for [conductance](#), which can be formalized as a [product instance](#).

**Note.** [Algorithm 2.6](#) is intuitive and simple in retrospect: we simply compute the sparsest cut and recurse. The caveat is to keep track of the final termination condition is based on the original graph parameter  $M = m$ .

We now prove that [Algorithm 2.6](#) indeed computes the [expander decomposition](#).

**Theorem 2.4.2.** Let  $G = (V, E)$  be a graph and  $\epsilon \in (0, 1)$ . Suppose there is an  $\alpha$ -approximation for the [uniform sparsest cut](#), then there is an efficient algorithm that outputs an  $(\Omega(\frac{\epsilon}{\alpha \log m}), \epsilon)$ -[expander decomposition](#) of  $G$ .

**Proof.** We prove that [Algorithm 2.6](#) achieves the desired properties.

**Claim.** The **conductance** of each subgraph output by **Algorithm 2.6** is at least  $\epsilon/10\alpha \log M$ .

**Proof.** For the base case, if  $G$  is connected and has at most  $10 \log M/\epsilon$  edges, then the **conductance** of  $G$  is at least  $\epsilon/10 \log M$  since at least one edge crosses any cut, and the volume of the smaller side is at most  $10 \log M/\epsilon$ .

On the other hand, if the  $\alpha$ -approximation algorithm of the **uniform sparsest cut** for **conductance** outputs a cut  $(S, V \setminus S)$  with **sparsity** at least  $\epsilon/10 \log M$ , we know that the actual **sparsity** of  $G$  is at least  $\epsilon/10\alpha \log M$  as desired.  $\circledast$

Next, we analyze the total number of edges cut, which needs to be at most  $\epsilon m$ .

**Intuition.** If  $G$  is of constant size (and connected) or it does not have a **sparse cut**, **Algorithm 2.6** does not cut any edges.

Let  $T(m)$  be the total number of edges cut by **Algorithm 2.6** on a graph with  $m$  edges. **Algorithm 2.6** removes edges between  $S$  and  $V \setminus S$  only when  $|\delta(S)| \leq \epsilon \text{vol}(S)/10 \log M$  where  $\text{vol}(S) \leq \text{vol}(V \setminus S)$ . Let  $m' := |\delta(S)|$ ,  $m_1 := |E(G[S])|$ , and  $m_2 := |E(G[V \setminus S])|$ , then

$$m' \leq \frac{\epsilon}{10 \log M} (2m_1 + m') \Rightarrow m' \leq (1 - o(1)) \frac{\epsilon}{5 \log M} m_1 \leq \frac{\epsilon}{4 \log M} m_1.$$

With  $m_1 \leq m_2$ , the recurrence can be written as

$$T(m) \leq T(m_1) + T(m_2) + \frac{\epsilon}{4 \log M} \min(m_1, m_2) = T(m_1) + T(m_2) + \frac{\epsilon}{4 \log M} m_1$$

where  $m_1 + m_2 \leq m$ , which gives  $T(m) \leq \epsilon m$ .  $\blacksquare$

If we don't care about efficiency, we can set  $\alpha = 1$  and solve the **uniform sparsest cut** exactly. In particular, **Theorem 2.4.2** guarantees that the decomposed pieces have **conductance**  $\Omega(1/\log m)$  while cutting only a constant fraction of the edges.

**Note.** The bound  $\Omega(1/\log m)$  is tight as shown by the hypercube [Ale+17].

We can rephrase **Theorem 2.4.2** in a different form where we want a lower bound on the **conductance** of the pieces and express the number of edges cut as a function that parameter:

**Corollary 2.4.1.** Let  $G = (V, E)$  be a graph and  $\phi$  be a parameter. Suppose there is an  $\alpha$ -approximation for the **uniform sparsest cut**, then there is an efficient algorithm that computes a  $(\phi, O(\alpha \cdot \phi \cdot \log m))$ -**expander decomposition**.

**Note.** Number of edges cut is less than  $m$  only if  $\alpha\phi \log m < 1$ , so one should think of  $\phi \leq 1/\alpha \log m$ .

**Remark.** **Theorem 2.4.2** is phrased in terms of  $m$ , the number of edges. Capacitated graphs can be handled by scaling since we do not assume that  $G$  is simple. However, the dependence on  $\log m$  means that when capacities are large, we are not guaranteed a strongly polynomial bound. One can handle this issue in various ways depending on the application. In most applications of **expander decomposition**, it is the case that the total capacity of the edges can be assumed to be polynomially bounded in  $n$  and in this case, the  $\log m$  factor is typically replaced with  $\log n$ .

We remark that **Algorithm 2.6** is based on **sparsest cut** algorithms. Traditionally, these algorithms were quite slow. There have been several developments in the last few years which enabled **sparsest cut** to be reduced to a poly-logarithmic number of **s-t flows** via the so-called *cut-matching game* [KRV09; Ore+08], which in turn enabled faster **flow** algorithms. There are now near-linear time randomized algorithms for **expander decomposition** (with slightly weaker parameters than the ideal one) for the regimes of interest [SW19]. In some applications the randomized algorithm is not adequate and there has been considerable effort to obtain deterministic algorithms. There are now almost-linear time deterministic algorithm [Chu+20; SL21].

### 2.4.4 Well-Linked Set

Consider the following generalization of [Definition 2.4.1](#), where we only care about [expansion](#) of a subset.

**Definition 2.4.5 (Well-linked).** A set  $X \subseteq V$  is *well-linked* in a graph  $G = (V, E)$  if for all  $S \subseteq V$ ,

$$|\delta(S)| \geq \min(|S \cap X|, |S \cap (V \setminus X)|).$$

On the other hand, recall that  [\$\alpha\$ -expansion](#) means that for all sets  $S \subseteq V$  with  $|S| \leq |V|/2$ ,  $|\delta(S)| \geq \alpha|S|$ . This is a cut condition. Suppose  $A, B$  are two disjoint sets of vertices of equal size  $|A| = |B|$ , clearly we have  $|A|, |B| \leq |V|/2$ . We can ask for a similar guarantee as the same cut condition. This turns out to be another generalization of [expander](#):

**Definition 2.4.6 (Linkage).** Let  $A, B \subseteq V$ ,  $A \cap B = \emptyset$ , and  $|A| = |B|$ . An  $A$ - $B$  *linkage* is a set of edge-disjoint paths connecting  $A$  to  $B$  with each vertex in  $A \cup B$  in exactly one path. In this case, we say  $A$  and  $B$  are *linked* in  $G$ .

**Note.** We do not have to insist on  $A \cap B = \emptyset$ . If not and we allow each vertex in  $A \cap B$  to connect to itself via an empty path, then it is the same as asking  $A \setminus B$  and  $B \setminus A$  to be [linked](#). Thus, requiring  $|A| = |B|$  suffice.

We can view [linkage](#) as sending [flows](#):

**Definition 2.4.7 (Fractional linkage).** An  $A$ - $B$  *linkage* is *fractional* if there is a [flow](#) in  $G$  with that satisfies demand of 1 on each vertex in  $A$  and a demand of  $-1$  on each vertex of  $B$ .<sup>a</sup> In particular, we say that  $A, B$  are  $\alpha$ -*linked* for some parameter  $\alpha$  if there is a [flow](#) in  $G$  that satisfies the demand of  $\alpha$  on each vertex in  $A$  and a demand of  $-\alpha$  on each vertex of  $B$ .

<sup>a</sup>Note that this corresponds to a single-commodity [flow](#).

**Lemma 2.4.2.** Suppose  $G$  is an  [\$\alpha\$ -expander](#) with  $\alpha \geq 1$ . Then there is an  $A$ - $B$  [linkage](#) in  $G$  for every pair of disjoint equal sized sets  $A, B$ .

**Proof.** Let  $|A| = |B| = k$ . To check whether there are  $k$  desired edge-disjoint paths, consider creating a [flow](#) problem by adding two new vertices  $s$  and  $t$  such that  $s$  and  $t$  are connected to all vertices in  $A$  and  $B$  of capacity 1, respectively. Let  $H = (V \cup \{s, t\}, E_H)$  be this new graph. If there is an  $s$ - $t$  [flow](#) of value  $k$  in  $H$ , then these correspond to the desired paths. Suppose otherwise, i.e., the [flow](#) is strictly less than  $k$ . Then by [max-flow min-cut theorem](#), there is a set  $S' \subseteq V_H$  with  $s \in S'$  and  $t \notin S'$  such that  $|\delta_H(S')| < k$ . Let  $S := S' - s$  be the corresponding set of vertices in  $G$ . It's easy to see that

$$|\delta_H(S')| = |\delta_G(S)| + |A \cap (V \setminus S)| + |B \cap S|.$$

This implies that  $S \neq \emptyset$  since otherwise  $|A \cap (V \setminus S)| = |A| = k$ , and the above will imply  $|\delta_H(S')| \geq k$ , a contradiction. Similarly,  $V \setminus S \neq \emptyset$ . Now, suppose  $|S| \leq |V \setminus S|$ , i.e., the smaller side. Then by the [expansion](#) guarantee,  $|\delta_G(S)| \geq |S| \geq |A \cap S|$ , implying

$$|\delta_H(S')| = |\delta_G(S)| + |A \cap (V \setminus S)| + |B \cap S| \geq |A \cap S| + |A \cap (V \setminus S)| = k,$$

which is again a contradiction. The same proof applies if  $|V \setminus S| \leq |S|$ . ■

One can scale capacities or directly prove the following.

**Corollary 2.4.2.** Suppose  $G$  is an  [\$\alpha\$ -expander](#). Then if  $A, B$  are disjoint vertex sets with  $|A| = |B|$ , then  $A, B$  are  [\$\alpha\$ -linked](#).

Interestingly, the converse is also true.

**Lemma 2.4.3.** Suppose  $G$  is a graph and for any two disjoint set  $A, B$  of equal size,  $A, B$  are  [\$\alpha\$ -linked](#). Then  $G$  is an  [\$\alpha\$ -expander](#).

**Proof.** We cannot have  $|\delta_G(A)| < \alpha|A|$  due to the [max-flow min-cut theorem](#). ■

The following shows the connection of [linkage](#) and [well-linked](#).

**Claim.** A set  $X$  is [well-linked](#) if for all  $A, B \subseteq X$  and  $|A| = |B|$ ,  $A$  and  $B$  are [linked](#).

**Definition 2.4.8 (Fractinoal well-linked).** A set  $X$  is  $\alpha$ -well-linked in a graph  $G$  is for any two  $A, B \subseteq X$  with  $|A| = |B|$ , the sets  $A, B$  are  [\$\alpha\$ -linked](#).

More generally, we have the following.

**Lemma 2.4.4.** A set  $X \subseteq V$  is  [\$\alpha\$ -well-linked](#) in  $G$  if and only if for any set  $S \subseteq V$ ,  $|\delta(S)| \geq \alpha \min(|S \cap X|, |S \cap (V \setminus X)|)$ .

**Corollary 2.4.3.** A graph  $G = (V, E)$  is an  [\$\alpha\$ -expander](#) if and only if  $V$  is  [\$\alpha\$ -well-linked](#) in  $G$ .

Thus, the notion of [well-linked](#) sets extends the definition of [expansion](#) to subsets of the graph. This is very useful in a number of settings.

**Example (Star).** A start on  $n$  vertices is an [expander](#) and has a [well-linked](#) set of size  $n$ . This strange artifact is because of the large degree of the center vertex.

This artifact disappears if we ask for constant degree graphs or if we insist on [node linkage](#), i.e., we now want *node-disjoint paths*.

**Definition 2.4.9 (Node linkage).** Let  $A, B \subseteq V$ ,  $A \cap B = \emptyset$ , and  $|A| = |B|$ . An  $A$ - $B$  node linkage is a set of node-disjoint paths connecting  $A$  to  $B$  with each vertex in  $A \cup B$  in exactly one path.<sup>a</sup>

<sup>a</sup>We also skip the definition of  $\alpha$ -linkage for now. The definition is basically the same where we want [flow](#) with node capacities rather than edge capacities.

**Definition 2.4.10 (Node well-linked).** A set  $X$  is  $\alpha$ -node-well-linked in  $G$  if for any two  $A, B \subseteq X$  with  $|A| = |B|$ ,  $A, B$  are  [\$\alpha\$ -linked](#).

**Intuition.** If  $X$  is [node-well-linked](#) in a graph, then  $X$  cannot have a sparse node separator, i.e., if  $S$  separates  $G \setminus S$  into components and  $S$  does not have any vertices of  $X$  then no component of  $G \setminus S$  can have more than  $|S|$  vertices of  $X$ .

In graph theory literature on [treewidth](#)  $\text{tw}(G)$  (formally introduced in [Definition 5.4.3](#)), the notion of [linkages](#) is defined primarily via node-disjoint paths. We will not use this very often in this course hence we overload [edge](#) and [node well-linked](#) notations. In particular, its connection to [node-well-linkedness](#) is the following.

**Theorem 2.4.3 ([Ree97]).** Let  $k$  be the cardinality of the largest [node-well-linked](#) set in a graph  $G$ . Then  $k \leq \text{tw}(G) \leq 4k$ .

**Remark.** In fact, most algorithmic approaches to computing  $\text{tw}(G)$  are based on algorithms for sparse node separator computations.

[Node-well-linkedness](#) is connected to vertex-[expanders](#). Sometimes people do not distinguish between these two notions too much in the [expansion](#) literature because of the following.

**Claim.** If  $G$  is an  [\$\alpha\$ -edge-expander](#) with maximum degree  $d$ , then  $G$  is an  $\Omega(\alpha/d)$ -vertex-[expander](#).

Thus, if one is working with constant degree graphs, the two notions are not very far.



**Example (Star).** Consider the star graph again. One can see that it is an [edge-expander](#), but it is very far from being a [vertex expander](#). In fact, the largest [node-well-linked](#) set in a star is of size 2.

On the other hand, the grid is not only [edge-well-linked](#), but also [node-well-linked](#).

**Example (Grid).** A  $\sqrt{n} \times \sqrt{n}$  grid is a planar graph with  $n$  vertices. It has a bisection with  $O(\sqrt{n})$  edges hence it is at best a  $1/\sqrt{n}$ -[expander](#) (which in fact it is). It has a [well-linked](#) set of size  $\Omega(\sqrt{n})$ , i.e., rows or columns  $X$  of  $\sqrt{n}$  vertices. We see that although  $G$  is not a good [expander](#), but it has good [expansion](#) w.r.t.  $X$ . Actually, it's even [node well-linked](#) (right).



In some sense, grid is the best planar graph in terms of [node-well-linkedness](#). First, we need to understand the following:

**Definition 2.4.11 (Balanced separator).** Given a graph  $G = (V, E)$ , a vertex subset  $S$  is a *balanced separator* if every connected component of  $G - S$  is of size less than  $2n/3$ .

**Theorem 2.4.4.** Every planar graph has a [balanced separator](#) of size  $O(\sqrt{n})$ . Hence, no planar graph on  $n$  vertices have a [node-well-linked](#) set of size more than  $c\sqrt{n}$  for some fixed constant  $c$ .

### 2.4.5 Well-Linked Decomposition

Finally, we note that the same decomposition can be done for [well-linked](#) sets, just as the [expander decomposition](#). This is a generalization since [well-linkedness](#) generalizes the notion of [expansion](#) and [conductance](#) from the whole vertex set to subset of vertices. To state the desired result, we need to introduce the notion of [weighted well-linked](#):

**Definition 2.4.12 (Weighted-well-linked).** Let  $\pi: X \rightarrow \mathbb{R}_+$  be non-negative weights on  $X \subseteq V$ . We say that  $X$  is  $\pi$ -*weighted well-linked* in  $G$  if for any  $S \subseteq V$ ,  $|\delta_G(S)| \geq \min(\pi(S \cap X), \pi(S \cap (V \setminus X)))$ .

Note that if  $\pi(v) = 1$  for all  $v \in X$ , we get back to [Definition 2.4.5](#). If  $\pi(v) = \deg(v)$  for each  $v \in V$ , we obtain the [conductance](#).

**Remark.** Typically, it is assumed that  $\pi(v) \leq c\pi(V)$  for some constant  $c < 1$ , e.g.,  $c = 1/3$ . That is, no single vertex is too heavy compared to the total weight.

Now, suppose we have a multi-commodity [flow](#) between vertices in  $V$  that is routed in  $G$ . Let  $D(uv)$  be the [flow](#) routed for the unordered pair  $uv$ . For each  $u \in V$ , let  $\pi(u) = \sum_{v \in X, v \neq u} D(uv)$  be the total [flow](#) from  $u$  to some vertex in  $X$ .

**Intuition.** We would like the terminals (vertices) to be  $\pi$ -[wel-linked](#), but if they are not, we would like to do a decomposition into vertex induced subgraphs such that they are.

Furthermore, we would like to do it in such a way that we lose as little amount of the original [flow](#) as possible. Since it is difficult to bound the total [flow](#) lost, so we will upper bound it by the total number of edges crossing the partitions.

**Note.** Consider each edge  $uv \in E$  and send one unit of [flow](#) along that edge for the pair  $uv$ , then  $\pi(u) = \deg(u)$  for each  $u \in V$ . This recovers the [expander decomposition](#) that we saw earlier.



**Theorem 2.4.5 (Well-linked decomposition).** Let  $G = (V, E)$  be a graph and  $\epsilon \in (0, 1)$  be a parameter. Suppose there is an  $\alpha$ -approximation algorithm for the [uniform sparsest cut](#). Let  $\pi: V \rightarrow \mathbb{R}_+$  be induced by a multi-commodity [flow](#) in  $G$ , where  $\pi(u)$  is the total [flow](#) originating at  $u$ . Then there is an efficient algorithm that decomposes  $G$  into vertex induced subgraphs  $\{G_i = G[V_i]\}_{i=1}^h$  for  $V_i \subseteq V$  such that in each  $G_i$ ,  $V_i$  is  $\beta\pi'$ -well-linked where  $\beta = \Omega(\frac{\epsilon}{\alpha \log m})$  and  $\pi'(u)$  is the total [flow](#) incident to  $u$  in  $G_i$  (that remains from the original [flow](#)). Furthermore, the number of inter-cluster edges is at most  $\epsilon\pi(V)$ , i.e.,

$$\frac{1}{2} \sum_{i=1}^h |\delta(V_i)| \leq c\pi(V).$$

**Proof.** The corresponding algorithm and the proof is essentially the same as in [Theorem 2.4.2](#). Let  $M = \pi(V)$  to keep it fixed throughout the algorithm as before. We use the  $\alpha$ -approximation algorithm on  $V$  with weights  $\pi$ . It outputs a cut  $(S, V \setminus S)$  with  $\pi(S) \leq \pi(V \setminus S)$ . If  $|\delta(S)| \geq \epsilon\pi(S)/(10 \log M)$ , we output  $G$ , and we are guaranteed that  $V$  is  $\pi/(10\alpha \log M)$ -well-linked in  $G$  since we used an  $\alpha$ -approximation algorithm for [uniform sparsest cut](#). Otherwise, we remove the edges in  $\delta(S)$  and recurse on  $G[S]$  and  $G[V \setminus S]$  after throwing out all the [flow](#) that is lost by the edges in  $\delta(S)$ . The recursion analysis is the same as before, and we can guarantee that the total number of edges cut is at most  $c\pi(V)$ . ■

**Remark.** In routing problems, it is more useful to work with the notion of [flow-well-linkedness](#) rather than [cut-well-linkedness](#). Theorems similar to [Theorem 2.4.5](#) can be shown via the known flow-cut gap results [[CKS05](#)].

The notion of [well-linked decomposition](#) can be generalized to node-capacitated setting naturally. It can also be generalized to directed graphs (only symmetric demands) [[CE15](#)], which has found several applications recently in fast algorithms [[BGS20](#)]. Via duality, they are related to [low-diameter decomposition in directed graphs](#), which have played a key role in recent fast algorithms for shortest paths with negative lengths. We will see this later ([Section 3.3](#)).

## Lecture 10: Tree-Based Oblivious Routing

### 2.5 Oblivious Routing

26 Sep. 11:00

Consider routing demands between source-sink pairs in a network  $G = (V, E)$  with capacity  $c: E \rightarrow \mathbb{R}_+$ . In particular, the actual demands are not known in advance and come and go in an online fashion. Hence, the problem is, which routes should the demand for some specific pair  $(s, t)$  be routed on?

**Intuition.** To feasibly route a given set of demands, we need to essentially solve a multi-commodity [flow](#), which helps to balance the network's capacity among many competing pairs.

This is a non-trivial problems and leads to lots of the breakthroughs. [Oblivious routing](#) is one of which that we will be interested in. The goal in [oblivious routing](#), in some sense, is to bridge the static setting where the demands are all fully known to the fully online setting where we specify a route to a new demand when it arrives.

**Problem 2.5.1 (Oblivious routing).** Given a directed graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , the goal of *oblivious routing* is to output a distribution of paths  $\mathcal{P}_{u,v}$  between every pair of  $u, v \in V$  such that when the demand  $D: V \times V \rightarrow \mathbb{R}_+$  come, specifically,  $D(u, v)$  for a pair  $u, v \in V$ , a [flow](#) is routed on a path  $P \in \mathcal{P}_{u,v}$  according to the distribution.

Alternatively, we can also think of sending  $D(u, v)$  demand fractionally along the paths in proportion to the probability. Clearly, [Problem 2.5.1](#) is oblivious since the probability distribution over  $\mathcal{P}_{u,v}$  is specified before knowing any of the actual demands. In particular, if we know the entire set of demands in advance, we can compute an optimal routing via some sort of multi-commodity [flow](#) computation. The question now is that, how well can an [oblivious routing](#) do when compared to a static routing, and how to measure the quality. More fundamentally, do good [oblivious routing](#) even exists?

The initial work that motivated the problem of **oblivious routing** is that good deterministic **oblivious routing** exist in special classes of graphs such as hypercubes [VB81]; later the need for randomization in the sense of picking paths randomly as we described was realized.

### 2.5.1 Routable Demands and Congestion

We first digress towards defining **routable** demands and some properties.

**Definition 2.5.1 (Routable).** A demand matrix  $D \in \mathbb{R}_+^{n \times n}$  is *routable* in a directed graph  $G = (V, E)$  with capacity  $c: E \rightarrow \mathbb{R}_+$  if there exists a feasible multi-commodity **flow** for  $D$  in  $G$ .

It's easy to see that  $\mathcal{D}_G := \{D \in \mathbb{R}_+^{n \times n} \mid D \text{ is routable}\}$  is a convex set. Moreover, given a demand matrix  $D$ , we can efficiently check if  $D$  is **routable** in  $G$  via solving a linear program for multi-commodity **flow**. Thus, we have a membership oracle for the convex set  $\mathcal{D}_G$ .

**Intuition.** The **flow** linear program is nice because it is also linear in  $D(u, v)$  values, hence we can claim something stronger.

Suppose we have  $w(u, v)$  for all  $u, v \in V$ . We ask the following question.

**Problem 2.5.2 (Optimizng demand polytope).** Can we solve  $\max_{D \in \mathcal{D}_G} \sum_{u, v} w(u, v) D(u, v)$ ?

**Answer.** By the equivalence of optimization and separation, we have an efficient separation oracle over  $\mathcal{D}_G$ ! That is, there is an efficient algorithm that given  $D \in \mathbb{R}_+^{n \times n}$ , either outputs that  $D \in \mathcal{D}_G$  or outputs a separating hyperplane that separates  $D$  from the convex set  $\mathcal{D}_G$ .  $\circledast$

The above is a convoluted but easy way of deriving a useful fact. A more direct way is often referred to as the **Japanese theorem on multi-commodity flow**, which can be derived from linear program duality, as we saw in **non-uniform sparsest-cut**.

**Lemma 2.5.1 (Japanese theorem).** The demand  $D$  is **routable** if and only if for all  $\ell: E \rightarrow \mathbb{R}_+$ ,

$$\sum_{e \in E} c(e) \ell(e) \geq \sum_{u, v \in V} D(u, v) d_\ell(u, v)$$

where  $d_\ell$  is the shortest path distance induced by  $\ell$ .

Hence, to prove that  $D$  is not **routable** in  $G$ , it suffices to produce one length function  $\ell: E \rightarrow \mathbb{R}_+$  that violates the inequality in **Lemma 2.5.1**.

**Intuition.** One can view  $\mathcal{D}_G$  as being defined by an infinite set of linear inequalities, one for each non-negative length function.

Next, to measure the quality of **oblivious routing**, the central notion is the **congestion**.

**Definition 2.5.2 (Congestion).** A demand matrix  $D \in \mathbb{R}_+^{n \times n}$  is *routable with congestion*  $\rho > 0$  in a directed graph  $G = (V, E)$  with capacity  $c: E \rightarrow \mathbb{R}_+$  if  $D$  is **routable** in  $G$  where edge capacities are multiplied by  $\rho$ .

When  $\rho = 1$ , it corresponds to the **routable** case. Moreover, the set of all **routable** demand metrics in  $G$  with a fixed **congestion**  $\rho$  is also convex.

### 2.5.2 Oblivious Routing to Minimize Congestion

In **oblivious routing**, we want to specify a probability distribution over  $\mathcal{P}_{u, v}$  for each pair  $(u, v)$ . Naively, since there are exponential many paths, specifying a probability on each path can be tricky.

**Intuition.** One can view such a probability distribution as a **flow** of one unit from  $u$  to  $v$  where the **flow** on a path  $P \in \mathcal{P}_{u, v}$  is equal to the probability of  $P$ .

Thus, one compact way to represent a probability distribution over paths is via a unit **flow** via **flow** values on edges, which can be specified using only  $m$  numbers. Formally, consider the following:

**Definition 2.5.3** (Oblivious routing scheme). Given a graph  $G = (V, E)$ , consider the following.

**Definition 2.5.4** (Edge-based oblivious routing). An *edge-based oblivious routing* is a collection of unit **flows** in  $G$ , one for each  $(u, v) \in V \times V$ , specified via edge-based **flows**  $f_e^{(u,v)}$ .

**Definition 2.5.5** (Path-based oblivious routing). A *path-based oblivious routing* is similar to **edge-based** where the unit **flow** is specified via a path based **flow**.

However, edge-based **flow** does not uniquely specify a path-based **flow**, so we are losing information by using the compact representation, but it is helpful in some computational situations.

**Note.**  $f_e^{(u,v)}$  is a number in  $[0, 1]$ , and whether the **oblivious routing** is specified via **edge-based** or **path-based**, this quantity is well-defined.

Now, for **oblivious routing**, we can define its **congestion** as follows.

**Definition 2.5.6** (Congestion of oblivious routing). Let  $D$  be a demand matrix. The *congestion* incurred for  $D$  via the **oblivious routing** specified by  $f_e^{(u,v)}$  for  $e \in E$  and  $(u, v) \in V \times V$  is

$$\max_{e \in E} \frac{\sum_{u,v \in V} D(u,v) f_e^{(u,v)}}{c(e)}.$$

Moreover, the *congestion* of an **oblivious routing** is the smallest  $\rho \geq 1$  such that every **routable** demand  $D \in \mathcal{D}_G$  is routed with congestion at most  $\rho$  by the **oblivious routing**.

We say that an optimal **oblivious routing** for  $G$  is the one which achieves the smallest  $\rho$  among all **oblivious routings**. In other words, we are asking how much should we scale up the capacities of  $G$  so that any demand matrix  $D$  that is **routable** in  $G$  can be routed in  $G$  via the **oblivious routing**, i.e.,

$$\max_{D \in \mathcal{D}_G} \max_{e \in E} \frac{\sum_{u,v \in V} D(u,v) f_e^{(u,v)}}{c(e)}.$$

**Claim.** Every graph on  $n$  vertices has an **oblivious routing scheme** with **congestion**  $n^2$ .

**Proof.** Consider computing the **max-flow** for every pair and then scale it down to a unit **flow**.  $\otimes$

We give some pointers to the fundamental results on **oblivious routing** in a chronological order:

- Harald Räcke proved that every undirected graph admits an **oblivious routing** with **congestion**  $O(\log^3 n)$  [Räc02]. The initial proof is based on an optimal algorithm for **non-uniform sparsest cut**, hence does not lead to an efficient algorithm to construct the **oblivious routing**. Soon after, efficient algorithms are known with the **congestion** being  $O(\log^2 n \log \log n)$  [BKR03; HHR03]. These results are based on the **hierarchical expander decomposition** of the graph which results in a compact cut representation of every graph.
- Via a simple idea in retrospect, that the optimal **oblivious routing** can be computed efficiently via linear program techniques even for directed graphs [Aza+03]. Note that being able to compute an optimum **oblivious routing** for any given graph does not tell us an easy way to understand a universal bound. We will see this next.
- **Oblivious routing** in directed graphs requires **congestion**  $\Omega(\sqrt{n})$  [Aza+03]; this lower bound holds even for the restricted case of single-source demands. A similar lower bound was shown to hold also for undirected graphs with node capacities [Haj+07]. Recently, the lower bound for directed graphs has been improved to  $\Omega(n)$  [Ene+16].

- In another breakthrough, Harald Räcke made a beautiful and surprising connection between [oblivious routing](#) and [probabilistic tree embeddings](#) for metric distortion [Räc08] and obtain an optimal  $O(\log n)$ -[congestion tree-based oblivious routing scheme](#). Via this algorithm, an  $O(\log n)$ -approximation for the minimum bisection problem in graphs is developed.

### 2.5.3 Efficient Algorithm for Optimal Oblivious Routing

In this section, we will prove that one can find an optimal [edge-flow based oblivious routing](#) in polynomial time [Aza+03]. In particular, we will consider directed graphs since it is easier to define edge-based flows. To find an [edge-flow based oblivious routing](#) with minimum [congestion](#), we write the following linear program which essentially follows from the definition:

$$\begin{aligned}
 \min \quad & \rho \\
 & f_e^{(u,v)} \text{ defines a unit flow from } u \text{ to } v \quad \forall e \in E, \forall (u,v) \in V \times V; \\
 & \sum_{(u,v) \in V \times V} D(u,v) f_e^{(u,v)} \leq \rho c(e) \quad \forall e \in E, \forall D \in \mathcal{D}_G; \\
 & f_e^{(u,v)} \geq 0 \quad \forall e \in E, \forall (u,v) \in V \times V.
 \end{aligned} \tag{2.5}$$

**Note.** We omit writing down the linear constraints for the unit flow from  $u$  to  $v$  since it's easy.

The only catch in solve Equation 2.5 is that it has an infinite number of constraints, with polynomially many  $(mn^2)$  variables. Hence, one can use the ellipsoid method if we have an efficient separation oracle.

**Claim.** There is an efficient separation oracle for Equation 2.5.

**Proof.** Given  $f_e^{(u,v)}$  and  $\rho$ , we see that checking unit flow is easy. For the second constraint, given any fix edge  $e$ , we simply maximize  $\sum_{u,v \in V} D(u,v) f_e^{(u,v)}$  via Problem 2.5.2 (efficiently!) and compare it with  $\rho c(e)$ . Hence, by iterating through all  $e \in E$ , we're done.  $\circledast$

While the resulting algorithm is not very efficient in practice, but it shows the power of the ellipsoid method and working with large “implicit” linear programs (Definition 4.1.5) fearlessly. One can use multiplicative weight updates and other techniques to obtain fast approximation algorithms for some of these problems.

**Remark.** This technique to design optimum [oblivious routing](#) is fairly general. We do not need to consider the full set of demand matrices  $\mathcal{D}_G$ , as long as we have a nice convex set of demand matrices that we can optimize over, we can find an optimum [oblivious routing](#) when restricted to those demand matrices.

### 2.5.4 Tree-Based Oblivious Routing via Duality and Low Stretch Trees

We now prove that in undirected graphs, there is always an [oblivious routing](#) with [congestion](#)  $O(\log n)$ , which is an optimal bound. Räcke prove this result via an elegant connection to [tree embeddings](#) for distance preservation [Räc08]. The bound, the connection, and the tree-based aspect are all important.

**As previously seen.** Probabilistic approximation of a graph by [spanning trees](#) for distances incurs an  $O(\log n \log \log n)$  [distortion](#) [ABN08] while we can obtain an optimal  $O(\log n)$  [distortion](#) if we allow [dominating tree metrics](#) (recall the differences).

It is easier to understand the ideas, and in particular the notation, by working with [spanning trees](#) than with general hierarchical tree representations (Theorem 3.1.1) of graphs (in some cases, the [spanning tree](#) result is useful and needed). The difference in the [distortion](#) is insignificant for our purposes here, and we will use these results in a black-box fashion.

Now, let  $\alpha(n)$  denote the best bound that we can obtain for approximating the distance in an  $n$ -node graph  $G = (V, E)$  with edge length  $\ell: E \rightarrow \mathbb{R}_+$  by a probability distribution over [spanning tree](#)  $p: \mathcal{T}_G \rightarrow [0, 1]$  such that  $\sum_{T \in \mathcal{T}} p_T = 1$ . A simple implication of  $\alpha(n) = O(\log n \log \log n)$  is the following.

**Lemma 2.5.2.** Let  $G = (V, E)$  be a graph with non-negative edge lengths  $\ell: E \rightarrow \mathbb{R}_+$ . Let  $w: E \rightarrow \mathbb{R}_+$  be any set of non-negative edge weights. Then there is a **spanning tree**  $T \in \mathcal{T}_G$  such that

$$\frac{\sum_{uv \in E} w(uv) d_T(u, v)}{\sum_{e \in E} w(e) \ell(e)} \leq \alpha(n),$$

where  $d_T(u, v)$  is the length of the unique path from  $u$  to  $v$  in  $T$ .

**Proof.** Recall that there is a probability distribution  $p: \mathcal{T}_G \rightarrow [0, 1]$  such that for every pair of vertices  $u, v \in V$ ,  $\mathbb{E}[d_T(u, v)] \leq \alpha(n) d_G(u, v)$  where distances are induced by the edge lengths  $\ell: E \rightarrow \mathbb{R}_+$ . Now, fix any weight function  $w$  over pairs of vertices. Then by linearity of expectation,

$$\mathbb{E} \left[ \sum_{u, v \in V} w((u, v)) d_T(u, v) \right] \leq \alpha(n) \sum_{u, v \in V} w((u, v)) d_G(u, v).$$

Thus, there exists a tree  $T$  such that  $\sum_{u, v \in V} w((u, v)) d_T(u, v) \leq \alpha(n) \sum_{u, v \in V} w((u, v)) d_G(u, v)$ . Now, consider the case when the support of the weights  $w$  is only on the edges of  $G$ , i.e.,  $w((u, v)) = 0$  when  $uv \notin E$ . In this case, we write  $w(uv)$  for  $uv \in E$ . Furthermore, we let  $d_T(u, v)$  is the shortest path length along the path in  $T$ . Since  $d_G(u, v) \leq \ell(u, v)$ ,

$$\sum_{uv \in E} w(uv) d_T(u, v) \leq \alpha(n) \sum_{uv \in E} w(uv) d_G(u, v) \leq \alpha(n) \sum_{uv \in E} w(uv) \ell(uv),$$

which gives the desired result. ■

We now introduce a specific type of **oblivious routing**. Consider any probability distribution  $p$  over the **spanning trees** of  $G$ . Observe that this distribution induces an **oblivious routing** since each tree  $T \in \mathcal{T}_G$  gives a unique path between any pair  $(u, v) \in V \times V$ . We sample a **tree** from the distribution and whenever a demand arrives, we simply route it along the unique path in the **tree**.

**Notation** (Tree-based oblivious routing). The above scheme is what we called *tree-based oblivious routing*, which is indeed **path-based**.

**Note.** The distribution over  $\mathcal{T}_G$  induces a distribution for every pair of vertices simultaneously.

While this is a restricted class of **oblivious routing**, but it's particularly nice, at least from a theoretical point of view. Two natural questions arise.

**Problem.** How good is the best **tree-distributions-based oblivious routing**?

**Problem.** Can the best **tree-distributions-based oblivious routing** be efficiently computed?

Räcke showed that **tree-based oblivious routing** have a nice structural property that allows one to characterize the **congestion** in a simple way [Räc08]. Let  $T$  be a **spanning tree** and consider an edge  $e \in E_T$ .  $T - e$  induces a partition of the vertex set of  $G = (V, E)$  into two sets  $(S_e, V \setminus S_e)$ . We define the load  $L(T, e)$  on edge  $e$  to be  $c(\delta(S_e))$ . If  $e$  is not in  $T$ , then we let  $L(T, e) = 0$ .

