MATH595
Stochastic Processes on Graphs

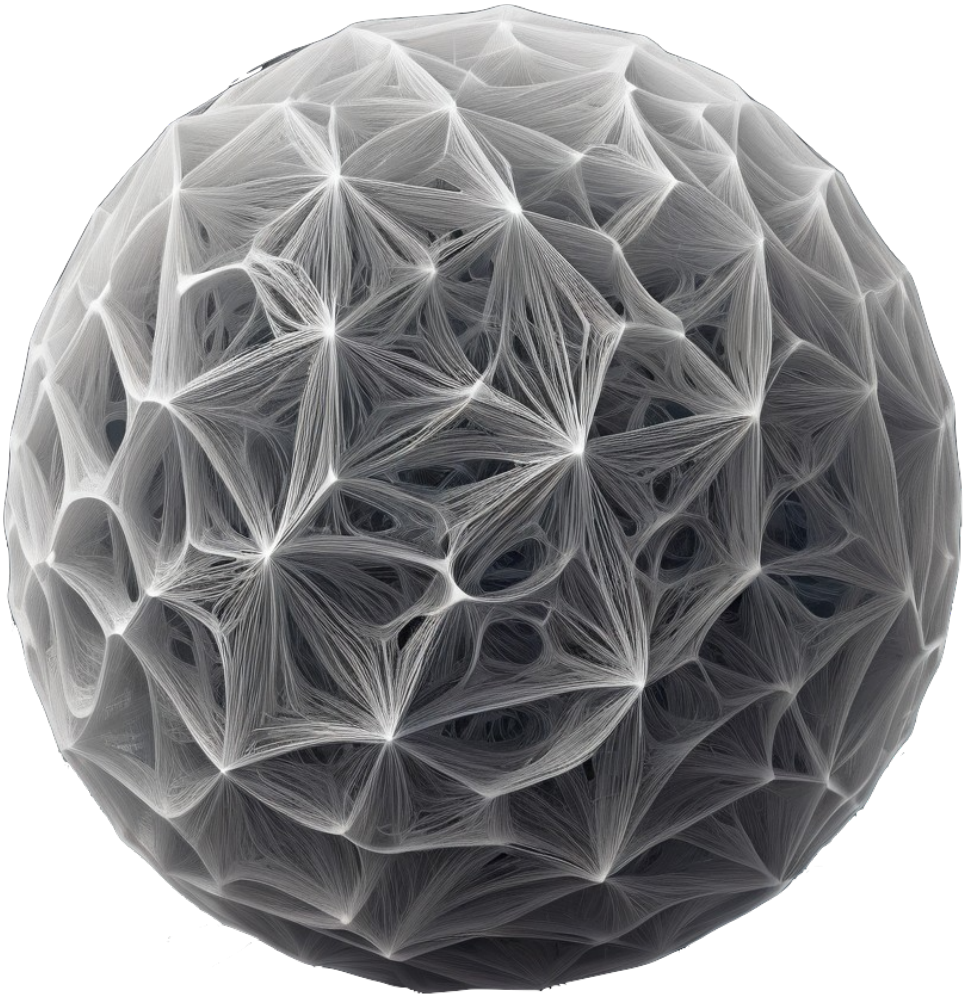
Pingbang Hu

January 22, 2025

Abstract

This is an advanced graduate-level math course taught by Partha Dey at University of Illinois Urbana-Champaign.

We list some references of this course, although we will not follow any particular book page by page: *Random Graph Dynamics* [Dur10], *Random Graphs* [JLR11], *Random Graphs and Complex Networks* [Van24].



This course is taken in Spring 2025, and the date on the cover page is the last updated time.

Contents

Chapter 1

Introduction

Lecture 1: Overview

21 Jan. 9:30

1.1 Emergence of Graph Structure

In this course, we will consider undirected, unweighted, and finite graph $G = (V, E)$. Given a graph $G = (V, E)$, for any $x, y \in V$, we define $\omega_{xy} \coloneqq \mathbb{1}_{(x,y)\in E}$ as the indicator of (x, y) in E.

1.1.1 Structure

One of the fundamental structures in a graph is the connected component, where we now define.

Definition 1.1.1 (Connected)**.** Given a graph $G = (V, E)$, we say $x, y \in V$ is *connected* if there exists a path $x = v_1, \dots, v_k = y$ such that $\omega_{v_i v_{i+1}} = 1$ for all $1 \le i \le k - 1$.

Notation. We will write $x \to y$ if x is connected to y.

It's easy to see that $\to$ is an equivalent relation, hence, one can define the so-called connected component, which is an equivalent class of G with $\to$.

Definition 1.1.2 (Connected component)**.** Given a graph G, a *connected component* $C \subseteq V$ is a maximal[a] size subset of V such that for all $x, y \in C$, $x \to y$.

[a] Note the wording: it's not equivalent to maximum.

Notation. For a particular vertex $v \in V$, we define $C(v, G) \coloneqq \{u \mid u \to v \text{ in } G\}$ as the connected component containing v. If G is realized, we simply write $C(v)$.

