

CS598
Topics in Graph Algorithms

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Abstract

This is an advanced graduate-level graph algorithm course taught by [Chandra Chekuri](#) at University of Illinois Urbana-Champaign.



This course is taken in Fall 2024, and the date on the cover page is the last updated time.

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Chapter 1

Introduction

Lecture 1: Overview

Throughout the course, we consider a graph $G = (V, E)$ such that $n := |V|$ and $m := |E|$. Let's see some examples about the recent breakthroughs. 27 Aug. 11:00

Example (Shortest paths with negative length). The classical algorithm runs in $O(mn)$. In 2022, [BNW] came up with an algorithm $O(m \log^3 n C)$, where C is the largest absolute value of the integer length.

cite

This is not a strongly polynomial time algorithm. In 2024 [Fineman] come up with $\tilde{O}(mn^{8/9})$, and soon after 2024 [HJQ] improve this to $\tilde{O}(mn^{4/5})$.

cite

cite

Example (s - t max-flow). The tradition running time is $O(mn \log m/n)$, and it's later improved to be $O(m\sqrt{n} \log n C)$. Recently, [Chen et-al] improve to $O(m^{1+o(1)})$, which is almost-linear.^a

cite

^aThis can be also applied to min-cost flow and quadratic-cost flow.

1.1 Minimum Spanning Tree

Finding the minimum cost **spanning tree** (MST) in a connected graph is a basic algorithmic problem that has been long-studied. We introduce the problem formally.

Definition 1.1.1 (Spanning tree). A *spanning tree* T of a connected graph $G = (V, E)$ is an induced subgraph of G which spans G , i.e., $V(T) = V$ and $E(T) \subseteq E$.

Then, the problem can be formalized as follows.

Problem 1.1.1 (Minimum spanning tree). Given a connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, find the min-cost **spanning tree**.

Remark. The edge costs need not be positive, but we can make them positive by adding a large number without affecting correctness.

Standard algorithm that are covered in most undergraduate courses are **Kruskal's algorithm**, **Jarnik-Prim's (JP) algorithm**,¹ and (sometimes) **Borůvka's algorithm**. There are many algorithms for **MST** and their correctness relies on two simple rules (structural properties). The first one is about **cuts**:

Lemma 1.1.1 (Cut rule). If e is a minimum cost edge in a cut $\delta(S)$ for some $S \subseteq V$, then e is in some **MST**. In particular, if e is the unique minimum cost edge in the cut, then e is in every **MST**.

¹This is typically attributed usually to Prim but first described by Jarnik

Definition 1.1.2 (Light). An edge e is *light* or *safe* if there exists a cut $\delta(S)$ such that e is the cheapest cost edge crossing the cut. We also say that e is *light* w.r.t. a set of edges $F \subseteq E$ if e is light in (V, F) .

Another one is about [cycles](#):

Lemma 1.1.2 (Cycle rule). If e is the highest cost edge in a cycle C , then there exists an [MST](#) that does not contain e . In particular, if e is the unique highest cost edge in C , then e cannot be in any [MST](#).

Definition 1.1.3 (Heavy). An edge e is *heavy* or *unsafe* if there exists a cycle C such that e is the highest cost edge in C . We also say that e is *heavy* w.r.t. a set of edges $F \subseteq E$ if e is heavy in (V, F) .

Corollary 1.1.1. Suppose the edge costs are unique and G is connected. Then the [MST](#) is unique and consists of the set of all [light](#) edges.

Remark. Without loss of generality, we can assume that the cost are unique by, e.g., perturbation or consistent tie-breaking rule.

1.1.1 Standard Algorithms

Let's review the basic algorithms, the data structures they use, and the run-times that they yield.

Kruskal's Algorithm

Intuitively speaking, [Kruskal's algorithm](#) sorts the edges in increasing cost order and greedily inserts edges in this order while maintaining a maximal forest F at each step. When considering the i^{th} edge e_i , the algorithm needs to decide if $F + e_i$ is a forest or whether adding e creates a cycle.

Algorithm 1.1: Kruskal's Algorithm

Data: A connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$

Result: A [MST](#) $T = (V, F)$

```

1 Sort the edges such that  $c(e_1) \leq c(e_2) \leq \dots \leq c(e_m)$  //  $O(m \log n)^a$ 
2  $F \leftarrow \emptyset$  // Initialize the tree
3 for  $i = 1, \dots, m$  do
4   if  $e_i + F$  has no cycle then
5      $F \leftarrow F + e_i$ 
6 return  $(V, F)$ 
```

^aSince the graph is connected, $O(m \log m) = O(m \log n)$ as $n/2 \leq m \leq n^2$.

Theorem 1.1.1. [Kruskal's algorithm](#) takes $O(m \log n)$.

Proof. Sorting takes $O(m \log n)$ time. The standard solution for [line 4](#) is to use a [union-find](#) data structure. Union-find data structure with path compression yields a total run time, after sorting, of $O(m\alpha(m, n))$ where $\alpha(m, n)$ is [inverse Ackerman function](#) which is extremely slowly growing. Thus, the bottleneck is sorting, and the run-time is $O(m \log n)$. ■

Jarnik-Prim's Algorithm

[Jarnik-Prim's algorithm](#) grows a tree starting at some arbitrary root vertex r while maintaining a tree T rooted at r . In each iteration it adds the cheapest edge leaving T until T becomes [spanning](#). Thus, the [Jarnik-Prim's algorithm](#) takes $n - 1$ iterations.

Algorithm 1.2: Jarnik-Prim's Algorithm**Data:** A connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$ **Result:** A **MST** $T = (V, F)$

```

1  $r \leftarrow \text{uniform}(V)$  // Sample a root
2  $V' \leftarrow \{r\}, F \leftarrow \emptyset$  // Initialize the tree
3 while  $V' \neq V$  do
4    $e \leftarrow \arg \min_{e=(u,v) \in \delta(V'), u \in V'} c(e)$ 
5    $F \leftarrow F + e, V' \leftarrow V' + v$  // Update the tree
6 return  $(V, F)$ 

```

Theorem 1.1.2. Jarnik-Prim's algorithm takes $O(m + n \log n)$.

Proof. To find the cheapest edge leaving T (line 4), one typically uses a priority queue where we maintain vertices not yet in the tree with a key for v equal to the cost of the cheapest edge from v to the current tree. When a new vertex v is added to T the algorithm scans the edges in $\delta(v)$ to update the keys of neighbors of v . Thus, one sees that there are a total of $O(m)$ decrease-key operations, $O(n)$ extract-min operations, and initially we set up an empty queue. Standard priority queues implement decrease-key and extract-min in $O(\log n)$ time each, so the total time is $O(m \log n)$. However, Fibonacci heaps and related data structures show that one can implement decrease-key in amortized $O(1)$ time which reduces the total run time to $O(m + n \log n)$. ■

Remark. The Jarnik-Prim's algorithm runs in linear-time for moderately dense graphs!

Borůvka's Algorithm

Borůvka's algorithm seems to be the first **MST** algorithm, which has very nice properties and essentially uses no data structures. The algorithm works in phases. We describe it recursively to simplify the description, while refer to Algorithm 1.3 for the real implementation. In the first phase the algorithm finds, for each vertex v the cheapest edge in $\delta(v)$. By the cut rule this edge is in every **MST**.

Note. An edge $e = uv$ may be the cheapest edge for both u and v .

The algorithm collects all these edges, say F , and adds them to the tree. It then shrinks the connected components induced by F and recurses on the resulting graph $H = (V', E')$. It's easy to see that Borůvka's algorithm can be parallelized, unlike the other two algorithms.

Algorithm 1.3: Borůvka's Algorithm**Data:** A connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$ **Result:** A **MST** $T = (V, F)$

```

1  $F = \emptyset$  // Initialize the tree
2  $\mathcal{S} \leftarrow \{S_v = \{v\}\}$  // Collection of all sets
3 while  $|\mathcal{S}| > 1$  do
4    $\mathcal{S}' \leftarrow \mathcal{S}$  // Make a copy
5   for  $S \in \mathcal{S}$  do
6      $e_S = (u, v) \leftarrow \arg \min_{e \in \delta(S)} c(e)$ 
7      $\mathcal{S}' \leftarrow \mathcal{S}' - \{S_u, S_v\} + S_u \cup S_v$  // Merge (i.e., shrink)
8      $F \leftarrow F + e_S$  // Update the tree
9    $\mathcal{S} \leftarrow \mathcal{S}'$  // Update  $\mathcal{S}$ 
10 return  $(V, F)$ 

```

Notation. In line 7, S_u and S_v both refer to $S := S_u \cup S_v$ later in the algorithm.

Theorem 1.1.3. Borůvka's algorithm takes $O(m \log n)$.

Proof. The first phase needs $O(m)$ from a linear scan of the adjacency lists, and also computing H (i.e., shrinking) can be done in $O(m)$ time. The main observation is that $|V'| \leq |V|/2$ since each vertex v is in a connected component of size at least 2 as we add an edge leaving v to F . Thus, the algorithm terminates in $O(\log n)$ phases for a total of $O(m \log n)$ time. ■

1.1.2 Faster Algorithms

A natural question is whether there is a linear-time, i.e., $O(m)$, **MST** algorithm. The following is the history of fast **MST** algorithms:

- Very early on, Yao, in 1975, obtained an algorithm that ran in $O(m \log \log n)$ [Yao75], which leverages the idea developed in 1974 for the linear-time Selection algorithm.
- In 1987, Fredman and Tarjan [FT87] developed the Fibonacci heaps and give an **MST** algorithm which runs in $O(m \log^* n)$.² This was further improved to $O(m \log \log n)$ [Gab+86].
- Karger, Klein, and Tarjan [KKT95] obtained a linear time randomized algorithm that will be the main topic of this lecture.
- Chazelle's algorithm [Cha00] that runs in $O(m \alpha(m, n))$ is the fastest known deterministic algorithm.

Note. Pettie and Ramachandran gave an optimal deterministic algorithm in the comparison model without known what its actual running time is [PR02]!

Perhaps an easier question is the following.

Problem 1.1.2 (MST verification). Given a graph G and a tree T , decide T is an **MST** of G or not.

One can always use an **MST** algorithm to solve the **verification** problem, but not necessarily the other way around. Interestingly, there is indeed a linear-time **MST verification** algorithm based on several non-trivial ideas and data structures and was first developed in the RAM model by Dixon, Rauch, and Tarjan [DRT92] with insights from Komlós [Kom85]. Simplification is done by King [Kin97].

Note (RAM model). The RAM model allows bit-wise operation on $O(\log n)$ bit words in $O(1)$ time.

Theorem 1.1.4 (MST verification). There is a linear-time **MST verification** algorithm in the RAM model. In fact, the algorithm is based on a more general result that we will need: Given a graph $G = (V, E)$ with edge costs and a **spanning tree** $T = (V, F)$, there is an $O(m)$ -time algorithm that outputs all the **F-heavy** edge of G .

Proof. The original complicated algorithm has been simplified over the years. See lecture notes of Gupta and Assadi for accessible explanation, also the MST surveys [Eis97; Mar08]. ■

Fredman-Tarjan's Algorithm

Here we briefly describe Fredman and Tarjan's algorithm [FT87; Mar08] via Fibonacci heaps, which is reasonably simple to describe and analyze modulo a few implementation details that we will gloss over for the sake of brevity. First, we develop a simple $O(m \log \log n)$ time algorithm by combining **Borůvka's algorithm** and **Jarník-Prim's algorithm**.

As previously seen. **Jarník-Prim's algorithm** takes $O(m + n \log n)$ time via Fibonacci heaps where the bottleneck is when $m = o(n \log n)$. On the other hand, **Borůvka's algorithm** starts with a graph on n nodes and after i^{th} phases, reduces the number of nodes to $n/2^i$; each phase takes $O(m)$ times.

²Formally, it runs in $O(m \beta(m, n))$, where $\beta(m, n)$ is the minimum value of i such that $\log^{(i)} n \leq m/n$, where $\log^{(i)} n$ is the logarithmic function iterated i times. Since $m \leq n^2$, $\beta(m, n) \leq \log^* n$.

Intuition. Suppose we run [Borůvka's algorithm](#) for k phases and then run [Jarnik-Prim's algorithm](#) once the number of nodes is reduced. We can see that the total run time is $O(mk)$ for the k phases of [Borůvka's algorithm](#), and $O(m + n/2^k \log n/2^k)$ for the [Jarnik-Prim's algorithm](#) on the reduced graph. Thus, if we choose $k = \log \log n$, we obtain a total run-time of $O(m \log \log n)$.

Tarjan and Fredman obtained a more sophisticated scheme based on the [Jarnik-Prim's algorithm](#), but the basic idea is to reduce the number of vertices. The algorithm runs again in phases. We describe the first phase here.

Intuition (First phase). Start growing the tree. If the heap gets too big, we stop.

Consider an integer parameter t such that $1 < t \leq n$. Pick an arbitrary root r_1 and grow a tree T_1 via [Jarnik-Prim's algorithm](#) with a Fibonacci heap. We stop the tree growth when the heap size exceeds t for the first time or if we run out of vertices. All the vertices in the tree are marked as visited. Now pick an arbitrary, unmarked vertex as root $r_2 \in V - T$ and grow a tree T_2 , and we stop growing T_2 if it touches T_1 , in which case it merges with it, or if the heap size exceeds t or if we run out of vertices. The algorithm proceeds in this fashion by picking new roots and growing them until all nodes are marked.

Note. While growing T_2 , the heap may contain previously marked vertices. It is only when the algorithm finds one of the marked vertices as the cheapest neighbor of the current tree that we merge the trees and stop.

It's easy to see that the [first phase of Fredman-Tarjan algorithm](#) correctly adds a set of [MST](#) edges F . After this, we simply shrink these trees and recurse on the smaller graph.

Algorithm 1.4: Fredman-Tarjan's Algorithm

Data: A connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$

Result: A [MST](#) $T = (V, F)$

```

1   $V' \leftarrow V, F \leftarrow \emptyset$                                 // Initialize the tree
2  while  $|V'| > 1$  do
3       $T \leftarrow \text{Grow}(G)$                                     // First phase
4       $F \leftarrow F \cup E(T)$                                   // Update the tree
5      Shrink  $G$  w.r.t.  $T$ , update  $V'$  and  $E$                     // Second phase
6  return  $(V', F)$ 
7
8  Grow( $G$ ):
9       $V' \leftarrow \emptyset, F \leftarrow \emptyset, T \leftarrow (V', F)$  // Initialize the forest
10     while  $V' \neq V$  do
11          $r \leftarrow \text{uniform}(V - V')$                         // Pick an unmarked vertex
12          $T' \leftarrow (\{r\}, \emptyset)$                         // Initialize a tree
13         while  $|N(T')| < t$  or  $V(T') \cap V' \neq \emptyset$  do
14             Run one more step of Jarnik-Prim( $r, T'$ )        // Starting at  $r$ , maintaining  $T'$ 
15              $V' \leftarrow V' \cup V(T')$                         // Mark
16              $F \leftarrow F \cup E(T')$                         // Update the forest by merging the tree
17     return  $(V, F)$                                            // Return a forest of  $G$ 

```

Note. This can be seen as a parameterized version of [Borůvka's algorithm](#).

The difficult part is to determine its runtime. We have the following.

Theorem 1.1.5. [Fredman-Tarjan's algorithm](#) takes $O(m\beta(m, n))$.

Proof. Firstly, the total time to scan edges and insert vertices into heaps and do **decrease-key** is $O(m)$ since an edge is only visited twice, once from each end point. Since each heap is not allowed to grow to more than size t , the total time for all the **extract-min** operations take $O(n \log t)$. With the fact that the initialization of each data structure is easy as it starts as an empty one, hence, the

first phase takes $O(m + n \log t)$. We claim that it also reduces the number of vertices to $2m/t$.

Claim. The number of connected components induced by F is $\leq 2m/t$ after the first phase.

Proof. Let C_1, \dots, C_h be the connected components of F . If for every C_i , $\sum_{v \in C_i} \deg(v) \geq t$,

$$2m = \sum_{v \in V} \deg(v) = \sum_{i=1}^h \sum_{v \in C_i} \deg(v) \geq ht \Rightarrow h \leq \frac{2m}{t}.$$

To see why the assumption holds, consider the growth of a tree T' in line 14:

- If we stop T' because heap size $|N(T')|$ exceeds t , then each of the vertex in the heap is a witness to a unique edge incident to T' , hence the property holds.
- If T' merged with a previous tree, then the property holds because the previous tree already had the property and adding vertices can only increase the total degree of the component.

The only reason the property may not hold is if line 17 terminates a tree because all vertices are already included in it, but then that phase finishes the algorithm. \otimes

The question reduces to choosing t .

Intuition. We want linear time in the first phase, i.e., $n \log t$ to be no more than $O(m)$, leading to $t = 2^{2m/n}$. If we do this in every iteration, then this leads to $O(m)$ time per iteration.

We now bound the number of iteration. Consider $t_1 := 2^{2m/n}$ and $t_i := 2^{2m/n_i}$,^a where n_i and m_i are the number of vertices and edges at the beginning of the i^{th} iteration, with $m_1 = m$ and $n_1 = n$. From the previous claim, $n_{i+1} \leq 2m_i/t_i$, which gives

$$t_{i+1} = 2^{2m/n_{i+1}} \geq 2^{\frac{2m}{2m_i/t_i}} \geq 2^{t_i}.$$

Thus, t_i is a power of twos with $t_1 = 2^{2m/n}$, and the Fredman-Tarjan's algorithm stops if $t_i \geq n$ since it will grow a single tree and finish. Thus, the algorithm needs at most $\beta(m, n)$ iterations, giving the total time $O(m\beta(m, n))$. \blacksquare

^aTechnically, we need to choose $t_i := 2^{\lceil 2m/n_i \rceil}$, but we will be a bit sloppy and ignore the ceilings here.

Lecture 2: MST and Tree Packing

Linear-Time Randomized Algorithm

29 Aug. 11:00

Using randomization, it's possible to derive a linear-time algorithm for MST.

Theorem 1.1.6 ([KKT95]). Karger-Klein-Tarjan's algorithm takes $O(m)$ time that computes the MST with probability at least $1 - 1/\text{poly}(m)$.

Karger-Klein-Tarjan's algorithm relies on the so-called sampling lemma, which we first discussed.

Lemma 1.1.3 (Sampling lemma). Given a graph $G = (V, E)$, and let $E' \subseteq E$ be obtained by sampling each edge e with probability $p \in (0, 1)$. Let F be a minimum spanning forest^a in $G' = (V, E')$. Then the expected number of F -light edge in G is less than $(n - 1)/p$.

^aAs G' can be disconnected.

