VV186 RC Part I

Logic

"Without logic, mathematics will falls apart... "

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Course Description

- What do Vv186-Vv285-Vv286 course series teach in general?
- What are the difference between Vv186 and high school math?
- How to learn well in Vv186?

How to study Vv186 well?

- Preview and review course materials. Please always DO refer to Horst's slides.
- Concept is important. There will be concept checking paper available.
- Do assignments by your own.
- Don't do Too Much math exercise!
- Leave at least half a week to prepare for Vv186 exams.

Pay attention to following...

For Vv186 series...

- **Forget EVERYTHING you have learned in school before.**

For this RC class section...

- Do think more about the question in "()".
  e.g. "(How to prove?)"
- You are welcome to ask questions in an adequate manner.
- The class is designed to be interactive. Don't be so shy!
- Focus more on the idea of a proof. Don't just "recite" everything.

Overview

Notation

Just one thing you need to pay attention to.

Always refer to Horst's convention.

- True statement
- False statement
- *Vacuous Truth*
- Statement variable
- Statement structure
  e.g. quantifier, predicate, specific value

Logical Operation

Logical Operation	
$\neg$	Negation
$\wedge$	Conjunction
$\vee$	Disjunction

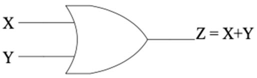
Truth Table

How to use Truth Table?

- Understand what the problem is about.
- Set up a Truth Table.
- Always cover all possible cases.

Just to show you how powerful the concept of Truth Table is, here is an example.

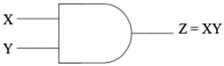
Have you ever consider how a computer add two number?
A: It is implemented by logic gates, and then circuits.
Introduce some logic gates.



2-input OR gate

X	Y	Z = X + Y
0	0	0
0	1	1
1	0	1
1	1	1

Truth Table



2-input AND gate

X	Y	Z = XY
0	0	0
0	1	0
1	0	0
1	1	1

Truth Table

Adder

Then, we can consider following:

Given two numbers as input, how to add them together?

As we all know, circuits work with binary number, e.g. 01011...
So, the question becomes

How to add two one digit binary numbers?

The idea is quite simple, that is let the circuit *mimic* the action when we perform addition. In other words, we want this circuit to cover all possible cases when we do the basic addition, namely

$$00 + 00 = 00 \quad 00 + 01 = 01 \quad 01 + 00 = 01 \quad 01 + 01 = 10$$

Adder

We consider how each result's digit will react to the changing of input by constructing a Truth Table as follows:

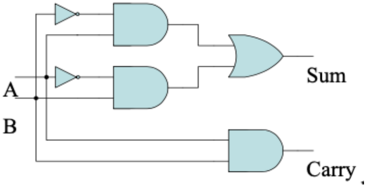
A	B	Sum	Carry
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Then if you done your assignment 1.4, you should know you can construct a *logic equation* from this Truth Table for **Sum** and **Carry**.

Logic gates for Adder

$$Sum = A'B + AB' \quad Carry = AB$$

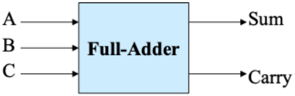
Then we can get following circuit



Which is so-called *Half Adder*.

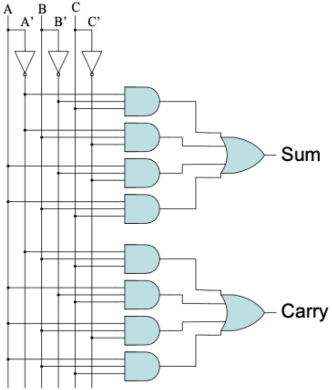
More about Adder . . .

You can design a more complicated circuit by using same method, like:



A	B	C	Sum	Carry
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

$$\begin{aligned} Sum &= A'B'C + A'BC' + AB'C' + ABC \\ &= \Sigma\, m(1, 2, 4, 7) \\ Carry &= A'BC + AB'C + ABC' + ABC \\ &= \Sigma\, m(3, 5, 6, 7) \end{aligned}$$



Which is so-called *Full Adder*. You will learn these interesting concept in VE270.