**Intuition.** Think of all the edges crossing the cut  $(S_e, V \setminus S_e)$ . For each of those edges  $e' = (s, t) \in \delta(S_e)$ , the path from  $s$  to  $t$  in  $T$  has to go via  $e$ .

Hence, if we want to route demands corresponding to edges, then the **congestion** on  $e$  will be  $L(T, e)/c(e)$ . Formally, since we now have a probability distribution over  $\mathcal{T}_G$ , hence we consider the **expected load** and **expected congestion**:

**Definition.** Let  $p: \mathcal{T}_G \rightarrow [0, 1]$  be a probability distribution that induces a **tree-based oblivious routing**. Consider a given edge  $e \in E$ .

**Definition 2.5.7** (Expected load). The *expected load* on  $e$  is  $L(e) = \sum_{T \in \mathcal{T}_G} p(T) L(T, e)$ .

**Definition 2.5.8** (Expected congestion). The *expected congestion* on  $e$  is  $\rho(e) = L(e)/c(e)$ .

A simple yet important observation is the following, which characterizes the quality of the [tree-based oblivious routing](#) based on the [expected congestion](#) of the edges.

**Lemma 2.5.3.** Given  $p: \mathcal{T} \rightarrow [0, 1]$ , the maximum [congestion](#) of the [tree-based oblivious routing](#) induced by  $p$  is at most  $\max_{e \in E} \rho(e)$ .

**Proof.** Fix a demand matrix  $D \in \mathcal{D}_G$ . [Congestion](#) of  $e$  is less than  $\rho(e)$  since

$$\frac{\sum_{T \in \mathcal{T}_G, T \ni e} p_T \sum_{|S_e \cap \{u, v\}|=1} D(u, v)}{c(e)} \leq \frac{\sum_{T \in \mathcal{T}_G, T \ni e} p_T L(T, e)}{c(e)}.$$

Taking maximum over  $e \in E$  gives the result.  $\blacksquare$

[Lemma 2.5.3](#) allows us to write an linear program to find the best [tree-based oblivious routing](#). Let  $x_T$  to denote the probability that  $T \in \mathcal{T}_G$  is chosen in the probability distribution, and  $\rho$  to denote the [congestion](#) that we wish to minimize. We write down constraints that express  $x$  as a probability distribution and to express the [load](#) on each edge being bounded by  $\rho c(e)$ :

$$\begin{array}{ll} \min \rho & \max \beta \\ \sum_{T \in \mathcal{T}_G} x_T = 1 & \sum_{e \in E} c(e) z_e = 1 \\ \sum_{T \in \mathcal{T}_G} x_T L(T, e) \leq \rho c(e) \quad \forall e \in E; & \sum_{e \in T} L(T, e) z_e \geq \beta \quad \forall T \in \mathcal{T}_G; \\ \text{(P)} \quad x_T \geq 0 \quad \forall T \in \mathcal{T}_G; & \text{(D)} \quad z_e \geq 0 \quad \forall e \in E. \end{array}$$

We see that the dual is equivalent to

$$\max_{z: E \rightarrow \mathbb{R}_+} \min_{T \in \mathcal{T}_G} \frac{\sum_{e \in T} L(T, e) z_e}{\sum_{e \in E} c(e) z_e}. \quad (2.6)$$

The observation is that [Lemma 2.5.2](#) implies that the optimal dual value is  $\alpha(n)$ , which corresponds to the bound for [tree-based distance approximation](#)! Suppose this was true then we have shown that there exists a [tree-based oblivious routing](#) with [congestion](#)  $O(\log n \log \log n)$ . We now prove this formally.

**Theorem 2.5.1.** The optimal dual value  $\beta$  is at most  $\alpha(n)$ .

**Proof.** Observe that by the definition of  $L(T, e)$ , after interchanging the order of summation,

$$\sum_{e \in T} L(T, e) z_e = \sum_{uv \in E} c(uv) \sum_{e \in P_T(u, v)} z_e,$$

where  $P_T(u, v)$  is the unique path from  $u$  to  $v$  in  $T$ . Note that  $\sum_{e \in P_T(u, v)} z_e$  can be thought of as the length of the path from  $u$  to  $v$  in  $T$  according to lengths given by  $z$ , which we denote as  $d_T(u, v)$ . Now, think of  $c(e)$  as a weight  $w(e)$  and think of  $z_e$  as length  $\ell(e)$ , [Equation 2.6](#) can be written as

$$\max_{\ell \in \mathbb{R}_+^n} \min_{T \in \mathcal{T}_G} \frac{\sum_{uv \in E} w(uv) d_T(u, v)}{\sum_{uv \in E} w(uv) \ell(uv)},$$

and by [Lemma 2.5.2](#), this is at most  $\alpha(n)$ .  $\blacksquare$

[Theorem 2.5.1](#) implies that we have  $\alpha(n) = O(\log n \log \log n)$  [expected congestion](#) for [tree-based oblivious routing](#).<sup>2</sup> Since [Theorem 2.5.1](#) is based on duality, it is not immediately clear that it leads to an efficient algorithm. As one would expect, we need an efficient algorithm for the dual separation

<sup>2</sup>See [note](#) for how to obtain the optimum bound  $O(\log n)$ .

oracle. This is the problem of approximating distances in the graph by [spanning trees](#), and we have seen efficient algorithms for it. However, it is still not obvious that we can use such approximation algorithms in the dual, but there are standard techniques via the multiplicative weight update method and related ideas [[Räc08](#)].



# Chapter 3

## Some Cool Stuffs

### Lecture 11: Expander Hierarchy and Its Sufficiency

In this chapter, we present some recent non-trivial results. The first one is a [hierarchical decomposition](#), which is also known as *expander hierarchy*. The second and the third are about [single-source shortest path](#), but with negative lengths.

1 Oct. 11:00

#### 3.1 Cut-Based Hierarchical Decomposition

Räcke proved the existence of a cut-based [hierarchical decomposition](#) of undirected graphs as a tool to prove the existence of good [oblivious routings](#) [Räc02], which is different from the [tree-based oblivious routing](#) construction we saw last time. This cut-based [hierarchical decomposition](#) has found several applications outside the original motivation for [oblivious routing](#), and is also a fundamental structural result in graph theoretic terms.

**Note.** The construction and proof are technical, and this is the first attempt to teach it and provide some additional commentary along the way, which we hope is of pedagogical value for those interested in understanding the details of a construction from [BKR03].

We first set up notations to state the result where the statement is tailored towards cut-approximation rather than the [oblivious routing](#) aspect.

##### 3.1.1 Statement of the Hierarchical Decomposition

Given a graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{Z}_+$ , consider the [cut-tree](#).

**Definition 3.1.1 (Laminary).** A family of subsets is *laminary* if no two sets  $A, B$  in the family cross, i.e.,  $A \cap B = \emptyset$  or  $A \subseteq B$  or  $A \supseteq B$ .

**Definition 3.1.2 (Hierarchical decomposition).** Given a graph  $G = (V, E)$ , a *hierarchical decomposition* is a [laminary](#) family of subsets of the vertex set  $V$ .

Any [laminary](#) family over  $V$ , augmented with the entire set  $V$  if needed, defines a way to decompose  $V$ , and hence implicitly the graph, in a recursive fashion. Each such decomposition has an associated rooted tree  $T$ , where the leaves of  $T$  are labeled with the vertex set  $V$  and each internal node  $v_t$  corresponds to a subset  $S_{v_t}$  of the vertices of  $G$  which are the leaves in the subtree  $T_{v_t}$ . We associate the graph  $H_{v_t} = G[S_{v_t}]$  with each  $v_t$ . [Hierarchical decompositions](#) of graphs are used in several settings and the meaning and applications of them depend on the application and context. Here, we will be interested in decompositions that preserve cuts of the graph, in particular for routing demands in  $G$ .

**Definition 3.1.3 (Cut-tree).** A [hierarchical decomposition](#)  $T$  is a *cut-tree* if for each edge  $(v_t, v_{t'})$  of  $T$  where  $v_{t'}$  is the parent of  $v_t$ , we assign a capacity equal to  $c(\delta_G(S_{v_t}))$ .

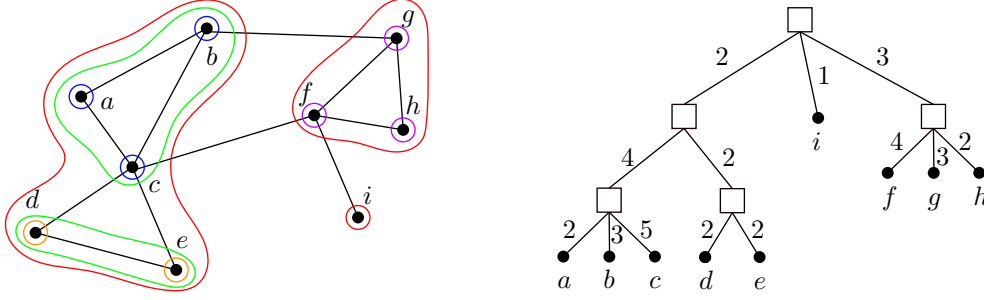


Figure 3.1: A graph with a **hierarchical decomposition** and its representation as a **cut tree**.

**Remark.** A **cut-tree** has exactly  $n$  edges, so it's keeping track only a very small number of cut values.

Surprisingly, every graph admits a **cut-tree** which can approximate all **sparse cuts** in the graph. To formalize the statement, we start with the following simple observation.

**Claim.** Let  $G = (V, E)$  be a graph with edge capacities  $c: E \rightarrow \mathbb{R}_+$ . Suppose  $D \in \mathbb{R}_+^{V \times V} \in \mathcal{D}_G$  is a non-negative demand matrix that is routable in  $G$ . Then, it is routable in any **cut-tree**  $T$  of  $G$  ( $D$  is now demand matrix over the leaves of  $T$ ).

**Proof.** Recall that a demand matrix is routable in a capacitated tree if and only if it satisfies the cut condition. The only **sparse cuts** of interest in a tree are the single edge cuts. If  $D$  is routable in  $G$ , then for any cut  $(S, V \setminus S)$  we have  $D(S, V \setminus S) \leq c(S, V \setminus S)$  in  $G$ . Since  $T$  is a **cut-tree**, and  $D$  is only between leaves of  $T$ , each edge  $v_t, v_{t'}$  in  $E_T$  corresponds to the cut  $(S_{v_t}, V \setminus S_{v_t})$  in  $G$  and the capacity of the edge in  $T$  is equal to  $fmc(S, V \setminus S)$ . Thus,  $D$  satisfies the cut condition in  $T$ , hence is routable in  $T$ .  $\otimes$

Now, we're ready to state the result.

**Theorem 3.1.1** (Hierarchical expander decomposition [Räc02]). Given a graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , there is a **cut-tree**  $T = (V_T, E_T)$  such that any demand matrix  $D \in \mathbb{R}_+^{V \times V}$  that is routable in  $T$  is also routable in  $G$  with **congestion**  $O(\log^3 n)$ .

**Note.** **Hierarchical expander decomposition** has found many applications in fast graph algorithms recently [Gor+21].

In other words, there is a very compact data structure, i.e., the **cut-tree**, that captures *all* the relevant cuts for routing (in particular, the **sparse cuts**) in  $G$  approximately within a poly-logarithmic factor. The first proof was based on finding **sparsest cuts** and hence did not lead to an efficient algorithm [Räc02]. Soon after, efficient algorithm [BKR03] with an improved **congestion** bound of  $O(\log^2 n \log \log n)$  [HHR03], which is still the best known.

We will give a proof outline of the construction of [BKR03] that achieves a weaker **congestion** bound of  $O(\log^4 n)$ . If one wants to approximate only cuts but not the routing aspect, then an improved bound of  $O(\log^{3/2} n \log \log n)$  is known [RS14].

### 3.1.2 Intuition, Connection to Expanders and Well-Lined Decomposition

Firstly, we say that a **cut-tree** is  $\rho$ -approximate if any demand matrix  $D$  routable in  $T$  can be routed in  $G$  with **congestion**  $\rho$ . Moreover, throughout we will be working with induced subgraphs of the original graph that arise as the algorithm constructs the **hierarchical decomposition**. Given  $S \subseteq V$ , its induced subgraph  $G[S]$  can also be referred as a cluster, and we associate it with a node of the **cut-tree**  $T$ . We will be interested in how well can  $S$  interface with the rest of the graph in terms of the edges between  $S$  and  $V \setminus S$ . Since we will be interested in  $G$  and various induced subgraphs, consider the following.

**Notation.** The set of all edges with one end point in  $A$  and the other in  $B$  is denoted as  $E(A, B)$ .<sup>a</sup>

<sup>a</sup>Note that  $A$  and  $B$  need not be disjoint.

Thus, to represent  $\delta_G(S)$ , we will often use  $E(S, V \setminus S)$ . Finally, we will use  $c(A, B)$  as a shortcut for the total capacity of edges in  $E(A, B)$ :

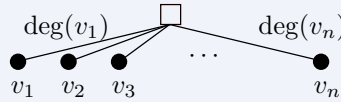
**Notation.** We let  $c(A, B) := \sum_{e \in E(A, B)} c(e)$ , and will refer to  $c(\delta_G(S))$  sometimes as  $\text{out}(S)$  for compactness.

Most of the time, we are interested in **conductance**, but we use the word **expansion**. Suppose  $G = (V, E)$  has good **conductance**  $\phi(G) = \Omega(1)$ , i.e., for all  $S \subseteq V$ ,  $c(\delta(S)) \geq \phi \text{vol}(S)$ . We know the following from the flow-cut gap.

**Claim.** Suppose  $G$  has **conductance**  $\phi$ . Let  $D$  be any demand matrix such that the following holds for each  $u \in V$ :  $b(u) = \sum_{v \in V} D(u, v) \leq c(\delta(u))$ . Then,  $G$  satisfies the cut condition for  $\phi D$  and hence  $D$  is routable in  $G$  with **congestion**  $O(\sigma(n)/\phi)$  where  $\sigma(n)$  is the flow-cut gap for product multi-commodity **flow** in graphs of at most  $n$  nodes. In particular,  $\sigma(n) = O(\log n)$  in general graphs and is  $O(1)$  in planar graphs.

Thus, if  $\phi = \Omega(1)$ , the set of routable demand matrices can be approximately characterized by stating that they should respect the trivial cuts at the vertices.

**Example (Expander).** If  $G$  is a graph with **conductance**  $\phi$ , then the star graph with  $V$  as the leaves is a **cut-tree** with approximation  $O(\log n/\phi)$ .

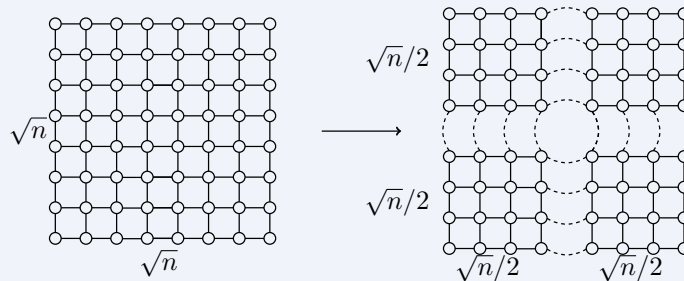


What should one do if  $G$  does not have good **conductance**?

**Intuition.** A natural approach is to decompose the graph into well-connected pieces, i.e., subgraphs with high **conductance**, along **sparse cuts**.

The question is how to. It is also not difficult to see that decomposing the graph into one or a few levels is not sufficient by looking at the grid graph.

**Example (Grid).** Consider a planar graph such as the  $\sqrt{n} \times \sqrt{n}$  grid. Then no subgraph has good **conductance** unless it is essentially of constant size. Thus, the only reasonable thing is to decompose the graph recursively into several levels. For example, the grid can be decomposed naturally into say four equal sized grids and then recursively to obtain a  $O(\log n)$  depth **cut-tree** which one can prove has poly-logarithmic approximation.



Although grids are easy to decompose due to their nice symmetric properties, it is still non-obvious to formally understand why it yields a good **cut-tree**.

**Intuition.** A key is to see that the boundary of the grid is **well-linked**.

How should one do this for a general graph whose structure is not apparent to us? The key concepts

that one needs are being able to understand how a given cluster  $H = G[S]$  interfaces with the rest of the graph through its boundary edges (those edges that cross  $S$ ) and how a given cluster is decomposed into sub-clusters. We formalize these notations as follows. They are the refined versions of [well-linkedness](#).

**Intuition.** Since [well-linkedness](#) allows us to use the notion of [expansion/conductance](#) for subsets of vertices and with arbitrary weights so that we can use it in a refined way.

We start from the following.

**Definition 3.1.4.** Let  $G = (V, E)$  and  $\pi: V \rightarrow \mathbb{R}_+$  be bounds on vertices. Let  $\alpha > 0$  be a scalar.

**Definition 3.1.5 (Cut well-linked).**  $G$  is  $(\pi, \alpha)$ -cut-well-linked if for all  $S \subseteq V$ ,  $c(\delta(S)) \geq \alpha \cdot \min(\pi(S), \pi(V \setminus S))$ .

**Definition 3.1.6 (Flow well-linked).**  $G$  is  $(\pi, \alpha)$ -flow-well-linked if for any demand matrix  $D$  such that  $\sum_{v \in V} D(u, v) \leq \pi(u)$  for all  $u \in V$  is routable in  $G$  with [congestion](#)  $1/\alpha$ .

We define [flow-well-linkedness](#) in a stronger form since this is useful for the application in this topic. It can be defined in a slightly different form as we only consider the demand matrix  $D_\pi$  being  $D_\pi(u, v) = \pi(u)\pi(v)/\pi(V)$ . The advantage of this definition is that we can check whether  $G$  satisfies this requirement efficiently via a multi-commodity [flow](#) computation. Indeed, these two do not differ much:

**Claim.** Let  $\pi: V \rightarrow \mathbb{R}_+$  be non-negative bounds on vertices of  $G = (V, E)$ . Suppose  $G$  can route the product multi-commodity [flow](#) induced by  $\pi$ , then  $G$  can route any demand matrix  $D$  with [congestion](#) 2 if  $\sum_{v \in V} D(u, v) \leq \pi(u)$  for all  $u \in V$ .

From the flow-cut gap, we obtain the following.

**Claim.** If  $G$  is  $(\pi, \alpha)$ -cut-well-linked, then  $G$  is  $(\pi, \alpha/\sigma(n))$ -flow-well-linked.

**Note.** If  $G$  has [conductance](#)  $\phi$  then it is  $(\pi, \phi)$ -cut-well-linked where  $\pi(u) = c(\delta(u))$  is the capacitated degree of  $u$  in  $G$ . As we saw, it also implies that  $G$  is  $(\pi, \phi/\sigma(n))$ -flow-well-linked.

From the above discussion, we see that the key advantage of allowing arbitrary  $\pi$  is the following:

**Intuition.** Allowing  $\pi$  to be arbitrary allows us to generalize the notation of [conductance](#) and routability to subsets of vertices and to control the amount of [flow](#) incident to a vertex.

IN the context of [cut-trees](#), we are interested in the following two types of well-linkedness:

**Definition 3.1.7 (Boundary-well-linked).** Let  $G = (V, E)$  be a graph and  $H = G[S]$  be an induced subgraph (cluster)  $S \subseteq V$ . We say  $S$  is  $\alpha$ -cut/flow-boundary-well-linked if  $H$  is  $(\pi, \alpha)$ -cut/flow-well-linked where  $\pi(u) = c(u, V \setminus S) = c(\delta(u) \cap \delta(S))$  for all  $u \in S$ .

Note that we require  $H = G[S]$  to be [well-linked](#) as a separate graph, implying the following:

**Intuition.** If a cluster is [boundary-well-linked](#), then it acts as a good router as far as its external interface is concerned.

This is an important property that is essentially required of any cluster in a [cut-tree](#) since the capacity of the edge from a cluster to its parent is equal to  $c(S, V \setminus S)$ , the boundary capacity. Another important property that is needed for a cluster is with respect to its children in the [cut-tree](#), i.e., how it is partitioned into sub-clusters for the next level. This motivates the following:

**Definition 3.1.8 (Partition-and-boundary-well-linked).** Let  $G = (V, E)$  be a graph and  $H = G[S]$  be an induced subgraph (cluster)  $S \subseteq V$ . Let  $\mathcal{D} = \{S_i\}_{i=1}^r$  be a partition of  $S$  into sub-clusters

$\{H_i = G[S_i]\}_{i=1}^r$ . We say  $S$  is  $\alpha$ -cut/flow-partition-and-boundary-well-linked if  $H$  is  $(\pi', \alpha)$ -cut/flow-well-linked where  $\pi'(u)$  is the capacity of edges going outside its sub-cluster in  $\mathcal{D}$  for each  $u \in S$ .

**Intuition.** Given a cluster  $S$  in the cut-tree and its children  $\{S_i\}_{i=1}^r$ , the edge connecting  $S_j$  to  $S$  in the tree has capacity equal to  $c(\delta_G(S_j))$ . In a sense the cluster  $S$  acts as a single node in the tree connecting the children together. Thus,  $S$  should be able to route any set of demands that go between the children as long as the total demand leaving each  $S_j$  is at most  $c(\delta_G(S_j))$ .

Consider the same example as in Figure 3.1, we see that for  $S = \{f, g, h\}$ , their corresponding  $\pi$  values are 2, 1, 0 when considering boundary-well-linkedness, while they are 4, 3, 2 when considering partition-and-boundary-well-linkedness (i.e.,  $\pi'$ ):

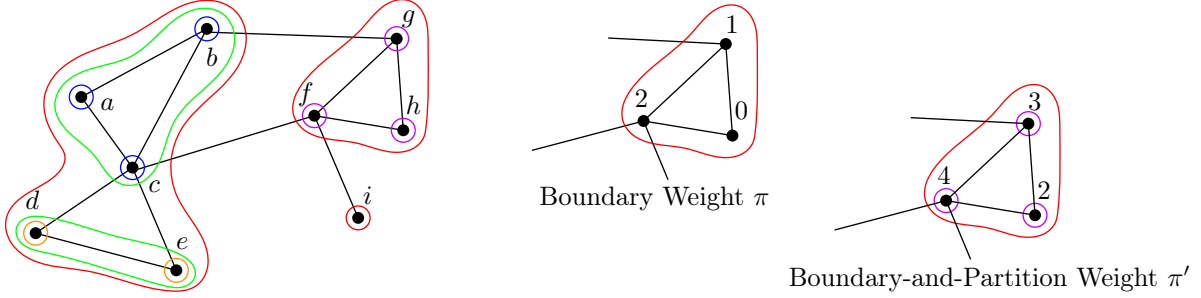


Figure 3.2: Corresponding weight values when considering Definition 3.1.7 ( $\pi(u)$ ) and Definition 3.1.8 ( $\pi'(u)$ ) for  $u \in S = \{f, g, h\}$ .

To simplify the notation, consider the following:

**Notation** (Partition-well-linked). We say that  $S$  is  $\alpha$ -partition-well-linked w.r.t.  $\mathcal{D}$  and omit  $\mathcal{D}$  when it is implicit in the context.

It is sometimes useful to view the boundary-well-linkedness and partition-well-linkedness by sticking dummy vertices as leaves and saying that the dummy vertices are well-linked (with weight 1). E.g., in Figure 3.2, imagine adding a dummy vertex to edge sticking out of the cluster.

### 3.1.3 Sufficient Condition of Being a Good Cut-Tree

Now that we have set up the definitions, the first part of the proof is to understand what conditions should the tree  $T$  satisfy such that it has good properties. Interestingly, only two properties are required: low depth/height and partition-well-linkedness of each cluster in the decomposition.

**Lemma 3.1.1.** Suppose  $T$  is a cut-tree for  $G$  of height  $h$ . Suppose each cluster  $S$  of  $T$  (corresponds to an internal node of  $T$ ) is  $\alpha$ -flow-partition-well-linked, then  $T$  is  $O(h/\alpha)$ -approximate.

**Proof.** Recall that each cluster  $H = G[S]$  in the tree  $T$  is  $\alpha$ -flow-partition-well-linked. For each  $S$ :

- $\pi_S$ : the weights on vertices of  $S$  which correspond to the out-degree of the vertices;
- $\pi'_S$ : the weights corresponding to the partition  $\mathcal{D}$  of  $S$  induced by its children in  $T$ .

Essentially, this is Figure 3.2 where we specify  $S$ . Clearly,  $\pi'_S(u) \geq \pi(u)$  for all  $u \in S$ . Also, note that  $\pi_S(S) = c(S, V \setminus S)$  is the capacity of the edges leaving  $S$ . When  $S$  is clear from the context, we simply use  $\pi$  and  $\pi'$ . Now, suppose  $D$  is a demand matrix that is routable in  $T$ . We need to show that it is routable in  $G$  with congestion  $O(h/\alpha)$ .

**Intuition.** The height  $h$  comes into the picture since  $T$  is a **hierarchical decomposition**, each edge  $e = uv \in E$  can be in up to  $h$  clusters along a path from the root of  $T$  until  $u$  and  $v$  get separated from the first time. The proof works by constructing a series of multi-commodity **flows** that can be stitched together to route all the demands.

We assume that each cluster  $S$  in  $T$  has its own private copy of each edge  $e \in E[S]$  when we compute multi-commodity **flows** in the cluster. We will bound the **congestion** of routing per cluster and then use the fact that each edge is in at most  $h$  clusters to derive the final **congestion**. Consider routing the demands from bottom up. Let  $(a, b)$  be a vertex-pair with demand  $D(a, b)$ . Let  $H = G[S]$  be the cluster which corresponds to the last common ancestor of  $a$  and  $b$  in  $T$ . We call the cluster  $H$  the **meetup cluster** for the pair  $(a, b)$ . Let  $\{H_i\}_{i=1}^r$  be the children of  $H$  in  $T$ . Without loss of generality, let  $a \in H_1 = G[S_1]$  and  $b \in H_2 = G[S_2]$ . Consider any cluster  $H' = G[S']$  along the path from  $a$  to  $H$  not including  $H$  itself. Such a cluster is called a **transition cluster** for pair  $(a, b)$ . Then,  $a$  will distribute its  $D(a, b)$  **flow** such that each node  $u \in S'$  receives a fraction  $\pi_{S'}(u)/\pi_{S'}(S')$  of the **flow**.

**Remark.** Recall that  $\pi_{S'}(S') = c(S', V \setminus S')$  is the boundary capacity of the cluster  $S'$  and  $\pi_{S'}(u)$  is the amount of that capacity incident to  $u \in S'$ . So for nodes  $u$  that is not on the boundary,  $\pi_{S'}(u) = 0$ , i.e., *we're sending **flow** to the boundary of  $S'$* .

**Intuition.** All of  $a$ 's **flow** has to leave the cluster  $S'$  to eventually reach  $b$ , so we spread the **flow** in proportion to the boundary capacity.

Note that the actual **flow** from  $a$  for the pair  $(a, b)$  that reaches  $u \in S'$  is  $D(a, b)\pi_{S'}(u)/\pi_{S'}(S')$ . For pair  $(a, b)$  this stops at  $H_1$  for  $a$  and at  $H_2$  for  $b$ , before the meetup cluster  $H = G[S]$ . It then comes the responsibility of the meetup cluster  $H$  to ensure that these **flows** match up in  $S$ . Thus, the **flow** for pair  $(a, b)$  is fully satisfied in the clusters of the subtree at their least common ancestor which corresponds how it is routed in the tree  $T$ .

To complete the description, since the **flow** is sent bottom up, we need to describe how a cluster  $H = G[S]$  with children  $\{H_i\}_{i=1}^r$  maintains the invariant. It has two parts:

- For all demand pairs  $(a, b)$  such that  $H$  is the meetup cluster,  $H$  has to ensure that the **flow** for the pairs which have reached the children's boundary are exchanged in the graph  $G[S] = H$ .
- For all demand pairs  $(a, b)$  for which  $H$  is a transition cluster (meaning that their least common ancestor is above  $H$  in  $T$ ), their **flow** has reached exactly one of the children of  $H$  inductively.  $H$  needs to redistribute this **flow** using edges only in  $G[S] = H$  such that for each such demand pair  $(a, b)$ , the **flow** is distributed over nodes  $u \in S$  in proportion to  $\pi_S(u)$ .

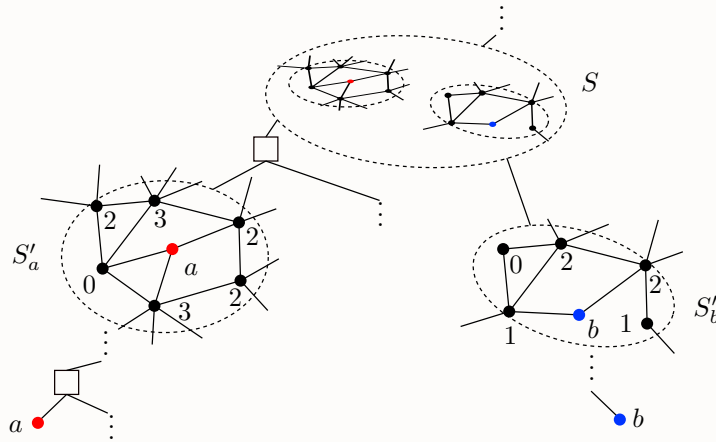


Figure 3.3: The cluster  $S'_a$  is on the path from  $a$  in  $T$  to the least common ancestor  $S$  with  $b$ , where  $a$  distributes  $D(a, b)$  to each node  $u \in S'_a$  in proportion to their boundary capacity  $\pi_{S'_a}$ . Same for  $b$ .



We show that each of the above two steps can be accomplished in  $H = G[S]$  with [congestion](#)  $O(1/\alpha)$  by using the  [\$\alpha\$ -flow-partition-well-linkedness](#) property. It turns out to be reasonably simple because we have set up the definitions properly. Consider the first part. Fix a pair  $(a, b)$  for which  $H$  is the meetup cluster. As we discussed before, say  $a \in H_1 = G[S_1]$  and  $b \in H_2 \in G[S_2]$  where  $H_1$  and  $H_2$  are children of  $H$ ,  $a$  has already sent its [flow](#) to vertices in  $S_1$  where  $u \in S_1$  has received a [flow](#) of  $D(a, b)\pi_{S_1}(u)/\pi_{S_1}(S_1)$ , and similarly,  $b$  has sent its [flow](#) to  $S_2$  in proportion to  $\pi_{S_2}$ . We exchange this [flow](#) by setting up a demand matrix. It is not hard to see that there is one which will ensure that the [flow](#) is matched up. We do this for every pair for which  $H$  is the meetup cluster we can exchange the [flows](#) that have reached the children clusters by setting up their own demand matrix. Routing the union of these demand matrices in  $H$  would suffice to exchange the [flow](#), and the problem now is whether we can do it and what is the [congestion](#) incurred.

**Claim.** All these demand matrices can be routed in  $H$  with [congestion](#)  $1/\alpha$ .

**Proof.** Consider any child cluster  $H_i = G[S_i]$  of  $H$ . Since  $D$  is routable in  $T$ , the total demand that needs to meetup in  $H$  and which originates in  $H_i$  is at most the capacity of the edge  $(H, H_i)$  in  $T$ , which has capacity  $c(S_i, V \setminus S_i) = \pi_{S_i}(S_i)$ , i.e., the boundary capacity of  $S_i$ . This means that for each  $u \in S_i$ , the total [flow](#) that has reached  $u$  for all the demands that need to be matched up in  $H$  is at most  $\pi_{S_i}(u)/\pi_{S_i}(S_i) \cdot c(S_i, V \setminus S_i) = \pi_{S_i}(u)$ . Thus, for exchanging the meetup [flow](#), we have set up many demand matrices, but the total demand from these matrices that originate at  $u \in S_i \in S$  is at most  $\pi_{S_i}(u)$ . This is true for all the vertices in  $S$ . But recall that any demand matrix that satisfies this degree condition, by definition of the  [\$\alpha\$ -flow-partition-well-linkedness](#), can be routed in  $H = G[S]$  with [congestion](#)  $1/\alpha$ .<sup>a</sup>  $\otimes$

<sup>a</sup>Indeed, this is the reason we required this condition on the clusters.

Hence, we're done for the first part. The second part is similar. Consider any demand pair  $(a, b)$  for which  $H$  is a transition cluster. This means that one of  $a, b$  is in  $H$  and the other is not. Say  $a \in H$ , and it has already distributed the [flow](#) to the boundary of the child  $H_i$  of  $H$  where  $a \in H_i$ , i.e., for each  $u \in S_i$  in proportion to  $\pi_{S_i}(u)/\pi_{S_i}(S_i)$ . To maintain the invariant, in cluster  $H$  we need to distribute  $a$ 's [flow](#) to nodes in  $S$  such that each  $v \in S$  gets [flow](#) in proportion to  $\pi_S(v)/\pi_S(S)$ . We can set up a demand matrix to exchange this [flow](#) again. For any node  $u \in S_i$ , the total demand originating at  $u$  is at most  $\pi_{S_i}(u)$  since the total demand crossing  $S_i$  is at most  $c(S_i, V \setminus S_i)$ , and for each node  $v \in S$ , the total demand that it receives is at most  $\pi_S(v) \leq \pi'_S(v)$ . Thus, the union of the demand matrices respect the total bounds imposed by  $\pi'$  at each node of  $S$ , and hence by the  [\$\alpha\$ -flow-partition-well-linkedness](#), can be routed in  $S$  with [congestion](#)  $1/\alpha$ .

For the base case of the induction, we see that the [flow](#) originates at the singleton leaves which is its own boundary and hence there is nothing to do there. As we argued earlier, the total [congestion](#) on an edge is the union of the [congestions](#) of all the clusters it participates in, and an edge participates in at most  $h$  clusters, which proves the result.  $\blacksquare$

**Remark.** We can also construct an [oblivious routing](#) from [Lemma 3.1.1](#) as well, but we will not do so to keep the presentation simpler.

The power of [Lemma 3.1.1](#) is that it has essentially captures what we need from the [hierarchical decomposition](#). We will see a construction that guarantees that  $h = O(\log n)$  and  $\alpha = \Omega(1/\log^3 n)$ , and this gives us a [cut-tree](#) that is  $O(\log^4 n)$ -approximate, proving (a weaker version of) [Theorem 3.1.1](#).

## Lecture 12: Construction of Expander Hierarchy

### 3.1.4 Top-Down Algorithm for Constructing a Cut-Tree

3 Oct. 11:00

We now describe a top-down divide-and-conquer algorithm that starts with  $V$  and creates a [cut-tree](#) satisfying the conditions (i.e., the [partition-well-linkedness](#) property) of [Lemma 3.1.1](#) with  $h = O(\log n)$  and  $\alpha = \Omega(1/\log^3 n)$ , which gives an  $O(\log^4 n)$ -approximation, as we promised.

The algorithm does not backtrack, i.e., once it partitions a cluster  $S$  into its children, it simply recurses on the children. For this to work, the algorithm requires that each cluster satisfy an additional property before it can be successfully partitioned into sub-clusters that satisfy the [partition-well-linkedness](#)



property. This requires something called [precondition](#) [BKR03].

**Notation** (Parameter). We will use some parameters for formal proofs and to help see the ideas.

- $\sigma$ : the upper bound on the flow-cut gap in an  $n$ -vertex graph for product multi-commodity [flow](#) instances, where we know that  $\sigma = O(\log n)$ .<sup>a</sup>
- $\lambda$ :  $64\sigma \log n$  with  $\lambda = \Theta(\log^2 n)$  for general graph.
- $\alpha$ :  $1/24\sigma\lambda = \Omega(1/\log^3 n)$ , which is the goal. Note that  $1/\alpha$  is a log factor smaller than  $\lambda$ .

<sup>a</sup>Recall that we can get an  $O(\sigma(n))$  approximation for [non-uniform sparsest cut](#) when needed.

We will assume that the capacities are integer valued. The algorithm is polynomial in the sum of the capacities, so technically we can only apply it for capacities that are poly-bounded, but we will ignore this issue since we are more interested in the proof of the existence for now.

**Definition 3.1.9 (Precondition).** Let  $H = G[S]$  for  $S \subseteq V$  be a cluster. Let  $\pi: S \rightarrow \mathbb{R}_+$  be boundary capacity weights, i.e.,  $\pi(u)$  is the capacity of the edges leaving  $S$  that are incident to  $u$ . We say  $S$  satisfies the *precondition* if for all  $U \subseteq S$  such that  $|U| \leq 3|S|/4$ ,

$$c(U, S \setminus U) \geq \frac{\pi(U)}{\lambda}.$$

**Intuition.** We want each cluster  $S$  to be [boundary-well-linked](#). On the other hand, to ensure the recursion depth, the algorithm will obtain a tree of height  $O(\log n)$  by ensuring that each cluster  $S$  is decomposed into sub-clusters such that no sub-cluster  $S_i$  has more than  $2|S|/3$  vertices.