Proof. The proof is based on the *principle of deferred decisions* in randomized analysis. Let A be the set of F -light edges. Note that both A and F are random sets that are generated by the process of sampling E' . To analyze $\mathbb{E}[|A|]$, we consider Kruskal's algorithm to obtain F from E' , where we generate E' on the fly:

Algorithm 1.5: Sampling Process**Data:** A connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, probability $p \in (0, 1)$ **Result:** A minimum spanning forest F and the set of *F-light* edges A

```

1 Sort the edges such that  $c(e_1) \leq c(e_2) \leq \dots \leq c(e_m)$ 
2  $A \leftarrow \emptyset, F \leftarrow \emptyset, E' \leftarrow \emptyset$ 
3 for  $i = 1, \dots, m$  do
4    $r \leftarrow \text{Ber}(p)$  // Toss a biased coin
5   if  $r = 1$  then
6      $E' \leftarrow E' + e_i$ 
7     if  $F + e_i$  is a forest then
8        $F \leftarrow F + e_i$ 
9        $A \leftarrow A + e_i$ 
10  else if  $e_i$  is F-light then
11     $A \leftarrow A + e_i$ 
12 return  $F, A$ 

```

The following is exactly the same as the above, but easier to analyze:

Algorithm 1.6: Sampling Process with Tweaks**Data:** A connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, probability $p \in (0, 1)$ **Result:** A minimum spanning forest F and the set of *F-light* edges A

```

1 Sort the edges such that  $c(e_1) \leq c(e_2) \leq \dots \leq c(e_m)$ 
2  $A \leftarrow \emptyset, F \leftarrow \emptyset$ 
3 for  $i = 1, \dots, m$  do
4   if  $e_i$  is F-light then // Sorting implies  $F + e_i$  is a forest  $\Leftrightarrow e_i$  is F-light
5      $A \leftarrow A + e_i$ 
6      $r \leftarrow \text{Ber}(p)$  // Toss a biased coin
7     if  $r = 1$  then
8        $F \leftarrow F + e_i$ 
9 return  $F, A$ 

```

The second algorithm makes the following observation clear.

Intuition. An edge e_i is added to A implies that it is added to F with probability p .Hence, $p\mathbb{E}[|A|] = \mathbb{E}[|F|] \leq n - 1$, hence $\mathbb{E}[|A|] \leq (n - 1)/p$. ■

With the [sampling lemma](#), we know that when $p = 1/2$, the number of *F-light* edges from E is at most $2n$. Hence, we can eliminate most of the edges from $E \setminus E'$ from consideration given the fact that we can efficiently compute the *F-heavy* edges via the [MST verification theorem](#). It's worth noting that to work with the [sampling lemma](#) via the natural recursion that it implies means that we need to work with potentially disconnected graph. That is, we will need to consider disconnected graph. Hence, we make the following generalization.

Definition 1.1.4 (Spanning forest). A *spanning forest* T of a graph $G = (V, E)$ (potentially disconnected) is an induced subgraph of G which spans G , i.e., $V(T) = V$ and $E(T) \subseteq E$.

Problem 1.1.3 (Minimum spanning forest). Given a graph $G = (V, E)$ (potentially disconnected) with edge capacity $c: E \rightarrow \mathbb{R}_+$, find the min-cost [spanning forest](#).

Note. [MST](#) and [MSF](#) are closely related and one is reducible to the other in linear time, and the [cut](#) and [cycle rules](#) can be generalized to [MSF](#) easily.

Now, consider the following natural recursive divide and conquer algorithm for computing [MSF](#).

Algorithm 1.7: Natural Recursive Algorithm from [Sampling Lemma](#)

Data: A graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$
Result: A [MSF](#) $T = (V, F)$

```

1 if  $|V| < n_0$  then                                     //  $n_0$  is some constant
2   return Standard-MST( $G, c$ )                             // Use a standard deterministic algorithm
3
4 Sample each edge i.i.d. from  $\text{Ber}(1/2)$  to obtain  $E_1 \subseteq E$ 
5  $(V, F_1) \leftarrow \text{Karger-Klein-Tarjan}((V, E_1))$         // Recursively compute MSF
6  $E_2 \leftarrow \text{Light-Edge}(G, F_1)$                        // Compute all F1-light edges with Theorem 1.1.4
7  $(V, F_2) \leftarrow \text{Karger-Klein-Tarjan}((V, E_2))$         // Recursively compute MSF
8 return  $(V, F_2)$ 

```

The correctness of [Algorithm 1.7](#) is clear from the [cut](#) and [cycle rules](#). The issue is the running time:

Claim. [Algorithm 1.7](#) is not efficient enough.

Proof. The expected number of edges in $G_1 := (V, E_1)$ is $m/2$, and the expected number of edges in $G_2 := (V, E_2)$, via the [sampling lemma](#), is at most $2n$. We see that the algorithm does $O(m+n)$ work outside the two recursive calls ([line 5](#), [line 7](#)). Let $T(m, n)$ be the expected running time of the algorithm on an m -edge n -node graph. Informally, we see the following recurrence:

$$T(m, n) \leq c(m+n) + T(m/2, n) + T(2n, n).$$

If we take the problem size to be $n+m$, then [Algorithm 1.7](#) generates two sub-problems of expected size $m/2 + n$ and $2n + n$, with the total size being $4n + m/2$. If $m > 10n$, say, then the total problem size is shrinking by a constant factor, and we obtain a linear-time algorithm. However, this is generally not the case. *

The problem becomes reducing the graph size, which is the trick of [Karger-Klein-Tarjan's algorithm](#): we run [Borůvka's algorithm](#) for a few iterations as a preprocessing step, reducing the number of vertices:

Algorithm 1.8: Karger-Klein-Tarjan's Algorithm [[KKT95](#)]

Data: A connected graph $G = (V, E)$ ^a with edge capacity $c: E \rightarrow \mathbb{R}_+$
Result: A [MSF](#) $T = (V, F)$

```

1 if  $|V| < n_0$  then                                     //  $n_0$  is some large constant
2   return Standard-MST( $G, c$ )                             // Use a standard deterministic algorithm
3
4  $G' = (V', E'), T' = (V', F') \leftarrow \text{Borůvka}(G, c, 2)$  // Run two iterations with  $|V'| \leq |V|/4$ .
5
6 Sample each edge in  $G'$  i.i.d. from  $\text{Ber}(1/2)$  to obtain  $E_1 \subseteq E'$ 
7  $(V', F_1) \leftarrow \text{Karger-Klein-Tarjan}((V', E_1))$         // Recursively compute MSF
8  $E_2 \leftarrow \text{Light-Edge}(G_1, F_1)$                        // Compute F2-light edges with Theorem 1.1.4
9  $(V', F_2) \leftarrow \text{Karger-Klein-Tarjan}((V', E_2))$         // Recursively compute MSF
10 return  $(V, F' \cup F_2)$ 

```

^aAssume no connected component of G is small.

Now, we provide the proof sketch of [Theorem 1.1.6](#), which can be made precise with expectation.

Proof Sketch of Theorem 1.1.6. The correctness is easy to see as before. As for the running time, we see that [Borůvka's algorithm](#) takes $O(m)$ time for each phase, so the total time for the preprocessing ([line 4](#)) is $O(m)$. Then, the recurrence for $T(m, n)$ is

$$T(m, n) \leq c(m+n) + T(m/2, n/4) + T(2n/4 + n/4),$$

i.e., the resulting sub-problem is of size $n/4 + m/2 + n/4 + n/2 = n + m/2$, which is good enough assuming $m \geq n - 1$.^a By a simple inductive proof, we can show that $T(m, n) = O(n+m)$. ■

^aSince we eliminate small components including singletons.

Remark. A more refined analysis of the [sampling lemma](#) can be used to show that the running time is linear with high probability as well.

Many properties of forests and spanning trees can be understood in the more general context of *matroids*. In many cases this perspective is insightful and also useful. The [sampling lemma](#) applies in this more general context and has various applications [Kar95; Kar98]. Obtaining a deterministic $O(m)$ time algorithm is a major open problem. Obtaining a simpler linear-time [MST verification](#) algorithm, even randomized, is also a very interesting open problem.

1.2 Tree Packing

We turn to another interesting problem, [tree packing](#).

Problem 1.2.1 (Tree packing). Given a multigraph $G = (V, E)$, find all the edge-disjoint [spanning trees](#) in G . In particular, find the maximum number, $\tau(G)$, of edge-disjoint [spanning trees](#) of G

1.2.1 Bound on the Tree Packing Number

There is a beautiful theorem that provides a min-max formula for this. We first introduce some notation.

Notation. Let \mathcal{P} be the collection of partitions of V , and E_P is the edge between connected components induced by a partition $P \in \mathcal{P}$, i.e., $e \in E_P$ if its endpoints are in different parts of P .

It's easy to see that any [spanning tree](#) must contain at least $|P| - 1$ edges from E_P . Thus, if G has k edge-disjoint [spanning trees](#), then

$$k \leq \frac{|E_P|}{|P| - 1}.$$

More generally, we have the following.

Theorem 1.2.1. The maximum number of edge-disjoint [spanning trees](#) in a graph G is given by

$$\tau(G) = \left\lfloor \min_{P \in \mathcal{P}} \frac{|E_P|}{|P| - 1} \right\rfloor.$$

Remark. [Theorem 1.2.1](#) is a special case of a theorem on matroid base packing where it is perhaps more natural to see [Sch+03].

A weaker version of the theorem is regarding fractional packing. In fractional packing, we allow one to use a fraction amount of a tree. The total amount to which an edge can be used is at most 1 (or $c(e)$ in the capacitated case). Clearly, an integer packing is also a fractional packing. The advantage of fractional packings is that one can write a linear program for it, and they often have some nice properties. Let $\tau_{\text{frac}}(G)$ be the fraction [tree packing](#) number. Clearly, we have $\tau_{\text{frac}}(G) \geq \tau(G)$.

Corollary 1.2.1. Given a graph G , we have

$$\tau_{\text{frac}}(G) = \min_{P \in \mathcal{P}} \frac{|E_P|}{|P| - 1}.$$

Proof. Assuming [Theorem 1.2.1](#), then with $c := |P^*| - 1$ for $P^* = \arg \min_{P \in \mathcal{P}} |E_P| / (|P| - 1)$,

$$\tau(G_c) - \min_{P \in \mathcal{P}} \frac{c|E_P|}{|P| - 1} = \lfloor |E_{P^*}| \rfloor - |E_{P^*}| = 0,$$

where G_c is with edge capacity scaled up by c . This implies that $\tau_{\text{frac}}(G_c) = \tau(G_c)$. As this holds for every c (with different graphs), this can only happen if $\tau_{\text{frac}}(G) = \min_{P \in \mathcal{P}} |E_P| / (|P| - 1)$. ■

The second important corollary that is frequently used is about the min-cut. We see that while the min-cut size $\lambda(G)$ of G is upper-bounding $\tau(G)$, i.e., $\tau(G) \leq \lambda(G)$, this is not tight at all.

Corollary 1.2.2. Let G be a capacitated graph and let $\lambda(G)$ be the global min-cut size. Then

$$\tau_{\text{frac}}(G) \geq \frac{\lambda(G)}{2} \frac{n}{n-1}.$$

Proof. Let P^* be the optimum partition that induces $\tau_{\text{frac}}(G)$. Then, $\tau(G) = |E_{P^*}|/(|P^*| - 1)$. Since for every connected component induced by P^* , at least $\lambda(G)$ edges are going out, hence

$$\tau_{\text{frac}}(G) = \frac{|E_{P^*}|}{|P^*| - 1} \geq \frac{\lambda(G)/2 \cdot |P^*|}{|P^*| - 1} \geq \frac{\lambda(G)}{2} \frac{n}{n-1},$$

where we use the fact that $|P^*| \leq n$ and $i/(i-1)$ is decreasing. \blacksquare

We first see a tight example.

Example (Cycle). Consider the n -node cycle C_n . Clearly, $\tau(C_n) = 1$, and $\tau_{\text{frac}}(C_n) \leq n/(n-1)$ since each tree has $n-1$ edges and there are n edges in the graph. Indeed, we have $\tau_{\text{frac}}(C_n) = n/(n-1)$. Finally, we see that $\lambda(G) = 2$.

Proof. Consider the n trees in C_n (corresponding to deleting each of the n edge) and assigning a fraction value of $1/(n-1)$ for each of them, with the corresponding tight partition consists of the n singleton vertices. \circledast

Note. Theorem 1.2.1 and its corollaries naturally extend to the capacitated case. For integer packing, we can assume c_e is an integer for each edge e , and the formula is changed to

$$\tau(G) = \left\lfloor \min_{P \in \mathcal{P}} \frac{c(E_P)}{|P| - 1} \right\rfloor.$$

Corollary 1.2.1 can also be proved in the same way when the edge capacity is rational.

Typically, one uses the connection between [tree packing](#) and min-cut to argue about the existence of many disjoint [trees](#), since the global minimum cut is easier to understand than $\tau(G)$. However, we will see that one can use [tree packing](#) to compute $\lambda(G)$ exactly which may seem surprising at first due to the approximate relationship [Corollary 1.2.2](#).

1.2.2 Proof of Corollary 1.2.1

Now, we give a different proof for [Corollary 1.2.1](#) via LP duality without relying on [Theorem 1.2.1](#).³

Proof of Corollary 1.2.1 [CQ17]. Consider $\mathcal{T}_G := \{T \mid T \text{ is a spanning tree of } G\}$. Then, consider the following primal and the dual linear program:

$$\begin{array}{ll} \max & \sum_{T \in \mathcal{T}_G} y_T \\ & \sum_{T \ni e} y_T \leq c(e) \quad \forall e \in E; \\ \text{(P)} & y_T \geq 0 \quad \forall T \in \mathcal{T}_G; \end{array} \quad \begin{array}{ll} \min & \sum_{e \in E} c(e)x_e \\ & \sum_{e \in T} x_e \geq 1 \quad \forall T \in \mathcal{T}_G; \\ \text{(D)} & x_e \geq 0 \quad \forall e \in E. \end{array}$$

Let y^* and x^* be the optimal solution to the primal and the dual. Then from the strong duality,

$$\sum_{T \in \mathcal{T}_G} y_T^* = \tau_{\text{frac}}(G) = \sum_{e \in E} c(e)x_e^*.$$

We see that if there exists e such that $x_e^* = 0$, then we can just contract all these edges, so without

³Indeed, this is a hard theorem to prove so we will not touch on this.

loss of generality, $x_e^* > 0$ for all $e \in E$.

Intuition. If $x_e^* = 0$, we can effectively increase $c(e)$ to ∞ without affecting the value of the dual solution, i.e., e is not a bottleneck in the primal **tree packing**, hence safe to contract.

Claim. If $x_e^* > 0$ for all $e \in E$, then $\tau_{\text{frac}}(G)$ is achieved via the singleton partition P . In particular,

$$\tau_{\text{frac}}(G) = \frac{\sum_{e \in E} c(e)}{n-1}.$$

Proof. From complementary slackness, we know that $\sum_{T \ni e} y_T^* = c(e)$ for all $e \in E$. Hence,

$$(n-1) \sum_{T \in \mathcal{T}_G} y_T^* = \sum_{T \in \mathcal{T}_G} \sum_{e \in T} y_T^* = \sum_{e \in E} \sum_{T \ni e} y_T^* = \sum_{e \in E} c(e),$$

implying that $\sum_{T \in \mathcal{T}_G} y_T^* = \sum_{e \in E} c(e)/(n-1)$. ⊗

Finally, we recall that $\tau_{\text{frac}}(G) \leq \min_P |E_P|/(|P|-1)$, hence, the above claim gives us the desired conclusion via induction: this is true if $x_e^* > 0$ for all $e \in E$; otherwise, we contract edges with $x_e^* = 0$ and reduce to this case. ■

Remark. In the above proof, the dual can be interpreted as a relaxation for the min-cut problem. In fact, if $x_e \in \{0, 1\}$, then this is exact.

1.2.3 Finding an Optimum Tree Packing and Approximating Tree Packing

If the linear program in the **proof** of **Corollary 1.2.1** can be solved efficiently to get $\tau_{\text{frac}}(G)$, then it will also yield an algorithm for the value of the integer packing $\tau(G)$ since it's just the floor of which. The problem is that while the primal has an exponentially many variables, the dual has an exponentially many constraints. We recall the following fact.

As previously seen. The Ellipsoid method needs a *separation oracle*. For example, applying it to the dual, we need to answer the following question efficiently:

- Given $x \in \mathbb{R}^E$, is it the case that $\sum_{e \in T} x_e \geq 1$ for all $T \in \mathcal{T}_G$?
- If not, find a tree T such that $\sum_{e \in T} x_e < 1$.

We see that this corresponds to solving **MST**, hence, the dual admits an efficient solution via the Ellipsoid method. One can convert an exact algorithm for the dual to an exact algorithm for the primal.

Remark. There are combinatorial algorithms for solving **tree packing** (both integer version and fraction versions) in strongly polynomial time [Sch+03].

On the other hand, we're also interested in whether we can find a faster algorithm for **tree packing** if one allows approximation. With an adaption of the *multiplicative weights update* (MWU) method and data structures for **MST** maintenance, there is a near-linear time algorithm:

Theorem 1.2.2 ([CQ17]). There is a deterministic algorithm to compute a $(1 - \epsilon)$ -approximate fractional **tree packing** in $O(m \log^3 n / \epsilon^2)$.

Lecture 3: Global Min-Cut with Tree Packing

1.3 Min-Cuts and Steiner Min-cuts via Tree Packing

3 Sep. 11:00

Consider the following famous problems about min-cuts.

Problem 1.3.1 (*s-t* min-cut). Given a graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, the *s-t* min-cut problem aims to find $\min_{S \subseteq V: s \in S, t \in V \setminus S} c(\delta(S))$.

Problem 1.3.2 (Global min-cut). Given a graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, the global min-cut problem aims to find $\min_{\emptyset \neq S \subset V} c(\delta(S))$.

In what follows, we will simply use *min-cut* to refer to *Problem 1.3.2* problem. A naive way to solve it is to first fix one end $s \in V$, and compute the *s-t* min-cut for all $t \in V - s$. Fairly recent work shows how one can do it with only poly-log max-flow computations.

Over the years, several very different algorithmic approaches have been developed for these problems. One of the surprising ones is based on MA-orderings [NI92], which is a combinatorial $O(mn + n^2 \log n)$ time algorithm that does not rely on flow at all.⁴ Another approach is to combine several flow computations together via the push-relabel method [HO94], which also works for directed graphs. Karger developed elegant and powerful random contraction based algorithms for global min-cuts [Kar95], leading to many results. Two notable consequences are the following.

Theorem 1.3.1 ([KS96]). There is a randomized algorithm that runs in $O(n^2 \log n)$ time and outputs the min-cut with high probability.^a

^aThis is a Monte-Carlo algorithm, so we cannot guarantee that the min-cut found is the correct one.

The following is a consequence of Karger's contraction algorithm [Kar95].

Theorem 1.3.2 (Approximate min-cut [Kar00]). The number of α -approximate min-cuts in a graph is at most $O(n^{2\alpha})$.

Karger then developed another approach via *tree packing* to obtain a randomized near-linear time algorithm for min-cut. He also was able to refine the bound on approximate min-cuts via this approach.

Theorem 1.3.3 ([Kar00]). There is a randomized algorithm that runs in time $O(m \log^3 n)$ and outputs the min-cut with high probability.

While the random contraction based algorithm is taught quite frequently due to its elegance and simplicity, the *tree packing* approach is more technical. More recently, the *tree packing* approach has led to several new results, which we now discuss.

1.3.1 Tree Packing-Based Algorithm for Min-Cut

Recall *Corollary 1.2.2*, which gives $\tau_{\text{frac}}(G) \in [\frac{\lambda(G)}{2} \frac{n}{n-1}, \lambda(G)]$. Intuitively, even if we can compute $\tau_{\text{frac}}(G)$ exactly, we have a 2-approximation to $\lambda(G)$. However, this already leads a crucial observation:

Intuition. On average, each tree can't cross the min-cut more than twice.