Relations between Statements

- Implication

$$A \Rightarrow B$$

- Equivalence

$$A \Leftrightarrow B$$

- Contraposition

$$(A \Rightarrow B) \Leftrightarrow (\neg A \Leftarrow \neg B)$$

Proof of the contraposition (**de Morgan rules**):

A	B	$\neg A$	$\neg B$	$\neg B \Rightarrow \neg A$	$A \Rightarrow B$	$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

And this is the concept of *tautology*

Exercise

Let P, Q be two sets such that $P \subseteq Q$. Then what is the relation between these two statements?

$$\text{Statement A: } x \in P. \text{ Statement B: } x \in Q.$$

Logical Quantifiers

Logical Quantifiers

Sign	Type	Interpretation
$\forall$	universal	for any; for all
$\exists$	existential	there exist; there is some
$\forall \ldots \forall \ldots$	nesting quantifier	for all ... for all ...
$\exists \ldots \exists \ldots$	nesting quantifier	there exists ... (such that) there exist ...
$\vdots$	$\vdots$	$\vdots$

Sets

1. What is a set?
2. Common set types
   - Empty set: $\emptyset := \{x : x \neq x\}$
   - Total set
   - Subset
   - Proper subset
   - Power set

Simple question:

Why is $\emptyset$ a subset for any set X?

Example

let $A := \{4, 5, 6\}$ be a set.

- The total set can be $\mathbb{N}$
- $B := \{4, 5, 5, 6, 6\} = A$
- $C = \{1, 5\} \subseteq A$
- $P := \{\emptyset, \{4\}, \{5\}, \{6\}, \{4, 5\}, \{5, 6\}, \{4, 6\}, \{4, 5, 6\}\}$ is the power set of A. (What is the cardinality of A ?)

Set Operations

Define

$$A := \{1, 2\} \quad B := \{2, 3\} \quad M := \{1, 2, 3, 4, 5\}$$

Set Operations		
$A \cup B$	Union	$\{1, 2, 3\}$
$A \cap B$	Intersection	$\{2\}$
$A \setminus B$	Difference	$\{1\}$
A^c	Complement	$\{3, 4, 5\}$

Simple question:

What is M^c ? Also, what is $\emptyset^c$?

Ordered Pairs

- What is an ordered pair?
- What is the difference between ordered pair and set?
- Concept of *Cartesian product*.

Russel Antinomy

There exist several paradox in native set theorey, including:

1. Russel Antinomy
2. Cantor's paradox
3. Burali-Forti paradox

The above paradoxes illustrate the fundemental flaw of our naive theorey, namely it's not **well-defined**.

However, these problem can be solved if we replaced naive set theorey by a *modern axiomatic set theory*, but the detail about it is beyond our scope.

1. Let A, B, C be three statements. Use truth table to prove that

$$(A \vee B) \wedge C \equiv (A \wedge C) \vee (B \wedge C)$$

Exercises

2. Let A, B, C be three sets. Prove that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

From Exercise 1 & 2, we can see that sets and statements are similar.

3. Check whether the following sentences are true statement, false statement, or not a statement.

- $\forall x, y \in \mathbb{R}, x^2 + y^3 \geq 0$
- Let $f(a) = a^4$, then $f(0) > 0$
- For any $a \in \mathbb{R}, a^4 > 0$
- An African Elephant is very big.
- Let A, B be two statements, then $(A \vee B) \Leftrightarrow \neg(\neg A \wedge \neg B)$

Simple question

Rewrite above sentences in quantifiers form if they are statement.

4. Use quantifiers to rewrite the following definition of covergence:

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. If for some fixed $c \in \mathbb{R}$, for any $a > 0$, there is an $N \in \mathbb{N}$, such that for all $n > N$, $|a_n - c| < a$, then we say (a_n) converges.

Reference

Reference.

- Exercises from 2019–Vv186 TA-Zhang Leyang.
- Figure for circuits from Ve270 T2-Logic Gate Slides.

Have Fun and Learn Well!