[Definition 3.1.9](#) resembles a [cut-well-linkedness](#) condition, but it is not quite the same since the numerator is about  $\pi$  while the denominator is about the cardinality of  $U$ . As we will see, this actually corresponds to [sparsity](#) of a related demand matrix.

The motivation for this [precondition](#) comes from the eventual divide-and-conquer algorithm for constructing a [cut-tree](#) which requires each part to be a constant factor smaller than its parent cluster. Note that the entire vertex set  $V$  will trivially satisfy this [precondition](#) and hence the algorithm can get started. To enable recursion, the algorithm will ensure that when it partitions  $S$  into sub-clusters, each of the resulting sub-clusters also satisfies this [precondition](#).

**Intuition.** We need this [precondition](#) because it turns out that not every cluster can be successfully partitioned to satisfy the [partition-well-linkedness](#) property.

However, since it's not feasible to efficiently check whether a given cluster satisfies the [precondition](#), so we will use approximation algorithms via flow-cut gap to indirectly guarantee that the [precondition](#) is satisfied. Now, to ensure [precondition](#), we consider something like [well-linked](#) style decomposition. Specifically, during the algorithm, we will need to ensure that the sub-clusters that are created satisfy the [precondition](#) for the recursion. As we remarked, we cannot directly guarantee it. For this reason, we will use a decomposition procedure called [assure precondition](#), that will decompose *any* given cluster  $S$  such that the resulting pieces after the decomposition satisfy the [precondition](#). We first see the lemma.

**Lemma 3.1.2.** Let  $S \subseteq V$  with  $|S| \geq 2$  be any cluster. Then [assure precondition](#) partitions  $S$  into sub-clusters  $\{S_i\}_{i=1}^p$  for some  $p \geq 1$  such that

- each  $S_i$  satisfies the [precondition](#), and
- $\sum_{i=1}^p c(\delta_G(S_i)) \leq 2c(\delta_G(S))$ .

We postpone the proof of [Lemma 3.1.2](#), and note that the algorithm to achieve the above is very similar to the [expander decomposition](#), but not exactly the same due to the nature of the condition.

**As previously seen.** In [expander decomposition](#), we wish to decompose a given graph into pieces such

that each piece has good **conductance** while cutting as few edges as possible. From [Theorem 2.4.2](#), we know that each piece has **conductance**  $\Omega(1/\sigma \log n)$  via an  $O(\sigma)$ -approximation algorithm for **non-uniform sparsest cut** based on flow-cut gap, while cutting only a constant fraction of edges.

Specifically, **assure precondition algorithm** first sets up a demand matrix  $D$ :<sup>1</sup> for each vertex  $u \in S$ , it creates a demand  $D(u, v)$  for each  $v$  where  $D(u, v) = \pi(u)/|S|$ .

**Intuition.**  $u$  wants to distribute  $\pi(u)$  uniformly among all vertices. The final demand matrix is then the union of these demand matrices, one for each  $u$ .

From [Lemma 3.1.2 \(b\)](#), the increase in the number of cut edges is upper bounded by  $2c(\delta_G(S))$  since the sum counts each such newly cut edge twice. But we are only interested in each piece being **boundary-well-linked**, so this is similar to **well-linked** decomposition with initial weight equal to  $\pi_S(u)$ .

---

**Algorithm 3.1:** Assure **Precondition**


---

**Data:** A connected graph  $G = (V, E)$ , cluster  $S \subseteq V$   
**Result:** A partition  $\{S_i\}_{i=1}^p$  of  $S$  satisfying [Lemma 3.1.2](#)

```

1 for  $(u, v) \in V \times V, u \neq v$  do // Ordered pair
2    $D(u, v) \leftarrow \pi_S(u)/|S|$ 
3
4  $((A, B), \psi) \leftarrow \alpha\text{-Non-Uniform-Sparsest-Cut}(G[S], H = (V, D))^a$  // sparsity  $\psi$  of  $(A, B)$ 
5
6 if  $\psi > 4\sigma/\lambda$  then // If not sparse
7   return  $\{S\}$ 
8 else
9    $\{S_i^{(1)}\}_{i=1}^{p_1} \leftarrow \text{Assure-Precondition}(G, A)$ 
10   $\{S_i^{(2)}\}_{i=1}^{p_2} \leftarrow \text{Assure-Precondition}(G, B)$ 
11  return  $\{S_i^{(1)}\}_{i=1}^{p_1} \cup \{S_i^{(2)}\}_{i=1}^{p_2}$ 

```

---

<sup>a</sup>We assume that this is a flow-cut gap based  $\alpha$ -approximation algorithm.

**Proof of Lemma 3.1.2.** Our goal is to relate the **sparsest cut** for the special demand matrix we design to the **precondition** so that we can guarantee that  $S$  will satisfy the **precondition** if there is no **sparse cut** with respect to  $D$ . Let  $\phi$  be the overall minimum **sparsity** according to  $D$ . Via the flow-cut gap, we can find a cut  $(A, B)$  such that its **sparsity** according to  $D$  is at most  $\sigma \cdot \phi$ .

**Claim.** Suppose  $S$  does not satisfy the **precondition**, then  $\phi \leq 4/\lambda$ .

**Proof.** If  $S$  does not satisfy the **precondition**, then there is a partition  $(U, S \setminus U)$  of  $S$  such that

$$c(U, S \setminus U) \leq \frac{\pi(U)}{\lambda}$$

from the definition (where  $|U| \leq 3|S|/4$ ). Consider the demand according to  $D$  crossing this partition, which is

$$\pi(U) \cdot \frac{|S \setminus U|}{|S|} + \pi(S \setminus U) \cdot \frac{|U|}{|S|} \geq \pi(U) \cdot \frac{|S \setminus U|}{|S|},$$

which is greater than  $\pi(U)/4$  since  $|S \setminus U| \geq |S|/4$ . Thus, the **sparsity** of  $(U, S \setminus U)$  is at most  $4c(U, S \setminus U)/\pi(U) \leq 4/\lambda$  by our assumption. Hence,  $\phi \leq 4/\lambda$ .  $\otimes$

This ensures that **assure precondition** guarantees that each cluster in the final partition satisfies the **precondition** because of the termination condition and the recursion. The main issue is why it does not cut too many edges in creating the partition.<sup>a</sup> We will write a recursion to understand the total number of edges cut. If  $S$  satisfies the minimum **sparsity** condition ([line 6](#)), i.e., we return  $S$  and don't cut any edges, then [\(b\)](#) is trivial. Now, suppose we find a partition  $(A, B)$  of  $S$  with

<sup>1</sup>This is very different from a product multi-commodity flow!

**sparsity**  $\psi < 4\sigma/\lambda < 1/16 \log n$ , then we recurse on  $A$  and  $B$ . The total demand crossing  $(A, B)$  is

$$\pi(A) \frac{|B|}{|S|} + \pi(B) \frac{|A|}{|S|} \leq \pi(A) + \pi(B) = \pi(S),$$

implying

$$c(A, B) \leq \frac{\pi(S)}{16 \log n},$$

resulting in the total edges cut is at most  $\pi(S)$  with a careful analysis, hence we're done. ■

<sup>a</sup>The analysis is similar to [Theorem 2.4.2](#), bit a bit more involved.

**Remark.** Unlike the recursion analysis in [Theorem 2.4.2](#), these new cut-edges  $c(A, B)$  create two new problems whose total size in terms of outgoing edges is slightly larger compared to the original problem. Nevertheless,  $\log n$  is sufficiently large that this increase can also be absorbed.

We now describe the main algorithm called [partition](#), which achieves the following guarantee:

**Lemma 3.1.3.** Let  $S \subseteq V$  with  $|S| \geq 2$  with  $|S| \geq 2$  be any cluster that satisfies the [precondition](#). Then [partition algorithm](#) partitions  $S$  into sub-clusters  $\{S_i\}_{i=1}^p$  for some  $p > 1$  such that

- (a)  $|S_i| \leq 2|S|/3$  for each sub-cluster  $S_i$ ,
- (b) each  $S_i$  satisfies the [precondition](#),
- (c)  $S$  satisfies the [α-flow-partition-well-linkedness](#) with respect to the decomposition.

Suppose we have such an algorithm. Then the top-down recursive algorithm is straightforward. In each step, it ensures that the sizes of the sub-clusters goes down by a constant factor, and they satisfy the [precondition](#) to enable recursion. Moreover, the cluster itself satisfies the desired [partition-well-linkedness](#) property with respect to the decomposition.

**As previously seen.**  $V$  trivially satisfies the precondition because it has no outgoing edges as we mentioned, hence the algorithm can get started. Moreover, given a cluster  $S$  and its decomposition into sub-clusters, we can check efficiently whether it satisfies the [partition-well-linkedness](#) property.

**Problem.** How should [partition](#) work?

**Answer.** Suppose  $S = V$  and  $G$  has good [conductance](#). Then, as we argued, the right decomposition is to simply take all the singleton vertices, and it satisfies the desired properties. Hence, naturally, the algorithm starts with an *optimistic* guess: the trivial singletons' decomposition of  $S$ . The nice aspect of this starting decomposition is that the properties (a) and (b) are automatically satisfied.

**Note.** Property (c) can be checked efficiently, and if we got lucky to satisfy it, we're done!

Of course this is too optimistic. What the algorithm actually does is to maintain a current partition that satisfies the properties (a) and (b). Let this partition be  $\mathcal{P} = \{H_i\}_{i=1}^p$ . It first checks if  $S$  satisfies property (c) for  $\mathcal{P}$ . If it does, we're done. Otherwise, it finds a [sparse cut](#)  $(A, B)$  of  $S$  w.r.t.  $\pi'$  as  $S$  is not [partition-well-linkedness](#). It then uses the partition  $(A, B)$  to compute another partition  $\mathcal{P}'$  such that once again,  $\mathcal{P}'$  satisfies the properties (a) and (b).

**Note.** We make sure it's making progress by ensuring that the total number of inter-cluster edges in  $\mathcal{P}'$  is strictly less than the inter-cluster edges in  $\mathcal{P}$ .<sup>a</sup> This is the crux of the proof.

<sup>a</sup>The algorithm is only guaranteed to reduce the number of inter cluster edges by one, so this is the reason for the dependence of the running time on  $\sum_{e \in E} c(e)$ .

Thus, the algorithm can repeat this process, and it is guaranteed to terminate with a partition that satisfies all three properties. \*

Now, we need to describe how to compute the new partition  $\mathcal{P}'$  from  $\mathcal{P}$ . We develop some intuition before giving a formal description.

**Intuition.** Given that  $(A, B)$  is a **sparse cut** for the **partition-well-linkedness** condition, it means that the number of edges crossing  $A$  is too small in comparison to  $\pi'(A)$ , i.e., the total number of inter-cluster edges in the current decomposition and the total number of external edges. Since  $S$  satisfy the **precondition**, it is already  **$1/\lambda$ -boundary-well-linked**. Thus, the reason we have a **sparse cut** is that there are too many inter-cluster edges inside  $A$  due to the current partition  $\mathcal{P}$ .

Suppose we get lucky and  $A$  is the union of some sub-collection of clusters from  $\mathcal{P}$ . Then it makes sense to merge all the clusters in  $A$  and make  $A$  the new cluster since it reduces the number of inter-cluster edges. But this new cluster  $A$  may not satisfy the **precondition**, but we can use **assure precondition** on  $A$  to partition it into clusters that satisfy the **precondition**.

**Remark.** We will introduce new inter-cluster edges in this process, but not too many.

Since  $A$  may not be a clean union of current clusters, the algorithm employs a simple heuristic to find a union of existing clusters: it takes the union of all clusters that have at least  $3/4$  of their vertices inside  $A$ .

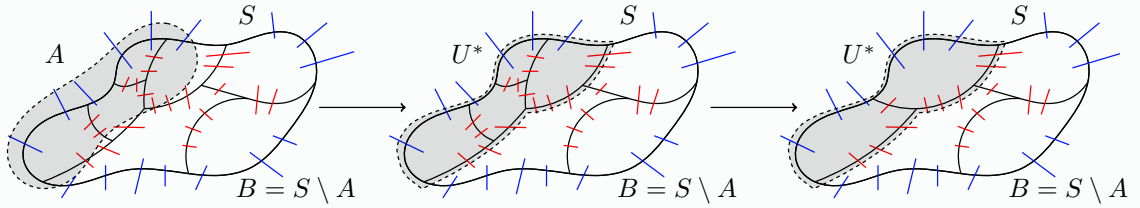


Figure 3.4: The algorithm should first find a **sparse cut**  $(A, B)$  if the current decomposition does not satisfy the desired **partition-well-linkedness** property. It converts  $A$  to a set  $U^*$  that is a union of existing clusters. It then uses **assure precondition** to find new clusters within  $U^*$  that satisfy the **precondition**. Note how new inter-cluster edges are introduced in the last step.

Such a heuristic is called the *rounding step*, which we denote as  $U^* = \text{round}(A)$  [ABN08]. While intuitive, the main technical difficulty is to show that this works.<sup>a</sup>

<sup>a</sup>It may not be clear at this stage that  $\text{round}(A)$  is non-empty!

We now formally describe the algorithm.

---

**Algorithm 3.2:** Partition

---

**Data:** A connected graph  $G = (V, E)$ , cluster  $S \subseteq V$  satisfying **precondition**

**Result:** A partition  $\mathcal{P}$  of  $S$  satisfying **Lemma 3.1.3**

---

```

1  $\mathcal{P} \leftarrow \{\{v\} : v \in S\}$ 
2
3 for  $(u, v) \in V \times V, u \neq v$  do                                     // Ordered pair
4    $D(u, v) \leftarrow \pi'_S(u)/|S|$ 
5
6 while  $S$  is not  $\alpha$ -partition-well-linkedness w.r.t.  $\mathcal{P}$  do           // Via flow computation
7    $((A, B), \psi) \leftarrow \alpha\text{-Non-Uniform-Sparsest-Cut}(G[S], H = (V, D))$  //  $|A| \leq |B|, \psi \leq \sigma\alpha$ 
8    $U^* \leftarrow \emptyset$ 
9   for  $R_i \in \mathcal{P}$  do                                                 // Compute round(A)
10    if  $|R_i \cap A| > 3|R_i|/4$  then
11       $U^* \leftarrow U^* \cup R_i$ 
12       $\mathcal{P} \leftarrow \mathcal{P} - \{R_i\}$ 
13    $\mathcal{Q} \leftarrow \text{Assure-Precondition}(G, U^*)$ 
14    $\mathcal{P} \leftarrow \mathcal{P} \cup \mathcal{Q} - \{U^*\}$ 
15 return  $\mathcal{P}$ 

```

---

To summarize, **Partition** starts with the partition consisting of the singletons and either terminates or updates the partition. When **Algorithm 3.2** does not terminate, it then computes  $U^*$  by taking the union of some existing clusters and then uses **assure precondition** to re-cluster it. We now prove **Lemma 3.1.3**, i.e., the formal guarantee of the **partition algorithm**.

**Proof of Lemma 3.1.3.** Firstly, when **partition** terminates, the output clearly satisfies property (c) as well since it explicitly checks for it. Hence, we will show that it maintains the properties (a) and (b) for the partition at the start of each iteration. The initial partition satisfies the properties (a) and (b), so we only need to show that this is maintained. First, we observe the following.

**Claim.** Suppose  $\mathcal{P}$  fails the **partition-well-linkedness** property. Let  $U^*$  be the union of clusters that have large intersection with  $A$ . Then  $|U^*| \leq 2|R|/3$ , i.e., property (a) is maintained by **partition**.

**Proof.** Since  $|A| \leq |R|/2$ , and we only include in  $U^*$  clusters that have large intersection with  $A$ , and they satisfy property (a) previously.  $\otimes$

Next, since we run **assure precondition** on  $U^*$  (line 13), the resulting sub-clusters  $\mathcal{Q}$  satisfy the **precondition** and their size can only decrease. Since the clusters outside  $U^*$  are not touched, and they already satisfied the properties, thus, **partition** indeed maintains the invariant for  $\mathcal{P}$  such that each cluster is not too big and that it satisfies the **precondition**, i.e., properties (a) and (b). Hence, the remaining part is to ensure that **partition** is efficient.

The main technical part that ensures termination is the following, which relates the **sparsity** of the cut  $(A, B)$  with the **sparsity** of the cut  $U^*$ .

**Claim.** Given a **sparse cut**  $(A, B)$  w.r.t.  $\pi'$  for the current  $\mathcal{P}$  and  $U^*$  outputted by **partition**,

$$\frac{c(\delta_G(U^*))}{\pi'(U^*)} \leq 4\lambda \frac{c(A, B)}{\pi'(A)}.$$

In other words, converting  $A$  to a union of the existing clusters costs a factor  $4\lambda$  in the **sparsity**.

**Proof.**  $\otimes$

Now, since  $(A, B)$  is a **sparse cut** with **sparsity** at most  $\sigma\alpha$ ,

$$\frac{c(A, B)}{\pi'(A)} \leq \sigma\alpha \leq \frac{1}{24\lambda}.$$

Combined with the claim, we have  $c(\delta_G(U^*)) \leq \pi'(U^*)/6$ . We now see how to use this to prove termination of **partition**. In particular, we prove that **partition** terminates in time polynomial in  $|S|$  and the maximum integer edge capacity  $\max_{e \in E} c(e)$ .

Let  $E(\mathcal{P}) := \{uv \in E \mid u, v \in S \text{ and } u, v \text{ in different clusters}\}$  be the set of inter-cluster edges induced by the partition  $\mathcal{P}$ . Observe that since we count each inter-cluster edge twice and the edges leaving  $S$  to outside once,

$$\pi'(S) = 2c(E(\mathcal{P})) + c(\delta_G(S))$$

Let  $\mathcal{P}'$  be the partition at the end of the iteration (line 14). It suffices to prove that the total capacity of  $E(\mathcal{P}')$  is strictly less than that of  $E(\mathcal{P})$ . We first recall how  $\mathcal{P}'$  differs from  $\mathcal{P}$ :

**As previously seen.** We remove all clusters of  $\mathcal{P}$  contained inside  $U^*$  and add clusters of the new partition  $\mathcal{Q}$  of  $S$ , obtained by running **assure precondition** on  $U^*$ .

**Claim.** Indeed,  $E(\mathcal{P}') < E(\mathcal{P})$ .

Not done yet

**Proof.** From [Lemma 3.1.2 \(b\)](#), we have

$$\sum_{S' \in \mathcal{Q}} c(\delta_G(S')) \leq 2c(\delta_G(U^*)).$$

Let  $\pi'': S \rightarrow \mathbb{R}_+$  be the new weights induced by the partition  $\mathcal{P}'$  that counts the inter-cluster and outgoing edges of  $\mathcal{P}'$ . We see that by counting, we have

$$\pi''(S) = \pi'(S) - \pi'(U^*) + \sum_{S' \in \mathcal{Q}} c(\delta_G(S')) \leq \pi'(S) - \pi'(U^*) + 2c(\delta_G(U^*)) < \pi'(S),$$

where the last inequality follows from the implication of the previous claim, i.e.,  $\pi'(U^*) \geq 6c(\delta_G(U^*))$ , and  $c(\delta_G(U^*)) > 0$ . Finally, from the same counting argument as before,  $\pi''(S) = 2c(E(\mathcal{P}')) + c(\delta_G(S))$ , we see that  $E(\mathcal{P}') < E(\mathcal{P})$ .  $\otimes$

Hence, the number of inter-cluster edges reduces in each iteration, and we're done.  $\blacksquare$

## Lecture 13: Real Negative Weights Shortest Path

### 3.2 Single-Source Shortest Path with Negative Real Weight<sup>2</sup>

8 Oct. 11:00

In the following two lectures, we will discuss a recent breakthrough of [SSSP](#) with negative length [[HJQ24](#)], which can now be solved in  $\tilde{O}(mn^{4/5})$ . This is built upon the recent work that first break the  $\tilde{O}(mn)$  bound that achieves  $\tilde{O}(mn^{8/9})$  [[Fin24](#)]. Before we start, we first formally introduce the [single-source shortest path](#) problem.

**Problem 3.2.1 (Single-source shortest path).** Given a graph  $G = (V, E)$  with edge capacity  $w: V \rightarrow \mathbb{R}$  and a source vertex  $s \in V$ , the *single-source shortest path* problem, or *SSSP*, aims to find the shortest path from a source  $s \in V$  to  $t$  for all  $t \in V - s$ .

#### 3.2.1 Johnson's Algorithm

There are several classical algorithms for solving the [SSSP](#), including [Dijkstra's algorithm](#) (which runs in  $O(m + n \log n)$  with Fibonacci heap) and [Bellman-Ford algorithm](#) (which runs in  $O(mn)$ ). However, we know that Dijkstra's algorithm can only handle the case when edge lengths are all positive, and hence, for general real edge lengths that are potentially negative, Bellman-Ford algorithm remains the state-of-the-art for decades. The key ingredient of the breakthrough is the classical Johnson's potential re-weighting algorithm. To be precise, let's give a quick review.

**As previously seen** (Johnson's algorithm). A natural strategy to achieve re-weighting is to consider a *potential*  $\phi: V \rightarrow \mathbb{R}$  and re-weight a (directed) edge  $u \rightarrow v$  to be  $w'(u, v) := w(u, v) + \phi(u) - \phi(v)$ .

**Intuition.** Reweight edge lengths to preserve shortest paths, and also make weights non-negative.

For any  $u$ - $v$  path  $u = v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k = v$ , the new weight of this path is

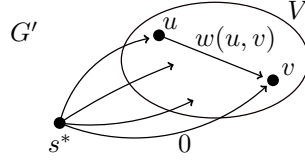
$$\begin{aligned} & w'(v_0, v_1) + w'(v_1, v_2) + \dots + w'(v_{k-1}, v_k) \\ &= w(v_0, v_1) + \phi(v_0) - \phi(v_1) + w(v_1, v_2) + \phi(v_1) - \phi(v_2) + \dots + w(v_{k-1}, v_k) + \phi(v_{k-1}) - \phi(v_k) \\ &= w(v_0, v_1) + w(v_1, v_2) + \dots + w(v_{k-1}, v_k) + \phi(v_0) - \phi(v_k). \end{aligned}$$

Hence, all  $u$ - $v$  paths change by  $\phi(v_0) - \phi(v_k)$ , i.e., the structure of the shortest paths are preserved. Furthermore, consider adding a dummy source vertex  $s^*$  with edge weight 0 for all new edge  $(s^*, v)$  for all  $v \in V$ . Then, consider  $\phi(v) := d_{G'}(s^*, v)$ . We see that for any  $u \rightarrow v \in E$ ,

$$w'(u, v) = w(u, v) + \phi(u) - \phi(v) = w(u, v) + d_{G'}(s^*, u) - d_{G'}(s^*, v) \geq 0$$

<sup>2</sup>This section is taught by [Kent Quanrud](#). Guest lectures!

if and only if  $w(u, v) + d_{G'}(s^*, u) \geq d_{G'}(s^*, v)$ , which is obviously true.



More generally, if we set  $\phi(v) := d_G(s, v)$  from any source  $s \in V$  rather than that dummy extra source, the above still holds. Hence, distance is a good potential function; on the other hand, good potential also helps computing distances: if  $w_\phi(e) \geq 0$  for all  $e \in E$ , then we can use Dijkstra's algorithm on  $G_\phi = (V, E)$  with edge weight  $w_\phi$  and solve SSSP.

With the notion of potential, the goal is to find such a “good” potential  $\phi$  such that  $w_\phi(e) \geq 0$  for all  $e \in E$ . We call this **neutralization**.

**Definition 3.2.1 (Neutralize).** A potential  $\phi$  *neutralize* a set of negative length edges  $F \subseteq E$  if  $w_\phi(e) \geq 0$  for all  $e \in F$ .

With potential, a crucial observation is the following.

**Lemma 3.2.1.** A graph  $G$  has no negative weight cycle if and only if there is a potential  $\phi$  such that  $w_\phi(e) \geq 0$  for all  $e \in E$ .

Thus, the goal is to either detect that  $G$  has a negative weight cycle or find a potential that **neutralizes** all the negative weight edges while not introducing any new negative weight edges. We see that finding good potential and SSSP is pretty much the same problem, and we will switch our perspective between these two. Finally, we make the following remark.

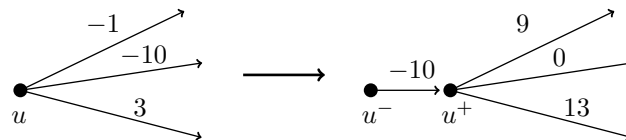
**Claim (Verification).** One of the useful facts about **single-source shortest path** trees and distances is that we can check in linear time whether a given tree or set of distances are valid or not.

**Proof.** We simply relax (update rule of Dijkstra's algorithm) all edges and see if any of the distances change. If they do not change then the distances are valid, otherwise they are obviously not.  $\circledast$

This verification is useful when running randomized algorithms that may work only under the assumption that  $G$  has no negative cycle but may still produce some (potentially incorrect) output when  $G$  has a negative cycle. Hence, in the following sections about SSSP with negative edge weights, we can assume that there is no negative weight cycle in the graph.

### 3.2.2 Some Preprocessing

We first note that without loss of generality, we can make sure that the number of negative weight edges is  $O(n)$  by splitting each vertex  $u$  into  $u^-$  and  $u^+$  and making  $w(u^-, u^+)$  equal the weight of the most negative weight out-going edges of  $u$  (0 if there is no negative weight edge), and adjusting the weights going out of  $u$  (now  $u^+$ ) appropriately, which now are all non-negative.



### 3.2.3 Preliminary

Before we get started, we need to establish some common terminologies used in this line of works.

**Notation (Hop).** A *hop* in a walk means a negative edge. Hence, we will say an  $h$ -hop shortest path to indicate that this path contains  $h$  negative edges.



## Lecture 14: Real Negative Weights Shortest Path Continued

Next, we need to deal with the so-called *betweenness reduction*.

10 Oct. 11:00

## Lecture 15: Scaling Algorithm for Negative Integral SSSP

### 3.3 Single-Source Shortest Path with Negative Integral Weight

15 Oct. 11:00

As we have seen how to deal with real weights, we consider the case of integer weights. The goal is to check if a graph  $G$  has a negative weight cycle or to output shortest path distances from a source  $s$  to any other vertex  $v \in V$  (which are well-defined when there is no negative weight cycle). As we have mentioned, developed from 50's and 60's, the classical Bellman-Ford algorithm runs in  $O(mn)$  time and the Floyd-Warshall runs in  $O(n^3)$  time. Similar to the real weighted case, the general strategy of [Johnson's algorithm](#) still applies, i.e., we want to find potentials  $\phi(v)$  such that each edge  $(u, v)$  has non-negative modified weight  $w_\phi(u, v) = w(u, v) + \phi(u) - \phi(v) \geq 0$ .

**As previously seen.** This can be done by adding a dummy source  $s^*$  in the new graph  $G'$  and compute  $\phi(v) := d_{G'}(s^*, v)$  for all  $v \in V$ .

Unlike the real weighted case, a trick called *scaling* can be exploited, where we don't need to obtain a potential that makes every edge non-negative exactly.<sup>3</sup> Goldberg described such a scaling algorithm for integer weighted case that runs in  $O(m\sqrt{n} \log W)$ , where  $W = \max_{e \in E, w(e) < 0} |w(e)|$  is the absolute value of the most negative weight [Gol95]. In a recent breakthrough, a randomized near-linear time scaling algorithm is developed [BNW22; BCF23], which we will now discuss.

#### 3.3.1 Scaling Algorithm

The high-level intuition of the scaling trick is the following.

**Intuition (Scaling).** When considering integer weights, paths between  $s$  and  $v$  are more structured in terms of *separation*: the weights of two paths are either the same or differ by at least 1.

In particular, given a graph  $G = (V, E)$  with edge weights  $w: V \rightarrow \mathbb{Z}$ , consider the scaled weight  $w'(e) := 2n \cdot w(e)$  for all  $e \in E$ . Note that the graph with  $w'$  will have a negative weight cycle if and only if the graph with the original weight  $w$  has.

**Claim.** Let  $P$  and  $Q$  be any two  $s$ - $v$  paths. Then either  $|w'(P) - w'(Q)| = 0$  or  $|w'(P) - w'(Q)| \geq 2n > n$ . This is also true for any potential  $\phi$ , i.e., either  $|w'_\phi(P) - w'_\phi(Q)| = 0$  or  $|w'_\phi(P) - w'_\phi(Q)| \geq n$ .

**Proof.** If two  $s$ - $v$  paths  $P$  and  $Q$  have different weights, since  $w(P), w(Q) \in \mathbb{Z}$ , we know that they differ by at least 1. By scaling,  $w'(P), w'(Q)$  differ by at least  $2n$ .  $\otimes$

With this simple observation, we can iteratively apply the following to increase the edge lengths in each iteration with suitable potentials:

**Theorem 3.3.1 (Scale down).** Let  $G = (V, E)$  be a graph with edge lengths  $w: V \rightarrow \mathbb{R}$  such that  $w(e) \geq -2B$  for all  $e \in E$ . There is a randomized algorithm called *scale down* such that:

- If it terminates, it outputs a *halving potential*  $\phi: V \rightarrow \mathbb{R}$  such that  $w_\phi(e) \geq -B$  for all  $e \in E$ .
- If  $G$  does not contain a negative weight cycle, then the expected runtime is  $O(m \log^4 n)$  and the output is correct. If  $G$  has a negative weight cycle, then the algorithm may not terminate.

**Note.** Even if there is a negative weight cycle, the subroutine from [Theorem 3.3.1](#) can still terminate and will be a valid halving potential.

<sup>3</sup>We should note once again that, if we solve [SSSP](#), a good potential can be obtained. So we can still find a good potential at the end, although our original goal is to solve [SSSP](#).

Assuming we have the subroutine from [Theorem 3.3.1](#), furthermore, if we assume that it will either detect that there is a negative weight cycle, or outputting a halving potential, we can solve [SSSP](#):

---

**Algorithm 3.3:** Scaling Algorithm for [SSSP](#)


---

**Data:** A directed graph  $G = (V, E)$  with integral edge length  $w: V \rightarrow \mathbb{Z}$ , source  $s \in V$

**Result:** [SSSP](#) from the source  $s$  or  $\perp$ , indicating a negative weight cycle

---

```

1  $w' \leftarrow 2nw$ 
2  $W' \leftarrow \max_{e \in E: w(e) < 0} |w'(e)|$ 
3  $\phi \leftarrow 0$  // Initialize potential for all  $v \in V$ 
4
5 for  $i = 1, \dots, O(\log W')$  do
6    $\phi' \leftarrow \text{Scale-Down}(G, w'_\phi)$ 
7   if  $\phi' = \emptyset$  then // Check negative weight cycle
8     return  $\perp$ 
9    $\phi \leftarrow \phi + \phi'$  // Also update  $w'_\phi$ 
10
11  $w'' \leftarrow \max(0, w'_\phi)$ 
12  $T \leftarrow \text{Dijkstra}(G, w'', s)$  // Compute shortest path tree
13  $d(s, \cdot) \leftarrow \text{Shortest-Path-Distance}(T, w)$  // Trivial to compute
14 return  $\{d(s, v)\}_{v \in V}$ 

```

---

**Theorem 3.3.2** ([BNW22]). There is a randomized algorithm ([Algorithm 3.3](#)) that takes as input a graph  $G = (V, E)$  with integral edge weights  $w: V \rightarrow \mathbb{Z}$  and a source  $s$  such that

- if  $G$  contains a negative cycle, it does not terminate;
- otherwise, it returns valid [SSSP](#) distances for  $s$  in expected time  $O(m \log^4 n \log(nW))$ .

**Proof.** Let  $B := 2^{\lceil \ln W' \rceil}$ .<sup>a</sup> We see that if we run the [scale down](#) subroutine  $O(\log W') = O(\log nW)$  times, we will find a potential  $\phi: V \rightarrow \mathbb{Z}$  such that  $w'_\phi(e) \geq -1$  for all  $e \in E$  (if there is a negative weight cycle, it'll be detected by running this subroutine as well). Then, by considering  $w'(e) := \max(0, w'_\phi(e))$  for all  $e \in E$ , running the Dijkstra's algorithm on  $G$  with edge length  $w'$ , we can find the correct shortest path in  $G$  since  $w'$  only perturbs shortest path lengths by at most  $n - 1$ , while we know that any two paths will have difference at least  $n$  if they're not of the same length. This is exactly [Algorithm 3.3](#). ■

<sup>a</sup>Hence,  $w'(e) = 2nw(e) \geq -2B$  for all  $e \in E$ .

With some slight tweaks, we obtain the following.

**Theorem 3.3.3.** There is a randomized Monte-Carlo algorithm that takes as input a graph  $G = (V, E)$  with integral edge weights  $w: V \rightarrow \mathbb{Z}$  and a source  $s \in V$  such that

- if  $G$  contains a negative weight cycle, it returns an error message;
- if  $G$  does not contain a negative weight cycle, then it returns valid [SSSP](#) distances for  $s$  with high probability. It may return an error message even if there is no negative weight cycle.

The algorithm runs in  $O(m \log^5 n \log(nW))$  time.

**Proof.** We first consider the following intermediate algorithm. It runs [Algorithm 3.3](#) on input  $(G, w, s)$ ; if it terminates before twice of its expected runtime and returns [SSSP](#) distances, then this intermediate algorithm returns the same output. If [Algorithm 3.3](#) takes more than twice of its expected runtime, then the intermediate algorithm is stopped and returns error. By Markov's inequality, this intermediate algorithm has the following properties:

- if  $G$  has no negative weight cycle, it returns a correct output with probability  $1/2$ ; and
- if  $G$  has a negative weight cycle, it always returns error; and

- it may return error even if  $G$  has no negative cycle with probability at most  $1/2$ .

Then, we can simply run this intermediate algorithm  $O(\log n)$  times independently and returning error if all invocations return error. It is easy to see that it has the desired properties. The runtime is then  $O(\log n)$  times the expected run-time of [Algorithm 3.3](#), which is  $O(m \log^5 n \log(nW))$ . ■

**Remark.** The Monte-Carlo algorithm described in [Theorem 3.3.3](#) can be viewed as almost what we would like because it returns valid SSSP distances with high probability when they exist. But it may also return error with a small probability even when there is no negative weight cycle. Ideally, we would also like an algorithm that can detect and output a negative cycle with high probability when  $G$  has one. Suppose we had such an algorithm, then we can run the two algorithms and get what we want. We will discuss such an algorithm that finds a negative weight cycle at the end.

Hence, the only difficult (and remaining) part is to prove [Theorem 3.3.1](#).

### 3.3.2 Preliminary Tools

Now, we can start building up toward proving [Theorem 3.3.1](#). This requires some preliminary facts. Firstly, the intuition is to reduce the problem to directed acyclic graph (DAG).

**Intuition.** DAG has an equivalent role for directed graphs as tree for undirected graphs in some sense. In particular, we know that for shortest path, DAG is easy.

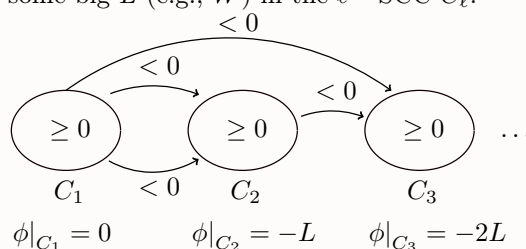
To further motivate making a directed graph a DAG, consider the following.

**Problem 3.3.1 (Feedback arc set).** Given a directed graph  $G = (V, E)$  with positive edge weight  $w: V \rightarrow \mathbb{R}_+$ . The *feedback arc set* problem asks for removing the least weight set of edges such that the resulting graph is a DAG.

The first fact is that if the graph induces a nice DAG, then we can find a good potential efficiently.

**Lemma 3.3.1 (Fix DAG edge).** Suppose  $G = (V, E)$  is a directed graph with edge weight  $w$  such that in each strongly connected component (SCC), the edge weights are non-negative. Then one can compute a potential  $\phi: V \rightarrow \mathbb{R}$  such that  $w_\phi(e) \geq 0$  for all  $e \in E$  in  $O(m + n)$  time.