To formalize the above intuition, consider the following definition.

Definition 1.3.1 (Respecting). Let $T = (V, E_T)$ be a spanning tree and $(S, V \setminus S)$ be a cut. The for an integer $h \geq 1$, we say T is *h-respecting* w.r.t. S if $|E_T \cap \delta(S)| \leq h$.

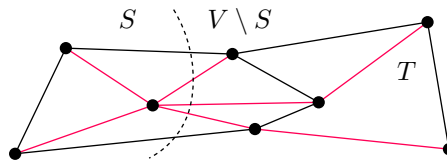


Figure 1.1: The spanning tree T is shown in red edges. T is 3-respecting the cut $(S, V \setminus S)$.

⁴This approach generalizes to symmetric submodular functions.

We can now formalize the intuition in [Lemma 1.3.1](#).

Lemma 1.3.1. Suppose $\{y_T\}_{T \in \mathcal{T}_G}$ is a $(1 - \epsilon)$ -approximate [tree packing](#) of G , and $\delta(S)$ is a [min-cut](#) of G . Let $\ell_T := |E_T \cap \delta(S)|$ be the number of edges of T that cross the cut S . Furthermore, let $p_T = y_T / \sum_{T \in \mathcal{T}_G} y_T$ and $q := \sum_{T: \ell(T) \leq 2} p_T$. Then,

$$q \geq \frac{1}{2} \left(3 - \frac{2}{1 - \epsilon} \left(1 - \frac{1}{n} \right) \right).$$

In particular, if $\epsilon = 0$, then $1 \geq 1/2 + 1/n$, and if $\epsilon < 1/5$, then $q > 1/4$.

Proof. From the assumption, $\sum_{T \in \mathcal{T}_G} y_T \geq (1 - \epsilon) \tau_{\text{frac}}(G)$. With [Corollary 1.2.2](#), we have

$$\sum_{T \in \mathcal{T}_G} y_T \geq (1 - \epsilon) \frac{n}{n - 1} \frac{\lambda(G)}{2}.$$

Let $S \subseteq V$ be a [min-cut](#), we have $1 = \sum_{T \in \mathcal{T}_G} p(T) = \sum_{T: \ell(T) \leq 2} p_T + \sum_{T: \ell(T) \geq 3} p_T$. Observe that

- each tree T with $\ell(T) \geq 3$ uses up at least 3 edges from $\delta(S)$; while
- each tree T with $\ell(T) \leq 2$ uses up at least 1 edge from $\delta(S)$.

Since the total capacity of $\delta(S)$ is $\lambda(G)$, and the [tree packing](#) solution is valid, we have

$$\sum_{T: \ell(T) \leq 2} y_T + 3 \sum_{T: \ell(T) \geq 3} y_T \leq \lambda(G) \Rightarrow q + 3(1 - q) \leq \frac{\lambda(G)}{\sum_{T \in \mathcal{T}_G} y_T} \leq \frac{2}{1 - \epsilon} \left(1 - \frac{1}{n} \right),$$

where the last inequality follows from the very first inequality we have derived. ■

Remark. [Lemma 1.3.1](#) states that if the [tree packing](#) is sufficiently good, then a constant fraction of the trees in the [packing](#) will cross the [min-cut](#) at most twice.

Now, we're ready to see Karger's algorithm for [min-cut](#) [Kar00]. However, the original algorithm was more involved since at that time, there was no near-linear time approximation algorithm for [tree packing](#), so he used a form of sparsification and then applied an approximation [tree packing](#) algorithm on the sparsified graph which is quite a feat. In our case, recall that following.

As previously seen. [Theorem 1.2.2](#) states that we can compute a $(1 - \epsilon)$ -approximate [tree packing](#) of G , given by $\{y_T\}_{T \in \mathcal{T}_G}$, in $O(m \log^3 n / \epsilon^2)$ time.

By black-boxing this near-linear time [tree packing](#) algorithm, consider the following.

Algorithm 1.9: [Tree Packing](#)-Based [Min-Cut](#) Algorithm [Kar00; CQ17]

Data: A connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, $\epsilon_0 \in (0, 1/5)$

Result: A cut S

- 1 $\{y_T\}_{T \in \mathcal{T}_G} \leftarrow \text{Approximate-Tree-Packing}(G, c, \epsilon_0)$ // $O(m \log^3 n)$
 - 2 Sample a tree T with probability $p_T = y_T / \sum_{T \in \mathcal{T}_G} y_T$
 - 3 Find the cheapest cut $(S, V \setminus S)$ in G such that T is [2-respecting](#) w.r.t. S
 - 4 **return** S
-

Firstly, we see that [Algorithm 1.9](#) admits the following.

Lemma 1.3.2. [Algorithm 1.9](#) outputs the [min-cut](#) of G with probability at least $1/4$.

Proof. It's immediate from [Lemma 1.3.1](#). ■

To boost the success probability, we can simply repeat the last two steps ([line 2](#), [line 3](#)) $\Theta(\log n)$ times, which results in a success probability to at least $1 - 1/n^c$ for any constant c . To analyze the running time, a key ingredient is [line 3](#). Karger showed that one can implement [line 3](#) via a clever dynamic programming coupled with link-cut tree data structure:

Theorem 1.3.4 ([Kar00]). Given a graph $G = (V, E)$ and a **spanning tree** $T = (V, E_T)$. There is a deterministic algorithm that computes a minimum cut $(S, V \setminus S)$ such that T is **2-respecting** w.r.t. S in $O(m \log^2 n)$ time.

We can now prove **Theorem 1.3.3**.

Proof of Theorem 1.3.3. Since **line 1** takes $O(m \log^3 n)$ for ϵ_0 being a constant, and observe that once the approximated **tree packing** $\{y_T\}_{T \in \mathcal{T}}$ is computed, we can reuse them and apply the repetition for **line 2** and **line 3** to boost the probability of success. With $\Theta(\log n)$ repetitions, we obtain an $O(m \log^3 n)$ time algorithm as desired with the running time guaranteed by **Theorem 1.3.4**. ■

1.3.2 Bounding the Number of Approximate Min-Cuts

As hinted in **Theorem 1.3.2**, we're now interested in how many distinct **min-cuts** can an undirected graph have. The following theorem was shown a long time ago:

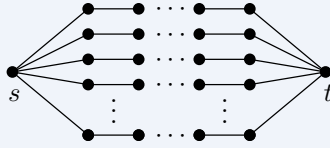
Theorem 1.3.5 ([DKL76]). The number of distinct **min-cuts** in an undirected graph is at most $\binom{n}{2}$.

Example (Cycle). The worst case example is an n -cycle C_n .

Remark. All the **min-cuts** of a graph can be represented in a nice and compact data structure called the cactus (cactus representation), which was also shown in [DKL76].

In contrast, for **s - t min-cuts**, it can be exponentially many in n .

Example. Consider the following multi-highway-like graph, which has exponentially many **s - t min-cuts** since if we choose one of the road section in each line of the road, it'll be a **s - t min-cut**.



Hence, we're interested in the number of **α -approximation min-cut**:

Definition 1.3.2 (Approximate min-cut). For $\alpha \geq 1$ an α -approximate min-cut is a cut $(S, V \setminus S)$ such that $c(\delta(S)) \leq \alpha \lambda(G)$.

Recall **Theorem 1.3.2**, where Karger used **tree packing** to prove that the number of **α -approximation min-cuts** is at most $O_\alpha(n^{\lfloor 2\alpha \rfloor})$. Before we prove **Theorem 1.3.2**, we recall some basic facts from linear programming.

As previously seen. A solution x^* to a linear program which has n non-trivial constraints means that the support size of x is at most n , i.e., $x_i > 0$ for at most n many i 's.

We're now ready to prove **Theorem 1.3.2**, which is based on [CQX20].

Proof of Theorem 1.3.2. Consider an optimum fraction **tree packing** solution $\{(T, y_T^*)\}_{T \in \mathcal{T}_G}$. In the **proof** of **Corollary 1.2.1**, where we define the fractional **tree packing** linear program, we know that there are only m non-trivial constraints, hence there are only m many T 's such that $y_T^* > 0$.

Consider an **α -approximate min-cut** $S \subseteq V$, and let $h = \lceil 2\alpha \rceil$. Now, let $q_{h,\alpha}$ be the fraction of **tree packing** that h -respects $S \subseteq V$, i.e.,

$$q_{h,\alpha} := \sum_{T: \ell(T) \leq h} p_T.$$

Using a similar analysis as the one in [Lemma 1.3.1](#), we can argue that

$$q_{h,\alpha} \geq \frac{1}{h}(1 - (2\alpha - \lfloor 2\alpha \rfloor)) \left(1 - \frac{1}{n}\right).$$

The main intuition is the following:

Intuition. Say at least one tree in the [packing](#) *h-respects* the cut (which is the case). Then, the total number of *α -approximate min-cuts* is at most $m \cdot n^h \leq m \cdot n^{\lfloor 2\alpha \rfloor}$.

But we can do better by noticing that $q_{h,\alpha} > 0$ is a fixed constant for any fixed α . Suppose N is the number of *α -approximate min-cuts*. For any fixed *α -approximate min-cut*, $q_{h,\alpha}$ fraction of the *tree packing* is *h-respecting* w.r.t. the cut. Consider the following question:

Problem. Fix a single tree T , how many distinct cuts are there such that T *h-respects* w.r.t.?

Answer. We can remove at most h edges from T to create at most $h + 1$ components and combine these components into two sides of a cut, hence, each tree T correspond to at most $2^{h+1} \binom{n-1}{h} \leq 2^{h+1} n^h$ cuts. \otimes

Thus, the number of *α -approximate min-cuts* is at most $2^{h+1} n^h / q_{h,\alpha}$. \blacksquare

Lecture 4: Steiner Min-Cut with Isolating Cuts

1.3.3 Steiner Min-Cut

5 Sep. 11:00

Consider the following problem that generalizes the *s-t min-cut* and *global min-cut*.

Problem 1.3.3 (Steiner min-cut). Given a graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$ and a set $T \subseteq V$ of terminals, the *Steiner min-cut* problem aims to find the min-cut $(S, V \setminus S)$ which separates some pair of terminals, i.e., $S \cap T \neq \emptyset$ and $(V \setminus S) \cap T \neq \emptyset$.

Remark. *Steiner min-cut* generalizes both *s-t min-cut* and *global min-cut*.

Proof. *s-t min-cut* corresponds to $T = \{s, t\}$, while *global min-cut* corresponds to $T = V$. \otimes

A simple algorithm for the *Steiner min-cut* is the same as the *global min-cut* by solving *s-t min-cut*: for $T = \{t_1, \dots, t_k\}$, fix a terminal, say t_1 , then compute *t_1 - t_i min-cut* for all $i \geq 2$. This requires $|T| - 1$ max-flow computations. In fact, this is the best known algorithm even for the *global min-cut* till [NI92].

Quite recently, a simple yet striking approach that computes the *Steiner min-cut* with high probability using only $O(\log^3 n)$ *s-t cut* computations is developed [LP20], which is based on *isolating cut*.

Submodular Function

The main interest here, i.e., solving *isolating cut*, will be essentially based on properties of symmetric *submodular functions*. Although we can prove various properties by appealing to only graph theoretic facts, it's useful to see the proofs via *submodularity*. Here, we give some background, and specifically, for the cut function of graphs.

Definition. Given a finite ground set V , consider a real-valued set function $f: 2^V \rightarrow \mathbb{R}$.

Definition 1.3.3 (Modular). The function f is *modular* if for all $A, B \subseteq V$,

$$f(A) + f(B) = f(A \cap B) + f(A \cup B).$$

Definition 1.3.4 (Submodular). The function f is *submodular* if for all $A, B \subseteq V$,

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B).$$

Definition 1.3.5 (Supermodular). The function f is *supermodular* if for all $A, B \subseteq V$,

$$f(A \cap B) + f(A \cup B) \geq f(A) + f(B).$$

Definition 1.3.6 (Posi-modular). The function f is *posi-modular* if for all $A, B \subseteq V$,

$$f(A - B) + f(B - A) \geq f(A) + f(B).$$

We note that perhaps a more common definition of *submodularity* is *diminishing marginal utility*, i.e., if $f(A + v) - f(A) \geq f(B + v) - f(B)$ for all $A \subseteq B$. Here, we see some examples.

Example (Modular function as weight function). f is *modular* if and only if there exists some $w: V \rightarrow \mathbb{R}$ such that $f(A) = \sum_{v \in A} w(v) + c$ for some shift c .

Example. If f and g are *submodular*, then so is $\alpha f + \beta g$ for some $\alpha, \beta \geq 0$.

One of the reason that *submodularity* is important for graphs is because of the following.

Example (Cut). Given a graph $G = (V, E)$, the cut size function $|\delta_G(\cdot)|: 2^V \rightarrow \mathbb{R}_+$ is *submodular*.

Proof. We simply note that for any $A, B \subseteq V$,

$$|\delta_G(A)| + |\delta_G(B)| = |\delta_G(A \cap B)| + |\delta_G(A \cup B)| + 2|E(A \setminus B, B \setminus A)| \geq |\delta_G(A \cap B)| + |\delta_G(A \cup B)|,$$

where $E(X, Y)$ is the set of edges crossing X and Y for some $X, Y \subseteq V$. ⊗

The above argument extends naturally to non-negative capacitated graph. Moreover, this is also true for directed graph.

Example. Let $G = (V, E)$ be a directed graph. $|\delta^+(\cdot)|$, and hence by symmetry $|\delta^-(\cdot)|$ are *submodular*.

We're also interested in the following property.

Definition 1.3.7 (Symmetric). A set function is *symmetric* if $f(A) = f(V \setminus A)$ for all $A \subseteq V$.

Clearly, $|\delta_G(\cdot)|$ is *symmetric*. However, for directed graph, this is not necessarily the case. Finally, we see that *symmetric submodular* function satisfies another important property.

Example. A *symmetric submodular* function is automatically *posi-modular*.

Now, we discuss uncrossing, a common and powerful technique that is frequently used in working with *submodular functions*. We illustrate this in the context of *min-cuts*.

Lemma 1.3.3. Let $G = (V, E)$ be a graph and $(A, V \setminus A)$, $(B, V \setminus B)$ be two *s-t min-cuts*. Then $(A \cap B, V \setminus (A \cap B))$ and $(A \cup B, V \setminus (A \cup B))$ are also *s-t min-cuts*.

Proof. From *submodularity*, we have $|\delta(A)| + |\delta(B)| \geq |\delta(A \cap B)| + |\delta(A \cup B)|$. However, as both $A \cup B$ and $A \cap B$ are themselves *s-t cuts*, all terms need to be equal. ■

Corollary 1.3.1. For any graph $G = (V, E)$, there is a unique (inclusion-wise) minimal *s-t min-cut*.^a

^aWhile maybe not that useful, from the same logic, there is a unique maximal *s-t min-cut*.

Proof. If there are two s - t min-cuts A, B that are both minimal and distinct, then $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$ since otherwise one will be contained in the other, contradicting the minimality. From Lemma 1.3.3, $A \cap B$ is also a s - t min-cut and $A \cap B$ is a strict subset of A and B , again contradicting minimality of A and B . ■

The above proof applies to directed graph as well since we only used submodularity.

Remark (Graphic matroid). A second aspect of submodularity in graphs comes via *matroids*. We will not discuss it here but the rank function of a matroid is a special class of submodular functions; and in a formal sense, matroid rank functions are building blocks for all submodular functions. Given an undirected graph $G = (V, E)$ there is a fundamental matroid associated with the edge set of G called the *graphic matroid*.^a Several properties of trees and forests can be better understood in the context of the graphic matroid including the *Tutte-Nash-Williams theorem*.

^aThere are other matroids that are also defined from graphs including the dual graphic matroid for instance.

Isolating Cuts via Poly-log Max-flow Computations

We can now formally introduce the isolating cut problem.

Problem 1.3.4 (Isolating cut). Given a graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$ and a set $T \subseteq V$ of terminals. The t_i -isolating cut problem aims to find a cut $(S_i, V \setminus S_i)$ such that $t_i \in S_i$ and $t_j \notin S_i$ (i.e., $t_j \in V \setminus S_i$) for all $j \neq i$.

Intuition. In other words, the cut isolates t_i from the rest of the terminals.

The minimum capacity t_i -isolating cut can be found by a single max-flow computation: by shrinking the terminals in $T - t_i$ into a single vertex s and computing the s - t_i min-cut. Thus, naively, computing all isolating cuts require k max-flow computations. The upshot is that this can be done in only $O(\log k)$ max-flow. Before we describe the algorithm, we first note that from submodularity, we also have a similar structural result, just like Corollary 1.3.1.

Lemma 1.3.4. There is a unique minimal t_i -isolating min-cut $(S_i^*, V \setminus S_i^*)$ such that if $(S_i, V \setminus S_i)$ is any t_i -isolating min-cut, then $S_i^* \subseteq S_i$.

We now describe the algorithm for computing the isolating cuts. Basically, we consider h bi-partitions $(A_1, B_1), \dots, (A_h, B_h)$ with $h = \lceil \log k \rceil$, and compute a cut separating each bi-partition. Then, we take the intersection among the resulting cut sets, which will be isolating cuts as we will see. Finally, with the structural property Lemma 1.3.4, we can then find the minimum isolating cuts from them.

Algorithm 1.10: Isolating Cut [LP20] (also developed independently in [AKT21])

Data: A connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, terminal $T = \{t_i\}_{i=1}^k$
Result: A set of isolating cuts $\{(S_i^*, V \setminus S_i^*)\}_{i=1}^k$ isolating t_i 's

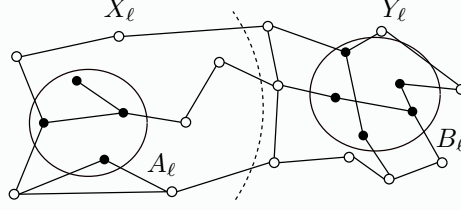
```

1  $h \leftarrow \lceil \log k \rceil$ 
2 for  $\ell = 1, \dots, h$  do                                     //  $h$  bi-partitions  $(A_\ell, B_\ell)$  of  $T$ 
3    $A_\ell \leftarrow \{t_i \mid \text{binary representation of } i \text{ has } 1 \text{ in } \ell^{\text{th}} \text{ bit}\}$ 
4    $B_\ell \leftarrow T \setminus A_\ell$ 
5    $(X_\ell, Y_\ell) \leftarrow s\text{-}t\text{-min-cut}(G, A_\ell, B_\ell)^a$            //  $Y_\ell = V \setminus X_\ell$ 
6 for  $i = 1, \dots, k$  do
7    $S_i \leftarrow \bigcap_{\ell: t_i \in A_\ell} X_\ell \cap \bigcap_{\ell: t_i \in B_\ell} Y_\ell$ 
8    $H_i = (V_i, E_i) \leftarrow \text{shrinking } V \setminus S_i \text{ to a single vertex } s_i$ 
9    $(S_i^*, V \setminus S_i^*) \leftarrow s\text{-}t\text{-min-cut}(H_i, t_i, s_i)$ 
10 return  $\{(S_i^*, V \setminus S_i^*)\}_{i=1}^k$ 

```

^aThis can be done via shrinking A_ℓ and B_ℓ into two separate nodes, and compute the s - t min-cut.