Connected component is an example of *structure*. Other common structures include triangle, for-cycle, clique, etc. A central problem we will be asking is the following:

Problem (Property)**.** Given a graph G, whether it contains a certain structure; if yes, how many?

1.1.2 Random Graph and Random Graph Process

In this course, instead of any graph, we are interested in certain graph models such that when the number of vertices grow, some kinds of structure emerges. The most famous (and simple) random graph model is the Erdős-Rényi random graph model.

Definition 1.1.3 (Erdős-Rényi random graph)**.** The *Erdős-Rényi random graph* model, denoted as $G(n, p)$, is a random graph generated on n vertices such that any two vertices are connected with

probability $p \in (0,1)$ independently.

Note. There are lots of independence and symmetry, leading to closed forms for many calculations.

To get a less restrictive model, one can also consider inhomogeneous model, where we let p_{xy} differ for different pairs of $(x,y) \in V \times V$. On the other hand, to relax edge independence, the so-called *exponential random graph model* exists.

Remark. These model all have light-tail. There are also models with heavy tail behavior, e.g., random graph with specified degree distribution, and preferential attachment model.

It's natural to view these random graph model by a random sequence of graphs, which we call graph process. People are interested in several optimization problems of such a graph process.

Example (Optimization on graph process)**.** Given a graph process, what's the (expected) number of the largest cycle, or what's the minimum spanning tree, or some maximum weight problem.

On the other hand, we can also consider another layer of randomness, where we are given a fixed graph, and consider stochastic processes on this graph.

Example. Infection model on a social network, or a growth process.

Some other more advanced topics include Gibbs measures, spin model (Ising model and its generalization Potts model), and spin glass model.

1.2 Erdős-Rényi Random Graph

Recall the Erdős-Rényi random graph model.

As previously seen. Let $V = [n] \coloneqq \{1, \dots, n\}$ and $p \in [0,1]$. For every $1 \le i < j \le n$, we let $\omega_{ij} \overset{\text{i.i.d.}}{\sim} \mathrm{Ber}(p)$, which induces $E \coloneqq \{(i,j) \mid \omega_{ij} = 1, 1 \le i < j \le n\}$.

Due to the independence and the simplicity, we get several immediate results.

Claim. The number of edges converges in distribution to a standard normal, in particular,

$$\frac{|E| - \binom{n}{2}p}{\sqrt{\binom{n}{2}p(1-p)}} \xrightarrow{D} \mathcal{N}(0,1),$$

if and only if $\binom{n}{2}p(1-p) \to \infty$. As a corollary, we have $|E| / \binom{n}{2}p \approx 1$.

Proof. We see that $|E| = \sum_{1 \le i < j \le n} \omega_{ij} \sim \mathrm{Bin}(\binom{n}{2}, p)$, hence, $\mathbb{E}[|E|] = \binom{n}{2}p = n(n-1)p/2$. Then, the result follows directly from the central limit theorem. ⊛

Now it's a good time to bring up another random graph model, $G(n,m)$, where we sample a graph with n vertices and m edges uniformly. This is actually the original Erdős-Rényi random graph model.

Remark. If we choose p such that $m \approx \binom{n}{2}p$, the results often transfer between $G(n,p)$ and $G(n,m)$.

1.2.1 Density and Phase Transition

We now introduce the concept of dense and sparse graph, which is decided by the parameter $|E| / \binom{n}{2}$.

Definition 1.2.1 (Dense graph)**.** A graph $G = (V,E)$ is *dense* if there exists a constant $\epsilon > 0$ such that $|E| / \binom{n}{2} > \epsilon$.

Definition 1.2.2 (Sparse graph). A graph $G = (V, E)$ is *sparse* if it is not dense.

Putting the motivation of defining Definition 1.2.1 and Definition 1.2.2 in this way aside, let's first observe an interesting property for the Erdős-Rényi random graph model. Firstly, note that the typical degree of the Erdős-Rényi random graph is some constant since for $G(n, p)$,

$$\frac{1}{|V|}\sum_{v\in V}\deg(v) = \frac{2|E|}{|V|} \approx \frac{2n(n-1)}{2}\frac{p}{n} = (n-1)p.$$

Note. Regime we consider hence depend on $\lambda := np$ for some $\lambda \in (0, \infty)$.

Then, given a particular vertex, its degree follows $\mathrm{Bin}(n-1, p) = \mathrm{Bin}(n-1, \lambda/n)$.

Claim. $\mathrm{Bin}(n-1, \lambda/n) \xrightarrow{D} \mathrm{Pois}(\lambda)$ as $n \to \infty$.

Proof. We see this in a straightforward way: for any k, $X \sim \mathrm{Bin}(n-1, \lambda/n)$ has a pmf

$$\Pr(X = k) = \binom{n-1}{k}\cdot\left(\frac{\lambda}{n}\right)^k\cdot\left(1-\frac{\lambda}{n}\right)^{n-1-k} \to \frac{\lambda^k}{k!}e^{-\lambda},$$

which is the pmf of $\mathrm{Pois}(\lambda)$. Hence, by definition, $\mathrm{Bin}(n-1, \lambda/n) \xrightarrow{D} \mathrm{Pois}(\lambda)$. Another proof is based on the total variational distance d_{TV}.