**Proof.** Consider the DAG<sup>a</sup>  $G^{\text{SCC}}$  associated with contracting all the SCCs  $\{C_i\}_{i=1}^\ell$  to nodes  $\{v_i\}_{i=1}^\ell$ . Then by sorting  $v_i$ 's in  $G^{\text{SCC}}$  in the **topological** order in  $O(m + n)$ , we can neutralize edges between  $C_i$ 's by adding  $-(\ell - 1)L$  for some big  $L$  (e.g.,  $W$ ) in the  $\ell^{\text{th}}$  SCC  $C_\ell$ .



This works since for any  $u \rightarrow v$  in  $C_i$ , change of  $u$  and  $v$ 's potential are the same, hence  $w(u, v) = w_\phi(u, v)$ . On the other hand, for edges crossing clusters, say from  $C_i$  to  $C_j$  for  $i < j$ ,  $w_\phi$  will increase by  $(j - i)L$ . Hence, if  $L$  is big enough, these negative edges can be all made non-negative. ■

<sup>a</sup>It's a fact that this will form a DAG.

Another useful tool turns out to be the low-diameter decomposition. We have seen **low-diameter decomposition** in the context of undirected graphs and metrics. Here, we're interested in directed graphs and distances. There were implicit in algorithm for cut and **flow** problems for symmetric demands [[Kle+97](#); [Sey95](#); [Eve+00](#)], but the utility of their randomized variants have only recently been explored.

**Intuition.** Since reachability is asymmetric in directed graphs, in several problems of interest, we are focus on diameter of SCC.

If we also think of cuts as removing subsets of edges rather than edges leaving a set, then a nature formulation of the low-diameter decomposition of a directed graph with *non-negative* weights  $w: E \rightarrow \mathbb{R}_+$  is to remove a subset of edges  $E' \subseteq E$  such that the diameter of each SCC in  $G - E'$  is at most some given parameter  $D$ . The diameter of an SCC  $C \subseteq V$  is  $\max_{u,v \in C} d_G(u,v) \leq D$  for all  $u, v \in C$  in  $G$ . This is the notion of “weak diameter” since we are using distances in  $G$ .

**Notation** (Strong diameter). The *strong diameter* guarantee is where  $d_C(u,v) \leq D$  for all  $u, v \in C$  where  $d_C(u,v)$  is the distance using only edges in  $G[C]$ .

The following states that there exists an efficient randomized algorithm to compute the low-diameter decomposition for directed graph.

**Theorem 3.3.4** (Low-diameter decomposition for directed graph [BNW22]). There is a randomized algorithm such that given a directed graph  $G = (V, E, w)$  where  $w: E \rightarrow \mathbb{R}_+$  and a diameter bound  $D \geq 0$ , outputs a set of edges  $E' \subseteq E$  such that in  $O(m \log^2 n + n \log^3 n)$  time,

(i) all the SCCs in  $G - E'$  has weak diameter  $\leq D$ , and

(ii)  $\Pr(e \in E') \leq O(w(e) \log^2 n / D + n^{-10})$ .<sup>a</sup>

<sup>a</sup>The  $n^{-10}$  is negligible and is for technical reasons to obtain a fast algorithm. We will ignore it for convenience.

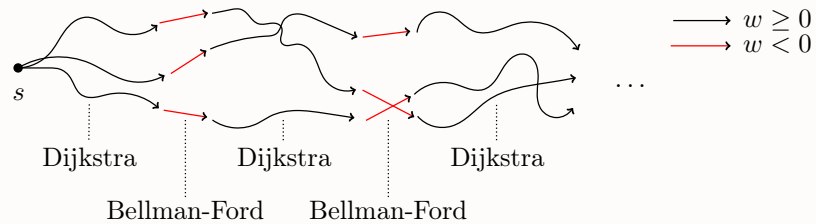
**Intuition.** The *low-diameter decomposition* allows us to get to the case of Lemma 3.3.1.

**Remark.** [BCF23] describes an algorithm with *strong diameter* guarantee with the cutting probability increased to  $O(\log^3 n w(e) / D)$ . They also describe a different decomposition procedure which gives a faster algorithm while not necessarily giving a low-diameter decomposition.

Lastly, we note that if we have some bound on the number of negative edges shortest paths will have, we can actually solve SSSP by a combination of Dijkstra’s algorithm and Bellman-Ford algorithm.

**Lemma 3.3.2** (Fix bad edge [BNW22]). Suppose  $G = (V, E)$  is a directed graph with edge weight  $w: V \rightarrow \mathbb{R}$  has no negative weight cycle. Suppose every  $s$ - $v$  shortest path is at most  $\kappa_v$ -hops for every  $v \in V$  from the source  $s$ . Then SSSP (with the corresponding potential) can be solved in  $O((n + \sum_{v \in V} |\delta^+(v)| \kappa_v) \log n)$  time.

**Proof.** We essentially run Dijkstra and Bellman-Ford alternatively on a modified graph. The intuition is that Dijkstra computes the correct shortest path length for positive edges, while one iteration of Bellman-Ford fix one negative edge.



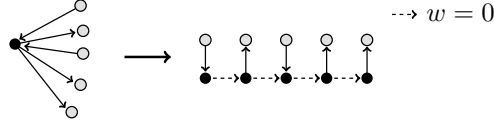
To actually bound the running time to be the sum of  $\sum_{v \in V} \kappa_v$ , one will need to implement this algorithm carefully with *laziness*, i.e., nodes participate in the computation only when necessary:

**Claim.** After  $i$  iterations, we get correct distance  $d(s, v)$  if  $\kappa_v \leq i$ . Hence,  $v$  contributes to runtime for  $\kappa_v$  iterations.

This leads to the desired bound.<sup>a</sup> ■

<sup>a</sup>This can be proved with only Dijkstra by the layering graph construction as we have seen in the Homework 1.

**Remark.** We can make the graph to have in and out-degree  $O(1)$  for all  $v$  by blow-up  $n$  to  $\Theta(m)$ :



Hence, the algorithm described in [Lemma 3.3.2](#) runs in  $O((m + \sum_{v \in V} \kappa_v) \log n)$  time.

## Lecture 16: Low-Diameter Decomposition for Directed Graph

### 3.3.3 The Scale Down Algorithm via Hop Reduction and Fixing

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Now, we can start proving [Theorem 3.3.1](#), i.e., implementing the [scale down algorithm](#). Recall that we're given a directed graph  $G = (V, E)$  with edge lengths  $w: V \rightarrow \mathbb{R}$  such that  $w(e) \geq -2B$  for all  $e \in E$ . We want to design an algorithm such that it outputs a potential where  $w_\phi(e) \geq -B$  for all  $e \in E$ .

**Note.** Following the earlier discussion, we will assume that there is no negative weight cycle.

Consider two new graphs  $G^B = (V, E, w^B)$  such that

$$w^B(e) = \begin{cases} B + w(e), & \text{if } w(e) < 0; \\ w(e), & \text{if } w(e) \geq 0, \end{cases}$$

and  $G_+^B = (V, E, w_+^B)$  where  $w_+^B(e) = \max(0, w^B(e))$ . For any subgraph  $H$  of  $G$ , we will use  $H^B$  and  $H_+^B$  to refer to the corresponding subgraphs of  $G^B$  and  $G_+^B$ .

**Claim.** If a potential  $\phi$  [neutralizes](#) all negative edges in  $G^B$  without introducing additional negative edges, then  $w_\phi(e) \geq -B$  for all  $e \in E$ .

**Proof.** Fix an edge  $e = (u, v)$ . If  $w(e) < 0$ , there are two cases:

- If  $w(e) \geq -B$ : then  $w^B(e) \geq 0$  and by our assumption,  $w_\phi^B(e) \geq 0$ , implying

$$w_\phi(e) = \phi(u) + w(e) - \phi(v) = \phi(u) + w^B(e) - B - \phi(v) = w_\phi^B(e) - B \geq -B.$$

- If  $w(e) < -B$ : then  $w^B(e) = w(e) + B < 0$  and  $w_\phi^B(e) \geq 0$ . It's obvious that  $w_\phi(e) \geq -B$ .

If  $w(e) \geq 0$ , then from our assumption this will not be affected since  $w^B(e) = w(e)$ . ⊛

Hence, our goal now is to [neutralize](#) all negative edges in  $G^B$  without introducing any new ones. It is not quite obvious why it'll be easier to work with  $G^B$ .

**Intuition.** If  $G$  does not contain a negative weight cycle, then  $G^B$  is making the problem easier. [\[BCF23\]](#) uses a slightly different scheme which ensures that the minimum mean cycle length is  $\geq 1$ , which formalizes more explicitly the change in the graph by adding  $B$  to the negative weight edges.

To [neutralize](#) negative edges in  $G^B$ , we again take inspiration from [Johnson's algorithm](#): i.e., let  $s^*$  be a dummy node with 0 weight edges from  $s^*$  to every  $v \in V$  in  $G^B$ , which we denote this new graph as  $G^B + s^*$ . Since  $s^*$  is connected to each vertex  $v$  with edge of weight 0 the shortest path distances from  $s^*$  are at most 0. Thus, consider an  $s^*$ - $v$  path  $P_{s^*,v}$  of non-zero [hops](#)  $\kappa_v > 0$ , the weight of any prefix (say,  $P_{s^*,u} \subseteq P_{s^*,v}$ ) of  $P_{s^*,v}$  is at most 0, since otherwise we can go from  $s^*$  directly to that intermediate vertex  $u$  to reduce the cost.

Now, the key idea is to use [low-diameter decomposition for directed graph](#) with an appropriate parameter such that the number of [hops](#) needed is reduced, i.e., hop reduction.

**Intuition.** From Lemma 3.3.2, if the number of hops for  $s^*-v$  shortest paths is small in  $G^B + s^*$ , or even if the average is small, then we can solve the problem in near-linear time. We aim to reduce to this with low-diameter decomposition for directed graph, recursion, and also fixing edges that are easy to fix along the way (i.e., the case of Lemma 3.3.1).

Since low-diameter decomposition requires non-negative edge lengths, we consider applying it to  $G_+^B$ .

**Lemma 3.3.3.** Let  $G = (V, E)$  be a graph with edge length  $w: V \rightarrow \mathbb{R}$  such that  $w(e) \geq -2B$  for all  $e \in E$ . Let  $\kappa_{G^B}(v)$  to be the maximum number of hops of any  $s^*-v$  shortest path in  $G^B + s^*$ , and  $\kappa_{G^B} := \max_{v \in V} \kappa_{G^B}(v)$ . Suppose  $\kappa \geq \kappa_{G^B}$ , and we run low-diameter decomposition for directed graph on  $G_+^B$  with  $D = \kappa B/2$  to get an output  $E'$ . Then,

- (i) for all  $v \in V$ ,  $\mathbb{E}[|P_v \cap E'|] \leq O(\log^2 n)$ , where  $P_v$  is a  $\kappa_{G^B}(v)$ -hop  $s^*-v$  shortest path in  $G^B$ ;
- (ii) for all SCC  $H$  in  $G - E'$ ,  $\kappa_{H^B} \leq \kappa/2$ , where  $\kappa_{H^B}$  concerns with  $H^B + s$ , similar to  $\kappa_{G^B}$ .

**Proof.** We prove (ii) first. Assume that  $H$  is an SCC in  $G - E'$  such that there is one shortest path  $P_v^{H^B} = s^* \rightarrow u \rightarrow \dots \rightarrow v$  contains entirely in  $H^B$  using more than  $\kappa/2$  negative edges w.r.t.  $w^B|_H$ . Denote the corresponding  $u-v$  path as  $P := P_v^H \setminus \{s^*\}$ . Observe that there is a  $v-u$  walk  $Q$  of length at most  $D$  w.r.t.  $w_+^B$  from the low-diameter decomposition guarantee, i.e.,  $w_+^B(Q) \leq D$ . Hence,  $w(Q) \leq D$  as well. However, we see that  $Q \cup P$  is a cycle, furthermore, its weight is negative:

$$w(Q \cup P) = w(Q) + w(P) \leq D - B \cdot \left(\frac{\kappa}{2} + 1\right) = \frac{\kappa B}{2} - \frac{\kappa B}{2} - B < 0,$$

since we assume that  $P$  contains more than  $\kappa/2$  negative edges w.r.t.  $w^B$ , with the fact that  $w^B(P) = w^B(P_v^H) \leq w^B(\{s^* \rightarrow v\}) = 0$ , hence its original weight is at most  $(\kappa/2 + 1) \cdot (-B)$ .

We now prove (i), consider any  $s^*-v$  shortest path  $P_v$  for some  $v$  in  $G^B$  of hops  $\kappa_{G^B}(v)$ . From the same reason as above, we have  $w^B(P_v) \leq 0$ , hence  $w_+^B(P_v) \leq \kappa B$ . By linear of expectation,

$$\mathbb{E}[|P_v \cap E'|] \leq \frac{w_+^B(P_v)}{D} O(\log^2 n) \leq \frac{\kappa B}{D} O(\log^2 n) = O(\log^2 n)$$

since  $\Pr(e \in E') \leq O(w(e) \log^2 n / D)$  from the low-diameter decomposition guarantee. ■

**Intuition.** Lemma 3.3.3 guarantees not only the hop-reduction, but also that (in expectation) the number of edges that prevent the reduction to Lemma 3.3.1 is small per shortest path.

**Remark.** Consider a vertex  $v$  and an  $s^*-v$  path  $P_v$  with  $\kappa$  negative weight edges in  $G^B$ . Why does that path not exist in  $H^B$  since we proved that in  $H^B$  the shortest path to  $v$  can only contain  $\kappa/2$  negative weight edges? The point is that suppose  $P_v$  has  $(s, u')$  as its first edge. Then the above proof shows that there cannot be a  $(v, u')$  path in  $G$  with weight at most  $D = \eta B/2$ . Thus,  $u'$  and  $v$  cannot be in the same SCC with diameter bound  $D$ . Thus, the shortest path to  $v$  inside  $H^B$  is not going to be able to use  $u'$ .

With Lemma 3.3.3, it's quite natural in retrospect to consider a recursive algorithm that reduces the hop length, recurses and fixes the remaining edges. Recall that the goal is to find a potential  $\phi$  that neutralizes all negative weight edges in  $G^B$  without adding new ones. The algorithm starts with an upper bound of  $\kappa_{G^B}$ , say  $n$ , and computes a set of edges  $E'$  using the low-diameter decomposition for with  $D = \kappa B/2$ . For each SCC  $H$  in  $G - E'$ , we recurse since  $\kappa_{G^B} \leq \kappa/2$ . This recursively yields a potential  $\phi_1^H$  on vertices of  $H^B$  that neutralizes all negative edges inside  $H^B$  (without introducing new ones). Since the potentials are found separately for each SCC, and these SCCs partition the vertex set, we can find an overall potential  $\phi_1$  that neutralizes all negative edges inside the SCCs. Note that  $G - E'$  consists of other edges which are not inside any SCC, and we call them the DAG edges since they induce a DAG on the SCCs, as mentioned in Lemma 3.3.1. Then, we simply use Lemma 3.3.1 to fix these DAG edges, leaving edges in  $E'$  that are still potentially negative. For those edges, we simply fix them using some sort of brute-force way: with Lemma 3.3.2. But as shown in Lemma 3.3.3 (i), in  $G^B$ , there can be at most  $O(\log^2 n)$  negative weight edges in expectation, which suffice for efficiency.



**Algorithm 3.4:** Scale-Down Algorithm**Data:** A directed graph  $G = (V, E)$  with integral edge length  $w: V \rightarrow \mathbb{Z}$ ,  $\kappa \geq \kappa_{G^B}$ **Result:** A potential  $\phi$ 

```

1 if  $\kappa \leq 2$  then // Base case
2    $d(s^*, \cdot) \leftarrow \text{Brute-Force}(G^B + s^*, w^B, s^*)$  // Lemma 3.3.2
3    $\phi \leftarrow d(s^*, \cdot)$ 
4   return  $\phi$ 
5
6  $E' \leftarrow \text{Low-Diameter-Decomposition}(G_+^B, \kappa B/2)$ 
7  $\{H_i\}_{i=1}^\ell \leftarrow \text{Strongly-Connected-Component}(G - E')$ 
8
9 for  $i = 1, \dots, \ell$  do // Recurse
10   $\phi_1^{H_i} \leftarrow \text{Scale-Down}(H_i, w|_{H_i}, \kappa/2)$ 
11
12  $\phi_1 \leftarrow \sum_{i=1}^\ell \phi_1^{H_i}$ 
13  $\phi_2 \leftarrow \phi_1 + \text{Fix-DAG-Edges}((G^B - E')_{\phi_1}, w_{\phi_1}^B)$  // Lemma 3.3.1
14  $\phi_3 \leftarrow \phi_2 + \text{Fix-Bad-Edges}((G^B)_{\phi_2}, w_{\phi_2}^B)$  // Lemma 3.3.2
15
16 if  $w_{\phi_3}^B(e) < 0$  for any  $e \in E$  then // Check validity
17   while 1 do // Infinite loop
18     1 + 1
19 else
20   return  $\phi_3$ 

```

We now prove [Theorem 3.3.1](#).

**Proof of Theorem 3.3.1.** We will show that running [Algorithm 3.4](#) with  $\kappa = n$  satisfies the requirement. Firstly, the correctness essentially follows from [Lemma 3.3.3](#), the properties of potentials, and the induction. Note that, under the assumption that  $G$  has no negative weight cycle, each step of [Algorithm 3.4](#) correctly computes its output and hence [Algorithm 3.4](#) becomes a correct Las Vegas algorithm. If  $G$  has a negative weight cycle, then the algorithm is not guaranteed to terminate. Since we check validity of  $\phi_3$  before returning it ([line 16](#)), we ensure that it returns a valid output if it terminates.

We now analyze the time complexity. Note that [Algorithm 3.4](#) has recursion depth  $O(\log n)$ . We consider the expected time outside the recursion. In the base case ([line 1](#)), it is easy to see that the running time is  $O(m \log n)$  via [Lemma 3.3.2](#). The low-diameter decomposition takes  $O(m \log^3 n)$  ([line 6](#)) as stated in [Theorem 3.3.4](#).<sup>a</sup> When fixing the DAG edges in [line 13](#), we consider the linear time ( $O(m + n) = O(m)$ ) algorithm implied by [Lemma 3.3.1](#). In the last step ([line 14](#)), we use the algorithm implied in [Lemma 3.3.2](#). Recall that by [Lemma 3.3.3](#), we have  $\mathbb{E}[|P_v \cap E'|] = O(\log^2 n)$ . Thus, by linearity of expectation,  $\mathbb{E}[\sum_{v \in V} |P_v \cap E'|] = O(n \log^2 n)$ . Thus, the expected running time of the algorithm from [Lemma 3.3.2](#) is  $O(m \log^3 n)$ .

Finally, in each recursive call, we reduce the parameter  $\kappa$  by half and the total sum of the edges inside the SCCs is at most  $m$ . Thus, the total expected time of the algorithm is simply the depth of the recursion times the time outside the recursion, i.e.,  $O(m \log^4 n)$ . ■

<sup>a</sup>Note that we have the preprocessing step such that  $n = \Omega(m)$ .

### 3.3.4 Finding a Negative Cycle

Finally, we discuss a scaling algorithm to find a negative weight cycle, assuming we have an algorithm  $\mathcal{A}$  such that it will detect when  $G$  has a negative weight cycle, and otherwise outputs a valid potential that neutralizes all negative weight edges without introducing new ones. While we don't have a deterministic algorithm of  $\mathcal{A}$ , but we do have the algorithm implied in [Theorem 3.3.3](#) which we can act as a substitute [[BNW22](#); [BCF23](#)]. Let  $G = (V, E)$  be a directed graph with integral edge weights  $w: V \rightarrow \mathbb{Z}$ . Say  $\mathcal{A}$  has detected that  $G$  contains a negative cycle. We start scaling all weights by  $n^3$ , i.e., consider  $w_0(e) = n^3 w(e)$  for all  $e \in E$ . Let  $G_0$  be this new graph.



**Note.** If  $G$  has a negative cycle, then  $G_0$  has a negative cycle with weights  $\leq -n^3$ .

Given  $G$  and an integer  $M \geq 0$ , we define a graph  $G^{+M}$  as the graph obtained by adding  $M$  to each edge weight. Note that we add  $M$  to all edges, not just negative weight edges as we did earlier in defining  $G^B$ . Let  $M^*$  be the smallest integer such that  $G_0^{+M^*}$  does not have a negative cycle. It means that  $G_0^{+(M^*-1)}$  has a negative cycle.

**Claim.**  $M^* \geq n^2$  and  $M^* \leq n^3W$ . Given access to  $\mathcal{A}$  that can detect a negative cycle using binary search, one can find  $M^*$  using  $O(\log(nW))$  call to which.

**Proof.** Since there is a negative weight cycle  $C$  in  $G_0$  of weight  $\leq -n^3$ , to make it positive, we need to add at least  $n^2$  to each edge since  $|C| \leq n$ . The second part is obvious.  $\otimes$

The following claim is also easy to see.

**Claim.** Since  $G_0^{+M^*}$  does not have a negative weight cycle, there is a potential  $\phi: V \rightarrow \mathbb{Z}$  such that  $\phi$  **neutralizes** all negative weight edges in  $G_0^{+M^*}$  without introducing new ones and this can be computed by  $\mathcal{A}$ .

The main lemma that leads to the algorithm is the following.

**Lemma 3.3.4.** Let  $\phi$  be a potential that **neutralizes** the negative weight edges in  $G_0^{+M^*}$ , and  $E' = \{e \in E \mid (w_0^{+M^*})_\phi(e) > n\}$ . Then  $G_1 := (V, E \setminus E')$  contains a cycle, and any cycle in  $G_1$  is a negative cycle in  $G$ .

**Proof.** There is a negative cycle  $C$  in  $G_0^{+(M^*-1)}$  which clearly is also a negative weight cycle in  $G$ . We see that  $w_0^{+M^*}(C) = w_0^{+(M^*-1)} + |C|$ , thus,  $w_0^{+M^*}(C) \leq n$ . Note that  $\phi$  **neutralizes** all negative weights in  $G_0^{+M^*}$ , hence all edge weights of  $C$  become non-negative w.r.t.  $\phi$ , but the total weight of  $C$  does not change w.r.t.  $\phi$  as potentials do not change cycle weights. This means that  $(w_0^{+M^*})_\phi(C) \leq n$ , and it contains only non-negative edge weights. All these imply that any edge in  $C$  must have weight at most  $n$  w.r.t.  $(w_0^{+M^*})_\phi$ , hence  $C \cap E' = \emptyset$ , which means that all edges of  $C$  survives in  $G_1$ , hence  $G_1$  must have a cycle.

Now, we argue that any cycle in  $G_1$  is a negative weight cycle in  $G$ . Let  $C$  be an arbitrary cycle in  $G_1$ . Since  $C$  does not have any edges from  $E'$ , we have  $(w_0^{+M^*})_\phi(C) \leq n|C| \leq n^2$ . Since potentials do not change cycle weights, we therefore have  $w_0^{+M^*}(C) \leq n^2$ . However, this implies

$$w_0(C) = w_0^{+M^*}(C) - M^*|C| \leq w_0^{+M^*}(C) - 2M^* \leq n^2 - 2n^2 < 0,$$

where we use the fact that  $|C| \geq 2$  and  $M^* \geq n^2$ . Hence, the weight of  $C$  in  $G_0$  is negative, which implies that it is also negative in  $G$  as well.  $\blacksquare$

Based on [Lemma 3.3.4](#), we see that the following algorithm outputs a negative cycle assuming access to  $\mathcal{A}$ . It invokes  $\mathcal{A}$  only  $O(\log(nW))$  times.

---

**Algorithm 3.5:** Find Negative Weight Cycle

---

**Data:** A directed graph  $G = (V, E)$  with integral edge length  $w: V \rightarrow \mathbb{Z}$  containing a negative weight cycle

**Result:** A negative weight cycle  $C$

```

1  $w_0 \leftarrow n^3w$ 
2  $G_0 \leftarrow (V, E, w_0)$ 
3 Find the smallest  $M^* > 0$  such that  $G_0^{+M^*}$  has no negative weight cycle      // Binary search
4  $\phi \leftarrow \mathcal{A}(G_0^{+M^*}, w_0^{+M^*})$                                            // Neutralize  $G_0^{+M^*}$ 
5  $E' \leftarrow \{e \in E \mid (w_0^{+M^*})_\phi(e) > n\}$ 
6  $C \leftarrow \text{Find-Cycle}(G - E')$ 
7 return  $C$ 
```

---

If we use the algorithm implies in [Theorem 3.3.3](#) in place of  $\mathcal{A}$ , the resulting algorithm may make

mistakes and fail, but we can detect failure in finding a negative cycle, and thus we can obtain a Monte Carlo algorithm that can find a negative cycle with high probability if  $G$  has one.

# Chapter 4

## Multiplicative Weight Update

### Lecture 17: Approximating Linear Programs

In this chapter, we try to bridge the field of combinatorial (discrete) optimization and continuous optimization from the lens of approximating positive linear programs using [multiplicative weight update](#). In particular, we will look into the class of *positive* linear programs, which is a large and interesting class of linear programs that arise in combinatorial optimization.

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**Intuition.** Some useful properties that can be exploited algorithmically via iterative methods coming from first order optimization methods and this leads to fast approximation solutions.

Although these methods are old in optimization, the formal analysis with provable worst-case guarantees for concrete problems can be traced to the work of efficient algorithms for multi-commodity flows [SM90] which was later abstracted to general classes of linear programs [PST95; GK94]. Parallel algorithms were also derived via these methods [LN93], and there have been many developments since then.

#### 4.1 Positive Linear Program

Positive linear programs, as the name suggests, are linear programs where the input to the linear program consists of positive numbers. This allows one to discuss relative approximations.

##### 4.1.1 Packing, Covering and Mixture of Both

There are three types of positive linear programs that we commonly work with.

**Definition 4.1.1 (Packing linear program).** Given data  $c \in \mathbb{R}_+^n$ ,  $A \in \mathbb{R}_+^{m \times n}$ , and  $b \in \mathbb{R}_+^m$ , a *packing linear program* with  $n$  variables and  $m$  non-trivial constraints is of the form

$$\begin{aligned} \max \quad & c^\top x \\ & Ax \leq b; \\ & x \geq 0. \end{aligned}$$

**Definition 4.1.2 (Covering linear program).** Given data  $c \in \mathbb{R}_+^n$ ,  $A \in \mathbb{R}_+^{m \times n}$ , and  $b \in \mathbb{R}_+^m$ , a *covering linear program* with  $n$  variables and  $m$  non-trivial constraints is of the form

$$\begin{aligned} \min \quad & c^\top x \\ & Ax \geq b; \\ & x \geq 0, \end{aligned}$$

Oftentimes, we have a mixture of both.

**Definition 4.1.3** (Mixed packing and covering linear program). Given data  $A \in \mathbb{R}_+^{m_1 \times n}$ ,  $b \in \mathbb{R}_+^{m_1}$ ,  $C \in \mathbb{R}_+^{m_2 \times n}$ , and  $d \in \mathbb{R}_+^{m_2}$ , a *mixed packing and covering linear program* with  $n$  variables and  $(m_1 + m_2)$  non-trivial constraints is in the form of feasibility problem of inequalities<sup>a</sup>

$$\begin{aligned} \max \quad & 0 \\ & Ax \leq b; \\ & Cx \leq d; \\ & x \geq 0. \end{aligned}$$

<sup>a</sup>This is general enough since we can put the objective (either maximization or minimization) into the constraints.

#### 4.1.2 Explicit and Implicit Linear Program

**Definition 4.1.4** (Explicit linear program). An *explicit linear program* is one in which all the data is specified in terms of the variables, constraints and the non-negative entries in the inequalities.

For an *explicit linear program*, we will assume that the representation is given in the sparse form, and we will usually let  $n$  denote the number of variables,  $m$  the number of constraints, and  $N$  the number of non-zeros in the constraint matrices.

**Remark.** Hence, the input size of an explicit specified positive linear program is  $\Theta(N + m + n)$ .

On the other hand, we also have the so-called *implicit linear program*.

**Definition 4.1.5** (Implicit linear program). An *implicit linear program* is one in which we have an underlying data such as a graph or geometric object and a linear program formulation that is specified implicitly based on that data.

We note that for *implicit linear programs*, they can have exponentially many variables or constraints, and often they can be solved efficiently or approximately via Ellipsoid method. In such settings, we will not write down the *explicit linear program*, and we will seek iterative methods that have a running time that depends on the size of the *original input* that defines the linear program.

**Note.** The disadvantage of the *explicit linear program* is that it has many variables and constraints and typical *explicit linear program* solvers use memory that is quadratic in the number of variables to do matrix operations while solving the linear program via the simplex or interior point method, and this makes the memory a bottleneck.

#### 4.1.3 Examples

Let's see some examples of *packing linear programs*.

**Example** (Maximum weight bipartite matching). Given a bipartite graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , the following linear program gives an exact solution to the maximum weight bipartite matching:

$$\begin{aligned} \max \quad & \sum_{e \in E} c(e)x_e \\ & \sum_{e \in \delta(u)} x_e \leq 1 \quad \forall u \in V; \\ & x_e \geq 0 \quad \forall e \in E. \end{aligned}$$

One can see that this is a *explicit packing linear program* with  $N = 2|E|$ ,  $n = |E|$ , and  $m = |V|$ .<sup>a</sup> Although this linear program is mainly used for bipartite matching, it is sometimes useful even for non-bipartite graphs. One can show that its integrality gap is  $2/3$  for general graphs.

<sup>a</sup>Note that the notation of  $n, m$  is different from the usual graph usage.

**Example (Tree packing).** Recall the [tree packing linear program](#), where the goal is to pack [spanning trees](#) into the capacity of a given graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ :

$$\begin{aligned} \max \quad & \sum_{T \in \mathcal{T}_G} y_T \\ & \sum_{T \ni e} y_T \leq c(e) \quad \forall e \in E; \\ & y_T \geq 0 \quad \forall T \in \mathcal{T}_G. \end{aligned}$$

This is an [implicit packing linear program](#) with an exponential number of variables and  $m = |E|$  constraints. As mentioned in [Theorem 1.2.2](#), this can be (approximately) solved efficiently without the Ellipsoid method (which runs in polynomial time, but it is considered impractical).

**Example (Maximum multi-commodity flow).** We considered the maximum multi-commodity [flow](#) problem as a relaxation for the [multi-min-cut](#) problem. Given a graph  $G = (V, E)$  with non-negative edge capacities  $c: E \rightarrow \mathbb{Q}_+$  and  $k$  source-sink pairs  $\{(s_i, t_i)\}_{i=1}^k$ . Let  $\mathcal{P}_{s_i, t_i}$  denote the set of all  $s_i$ - $t_i$  paths in  $G$ . We have a variable  $x_P$  for each path  $P \in \bigcup_{i=1}^k \mathcal{P}_i$  to denote the amount of [flow](#) that is being routed on path  $P$ , and we want to maximize the total [flow](#) sent between all pairs:

$$\begin{aligned} \max \quad & \sum_{i=1}^k \sum_{P \in \mathcal{P}_{s_i, t_i}} x_P \\ & \sum_{i=1}^k \sum_{\substack{P \in \mathcal{P}_{s_i, t_i} \\ P \ni e}} x_P \leq c(e) \quad \forall e \in E; \\ & x_P \geq 0 \quad P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}. \end{aligned}$$

This is also an [implicit packing linear program](#). We can also write it as an [explicit linear program](#) with variables  $f(e, i)$ , which is the amount of [flow](#) on  $e$  for pair  $i$ :

$$\begin{aligned} \max \quad & \sum_{i=1}^k \left[ \sum_{e \in \delta^+(s_i)} f(e, i) - \sum_{e \in \delta^-(s_i)} f(e, i) \right] \\ & \sum_{i=1}^k f(e, i) \leq c(e) \quad \forall e \in E; \\ & \sum_{e \in \delta^-(u)} f(e, i) = \sum_{e \in \delta^+(u)} f(e, i) \quad u \neq s_i, t_i \text{ for } i \in [k]. \end{aligned}$$

We see that the [flow](#) conservation constraints make this linear program not positive anymore.

As for the [covering linear programs](#), we first introduce the well-known [set cover](#) problem and its special case called [vertex cover](#).

**Problem 4.1.1 (Set cover).** Given a collection of  $n$  subsets  $\mathcal{S} = \{S_i\}_{i=1}^n$  of  $S$  such that each  $S_i \subseteq S$  is of size  $m$  and has a cost  $c_i$ . The *set cover* asks to find a minimum cost sub-collection of  $\mathcal{S}$  whose union is  $S$ .

**Problem 4.1.2 (Vertex cover).** Given a graph  $G = (V, E)$  with edge capacity  $w: E \rightarrow \mathbb{R}_+$ , the *vertex cover* is a special case of [set cover](#) with  $S = E$  and  $\mathcal{S} := \{S_v = \delta(v)\}_{v \in V}$  with unit cost for each  $S_v$ .

**Example (Set cover).** Define an  $m \times n$  matrix  $A$  where  $A_{ij} = 1$  if  $j \in S_i$ , 0 otherwise. Then, the

linear program relaxation for [set cover](#) is of the form

$$\begin{aligned} \min \quad & c^\top x \\ & Ax \geq 1; \\ & x_v \geq 0 \quad \forall v \in V. \end{aligned}$$

This is clearly a [covering linear program](#). It is known that the integrality gap is  $O(\log m)$ , and the integrality gap for [vertex cover](#) is 2.

**Example (Covering integer program).** The *covering integer program* (CIP) generalizes [set cover](#). A CIP is essentially an integer version of a [covering linear program](#):

$$\begin{aligned} \min \quad & c^\top x \\ & Ax \geq b; \\ & x \in \{0, 1\}^n. \end{aligned}$$

We work with the linear program relaxation which is a [covering](#) problem. When  $A, b$  are arbitrary, the integrality gap of the natural linear program for CIP can be as large as  $m$ . One needs to add knapsack cover inequalities to strengthen the linear program and then the linear program becomes much more complex but nevertheless one can solve it fast using the [multiplicative weight update](#) methods. See [CQ19] and references to work on CIPs.

Finally, we see some examples on the [mixed packing and covering linear programs](#).

**Example (Covering integer program with bounded constraints).** One can also add bound constraint on CIPs where a variable  $x_i$  can be at most  $d_i$ . Then we get packing constraints and the linear program becomes a [mixed packing and covering linear program](#), albeit in a simple form

$$\begin{aligned} \min \quad & c^\top x \\ & Ax \geq b; \\ & x \leq d; \\ & x \in \mathbb{Z}_+^n. \end{aligned}$$

This naive formulation is bad in terms of integrality gap; one needs knapsack cover inequalities.

**Example (Maximum concurrent flow).** We have seen the [maximum concurrent flow linear program](#) in the context of the [sparsest cut problem](#). If we guess  $\lambda$ , then it becomes a constant, and we get a [mixed packing and covering linear program](#). To do this, we can do a binary search for  $\lambda$ .

$$\begin{aligned} \max \quad & \lambda \\ & \sum_{P \in \mathcal{P}_{s_i, t_i}} y_P \geq \lambda D_i \quad \forall i = 1, \dots, k; \\ & \sum_{i=1}^k \sum_{\substack{P \in \mathcal{P}_{s_i, t_i} \\ P \ni e}} y_P \leq c(e) \quad \forall e \in E; \\ & y_P \geq 0 \quad \forall P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}; \\ & \lambda \geq 0, \end{aligned}$$

**Example (Densest subgraph).** Another interesting [mixed packing and covering linear program](#) comes up when solving the densest subgraph problem [BGM14].

#### 4.1.4 Known Approximation Results for Explicit Packing Linear Programs

Finally, we list out several results known for explicit positive linear programs. We refer to [Qua19; Wan17] for a good overview. Here, we state a few high-level results for sequential models.

- For **mixed packing and covering linear programs**, there is a **multiplicative weight update** based algorithm that runs in deterministic  $O(N \log N / \epsilon^2)$  time and outputs a  $(1 - \epsilon)$ -approximate feasible solution [You14]. This implies a  $(1 - \epsilon)$ -approximation algorithm for **packing** and **covering** in  $O(N \log N / \epsilon^2)$  time.
- For **packing**, there is a randomized algorithm that yields a  $(1 - \epsilon)$ -approximation feasible solution in  $O(N \log N \log(1/\epsilon)/\epsilon)$  time [AO15]. A similar time bound for **covering** is also known [WRM15].