Intuition. The following illustrates [line 5](#), where terminals are black vertices. (A_ℓ, B_ℓ) is a bi-partition of T , while (X_ℓ, Y_ℓ) is a [min-cut](#) that separates (A_ℓ, B_ℓ) .



Additionally, [line 7](#) is created by considering the intersections of all X_ℓ (or Y_ℓ) that includes t_i .

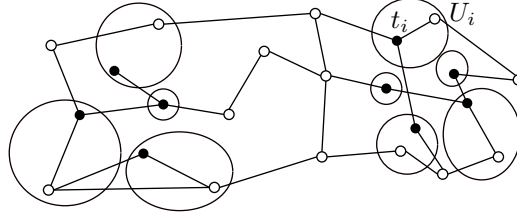
To see the correctness of the [Algorithm 1.10](#), we want to say that [s_i-t_i min-cut](#) is exactly the minimum cost [t_i-isolating cut](#) in G . This is due to [Lemma 1.3.5](#).

Lemma 1.3.5. For each i , $(S_i, V \setminus S_i)$ is a [t_i-isolating cut](#). Furthermore, S_i 's are pairwise disjoint.

Proof. Firstly, $t_i \in A_\ell \subseteq X_\ell$ or $t_i \in B_\ell \subseteq Y_\ell$, implying $t_i \in S_i$. Consider t_j with $j \neq i$. As $i \neq j$, there is some index ℓ in the binary representation of i and j differ in the bit position. Suppose i has 1 in the ℓ^{th} position and j has 0, then $t_i \in A_\ell$ and $t_j \in B_\ell$, implying $t_i \in X_\ell$ and $t_j \notin X_\ell$ as $t_j \in B_\ell \subseteq Y_\ell$ and $Y_\ell \cap X_\ell = \emptyset$. This means $t_j \notin S_i$.

We now prove that $S_i \cap S_j = \emptyset$ for all $i \neq j$. Firstly, there exists some ℓ such that $t_i \in A_\ell$ and $t_j \in B_\ell$ (or $t_i \in B_\ell$ and $t_j \in A_\ell$). Suppose $v \in X_\ell$, then v can't be in $S_j \subseteq Y_\ell$ and if $v \in Y_\ell$, then v can't be in $S_i \subseteq X_\ell$, hence v can't be in both S_i and S_j . ■

[Lemma 1.3.5](#) gives the following picture, where each t_i lives in exactly one S_i .



Hence, for each i , we have a [t_i-isolating cut](#) $(S_i, V \setminus S_i)$. Now, [Lemma 1.3.4](#) states that there is a [t_i-isolating min-cut](#) $(S_i^*, V \setminus S_i^*)$ where S_i^* is a subset of any [t_i-isolating min-cut](#), it doesn't say it will be a subset of S_i in particular, as $(S_i, V \setminus S_i)$ is only a [t_i-isolating cut](#). However, we do not lose anything:

Lemma 1.3.6. The minimal [t_i-isolating min-cut](#) $(S_i^*, V \setminus S_i^*)$ is in $(S_i, V \setminus S_i)$, i.e., $S_i^* \subseteq S_i$.

Proof. It suffices to prove that if $t_i \in A_\ell$ then $S_i^* \subseteq X_\ell$. Assume not, then $S_i^* \cap (V \setminus X_\ell) \neq \emptyset$. But since $S_i^* \cap X_\ell$ is a [t_i-isolating cut](#), while S_i^* is the minimal [t_i-isolating min-cut](#), $|\delta(S_i^* \cap X_\ell)| > |\delta(S_i^*)|$.

Moreover, it's trivial to see that $S_i^* \cup X_\ell$ is a A_ℓ - B_ℓ cut (not necessarily minimum, just a cut), hence we also have $|\delta(S_i^* \cup X_\ell)| \geq |\delta(X_\ell)|$. From [submodularity](#) of $|\delta(\cdot)|$, we have

$$|\delta(S_i^*)| + |\delta(X_\ell)| \geq |\delta(S_i^* \cap X_\ell)| + |\delta(S_i^* \cup X_\ell)|,$$

which is a contradiction. ■

With all the lemmas, it's now easy to see that [Algorithm 1.10](#) is at least correct. Firstly, from [Lemma 1.3.6](#), we know that there the optimal [t_i-isolating min-cut](#) $S_i^* \subseteq S_i$ (here, S_i^* is not necessary the one found by [Algorithm 1.10](#): indeed, we're trying to argue this). As S_i 's are disjoint, each terminal t_i lives in exactly one S_i , hence computing [s_i-t_i min-cut](#) will indeed recover S_i^* .

Intuition. When we contract $V \setminus S_i$, we do not lose the optimal [isolating cut](#) S_i^* .

Theorem 1.3.6 ([LP20]). **Algorithm 1.10** is a deterministic algorithm that given $G = (V, E)$ and a terminal set $T \subseteq V$ with $|T| = k$, computes all the **isolating cuts** using $O(\log k)$ max-flow computations on graphs with $|V|$ vertices and $|E|$ edges each.

Proof. We analyze the runtime. It's easy to see that **line 2** requires $O(\log k)$ max-flow computations on G . It's also easy to show that computing S_i 's in **line 7** can be done in $O((m+n)\log k)$ time given (X_ℓ, Y_ℓ) for $\ell \in [h]$. However, **line 8** and **line 9** seem to require k max-flow computations.

Claim. In total, **line 9** only requires $O(\log k)$ max-flow computations.

Proof. Let us understand the size of H_i . It has $n_i + 1$ vertices where $n_i = |S_i|$, and it has m_i edges where $m_i = |E(S_i)| + |\delta(S_i)|$. Thus, the running time of max-flow on H_i is $T(n_i + 1, m_i)$ where $T(a, b)$ is the running time of max-flow on graph with a nodes and b edges. We observe that $\sum_i (n_i + 1) \leq 2n$ since S_i 's are disjoint, while $\sum_i m_i \leq 2m$: consider any edge $uv \in E$. If $uv \in E(S_i)$ for some i , then it does not contribute to any other H_j . If $uv \in \delta(S_i)$ for some i , then it can be in $\delta(S_j)$ for only one more index $j \neq i$.

Thus, the total time to compute all k max-flows is $\sum_i T(n_i, m_i) \leq T(2n, 2m)$ under reasonable assumption, specifically, $T(a, b)$ is super-additive.^a ⊗

^aFormally, we first create a single H that includes each H_i as a copy in it, and we can run a single max-flow on H to recover all the max-flow values in each H_i . H will have $O(n)$ vertices and $O(m)$ edges.

With the correctness of **Algorithm 1.10**, the theorem is proved. ■

We see that this could have been discovered many years ago in terms of its simplicity. **Algorithm 1.10** has been very influential in the last few years for a number of problems.

Note. Another perspective of the bi-partitions is that they are a way to *derandomize* a natural randomized algorithm that picks some $O(\log k)$ bi-partitions of T at random and computes the cuts between them. With high probability, every t_i, t_j with $i \neq j$ will be separated in at least one of the random bi-partitions.

Remark. The core idea of **isolating cuts** relies only on **submodularity** and symmetry, thus, this applies in much more generality and to several other problems. This is explicitly discussed in [CQ21], though the ideas are implicit in [LP20].

Randomized Algorithm for Steiner Min-Cut via Isolating Cuts

Isolating cut naturally lead to a simple randomized algorithm for **Steiner min-cut**. The basic idea is quite simple. Consider an optimum **Steiner min-cut** $(S, V \setminus S)$ and let $T_1 := S \cap T$ and $T_2 := (V \setminus S) \cap T$, with $k_1 = |T_1|$ and $k_2 = |T_2|$. We may assume that $1 \leq k_1 \leq k_2$.

Note. $(S, V \setminus S)$ is a t_i - t_j **min-cut** for any $i \neq j$ since otherwise, it induces a lower-cost cut.

The basic intuition is the following.

Intuition. If we can sample exactly one terminal in one side of the **Steiner min-cut**, then we can simply use the **isolating cut** to recover the **Steiner min-cut**.

Say we know k_1 . We can sample each terminal in T independently with probability $1/k_1$ to obtain $T' \subseteq T$ such that with constant probability, $|T' \cap T_1| = 1$ and $|T' \cap T_2| \geq 1$ (recall $k_1 \leq k_2$). Suppose T' satisfies these properties and let $T' \cap T_1 = \{t_i\}$. Then, $(S, V \setminus S)$ is a minimum cost t_i -**isolating cut** w.r.t. T' . Hence, by computing t_i -**isolating cuts** for all $t_i \in T'$ and choosing the cheapest one identifies the **Steiner min-cut** for T .

The problem is that we don't know k_1 , and trying all possible values for k_1 (from 1 to $k/2$) will be too expensive. The idea is that the above sampling procedure is robust: say if we sample with probability, say, $1/2k_1$, everything still happens with constant probability. Hence, we only need to try $k_i = 2^i$, i.e., $O(\log k)$ different sampling probabilities.

Algorithm 1.11: Steiner Min-Cut

Data: A connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, terminal $T = \{t_i\}_{i=1}^k$
Result: A possible **Steiner min-cut** $(U^*, V \setminus U^*)$

```

1  $U^* \leftarrow \emptyset$  // Initialize Steiner min-cut
2 for  $i = 0, \dots, \lceil \log k \rceil$  do
3    $T' \leftarrow \text{Sample}(T, 1/2^i)$  // Sample each terminal in  $T$  with probability  $1/2^i$ 
4    $\{(S_i^*, V \setminus S_i^*)\}_{i=1}^{|T'|} \leftarrow \text{Isolating-Cut}(G, c, T')$ 
5    $U^* \leftarrow \text{Min-Cost}(\{(S_i^*, V \setminus S_i^*)\}_{i=1}^{|T'|} \cup \{(U^*, V \setminus U^*)\})$  // Update minimum cost cut
6 return  $(U^*, V \setminus U^*)$ 

```

We now formally prove the robustness we have mentioned.

Lemma 1.3.7. Algorithm 1.11 finds the **Steiner min-cut** for T with a constant probability.

Proof. We see that for $k_1 = 1$, Algorithm 1.11 is correct (deterministically) since i can only be 0 and $T' = T$. Hence, let $k_1 > 1$. Consider the case that $1/2^{i+1} < 1/k_1 \leq 1/2^i$, where i will be tried at some point during $i = 0$ to $\lceil \log k \rceil$ since $1 \leq k_1 \leq k/2$. Let $\ell = 2^i$, i.e., $\ell \leq k_1 \leq 2\ell$.

Now, let \mathcal{E}_1 be the event that $|T_1 \cap T'| = 1$, i.e., exactly one terminal from T_1 is chosen. Then

$$\Pr(\mathcal{E}_1) = k_1 \cdot \frac{1}{\ell} \cdot \left(1 - \frac{1}{\ell}\right)^{k_1 - 1} \geq \left(1 - \frac{1}{\ell}\right)^{2\ell} \geq \frac{1}{e^2}.$$

On the other hand, let \mathcal{E}_2 be the event that $T_2 \cap T' \neq \emptyset$. We see that

$$\Pr(\mathcal{E}_2) \geq 1 - \left(1 - \frac{1}{\ell}\right)^{k_2} \geq \left(1 - \frac{1}{\ell}\right)^{\ell} \geq 1 - \frac{1}{e}.$$

Since T_1 and T_2 are disjoint, \mathcal{E}_1 and \mathcal{E}_2 are independent, we have

$$\Pr(\mathcal{E}_1 \cap \mathcal{E}_2) \geq \left(1 - \frac{1}{e}\right) \cdot \frac{1}{e^2},$$

which is a constant. ■

Theorem 1.3.7. There is a randomized algorithm that given $G = (V, E)$ and terminal set $T \subseteq V$ with $|T| = k$, outputs the **Steiner min-cut** with high probability using in $O(\log^2 k \log n)$ max-flow computations.^a

^aAgain, potentially on graphs with $|V|$ vertices and $|E|$ edges each from Theorem 1.3.6.

Proof. From Lemma 1.3.7, Algorithm 1.11 succeeds with a constant probability. We further boost the overall success probability by rerunning Algorithm 1.11 $\Theta(\log n)$ times. With Theorem 1.3.6, this requires $O(\log^2 k \log n)$ max-flow computations. ■

Remark (Deterministic algorithm). Li and Panigraphy [LP20] also developed deterministic **min-cut** and **Steiner min-cut** algorithms using additional ideas based on expander decomposition.

Lecture 5: Metric Embedding and Multi-Cut Problem

1.3.4 Max-Flow Min-Cut Theorem

10 Sep. 11:00

Finally, we conclude this section by proving the well-known **max-flow min-cut theorem**. Given a directed graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, consider the following linear program relaxation of the

s - t min-cut where s, t are two distinct vertices:

$$\begin{aligned}
 \min \quad & \sum_{e \in E} c(e)x_e \\
 \sum_{e \in P} x_e & \geq 1 \quad P \in \mathcal{P}_{s,t}; \\
 (P) \quad & x_e \geq 0 \quad \forall e \in E;
 \end{aligned}
 \qquad
 \begin{aligned}
 \max \quad & \sum_{P \in \mathcal{P}_{s,t}} y_P \\
 \sum_{P \ni e} y_P & \leq c(e) \quad \forall e \in E; \\
 (D) \quad & y_P \geq 0 \quad \forall P \in \mathcal{P}_{s,t},
 \end{aligned}
 \tag{1.1}$$

where $\mathcal{P}_{s,t}$ is the set of all s - t paths. The integer version is with constraints $x_e \in \{0, 1\}$.

Remark. An s - t cut is often also defined as $\delta^+(S)$ for some $S \subseteq V$ where $s \in S$ and $t \in V \setminus S$.

Proof. Suppose E' is an s - t cut and S is the set of nodes reachable from s in $G - E'$. Then, $\delta(S) \subseteq E'$ and $\delta(S)$ is an s - t cut. Hence, it suffices to focus on such limited type of cuts.^a \circledast

^aIn some more general settings, it is useful to keep these notions separate.

It is well-known that s - t min-cut can be computed efficiently via s - t max-flow, establishing the **max-flow min-cut theorem**. This fundamental theorem in combinatorial optimization has many applications, and is typically established via the augmenting path algorithm. Here, we give another proof.

Theorem 1.3.8 (Max-flow min-cut). Let $G = (V, E)$ be a directed graph with rational edge capacities $c: E \rightarrow \mathbb{Q}_+$ and let $s, t \in V$ be two distinct vertices. The s - t max-flow value in G is equal to the s - t min-cut value, and both can be computed in strongly polynomial time. Furthermore, if c is integer valued, then there exists an integer-valued max-flow as well.

Proof. To start, we observe that following.

Intuition. The primal **linear program** assigns lengths to edges such that the s - t shortest path according to which is at least 1. This is a fractional relaxation of the cut.

We claim that it's possible to round the fractional solution of the primal to the exact s - t min-cut without any loss. Consider the following rounding algorithms for the primal **linear program**.

Algorithm 1.12: θ -Rounding Algorithm

Data: A directed graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, $s, t \in V$

Result: A s - t min-cut F

```

1  $\{x_e\}_{e \in E} \leftarrow \text{LP-Solve}(\text{Min-Cut-LP}(G, c, s, t))$  // Solve the primal
2  $\theta \leftarrow \text{Uniform}((0, 1))$ 
3 for  $v \in V$  do
4    $d_x(s, v) \leftarrow \text{Shortest-Path-Dist}(s, v, G, x)$ 
5 return  $F = \delta^+(B_x(s, \theta))$  //  $B_x(s, \theta) = \{v \in V \mid d_x(s, v) \leq \theta\}$ 

```

Firstly, **Algorithm 1.12** will output a valid s - t cut since $d_x(s, t) \geq 1$ by feasibility of the linear program solution x and hence $t \notin B_x(s, \theta)$ for any $\theta < 1$.

Claim. For any $e \in E$, $\Pr(e \text{ is cut by Algorithm 1.12}) \leq x_e$.

Proof. The edge $e = (u, v)$ is cut if and only if $d_x(s, u) \leq \theta < d_x(s, v)$. Hence, the edge is not cut if $d_x(s, v) \leq d_x(s, u)$. If $d_x(s, v) > d_x(s, u)$, we have $d_x(s, v) - d_x(s, u) \leq x_{(u,v)}$. Since θ is chosen uniformly at random from $(0, 1)$, the probability that θ lies in the interval $[d_x(s, u), d_x(s, v)]$ is at most $x_{(u,v)}$. \circledast

With this claim, from linearity of expectation, we see that $\mathbb{E}[c(\delta^+(B_x(s, \theta)))] \leq \sum_{e \in E} c(e)x_e$. As $B_x(s, \theta)$ will always be a valid s - t cut, this implies that there is an integral cut whose cost is at most that of the linear program relaxation, implying that the linear program relaxation yields an optimum solution.

Finally, observe that the dual is the *path version* of the s - t max-flow. Hence, from strong duality, the optimal value of s - t min-cut is the same as the s - t max-flow. Moreover, we note that the primal

is strongly polynomial-time solvable if we have a separation oracle. In this case, given $\{x_e\}_{e \in E}$, we need to answer either this is a feasible solution, or outputs some path p such that $\sum_{e \in p} x_e < 1$, which is exactly the shortest s - t path algorithm and can be solved efficiently. ■

Intuition (Line embedding). The rounding can be thought as putting every vertex on a line from s to t , sorting by their distances given by x_e 's. Then, observe that on that line, any two vertices $u, v \in V$ has distance at most $x_{(u,v)}$, and picking θ corresponds to picking a threshold on the line.

The above intuition not only helps the analysis, but also gives a way to derandomize [Algorithm 1.12](#). Basically, we can try all possible θ 's, and if we adapt this line embedding viewpoint, the only interesting θ 's are given by the n values $d_x(s, v)$.

Chapter 2

Metric Methods

In this chapter, we will see a series of techniques that are based on the structure of the metric spaces underlying the graphs.

2.1 Multi-cut via Metric Decomposition

Recall that [s-t min-cut problem](#) we just saw. Now, consider a more general problem called [multi-min-cut](#).

Problem 2.1.1 (Multi-min-cut). Given a graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$ and k pairs of vertices $\{(s_i, t_i)\}_{i=1}^k$, the *multi-min-cut problem* aims to find a minimum capacity cut that separates all pairs.

[Multi-min-cut](#) is NP-hard even on trees. In general, it is a NP-complete problem. Hence, we ask for an approximation algorithm instead. An $O(k)$ -approximation algorithm is trivial by simply outputting the union of all [s_i-t_i min-cuts](#). The goal is an $O(\log k)$ -approximation algorithm, which also proves the multi-commodity flow-cut gap.

Note. It turns out that $O(\log k)$ is tight in general graphs. For planar graphs, one can get an $O(1)$ -approximation and flow-cut gap. These results are only for undirected graphs since the situation is more complicated in directed graphs, and we will discuss that later.

Again, we can write the following linear program relaxation for the [multi-min-cut problem](#):

$$\begin{aligned}
 \min \quad & \sum_{e \in E} c(e)x_e & \max \quad & \sum_{i=1}^k \sum_{P \in \mathcal{P}_{s_i, t_i}} y_P \\
 \sum_{e \in P} x_e \geq 1 \quad & P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}; & \sum_{i=1}^k \sum_{P \ni e} y_P \leq c(e) & \quad \forall e \in E; \\
 \text{(P)} \quad x_e \geq 0 & \quad \forall e \in E; & \text{(D)} \quad y_P \geq 0 & \quad \forall P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}.
 \end{aligned} \tag{2.1}$$

The primal assigns distance labels x_e to edges so that, on each path P between s_i and t_i , the distance labels of these edges on P sum up to at least one, just like the [s-t min-cut](#).

