上 海 交 通 大 学 试 卷

2020 – 2021 Academic Year (Fall Term)

Vv186 Honors Mathematics II
First Midterm Exam – Solutions

Exercise 1

In the following exercises, mark the boxes corresponding to true statements with a cross (☒). In each case, it is possible that none of the statements are true or that more than one statement is true.

i) Let $A \subset \mathbb{R}$ be a non-empty set.

- ☒ If $x \in \mathbb{R}$ is an interior point of A, then it is also an accumulation point of A.
- ☐ If $x \in \mathbb{R}$ is an accumulation point of A, then it is also an interior point of A.
- ☐ If $x \in \mathbb{R}$ is a boundary point of A, then x can not be an accumulation point of A.
- ☐ If $x \in \mathbb{R}$ is a boundary point of A, then x must be an accumulation point of A.

ii) Let (a_n) be a sequence of real numbers.

- ☒ If (a_n) is convergent, then (a_n) is Cauchy.
- ☐ If (a_n) is Cauchy, then (a_n) is bounded.
- ☐ If (a_n) has an accumulation point, then (a_n) is bounded.
- ☒ If (a_n) is bounded, then (a_n) has an accumulation point.

iii) Let $f\colon [0,\infty) \to \mathbb{R}$ be given by $f(x) = \dfrac{\sqrt{x+3}-\sqrt{3x+1}}{x-1}$. Then

- ☐ $\lim\limits_{x\to 1} f(x) = 0$.
- ☐ $\lim\limits_{x\to 1} f(x) = 1$.
- ☐ $\lim\limits_{x\to 1} f(x) = 1/2$.
- ☒ $\lim\limits_{x\to 1} f(x) = -1/2$.

iv) Suppose that $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfy $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$ and $g(x) \not\to 0$ as $x \to \infty$. Then

- ☐ $\lim\limits_{x\to\infty} \big(f(x) - g(x)\big) = 0$.
- ☒ $\lim\limits_{x\to\infty} \frac{f(x)-g(x)}{g(x)} = 0$.
- ☒ $f(x) = g(x) + o(g(x))$ as $x \to \infty$.
- ☐ $f(x) = g(x) + o(x)$ as $x \to \infty$.

v) Let $I \subset \mathbb{R}$ be an interval and $f\colon I \to \mathbb{R}$ a real function.

- ☐ If I is open and f is continuous, then f is bounded.
- ☒ If I is closed and f is continuous, then f is bounded.
- ☐ If I is open and f is bounded, then f is continuous.
- ☐ If I is closed and f is bounded, then f is continuous.

(4 Marks)

Exercise 2
Describe in words the difference between **continuity of a function on an interval** and **uniform continuity of a function on an interval**. How are these two concepts different? If you can, state some properties that they have in common or that serve to differentiate them from each other. Is one of them also always an example of the other? Give examples. **(5 Marks)**

Solution. Give **(1 Mark)** for any of the following:

- A function is continuous on an interval if it is continuous at every point of that interval.
- A function that is uniformly continuous on an interval has the property that its values at two points can be forced to be arbitrarily close together by requiring the two points to be sufficiently close together. The required proximity of the two points does not depend on the particular points, only on the desired distance of their values.
- For a function that is simply continuous on an interval, the actual points may also affect the required proximity.
- A function that is uniformly continuous on an interval is always also continuous on that same interval.
- A function that is continuous on a closed interval is also uniformly continuous on that interval.

Give up to **(2 Marks)** for examples that highlight similarities and differences, with **(1 Mark)** for each example, e.g.,

- The function given by $f(x) = 1/x$ is continuous on $(0, 1)$ but not uniformly continuous.
- The function given by $f(x) = 1/x$ is uniformly continuous on $(1, 2)$, showing that a continuous function may, but does not have to, be continuous on an open interval.
- The function given by $f(x) = 1/(1 + x^2)$ is uniformly continuous on $\mathbb{R}$, showing that functions may be uniformly continuous on unbounded intervals.
- The function given by $f(x) = x$ is uniformly continuous on $\mathbb{R}$.
- The function given by $f(x) = x^2$ is uniformly continuous on any bounded interval $I \subset \mathbb{R}$, but is not uniformly continuous on $\mathbb{R}$.

Exercise 3
Prove the following statement using induction in n:

$$\prod_{j=1}^{n-1} \left(1 + \frac{1}{j}\right)^j = \frac{n^n}{n!}, \qquad\qquad n \in \mathbb{N} \setminus \{0, 1\}.$$

(3 Marks)

Solution. Award **1 Mark** for checking that the statement is true for $n = 2$ and **1 Mark** for saying that "Assuming the statement is true for n, we now show that it is true for $n + 1$" or some equivalent remark. Award **1 Mark** for then successfully proving this.

- $A(n = 2)$:
$$\prod_{j=1}^{2-1}\left(1+\frac{1}{j}\right)^j = (1+\frac{1}{1})^1 = 2 = \frac{2^2}{2!}$$

- $A(n) \Rightarrow A(n+1)$:
$$\prod_{j=1}^{n}\left(1+\frac{1}{j}\right)^j = \left(1+\frac{1}{n}\right)^n)\prod_{j=1}^{n-1}\left(1+\frac{1}{j}\right)^j = \frac{(n+1)^n}{n^n}\frac{n^n}{n!} = \frac{(n+1)^{n+1}}{(n+1)!}$$

Exercise 4

Consider the sequence (a_n) defined recursively by

$$a_1 = 1, \qquad a_{n+1} = \frac{1}{4}a_n^2 + 1.$$

i) Discuss the monotonicity of the sequence.

ii) Prove that the sequence converges, i.e., a limit a exists.

iii) Find the limit a.