## 4.2 Multiplicative Weight Update Method

The *multiplicative weight update* (MWU) is a meta-algorithm that has many applications and arises in a number of areas, even though the central analysis is very similar. The survey of Arora, Hazan, and Kale [AHK12] outlines the utility of seeing the general approach. Here, we are interested in applications of MWU to solving positive linear programs. We will follow the ideas and exposition from [CJV15].

**Note.** Even within offline optimization, the use of MWU is varied, and it is not always easy to figure out the similarities and differences even among closely related papers. For example, although one can derive some of these via the expert framework outlines in [AHK12], there are some limitations as well as the overhead of introducing the expert framework.

### 4.2.1 Convex Optimization, Lagrangian Relaxation, and Soft-Max Trick

While our goal is to solve linear programs, we start by considering convex programs.

**Intuition.** It makes some ideas and notations cleaner and more general, and also indicates where we use linearity and positivity when we ant refined results.

Let  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  be a concave function and let  $g_i: \mathbb{R}^n \rightarrow \mathbb{R}$  for all  $i \in [m]$  be a set of convex functions, and let  $P$  be a convex set in  $\mathbb{R}^n$ .<sup>1</sup>

**Note.** We will assume that all functions are continuous and bounded in the domain of interest, and ignore the technical difficulties that arise when considering unbounded functions and domains.

Consider the following optimization problem which generalizes the setting of **packing linear programs**:

$$\begin{aligned} \max \quad & f(x) \\ & g_i(x) \leq 1 \quad \forall i \in [m]; \\ & x \in P. \end{aligned}$$

This is a constrained convex program. Unconstrained convex minimization is itself a non-trivial problem and typically addresses it via gradient descent and other methods. Here, we use it as a modeling tool for explanation purposes and eventually work with linear functions. The first idea is to view the multiple constraints as a single constraint  $h(x) \leq 1$  where  $h(x) = \max_{i \in [m]} g_i(x)$ :

$$\begin{aligned} \max_{\substack{x \in P; \\ \max_{i \in [m]} g_i(x) \leq 1;}} f(x) & \Leftrightarrow \max_{\substack{x \in P; \\ h(x) \leq 1;}} f(x) \end{aligned}$$

From convex analysis, we know that  $h(x) = \max_{i \in [m]} g_i(x)$  is still convex, but not smooth anymore. A heuristic way is to make it smooth by considering the soft-max, a smooth-out version of max. Specifically,

<sup>1</sup>Even though we typically use  $P$  as the non-negative orthant in  $\mathbb{R}^n$ , we can usually restrict the domain to be a box of sufficiently large size in the non-negative orthant. We will come back to this later.



for a parameter  $\eta > 0$ , the soft-max function is defined as

$$g_\eta(x) := \text{soft-max}_\eta(\{g_i(x)\}_{i=1}^m) := \frac{1}{\eta} \ln \left( \sum_{i=1}^m \exp(\eta g_i(x)) \right).$$

Again, from convex analysis,  $g_\eta(x)$  is convex.

**Intuition.** The soft-max function tries to blow-up the maximum value among its argument by some (large) positive number  $\eta$  and applying the exponential function to which. By doing so, the maximum among its arguments will stand out and dominate others.

It turns out that the error incurs by replacing maximum with the soft-max function is small:

**Claim.** For all  $x \in \mathbb{R}^n$ , we have

$$\max_{i \in [m]} g_i(x) \leq g_\eta(x) \leq \max_{i \in [m]} g_i(x) + \frac{\ln m}{\eta}.$$

Then, if we choose  $\eta = \ln m / \epsilon$ , we get a smooth proxy for  $h(x)$  that has an additive error of at most  $\epsilon$ . Specifically, considering

$$\begin{aligned} \max f(x) \\ g_\eta(x) \leq 1; \\ x \in P, \end{aligned}$$

we see that since  $h(x) \leq g_\eta(x) \leq h(x) + \ln m / \eta = h(x) + \epsilon$ , we have  $g_\eta(x) - \epsilon \leq h(x) \leq g_\eta(x)$ .

**Remark.** We can compress multiple constraints into one with a small loss of factor  $(1 + \epsilon)$ .

Another standard approach in constrained continuous (convex and non-linear) optimization is to replace a constrained problem by its Lagrangian relaxation. In the context of the above, we would replace the constrained optimization problem by the following concave optimization problem:

$$\max_{x \in P} f_w(x) := \max_{x \in P} f(x) - \sum_{i=1}^m w_i (g_i(x) - 1),$$

where  $w_1, w_2, \dots, w_m \geq 0$  are Lagrange multipliers.

**Intuition.** We penalize the objective for violating the constraints.

The following weak duality claim is easy to check.

**Claim.** For any non-negative weights  $w_1, w_2, \dots, w_m \geq 0$ ,  $\max_{x \in P} f_w(x) \geq \text{OPT}$  where  $\text{OPT}$  is the optimum value of the original optimization problem. Thus,  $\min_{w \geq 0} \max_{x \in P} f_w(x) \geq \text{OPT}$ .

For convex optimization problems, the duality theorem says that under some mild conditions (e.g., **Slater conditions**), strong duality holds, i.e.,  $\min_{w \geq 0} \max_{x \in P} f_w(x) = \text{OPT}$ .

Although Lagrangian relaxation is nice, it is not obvious how to find good weights that lead to good upper and lower bounds. One can combine the idea of soft-max function and Lagrangian relaxation to reduce the problem to have a single constraint, namely  $g_\eta(x) \leq 0$ . Thus, we want to solve the problem  $\max_{x \in P} f(x)$  such that  $g_\eta(x) \leq 0$ . One can use some gradient descent-like approaches for solving this problem and the gradient of  $g_\eta(x)$  is relevant in this context. We will elaborate on this later.

Finally, motivated by the Lagrangian relaxation approach, we assume that we have a black-box oracle for the following problem for any given non-negative weights  $w_1, w_2, \dots, w_m \geq 0$ :

$$\begin{aligned} \max f(x) \\ \sum_{i=1}^m w_i g_i(x) \leq \sum_{i=1}^m w_i; \\ x \in P. \end{aligned} \tag{4.1}$$

The upshot is the following:

**Remark.** When  $f$  and  $g_i$ 's are positive linear functions and  $P$  is the non-negative orthant, the above **oracle** can be easily obtained.

### 4.2.2 Continuous Time Multiplicative Weight Update Algorithm

We consider an adaptive iterative algorithm that uses a notion of time that goes from 0 to 1.

**Intuition.** We're essentially solving a sequence of Lagrange relaxation and penalize the violated constraints more along the way.

From a discrete point of view, we can think of taking  $T$  steps of size  $\delta = 1/T$  where  $\delta$  is small. Thus, after  $\ell$  steps, the time is  $t = \ell\delta$ . Later, for ease of analysis, we will make this a continuous process.

The algorithm will maintain a set of non-negative weights  $w_i(t)$ , one for each constraint. The non-negative weights represent the relative importance of each constraint at time  $t$ . Initially, we have no information, so all weights are equally important, hence we set  $w_i(0) = 1$ .

**Intuition.** An alternative view is to think of the normalized weights  $w_i(t) / \sum_{k=1}^m w_k(t)$  as a probability distribution on the constraints.

In each step, we will use the oracle to compute a solution  $v(t)$  for the Lagrangian relaxation defined by the weights, i.e.,

$$\begin{aligned} v(t) &:= \arg \max_y f(y) \\ &\quad \sum_{i=1}^m w_i(t) g_i(y) \leq \sum_{i=1}^m w_i(t); \\ &\quad y \in P. \end{aligned}$$

Clearly,  $f(v(t)) \geq \text{OPT}$  since we are solving a relaxation. At the end, we will take an average of convex combination of these solutions and concavity of  $f$  will guarantee that we are doing well on the objective value. We think  $x(t)$  the current solution as  $\int_0^t v(t) dt$  as the accumulation of the solutions computed so far with initial point at  $t = 0$  being  $x(0) = 0$ .

The main question is to deal with the constraint violation. For this, we will maintain the invariant that  $w_i(t) = \exp\left(\eta \int_0^t g_i(v(t)) dt\right)$ , which tries to keep track of the violation of constraint  $i$ .

---

#### Algorithm 4.1: Multiplicative Weight Update Uniform Step

---

**Data:** An objective  $f$ , constraints  $\{g_i\}_{i=1}^m$ , feasible set  $P$ , step size  $\delta$ , parameter  $\eta$

**Result:** A solution  $x_{\text{out}}$

```

1 for  $i = 1, \dots, m$  do
2    $w_i(0) \leftarrow 1$ 
3  $x(0) = 0$  //  $x \in \mathbb{R}^n$ 
4  $t = 0$ 
5
6 while  $t < 1$  do
7    $v(t) \leftarrow \text{Oracle}(f, \{g_i\}_{i=1}^m, \{w_i(t)\}_{i=1}^m, P)$ 
8   for  $i = 1, \dots, m$  do
9      $w_i(t + \delta) \leftarrow w_i(t) \cdot e^{\eta \delta g_i(v(t))}$ 
10   $x(t + \delta) \leftarrow x(t) + \delta v(t)$ 
11   $t \leftarrow t + \delta$ 
12 return  $x(1)$ 

```

---

By taking  $\delta \rightarrow 0$ , we obtain a continuous time algorithm whose analysis can be done simply and is illuminating. Specifically, we will get a differential equation on the evolution of the weights: from

$$w_i(t + \delta) = w_i(t) \exp(\eta \delta g_i(v(t))),$$

we see that

$$\frac{dw_i(t)}{dt} = \lim_{\delta \rightarrow 0} \frac{w_i(t + \delta) - w_i(t)}{\delta} = \lim_{\delta \rightarrow 0} \frac{w_i(t)(\exp(\eta \delta g_i(v(t))) - 1)}{\delta} = w_i(t) \eta g_i(v(t))$$

by using the fact that  $e^x - 1 \rightarrow x$  when  $x \rightarrow 0$ . Finally, we also see that the final return is

$$x_{\text{out}} = \int_0^1 v(t) dt.$$

This results in the following continuous version of [Algorithm 4.1](#).

---

**Algorithm 4.2:** Multiplicative Weight Update Continuous

---

**Data:** An objective  $f$ , constraints  $\{g_i\}_{i=1}^m$ , feasible set  $P$ , parameter  $\eta$

**Result:** A solution  $x_{\text{out}}$

```

1 for  $i = 1, \dots, m$  do
2    $w_i(0) \leftarrow 1$ 
3
4 while  $t = 0, \dots, 1$  continuously do
5    $v(t) \leftarrow \text{Oracle}(f, \{g_i\}_{i=1}^m, \{w_i(t)\}_{i=1}^m, P)$ 
6   for  $i = 1, \dots, m$  do
7      $\text{Evolve } w_i(t) \text{ as } dw_i(t)/dt = w_i(t)\eta g_i(v(t))$ 
8 return  $\int_0^1 v(t) dt$ 

```

---

The following is one reason to choose the time interval to be  $[0, 1]$  to naturally obtain a convex combination of solutions as the output.

**Lemma 4.2.1.** In [Algorithm 4.2](#), if  $f$  is concave, then  $f(x_{\text{out}}) \geq \text{OPT}$ .

**Proof.** From the concavity of  $f$ , we have

$$f(x_{\text{out}}) = f\left(\int_0^1 v(t) dt\right) \geq \int_0^1 f(v(t)) dt \geq \int_0^1 \text{OPT} dt = \text{OPT},$$

where  $f(v(t)) \geq \text{OPT}$  since  $v(t)$  is an optimum solution to a Lagrangian relaxation. ■

We now express the constraint values in terms of the weights.

**Lemma 4.2.2.** In [Algorithm 4.2](#), For each  $i \in [m]$ , we have  $g_i(x_{\text{out}}) \leq \ln w_i(1)/\eta$ .

**Proof.** By concavity, we have

$$\begin{aligned} g_i(x_{\text{out}}) &= g_i\left(\int_0^1 v(t) dt\right) \leq \int_0^1 g_i(v(t)) dt \\ &= \int_0^1 \frac{1}{\eta} \frac{1}{w_i(t)} \frac{dw_i(t)}{dt} dt = \frac{1}{\eta} (\ln w_i(1) - \ln w_i(0)) = \frac{\ln w_i(1)}{\eta}, \end{aligned}$$

where we know that  $dw_i(t)/dt = \eta w_i(t) g_i(v(t))$  in [Algorithm 4.2](#). ■

While we do not have a direct way to bound  $w_i(1)$ , but we can bound  $w_i(1)$  by  $\sum_{i=1}^m w_i(1)$ . This is a very crude bound, but since we will see how the total sum of weights evolves, it turns out to be enough.

**Lemma 4.2.3.** In [Algorithm 4.2](#), for all  $t$ ,  $\sum_{i=1}^m w_i(t) \leq e^{\eta t} \sum_{i=1}^m w_i(0)$ . Hence,  $\sum_{i=1}^m w_i(1) \leq me^{\eta}$ .

**Proof.** Here is where we use the fact that we solve the Lagrangian relaxation. Note that  $v(t)$  satisfies the constraint that  $\sum_{i=1}^m w_i(t) g_i(v(t)) \leq \sum_{i=1}^m w_i(t)$ . Moreover, we have  $dw_i(t)/dt = \eta w_i(t) g_i(v(t))$  for  $i \in [m]$ . Hence,

$$\frac{1}{\eta} \sum_{i=1}^m \frac{dw_i(t)}{dt} \leq \sum_{i=1}^m w_i(t) g_i(v(t)) \leq \sum_{i=1}^m w_i(t).$$

Thus, the total weight  $w(t) := \sum_{i=1}^m w_i(t)$  satisfies the differential equation  $dw(t)/dt \leq \eta w(t)$ , which implies that  $w(t) \leq e^{\eta t} w(0) = me^{\eta}$ . ■

Putting everything together, we have the following.

**Theorem 4.2.1.** The output  $x_{\text{out}}$  of [Algorithm 4.2](#) satisfies the following properties:

- (i)  $f(x_{\text{out}}) \geq \text{OPT}$ ;
- (ii)  $g_i(x_{\text{out}}) \leq 1 + \ln m / \eta$  for all  $i \in [m]$ .

If there is a point  $x_0 \in P$  such that  $g_i(x_0) = 0$  for all  $i \in [m]$ , then  $x'_{\text{out}} = \theta x_{\text{out}} + (1 - \theta)x_0$  is feasible, in that  $g_i(x'_{\text{out}}) \leq 1$  for all  $i \in [m]$  and  $x'_{\text{out}} \in P$ .

**Proof.** Firstly, (i) is shown in [Lemma 4.2.1](#). For (ii), by combining [Lemma 4.2.2](#) and [Lemma 4.2.3](#),

$$g_i(x_{\text{out}}) \leq \frac{1}{\eta} \ln(w_i(1)) \leq \frac{1}{\eta} \ln \left( \sum_{i=1}^m w_i(1) \right) \leq \frac{1}{\eta} \ln(e^\eta m) \leq 1 + \frac{\ln m}{\eta}.$$

The second part is easy to verify. ■

We see that to obtain a  $(1 + \epsilon)$ -approximation in terms of the constraint violation, we can choose  $\eta = \ln m / \epsilon$ . We note that the second part of [Theorem 4.2.1](#) allows us to obtain a feasible solution  $x'_{\text{out}}$  since  $x_{\text{out}}$  from [Algorithm 4.2](#) might be infeasible.

**Intuition.** The impact of using  $x'_{\text{out}}$  instead of  $x_{\text{out}}$  in terms of the objective function  $f$  is less easy to see. However, for non-negative linear objectives, we see that  $f(x_{\text{out}})' \geq \theta f(x_{\text{out}})$ , and thus we get an approximation to the objective while making the solution feasible.

**Remark.** The analysis in [Theorem 4.2.1](#) does not use any property of  $f, g_i$  other than continuity. In particular, it does not use non-negativity of  $g_i$ 's. Hence, the analysis can also be applied to have constraints of the form  $g_i(x) \geq 1$  where  $g_i$  is concave, which is equivalent to  $-g_i(x) + 2 \leq 1$ .

**Note (Potential approach).** Our analysis relied on keeping track of the total weight  $w(t)$  carefully. In some works, the potential function  $\phi(t) = \ln w(t) / \eta$  or proxies for it are used. Potential functions can be powerful tools that can yield strong results though they can sometimes be mysterious [\[BG17\]](#).

## Lecture 18: Discrete Multiplicative Weight Update Algorithm

### 4.2.3 Discrete Time Multiplicative Weight Update Algorithm

24 Oct. 11:00

We want to obtain an algorithm that terminate in a few iterations.

**As previously seen.** Recall that we're working with a convex program

$$\begin{aligned} \max \quad & f(x) \\ & g_i(x) \leq 1 \quad \forall i \in [m] \\ & x \in P, \end{aligned}$$

where  $f$  is concave and  $g_i$ 's are convex. The underlying linear programs we're interested in are

$$\begin{aligned} \max \quad & c^\top x \\ & Ax \leq b \\ & x \geq 0, \end{aligned}$$

where  $c \in \mathbb{R}_+^n$ ,  $A \in \mathbb{R}_+^{m \times n}$ , and  $b \in \mathbb{R}_+^m$ .

For this, we will work with the extra condition that  $g_i(x) \geq 0$  for all  $x \in P$ . This models the [packing](#) constraints of interest and is needed for the algorithms and analysis, unlike the [continuous algorithm](#).

Firstly, consider the simple uniform step size algorithm [Algorithm 4.1](#). For simplicity,  $\delta$  is chosen that there is an integer  $T$  such that  $T\delta = 1$ . Thus, the algorithm runs for  $T$  time steps, and the goal

is to minimize  $T$  such that the discrete implementation ensures that the constraints are approximately satisfied. It turns out that  $T$  is related to a parameter called the **width** of the given instance.

**Definition 4.2.1 (Width).** The *width*  $\rho$  of the given instance is defined as  $\rho := \sup_{x \in P, i \in [m]} g_i(x)$ .

In other words, it is asking how much can some arbitrary point in  $P$  violate a specific constraint in a *relative sense*.<sup>2</sup> Recall that we solve a Lagrangian relaxation in each step and find a point  $v(t)$ , which can violate some specific constraint  $i$  by significant factor since we are only solving the relaxation by taking weighted sum of the constraints. Hence,  $\rho$  allows us to bound the worst-case violation.

It may seem that  $\rho$  is unbounded for even simple instances because  $g_i$  may be an increasing function and  $P$  is typically the non-negative orthant. However, for many problems of interest, there are **implicit** bounds on the variables and  $P$  can be implicitly taken to be a bounding box that restrict  $\rho$ . We will give some examples later on to illustrate this. Suppose for now that  $\rho$  is some bounded number.

**Theorem 4.2.2.** Suppose  $\eta = \ln m/\epsilon$ ,  $\delta = \epsilon/\eta\rho$ , and  $\epsilon \in (0, 1/2)$ . Then in  $O(\rho \ln m/\epsilon^2)$  iterations, **Algorithm 4.1** will output a point  $x_{\text{out}}$  such that

- (i)  $f(x_{\text{out}}) \geq \text{OPT}$ , and
- (ii)  $g_i(x_{\text{out}}) \leq 1 + 2\epsilon$  for all  $i \in [m]$ .

Instead of proving **Theorem 4.2.2**, we consider the non-uniform step size case, which turns out to be stronger and **Theorem 4.2.2** can be reduced to which. Specifically, Garg and Konemann [GK07] obtained a **width**-independent algorithm and analysis by a seemingly simple but important tweak to **Algorithm 4.1**:

**Intuition.** We choose the maximum step size at any point greedily so that the analysis goes through.

In particular, all we need is that we want to find  $\delta$  such that  $w_i(t + \delta) \leq w_i(t) \cdot e^\epsilon$ , i.e.,  $\delta$  is the maximum value such that  $\eta\delta g_i(v(t)) \leq \epsilon$ .

**Note.** The choice of the step size requires  $g_i \geq 0$  over the domain  $P$ , otherwise, it is not well-defined.

Surprisingly, this yields an algorithm that bounds the number of iterations to  $O(m \log m/\epsilon^2)$ , which depends only on the number of constraints and  $\epsilon$  and is **width**-independent. We see the algorithm directly.

---

**Algorithm 4.3:** Multiplicative Weight Update Non-Uniform Step

---

**Data:** An objective  $f$ , constraints  $\{g_i\}_{i=1}^m$ , feasible set  $P$ , parameter  $\eta$

**Result:** A solution  $x_{\text{out}}$

```

1 for  $i = 1, \dots, m$  do
2    $w_i(0) \leftarrow 1$ 
3  $x(0) = 0$  //  $x \in \mathbb{R}^n$ 
4  $t = 0$ 
5
6 while  $t < 1$  do
7    $v(t) \leftarrow \text{Oracle}(f, \{g_i\}_{i=1}^m, \{w_i(t)\}_{i=1}^m, P)$ 
8    $\delta \leftarrow \max \delta'$  such that  $\max_{i \in [m]} \delta' \eta g_i(v(t)) \leq \epsilon$  // Max step size
9    $\delta \leftarrow \min(\delta, 1 - t)$  // Handle the last iteration
10  for  $i = 1, \dots, m$  do
11     $w_i(t + \delta) \leftarrow w_i(t) \cdot e^{\eta\delta g_i(v(t))}$ 
12     $x(t + \delta) \leftarrow x(t) + \delta v(t)$ 
13     $t \leftarrow t + \delta$ 
14 return  $x(1)$ 
```

---

We will mimic the continuous time algorithm (**Algorithm 4.2**) analysis in some ways and point out where the discretization matters and how we bound the errors. First, let's set up some notations.

<sup>2</sup>Since we have normalized the right-hand-side of each constraint to 1 by considering  $g_i(x) \leq 1$  as the constraint.

**Notation.** The algorithm runs for some number  $T$  of iterations, and we use  $j$  to index the iterations and  $i$  as before to index the constraints. We think of the time before the start of iteration  $j + 1$  as  $t_j$ , and the step size in that iteration as  $\delta_j$ . We have  $t_0 = 0$  and  $t_{j+1} = t_j + \delta_j$  for  $j = 0, 1, \dots, T-1$  such that  $\sum_{j=0}^{T-1} \delta_j = 1$ . Note that in iteration  $j + 1$ , we compute  $v(t_j)$  and  $\delta_j$  w.r.t. weights  $w_i(t_j)$ 's. In other words,  $t_{j+1}$  is the time at the end of iteration  $j + 1$ .

The following is easy to prove even with a discrete step size (compared to [Lemma 4.2.1](#)).

**Lemma 4.2.4.** In [Algorithm 4.3](#), if  $f$  is concave, then  $f(x_{\text{out}}) = f(x(1)) \geq \text{OPT}$ .

**Proof.** From the definition and concavity of  $f$ , we have

$$f(x_{\text{out}}) = f\left(\sum_{j=0}^{T-1} \delta_j v(t_j)\right) \geq \sum_{j=0}^{T-1} \delta_j f(v(t_j)) \geq \sum_{j=0}^{T-1} \delta_j \text{OPT} \geq \text{OPT},$$

where we again use the fact that  $v(t_j)$  is the solution of the Lagrangian relaxation.  $\blacksquare$

A similar claim to [Lemma 4.2.2](#) still holds in the discrete setting, but the proof is useful to see.

**Lemma 4.2.5.** In [Algorithm 4.3](#), for each  $i \in [m]$ , we have  $g_i(x_{\text{out}}) \leq \ln w_i(1)/\eta$ .

**Proof.** We have  $x_{\text{out}} = \sum_{j=0}^{T-1} \delta_j v(t_j)$ , and hence by convexity of  $g_i$ ,  $g_i(x_{\text{out}}) \leq \sum_{j=0}^{T-1} \delta_j g_i(v(t_j))$ . We see that in iteration  $j + 1$ , the weight of constraint  $i$  is updated as

$$w_i(t_{j+1}) = w_i(t_j) \exp(\eta \delta_j g_i(v(t_j))) \Rightarrow \delta_j g_i(v(t_j)) = \frac{\ln w_i(t_{j+1}) - \ln w_i(t_j)}{\eta}.$$

Summing up two sides over  $j$  and using telescoping, we have

$$g_i(x_{\text{out}}) \leq \sum_{j=0}^{T-1} \delta_j g_i(v(t_j)) \leq \frac{\ln w_i(1) - \ln w_i(0)}{\eta},$$

with  $w_i(0) = 1$ , we're done.  $\blacksquare$

The main technical lemma is the following, which bounds the evolution of the total weight and this is where the choice of the step size is crucial.

**Lemma 4.2.6.** In [Algorithm 4.3](#), for all  $t_j$ ,  $\sum_{i=1}^m w_i(t_j) \leq e^{(1+\epsilon)\eta t_j} \sum_{i=1}^m w_i(0)$ . Hence,  $\sum_{i=1}^m w_i(1) \leq m e^{(1+\epsilon)\eta}$ .

**Proof.** Consider iteration  $j + 1$ . Since  $g_i$ 's are all positive on the domain, all numbers involved are positive. We see that

$$\sum_{i=1}^m w_i(t_j + \delta_j) = \sum_{i=1}^m w_i(t_j) \exp(\eta \delta_j g_i(v(t_j))),$$

where we ensure that  $\eta \delta_j g_i(v(t_j)) \leq \epsilon \leq 1/2$  in [Algorithm 4.3](#). Observe that it suffices to prove

$$\sum_{i=1}^m w_i(t_j + \delta_j) \leq e^{(1+\epsilon)\delta_j \eta} \left( \sum_{i=1}^m w_i(t_j) \right).$$

**Claim.** If  $a \in (0, 1/2)$ , we have  $e^a \leq 1 + a + a^2$ .

**Proof.** This is immediate from Taylor's expansion.  $\otimes$

Hence, by writing  $\eta\delta_j g_i(v(t_j)) = a_i$ , we have

$$\begin{aligned}
 \sum_{i=1}^m w_i(t_j) \exp(\eta\delta_j g_i(v(t_j))) &\leq \sum_{i=1}^m w_i(t_j) \exp(1 + a_i + a_i^2) \\
 &\leq \sum_{i=1}^m w_i(t_j) (1 + a_i(1 + \epsilon)) \\
 &\leq \sum_{i=1}^m w_i(t_j) + (1 + \epsilon)\eta\delta_j \underbrace{\sum_{i=1}^m w_i(t_j) g_i(v(t_j))}_{\leq \sum_{i=1}^m w_i(t_j)} \\
 &\leq \left( \sum_{i=1}^m w_i(t_j) \right) (1 + (1 + \epsilon)\eta\delta_j) \leq \left( \sum_{i=1}^m w_i(t_j) \right) \exp((1 + \epsilon)\eta\delta_j)
 \end{aligned}$$

since  $1 + b \leq e^b$  for all  $b \geq 0$ . Hence, we're done.  $\blacksquare$

**Remark.** The proof of [Lemma 4.2.6](#) will go through easily for a fixed step size as long as the step size guarantees that no weight of a constraint goes up by more than an  $e^\epsilon$  factor. This is satisfied by the choice of the step size based on the [width](#)! The rest of the analysis is the same.

**Theorem 4.2.3.** If  $\eta \geq \ln m / \epsilon$ , then the output  $x_{\text{out}}$  of [Algorithm 4.3](#) satisfies the following:

- (i)  $f(x_{\text{out}}) \geq \text{OPT}$ ;
- (ii)  $g_i(x_{\text{out}}) \leq 1 + 2\epsilon$  for all  $i \in [m]$ .

Moreover, [Algorithm 4.3](#) terminates in  $O(m \log m / \epsilon^2)$  iterations.

**Proof.** Firstly, (i) is shown in [Theorem 4.2.3](#). For (ii), from [Lemma 4.2.5](#), we have  $g_i(x_{\text{out}}) \leq \ln w_i(1) / \eta$  and that  $\ln \sum_{i=1}^m w_i(1) \leq (1 + \epsilon)\eta \ln m$  from [Lemma 4.2.6](#). This implies

$$g_i(x_{\text{out}}) \leq \frac{1}{\eta} \ln(w_i(1)) \leq \frac{1}{\eta} \ln \left( \sum_{i=1}^m w_i(1) \right) \leq \frac{1}{\eta} \ln(m e^{(1+\epsilon)\eta}) = 1 + \epsilon + \frac{\ln m}{\eta} \leq 1 + 2\epsilon.$$

Next, we switch to exponential weights instead of reasoning with logarithms since it will be useful later. We will also use a loose argument though one can do a tighter analysis. Let  $T$  be the total number of iterations [Algorithm 4.3](#) takes.

**Claim.** For some fixed small constant  $c$ ,  $T \leq cm \ln m / \epsilon^2$ .

**Proof.** Firstly, in the end, the total weight  $\sum_{i=1}^m w_i(1)$  is

$$e^{(1+\epsilon)\eta m} = m \cdot e^{(1+\epsilon)(\ln m)/\epsilon} = m^{1 + \frac{1+\epsilon}{\epsilon}} = m^{O(1/\epsilon)}.$$

Observe that since we're being greedy, in each iteration we choose  $\delta_j$  to be the maximum such that there is some  $i$  in that iteration with  $\eta\delta_j g_i(v(t_j)) = \epsilon$  exactly. But this implies that for that  $i$ , its weight  $w_i$  increases by a factor of  $e^\epsilon$ . Since each weight starts at 1, with the fact that the total weight at the end is  $m^{O(1/\epsilon)}$ , no weight  $w_i$  can have its weight updated by an  $e^\epsilon$  factor more than  $O(\log m / \epsilon^2)$  times. But there are only  $m$  constraints and each iteration must increase at least one constraint's weight by a factor of  $e^\epsilon$ , thus, we cannot have  $T$  more than  $cm \log m / \epsilon^2$  for some small but fixed constant  $c$ .  $\otimes$

This finishes the proof.  $\blacksquare$



#### 4.2.4 Scaling Weights

In the analysis so far, we have used constraints of the form  $g_i(x) \leq 1$  which makes the notation clean. Of course, any constraint of the form  $g_i(x) \leq b_i$  with  $b_i > 0$  can be scaled to achieve the standard form. However, in some combinatorial applications, when  $A$  the packing matrix corresponds to an incidence matrix of a combinatorial object and  $b_i$  represents some capacity, scaling makes the matrix unnatural.

We can avoid this by modifying [Algorithm 4.3](#) as follows. Suppose we have constraints of the form  $g_i(x) \leq b_i$  for all  $i \in [m]$  where  $b > 0$ . We start with  $w_i(0) = 1/b_i$  for each  $i$ . Then, we do the analysis with  $b_i w_i$  instead of  $w_i$ . Since the weights are updated in a multiplicative fashion, the whole analysis goes through cleanly.

**Note.** With weights set up like this, we still solve the [oracle](#) in each iteration with the constraint  $\sum_{i=1}^m w_i g_i(y) \leq \sum_{i=1}^m w_i$ ; the  $b_i$ 's are taken care of automatically.

### 4.3 Application to Packing Linear Program

We show how the generic scheme that we have described via [MWU](#) can be used to derive fast approximation algorithms for [packing linear programs](#). Recall that the general [packing linear program](#) looks like

$$\begin{aligned} \max \quad & c^\top x \\ & Ax \leq b; \\ & x \geq 0. \end{aligned}$$

In terms of the generic framework, we have  $f(x) = c^\top x$  and  $g_i(x) = \sum_{j=1}^n A_{ij} x_j$  corresponds to the  $i^{\text{th}}$  row of  $A$ , and  $P$  corresponds to  $\mathbb{R}_+^n$ .

**Note.** In some settings, we think of  $P$  as a bounding box implied by constraints with  $0 \leq x_j \leq u_j$  where  $u_j$  is an upper bound on  $x_j$  implicitly implied by the constraints in  $Ax \leq b$ . This often helps in figuring out the [width](#) of the system.

Firstly, we can interpret the weights as exponential in loads.

**Intuition.** In the [continuous MWU method](#), we are maintaining  $w_i(t)$  as  $\exp\left(\eta \int_0^t g_i(v_t) dt\right)$ . When  $g_i$  is linear and  $P$  is the non-negative orthant,  $\int_0^t g_i(v_t) dt$  is simply  $\sum_{j=1}^n A_{ij} x_j(t)$ , where  $x(t)$  is the current accumulated vector. We think of this as the total load  $\ell_i(t)$  on constraint  $i$  at time  $t$ , and hence  $w_i(t) = \exp(\eta \ell_i(t))$ .

Next, we note that in the case of linear objective, it is possible to transform between the approximation guarantees for constraints ([Theorem 4.2.3](#)) and objective value.

**Remark.** The [MWU method](#), as described, naturally give an optimum solution that violates the constraints. We can then scale the solution down to obtain a feasible solution while losing a bit in the objective. This is particularly easy for [packing](#) problems with linear objectives. Thus, we can obtain a  $(1 - \epsilon)$ -approximation in the objective while obtaining a feasible solution.

#### 4.3.1 Efficient Oracle for Positive Packing Linear Program

Recall that we need an [oracle](#) that given weights  $w_i$ 's, solve

$$\begin{aligned} \max \quad & f(x) \\ & \sum_{i=1}^m w_i g_i(x) \leq \sum_{i=1}^m w_i; \\ & x \geq 0, \end{aligned}$$

In the case of [packing linear programs](#), consider the normalized linear program with constraints  $Ax \leq 1$ , then the problem becomes

$$\begin{aligned} \max \sum_{j=1}^n c_j x_j \quad & \sum_{i=1}^m w_i \sum_{j=1}^n A_{ij} x_j \leq \sum_{i=1}^m w_i \Leftrightarrow \quad \max \sum_{j=1}^n c_j x_j \quad & \sum_{j=1}^n \left( \sum_{i=1}^m w_i A_{ij} \right) x_j \leq \sum_{i=1}^m w_i =: \quad \max \sum_{j=1}^n c_j x_j \\ & x \geq 0; \quad & x \geq 0; \quad & \sum_{j=1}^n \alpha_j x_j \leq 1 \\ & & & x \geq 0, \end{aligned} \quad (4.2)$$

where

$$\alpha_j := \frac{1}{\sum_{i=1}^m w_i} \sum_{i=1}^m w_i A_{ij} \geq 0$$

since all entries are positive.

**Intuition.** This is a fractional knapsack problem! We know that the optimum solution  $x^*$  is just to pick the coordinate  $j^* := \arg \max_{j \in [n]} c_j / \alpha_j$  and setting  $x_{j^*}^* := 1 / \alpha_{j^*}$  and the rest to 0.

We summarize the [MWU algorithm](#) for the case of positive [packing linear program](#) as follows.

---

**Algorithm 4.4:** Multiplicative Weight Update for Positive [Packing Linear Program](#)

---

**Data:** A cost vector  $c \in \mathbb{R}_+^n$ , constraint matrix-vector pair  $A \in \mathbb{R}_+^{m \times n}$ ,  $b \in \mathbb{R}_+^m$ , error  $\epsilon$

**Result:** A solution  $x_{\text{out}}$

```

1  $\eta \leftarrow \log m / \epsilon$ 
2
3 for  $i = 1, \dots, m$  do
4    $w_i(0) \leftarrow 1 / b_i$ 
5  $x(0) = 0$  //  $x \in \mathbb{R}^n$ 
6  $t = 0$ 
7
8 while  $t < 1$  do
9    $j^* \leftarrow \arg \max_{j \in [n]} c_j / (\sum_{i=1}^m A_{ij} w_i)$ 
10   $\delta \leftarrow \epsilon / (\eta \max_{i \in [m]} A_{ij^*})$  // Max step size
11   $\delta \leftarrow \min(\delta, 1 - t)$  // Handle the last iteration
12  for  $i = 1, \dots, m$  do
13     $w_i(t + \delta) \leftarrow w_i(t) \cdot e^{\eta \delta A_{ij^*}}$ 
14   $x(t + \delta) \leftarrow x(t) + \delta e_{j^*}$ 
15   $t \leftarrow t + \delta$ 
16 return  $x(1)$ 

```

---

We note that an important aspect of  $j^*$  is that there is an optimum solution  $x^*$  to the [oracle](#) whose support is a *single coordinate*.

**As previously seen.** The [width-independent analysis](#) ([Theorem 4.2.3](#)) shows that the number of iterations depends only on the number of constraints  $m$ .

Thus, we can apply the method to [implicit linear programs](#) with exponential (in some original problem size) number of variables but only a polynomial number of constraints as long as we can implement the [oracle](#) efficiently. Due to this structure, the sequence of [oracles](#) can often be maintained dynamically.