Remark. The primal (with exponentially many constraints) is efficiently solvable.

Proof. With the ellipsoid method, we just need a separation oracle. Consider setting the length of each edge to x_e and for each pair (s_i, t_i) , compute the length of the shortest path between s_i and t_i and check whether it is at least 1. This only takes k many times compared to the previous separation oracle for the [s-t min-cut](#) linear program relaxation, hence it's still polynomial time. \otimes

On the other hand, the dual variable can be interpreted as the amount of flow between s_i and t_i that is routed along the path P . This is called the *maximum throughput multi-commodity flow* problem,

where we don't care about individual demands, but only the overall flow. The dual tries to assign an amount of flow y_P to each path P so that the total flow on each edge is at most the capacity of the edge.

Note. The flow conservation constraints are automatically satisfied and the endpoints of the path P determine which kind of commodity is routed along the path.

2.1.1 Approximation via Randomized Decomposition

The first algorithm [GVY93] (see [Vaz01; WS11]) that achieved an $O(\log k)$ -approximation for **multi-min-cut** is based on the region growing technique [LR99]. Here, we present the randomized rounding algorithm due to its future application for metric embedding [CKR05], which in particular also shows the integrality gap of the primal **linear program** to be $O(\log k)$. The goal is to find a procedure to cut (i.e., decompose) the graph into components that satisfies the requirement of being a **multi-cut**, i.e.,

- each component has diameter at most 1 when the edge capacity is induced by the primal solution;
- the probability that edge e is cut is at most αx_e ,

then we will get an α -approximation algorithm. To do this, we consider the following algorithm.

Algorithm 2.1: Random Partition [CKR05]

Data: A connected graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, $\{(s_i, t_i) \mid s_i, t_i \in V\}_{i=1}^k$

Result: A **multi-min-cut** F

```

1  $\{x_e\}_{e \in E} \leftarrow \text{LP-Solve}(\text{Multi-Min-Cut-LP}(G, c, \{(s_i, t_i)\}_{i=1}^k))$  // Solve the primal
2  $\theta \leftarrow \text{Uniform}([0, 1/2))$ 
3  $\sigma \leftarrow \text{Uniform}(S([k]))$  // Random permutation from permutation group  $S([k])$ 
4 for  $i = 1, \dots, k$  do
5    $V_{\sigma(i)} \leftarrow B_x(s_{\sigma(i)}, \theta) \setminus \bigcup_{j < i} V_{\sigma(j)}$ 
6  $F \leftarrow \bigcup_{i=1}^k \delta(V_i)$ 
7 return  $F$ 

```

Intuition. It essentially reduces to *low-diameter decomposition* in a metric space.

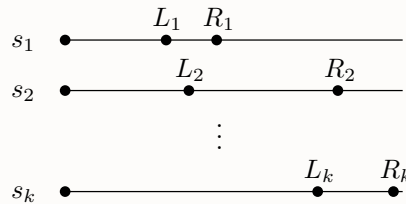
Lemma 2.1.1. Algorithm 2.1 outputs a feasible **multi-cut** for the given instance.

Proof. Suppose not, then there exists a pair (s_i, t_i) are still connected in $G - F$. This can only happen if s_i is “grabbed” by some terminals s_j which is proceeded before s_i , i.e., there exists some $V_j \subseteq B_x(s_j, \theta)$ that contains both s_i and t_i . However, if s_j grabs both s_i and t_i , it means the distance between s_i and t_i is at most $2\theta < 1$, a contradiction to the feasibility of $\{x_e\}_{e \in E}$. ■

Lemma 2.1.2. For any $e \in E$, $\Pr(e = uv \text{ is cut by Algorithm 2.1}) \leq 2H_k x_e \leq O(\log k)x_e$.^a

^a H_k is the k^{th} harmonic number.

Proof. Let $L_i := \min(d_x(s_i, u), d_x(s_i, v))$ and $R_i := \max(d_x(s_i, u), d_x(s_i, v))$. Without loss of generality, we can renumber s_i 's such that $L_1 \leq L_2 \leq \dots \leq L_k$.

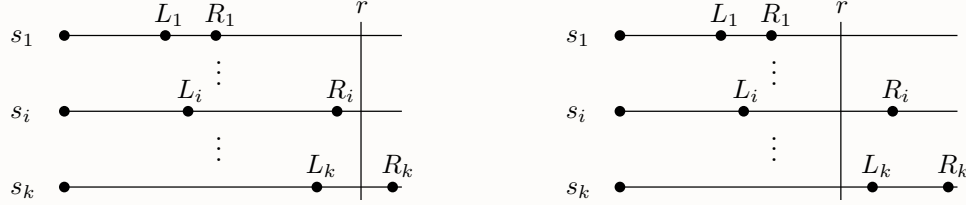


Let A_i be the event that s_i cut $e = uv$ first, i.e., A_i is the event that $|V_i \cap \{u, v\}| = 1$ and $|V_j \cap \{u, v\}| = 0$ for all j such that $\sigma(j) < \sigma(i)$, where $|V_i \cap \{u, v\}| = 1$ simply says that s_i cuts the edge e . If A_i happens, then for all j that come before i in σ , neither u nor v can be in V_j since:

- if only one of u and v is in V_j , then s_j cuts e ;
- if both u and v are in V_j , s_i can't cut e as the cut set only grabs the leftover vertices (line 5).

Let A be the event that e is cut, which is the union of the disjoint events A_i 's, hence $\Pr(A) = \sum_{i=1}^k \Pr(A_i)$. Now, for any fixed $r \in [0, 1/2)$, we see that

- $r \notin [L_i, R_i]$: This is easy to understand as if $r \notin [L_i, R_i]$, s_i is impossible to cut e ;
- $r \in [L_i, R_i]$: Consider some $j < i$ and suppose j comes before i in the permutation (i.e., $\sigma(j) < \sigma(i)$). Since $j < i$, $L_j \leq L_i \leq r$. Hence, at least one of u and v is inside the ball of radius r centered at s_j . Consequently, s_i can't be the first to cut e , resulting in the fact that s_i is the first to cut the edge e if $\sigma(i) < \sigma(j)$ for all $j < i$.



Since σ is a random permutation, i appears before j for all $j < i$ with probability $1/i$. Hence,

$$\begin{cases} \Pr(A_i \mid \theta = r) = 0, & \text{if } \theta \notin [L_i, R_i]; \\ \Pr(A_i \mid \theta = r) \leq 1/i, & \text{if } \theta \in [L_i, R_i]. \end{cases}$$

As θ is independent of σ , we have

$$\Pr(A_i) \leq \frac{1}{i} \cdot \mathbb{P}(\theta \in [L_i, R_i]) = \frac{2}{i}(R_i - L_i) \leq \frac{2x_e}{i}$$

from the triangle inequality $R_i \leq L_i + x_e$. This finally leads to

$$\Pr(A) = \sum_{i=1}^k \Pr(A_i) \leq \sum_{i=1}^k \frac{2x_e}{i} \leq 2H_k x_e,$$

and we conclude the theorem by noting that $H_k = O(\log k)$. ■

Theorem 2.1.1. Algorithm 2.1 is an $O(\log k)$ -approximation (in expectation) algorithm for the multi-min-cut problem. Furthermore, the integrality gap of the multi-min-cut linear program is $O(\log k)$.

Proof. Let F be the set of edges outputted by Algorithm 2.1. For each edge e , let $\xi_e = \mathbb{1}_{e \in F}$ in an indicator random variable. Hence, we have $\mathbb{E}[\xi_e] = \mathbb{P}(\xi_e = 1) \leq 2H_k x_e$ from Lemma 2.1.2. This leads to

$$\mathbb{E}[c(F)] := \mathbb{E}\left[\sum_{e \in F} c(e)\right] = \mathbb{E}\left[\sum_{e \in E} c(e)\xi_e\right] = \sum_{e \in E} c(e) \Pr(\xi_e) \leq 2H_k \sum_{e \in E} c(e)x_e = 2H_k \text{OPT}_{\text{LP}}$$

where OPT_{LP} is the optimal value of the linear program. Since $\text{OPT}_{\text{LP}} \leq \text{OPT}$ where OPT is the optimum value of the multi-min-cut problem, we have

$$\mathbb{E}[c(F)] \leq 2H_k \text{OPT}_{\text{LP}} \leq 2H_k \text{OPT} = O(\log k) \text{OPT}.$$

This also implies that there exists a set of edges F such that the total capacity of edges in F is at most $2H_k \text{OPT}_{\text{LP}}$, i.e., $\text{OPT}_{\text{LP}} \leq \text{OPT} \leq 2H_k \text{OPT}_{\text{LP}}$, which proves the integrality gap result. ■

The expected cost analysis can be used to obtain a randomized algorithm via repetition that outputs an $O(\log k)$ -approximation with high probability. The algorithm can also be derandomized, but it's not straight forward.

Remark (Flow-cut gap). Recall that when $k = 1$, we have the [max-flow min-cut theorem](#). The integrality gap of the standard linear program for [multi-min-cut](#) is the same as the relative gap between flow and cut when k is arbitrary. The upper bound on the integrality gap gives an upper bound on the gap.

Lecture 6: Low-Diameter Decomposition and Tree Embeddings

2.1.2 Low-Diameter Decomposition

12 Sep. 11:00

Given a graph $G = (V, E)$ and edge length $\ell: E \rightarrow \mathbb{R}_+$, which define a metric space (V, d) where $d(u, v)$ is the shortest path distances between u and v in G as before. As we have seen, a useful notion is to decompose or partition the graph into subgraphs (or clusters) of small diameter. More precisely, given a graph $G = (V, E)$, we would like to partition V into clusters with vertex sets $\{V_i\}_{i=1}^h$ such that each V_i has diameter at most some given parameter δ .

Example (Singleton). It's trivial to consider the singleton partition, where $V_i = \{v\}$ for all $v \in V$.

However, the goal in partitioning is to ensure that two vertices $u, v \in V$ that are close to each other, say $d(u, v) < \delta$, should ideally not be split apart into different clusters. However, as the graph (or metric space) is connected, it's impossible to do this deterministically.

Example (Line graph). Considered a line graph L_n . The most natural randomized algorithm is to shift the line by $\theta \in [0, \delta)$, and then we separate the line graph by δ -length clusters. In this way, the probability that any pair $u, v \in V$ is cut is at most $d(u, v)/\delta$.

This is the best we can hope for, i.e., u, v are separated only with probability proportional to $d(u, v)/\delta$.

Definition 2.1.1 (Low-diameter decomposition). Let $G = (V, E)$ be a graph with edge lengths $\ell: E \rightarrow \mathbb{R}_+$, which induces a distance metric $d: V \times V \rightarrow \mathbb{R}_+$. Let Δ be the diameter of the metric space (V, d) . For a given $\delta \in [0, \Delta]$, a *low-diameter decomposition* with cutting probability parameter α is a probability distribution \mathcal{D} over the set \mathcal{P} of all partitions of V such that

- for any partition $P = (V_1, \dots, V_h) \in \mathcal{P}$ in $\text{supp}(\mathcal{D})$, $\text{diam}((V_i, d)) \leq \delta$;
- for all $u, v \in V$, $\Pr(u, v \text{ are separated}) \leq \alpha d(u, v)/\delta$.

Given a partition $P \in \mathcal{P}$, we write E_P be the set of edges (pairs) that are separated by P . Hence, in [Definition 2.1.1](#), the second point is equivalent to $\Pr((u, v) \in E_P) \leq \alpha d(u, v)/\delta$.

Definition 2.1.2 (Low-diameter decomposition scheme). A *low-diameter decomposition scheme* with parameter α is a family of algorithms that given any $\delta \in [0, \Delta]$, generates a [low-diameter decomposition](#) with cutting (separation) probability parameter at most α .

Notation (Strong v.s. weak diameter guarantee). A [low-diameter decomposition](#) is said to have the *strong diameter* guarantee if the diameter of each cluster V_i in the induced graphs $G[V_i]$ is at most δ . Note that [Definition 2.1.2](#) does not require that because it is based on the metric closure (V, d) of the given graph. The standard definition is called the *weak diameter* guarantee. Some applications require the strong diameter guarantee. We will, by default, work with weak diameter guarantee and mention strong diameter guarantee when needed.

Note (Padded decomposition). [Definition 2.1.1](#) is often strengthened to require more. Given a point u , let $B_d(u, r) = \{v \in V \mid d(u, v) \leq r\}$. In *padded decomposition*, we require that for each u , and for each $r \leq \delta$, the probability of $B_d(u, r)$ being contained in the same part is at least $e^{-\beta r}$.

Remark (Sparse cover). A *sparse cover* consists of several clusters $\{V_i\}_{i=1}^h$. Each cluster should have weak/strong diameter at most δ . For each u, v with distance at most δ , there must be some cluster V_i that contains both u and v , and no vertex u must be in more than some number s of clusters. The techniques underlying sparse covers and [low-diameter decomposition](#) are related though we will mostly work only with the latter.

The main question here for the [low-diameter decomposition](#) is the smallest α that one can obtain. It turns out that for general metric spaces, $\alpha = O(\log n)$ is a tight bound for both strong and weak diameter guarantee [Bar96]. For planar graph metrics, $\alpha = O(1)$ is achievable, where weak diameter guarantee was shown in [KPR93], and the strong diameter guarantee was more difficult and was shown later. See [Fil24] for some recent work and pointers to literature on these topics.

Now, we present the algorithm for weak diameter guarantee. Let (V, d) be a metric space with $|V| = n$. Borrowing ideas from [Algorithm 2.1](#), we see that implicitly this is a [metric partitioning scheme](#). We simply modify the algorithm to ensure that the weak diameter of each cluster is at most a given parameter δ .

Algorithm 2.2: Random Partition [CKR05]

Data: A metric space (V, d) , δ
Result: E_P for the partition P

```

1  $\theta \leftarrow \text{Uniform}([0, \delta/2])$ 
2  $\sigma \leftarrow \text{Uniform}(S(V))$  // Random permutation from permutation group  $S(V)$ 
3 for  $i = 1, \dots, n$  do
4    $V_{\sigma(i)} \leftarrow B_d(v_{\sigma(i)}, \theta) \setminus \bigcup_{j < i} V_{\sigma(j)}$ 
5 return  $\bigcup_{i=1}^n \delta(V_i)$ 
```

From the same analysis as in [Lemma 2.1.2](#), claim the following.

Claim. [Algorithm 2.2](#) correctly outputs a partition of V into clusters, each of which has weak diameter at most δ .

Furthermore, the probability guarantee can also be stated.

Theorem 2.1.2. The probability that u and v are in different clusters outputted by [Algorithm 2.2](#) is at most $2H_n d(u, v)/\delta$, i.e., $\alpha = H_k = O(\log n)$.

Proof. We see that $\Pr(A_j) \leq 2d(u, v)/\delta \cdot 1/j$, hence $\Pr(A) \leq 2H_n d(u, v)/\delta$. ■

Finally, we consider a not so apparent modification of [Algorithm 2.2](#): we sample θ from $[\delta/4, \delta/2]$ instead of $[0, \delta/2]$.

Algorithm 2.3: Refined Random Partition [CKR05]

Data: A metric space (V, d) , δ
Result: E_P for the partition P

```

1  $\theta \leftarrow \text{Uniform}([\delta/4, \delta/2])$ 
2  $\sigma \leftarrow \text{Uniform}(S(V))$  // Random permutation from permutation group  $S(V)$ 
3 for  $i = 1, \dots, n$  do
4    $V_{\sigma(i)} \leftarrow B_d(v_{\sigma(i)}, \theta) \setminus \bigcup_{j < i} V_{\sigma(j)}$ 
5 return  $\bigcup_{i=1}^n \delta(V_i)$ 
```

Intuition. Intuitively, this will preserve closer points.

It's clear that the guarantee about the diameter remains the same.

Claim. [Algorithm 2.3](#) correctly outputs a partition of V into clusters, each of which has weak diameter at most δ .

The main difference is in the probability guarantee which is refinement of the previous bound.

Theorem 2.1.3. The probability that u and v are in different clusters outputted by Algorithm 2.3 is at most $\frac{4d(u,v)}{\delta} \log \frac{|B(u,\delta/2)|}{|B(u,\delta/8)|}$, i.e., $\alpha = \alpha(\delta) = 4 \log \frac{|B(u,\delta/2)|}{|B(u,\delta/8)|}$.

Proof. We sketch the proof based on the proof of Lemma 2.1.2. Assuming the exact same notation, and we fix $u, v \in V$ and think of V as v_1, \dots, v_n . If $d(u, v) \geq \delta/8$, then the edge is going to get cut with constant probability, and the bound is not giving anything interesting, so we are primarily interested in the case when $d(u, v) < \delta/8$.

We consider the event A_i which is that v_i is the first vertex to separate the pair u, v . We can argue as before that $\Pr(A_i) \leq 1/i \cdot \Pr(\theta \in [L_i, R_i])$, and this is at most $4d(u, v)/\delta i$ since we're choosing the radius from $[\delta/4, \delta/2]$. The new twist is that since we choose $\theta \in [\delta/4, \delta/2]$ and $d(u, v) < \delta/8$, no vertex $v_j \in B(u, \delta/8)$ can separate u, v because $L_j \leq d(v_j, u) < \delta/8$ and $R_j \leq L_j + d(u, v_j) \leq \delta/8 + \delta/8 = \delta/4$. Any such vertex will capture both u, v if they are not already separated. Similarly, any vertex $v_j \notin B(u, \delta)$ can cut the pair because $L_j \geq \delta - d(u, v) \geq \delta - \delta/8 \geq \delta/2$. Therefore, if A is the event of u, v being cut, then

$$\Pr(A) \leq \sum_{j \in B(u, \delta) \setminus B(u, \delta/8)} \Pr(A_j) \leq \frac{4d(u, v)}{\delta} \sum_{|B(u, \delta/8)| < j \leq |B(u, \delta)|} \frac{1}{j} \leq \frac{4d(u, v)}{\delta} \log \frac{|B(u, \delta)|}{|B(u, \delta/8)|},$$

proving the result. ■

2.2 Dominating Tree Metrics Embedding

Using tree representations of graphs is a powerful tool in algorithm design. Here, we are interested in representing the distances in an undirected graph via distances in a [spanning tree](#). Let $G = (V, E)$ be a graph with edge length $\ell: E \rightarrow \mathbb{R}_+$, which induces a metric space (V, d) via shortest path distances. The main question is the following:

Problem 2.2.1 (Tree embedding). Given a graph $G = (V, E)$ with edge length $\ell: E \rightarrow \mathbb{R}_+$, the *tree embedding* problem aims to find a [spanning tree](#) $T = (V, E_T)$ of G such that for any $u, v \in V$, $d_T(u, v) \leq \alpha d_G(u, v)$ where α is called the *distortion* or *stretch*.^a

^aClearly, $d_T(u, v) \geq d_G(u, v)$.

Example (Cycle). Consider a cycle C_n . We see that for a fixed edge (u, v) , there exists one spanning tree T such that $d_T(u, v) = n - 1$.

Motivated by applications of [spanning tree](#) based metric approximations, we observe that if we are allowed to pick a probability distribution over [spanning trees](#), then the expected distance for any pair of vertices can be much better than the above worst-case example.