> **As previously seen** (Total variational distance). For the discrete case, given two discrete probability distributions p, r with a finite support Ω,
>
> $$d_{\mathrm{TV}}((p_k)_{k\in\Omega}, (r_k)_{k\in\Omega}) := \frac{1}{2}\sum_{k\in\Omega}|p_k - r_k|.$$

Now, consider the empirical degree distribution defined as $d^{(n)} := \frac{1}{n}\sum_v \delta_{\deg(v)}$. We see that

$$d_{\mathrm{TV}}\left(d^{(n)}, \mathrm{Pois}(\lambda)\right) = \frac{1}{2}\sum_{k=0}^{n}\left|\frac{|\{v \mid \deg(v) = k\}|}{n} - \frac{e^{-\lambda}\lambda^k}{k!}\right|,$$

and by Jensen's inequality,

$$\mathbb{E}\left[d_{\mathrm{TV}}\left(p^{(n)}, \mathrm{Pois}(\lambda)\right)\right] \le \frac{1}{2}\sum_{k=0}^{n}\sqrt{\mathbb{E}\left[\left(\frac{|\{v \mid \deg(v) = k\}|}{n} - \frac{e^{-\lambda}\lambda^k}{k!}\right)^2\right]} \approx \sqrt{\frac{p_k}{n}} = O\left(\frac{1}{\sqrt{n}}\right),$$

where $p_k = e^{-\lambda}\lambda^k/k!$. ⊛

The above basically gives a distance-one neighborhood characterization of $G(n, p)$. However, this actually gives us a higher-level picture on larger neighborhoods.

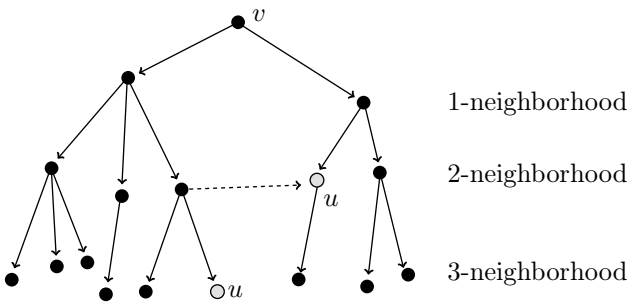
Notation. Given a graph G, let $C_{\max_i}$ denotes the i^{th} largest connected component in G.

Theorem 1.2.1. Consider the Erdős-Rényi random graph model $G(n, \lambda/n)$ for some $\lambda > 0$.

- If $\lambda < 1$, the graph is disconnected with high probability such that $|C_{\max_1}| = \Theta_p(\log n)$. In particular,
  $$\Pr\left(|C_{\max_1}| \ge a\log n\right) \to 0 \text{ as } n \to \infty \text{ if } a \dots.$$
- If $\lambda > 1$, $\frac{1}{n}|C_{\max_1}|$ converges to a constant, i.e., there exists a giant component. Moreover, $C_{\max_2}$ has size of $O(\log n)$.

- At $\lambda = 1$, the random vector $\frac{1}{n^{2/3}}(|C_{\max_1}|, |C_{\max_2}|, \dots)$ converges in distribution to a non-trivial limit.

Before we prove it formally, we first give a heuristic explanation. Consider the neighborhood structure of vertex 1, which should be tree-like, at least locally:



This is because, for any $k \geq 2$, the expected number of cycles of length k in this structure is

$$\binom{n}{k} \cdot k! \cdot \left(\frac{\lambda}{n}\right)^k \approx n^k \frac{\lambda^k}{n^k} = \lambda^k,$$

which implies

$$\sum_{k=2}^{n} \mathbb{E}[\#\text{cycle of length } k] \leq \frac{1}{1-\lambda}$$

when $\lambda < 1$. Hence, in this regime, for a random vertex v, up to any finite distance k, we will see no cycle. On the other hand, we can view this neighborhood structure as a branching process, and for $\lambda < 1$,

$$\begin{aligned}\Pr\left(|C_{\max_1}| > a \log n\right) &\leq \sum_{v=1}^{n} \Pr(|C(v)| > a \log n) \\ &= n \Pr(|C(1)| > a \log n) \leq n \Pr(|\mathrm{BP}(\mathrm{Pois}(\lambda))| > a \log n).\end{aligned}$$

Consider $s_0 = 1$, $s_1 = X_1 \sim \mathrm{Pois}(\lambda) - 1$, and $s_2 = s_1 + x_2$, etc. Then, the hitting time $T = H^{\{v\}}$ has the same distribution as $|\mathrm{BP}(\mathrm{Pois}(\lambda))|$.

Appendix

Bibliography

[Dur10] Rick Durrett. *Random graph dynamics*. Vol. 20. Cambridge university press, 2010.

[JLR11] Svante Janson, Tomasz Luczak, and Andrzej Rucinski. *Random graphs*. John Wiley & Sons, 2011.

[Van24] Remco Van Der Hofstad. *Random graphs and complex networks*. Vol. 54. Cambridge university press, 2024.