**Remark (Maintaining the oracle).** We see that at each time step  $t$ , we solve for  $v(t) = x^*$ , which is simply  $1 / \alpha_{j^*}$  at the  $j^*$ <sup>th</sup> coordinate and 0 otherwise. In the next round for time step  $t + \delta$ , only those  $w_i$ 's such that  $g_i(v(t)) > 0$  needs to be updated, i.e., for those  $i$  such that  $\sum_{j=1}^n A_{ij} x_j^* \neq 0$ . Since  $A$  is positive, this is equivalent to those  $i$  such that  $A_{ij^*} > 0$ .

Finally, another nice aspect of the [MWU method](#) is that it accumulates a series of solutions of Lagrangian relaxations by taking non-negative sums. In the positive settings, this means that we do not

use subtraction at all. This is convenient for approximation: Suppose we only have an  $\alpha$ -approximation [oracle](#) in the relative sense. Then, the process yields a  $(1-\epsilon)\alpha$ -approximation solution. This is convenient not only in obtaining approximate solutions, but also in speeding up [MWU method](#)-based algorithms substantially by using various tricks and data structures that can avoid updating information at every step since we are allowed to use an approximate [oracle](#).

**Note.** If  $\alpha = (1 - \epsilon)$ , then  $(1 - \epsilon)\alpha \geq 1 - 2\epsilon$ .

**Remark (Explicit packing linear program).** From [Theorem 4.2.3](#), the algorithm takes  $O(m \log m / \epsilon^2)$  iterations. The [oracle](#) for each iteration can be done in  $O(N)$  time easily and naively. Thus, we can obtain a  $(1 - \epsilon)$ -approximate solution in  $O(mN \log m / \epsilon^2)$  time. By being slightly careful and using a simple approximate [oracle](#), we can reduce the running time to  $O(N \log m / \epsilon^2)$ , which is near-linear.

## Lecture 19: Make Multiplicative Weight Update Faster

### 4.3.2 Examples

31 Oct. 11:00

It is instructive to see what the [oracle](#) corresponds to when considering combinatorial problems. Hence, in this section, we continue on the previous examples we have seen for the [packing linear programs](#), and see what are their corresponding [MWU oracle](#).

**Example (Maximum Weight Matching).** Continue from the [previous example](#), where we consider the maximum weight matching problem in bipartite graphs with the following linear program:

$$\begin{aligned} \max \quad & \sum_{e \in E} c(e)x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(u)} x_e \leq 1 \quad \forall u \in V; \\ & x_e \geq 0 \quad \forall e \in E, \end{aligned}$$

where the variables  $x_e$  correspond to the edges and the constraints correspond to the vertices. Thus, the [MWU method](#) maintains a weight  $w_u$  for each  $u \in V$ . The [oracle](#) is that given vertex weights  $w_u$ , it finds the edge  $e = uv \in E$  that maximizes  $c_e / (w_u + w_v)$  in each step.

**Intuition.** Updating the weights is simple and fast because we are only touching two vertex weights. Hence, the bottleneck is finding the edge with the maximum ratio in each iteration.

We can explore simple tricks that will help speed this up. First, we bucket edges into logarithmic groups by considering weights in powers of  $(1+\epsilon)$ ; this affects the approximation factor only a  $(1-\epsilon)$ -factor. Within each bucket, we are trying to find the weight with the minimum weight where the weight of an edge is the sum of the weights of its end points. We can keep a priority queue for edges based on their weight. How do we update these weights? Suppose in iteration  $j$  of the algorithm, we pick edge  $e_j = (u, v)$ . Then the algorithm updates the weights of edges that are incident to  $u$  and  $v$  and these will affect the priority queues. It is  $\deg(u) + \deg(v)$ . This seems like a lot, but we make a crucial observation.

**As previously seen.** The [width-independent MWU analysis](#) proves that the weight of any specific constraint  $i$  is updated only  $O(\log m / \epsilon^2)$  times!

In particular, the weight of a vertex  $u$  is updated only  $O(\log m / \epsilon^2)$  times, where  $m = |V|$ . Hence, the total number of updates is only  $O(\sum_{u \in V} \deg(u) \log m / \epsilon^2) = O(|E| \log |V| / \epsilon^2)$ . The priority queue operations have only a logarithmic overhead. Thus, we can see that with some basic observations we can implement the [MWU algorithm](#) in  $\tilde{O}(|E| / \epsilon^2)$  time.

**Example (Tree packing).** Continue from the [previous example](#), where we consider the tree packing problem with the following linear program:

$$\begin{aligned} \max \quad & \sum_{T \in \mathcal{T}_G} y_T \\ \sum_{T \ni e} y_T & \leq c(e) \quad \forall e \in E; \\ y_T & \geq 0 \quad \forall T \in \mathcal{T}_G. \end{aligned}$$

This is an [implicit linear program](#) with an exponential number of variables but only  $m$  constraints where  $m = |E|$ . Note that the coefficient of each variable  $y_T$  in the objective is 1. In this example, the right-hand side of the constraint corresponding to edge  $e$  is its capacity  $c(e)$  and these are non-uniform. It now makes sense to use weights that start with  $w_e = 1/c(e)$ .

The [oracle](#) is that given weights  $w_e$  on edges, we wish to find a tree  $T$  that maximizes  $\frac{1}{\sum_{e \in T} w_e}$ , i.e.,  $\min_{T \in \mathcal{T}_G} \sum_{e \in T} w_e$ . This is just the [MST](#) problem, which is the separation oracle for the [dual tree-packing linear program](#)! Thus, each iteration, we simply solve an [MST](#) problem and update the weights of the edges in the chosen [MST](#)  $T^*$ , which takes  $O(m + n)$  time from [Theorem 1.1.6](#). There are only  $O(m \log m / \epsilon^2)$  iterations, hence the total time is  $O(m^2 \log m / \epsilon^2)$ .

**Example (Maximum multi-commodity flow).** Continue from the [previous example](#), where we consider the maximum multi-commodity [flow](#) with the following linear program:

$$\begin{aligned} \max \quad & \sum_{i=1}^k \sum_{P \in \mathcal{P}_{s_i, t_i}} x_P \\ \sum_{i=1}^k \sum_{\substack{P \in \mathcal{P}_{s_i, t_i} \\ P \ni e}} x_P & \leq c(e) \quad \forall e \in E; \\ x_P & \geq 0 \quad P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}. \end{aligned}$$

Once again, we have an [implicit packing linear program](#) with an exponential number of variables corresponding to the paths for all the commodity pairs, and the number of constraints is  $m$ , corresponding to the edges. The weights correspond to exponential of the loads on the edges where the load on an edge is the total [flow](#) routed so far on the edge. The [oracle](#) explicitly in this case is the same as  $\max_{P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}} 1 / \sum_{e \in P} w_e$ , i.e., finding the shortest path among all the commodity pairs, according to the weights on edges! We can find for each pair  $(s_i, t_i)$  a shortest  $s_i$ - $t_i$  path and choose the minimum. The algorithm routes some [flow](#) on that path and updates the weights along that path and iterates. The bottleneck is to compute the shortest paths.

## 4.4 Speed Up Multiplicative Weight Update

Finally, we discuss how to combine what we have seen to some specific example of positive [packing linear program](#) problem to further speed up [Algorithm 4.4](#).

**As previously seen.** Naively, each iteration in [Algorithm 4.4](#) requires  $O(N)$ , hence the overall runtime is  $O(Nm \log m / \epsilon^2)$  with [Theorem 4.2.3](#).

### 4.4.1 General Idea

Without loss of generality, we may assume that  $c_j = 1$  for all  $j$  via scaling. Hence, solving  $j^*$  is equivalent to solving  $\min_{j \in [n]} \sum_{i=1}^m A_{ij^*} w_i$ . Let  $\lambda_j := \sum_{i=1}^m A_{ij} w_i$ . The idea of speeding up [Algorithm 4.4](#) is the following:

**Intuition.** Instead of solving  $j^*$  exactly, we allow to return some coordinate such that the returned  $\lambda_j$  is not too far from the optimal value for the particular problem at hand.

**Example.** Consider divide values into  $(1 + \epsilon)$ -buckets.

#### 4.4.2 Case Study: Tree Packing

As an example, consider the [tree packing](#) again:

$$\begin{aligned} \max \quad & \sum_{T \in \mathcal{T}_G} x_T \\ & \sum_{T \ni e} x_T \leq c(e) \quad \forall e \in E; \\ & x_T \geq 0 \quad \forall T \in \mathcal{T}_G. \end{aligned}$$

As we have seen, the optimal solution  $j^*$  corresponds to the [MST](#)  $T^*$  w.r.t. weights  $w_e$  maintained by [Algorithm 4.4](#), and the total algorithm takes  $O(m^2 \log m / \epsilon^2)$  from the [previous example](#). To speed up the algorithm, as we have hinted on, we avoid computing [MST](#) from scratch; instead, we maintain [MST](#) via dynamic data structure. The classical dynamic algorithm for maintaining [MST](#) takes  $\text{poly log } n$  per edge weight change [[HDT01](#)] (specifically,  $O(\log^2 n)$ ), hence the total time is

$$O\left(\frac{m \log m}{\epsilon^2} \cdot (n \text{poly log } n + n)\right) = O(mn \text{poly log } n / \epsilon^2)$$

if we implement this naively, i.e., we update  $n - 1$  edge weights at the end of each iteration and also maintain the corresponding new [MST](#), which takes  $O(n) + n \cdot \text{poly log } n$ .

Compared to the original runtime,  $O(m^2 \log m / \epsilon^2)$ , this is not that much of an improvement. In particular, as we have seen in the very beginning of the class, we know that it is possible to get a near-linear time algorithm:

**Theorem 4.4.1** ([Theorem 1.2.2 \[CQ17\]](#)). There is a deterministic  $O(m \log^3 n / \epsilon^2)$ -time algorithm that gives a  $(1 - \epsilon)$ -approximation for fractional [tree packing](#) in a capacitated undirected graph with  $m$  edges. Hence, a  $(1 - \epsilon)$ -approximate [packing](#) can be described in  $\tilde{O}(m / \epsilon^2)$  space.

The key idea of [Theorem 4.4.1](#) is twofold. Firstly, as suggested before, we can delay the update of edge weight  $w_e$  for *only* computing [MST](#):

**Intuition (Lazy update).** Report the update to the [MST](#) dynamic structure [[HDT01](#)] only if weight increases by  $(1 + \epsilon)$  factor. Hence, now [MST](#) is approximate.

Specifically, we can update weight lazily if it does not change much, say, less than an  $e^\epsilon \approx 1 + \epsilon$  factor. From [Theorem 4.2.3](#), the total number of updates we will report to the dynamic structure is only  $O(m \log m / \epsilon^2)$ . Hence, now, the total runtime becomes

$$O\left(\frac{m \log m}{\epsilon^2} \cdot (\text{poly log } n + n)\right) = O(mn \text{poly log } n / \epsilon^2)$$

due to the fact that we still maintain every actual (ground-truth) edge weight update after each iteration.

**Remark.** To get rid of  $n$  entirely, we cannot afford to update  $w_e$  exactly after each iteration, even if we only report the updates lazily.

The second idea is that on top of the lazy update to the dynamic [MST](#) data structure, we also update our ground-truth weight lazily if it does not change much:

**Intuition.** Maintain edge weight  $w_e$  within a  $(1 \pm \epsilon)$  multiplicative factor throughout.

It turns out that there exists such a data structure that can handle this type of lazy update as well [KY14; You14], but it is designed for **explicit** problems. By some adaption, the idea is then to combine it with the previous dynamic **MST** data structures [HDT01], which yields an amortized runtime throughout the algorithm of  $O(\text{poly log } n)$  per iteration, giving the desired near-linear runtime.

The high-level intuition of this adaption is the following.

**Intuition.** When a tree  $T$  is updated, its rate of change of  $w_e$  depends on  $c(e)$ :

- if all  $c(e)$  values are *uniform*, then  $n - 1$  edges increase weight by  $(1 + \epsilon)$  factor;
- if  $c(e)$  values are *non-uniform*, delay updating small weight changes.

The problem is how to handle the second case. If we allow randomization, then the idea is simply to touch (update)  $w_e$  *in proportion to*  $1/c(e)$ . Specifically, when  $T$  is the tree added in an iteration, let  $c_{\min} := \min_{e \in T} c(e)$  be the minimum edge capacity in  $T$ . Then, for each edge  $e \in E(T)$ , we update  $w_e$  as

$$w_e \leftarrow \begin{cases} \exp(\theta)w_e, & \text{if } \theta \leq c_{\min}/c(e); \\ w_e, & \text{otherwise,} \end{cases}$$

where  $\theta \sim \mathcal{U}([0, 1])$ . In expectation, this update is correct, although these updates are clearly correlated. The benefit of this is that these updates are easy to implement via data structures due to the correlation, with the catch being we need to prove that MWU analysis works with correlated rounding. This is done by the so-called *drift analysis* [KY14].

**Remark (Deterministic approach).** If we insist on a deterministic algorithm, the idea is to bucket  $c(e)$  values geometrically and lazily update using amortization. However, this is somewhat involved to describe, and it takes advantage of  $\{0, 1\}$  structure of  $A$ .

## Chapter 5

# Cut-Matching for Fast Sparsest Cut

### Lecture 20: Sparsest Cuts via $s$ - $t$ Flow

In this chapter, we show that one can approximate [expansion](#) to within an  $O(\log^2 n)$  factor via  $O(\log^2 n)$  calls to an approximate single-commodity  $s$ - $t$  flow routine. Since we have fairly reliable  $s$ - $t$  flow algorithms, both in practice and in theory, this gives us a fast approximation algorithm for computing [expansion](#) and other problems such as [uniform sparsest cut](#) and [conductance](#).

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**Note.** We now have near-linear time algorithms for a *constant factor approximation* to  $s$ - $t$  flow in undirected graphs [Pen16], and an almost-linear time *exact* algorithm for  $s$ - $t$  flow in directed graphs [Che+22].

## 5.1 Recap: Algorithm for Sparsest Cut

Previously, we have discussed how to check if a graph  $G$  is an [expander](#); more generally, how to estimate the [expansion](#), [conductance](#), and [uniform sparsest cut](#).<sup>1</sup>

**As previously seen.** There are three approximation algorithms that we have discussed or mentioned:

- Multi-commodity flow [LR99]:  $O(\log n)$ -approximation ([Theorem 2.3.4](#)).
- SDP relaxation [ARV09]:  $O(\sqrt{\log n})$ -approximation ([previous problem](#)).
- Spectral method:  $O(\sqrt{\text{OPT}})$ -approximation for [conductance](#) ([Theorem 2.4.1](#)).

The spectral method is very fast but does not yield good approximation in the regime when the [conductance](#) is low. The other two methods are based on solving slow (although polynomial) mathematical programming relaxation (e.g., [Equation 2.4](#)). While we have discussed the [multiplicative weight update](#)-based methods for fast solutions to linear programs, and one of the motivations for these algorithms was to speed up the maximum concurrent flow linear program for [uniform sparsest cut](#). However, even with several ideas, the fastest way we know how to solve the linear programs takes  $O(mn)$  time for a constant factor approximation.

## 5.2 Cut-Matching Game

The algorithm that achieves  $O(\log^2 n)$ -approximation for [expansion](#) using only  $O(\log^2 n)$  many  $s$ - $t$ -max-flow computations is based on the so-called [cut-matching game](#) framework [KRV09], which has since become a powerful tool in several applications. The framework has been further improved to yield an  $O(\log n)$ -approximation [Ore+08]. On another line of work, a more systematic approach via matrix-multiplicative weight updates obtains fast primal-dual algorithms that yield  $O(\log n)$  and  $O(\sqrt{\log n})$ -approximation algorithms, which also use appropriate flow sub-routines [AK07].

<sup>1</sup>The [expansion](#) is within a factor of 2 of the [sparsity](#) from the previous [remark](#).



### 5.2.1 Sparse Cut and Expanding Matching

For the sake of discussion, let us consider the decision version in terms of [expansion](#):

**Problem 5.2.1 (Expansion).** Given a graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$  and a parameter  $\varphi$ , decide if the [expansion](#) of  $G$  is at least  $\varphi/\alpha$  for some approximation factor  $\alpha \geq 1$ . If not, then find a corresponding [sparse cut](#), i.e.,  $S \subseteq V$  such that  $|\delta(S)| \leq \varphi \cdot \min(|S|, |V \setminus S|)$ .

**Note.** We assume that  $\varphi = 1$  since we can scale  $c$  correspondingly by  $1/\varphi$ .

**Remark.** If one can solve [expansion](#), then binary search can find an approximate [sparsest cut](#).

To motivate the [cut-matching game](#), we recall the following.

**As previously seen.** Recall that we have a “dual” notion of [expansion](#), i.e., [well-linked](#), from [Corollary 2.4.2](#) and [Lemma 2.4.3](#).

We further note that one can check if  $A, B$  are [linked](#) by an  $s$ - $t$  flow computation such that a [linkage](#) can be viewed as creating a perfect matching between  $A$  and  $B$  where the edges of the matching correspond to the paths in the  $A$ - $B$  linkage.

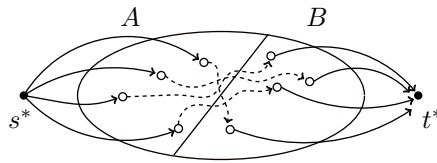


Figure 5.1: Adding a super source  $s^*$  to every  $s \in A$  and a super sink from  $t \in B$  to  $t^*$ .

From [Figure 5.1](#), it's clear that the [flow](#) paths of integral  $s^*$ - $t^*$  flow will implicitly match up vertices in  $A$  to vertices in  $B$ , hence inducing a perfect matching  $M_i$  if such a feasible [flow](#) exist.

**Problem.** What is the advantage of the notion of [well-linkedness](#) in this context?

**Answer.** To prove that  $G$  is not an [expander](#), it suffices to exhibit an equal-sized bipartition  $(S, V \setminus S)$  with  $|S| = n/2^a$  and one can check via an  $s^*$ - $t^*$  flow computation if  $S, V \setminus S$  are not [linked](#). Of course, it is much easier to exhibit a cut  $S$  which is not expanding. ⊗

<sup>a</sup>For simplicity, let  $n$  be even throughout this section.

On the other hand, if the  $s^*$ - $t^*$  flow does perfectly match  $S$  to  $V \setminus S$ , then this only says that  $G$  is not necessarily not an [expander](#). How to proceed?

**Intuition.** We think of the problem as generating a certificate for the [expansion](#) by generating a series of partition of  $V$ .

One idea is to keep selecting different sets  $S_i \subseteq V$ , each of size  $n/2$ , and using the corresponding  $s_i^*$ - $t_i^*$  max-flow  $f_{s_i^*, t_i^*}$  to either route a perfect matching  $M_i$  between  $S_i$  and  $V \setminus S_i$ , or prove that  $G$  has [expansion](#) at most a constant.

**Intuition.** If these matchings behave like *uniformly random matchings*, then after a constant number  $k$  (e.g., 3) of sets  $S_1, \dots, S_k$ , we will have embedded a random [expander](#) with [congestion](#)  $k$  into  $G$ , certifying that  $G$  has [expansion](#) at least  $1/k$  (e.g., [Lemma 2.4.1](#)).

The problem becomes that, since we treat  $s_i^*$ - $t_i^*$  flow computation as a black-box, and there is no reason that the matchings should behave like random matchings. The upshot is that Khandekar, Rao,

and Vazirani [KRV09] show how to adaptively select sets  $S_1, \dots, S_k$ , where  $S_i$  depends on the matchings  $M_1, \dots, M_{i-1}$  returned in rounds 1 through  $i-1$  such that with high probability, after  $k = O(\log^2 n)$  rounds, the multigraph  $H = (V, \bigcup_{i=1}^k M_i)$  induced by the union of  $M_1, \dots, M_k$ , is a **1-expander** by showing that  $H$  can route uniform multi-commodity flow: a demand of  $1/n$  between each ordered pair of vertices  $(u, v)$ , which suffice for  $H$  being a **1-expander**.

**Remark.** The above means that the **1-expander**  $H$  can be embedded with **congestion**  $O(\log^2 n)$  since each  $M_i$  is **routable** in  $G$ , implying that  $G$  has **expansion** at least  $1/k = \Omega(1/\log^2 n)$ .

Our goal is then to minimize  $k$  while ensuring that  $H$  is an **expander** without actually manually checking it. Other methods use different ways to certify that  $H$  is a **1-expander**.

**Example.** One can also use spectral methods via **Cheeger's inequality**.

### 5.2.2 Cut-Matching Game and Results

We now describe the **cut-matching game** formally.

**Definition 5.2.1 (Cut-matching game).** Given a graph  $G$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , the **cut-matching game** proceeds in iterations. In iteration  $i$ :

1. *cut player* generates a partition  $(S_i, V \setminus S_i)$  of  $V$  such that  $|S_i| = n/2$ ;
2. *matching player* gives a routable perfect matching  $M_i$  in  $G$  between  $S_i$  and  $V \setminus S_i$ .

The cut player's goal is to make  $H_i := (V, \bigcup_{j \leq i} M_j)$  an **expander** as quickly as possible, while the goal of the matching player is to delay it as much as possible.

We think of ourselves as the cut player, while the matching player is some black-box  **$s$ - $t$  flow** computation algorithms that we do not have control of.

**Intuition.** In reality, however, we compute the **flow** ourselves and hence we're essentially also the matching player as well, just that we do not have the control of the final matching  $M_i$  (if it exists).

Firstly, if there is no feasible matching **flow** between  $S$  and  $V \setminus S$ , then this should indicate that  $G$  is not a very good **expander**. Indeed, by multi-commodity flow-cut gap (**Theorem 2.3.4**), there is a cut with **expansion** at most  $O(\log n)$ . However, in this case, we can improve this bound to 1.

**Lemma 5.2.1.** Suppose there is no feasible matching **flow** between  $S$  and  $V \setminus S$  for some  $S \subseteq V$  with  $|S| = n/2$ . Then  $G$  has **expansion** at most 1.

**Proof.** Since we have an  $s^*$ - $t^*$  min-cut of size less than  $n/2$ , where the graph is now with the super source  $s^*$  and the super sink  $t^*$  with the corresponding auxiliary edges from  $s^*$  to  $S$  and  $t \in V \setminus S$  to  $t^*$ . In this cut, let  $a$  be the number of edges from  $s^*$  and let  $b$  be the number of edges to  $t^*$ , and let  $c$  be the remaining number of edges. We have  $a + b + c < n/2$ .

The  $c$  non-auxiliary edges is also a cut in  $G$ , separates at least  $n/2 - a$  vertices in  $S$  from  $n/2 - b$  vertices in  $V \setminus S$ . The **expansion** of this cut is at most

$$\frac{c}{\min(n/2 - a, n/2 - b)} < \frac{n/2 - a - b}{n/2 - \max(a, b)} \leq 1,$$

which implies that  $G$  has **expansion** at most 1. ■

On the other hand, if  $G$  has **expansion** at least 1, every bipartition produces a fractional perfect matching  $M_i$  via an  $s_i^*$ - $t_i^*$  **flow**. Repeatedly doing so only implies that  $G$  is not necessarily not an **expander** (which isn't saying much). As a thought experiment, supposed the  $M_i$ 's behaved like random matchings. As we have seen, a constant number of matchings forms an **expander**, and embedding each of these matchings in  $G$  implies that  $G$  is an **expander**. Unfortunately, we cannot say that  $M_i$ 's behave like random matchings since ultimately the matchings are out of our control. However, the following

theorem says that we can adaptively choose the bipartitions so that the union of  $M_i$ 's form a matching after  $O(\log^2 n)$  rounds.

**Theorem 5.2.1** ([KRV09]). Consider the **cut-matching game**, where the matching player returns a fractional matching  $M_i \in [0, 1]^{V \times V}$  between the cut  $(S_i, V \setminus S_i)$  produced by the cut player in iteration  $i$ . There is a randomized adaptive strategy for the cut player such that with high probability, the union of the first  $k$  matchings  $M_1, \dots, M_k$ <sup>a</sup> forms a **1/2-expander** with  $k = O(\log^2 n)$ . Moreover, the certificate of  $H$ 's **expansion** is a feasible routing of the uniform multi-commodity **flow** over  $V$ .

<sup>a</sup>Formally, each matching **flows** is treated as a weighted, undirected edge set. Then the union is just the sum.

It's clear that by combining **Lemma 5.2.1** and **Theorem 5.2.1**, we can prove the following.

**Corollary 5.2.1.** There is a randomized algorithm, that given a graph  $G$ , either proves that  $G$  is not an **expander**, or with high probability certifies that it is an  $\Omega(\frac{1}{\log^2 n})$ -**expander** using  $O(\log^2 n)$  single-commodity **flow** instances on  $G$ . Via binary search, there is a randomized algorithm for **uniform sparsest cut** via  $O(\log^3 n)$  single-commodity **flow**.

**Proof.** If any  $(S_i, V \setminus S_i)$  cannot be feasibly routed, then we have a cut with **expansion** at most 1 from **Lemma 5.2.1**. Otherwise, from **Theorem 5.2.1**, with high probability, we obtain a **1/2-expander**  $H = (V, \bigcup_{i=1}^k M_i)$ . Since each matching **flow** fits in  $G$ , so the  $k$  matching **flows** fit in  $G$  scaled up by a factor of  $k$ , i.e.,  $H$  can be embedded in  $G$  with congestion  $k = O(\log^2 n)$ , proving the corresponding **expansion**. The binary search for **uniform sparsest cut** is trivial. ■

Finally, we note that this also yields an alternate proof of the flow-cut gap, although it is slightly weaker than the  $O(\log n)$ -bound we saw in **Theorem 2.3.4**:

**Corollary 5.2.2.** The multi-commodity flow-cut gap for **uniform instances** is  $O(\log^2 n)$ .

## 5.3 A Randomized Cutting Strategy

Our goal now is to prove **Theorem 5.2.1**, specifically, come up with a randomized strategy of selecting the cut  $(S_i, V \setminus S_i)$  in each iteration  $i$  such that the union of the fractional matchings,  $H = (V, \bigcup_{i=1}^k M_i)$ , is a **1/2-expander**.

### 5.3.1 Intuition: Certifying Expansion

But how can we certify that  $H$  is a **1/2-expander** in the first place?

**Problem.** This seems circular: our original motivation is to decide whether  $G$  is a **1-expander**. However, it seems like we still require to answer the same problem for  $H$ .

**Answer.** This is not cyclical because we will do something very specific to the structure of the matchings, such that we do not need to do the actual checking. ⊛

The idea is inspired by random-walk, but we will not explicitly mention the motivation and instead think of routing:

**Intuition.** Use  $H_i := (V, \bigcup_{j \leq i} M_j)$  to route a directed multi-commodity **flow** where we send (about)  $1/n$  units of **flow** between every pair of vertices. This implies that we can embed the weighted clique,  $K_n/n$ , into  $H_i$ . Since  $K_n/n$  has **expansion** 1, which also certifies that for  $H_i$ .

We build out this multi-commodity **flow** incrementally, one matching at a time. We start with a trivial **flow** where each vertex sends one unit of **flow** to itself. Each successive matching  $M$  is used to disperse (and hopefully diversify) the **flow**. Specifically, at the beginning of a generic iteration, we have a multi-commodity **flow** for again each vertex sending and receiving one unit of **flow**. Given the new matching  $M$ , we update the **flow** as follows:

- for each pair  $(u, v)$ , out of the 1 total unit of **flow** terminating at  $u$  from various sources, we send an  $M(uv)/2$ -fraction of each commodity to  $v$ ;
- we also do the same for the 1 total unit of **flow** terminating at  $v$ , sending  $M(uv)/2$ -fraction of each commodity at  $u$ .

**Intuition.** Use the fractional perfect matching to mix up the existing **flow**, hoping to produce a multi-commodity **flow** closer to uniform. We then choose the bipartition that maximizes the mixing.

### 5.3.2 Potential Argument

Mathematically, at the beginning of the iteration  $i$ , for each  $u \in V$ , let  $b_u^{(i)} \in [0, 1]^V$  be the stochastic vector such that  $b_u^{(i)}(v)$  is the amount of **flow** that  $u$  has routed to  $v$  after the  $i^{\text{th}}$  iteration in  $H_i$ . Then the above strategy is essentially that for each edge  $uv \in M_i$ , let

$$b_u^{(i)} = b_v^{(i)} = \frac{1}{2}(b_u^{(i-1)} + b_v^{(i-1)}),$$

where we assume for simplicity that the perfect matching  $M_i$  is integral. We note that at the beginning,  $b_u^{(0)}(v) = \mathbb{1}_{u=v}$  is the indicator vector for all  $u \in V$ , i.e.,  $b_u^{(0)}(u) = 1$  and  $b_u^{(0)}(v) = 0$  for all  $v \neq u$ .

**Note.** For the case of fractional matching, we have  $b_u^{(i)} = b_u^{(i-1)}/2 + \sum_{v \in V} M(uv)b_v^{(i-1)}/2$ . This will not make any difference, hence we just focus on the integral case.

We first observe the following, which simply formalizes that the routing is feasible in each iteration.

**Claim.** For all  $u \in V$ ,  $\sum_{v \in V} b_u^{(i)}(v) = 1$  for all  $i$ . And If  $H_{i-1}$  routes the demand matrix implied by  $b_u^{(i-1)}$  for all  $u \in V$ , then  $H_i$  routes the demand matrix implied by  $b_u^{(i)}$  for all  $u \in V$  as well.

**Proof.** The first part is trivial. For the second part, since  $H_i$  differs from  $H_{i-1}$  via the matching  $M_i$ . For each  $uv \in M_i$ , we use the edge to exchange the **flow** at  $u$  and **flow** at  $v$ : 1/2 capacity to send  $b_u^{(i-1)}$  to  $b_v^{(i-1)}$  and 1/2 capacity to send  $b_v^{(i-1)}$  to  $b_u^{(i-1)}$ .  $\otimes$

Ideally, as we have discussed, we want  $b_u(v) = 1/n$  for all  $u$  and  $v$ . In a sense, we are taking the weighted averages of  $b_u^{(i)}$ 's as dictated by  $M$ .

**Intuition.** In principle, pairing up very “different”  $b_u^{(i)}$ 's should produce averages closer to uniform.

The problem is again that we do not have the control of how the pairing is done, as it is given by  $M_{i+1}$ . We can only choose the next partition  $(S_{i+1}, V \setminus S_{i+1})$ . To do that, let  $x_u^{(i)} = b_u^{(i)} - \vec{1}/n \in \mathbb{R}^V$ , and consider the potential  $\phi_i = \sum_{u \in V} \|x_u^{(i)}\|_2^2$ , i.e.,

$$\phi_i = \sum_{(u,v) \in V \times V} \left( b_u^{(i)}(v) - \frac{1}{n} \right)^2.$$

We note that  $\phi_0 = \sum_{(u,v) \in V \times V} (\mathbb{1}_{u=v} - 1/n)^2 = n(1 - 1/n) = n - 1$ .

**Intuition.**  $\phi_i$  measures the difference between the multi-commodity **flow** routed so far and a fully symmetric multi-commodity **flow**.

Note that in terms of  $x_u^{(i)}$ , we now have  $\sum_{u \in V} x_u^{(i)} = 0$ , and the update rule for  $x_u^{(i)}$  becomes

$$x_u^{(i)} = x_v^{(i)} = \frac{1}{2}(x_u^{(i-1)} + x_v^{(i-1)}).$$

This potential is useful due to the following.

**Claim.** If  $\phi_i \leq 1/4n^2$ , then  $b_u^{(i)}(v) \geq 1/2n$  for all  $u, v \in V$ , and hence  $H_i$  is a  $1/2$ -expander.

### 5.3.3 Reducing Potential Via Matching

Now the question is, how can we effectively minimize  $\phi_i$ . In this context, we can abstract out the graphs and flows, just focus on analyzing  $x_u^{(i)}$  with the abstract update rule  $x_u^{(i)} = x_v^{(i)} = (x_u^{(i-1)} + x_v^{(i-1)})/2$ . A natural greedy strategy is that at each iteration, choose a new bipartition  $(S_i, V \setminus S_i)$  that minimizes the maximum of  $\phi_i = \sum_{u \in V} \|x_u^{(i+1)}\|_2^2$  over all fractional perfect matchings  $M_i$  of  $(S_i, V \setminus S_i)$ . We work backwards by first analyzing  $\phi_i$  analytically as a function of  $M$ .

**Lemma 5.3.1.** Let  $M$  be an arbitrary perfect matching  $M$  on  $V$ , and let  $x_u \in \mathbb{R}^k$  for some  $k \in \mathbb{N}$  for all  $u \in V$ . Suppose for each matching edge  $uv \in M$ ,  $y_u = (x_u + x_v)/2$ , then

$$\sum_{u \in V} \|y_u\|_2^2 = \sum_{u \in V} \|x_u\|_2^2 - \frac{1}{2} \sum_{uv \in M} \|x_u - x_v\|_2^2.$$

**Proof.** For any  $u \in V$  and the matching edge  $uv \in M$ ,

$$\begin{aligned} \|y_u\|_2^2 &= \left\| \frac{1}{2}x_u + \frac{1}{2}x_v \right\|_2^2 \\ &= \frac{1}{4}\|x_u\|_2^2 + \frac{1}{4}\|x_v\|_2^2 + \frac{1}{2}\langle x_u, x_v \rangle \\ &= \frac{1}{4}\|x_u\|_2^2 + \frac{1}{4}\|x_v\|_2^2 + \frac{1}{4}(\|x_u\|_2^2 + \|x_v\|_2^2 - \|x_u - x_v\|_2^2) = \frac{1}{2}\|x_u\|_2^2 + \frac{1}{2}\|x_v\|_2^2 - \frac{1}{4}\|x_u - x_v\|_2^2. \end{aligned}$$

By summing up over  $u \in V$ , since  $M$  is a perfect matching, we have the result.  $\blacksquare$

Lemma 5.3.1 shows that, specifically for  $x_u^{(i-1)} \in \mathbb{R}^V$ , under the update rule (which is the same as our mixing rule), we can express the reduction of  $\phi_{i-1}$  explicitly:

$$\phi_i = \phi_{i-1} - \frac{1}{2} \sum_{uv \in M_i} \|x_u^{(i-1)} - x_v^{(i-1)}\|_2^2. \quad (5.1)$$

Hence, the potential reduction for an edge  $uv$  is proportional to  $\|x_u - x_v\|_2^2$ , which is the square of the Euclidean distance between  $x_u$  and  $x_v$ . Thus, the cut player should somehow force the matching player to pair vertices that are *far apart*.

**As previously seen.** The cut player can only give a partition  $(S_i, V \setminus S_i)$  and the matching player has control of the actual matching it generates.

It's quite hard to ensure this in high dimension. The intuition is to look at the easiest case:

**Intuition.** The key observation is that when  $k = 1$ , the potential goes down a lot.

We formalize this as follows.

**Lemma 5.3.2.** For each  $u \in V$ , let  $z_u \in \mathbb{R}$  be a scalar such that  $\sum_{u \in V} z_u = 0$ . Suppose we partition  $V$  to be  $(A, B)$  according to  $z_u$ , where  $A$  and  $B$  is the first and the second half, respectively.<sup>a</sup> Then for every perfect matching  $M$  between  $A$  and  $B$ , we have

$$\sum_{uv \in M} \|z_u - z_v\|_2^2 = \sum_{uv \in V} (z_u - z_v)^2 \geq \sum_{u \in V} z_u^2.$$

<sup>a</sup>I.e.,  $A = \{u : z_u \leq \text{median}(\{z_v\}_{v \in V})\}$  and  $B = V \setminus A$  such that  $|A| = |B| = n/2$ , with ties broken arbitrarily.