Example (Cycle). Again, consider a cycle C_n . If we allow randomization (picking trees randomly),

$$\mathbb{E}[d_T(u, v)] = \frac{n-1}{n} \cdot 1 + \frac{1}{n} \cdot (n-1) \leq 2.$$

It's showed that [Alo+95] for any weighted graph $G = (V, E)$, there is a distribution \mathcal{D} over [spanning trees](#) of G such that for any $u, v \in V$,

$$\mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] \leq \exp\left(\sqrt{\log n \log \log n}\right) \cdot d_G(u, v) < n^{o(1)} d_G(u, v).$$

Intuition. This is a probabilistic approximation of a graph metric by [spanning tree](#) metrics.

This can also be viewed as a metric *embedding* result. In keeping with metric embedding terminology, we're interested in the worst-case guarantee of how much the expected distance for any pair increases, i.e., minimizing α . In the above example $\alpha \leq \exp(\sqrt{\log n \log \log n})$.

Note (Lower bound). A lower bound of $\alpha = \Omega(\log n)$ is required for probabilistic [tree](#) approximation, and it's conjectured that this is tight [\[Alo+95\]](#).

2.2.1 Dominating Tree Metric

It turned out that this is quite difficult to obtain even a poly-logarithmic bound [\[Elk+05\]](#), and currently the best known bound is $O(\log n \log \log n)$ [\[ABN08\]](#). To make the problem easier, [\[Bar96\]](#) proposed to forget about the graph topology and just focus on (V, d) , i.e., consider the *metric embeddings* instead of [spanning tree embeddings](#). More generally, we work with the metric completion (V, d) and view it as a complete graph on V , where any [spanning tree](#) of the complete graph is now allowed. Moreover, we allow additional vertices. This is formalized as follows.

Definition 2.2.1 (Dominating tree metric). A tree $T = (V_T, E_T)$ with edge length $\ell_T: E_T \rightarrow \mathbb{R}_+$ is a *dominating tree metric* for a finite metric space (V, d) if $V \subseteq V_T$ and for all $u, v \in V$, $d_T(u, v) \geq d_G(u, v)$.

Then, we're interested in approximating metrics probabilistically by [dominating tree metrics](#):

Definition 2.2.2 (Probabilistic approximation). A *probabilistic approximation* of a metric space (V, d) by [dominating tree metrics](#) is a probability distribution \mathcal{D} over a collection of trees $\{T_i\}_{i=1}^h$ if each T_i is a [dominating tree metric](#) for (V, d) .

Furthermore, we say that the [probabilistic approximation](#) \mathcal{D} has *stretch* α if for all $u, v \in V$,

$$\mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] \leq \alpha d(u, v).$$

2.2.2 Tree Embedding with Low-Diameter Decomposition

One can use [low-diameter decomposition](#) to efficiently sample from a distribution that has stretch $\alpha = O(\log^2 n)$ [\[Bar96\]](#), and this was subsequently improved to $O(\log n \log \log n)$ [\[Bar98\]](#), and finally improved to the optimal $O(\log n)$ [\[FRT03\]](#) using [Algorithm 2.3](#).

Intuition. Recursively decompose (V, d) using [low-diameter decomposition](#).

Specifically, let (V, d) be a metric space with diameter Δ . We use [low-diameter decomposition](#) with parameter $\delta = \Delta/2$ to randomly partition V into clusters $\{V_i\}_{i=1}^h$ of diameter at most $\Delta/2$, then we recursively find a tree for each of the V_i , with root r_i . We create a new dummy root r and connect each r_i to r with an edge of length Δ .

Algorithm 2.4: Tree Embedding

Data: A metric space (V, d) , diameter D

Result: A rooted tree (T, r)

```

1 if  $|V| = 1$  then
2    $T \leftarrow (V, \emptyset)$ 
3   return  $(T, v)$                                      //  $V = \{v\}$ 
4
5 Create a tree  $T$  with root  $r$ 
6  $\{V_i\}_{i=1}^h \leftarrow \text{Low-Diameter-decomposition}((V, d), D/2)$ 
7 for  $j = 1, \dots, h$  do
8    $(T_j, r_j) \leftarrow \text{Tree-Embedding}((V_j, d), D/2)$ 
9   Connect  $T_j$  to  $T$  by adding edge  $(r, r_j)$  of length  $D$ 
10 return  $(T, r)$ 

```



Figure 2.1: Illustration of Algorithm 2.4.

Notation. We say $\delta = \Delta/2^i$ at level i to make the analysis cleaner.

We will now assume that the minimum distance is at least 1 by scaling, and let Δ be the diameter of the metric space with this assumption.

Remark. If the minimum distance is at least 1, Algorithm 2.4 yields a stretch of $O(\log n \log \Delta)$.

Theorem 2.2.1. Let Δ be the diameter of (V, d) . Algorithm 2.4 outputs a random dominating tree metric $T = (V_T, E_T)$ with length ℓ_T such that for each $u, v \in V$, $\mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] \leq O(\alpha \log \Delta) d(u, v)$ where α is the cutting probability of the low-diameter decomposition algorithm used.

Proof. We prove this by induction. It's clear that the base case is trivial. If we start with $D = \Delta$, then at depth i of the recursion, the parameter is $\Delta/2^{i-1}$, and it is the upper-bound on the diameter of the metric space in that recursive call. We note the following claims.

Claim. The length of the root to leaf path of a tree created at level i of the recursion is at most $\sum_{j \geq i} \Delta/2^{j-1} \leq 2\Delta/2^{i-1}$.

Suppose u and v are first separated at level i of the recursion. Then, $d_T(u, v) \leq 4\Delta/2^{i-1}$ from the above claim. We see that if u and v are separated in the first level of the recursion due to the low-diameter decomposition algorithm, its probability is at most $\alpha d(u, v)/(\Delta/2) \leq 2\alpha d(u, v)/\Delta$, in which case their distance in the tree is at most 4Δ . Otherwise, they are in the same cluster, and we can apply induction. Note that u and v are definitely separated by level t where t is the smallest integer such that $\Delta/2^{t+1} < d(u, v)$. Hence, the depth of the recursion is at most $1 + \lceil \log \Delta \rceil \leq 2 \log \Delta$. It's easy to unroll the induction and use the preceding claim to obtain

$$\mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] \leq \sum_{i=0}^{t+1} 2\alpha \frac{d(u, v)}{(\Delta/2^{i-1})} \cdot \left(4 \frac{\Delta}{2^{i-1}}\right) \leq O(\alpha \log \Delta) d(u, v) \quad (2.2)$$

since the depth of the recursion is $O(\log \Delta)$. ■

If we use Algorithm 2.2 as the low-diameter decomposition algorithm, then we have $\alpha = O(\log n)$. From Theorem 2.2.1, we see that the tree may require depth $\log \Delta$ to provide a good approximation, and in general $\log \Delta$ can be as large as n , so we get an $O(n \log n)$ approximation.

Example. Consider the metric induced by a path with n edges and edge lengths are 2^i for all $i = 1, \dots, n$. In such cases, the dependence of the stretch on $\log \Delta$ is undesirable.

Remark. One can alter Algorithm 2.4 to make the stretch bound $O(\log^2 n)$.

Proof. In applying the low-diameter decomposition algorithm with parameter δ , we ensure that any pair u, v such that $d(u, v) \leq \delta/n^2$ is not cut during the procedure. We can do this by contracting all

such pairs without changing the diameter of the resulting metric space too much. This will ensure that in the tree construction process, a pair u, v participates in only $O(\log n)$ levels and hence the expected stretch can be bounded by $O(\alpha \log n)$. \otimes

Remark (Hierarchically well-separated tree). The trees constructed by Algorithm 2.4 have an additional strong property: the edge lengths at each level are the same and the length from the root to the leaf go down by a factor of 2 at each level. A tree metric with such a property is called *hierarchically well-separated* tree metric and this additional property can be exploited in algorithms and comes for free in the construction.

Finally, we note that if we choose Algorithm 2.3 as the *low-diameter decomposition* algorithm specifically, the expected stretch is actually $O(\log n)$, which is optimal [FRT03].

As previously seen (Theorem 2.1.3). The probability that u and v are in different clusters outputted by Algorithm 2.3 is at most $\frac{4d(u,v)}{\delta} \log \frac{|B(u, \delta/2)|}{|B(u, \delta/8)|}$.

Thus, the guarantee $\alpha(\delta)$ from the *low-diameter decomposition* algorithm is no longer uniform but depends on the diameter. Plugging this to Equation 2.2, we have

$$\begin{aligned} \mathbb{E}_{T \sim \mathcal{D}}[d_T(u, v)] &\leq \sum_{i=0}^{t+1} 2 \log \frac{|B(u, \Delta/2^i)|}{|B(u, \Delta/2^{i+3})|} \frac{d(u, v)}{\Delta/2^{i-1}} \cdot \left(4 \frac{\Delta}{2^{i-1}}\right) \\ &\leq 8d(u, v) (\log |B(u, \Delta/2)| + \log |B(u, \Delta/4)| + \log |B(u, \Delta/8)|) \leq O(\log n)d(u, v). \end{aligned}$$

Remark (Lower bound). $O(\log n)$ bound for *low-diameter decomposition* algorithm and *tree embeddings* are near optimal (modulo precise constant factors).

Proof. We can use the existence of low-girth graphs which are closely tied to expanders in a direct fashion. Another way is via indirection. Previously, in the lecture note, we saw that the integrality gap of the *linear program relaxation* for *multi-min-cut* is $\Omega(\log k)$ (the upper-bound is proved in Theorem 2.1.1 via a *low-diameter decomposition* algorithm). In fact, if α is the factor for the *low-diameter decomposition* algorithm, then we get an $O(\alpha)$ -approximation for *multi-min-cut* via the *linear program*. Thus, one sees that $\alpha = \Omega(\log n)$ for general metrics (in the integrality gap example for *multi-min-cut* $k = \Omega(n^2)$). One can use similar approaches to prove for *tree embedding*. \otimes

Note (Efficient algorithms). While we focus on the quality of *low-diameter decomposition* algorithms and *tree embedding* but not so much on the running times, it is easy to see that the algorithms themselves can be implemented in polynomial time. The main computation is about the shortest paths. If one computes all pairs shortest paths (APSP), then the algorithms are pretty simple. However, APSP is slow, which takes $O(mn)$ times. It is possible to compute the *low-diameter decomposition* algorithm and metric *tree embeddings* in close to linear time on a weighted graph with m edges. This involves computing approximate shortest paths and several tricks, and sometimes we give up on the quality of the approximation by logarithmic factors.

Lecture 7: Linear Programming for Sparsest Cut

2.3 Sparsest Cut

17 Sep. 11:00

Consider building a network of n vertices. The best network might be the complete graph, which can do everything and is robust. However, the problem is that the degree is too high ($n-1$).

Example. For degree equal to 2, the best we can hope for is a cycle.

The magic happens whenever the degree goes up to 3.

Intuition. If we can down-weight edges in a complete graph K_n by $3/(n-1)$, then any cut S has

$$\delta(S) = |S| \cdot |V \setminus S| \cdot \frac{3}{n-1} \approx c|S|$$

for $|S| \ll |V \setminus S|$ and some $c \geq 0$.

This notion can be formalized as the so-called [expander](#) [HLW06]. We postpone the formal introduction of [expander](#), and first focus on a closely related problem, the [sparsest cut problem](#), specifically, the [non-uniform](#) version. It turns out that solving this helps us answer various questions for [expanders](#).

2.3.1 Uniform and Non-Uniform Sparsest Cut Problem

To introduce the problem, we first define the [sparsity](#) for a cut.

Definition 2.3.1 (Sparsity). Given a graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$ and a demand graph $H = (V, F)$ with demand capacity $D: F \rightarrow \mathbb{R}_+$, for any cut $S \subseteq V$, its *sparsity* is defined as

$$\frac{c(\delta(S))}{\sum_{i: |S \cap \{s_i, t_i\}|=1} D_i}.$$

That is, the [sparsity](#) of a cut is the ratio of the capacity of the cut and the total demand of the pairs separated by S . Now, we can introduce the [non-uniform sparsest cut problem](#).

Problem 2.3.1 (Non-uniform sparsest cut). Given a supply graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$ and k pairs of vertices $\{(s_i, t_i)\}_{i=1}^k$ along with non-negative demand values D_1, \dots, D_k .^a The *non-uniform sparsest cut problem* aims to find a cut S with minimum [sparsity](#).

^aIf G is undirected, then the demand pairs are unordered, i.e., we do not distinguish (s_i, t_i) from (t_i, s_i) .

Here, demands forms a demand graph $H = (V, F)$ with edges (s_i, t_i) and demand capacity $D: F \rightarrow \mathbb{R}_+$ such that $D((s_i, t_i)) = D_i$. With this representation, the [sparsity](#) of cut S is simply $c(\delta_G(S))/D(\delta_H(S))$ where $\delta_G(S)$ (respectively, $\delta_H(S)$) represents the supply (respectively, demand) edges crossing S .

Intuition. We're trying to find the best “bang per buck” cut, i.e., how much capacity do we need to remove per amount of demand separated to satisfy the demand?

Remark. We can define a cut as removing a set of edges, leading to more than two components. In the case of [sparsest cut](#) in undirected graphs, it suffices to restrict attention to cuts of the form $\delta(S)$ for some $S \subseteq V$. This is not necessarily true for directed graphs or even in undirected graphs with node-weights or in hypergraphs.

Problem 2.3.2 ((Uniform) sparsest cut). The *sparsest cut* problem is the same as [Problem 2.3.1](#) with all $D(u, v) = 1$ for each unordered pair of vertices (u, v) .^a

^aThat is, $\{(s_i, t_i)\}_{i=1}^k$ is the set of all unordered pairs of vertices.

We see that in the [uniform sparsest cut problem](#), the [sparsity](#) of a cut S is given by $c(\delta_G(S))/|S||V \setminus S|$. In this case, the demand graph H is a complete graph with unit demand values on each edge.

Finally, to further motivate the problem, we see that the [uniform sparsest cut](#) helps us directly and indirectly solve the [balanced separator problem](#), a central problem in graph algorithm:

Problem 2.3.3 (Balanced separator). The *balanced separator problem* aims to partition a graph $G = (V, E)$ into two pieces $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that $|V_1| \approx |V_2|$ while minimizing edges between V_1 and V_2 .

2.3.2 Linear Program Relaxation and Maximum Concurrent Flow

It's not clear how to start solving the [uniform sparsest cut problem](#) as writing a linear program relaxation for which is not obvious, compared to [multi-min-cut](#) and other cut problems where we have explicit terminal pairs that we wish to separate. Hence, to write an integer program for which, we let $y_i \in \{0, 1\}$ being the indicator variable of whether we want to separate (s_i, t_i) . Moreover, let $x_e \in \{0, 1\}$ for all $e \in E$ to be the cut indicator variables.

Intuition. If we decide to separate (s_i, t_i) , then for every path between s_i and t_i we should cut at least one edge on the path.

Hence, a natural integer program of [non-uniform sparsest cut](#) and its relaxation is given by:

$$\begin{aligned}
 \min \quad & \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k D_i y_i} \\
 \sum_{e \in P} x_e &\geq y_i \quad \forall P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}; \\
 x_e &\in \{0, 1\} \quad \forall e \in E; \\
 y_i &\in \{0, 1\} \quad \forall i = 1, \dots, k;
 \end{aligned}
 \quad \rightarrow \quad
 \begin{aligned}
 \min \quad & \sum_{e \in E} c(e)x_e \\
 \sum_{i=1}^k D_i y_i &= 1 \\
 \sum_{e \in P} x_e &\geq y_i \quad \forall P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}; \\
 x_e &\geq 0 \quad \forall e \in E; \\
 y_i &\geq 0 \quad \forall i = 1, \dots, k,
 \end{aligned}
 \tag{2.3}$$

where we use the standard trick, i.e., linearization to normalize the denominator to be 1, to make the ratio of the integer program into a linear program. With the dual variable z_P for each path such that it indicates the amount of “flow” sent on the path P , the dual of the above is

$$\begin{aligned}
 \max \quad & \lambda \\
 \sum_{P \in \mathcal{P}_{s_i, t_i}} z_P &\geq \lambda D_i \quad \forall i = 1, \dots, k; \\
 \sum_{i=1}^k \sum_{\substack{P \in \mathcal{P}_{s_i, t_i} \\ P \ni e}} z_P &\leq c(e) \quad \forall e \in E; \\
 z_P &\geq 0 \quad \forall P \in \bigcup_{i=1}^k \mathcal{P}_{s_i, t_i}; \\
 \lambda &\geq 0,
 \end{aligned}
 \tag{2.4}$$

which is a multi-commodity flow. In particular, it solves the *maximum concurrent multi-commodity flow* problem for the given instance, i.e., it finds the largest value of λ such that there is a feasible multi-commodity flow for the given pairs in which the flow routed for pair (s_i, t_i) is at least λD_i .

Notation (Concurrent flow). It is called *concurrent flow* since we need to route all demand pairs to the same factor which is in contrast to the [dual of multi-min-cut](#), which corresponds to the maximum throughput multi-commodity flow.^a

^aRecall that in this case, some pairs may have zero flow while others have a lot of flow.

We note that this dual can be solved efficiently via ellipsoid method since the separation oracle is just the shortest path problem. One can also write a compact linear program via distance variables as

$$\begin{aligned}
 \min \quad & \sum_{uv \in E} c(uv)d(uv) \\
 \sum_{i=1}^k D_i d(s_i, t_i) &= 1 \\
 d &\text{ is a metric on } V.
 \end{aligned}$$

In this context, we can understand the *flow-cut gap* of [non-uniform sparsest cut](#) by the following equivalent way of thinking about the problem:

Intuition (Cut-condition). Given a multi-commodity flow instance on G . A necessary condition to route all the demand pairs is that if G satisfies the *cut-condition*, i.e., for every $S \subseteq V$, the capacity $c(\delta(S))$ is at least the demand separated by S . However, the converse is not necessarily true, i.e., the cut-condition is not sufficient.

The cut-condition is sufficient when $k = 1$ but is not true in general even for $k = 3$ in undirected graphs. The question is the maximum value of λ such that we can route λD_i for each pair i . The worst-case integrality gap of the [proceeding linear program](#) is precisely the flow-cut gap.

2.3.3 Rounding Linear Program via ℓ_1 Embeddings

It's known that there is an $O(\log n)$ -approximation algorithm and the flow-cut gap for [uniform sparsest cut](#), together with a lower bound of $\Omega(\log n)$ on the flow-cut gap for [uniform sparsest cut](#) via [expanders](#) [LR99]. This leads to an $O(\log^2 n)$ -approximation for [non-uniform sparsest cut](#), and it was an open problem to obtain a tight conjectured bound of $O(\log n)$. It turns out to be possible via the optimal rounding algorithm for the [linear program relaxation](#). This goes via metric embedding theory [LLR95; AR98], hence we need some basics in metric embeddings to point out the connection and rounding.

Note. Even though the metric embedding machinery is powerful, it can seem like magic. The more basic ideas for [uniform sparsest cut](#) based on region growing is useful to know [WS11].

Remark (Rounding via multi-min-cut). There are also close connections between [sparsest cut](#) and [multi-min-cut](#). In particular, suppose there is an $\alpha(k, n)$ -approximation for [non-uniform sparsest cut](#), then we have an $O(\alpha(k, n) \ln k)$ -approximation for [multi-min-cut](#). The converse is also true.^a

^aSee the [note](#) for a reference.