(6 Marks)

Solution.

i) From the definition it is clear that $a_n \geq 1$ for all $n \in \mathbb{N}$. **(1/2 Mark)** Then

$$\begin{aligned} a_{n+1} - a_n &= \frac{1}{4}a_n^2 + 1 - a_n \\ &= 1 + \frac{a_n(a_n-4)}{4} \\ &> 1 + \frac{a_n - 4}{4} > 1 + \frac{-3}{4} > 0 \end{aligned}$$

so $a_{n+1} > a_n$ **(1 Mark)** and (a_n) is (strictly) increasing. **(1/2 Mark)**

ii) The sequence is bounded: we claim that $a_n < 2$ for all n. If $a_n < 2$, then

$$a_{n+1} = \frac{1}{4}a_n^2 + 1 < \frac{2^2}{4} + 1 < 2,$$

and with $a_1 = 1 < 2$, this proves $a_n < 2$ for all n by induction. **(1 Mark)** Since the sequence is bounded and monotonic, a limit exists. **(1 Mark)**

iii) The limit is given by

$$a := \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{1}{4}a_n^2 + 1\right) = \frac{1}{4}a^2 + 1$$

(1 Mark) which yields the unique solution

$$a = 2.$$

(1 Mark)

Exercise 5
Let $a, b \in \mathbb{R}$ with $a > 0$ be given. Find $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\frac{1+ax}{1+bx^2} = \alpha + \beta x + \gamma x^2 + o(x^2) \qquad \text{as } x \to 0.$$

(4 Marks)

Solution. We have

$$\frac{1+ax}{1+bx^2} = (1+ax)\frac{1}{1+bx^2}$$

Since $(1+y)(1-y) = 1 - y^2$, we have

$$\frac{1}{1+y} = 1 - y + \frac{y^2}{1+y} = 1 - y + o(y) \qquad \text{as } y \to 0$$

(1 Mark) Substituting $y = bx^2$, we obtain

$$(1+ax)\frac{1}{1+bx^2} = (1+ax)\frac{1}{1+bx^2} = (1+ax)\big(1 - bx^2 + o(x^2)\big) = 1 + ax - bx^2 + o(x^2)$$

so $\alpha = 1$, **(1 Mark)** $\beta = a$, **(1 Mark)** $\gamma = -b$. **(1 Mark)**

Exercise 6
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

- f is strictly increasing;
- $\lim_{x\to-\infty} f(x) = 0$;
- $\lim_{x\to\infty} f(x) = \infty$.

Show that $\operatorname{ran} f = (0, \infty)$.
(4 Marks)

Solution. We first show that $\operatorname{ran} f \subset (0, \infty)$ by showing that if $y \leq 0$, then $y \notin \operatorname{ran} f$.
Suppose that $f(x_0) = y_0 < 0$ for some $x_0 \in \mathbb{R}$. Since $\lim\limits_{x \to -\infty} f(x) = 0$ we can find some $x_1 < x_0$ such that $|f(x_1) - 0| < |y_0|$, so $f(x_1) > y_0$. But this contradicts the fact that f is increasing. $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Since f is strictly increasing, $f(x - 1) < 0$. But this again leads to a contradiction. Hence, $\operatorname{ran} f \subset (0, \infty)$. **(2 Marks)**
We now show that $(0, \infty) \subset \operatorname{ran} f$. Suppose that $y_0 \in (0, \infty)$ is given. Since $\lim\limits_{x \to -\infty} f(x) = 0$ there exists an x_1 such that $|f(x_1) - 0| < |y_0|$. Since $\lim\limits_{x \to \infty} f(x) = \infty$ there exists an $x_2 > x_1$ such that $|f(x_2)| > |y_0|$. By the Bolzano intermediate value theorem applied to the closed interval $[x_1, x_2]$, there exists an x_0 such that $f(x_0) = y_0$, so $y_0 \in \operatorname{ran} f$. **(2 Marks)**
This proves that $\operatorname{ran} f = (0, \infty)$

Exercise 7
Let $A \subset \mathbb{R}$ be a closed set and $x \in \mathbb{R}$. Show that $x \in A$ if and only if there exists a sequence (a_n) with $a_n \in A$ for all n such that $\lim\limits_{n \to \infty} a_n = x$.
(4 Marks)

Solution.

($\Rightarrow$) Suppose that $x \in A$. Then a sequence is given by $a_n = x$ for all $n \in \mathbb{N}$. **(1 Mark)**

($\Leftarrow$) Suppose that (a_n) is given with $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \to x$ as $n \to \infty$. We will show that $x \in A$. **(1 Mark)** Suppose that $x \notin A$. Since A is closed, the complement A^c is open. Then there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \in A^c$. Fix this $\varepsilon > 0$. **(1 Mark)** However, since $a_n \to x$, we can find an $n \in \mathbb{N}$ such that $|a_n - x| < \varepsilon$, i.e., $a_n \in B_\varepsilon(x) \subset A^c$. But this contradicts $a_n \in A$ for all $n \in \mathbb{N}$. Hence, $x \notin A$ gives a contradiction and we conclude $x \in A$. **(1 Mark)**