**Proof.** Let  $\beta \in \mathbb{R}$  to be the median of  $z_u$ 's. Hence,  $z_u \leq \beta$  for all  $u \in A$  and  $z_u \geq \beta$  for all  $u \in B$ . Then, for any perfect matching  $M$  between  $A$  and  $B$ , consider a matched edge  $uv \in M$ . We have  $(z_u - z_v)^2 = ((z_u - \beta) - (z_v - \beta))^2$ . Since  $z_u \leq \beta$  and  $z_v \geq \beta$ , we have  $((z_u - \beta) - (z_v - \beta))^2 =$

$(|z_u - \beta| + |z_v - \beta|)^2$ . Summing up over all pairs in  $M$ , we have

$$\sum_{uv \in M} \|z_u - z_v\|_2^2 = \sum_{uv \in M} (|z_u - \beta| + |z_v - \beta|)^2 \geq \sum_{u \in V} z_u^2 - 2\beta \sum_{u \in V} z_u + n\beta^2 \geq \sum_{u \in V} z_u^2,$$

where we use the fact that  $\sum_{u \in V} z_u = 0$  in the final inequality. ■

**Remark.** The potential reduction is huge, indeed, linear, at least for 1 dimension.

**Proof.** Recall that from [Lemma 5.3.1](#), the potential is reduced by  $\sum_{uv \in M} \|x_u - x_v\|_2^2/2$ , which in this case is  $\sum_{uv \in M} (z_u - z_v)^2/2$ . From [Lemma 5.3.2](#), this is at least half of the original potential ( $\sum_{u \in V} z_u^2 = \sum_{u \in V} \|z_u\|_2^2$ ), so we get a linear decay. \*

### 5.3.4 Random Projection Cutting Strategy

A natural idea is then to reduce the original  $x_u^{(i)} \in \mathbb{R}^V$  to one-dimensional.

**Intuition.** The general strategy inspired by [Lemma 5.3.2](#) is then to project onto a random line.

## Lecture 21: Finishing Cut-Matching Game

Specifically, in iteration  $i$ , we have vectors  $x_u^{(i)}$  for  $u \in V$ , where each  $x_u \in \mathbb{R}^n$  and  $\sum_{v \in V} x_u^{(i)}(v) = 0$ . We can then reduce the problem to the one dimensional case by using the well-known idea of random projection that maps points in high dimensions to a line. Recall the following simple facts:

**As previously seen.** If  $Z_i \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu_i, \sigma_i^2)$  for  $i = 1, 2$ , then  $Z_i + Z_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ .

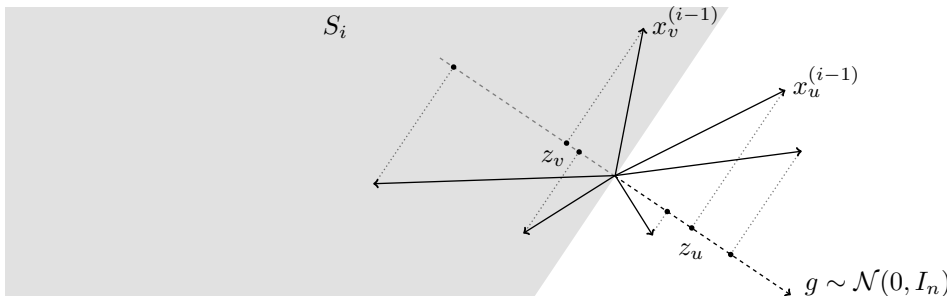
Furthermore, we also have the following.

**Lemma 5.3.3.** Let  $g \sim \mathcal{N}(0, I_n)$  and let  $x \in \mathbb{R}^n$  and  $z := \langle g, x \rangle$ . Then:

- (i)  $\mathbb{E}[z] = 0$ ;
- (ii)  $\mathbb{E}[z^2] = \sum_{i=1}^n x_i^2 = \|x\|_2^2$ ;
- (iii) for any  $t$ ,  $\Pr(z^2 \geq t\|x\|_2^2) \leq e^{-t/2}$ .

**Proof.** The first two are standard properties. We just make some additional remark on the third concentration bound for  $\chi^2$ . Suppose  $Z \sim \mathcal{N}(0, \sigma^2)$ . Then, it's well known that we have  $\Pr(Z > \beta) \leq e^{-\beta^2/2\sigma^2}$ . Now,  $Z^2$ , the square of the Gaussian random variable, is a positive random variable following the so-called  $\chi^2$  distribution. Note that its mean is  $\sigma^2$ . It also exhibits strong concentration around its mean: specifically, when  $Z \sim \mathcal{N}(0, 1)$ , we have  $\Pr(Z^2 \geq 1 + 4t) \leq e^{-t}$ . Using this, the third concentration bound can be easily proved. ■

Hence, in iteration  $i$ , a natural strategy for the cut player is to project  $x_u^{(i-1)}$  to one dimension via  $z_u^{(i)} = \langle g, x_u^{(i-1)} \rangle$  and partition  $V$  into  $(S_i, V \setminus S_i)$  by sorting  $z_u^{(i)}$  and select the first half to be  $S_i$ :



This leads to the following guarantee.

**Lemma 5.3.4.** Let  $M_i$  be any perfect matching between  $S_i$  and  $V \setminus S_i$  generated by the cut player with the linear projection strategy based on  $x_u^{(i-1)}$  for  $u \in V$ . Then

$$\mathbb{E} \left[ \sum_{uv \in M_i} (z_u - z_v)^2 \right] \geq \phi_{i-1},$$

where  $z_u := \langle g, x_u^{(i-1)} \rangle$  for all  $u \in V$ , where  $g \sim \mathcal{N}(0, I_n)$ .

**Proof.** From the proof of Lemma 5.3.2, we have<sup>a</sup>

$$\sum_{uv \in M_i} (z_u - z_v)^2 \geq \sum_{u \in V} z_u^2 - 2\beta \sum_{u \in V} z_u,$$

where  $\beta$  is the median of  $z_u$  for  $u \in V$ . We see that by taking expectation on both sides, with Lemma 5.3.3, we have

$$\mathbb{E} \left[ \sum_{uv \in M_i} (z_u - z_v)^2 \right] \geq \mathbb{E} \left[ \sum_{u \in V} z_u^2 \right] - 2\beta \mathbb{E} \left[ \sum_{u \in V} z_u \right] = \sum_{u \in V} \mathbb{E} [z_u^2] = \sum_{u \in V} \|x_u^{(i-1)}\|_2^2 = \phi_{i-1},$$

where we use the fact that the sum of mean zero Gaussian's is still a mean zero Gaussian. ■

<sup>a</sup>Since  $\sum_{u \in V} z_u$  is not necessarily 0 as  $z_u$  is random.

With Lemma 5.3.4, we can then combine Lemma 5.3.2 and Lemma 5.3.1 to prove the following:

**Lemma 5.3.5.** Given current flow vectors  $b_u^{(i-1)}$  and difference vectors  $x_u^{(i-1)}$  for all  $u \in V$ , there is a randomized algorithm that produces a partition  $(S_i, V \setminus S_i)$  such that for any perfect matching  $M_i$  between  $S_i$  and  $V \setminus S_i$ , the potential  $\phi_i$  after mixing according to  $M_i$  satisfies

$$\mathbb{E}[\phi_i] \leq \left(1 - \frac{1}{c \log n}\right) \phi_{i-1} + O(n^{-10}),$$

where  $c$  is some sufficiently large fixed constant.

**Proof.** Consider the linear projection strategy described above. Since  $(z_u - z_v)^2 = \langle g, x_u^{(i-1)} - x_v^{(i-1)} \rangle^2$ , then by Lemma 5.3.3, with high probability,

$$(z_u - z_v)^2 = \langle g, x_u^{(i-1)} - x_v^{(i-1)} \rangle^2 \leq c \log n \cdot \|x_u^{(i-1)} - x_v^{(i-1)}\|_2^2. \quad (5.2)$$

Assuming this happens deterministically. Then, from Lemma 5.3.4 and Equation 5.1,

$$\mathbb{E}[\phi_{i-1} - \phi_i] = \frac{1}{2} \mathbb{E} \left[ \sum_{uv \in M_i} \|x_u^{(i-1)} - x_v^{(i-1)}\|_2^2 \right] \geq \frac{1}{c \log n} \mathbb{E} \left[ \sum_{uv \in M_i} (z_u - z_v)^2 \right] \geq \frac{1}{c \log n} \phi_{i-1},$$

which implies  $\mathbb{E}[\phi_i] \leq (1 - 1/c \log n) \phi_{i-1}$ . To handle the case that Equation 5.2 doesn't happen deterministically, we need to do a little of extra work, which is why we get the weaker bound with a slight additive bound. We omit the exact detail for simplicity. ■

**Intuition.** Lemma 5.3.5 states that using random projection, we only loses a  $1/\log n$  factor compared to the true one-dimensional case.

**Remark.** Lemma 5.3.5 implies that the algorithm uses only the information about the flow routed so far and nothing about the previous matchings.

Lemma 5.3.5 immediately imply Theorem 5.2.1:



**Proof of Theorem 5.2.1.** From the previous [claim](#), we know that we want to reduce  $\phi_0 \leq n$  to  $1/4n^2$ . As  $\phi_i \leq (1 - 1/c \log n)\phi_{i-1}$  for some constant  $c > 0$  from [Lemma 5.3.5](#), we have

$$\phi_i \leq \left(1 - \frac{1}{c \log n}\right)^i \phi_0 \leq \left(1 - \frac{1}{c \log n}\right)^i n.$$

Setting the right-hand side to be less than  $1/4n^2$  and solve for  $i$ , we have  $i = O(\log^2 n)$ .  $\blacksquare$

This completes the whole proof. Finally, we describe the whole [cut-matching game](#) in [Algorithm 5.1](#) for completeness.

---

**Algorithm 5.1: Cut-Matching Game**


---

**Data:** A graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , iteration  $k$

**Result:** A certification graph  $H = (V, \bigcup_{i=1}^k M_i)$

```

1 for  $u \in V$  do                                     // Initialization
2    $x_u^{(0)} \leftarrow \mathbb{1}_{u=v} - \bar{l}/n$ 
3
4 while  $i = 1, \dots, k$  do
5    $S_i \leftarrow \text{Cut-Player}(\{x_u^{(i-1)}\}_{u \in V})$ 
6    $M_i \leftarrow \text{Matching-Player}(G, S_i)$ 
7   for  $uv \in M_i$  do
8      $x_u^{(i)} \leftarrow (x_u^{(i-1)} + x_v^{(i-1)})/2$ 
9      $x_v^{(i)} \leftarrow (x_u^{(i-1)} + x_v^{(i-1)})/2$ 
10 return  $(V, \bigcup_{i=1}^k M_i)$ 
11
12 Cut-Player( $\{x_u\}_{u \in V}$ ):
13    $g \leftarrow \text{Normal}(0, I_n)$                        //  $g \sim \mathcal{N}(0, I_n)$ 
14   for  $u \in V$  do
15      $z_u \leftarrow \langle g, x_u \rangle$ 
16    $\beta \leftarrow \text{Median}(\{z_u\}_{u \in V})$ 
17    $S \leftarrow \{u \in V : z_u \leq \beta\}$                 // Break ties arbitrarily such that  $|S| = n/2$ 
18   return  $S$ 
19
20 Matching-Player( $G, S$ ):
21    $(G', s^*, t^*) \leftarrow \text{Build-Linkage-Routing-Graph}(G, S, V \setminus S)$  // Figure 5.1
22    $M \leftarrow \text{Single-Commodity-Flow}(G', s^*, t^*)$  // Compute routable perfect matching
23   return  $M$ 

```

---

**Note.** Explicitly, we run [Algorithm 5.1](#) with  $k = O(\log^2 n)$ , and whenever  $M$  is not a valid perfect matching or is not [routable](#) (indicating that  $S$  and  $V \setminus S$  is not [well-linked](#)), we just report that  $G$  is not a [1-expander](#).

## 5.4 Application to Treewidth

Previously, we have mentioned [treewidth](#) briefly (e.g., [Theorem 2.4.3](#)). Now, we see another good application of the [cut-matching game](#) to [treewidth](#).

### 5.4.1 Tree Decomposition and Treewidth

We care about how “tree-like” does the graph look. One of the (among many) interesting properties of a tree is that we can separate the tree in a balanced way by removing only one node:

**Example (Tree).** For a tree  $T$ , there exists a single vertex such that after removing it, no connect component has more than  $2n/3$  vertices, i.e., there exists a singleton **balanced separator**.

What if we allow removing more vertices?

**Example (Series-parallel graph).** **Series-parallel graph** has the same property when allowing removing 2 vertices.

Taking these two examples into account, we consider the following notion of **tree decomposition**.

**Definition 5.4.1 (Tree decomposition).** Given a graph  $G = (V, E)$ , a *tree decomposition* of  $G$  is a tree  $T = (V_T, E_T)$  and a function  $B: V_T \rightarrow 2^V$  such that

- (i) for all  $u \in V$ , the set of “bags”  $B(t)$  that contains  $u$  form a connected sub-tree of  $T$ ;
- (ii) for all  $uv \in E$ , there exists some  $t \in V_T$  such that both  $u, v \in B(t)$ .

**Definition 5.4.2 (Width).** The *width*  $\text{width}(T)$  of a **tree decomposition**  $T$  of  $G$  with the function  $B: V_T \rightarrow 2^V$  is defined as  $\max_{t \in V_T} |B(t)|$ .

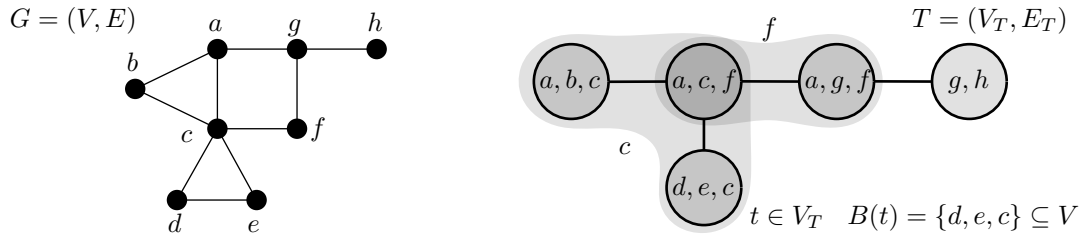


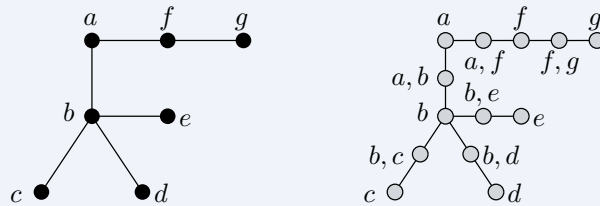
Figure 5.2: The **tree decomposition** of  $G$  with its “bag” plotted explicitly.

Then, we can define the **treewidth** formally.

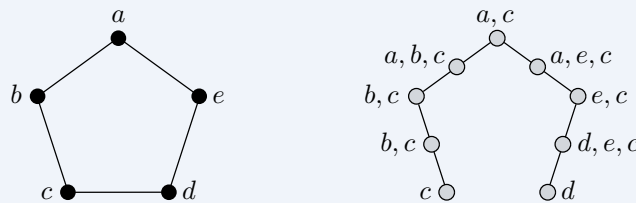
**Definition 5.4.3 (Treewidth).** Given a graph  $G$ , the *treewidth*  $\text{tw}(G)$  is defined as  $\min_T \text{width}(T) - 1$ , where the minimum is taking over all possible **tree decompositions** of  $G$ .

The minus one is because the minimum maximum **width** of the **tree decomposition** for a tree is 2:

**Example (Tree).** For any tree  $T$ , there exists a **tree decomposition** with **width** 2, i.e.,  $\text{tw}(T) = 1$ .



**Example (Cycle).** For any cycle  $C_n$ , there exists a **tree decomposition** with **width** 3, i.e.,  $\text{tw}(C_n) = 2$ .



**Example (Series-Parallel).** We have that  $\text{tw}(G) \leq 2$  if and only if  $G$  is series-parallel.

**Example (Clique).** For any clique  $K_n$ , we have  $\text{tw}(K_n) = n - 1$  with the trivial [tree decomposition](#), i.e.,  $T = (V_T, E_T)$  with  $V_T = \{t\}$ ,  $E_T = \emptyset$  such that  $B(t) = V$ .

### 5.4.2 Balanced Separator and Treewidth

Essentially, [Definition 5.4.3](#) captures the desired [balanced separator](#) properties we mentioned above:

**Intuition.** Any graph  $G$  with  $\text{tw}(G) \leq k$  implies that  $G$  can be recursively decomposed via [balanced separators](#) of size  $k$ . Approximate converse also holds, i.e., if there is a subgraph  $H$  of  $G$  with no [balanced separator](#) of size  $k$ , then  $\text{tw}(G) \geq k/c$  for some  $c > 0$ .

Formally, we have the following:

**Lemma 5.4.1.** Suppose  $\text{tw}(G) \leq k$ . Then for any  $X \subseteq V$ , there exists a [balanced separator](#)  $S$  for  $X$  of size  $|S| \leq k$ , i.e.,  $G - S$  has no component with more than  $2|X|/3$  vertices from  $X$ .

The idea is that for any edge  $(t, t') \in E_T$  of the [tree decomposition](#)  $T$  that achieves the optimal [width](#) (i.e.,  $\text{tw}(G) + 1$ ),  $B(t) \cap B(t')$  is a separator of  $G$ . Hence,  $\text{tw}(G) \leq k$  implies  $G$  has a [balanced separator](#)  $S$  of size  $k$ . The recursive property follows from that fact that  $\text{tw}(H) \leq \text{tw}(G)$  for any subgraph  $H$  of  $G$ . Let us rephrase [Theorem 2.4.3](#) as follows:

**Lemma 5.4.2.** For any graph  $G$ ,  $\text{tw}(G) \geq k$  if and only if there exists  $X$  such that  $|X| \geq k/4$  such that  $X$  is [node-well-linked](#).

With these characterizations, we have the following examples.

**Example (Grid).** For  $\sqrt{n} \times \sqrt{n}$  grid  $G$ ,  $\text{tw}(G) = \sqrt{n} - 1$ .

Indeed, grid the worst case for planar graphs.

**Example (Planar separator theorem).** For any planar graph  $G$ ,  $\text{tw}(G) = O(\sqrt{n})$  [[LT79](#)].

**Example (Random graph).** For random  $d$ -regular graph  $G$ ,  $\text{tw}(G) = \Theta(n)$  with high probability.

**Proof.** Since random  $d$ -regular graph is an [expander](#) with high probability, and [balanced separator](#) is of size  $\Omega(n)$  for [expander](#). \*

### 5.4.3 Cut-Matching Game to Certify Treewidth

Now, we try to verify the [treewidth](#). Verifying a small [treewidth](#) is easy, we just ask for a [tree decomposition](#) as a certification. In general, given  $G$  and  $k$ , checking if  $\text{tw}(G) \leq k$  is NP-complete [[ACP87](#)]. However, it is fixed-parameter-tractable, i.e., for every fixed  $k$ , there is a linear-time algorithm (specifically,  $O(k^3 n)$ ), that, given  $G$ , decides whether  $\text{tw}(G) \leq k$  [[Bod93](#)]. As for an approximation algorithm for computing [treewidth](#), it takes  $O(2^k n)$  time for a 5-approximation [[Bod+16](#)], while for the best known polynomial-time approximation algorithm is  $O(k\sqrt{\log k})$ -approximate [[FHL05](#)].

Now, it turns out that instead of directly certifying/computing the [treewidth](#), from the understanding of the connection between the size of [balanced separator](#) and [treewidth](#), with the fact that [expander](#) has large [balanced separator](#), one natural idea is the following:

**Intuition.** We try to “embed” an [expander](#) using [cut-matching game](#).

Utilizing this idea, the following is known:

**Theorem 5.4.1.** Let  $G = (V, E)$  be a graph with  $\text{treewidth} \geq k$ . Then there is a randomized algorithm that computes a subgraph  $H = (V, E_H)$  such that with high probability,

- (i)  $\text{tw}(H) = \Omega(k/\text{poly log } k)$ , where  $\text{poly log } k$  is something like  $\log^2 k$ ;
- (ii)  $\deg_H(v) = O(\log^2 k)$ .

Theorem 5.4.1 is further improved as follows:

**Theorem 5.4.2** ([CE13; CC14]). Let  $G = (V, E)$  be a graph with  $\text{treewidth} \geq k$ . Then there is a randomized algorithm that computes a subgraph  $H = (V, E_H)$  such that with high probability,

- (i)  $\text{tw}(H) = \Omega(k/\text{poly log } k)$ , where  $\text{poly log } k$  is something like  $\log^{100} k$ ;
- (ii)  $\deg_H(v) = 3$ .

**Remark.** The full generality of Theorem 5.4.1 and Theorem 5.4.2 are stated as topological minor. We omit the exact details.

#### 5.4.4 Algorithm with Treewidth

The reason why we care about  $\text{treewidth}$  is that there are several “templates” for applications:

- if  $G$  has *small* (constant)  $\text{treewidth}$ , we can then solve the problem via dynamic programming on the  $\text{tree decomposition}$ ;
- if  $G$  has *large*  $\text{treewidth}$ , we can then use structure, in particular, obstructions such as grids:
  - answer is clear from obstruction, or
  - “reduce” the problem in some fashion and recurse.

Let’s just look at the case that when the  $\text{treewidth}$  is small. In this case, dynamic programming based algorithms for trees extends naturally to bounded  $\text{treewidth}$  graphs.

**Example (Problems on planar graph).** From the  $\text{planar separator theorem}$ , we can solve many problems in  $2^{\sqrt{n}}$  using dynamic programming for planar graphs.

One such problem is the  $\text{maximum (weighted) independent set}$ :

**Problem 5.4.1 (Maximum weighted independent set).** Given a graph  $G = (V, E)$  with vertex capacity  $w: V \rightarrow \mathbb{R}$ , the *maximum weighted independent set* (MWIS) problem asks for computing  $\max_{S \subseteq V} w(S)$  such that  $S \subseteq V$  is an independent set.

MWIS is NP-hard, even for planar graphs, and more generally, is also very hard even to approximate in general graphs. However, we can still solve it efficiently (in polynomial time) in bounded  $\text{treewidth}$  graphs. To get some intuition, let’s first see how to solve it for tree using dynamic programming.

**Example (MWIS on tree).** Given a tree  $T$  with vertex capacity  $w: V \rightarrow \mathbb{R}$ , let  $T_v$  be the sub-tree of  $T$  rooted at node  $v \in V_T$ , and let  $\text{OPT}(v)$  be the optimum value of MWIS on  $T_v$ . Furthermore, let’s define  $\text{OPT}(v, 1)$  be the optimum value of MWIS on  $T_v$  that includes  $v$ , and  $\text{OPT}(v, 0)$  is defined similarly such that it does not include  $v$ . Then, we have  $\text{OPT}(v) = \max(\text{OPT}(v, 1), \text{OPT}(v, 0))$ .

After some observation, we see that

$$\begin{cases} \text{OPT}(v, 1) = w(v) + \sum_{u \text{ child of } v} \text{OPT}(u, 0); \\ \text{OPT}(v, 0) = \sum_{u \text{ child of } v} \text{OPT}(u). \end{cases}$$

Using this, we can solve the corresponding dynamic programming problem efficiently.

Let's now see how to use the above idea to solve **MWIS** for graphs with low **treewidth**.

**Example (MWIS on tree decomposition).** Given a **tree decomposition** of **width**  $k$  for a graph  $G$ , **MWIS** can be computed in  $O(k2^{k+1}n)$  time.

**Proof.** Let  $T = (V_T, E_T)$  be the corresponding **tree decomposition**. For  $t \in V_T$ , let  $G_t$  be the subgraph of  $G$  induced by  $S := \bigcup_{t \in V(T_t)} B(t) \subseteq V$ , i.e., nodes in bags of  $T_t$ .

Now, let  $\text{OPT}(t, S)$  be the value of **MWIS** in  $T_t$  among independent sets  $I$  such that  $I \cap X_t = S$ . We see that the number of values to compute at each node  $t \in V_T$  is  $\leq 2^{k+1}$ , where  $k$  is the **width** of  $T$ . Using dynamic programming, we can compute all values from leaves to root in  $O(k2^{k+1}N)$  time, where  $N$  is the number of nodes in  $T$ , i.e.,  $|V_T|$ .  $\circledast$

The above yield a  $2^{O(\sqrt{n})}$  time algorithm for planar graphs. However, it's worth noting that it's possible to get a fast  $(1 - \epsilon)$ -approximation algorithm in planar graphs. For that, we need the following.

**Theorem 5.4.3.** For any planar graph  $G = (V, E)$  and integer  $h \geq 3$ ,  $V$  can be colored with  $h$  colors such that for any  $i \in [h]$ ,  $\text{tr}(G - V_i) \leq 3h$  where  $V_i \subseteq V$  is the set of vertices colored with color  $i$ .

**Example (MWIS on planar graph).** Consider **MWIS** on planar graphs. Let  $\epsilon \in (0, 1)$  and  $h = 1/\epsilon$ . From **Theorem 5.4.3**, consider removing  $V_i$  from  $G$  and compute the corresponding **MWIS** to get  $\text{OPT}_i$  in  $G - V_i$  using the **previous example's** approach, which takes  $O(2^{1/\epsilon}n/\epsilon)$  since  $\text{tr}(G - V_i) \leq 3h = O(1/\epsilon)$ . In total, there are  $h = 1/\epsilon$  of  $V_i$ 's, hence the total running time is  $O(2^{1/\epsilon}n/\epsilon^2)$ .

Then, we simply output the final answer as  $\max_{i \in [h]} \text{OPT}_i$ , say it's  $\text{OPT}_{i^*}$ . From the pigeonhole principle, the output  $\text{OPT}_{i^*}$  for the graph  $G - V_{i^*}$  is  $(1 - \epsilon)$ -approximation compared to the original optimum value  $\text{OPT}$  for the whole graph  $G$ .

# Chapter 6

## Blocking Flow and Push-Relabel

### Lecture 22: Blocking Flow and Link-Cut Tree

We have mentioned the [max-flow](#) problem multiple times throughout the course, which is used as a primitive for several problems. For example, from [Theorem 1.3.6](#), we know that solving [Steiner min-cut](#) reduces to polynomial-logarithmically many [max-flow](#) computations via [isolating cut](#). Hence, in this chapter, we present several classical [max-flow](#) algorithms which are not usually taught.

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Notably, all the algorithms we will see are based on the [augmenting path](#) approach, first introduced by Ford and Fulkerson [\[FF56\]](#).

#### 6.1 Augmenting Path Framework

In this section, we first introduce the *augmenting path framework* formally, and briefly talk about a naive algorithm for solving [max-flow](#) in  $O(m^2n)$  time [\[Din70; EK72\]](#) (Dinitz includes additional techniques that reduce the running time to  $O(mn^2)$  as we will see later).

##### 6.1.1 Ford-Fulkerson Algorithm

Consider a directed graph  $G = (V, E)$  with capacities  $c: E \rightarrow \mathbb{R}_+$ . In the augmenting path framework, algorithms repeatedly find an [augmenting path](#) (or a collection of them) in the [residual graph](#):

**Definition 6.1.1 (Augmenting path).** An *augmenting path* is a simple path through the directed graph  $G = (V, E)$  using only edges with positive capacity from the source to the sink.

**Definition 6.1.2 (Residual graph).** Let  $G = (V, E)$  be a directed graph with capacities  $c: E \rightarrow \mathbb{R}_+$  and a [flow](#)  $f$ . The *residual graph*  $G_f = (V, E_f)$  with capacities  $c_f: E_f \rightarrow \mathbb{R}_+$  where  $c_f(u, v) = c(u, v) - f(u, v) + f(v, u)$  for all  $e = (u, v) \in E$ .<sup>a</sup>

<sup>a</sup>We let  $E_f := \{e = (u, v) \in V \times V \mid c_f(e) > 0\}$  for convenience.



Figure 6.1: Residual edge induced by  $f$  on  $e$ .

We then augment, i.e., add, the [flow](#) along this path with a value equal to the bottleneck of the augmenting path to the current flow. By repeated this until we cannot find [augmenting path](#) anymore.

**Algorithm 6.1:** Ford-Fulkerson Algorithm [FF56]

---

**Data:** A connected directed graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , source  $s$ , sink  $t$   
**Result:**  $s$ - $t$  max-flow  $f$

```

1  $f \leftarrow 0$ 
2
3 while  $\exists s$ - $t$  augmenting path  $P \in G_f$  do           // Update  $G_f$  implicitly
4    $b \leftarrow \min_{e \in P} c(e)$                        // Bottleneck
5   for  $e \in P$  do
6      $f(e) \leftarrow f(e) + b$ 
7 return  $f$ 

```

---

**6.1.2 Naive Algorithm**

There are several variants we can do when finding the augmenting paths in line 3. To be naive, one can literally run Algorithm 6.1 without any care. This will work already, by recalling the classical proof of max-flow min-cut theorem:

**Theorem 6.1.1** (Max-flow min-cut). If there is no augmenting path in the residual graph  $G_f$  for some  $s$ - $t$  flow  $f$ , then  $f$  is the  $s$ - $t$  max-flow.

**Proof.** Suppose there is no augmenting path in  $G_f$ , i.e., there is no valid simple  $s$ - $t$  path. Then, it's clear that there is an  $S$ - $T$  cut in  $G_f$  where  $s \in S$  and  $t \in T = V \setminus S$ , such that all edges from  $S$  to  $T$  are saturated, i.e.,  $c_f(u, v) = 0$  for  $u \in S$  and  $v \in T$ . Hence, we have  $f(u, v) = c(u, v)$ .

Furthermore, all edges from  $T$  to  $S$  have zero flow, i.e.,  $f(v, u) = 0$  for  $u \in S$  and  $v \in T$ . To see this, suppose there is some  $(v, u)$  such that it carries some non-zero flow  $f(v, u)$ . Then, there exists a backward edge from  $u$  to  $v$  in  $G_f$ , and hence there is a path from  $s$  to  $u$  and to  $v$  in  $G_f$ , a contradiction.

Combining these facts, the capacity of the cut  $(S, T)$  is equal to  $|f|$ , and from the fact that any flow has value less than the capacity of every possible cut,  $(S, T)$  is the  $s$ - $t$  min-cut that witness the  $s$ - $t$  max-flow as well. ■

With Theorem 6.1.1, we can prove the following.

**Theorem 6.1.2.** Algorithm 6.1 computes the  $s$ - $t$  max-flow on a directed graph with integral edge capacities in  $O(m|f^*|)$  time, where  $|f^*|$  is the  $s$ - $t$  max-flow value.

**Proof.** The correctness is from Theorem 6.1.1. For the runtime, since each augmenting path computation can be easily done in  $O(m)$  time with BFS, and for each augmentation, the flow value increased by at least 1. ■

**Corollary 6.1.1.** There exists an integral max-flow when the graph is with integral capacities.

**Proof.** It is obvious from the proof of Theorem 6.1.2. ■

**Corollary 6.1.2.** Let  $W = \max_{e \in E} c(e)$ , then Algorithm 6.1 actually runs in  $O(mnW)$ .

**Proof.** From max-flow min-cut theorem, we know that  $|f^*|$  is equal to the  $s$ - $t$  min-cut cost, which is less than  $c(\delta^+(\{s\}))$  for example. Then since  $c(\delta^+(\{s\})) \leq nC$ , we have  $|f^*| \leq nC$ , hence the algorithm runs in  $O(mnW)$ . ■

We see that Corollary 6.1.2 proves that the naive implementation of Algorithm 6.1 does not lead to a truly polynomial time algorithm since the running time depend linearly on  $W$ .

**6.1.3 Maximum Bottleneck Capacity Augmenting Path**

To be a bit more careful, the next idea is to always choose an augmenting path  $P$  that has the maximum bottleneck capacity  $\max_{e \in P} c_f(e)$  in line 3. Such an augmenting path can be found in  $O(m \log n)$  time



via binary search over the capacity of the bottleneck edge, and, for each guess, check for the  $s$ - $t$  path in  $G_f$  after all the edges with capacity less than the guess were removed.

**Note.** By being (much) more sophisticated, one can get  $O(\min(m + n \log n, m \log^* n))$  time.

To see this leads to an actual improvement, we want to know how much progress each such **augmenting path** guarantees to make.

**As previously seen.** The **flow** decomposition bound tells us that any **flow** can be decomposed into at most  $m$  **flow** paths.

**Theorem 6.1.3.** By choosing the maximum bottleneck capacity **augmenting path** in **Algorithm 6.1**, the algorithm computes the  **$s$ - $t$  max-flow** on a directed graph  $G = (V, E)$  with integral edge capacities in  $O(m^2 \log n \log nW)$  time, where  $W = \max_{e \in E} c(e)$ .

**Proof.** If we apply the above observation to the maximum “remaining” **flow**, i.e., the maximum **flow** in the **residual graph**  $G_f$  together with an averaging argument, we can conclude that at least one of these paths carries at least  $1/m$ -fraction of the remaining value of the **flow**. Hence, each **flow** augmentation reduces the remaining value of the **flow** by a factor of  $1 - 1/m$ .

After  $O(m \log |f^*|)$  augmentations, the remaining flow would be less than 1. Since we have integral capacities, the obtained solution has to be optimal. We conclude that the overall running time is  $O(m \log n \cdot m \log |f^*|) = O(m^2 \log n \log nW)$ . ■

**Theorem 6.1.3** proves that **Algorithm 6.1** with choosing maximum bottleneck capacity **augmenting path** leads to a weakly polynomial time algorithm since the running time now only depend on  $\log W$ , which is the same as the capacity representation. However, we can ask for more.

### 6.1.4 Shortest Augmenting Path

Unlike the maximum bottleneck approach, which depend on the value representation; instead, the next idea leads to an algorithm that runs in  $O(m^2 n)$  [Din70; EK72] via a naive implementation. The key is to implement **line 3** by choosing the shortest **augmenting path** (shortest w.r.t. the number of edges). From now on, unless specified, the capacities are not necessarily integral.

**As previously seen.** Maximum bottleneck capacity **augmenting path** is “primal greedy,” i.e., it applies a greedy strategy in which we improve our current solution in each step and then lower bounded the progress made by each such step in terms of the value of the  **$s$ - $t$  max-flow**.

Therefore, as the value of the  **$s$ - $t$  max-flow** directly depends on the value of capacities, this kind of “primal greedy” approach is inherently incapable of providing a strongly polynomial bound. Hence, we need a different way to measure our progress towards optimality. To see how can we improve upon this observation, consider the following.

**As previously seen.** The way we certified the optimality of our current **flow**  $f$  was by looking whether  $s$  and  $t$  are connected in  $G_f$ . If not,  $f$  was already a  **$s$ - $t$  max-flow**; otherwise, it was not optimal (and the  $s$ - $t$  path enables us to improve it).

The question then is, can we make this yes/no statement quantitative to also be able to differentiate between the **flows** that are “almost maximum” and the ones that might be “far” from being maximum? To this end, consider the  $s$ - $t$  distance  $d_f(s, t)$  in  $G_f$ , where each edge with positive residual capacity has length one, and every edge with zero residual capacity has length  $+\infty$ . The intuition of considering  $d_f(s, t)$  is the following.

**Intuition.** Any **flow** in  $G_f$  has a fixed total volume ( $\leq m$  for unit capacity graph). Hence, if  $s$  and  $t$  are far apart, then not much **flow** can fit in  $G_f$ , as each **flow** path has to utilize a lot of volume.

Specifically, observe that if  $d_f(s, t) \geq n$ , then  $s$  and  $t$  are disconnected in  $G_f$ , meaning that  $f$  is already a  **$s$ - $t$  max-flow**. Hence, naturally, our new goal is to design an **augmenting path** based algorithm that aims to increase the  $s$ - $t$  distance  $d_f(s, t)$  in  $G_f$ .

**Problem.** What are the best **augmenting paths** to use if we are interested in increasing  $d_f(s, t)$ ?

**Answer.** Shortest **augmenting paths** are the obstacles to  $d_f(s, t)$  being large. So, we simply try to destroy them by augmenting the **flow** with it.  $\otimes$

We note that finding the shortest **augmenting path** correspond to running BFS in  $G_f$ . Hence, it only takes  $O(m)$  time, hence the only challenge is how to ensure that this augmentation does not introduce new shortest paths in  $G_f$ .

**Lemma 6.1.1.** Let  $d_f(s, v)$  and  $d'_f(s, v)$  to be the distance from  $s$  to  $v \in V$  in  $G_f$  before and after augmenting the **flow** along some shortest **augmenting path**  $P$ , respectively, then  $d_f(s, v) \leq d'_f(s, v)$ .

**Proof.** Suppose the opposite, and let  $A \neq \emptyset$  to be the set of vertices  $v$  such that  $d_f(s, v) > d'_f(s, v)$ . Let  $v \in A$  be the one with the minimum  $d'_f(s, v)$ . Let  $P'$  be the shortest  $s$ - $v$  path in  $G_f$  after the augmentation, and let  $u$  be the vertex preceding  $v$  on this path (since  $v \neq s$ , such path and  $u$  exist). Hence, we have  $d'_f(s, v) = d'_f(s, u) + 1$ . Moreover, we must have that  $d_f(s, u) \leq d'_f(s, u)$  since otherwise,  $u \in A$  and  $d'_f(s, u) < d'_f(s, v)$ , contradicting the minimality of  $v$ .