Before we start, consider the following simple setting when G is a tree $T = (V, E)$.

Example (Tree). Given a tree $G = T = (V, E)$, for each edge $e \in T$, we can associate a cut S_e which is one side of the two components in $T - e$. The capacity of the cut $\delta(S_e)$ is $c(e)$. Let $D(e) = \sum_{i: |S_e \cup \{s_i, t_i\}|=1} D_i$ be the demand separated by e . The [sparsity](#) of the cut S_e is simply $c(e)/D_e$, hence finding the [sparsest cut](#) in this case is easy. Interestingly, the [linear program relaxation](#) give an optimum solution on a tree.

Proof. Let (x, y) be a feasible solution to the [dual](#) of the [linear program relaxation](#) with objective value λ . We want to prove that if G is a tree T , then there is an edge $e \in T$ such that $c_e/D_e \leq \lambda$. We note that the integer program in this case becomes

$$\begin{aligned} \min \quad & \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k D_i d_x(s_i, t_i)} \\ & \sum_{e \in P} x_e \geq y_i \quad \forall P \in \mathcal{P}_{s_i, t_i}; \\ & x_e \in \{0, 1\} \quad \forall e \in E; \\ & y_i \in \{0, 1\} \quad \forall i = 1, \dots, k, \end{aligned}$$

where $d_x(s_i, t_i)$ is the shortest path distance between s_i and t_i induced by x . Since there is a unique path P_{s_i, t_i} from s_i to t_i in T such that $d_x(s_i, t_i) = \sum_{e \in P_{s_i, t_i}} x_e$, we have

$$\lambda = \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k D_i d_x(s_i, t_i)} = \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k D_i \sum_{e \in P_{s_i, t_i}} x_e} = \frac{\sum_{e \in E} c(e)x_e}{\sum_{e \in E} x_e \sum_{i: e \in P_{s_i, t_i}} D_i} = \frac{\sum_{e \in E} c(e)x_e}{\sum_{e \in E} D(e)x_e}.$$

Finally, the result follows from $\sum_i a_i / \sum_i b_i \geq \min_i a_i / b_i$ for positive a_i and b_i 's. ⊛

Example (Ring). For a ring graph, the same technique works where we need to remove two edges.

The reason why the above proof works for trees is because of a more general phenomenon: the shortest path distances are ℓ_1 metrics, or equivalently, cut metrics.

Cut, Line, and ℓ_1 Metrics, and Metric Embedding

To explain the above phenomenon, consider a finite metric space (V, d) with following metrics:

Definition 2.3.2 (Cut metric). Let (V, d) be a finite metric space. The metric d is a *cut metric* if there is a set $S \subseteq V$ such that $d = d_S$, where d_S associated with the cut S is defined as

$$d_S(u, v) = \begin{cases} 1, & \text{if } |S \cap \{u, v\}| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The *cut-cone* consists of non-negative combination of cut metrics:

Definition 2.3.3 (Cut cone). Let (V, d) be a finite metric space. The metric d is in the *cut cone* if there exist non-negative scalars y_S where $S \subseteq V$ such that for all $u, v \in V$,

$$d(u, v) = \sum_{S \subseteq V} y_S d_S(u, v).$$

Beside the cut metric, another useful metric is the line metric and the well-known ℓ_1 metric:

Definition 2.3.4 (Line metric). Let (V, d) be a finite metric space. The metric d is a *line metric* if there is a mapping $f: V \rightarrow \mathbb{R}$ such that for all $u, v \in V$,

$$d(u, v) = |f(u) - f(v)|.$$

Definition 2.3.5 (ℓ_1 metric). Let (V, d) be a finite metric space. The metric d is an ℓ_1 metric if there is some integer d and a mapping $f: V \rightarrow \mathbb{R}^d$ such that for all $u, v \in V$,

$$d(u, v) = \|f(u) - f(v)\|_1$$

It might not be too surprising that the following holds.

Lemma 2.3.1. A metric d of a metric space (V, d) is an ℓ_1 metric if and only if it is a non-negative combination of line metrics (in the cone of the line metrics).

Proof. If d is an ℓ_1 metric then each dimension corresponds to a line metric. Conversely, any non-negative combination of line metrics can be made into an ℓ_1 metric where each line metric becomes a separate dimension (scalar multiplication for a line metric is also a line metric). ■

A more interesting observation is that any cut metric d_S is a simple line metric: map all vertices in S to 0 and all vertices in $V \setminus S$ to 1. This leads to the following.

Lemma 2.3.2. A metric d is an ℓ_1 metric if and only if d is in the cut cone.

Proof. If d is in the cut cone, then it is a non-negative combination of the cut metrics, hence it is a non-negative combination of line metrics by the above observation, hence an ℓ_1 metric.

For the converse, it suffices to argue that any line metric is in the cut cone. Let $V = \{v_i\}_{i=1}^n$ and let d be a line metric on V . Without loss of generality, assume that the coordinates x_i of the points for each v_i corresponding to the line metric d are $x_1 \leq x_2 \leq \dots \leq x_n$ on the real line. For $1 \leq i < n$, let $S_i = \{v_1, v_2, \dots, v_i\}$. It is not hard to verify that $\sum_{i=1}^{n-1} |x_{i+1} - x_i| d_{S_i} = d$. ■

Now we have introduced all the necessary metrics we will use. Consider a finite metric space (V, d) .

Claim. Any finite metric space can be viewed as one that is derived from the shortest path metric induced on a graph with some non-negative edge lengths.

If $G = (V, E)$ is a simple graph and $\ell: E \rightarrow \mathbb{R}_+$ are some edge-lengths, the metric induced on V depends both on the *topology* of G and the lengths as well, i.e., finite metrics can encode graph structure, hence it can be diverse. When trying to round we may want to work with simpler metric spaces.

Intuition (Embedding). Embed a given metric space (V, d) into a simpler host metric space (V', d') via an embedding $f: V \rightarrow V'$.

Note. Even though we may be interested in finite metric spaces, the host metric space can be continuous or infinite such as the \mathbb{R}^h for dimension h .

As embedding typically distorts the distances, thus, we want to find embeddings with small **distortion**.

Definition 2.3.6 (Distortion). Let (V, d) and (V', d') be two metric spaces and let $f: V \rightarrow V'$ be an embedding. The *distortion* of f is given by^a

$$\max_{\substack{u, v \in V \\ u \neq v}} \left(\frac{d'(f(u), f(v))}{d(u, v)}, \frac{d(u, v)}{d'(f(u), f(v))} \right).$$

^aAdditive version are also explored, although they are very restrictive due to lack of scale invariance.

Additionally, we're interested in the following kind of embeddings:

Definition. Let (V, d) and (V', d') be two metric spaces and let $f: V \rightarrow V'$ be an embedding.

Definition 2.3.7 (Isometric embedding). f is an *isometric embedding* if $d(u, v) = d'(f(u), f(v))$ for all $u, v \in V$.

Definition 2.3.8 (Contraction). f is a *contraction* if $d(u, v) \geq d'(f(u), f(v))$ for all $u, v \in V$.

Definition 2.3.9 (Non-contracting). f is *non-contracting* if $d(u, v) \leq d'(f(u), f(v))$ for all $u, v \in V$.

Of particular importance are embeddings of finite metric spaces into \mathbb{R}^h , where the distance in the host space is measured under a norm such as ℓ_p norm. The dimension h is also important in various applications but in some settings like with **non-uniform sparsest cut**, it is not. We assume the following:

Theorem 2.3.1 (Bourgain). Any n -point finite metric space can be embedded into ℓ_2 (hence also ℓ_1) with **distortion** $O(\log n)$. Moreover, the embedding is a **contraction** and can be constructed in randomized polynomial time and embeds points into \mathbb{R}^h where $h = O(\log^2 n)$.

In fact, one can obtain a refined version of **Theorem 2.3.1** that is useful for **non-uniform sparsest cut**.

Theorem 2.3.2 (Bourgain). Let (V, d) be an n -point finite metric space and let $S \subseteq V$ with $|S| = k$. Then there is a randomized polynomial time algorithm to compute an embedding $f: V \rightarrow \mathbb{R}^{O(\log^2 n)}$ such that the embedding is a **contraction**^a and for every $u, v \in S$, $\|f(u) - f(v)\|_1 \geq cd(u, v) / \log k$ for some universal constant c .

^aI.e., $\|f(u) - f(v)\|_1 \leq d(u, v)$ for all $u, v \in V$

We saw that the integrality gap of the **linear program** is 1 on trees since the shortest path metric on trees is in the **cut cone**, i.e., ℓ_1 -embeddable. More generally, one can prove that if the shortest path metric on a graph G embeds into ℓ_1 with **distortion** α , then the integrality gap of the **linear program** is at most α . This will imply an $O(\log n)$ -integrality gap via **Theorem 2.3.2** since any n -point finite metric space embeds into ℓ_1 with **distortion** $O(\log n)$.

Lecture 8: Randomized Rounding for Sparsest Cut and Expanders

Randomized Rounding Algorithm with Metric Embeddings

19 Sep. 11:00

Now, we see how to utilize [Theorem 2.3.2](#) to design a randomized rounding algorithm for the [non-uniform sparsest cut problem](#). The main theorem is the following.

Theorem 2.3.3. Let $G = (V, E)$ be a graph. Suppose any finite metric induced by edge lengths on E can be embedded into ℓ_1 with [distortion](#) α , then the integrality gap of the [linear program](#) for the [non-uniform sparsest cut](#) is at most α for any instance on G .

Proof. Let (x, y) be a feasible fraction solution of the [linear program relaxation](#), and let d be the metric induced by edge lengths given by x . Let λ be the value of the solution, i.e.,

$$\lambda = \frac{\sum_{uv \in E} c(uv)d(u, v)}{\sum_{i=1}^k D_i d(s_i, t_i)}.$$

Since d can be embedded into ℓ_1 with [distortion](#) at most α , and any ℓ_1 metric is in the [cut cone](#) from [Lemma 2.3.2](#), it implies that there are scalars z_S , $S \subseteq V$, such that for all $u, v \in V$,

$$\frac{1}{\alpha} \sum_{S \subseteq V} z_S d_S(u, v) \leq d(u, v) \leq \sum_{S \subseteq V} z_S d_S(u, v).$$

Without loss of generality, we assume that the embedding is a [contraction](#). Then, we have

$$\begin{aligned} \lambda &= \frac{\sum_{uv \in E} c(uv)d(u, v)}{\sum_{i=1}^k D_i d(s_i, t_i)} \geq \frac{1}{\alpha} \frac{\sum_{uv \in E} c(uv) \sum_{S \subseteq V} z_S d_S(u, v)}{\sum_{i=1}^k D_i \sum_{S \subseteq V} d_S(s_i, t_i)} \\ &= \frac{1}{\alpha} \frac{\sum_{S \subseteq V} z_S c(\delta_G(S))}{\sum_{S \subseteq V} z_S D(\delta_H(S))} \geq \frac{1}{\alpha} \min_{S \subseteq V} \frac{c(\delta_G(S))}{D(\delta_H(S))}. \end{aligned}$$

Hence, there is a cut whose [sparsity](#) is at most $\alpha\lambda$. ■

[Theorem 2.3.3](#) shows that one of the cuts with $z_S > 0$ has [sparsity](#) at most $\alpha\lambda$. Now, suppose we have an ℓ_1 embedding into h -dimensions, i.e., \mathbb{R}^h . Observe the following.

Intuition. First, each dimension in \mathbb{R}^h corresponds to a [line](#) embedding, and each [line](#) embedding is in the [cut cone](#) with only $n - 1$ cuts used to express it (recall [Lemma 2.3.2](#)). Thus, given an ℓ_1 embedding into \mathbb{R}^h with [distortion](#) α , we only need to try $d(n - 1)$ cuts and one of them will be guaranteed to have [sparsity](#) at most $\alpha\lambda$.

[Algorithm 2.5](#) exploits this intuition.

Algorithm 2.5: Non-Uniform Sparsest Cut via Embedding

Data: A supply graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, demand graph $H = (V, F)$ with demand capacity $D: F \rightarrow \mathbb{R}_+$

Result: The [sparsest cut](#) $S \subseteq V$

```

1  $(\{x_e\}_{e \in E}, \{y_i\}_{i=1}^k) \leftarrow \text{LP-Solve}(\text{Non-Uniform-Sparsest-Cut-LP}(G, c, H, D))$ 
2  $f \leftarrow \text{Bourgain-Embedding}((V, d_x))$  //  $f: V \rightarrow \mathbb{R}^h, h = O(\log^2 n)$ 
3 for  $\ell = 1, \dots, h$  do
4   Sort  $\ell^{\text{th}}$  coordinate of  $\{f(v)\}_{v \in V}$  as  $x_1^{(\ell)} \leq x_2^{(\ell)} \leq \dots \leq x_n^{(\ell)}$  //  $f(v)_\ell = x_i^{(\ell)}$  for some  $i$ 
5   for  $j = 1, \dots, n - 1$  do
6      $S_j^{(\ell)} \leftarrow \{v \in V \mid f(v)_\ell \leq x_j^{(\ell)}\}$ 
7  $S \leftarrow \arg \min_{\ell \in [h], j \in [n-1]} c(\delta_G(S_j^{(\ell)})) / D(\delta_H(S_j^{(\ell)}))$ 
8 return  $S$ 
```

The guarantee of [Algorithm 2.5](#) can be derived from [Theorem 2.3.3](#).

Theorem 2.3.4. Algorithm 2.5 outputs cuts of sparsity at most $\alpha\lambda^*$,^a in particular, it's a randomized $O(\log k)$ -approximation algorithm for non-uniform sparsest cut.

^a λ^* is the optimal solution of the dual linear program.

Proof. From Theorem 2.3.2, f is a contraction and with distortion $\alpha = O(\log k)$. From a similar argument as in Theorem 2.3.3, we see that for some z_S , $S \subseteq V$, we have

$$\begin{aligned}\lambda^* &= \frac{\sum_{e \in E} c(e)x_e}{\sum_{i=1}^k D_i d_x(s_i, t_i)} \\ &\geq \frac{\sum_{uv \in E} c(uv) \|f(u) - f(v)\|_1}{\alpha \sum_{i=1}^k D_i \|f(s_i) - f(t_i)\|_1} \\ &= \frac{1}{\alpha} \frac{\sum_{uv \in E} c(uv) \sum_{\ell=1}^h |f(u)_\ell - f(v)_\ell|}{\sum_{i=1}^k D_i \sum_{\ell=1}^h |f(s_i)_\ell - f(t_i)_\ell|} \\ &= \frac{1}{\alpha} \frac{\sum_{\ell=1}^h \sum_{j=1}^{n-1} |x_{j+1}^{(\ell)} - x_j^{(\ell)}| c(\delta_G(S_j^{(\ell)}))}{\sum_{\ell=1}^h \sum_{j=1}^{n-1} |x_{j+1}^{(\ell)} - x_j^{(\ell)}| D(\delta_H(S_j^{(\ell)}))} = \frac{1}{\alpha} \frac{\sum_{S \subseteq V} z_S c(\delta_G(S))}{\sum_{S \subseteq V} z_S D(\delta_H(S))} \geq \frac{1}{\alpha} \min_{S \subseteq V} \frac{c(\delta_G(S))}{D(\delta_H(S))},\end{aligned}$$

where the second last equality follows from the fact that $s_i, t_i \in V$ as well. ■

2.3.4 Line, ℓ_1 , and Tree Embeddings, and State-of-the-Art

Theorem 2.3.2 shows that any finite metric space on n points embeds into ℓ_1 with distortion $O(\log n)$. Here, we hint on the underlying algorithm of the construction: from Lemma 2.3.1, ℓ_1 embeddings are a non-negative combination of line embeddings. A particular type of line embedding is the following.

Definition 2.3.10 (Fréchet embedding). Let (V, d) be a metric space and let $S \subseteq V$. The *Fréchet embedding* is a contraction $f: V \rightarrow \mathbb{R}$ such that $f(v) = d(S, v)$.

Many results in embeddings into ℓ_p spaces are based on using Fréchet embeddings in various clever and often highly non-trivial ways. In particular, Theorem 2.3.2 is based on picking many random sets and combining the resulting Fréchet embeddings.

Now, we note that Theorem 2.3.2 can also be derived via probabilistic tree embeddings because every tree metric embeds into ℓ_1 isometrically. For general metrics, tree embeddings provide a more constrained space while yielding the same worst-case distortion. However, one can ask if ℓ_1 embeddings provide better distortion for concrete graph classes. This is indeed the case.

Example (Ring). Consider a ring graph (a cycle with capacities). One can prove that tree embeddings require a distortion 2, while the ring metric can be isometrically embedded into ℓ_1 . Thus, the flow-cut gap on ring is 1 which is not obvious.

Rather than looking at distortion, one can ask about the flow-cut gap obtained via different embeddings for a particular graph class. First, recall the followings for the non-uniform sparsest cut.

As previously seen (Theorem 2.3.4). The flow-cut gap in general undirected graphs is $O(\log k)$.

Additionally, for general graph, a lower bound is also known.

Remark (Lower bound). Expanders give a lower bound on the flow-cut gap to be $\Omega(\log k)$ even for uniform sparsest cut [LR99].

With these general bounds in mind, we now consider the flow-cut gap for planar graphs in particular.

Example (Planar graph). There is a famous conjecture that the flow-cut gap in planar graphs is $O(1)$ [Gup+04]. Interestingly, for tree embeddings, there is a lower bound of $\Omega(\log n)$ even on the special case of planar graphs called series parallel graphs. Hence, tree embeddings are not powerful

enough to prove the conjecture.

The best flow-cut gap so far is $O(\sqrt{\log n})$ via ℓ_1 embeddings, thus separating the general graph case from the planar graph case. For series parallel graphs, we know that the flow-cut gap is a tight bound of 2 and establishing this tight bound took a fair amount of work.

For **uniform sparsest cut**, the flow-cut gap in planar graphs is $O(1)$ [KPR93]. One can show a tight connection between embeddability into ℓ_1 and flow-cut gap [Gup+04].

On the other hand, we can ask for a better approximation guarantee. First, note the following.

Note. Approximating the **non-uniform sparsest cut problem** is not the same as establishing the flow-cut gap as the flow-cut gap relies on the **linear program relaxation**.

Problem. Can we obtain a better approximation than $O(\log k)$ for **non-uniform sparsest cut**?

Answer. Yes! By using semi-definite programming based relaxation, one can obtain an $O(\sqrt{\log n})$ -approximation for **uniform sparsest cut** [ARV09] and also the **product instances**. Based on this, an $O(\sqrt{\log n} \log \log n)$ -approximation for **non-uniform sparsest cut** is achieved [ALN05; ALN07].^a *

^aThere was a conjecture that the SDP based relaxation would yield an $O(1)$ -approximation, but it was shown that the integrality gap is essentially close to $\Omega(\sqrt{\log n})$.

2.3.5 Node Capacities¹

A slight generalization of the **uniform sparsest cut problem** is obtained by considering demands induced by weights on vertices, which we refer to **product instance**.

Problem 2.3.4 (Product instance of sparsest cut). Given a graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$ and vertex weight $\pi: V \rightarrow \mathbb{R}_+$. The *product instance of sparsest cut problem* is the **non-uniform sparsest cut problem** with demand $D(u, v)$ for edge $uv \in E$ setting to be $\pi(u)\pi(v)$.