**Claim.** The last edge of  $P'$ , i.e.,  $(u, v)$ , has zero residual capacity before augmented by  $P$ .

**Proof.** Otherwise, we can reach  $v$  from  $u$  before augmented by  $P$ , hence

$$d_f(s, v) \leq d_f(s, u) + 1 \leq d'_f(s, u) + 1 = d'_f(s, v),$$

contradicting to the fact that  $v \in A$ .  $\otimes$

Hence, the only way for  $(u, v)$  to have non-zero residual capacity after augmented by  $P$  would be if the edge  $(v, u)$  belongs to  $P$ . Since  $P$  is the shortest path before the augmentation, i.e.,  $d_f(s, u) = d_f(s, v) + 1$ . This means

$$d_f(s, v) = d_f(s, u) - 1 \leq d'_f(s, u) - 1 = d'_f(s, v) - 2 \leq d'_f(s, v),$$

which again contradicts to the assumption that  $v \in A$ .  $\blacksquare$

**Note.** Lemma 6.1.1 states that the distance from  $s$  doesn't decrease, not only for  $t$ , but for every  $v$ .

By symmetry, we can argue that the distance from any  $v$  to  $t$  is also non-decreasing. Hence, augmenting the **flow** using shortest paths indeed does not make things worse. We can further show that we're indeed making progress.

**Lemma 6.1.2.** At most  $mn/2$  shortest path augmentations can be made before  $d_f(s, t) \geq n$ .

**Proof.** We first note that each augmentation saturates at least one bottlenecking edge  $(u, v)$ . For an already saturated edge, before it can be saturated again in some subsequent augmentation, we must have pushed some **flow** via an **augmenting path** that contained the opposite edge  $(v, u)$ .

Let  $d(w)$  be the distance from  $s$  to  $w \in V$  in the **residual graph** just before the first saturation of  $(u, v)$ , and let  $d'(w)$  be the corresponding distance just before the **flow** is pushed along  $(v, u)$  in some subsequent augmentation that makes  $(u, v)$  has non-zero residual capacity again, as mentioned above. Since we always augment the shortest paths,  $d(v) = d(u) + 1$  and  $d'(u) = d'(v) + 1$ .

From Lemma 6.1.1, we know that  $d(w) \leq d'(w)$  for any  $w \in V$ . Hence, we have

$$d'(u) = d'(v) + 1 \geq d(v) + 1 = d(u) + 2.$$

Therefore, the distance from  $s$  to  $u$  has to increase by at least 2 by the time the edge  $(u, v)$  can again be saturated by some **augmenting path**. We conclude that each edge  $(u, v)$  can be saturated at most  $n/2$  times before  $d_f(s, u) \geq n$ , so, there is at most  $mn/2$  saturations and thus augmentations possible before  $d_f(s, t) \geq n$ .  $\blacksquare$

**Note.** We have no way of lower bounding how much flow a particular flow augmentation pushed. We can only argue that overall the  $mn/2$  augmentations we managed to push the whole max-flow value. This is an important feature of the so-called primal-dual algorithms.

Hence, we have the following.

**Theorem 6.1.4** ([Din70; EK72]). By choosing the shortest augmenting path (w.r.t.  $d_f(s, t)$ ) in Algorithm 6.1, the algorithm computes the  $s$ - $t$  max-flow on a directed graph with general capacities in  $O(m^2n)$  time.

**Proof.** From Lemma 6.1.2, we know that at most  $O(mn/2)$  augmentations is needed. Since finding each shortest augmenting path requires  $O(m)$ , the total running time takes  $O(m^2n)$ . ■

## 6.2 Blocking Flow

Dinitz then consider augmenting the so-called blocking flow in line 3, based on the analysis of the shortest augmenting path approach in Theorem 6.1.4. It turns out that this leads to a speedup of the running time from  $O(m^2n)$  to  $O(mn^2)$  [Kar73; Din70; GR98]. The idea is not to be wasteful:

**As previously seen.** Previously, we recompute the shortest augmenting path using  $O(m)$  time for each augmentation. Each search for a path via BFS gives us the whole shortest path tree, but we're only using one path from it.

The reason why this is wasteful is that augmenting along a single path can take  $m$  iterations before the current shortest  $s$ - $t$  path distance in  $G_f$  increases by one. Instead of augmenting along a single path, we try to do this for a collection of paths.

**Intuition.** If we find a flow  $g$  in  $G_f$  such that  $g$  blocks all  $s$ - $t$  shortest paths of length at most  $d$ , then after adding  $g$  to  $f$  in the residual graph, the shortest path distance increases by one. This can happen at most  $n$  times, since the shortest path distance between  $s$  and  $t$  is at most  $n$ .

This intuition leads to the so-called blocking flow, defined as follows.

**Definition 6.2.1** (Blocking flow). An  $s$ - $t$  flow  $f$  is a blocking flow if for every  $s$ - $t$  path  $p$ , at least one edge in  $p$  is saturated by  $f$ .

**Note.** In other words, a blocking flow is a maximal flow in that one cannot add greedily add flow to  $f$  in  $G$ , which is different from a max-flow.

To find a blocking flow in a general graph  $G$ , we can do a simple greedy search, where in each iteration we find an  $s$ - $t$  path and augment along the path to saturate one additional edge. The capacities along the path are reduced and those that saturated are not considered in future path selections.

**Note.** We are not computing a residual graph in this process.

It's clear that the greedy algorithm will terminate in  $O(m)$  iterations, and a naive implementation takes  $O(m^2)$  time. We claim the following while refer the proof to Lemma 6.2.1 for a special case.

**Claim.** Suppose  $f$  is a blocking flow in  $G$ , then  $d_{G_f}(s, t) > d_G(s, t)$ .

The above basically just a restatement of the intuition we have. This leads to an  $O(m^2n)$  algorithm, since we know that we only need to augment  $n$  times, where in each iteration, we find a blocking flow  $g$  in  $G_f$  and augment along  $g$ . We see that this algorithm's running time is the same as Theorem 6.1.4.

### 6.2.1 Blocking Flow in Layer Graph

The key observation that leads to an improvement from  $O(m^2n)$  to  $O(mn^2)$  is that instead of finding a **blocking flow** in the entire graph  $G_f$ , it suffices to find it only in the “layer graph” consisting of the “forward edges” in the BFS computation. We first formalize the notion of “forward edge” as follows.

**Definition 6.2.2 (Admissible).** Given a directed graph  $G = (V, E)$  and a source  $s \in V$ , an edge  $e \in E$  is *admissible* if it points away from  $s$ , i.e.,  $e$  is a part of some  $s$ - $v$  shortest path for some  $v \in V$ .

We can also talk about the **admissible** graph, i.e., the graph  $(V, E')$  where  $E'$  is the set of all **admissible** edges.<sup>1</sup> Meanwhile, the **admissible** path is a path in the **admissible** graph.

**Note.** The **admissible** edges “hop”<sup>a</sup> exactly one BFS layer, and all other edges either stay in the same layer or go back.

<sup>a</sup>This is different from the previous notion of “hop” for negative edges.

The whole reason why augmenting **blocking flows** on the **admissible** graph works is because we can still guarantee that we’re making progress:

**Lemma 6.2.1.** Suppose  $g$  is a **blocking flow** in the **admissible** graph of  $G_f$ , then  $d_{f'}(s, t) > d_f(s, t)$ , where  $f' = f + g$ .

**Proof.** Let  $f$  and  $f'$  be the **flow** before and after augmentation via the **blocking flow**  $g$ . Since every edge on a shortest  $s$ - $t$  path in  $G_f$  has to “hop” exactly one BFS layer, hence it has to be **admissible**. Furthermore, augmentation does not create in  $G_{f'}$  any edges that would be **admissible** in  $G_f$ , i.e., no new arcs in  $G_{f'}$  would “hop” a BFS layer in  $G_f$ .

After the augmentation, there is some  $s$ - $t$  cut in  $G_f$  such that all its **admissible** edges are saturated. Hence, every  $s$ - $t$  path in  $G_{f'}$  now has to use at least one edge that would not be **admissible** in  $G_f$ , so that edge would not hop a BFS layer in  $G_f$ . But, every edge in  $G_{f'}$  as well as  $G_f$  can hop at most one BFS layer in  $G_f$ , so there is no way for an  $s$ - $t$  path in  $G_{f'}$  to make up for not hopping a  $G_f$  BFS layer at least once. All such paths had to have length larger than the  $s$ - $t$  distance in  $G_f$ , hence the  $s$ - $t$  distance in the new graph had to increase by at least one. ■

Hence, we can again conclude that after at most  $n$  **blocking flow** computations, we will have an  $s$ - $t$  **max-flow**. But this time, the **blocking flow** computations only need to be done on a smaller **admissible** graph. This is faster since computing **blocking flow** in a layer DAG (e.g., the **admissible** graph) can be done faster than the naive greedy algorithm in a general graph by “adaptive DFS search”.

**Intuition.** Augmenting the **flow** using an **admissible** path does not create new **admissible** edges in the **residual** graph. Hence, when we saturate an edge on an **admissible** path, we can just discard it, as the reverse arc is never **admissible**.

Next, we discuss how to do this type of adaptive DFS search formally.

### 6.2.2 Unit Capacity Graph

Let’s first consider the case of unit capacity graph. In this case, observe that a **blocking flow** in the **admissible** graph corresponds to a maximal collection of edge-disjoint **admissible** path. Since we’re now finding a maximal set, greedy is usually the best strategy. Consider running a DFS starting at  $s$ :

1. In each step, *advance* along an unexplored outgoing **admissible** edge.
2. If reach  $t$ , retrace the path back to  $s$  and *block* it by adding it to the **blocking flow** and discarding all edges on this path.
3. If reach a vertex  $v \neq t$  with no outgoing **admissible** edges, *retreat* back along the edge we came from to  $v$  and discard it.
4. Once there is no **admissible** edges leaving  $s$ , return the **blocking flow**.

<sup>1</sup>Implicitly, we will discard edges (and nodes) with distance beyond  $d(s, t)$ .

This seems much like just search for an [augmenting path](#), but actually this is not.

**Intuition.** The key difference is that we can save information about which part of the graph we already explored, hence this is “adaptive.”

Crucially, as we never add new [admissible](#) edges, once a vertex is a dead end, it stays that way. This is why it is fine for us to discard an edge upon retreat.

**Theorem 6.2.1.** By choosing the [blocking flow](#) in [Algorithm 6.1](#), the algorithm computes the  $s$ - $t$  [max-flow](#) on a directed graph with unit capacities in  $O(mn)$  time.

**Proof.** With [Lemma 6.2.1](#), we only need to prove that the above adaptive DFS search takes  $O(m)$  for finding a [blocking flow](#) in the [admissible](#) graph. There are three types of operations:

- [Retreat](#): There is at most  $m$  retreats, as we always discard the edge when we do that. Hence, the total cost is  $O(m)$ .
- [Advance](#): Advances can be charged to the total cost of retreats and blocks corresponding to traversing back the edge.
- [Block](#): Each blocking of a path  $P$  takes  $O(|P|)$  time, but also discards  $|P|$  edges. Hence, with amortization, it is  $O(1)$  per edge. The total time is then  $O(m)$ .

Hence, the total cost is  $O(m)$ , which proves that the total runtime is  $O(mn)$ . ■

However, this is just like our naive algorithm from [Corollary 6.1.2](#), which takes  $O(mnW) = O(mn)$  as  $W = 1$  in this case. The upshot is that our analysis can be improved for *simple graph*:

**Corollary 6.2.1** ([ET75]). [Theorem 6.2.1](#) can be improved to  $O(m \cdot \min(\sqrt{m}, n^{2/3}))$  for simple graph.

**Proof.** We prove the result in two cases.

**Claim.** We can improve the bound to be  $O(m^{3/2})$ .

**Proof.** Suppose we already did  $k$  [blocking flows](#). Consider the [max-flow](#) in the [residual graph](#): if we decompose it into paths, then the number of these paths will be exactly the value of the remaining [flow](#). Since we have done  $k$  [blocking flows](#), the  $s$ - $t$  distance in the [residual graph](#) is greater than  $k$ , hence, each of these [flow](#) paths has length greater than  $k$ .

Since these [flow](#) paths are edge-disjoint, with their total volume being  $m$ , the number of these [flow](#) paths is less than  $m/k$ . Each [blocking flow](#) increases the value of the [flow](#) by at least one, as there always is at least one [admissible](#) path. Hence,  $m/k$  additional [blocking flows](#)<sup>a</sup> suffices. Combining the above, the total running time is then  $O(m(k + m/k))$  from [Theorem 6.2.1](#), which is minimized when  $k = \sqrt{m}$ , giving  $O(m^{3/2})$  as desired. ⊗

<sup>a</sup>Or even just [augmenting paths](#) can do the job.

**Claim.** We can improve the bound to be  $O(mn^{2/3})$ .

**Proof.** Suppose we already did  $k$  [blocking flows](#). It's clear that there are at most  $k/2$  different layers in the BFS [admissible](#) graph with vertices greater than  $2n/k$  from a counting argument. Then by the pigeonhole principle, there are two consecutive layers such that both have less than  $2n/k$  vertices. There can be at most  $O(n^2/k^2)$  edges between these two layers. Observe that these edges form an  $s$ - $t$  cut, hence the  $s$ - $t$  [max-flow](#) value is  $O(n^2/k^2)$ .

From a similar argument, we only need  $O(n^2/k^2)$  additional [blocking flows](#), hence the total running time is  $O(m(k + n^2/k^2))$  from [Theorem 6.2.1](#), which is minimized when  $k = n^{2/3}$ , giving  $O(mn^{2/3})$  as desired. ⊗

By combining the above two cases, we're done. ■

**Remark.** The  $O(m\sqrt{m})$  bound extends to multi-graphs, while  $O(mn^{2/3})$  bound does not.

**Proof.** This is clear from the counting argument for  $O(mn^{2/3})$ , where we assume for two consecutive layers with each less than  $2n/k$  vertices, there are only  $O(n^2/k^2)$  edges between them, which is not the case for multi-graphs.  $\otimes$

### 6.2.3 General Capacity Graph

For graph with general capacities, as we promised at the beginning of this section, we can achieve  $O(mn^2)$  running time [Kar73; Din70; GR98]. To prove this, we look into what breaks for the adaptive DFS search for unit capacity case when we try the same analysis as in Theorem 6.2.1 for general graphs.

From the previous adaptive DFS search algorithm, the basic idea of *advance*, *retreat*, and *block* are still valid. The key problem is that when augmenting, i.e., *blocking*:

**Problem.** We might saturate only one bottlenecking edge on the  $s$ - $t$  path we found!

**Answer.** Simply retrace back to the tail of the farthest saturated edge and repeat.  $\otimes$

Specifically, consider the following modified version of the adaptive DFS search, starting at  $s$ :

1. In each step, *advance* along an unexplored outgoing *admissible* edge.
2. If reach  $t$ , *block* the  $s$ - $t$  path by adding it to the *blocking flow* (with value to be the bottleneck) and discarding edges with 0 residual capacity on this path. Finally, retrace back to the head of the farthest saturated edge on this path.
3. If reach a vertex  $v \neq t$  with no outgoing *admissible* edges, *retreat* back along the edge we came from to  $v$  and discard it.
4. Once we retrace back to  $s$  and there is no *admissible* edges leaving  $s$ , return the *blocking flow*.

**Theorem 6.2.2.** By choosing the *blocking flow* in Algorithm 6.1, the algorithm computes the  $s$ - $t$  *max-flow* on a directed graph with general capacities in  $O(mn^2)$  time.

**Proof.** In the new adaptive DFS search, every *advance* is still paid for by costs of corresponding *retreat* or *block*. In this case, all *retreats* still cost  $O(m)$  in total, and each *block* now costs  $O(n)$  as only one bottlenecking edge can be discarded per *block*. Therefore, the *blocking flow* computation now takes  $O(mn)$  time, proving the result.  $\blacksquare$

**Intuition.** Using *blocking flows* enables us to charge  $m$  into  $n$  because of the resulting more efficient search for *augmenting paths*.

In fact, we can make the above intuition stronger via an even more careful analysis.

**Corollary 6.2.2.** By choosing the *blocking flow* in Algorithm 6.1, the algorithm computes the  $s$ - $t$  *max-flow* on a directed graph with general capacities in  $O(mn + |f^*|n)$  time, where  $|f^*|$  is the  $s$ - $t$  *max-flow* value.

**Proof.** Since each *block* corresponds to finding a new *augmenting path*, hence, each *block* increases the value of the *flow* by at least 1. This means that the total cost of *block* is only  $O(n|f^*|)$ . Thus, the running time is actually  $O(mn + n|f^*|) = O((m + |f^*|)n)$ .  $\blacksquare$

In other words, the *blocking flow* approach takes  $O(n)$  time “per unit of flow routed,” while the basic/naive *augmenting path* approach took  $O(m)$  time (this is true even if capacities are not unit) as shown in Theorem 6.1.4.



### 6.2.4 Scaling Technique

If we're dealing with integral capacities, we can utilize the idea of **scaling**, just like what we have seen in Algorithm 3.3 for the SSSP. The general idea is still the same: apply the “unit case” algorithm to obtain a “general numerical case” algorithm. This time, consider the bit representation of the capacities. We observe that bits can correspond to unit case by looking at only the  $i^{\text{th}}$  bit at a time for every  $c(e)$ 's! To exploit this idea, consider the following:

**Intuition.** Starting from an optimal solution to the rounded down instance (i.e., dropping some least significant bits for the capacities). One can repeatedly put back the next dropped bit, i.e., shifting/adding one bit to the right, and fix the solution along the way.

Such a shifting corresponds to a new capacity that is either doubled or doubled and plus 1. Hence, intuitively, the problem boils down to applying an algorithm that works well only for unit capacity case as every iteration we only need to deal with one unit of deviation in some sense.

**Note.** One can think of scaling as a reverse process of rounding.

For convenience, we use the following notation to refer to  $i^{\text{th}}$  bit.

**Notation.** Given  $a \in \mathbb{N}$ , let  $a_i$  be the  $i^{\text{th}}$  most significant bit in  $a$ 's bit representation.

Formally, the idea of scaling leads to a weakly polynomial algorithm.

**Theorem 6.2.3.** There exists a scaling-based algorithm that computes the  $s$ - $t$  max-flow on a directed graph  $G = (V, E)$  with integral capacities in  $O(mn \log W)$  time, where  $W = \max_{e \in E} c(e)$ .

**Proof.** We first describe the algorithm. In the first iteration, look at the most significant bit  $c^{(1)}(e) = c_1(e) \in \{0, 1\}$ , which corresponds to a unit capacity graph. We then obtain the  $s$ - $t$  max-flow  $f$  on the graph with capacity  $c^{(1)}$  via the unit capacity algorithm in Theorem 6.2.1, which takes  $O(mn)$  time. In the next iteration, we consider the new capacity  $c^{(2)}(e) \in \{0, 1, \dots, 2^2\}$  to be the integer represented by the first two most significant bits of  $c(e)$ , i.e.,  $c^{(2)}(e) = 2c^{(1)}(e) + c_2(e)$  for some  $c_2(e) \in \{0, 1\}$ . We also double  $f$  as  $f'(e) = 2f(e)$ , which is clearly feasible for  $c^{(2)}$ .

Now, consider the corresponding residual graph  $G_{f'}$ . The crucial observation is the following.

**Claim.** There is a min-cut in  $G_{f'}$  with residual capacity at most  $m$ .

**Proof.** Since our flow was optimal before, there must exist an  $s$ - $t$  cut with its residual capacity (w.r.t.  $f$ ) be 0. Hence, after shifting a new bit and doubling the flow to be  $f'$ , this cut has residual capacity being at most  $m$ .  $\otimes$

Hence, we see that after the bit shifting and doubling the current flow, finding the new optimum boils down to solving the  $s$ - $t$  max-flow in a graph with  $|f^*| \leq m$ . From Corollary 6.2.2, we know that this takes  $O(mn)$  time.<sup>a</sup> As there are only  $O(\log W)$  many shiftings required, the total running time of this algorithm is  $O(mn \log W)$ .  $\blacksquare$

<sup>a</sup>Observe that after the shifting,  $G_{f'}$  does not need to be unit capacity: we only have control on  $|f^*|$ .

**Remark.** The same idea with Theorem 6.1.2 improves Theorem 6.1.3 to  $O(m^2 \log W)$ .

### 6.2.5 Data Structure for Blocking Flow

So far, our best strongly polynomial time algorithm runs in  $O(mn^2)$  (Theorem 6.2.2), and weakly polynomial time algorithm runs in  $O(mn \log W)$  (Theorem 6.2.3). A natural question is that whether we need to pay this extra factor of  $n$  for being strongly polynomial, or maybe there is a better way to implement blocking flows in general graphs?



**As previously seen.** In the adaptive DFS search, we repeatedly find  $s$ - $t$  paths while discarding parts of the graph that we have finished searching. Suppose we find an  $s$ - $t$  path  $P$ . Then, we decrease the capacities along  $P$  as to saturate the bottleneck edge  $e$  along  $P$ , removing  $e$  from the search. Then, we start the searching over from  $s$ .

The key bottleneck is that **blocking** in the adaptive DFS search takes  $\Theta(n)$  time but might only discard a single edge from the whole **admissible** path we found, and then we start the searching all over again from  $s$ .

**Intuition.** This is a waste of the two partial paths of  $p - e$ , one from  $s$  and one to  $t$ , respectively.

Hence, the question is that, as the pieces of this path that were not discarded are still useful, can we somehow take advantage of this fact and maintain these pieces for efficient access later? This is a data structure question, and fortunately, there is the so-called **link/cut tree** [ST85] that supports all operations we need for implementing the adaptive DFS search!

Before we formally introduce the **link/cut tree**, we first need to define what an **arborescence** is.

**Definition 6.2.3 (Arborescence).** An *arborescence* is a rooted directed tree with edges always directed towards the root.

The **Link/cut tree** maintains a collection of **arborescences** with several supported operations:

**Definition 6.2.4 (Link/cut tree).** The *link/cut tree* data structure maintains a disjoint collection of **arborescences**, which is modified via the following operations:

- **maketree**( $v$ ): make a new **arborescence** with only a root  $v$ .
- **link**( $v, w$ ):  $v$  is the root of an arborescence, and  $w$  is a vertex in a different **arborescence** from  $v$ , add a directed edge  $(v, w)$ , i.e., making  $v$  a child of  $w$ .
- **cut**( $v$ ): detach  $v$  from its **arborescence** by deleting the edge toward  $v$ 's root, and make the sub-**arborescence** rooted at  $v$  a new **arborescence** in the collection.

For each vertex  $v$ , the following queries are supported:

- **root**( $v$ ): return the root of  $v$ 's arborescence.
- **min-capacity**( $v$ ): return the minimum capacity edge in the path from  $v$  to its root.
- **add-capacity**( $v, c$ ): add  $c$  to the capacity of every edge in the path from  $v$  to its root.

**Remark.** In general, the **link/cut trees** allows for values along edges and vertices and efficient aggregate queries and updates along node-to-root paths in the arborescence. In **Definition 6.2.4**, we have defined two such operations (**min-capacity** and **decrease-capacity**) specifically for computing **blocking flows** in **residual graphs**.

It has been shown that the **link/cut tree** can be implemented efficiently.

**Theorem 6.2.4 ([ST83]).** **Link/cut tree** can be maintained in amortized  $O(\log n)$  for each operation.

**Proof intuition.** To get a high level idea of the implementation, let us just consider the representation of an **arborescence** when it is just a path.

**Intuition.** A path can be represented as a dynamic binary search tree (BST) of height  $O(\log n)$ . Then all operations of a **link/cut tree** are trivial. For a general **arborescence**, the **heavy light decomposition** [ST83] can essentially break it into a heavy path and some small things.

Specifically, consider maintaining an ordered list of vertices on path in a **splay tree** (but we could use any balanced BST). The key trick is that instead of storing the current residual capacity of each edge, store only the “deltas”  $\Delta$ , i.e., their differences. This way, the true value of the residual

capacity of an edge is obtained by summing all the  $\Delta$ 's on the path from that edge node to the root of the tree. Note that this representation is easy to maintain under rotations.

To add a capacity of  $c$  on a path from some vertex  $v$  to the root of the path, splay successor of  $v$  to root, add  $c$  from the root of the left subtree. Similarly, one can maintain at each node the minimum capacity of its subtree (to help with min-capacity( $v$ )). ■

We now formally describe how to use [link/cut tree](#) to speed up the adaptive DFS search for a general capacity graph, as described just above [Theorem 6.2.2](#).

**Theorem 6.2.5.** By choosing the [blocking flow](#) with [link/cut tree](#) in [Algorithm 6.1](#), the algorithm computes the [s-t max-flow](#) on a directed graph with general capacities in  $O(mn \log n)$  time.

**Proof.** We maintain a forest of [arborescences](#) of “[block-able](#)” edges, i.e., [admissible](#) edges that are non-saturated. This means, we will discard parts of the graph that we have finished searching.

Specifically, these [arborescences](#) will always be subtrees of the [residual graph](#) such that  $s$  is always a leaf, and  $t$  is always a root (of some, potentially different, [arborescences](#)). Moreover, the path from  $s$  to  $s$ 's root (in  $s$ 's [arborescence](#)) represents a partial shortest path from  $s$  in the [residual graph](#). The goal is to find a new  $s$ - $t$  path to [block](#) by keep extending this path (at  $s$ 's root).

- Initially, all vertices are isolated by calling `maketree( $v$ )` for all  $v \in V$ .
- In each iteration, start from the current vertex, i.e.,  $s$ 's root  $r = \text{root}(s)$ .
- Unless  $r = t$ , scan the unexamined outgoing [admissible](#) edges from  $r$  in the [residual graph](#):
  - If there is no edge at all, then there is no  $s$ - $t$  path via  $r$ . We [retreat](#) from  $r$ :
    - \* Delete  $r$  via calling `cut( $e$ )` for all  $e$  incident to  $r$ . Discard  $r$  after this (which is doable since now  $r$  is just a singleton [arborescence](#)).
    - \* Or if  $r = s$ , we finish the search.
  - Otherwise, we find a directed [admissible](#) edge  $(r, u)$  to [advance](#):
    - \* Merge  $s$ 's [arborescence](#) with  $u$ 's [arborescence](#) via `link( $r, u$ )`. This makes  $s$ 's [arborescence](#) a sub-[arborescence](#) of  $u$ , and  $s$ 's new root becomes `root( $u$ )`.
    - \* Repeat.
- If  $r = t$ , then the  $s$ - $r$  path  $P$  in  $s$ 's [arborescence](#) is the desired  $s$ - $t$  path. We [block](#)  $P$ :
  - Find the bottleneck capacity  $b$  on  $P$  via  $b = c(e)$  for some  $e = \text{min-capacity}(s)$ .
  - Decrease all edge capacities on  $P$  by  $b$  via calling `add-capacity( $s, -b$ )`.
  - Cut all edges with capacity 0 now by calling `cut( $e$ )` for all  $e$  with  $c(e) = b$ . These edges can be identified via calling `min-capacity( $s$ )` repeatedly and checking their capacity.
  - Repeat.

We continue the search until  $r = s$  and there are no edges leaving  $s$  left to explore. We claim that constructing a [blocking flow](#) takes only constant number of [link/cut tree](#) operations.

**Claim.** Computing a [blocking flow](#) takes a constant number of [link/cut tree](#) operations.

**Proof.** Observe that the search does a constant number of operations per edge in the [residual graph](#) and per [augmenting path](#) obtained by the search. For each edge  $e$ , we `link( $e$ )` as it is searched, and `cut( $e$ )` as we either retrace from  $e$  or saturate  $e$ .<sup>a</sup> For each [augmenting path](#), we execute a constant number of operations (querying `min-capacity( $s$ )` or `add-capacity( $s, -b$ )`) to update the [residual graph](#). Each [augmenting path](#) saturates an edge that is then deleted from the [residual graph](#), so there are at most  $m$  [augmenting path](#). ⊛

<sup>a</sup>When  $e$  is `cut( $e$ )` by saturation, it is also preceded by a `min-capacity( $s$ )` query that identifies it.

Hence, one can construct a [blocking flow](#) in only  $O(m \log n)$  time, instead of  $O(mn)$  from [Theorem 6.2.2](#). We conclude that the whole algorithm now takes only  $O(mn \log n)$  time. ■

**Note.** Theorem 6.2.4 has a long line of history [Shi78; GN80; Sle81; ST81; ST83].

**Corollary 6.2.3.** There is an algorithm that computes the  $s$ - $t$  max-flow on a directed graph with integral capacities in  $O(mn \log n^2/m)$  time.

**Proof.** With more care, the  $\log n$  factor in Theorem 6.2.5 can be improved to  $\log n^2/m$  [GT90]. ■

Finally, for completeness, we summarized the algorithm described in Theorem 6.2.5 as follows.

---

**Algorithm 6.2:** Ford-Fulkerson Algorithm with Link/Cut Tree

---

**Data:** A connected directed graph  $G = (V, E)$  with edge capacity  $c: E \rightarrow \mathbb{R}_+$ , source  $s$ , sink  $t$

**Result:** A blocking flow  $f$

```

1  $c' \leftarrow c$  // Record the original capacity
2  $E' \leftarrow \text{Admissible}(G, c, s, t)$  // Set of admissible edges via DFS from  $s$  to  $t$ 
3  $T \leftarrow \text{Link/Cut Tree}()$  // Initialize an empty link/cut tree
4
5 for  $v \in V$  do
6    $T.\text{maketree}(v)$ 
7
8 while  $1 = 1$  do
9    $r \leftarrow T.\text{root}()$ 
10  if  $r = t$  then // Find an  $s$ - $t$  path to block
11     $e \leftarrow T.\text{min-capacity}(s)$ 
12     $b \leftarrow c(e)$ 
13     $T.\text{add-capacity}(s, -b)$  // Assuming  $T$  and  $G$  shares  $c$ 
14    for  $e \leftarrow T.\text{min-capacity}(s)$  do
15      if  $c(e) = 0$  then
16         $T.\text{cut}(e)$ 
17         $E' \setminus \{e\}$ 
18  else
19    if  $\nexists e = (r, v) \in E'$  then // Retreat from  $r$ 
20      if  $r = s$  then // Delete saturated edges
21         $f \leftarrow c - c'$  // Compute  $f$ 
22        return  $f$ 
23      for  $e = (v, r) \in E'$  do
24         $T.\text{cut}(e)$ 
25         $T \leftarrow T \setminus \{r\}$ 
26    else // Find a directed admissible edge  $(r, u)$  to advance
27       $T.\text{link}(r, u)$ 

```

---

**Remark.** In line 21, as described in Theorem 6.2.5, we're not actually augmenting the  $s$ - $t$  path we found, we only **block** (i.e., add) them to the **blocking flow** without introducing the residual edge, hence this step is necessary. One can also directly augment during the algorithm run.

## Lecture 23: Push-Relabel Algorithm

### 6.3 Push-Relabel

21 Nov. 11:00

On the other hand, another approach utilize the so-called **preflow**. [GT88]

**Definition 6.3.1 (Preflow).**  $f: E \rightarrow \mathbb{R}_+$  is a *preflow* if

- (i)  $0 \leq f(e) \leq c(e)$  for all  $e \in E$ ;
- (ii)  $\sum_{e \in \delta^-(v)} f(e) \geq \sum_{e \in \delta^+(v)} f(e)$  for all  $v \in V$ .

So just like [flow](#), [preflow](#) violates the [flow conservation](#) constraint.

**Notation.** For a [preflow](#)  $f$ , the *excess* at  $v$  is defined as  $\sum_{e \in \delta^-(v)} f(e) - \sum_{e \in \delta^+(v)} f(e)$ .

If the excess is 0 for all  $v \neq t$ , then  $f$  is a [flow](#). The residual graph w.r.t. [preflow](#)  $f$  is the same as before.

**Definition 6.3.2 (Label).**  $\ell: V \rightarrow \mathbb{Z}_+$  is a *label* if

- (i)  $\ell(s) = n$  and  $\ell(t) = 0$ ;
- (ii) if  $(u, v) \in E(G_f)$ , then  $\ell(v) \leq \ell(u) - 1$ .

The initial [preflow](#) is defined as  $f(e) = c(e)$  for all  $e \in \delta^+(s)$ , and 0 otherwise, and the initial [label](#) is defined as  $\ell(s) = n$  with  $\ell(t) = 0$ , and also  $\ell(v) = 0$  for all  $v \in V \setminus \{s, t\}$ .

**Intuition.** We always try to maintain the invariance that some cut is saturated, i.e., a [preflow](#) with value equal to some cut, hence more than the [max-flow](#).

Define an activate set to be  $A := \{v \neq s, t \mid \text{excess}(v) > 0\}$ . Consider two different operations:

- (i) Full push along forward edge  $e$  (w.r.t. the [label](#)) from  $v \in A$ .
  - *Saturating* push: when  $\text{excess}(v) > c(e)$ , then we push  $c(e)$ .
  - *Non-saturating* push: when  $\text{excess}(v) \leq c(e)$ , then we push  $\text{excess}(v)$ .
- (ii) If  $v \in A$  and has no forward arc, then relabel:  $\ell(v) \leftarrow \ell(v) + 1$ .

By greedily doing the above two operations, we can prove that it'll terminate in  $O(mn^2)$  time. With a bit more careful, this is actually  $O(n^3)$ . This can be further improved to  $O(n^2\sqrt{m})$ .

**Lemma 6.3.1.** We see that  $\ell(u) - \ell(v) \leq d_{G_f}(u, v)$ .

**Proof.** This is because we restrict that any forward arc can only have label more than 1. ■

We see that since  $\ell(s) = n$  and  $\ell(t) = 0$  always, hence  $\ell(s) - \ell(t) = n \leq d_{G_f}(s, t)$ , i.e.,  $s$  can't reach  $t$  in  $G_f$ .

**Lemma 6.3.2.** Suppose  $f$  is a [preflow](#) that the label invariants and  $\text{excess}(v) = 0$  for all  $v \neq s, t$ . Then there exists an index  $i$  such that  $U = \{v \in V \mid \ell(v) > i\}$  induces an [s-t min-cut](#).

**Proof.** Specifically, from the Pigeonhole principle, there exists some  $i$  such that no vertices are labeled as  $i$ . Then, by the flow-conservation, the cut  $U$  value is equal to the flow value (just as the classical proof of [max-flow min-cut theorem](#)), we're done. ■

**Lemma 6.3.3.** If  $v \in A$ , then there is a path from  $v$  to  $s$  in  $G_f$ .

**Proof.** This is by induction. ■

**Lemma 6.3.4.** For each  $v$ ,  $\ell(v)$  is non-decreasing, non-negative, and is at most  $2n - 1$ .

**Proof.** From [Lemma 6.3.3](#) and [Lemma 6.3.1](#), this is easy. ■

We see that the cost of relabeling is at most  $O(n^2)$  from [Lemma 6.3.4](#).

**Lemma 6.3.5.** The number of saturating pushes is  $O(mn)$ .

**Proof.** For any fixed arc  $(u, v)$ , to push, we know that  $\ell(v) = \ell(u) + 1$ . To do a saturating push again along  $(u, v)$ , the label of  $v$  needs to be relabeled above  $u$  in the first place, i.e.,  $\ell(v)$  needs to go up by at least 2. Hence, the total number of this can happen is  $O(n)$ . There are  $O(m)$  edges in the residual graph, so we're done. ■

**Lemma 6.3.6.** The number of non-saturating function is  $O(mn^2)$ .

**Proof.** Consider a potential  $\phi = \sum_{v \in A} \ell(v)$ . We see that for a saturating push, the potential can go up by at most  $2n - 2$ , hence the potential can go up to at most  $O(mn^2)$ . However, for every non-saturating function, one can argue that the potential decrease by at least 1. Hence, we're done. ■

**Lemma 6.3.7.** Consider only doing operation for the vertex with the highest label. Then this is  $O(n^3)$ .

**Proof.** We see that if we do this, then consider a fixed vertex  $u$ . At each label level  $i$ ,  $u$  can only do  $O(n)$  non-saturating push since some vertex will need to first go higher than  $i$  and push to  $u$  again to make it active. ■

# Appendix

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