Notation. In this case, the dual flow instances are called *product multi-commodity flow*.

We see that the **product instances** indeed generalizes **uniform sparsest cut**: If $\pi(u) = 1$ for all u , then this reduces to the **uniform sparsest cut problem**.

Remark (Subset sparsity). If $\pi(u) \in \{0, 1\}$ for all u , then we are focusing our attention on the **sparsity** w.r.t. the set $V' = \{v \in V \mid \pi(v) = 1\}$. Since vertices with $\pi(u) = 0$ play no role.

2.4 Expander and Well-Linked Set

We now introduce **expanders** [HLW06], which relate to the **non-uniform sparsest cut** in an intricate way.

Definition 2.4.1 (Expander). An *expander* with parameter α is a graph $G = (V, E)$ such that for all $S \subseteq V$ with $|S| \leq |V|/2$ such that $|\delta(S)| \geq \alpha|S|$.

Definition 2.4.2 (Expansion). The *expansion* of a graph $G = (V, E)$ is $\min_{S: |S| \leq |V|/2} |\delta(S)|/|S|$.

Another related notion called **conductance** also has a nice connection to **uniform sparsest cut**.

Definition 2.4.3 (Conductance). Given a graph $G = (V, E)$ and a cut $S \subseteq V$, the *conductance* $\phi(G)$ of G is defined as $|\delta(S)|/\text{vol}(S)$ where $\text{vol}(S) = \sum_{v \in S} \deg(v)$.

It's clear that G is an α -**expander** if the **expansion** of G is at least α . Initially, **expansion** arose from the **graph bisection problem**.

¹We refer to the **note** for further information on further generalizations to node capacitated graph and directed graph.

Problem 2.4.1 (Graph bisection). Given a graph $G = (V, E)$, the *graph bisection problem* aims to find a partition of G into $(S, V \setminus S)$ such that $|S| = |V \setminus S|$ while minimizing $|\delta(S)|$.

However, as we will soon see, *expander* itself is quite interesting. Firstly, we note that *expanders* do exist, and they are quite common.

Lemma 2.4.1. There exists *expanders* with degree 3 with *expansion* $\alpha = \Omega(1)$. More specifically, a random 3-regular graph is an *expander* with high probability.

Hence, with [Lemma 2.4.1](#), a natural way to generate an *expander* is to first sample a random regular graph, then check its *expansion*. However, computing the *expansion* is coNP-hard.

2.4.1 Expansion and Conductance via Sparsest Cut

The first connection of *expander* to the *uniform sparsest cut* is that the latter can be used to find the *expansion* of a graph. In particular, we see that when $|S| \leq |V|/2$, we have

$$\frac{1}{|V|} \frac{|\delta(S)|}{|S|} \leq \frac{|\delta(S)|}{|S||V \setminus S|} \leq \frac{2}{|V|} \frac{|\delta(S)|}{|S|}.$$

Remark. The *expansion* and the *uniform sparsest cut*'s sparsity are within a factor of 2 of each other. Hence, while determining the *expansion* exactly is coNP-hard, we can use *uniform sparsest cut* to certify the *expansion* of a graph within a factor of 2.

Sometimes it is useful to consider *expansion* with vertex weights $w: V \rightarrow \mathbb{R}_+$ as well. In this case, the expansion is defined as $\min_{S: w(S) \leq w(V)/2} |\delta(S)|/w(S)$.

Note. It's dual corresponds to the *product multi-commodity flow instances* where $\pi(v) = w(v)$.

As for *conductance*, it's easy to see that one can capture it by *expansion* via setting weights on vertices with $w(v) = \deg(v)$, which further reduces to the *product instance of sparsest cut*.

Claim. For regular graphs, *expansion* and *conductance* are the same.

2.4.2 Spectral Relaxation for Conductance

In several applications it is important to obtain constant-degree *expanders* with constant *expansion*.

Intuition. The $O(\sqrt{\log k})$ -approximation *algorithms* we saw are not useful in this regime.

It turns out that there is a very different method based on spectral graph theory that helps in this regime. For an undirected graph on n vertices, consider the Laplacian $\mathcal{L}_G := D - A$, where D is the diagonal degree matrix and A is the adjacent matrix. In particular, we have

$$(\mathcal{L}_G)_{ij} := \begin{cases} \deg(v_i), & \text{if } i = j; \\ -1, & \text{if } i \neq j \text{ and } A_{ij} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Since \mathcal{L}_G is symmetric, all its eigenvalues are real. Moreover, this matrix is also positive semi-definite, hence all its eigenvalues are actually non-negative. Let $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be its eigenvalues, then a well-known and famous result in spectral graph theory is the following.

Theorem 2.4.1 (Cheeger's inequality). Given a graph G with *conductance* $\phi(G)$,

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}.$$

Remark. λ_2 provides a constant factor approximation for the **conductance** when it is a constant!

Since the **expansion** and **conductance** are related by the maximum degree, when the degree is a small constant, one can use λ_2 to certify **expansion**. Due to its importance for certifying **expansion/conductance**, some use λ_2 as the definition of **expansion** since it is computable and also helps in construction of **expanders**.

2.4.3 Expander Decomposition

Even if a graph is not an **expander** at first, it is possible to decompose a graph into smaller subgraphs such that each of them has good **expansion/conductance**. More explicitly, the goal is to remove as few edges as possible such that the graph decomposes into **expanders**. This is useful since **expander** leads to good algorithms. The question is the trade-off between the number of edges that we remove and the **expansion** that we can guarantee for the pieces.

Notation. Technically the process works with **conductance**, but we use the terminology of **expander decomposition** for historical reasons.

Since one is often interested in finding fast algorithms for **expander decomposition**, it is common to explore trade-offs between the quality of the **conductance** of the pieces and the number of edges.

Definition 2.4.4 (Expander decomposition). A (ϕ, ϵ) -*expander decomposition* for some $\epsilon \in (0, 1)$ is a partition of the (connected) graph $G = (V, E)$ into vertex induced subgraphs $\{G_i = G[V_i]\}_{i=1}^h$ such that each G_i has **conductance** at least ϕ , and the number of inter-cluster edges is at most ϵm , i.e.,

$$\frac{1}{2} \sum_{i=1}^h |\delta(V_i)| \leq \epsilon m.$$

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Theorem 2.4.2. Let $G = (V, E)$ be a graph and $\epsilon \in (0, 1)$. Suppose there is an α -approximation for the **uniform sparsest cut**, then there is an efficient algorithm that outputs an $(\Omega(\frac{\epsilon}{\alpha \log m}), \epsilon)$ -**expander decomposition** of G .

Proof. Consider the following algorithm.

Algorithm 2.6: **Expander decomposition**

Data: A connected graph $G = (V, E)$, base graph edge set size M , parameter $\epsilon \in (0, 1)$

Result: An $(\Omega(\frac{\epsilon}{\alpha \log m}), \epsilon)$ -**expander decomposition** $\{G_i\}_{i=1}^h$

```

1 if  $m \leq 10 \log M / \epsilon$  then                                     // Base case
2   return  $G$ 
3
4  $(S, V \setminus S) \leftarrow \alpha$ -Uniform-Sparsest-Cut( $G$ )a         //  $\text{vol}(S) \leq \text{vol}(V \setminus S)$ 
5
6 if  $|\delta(S)| / \text{vol}(S) > c / 10 \log M$  then                       // Check sparsity
7   return  $G$ 
8 else
9    $\{G_i^{(1)}\}_{i=1}^{h_1} \leftarrow \text{Expander-Decomposition}(G[S], M, \epsilon)$ 
10   $\{G_i^{(2)}\}_{i=1}^{h_2} \leftarrow \text{Expander-Decomposition}(G[V \setminus S], M, \epsilon)$ 
11  return  $\{G_i^{(1)}\}_{i=1}^{h_1} \cup \{G_i^{(2)}\}_{i=1}^{h_2}$ 

```

^aRecall that this is for **conductance**, which can be formalized as a **project instance**.

Claim. The **conductance** of each subgraph output by **Algorithm 2.6** is at least $\epsilon / 10 \alpha \log M$.

Proof. For the base case, if G is connected and has at most $10 \log M / \epsilon$ edges, then the **conductance** of G is at least $c/10 \log M$ since at least one edge crosses any cut, and the volume of the smaller side is at most $10 \log M$.

On the other hand, if the α -approximation algorithm of the **uniform sparsest cut** for **conductance** outputs a cut $(S, V \setminus S)$ with **sparsity** at least $c/10 \log M$, we know that the actual **sparsity** of G is at least $c/10\alpha \log M$ as desired. \otimes

Next, we analyze the total number of edges cut, which needs to be at most ϵm .

Intuition. If G is of constant size (and connected) or it does not have a **sparse cut**, **Algorithm 2.6** does not cut any edges.

Let $T(m)$ be the total number of edges cut by **Algorithm 2.6** on a graph with m edges. **Algorithm 2.6** removes edges between S and $V \setminus S$ only when $|\delta(S)| \leq \epsilon \text{vol}(S)/10 \log M$ where $\text{vol}(S) \leq \text{vol}(V \setminus S)$. Let $m' := |\delta(S)|$, $m_1 := |E(G[S])|$, and $m_2 := |E(G[V \setminus S])|$, then

$$m' \leq \frac{\epsilon}{10 \log m} (2m_1 + m') \Rightarrow (1 - o(1))m' \leq \frac{\epsilon}{5 \log m} m_1 \leq \frac{\epsilon}{4 \log M} m_1.$$

With $m_1 \leq m_2$, the recurrence can be written as

$$T(m) \leq T(m_1) + T(m_2) + \frac{\epsilon}{4 \log M} \min(m_1, m_2) = T(m_1) + T(m_2) + \frac{\epsilon}{4 \log M} m_1$$

where $m_1 + m_2 \leq m$, which gives $T(m) \leq \epsilon m$. \blacksquare

If we don't care about efficiency, we can set $\alpha = 1$ and solve the **uniform sparsest cut** exactly. In particular, **Theorem 2.4.2** guarantees that the decomposed pieces have **conductance** $\Omega(1/\log m)$ while cutting only a constant fraction of the edges.

Note. The bound $\Omega(1/\log m)$ is tight as shown by the hypercube [Ale+17].

We can rephrase **Theorem 2.4.2** in a different form where we want a lower bound on the **conductance** of the pieces and express the number of edges cut as a function that parameter:

Corollary 2.4.1. Let $G = (V, E)$ be a graph and ϕ be a parameter. Suppose there is an α -approximation for the **uniform sparsest cut**, then there is an efficient algorithm that computes a $(\phi, O(\alpha \cdot \phi \cdot \log m))$ -expander decomposition.

Note. Number of edges cut is less than m only if $\alpha \phi \log m < 1$, so one should think of $\phi \leq 1/\alpha \log m$.

Remark. **Theorem 2.4.2** is phrased in terms of m , the number of edges. Capacitated graphs can be handled by scaling since we do not assume that G is simple. However, the dependence on $\log m$ means that when capacities are large, we are not guaranteed a strongly polynomial bound. One can handle this issue in various ways depending on the application. In most applications of **expander decomposition**, it is the case that the total capacity of the edges can be assumed to be polynomially bounded in n and in this case, the $\log m$ factor is typically replaced with $\log n$.

We remark that **Algorithm 2.6** is based on **sparsest cut** algorithms. Traditionally, these algorithms were quite slow. There have been several developments in the last few years which enabled **sparsest cut** to be reduced to a poly-logarithmic number of s - t flows via the so-called *cut-matching game* [KRV09; Ore+08], which in turn enabled faster flow algorithms. There are now near-linear time randomized algorithms for **expander decomposition** (with slightly weaker parameters than the ideal one) for the regimes of interest [SW19]. In some applications the randomized algorithm is not adequate and there has been considerable effort to obtain deterministic algorithms. There are now almost-linear time deterministic algorithm [Chu+20; SL21].

2.4.4 Well-Linked Set

Consider the following generalization of **Definition 2.4.1**, where we only care about **expansion** of a subset.

Definition 2.4.5 (Well-linked). A set $X \subseteq V$ is *well-linked* in a graph $G = (V, E)$ if for all $S \subseteq V$,

$$|\delta(S)| \geq \min(|S \cap X|, |S \cap (V \setminus X)|).$$

On the other hand, recall that *α -expansion* means that for all sets $S \subseteq V$ with $|S| \leq |V|/2$, $|\delta(S)| \geq \alpha|S|$. This is a cut condition. Suppose A, B are two disjoint sets of vertices of equal size $|A| = |B|$, clearly we have $|A|, |B| \leq |V|/2$. We can ask for a similar guarantee as the same cut condition. This turns out to be another generalization of *expander*:

Definition 2.4.6 (Linkage). Let $A, B \subseteq V$, $A \cap B = \emptyset$, and $|A| = |B|$. An *A - B linkage* is a set of edge-disjoint paths connecting A to B with each vertex in $A \cup B$ in exactly one path. In this case, we say A and B are *linked* in G .

Note. We do not have to insist on $A \cap B = \emptyset$. If not and we allow each vertex in $A \cap B$ to connect to itself via an empty path, then it is the same as asking $A \setminus B$ and $B \setminus A$ to be *linked*. Thus, requiring $|A| = |B|$ suffice.

We can view *linkage* as sending flows:

Definition 2.4.7 (Fractional linkage). An *A - B linkage* is *fractional* if there is a flow in G with that satisfies demand of 1 on each vertex in A and a demand of -1 on each vertex of B .^a In particular, we say that A, B are *α -linked* for some parameter α if there is a flow in G that satisfies the demand of α on each vertex in A and a demand of $-\alpha$ on each vertex of B .

^aNote that this corresponds to a single-commodity flow.

Lemma 2.4.2. Suppose G is an *α -expander* with $\alpha \geq 1$. Then there is an *A - B linkage* in G for every pair of disjoint equal sized sets A, B .

Proof. ■

One can scale capacities or directly prove the following.

Corollary 2.4.2. Suppose G is an *α -expander*. Then if A, B are disjoint vertex sets with $|A| = |B|$, then A, B are *α -linked*.

Interestingly, the converse is also true.

Lemma 2.4.3. Suppose G is a graph and for any two disjoint set A, B of equal size, A, B are *α -linked*. Then G is an *α -expander*.

The following shows the connection of *linkage* and *well-linked*.

Claim. A set X is *well-linked* if for all $A, B \subseteq X$ and $|A| = |B|$, A and B are *linked*.

Definition 2.4.8 (Fractional well-linked). A set X is *α -well-linked* in a graph G if for any two $A, B \subseteq X$ with $|A| = |B|$, the sets A, B are *α -linked*.

More generally, we have the following.

Lemma 2.4.4. A set $X \subseteq V$ is *α -well-linked* in G if and only if for any set $S \subseteq V$, $|\delta(S)| \geq \alpha \min(|S \cap X|, |S \cap (V \setminus X)|)$.

Corollary 2.4.3. A graph $G = (V, E)$ is an *α -expander* if and only if V is *α -well-linked* in G .

Thus, the notion of *well-linked* sets extends the definition of *expansion* to subsets of the graph. This is very useful in a number of settings.

Example (Star). A star on n vertices is an **expander** and has a **well-linked** set of size n . This strange artifact is because of the large degree of the center vertex.

This artifact disappears if we ask for constant degree graphs or if we insist on **node linkage**, i.e., we now want *node-disjoint paths*.

Definition 2.4.9 (Node linkage). Let $A, B \subseteq V$, $A \cap B = \emptyset$, and $|A| = |B|$. An A - B *node linkage* is a set of node-disjoint paths connecting A to B with each vertex in $A \cup B$ in exactly one path.^a

^aWe also skip the definition of α -linkage for now. The definition is basically the same where we want flow with node capacities rather than edge capacities.

Definition 2.4.10 (Node well-linked). A set X is α -*node-well-linked* in G if for any two $A, B \subseteq X$ with $|A| = |B|$, A, B are **α -linked**.

Intuition. If X is **node-well-linked** in a graph, then X cannot have a sparse node separator, i.e., if S separates $G \setminus S$ into components and S does not have any vertices of X then no component of $G \setminus S$ can have more than $|S|$ vertices of X .

In graph theory literature on **treewidth**, the notion of **linkages** is defined primarily via node-disjoint paths. We will not use treewidth very often in this course hence we overload **edge** and **node well-linked** notations. In particular, its connection to **node-well-linkedness** is the following.

Theorem 2.4.3. Let k be the cardinality of the largest **node-well-linked** set in a graph G . Then $k \leq \text{tw}(G) \leq 4k$.

Remark. In fact, most algorithmic approaches to computing treewidth are based on algorithms for sparse node separator computations.

Node-well-linkedness is connected to vertex-**expanders**. Sometimes people do not distinguish between these two notions too much in the **expansion** literature because of the following.

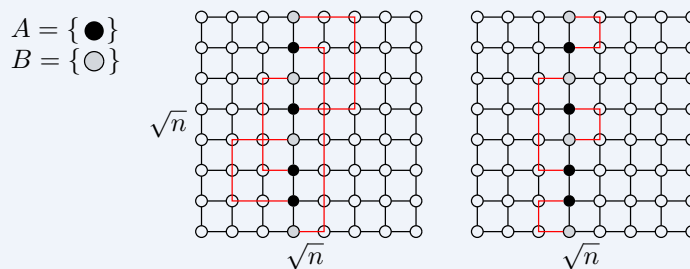
Claim. If G is an α -**edge-expander** with maximum degree d , then G is also an $\Omega(\alpha/d)$ -**vertex-expander**.

Thus, if one is working with constant degree graphs, the two notions are not very far.

Example (Star). Consider the star graph again. One can see that it is an **edge-expander**, but it is very far from being a vertex **expander**. In fact, the largest **node-well-linked** set in a star is of size 2.

On the other hand, the grid is not only **edge-well-linked**, but also **node-well-linked**.

Example (Grid). A $\sqrt{n} \times \sqrt{n}$ grid is a planar graph with n vertices. It has a bisection with $O(\sqrt{n})$ edges hence it is at best a $1/\sqrt{n}$ -**expander** (which in fact it is). It has a **well-linked** set of size $\Omega(\sqrt{n})$, i.e., rows or columns X of \sqrt{n} vertices. We see that although G is not a good **expander**, but it has good **expansion** w.r.t. X . Actually, it's even **node well-linked** (right).



In some sense, grid is the best planar graph in terms of **node-well-linkedness**.

Theorem 2.4.4. Every planar graph has a balanced separator of size $O(\sqrt{n})$. Hence, no planar graph on n vertices have a **node-well-linked** set of size more than $c\sqrt{n}$ for some fixed constant c .

Theorem 2.4.5. Given a graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$, there is a rooted tree $T = (V_T, E_T)$ such that

- (a) leaves of T are V ,
- (b) every edge $e \in E_T$ is associated with capacity of the cut induced by T_u , and
- (c) any demand matrix D that is reachable in T is reachable in G with congestion $O(\log^3 n)$.

Problem 2.4.2 (Oblivious routing). Given a directed graph $G = (V, E)$ with edge capacity $c: E \rightarrow \mathbb{R}_+$. The goal of *oblivious routing* is to output a distribution of paths between every pair of $u, v \in V$ such that when the demand $D: V \times V \rightarrow \mathbb{R}_{\geq 0}$ come, the congestion is low.

Appendix

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