VV186 RC Part III

Convergence

"Find the problem, understand it, then explore more."

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Overview

Let first look at the Exercise 7 from last time.

Let (x_n) be a bounded real sequence. Then define

$$a_n := \sup_{m \geq n}(x_m), \quad b_n := \inf_{m \geq n}(x_m)$$

1. Prove that (a_n) is decreasing, while (b_n) is increasing.
2. Since both $(a_n), (b_n)$ are monotonic and bounded, they are convergent. We denote $\underline{\lim} x_n = \lim b_n$; $\overline{\lim} x_n = \lim a_n$. Show that:

$$\underline{\lim} y_n + \underline{\lim} z_n \leq \underline{\lim}(y_n + z_n) \leq \overline{\lim} y_n + \underline{\lim} z_n \leq \overline{\lim} y_n + \overline{\lim} z_n$$

Any Questions?

Metric Space

- What is the definition of a metric?
- Why we want to introduce the idea of Metric Space?
- What new results can we explore from this new idea?

Metric Space

We want to generalize the idea of *convergence*, we want to define the most essential thing of convergence by ourselves, namely the **Length Function**.

What properties a usual length function should have?

1. Always positive.
2. Symmetric.
3. Followed *Triangle Inequality*.

Transform these into mathematical language. . .

Metric Space

A two variables functions

$$\rho(\cdot,\cdot) : M \times M \to \mathbb{R}$$

is called a metric if it satisfies:

1. $\forall x, y \in M,\ \rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$.
2. $\forall x, y \in M,\ \rho(x, y) = \rho(y, x)$.
3. $\forall x, y, z \in M,\ \rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

Examples

- $M = \mathbb{R}^n$, the usual metric is given by

$$\rho((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \sqrt{\sum_{i=k}^{n} (x_i - y_i)^2}$$

  and this is so-called *Euclidean distance*.
- $M = \mathbb{N}, \rho(x, y) = \#\{a : a \in [min\{x, y\}, max\{x, y\}]\ \}$
- $M = \mathbb{R}, \rho(x, y) = 1$ if $x \neq y$; $\rho(x, y) = 0$ if $x = y$

A simple question arise, is any metric **well-defined**?

Generalization of Convergence

Then, by replacing the usual matric $\rho(x, y) = |x - y|$ and choosing our universal set M, we get the natural definition for generalize convergence in metric space (M, ρ) for a sequence $(a_n) : \mathbb{N} \to M$, which is given by:

$$\lim_{n \to \infty} a_n = a \quad :\Leftrightarrow \quad \forall_{\epsilon > 0} \; \exists_{N \in \mathbb{N}} \; \forall_{n > \mathbb{N}} \, a_n \in B_\epsilon(a)$$

where

$$B_\epsilon(a) = \{x \in M : \rho(x, a) < \epsilon\}, \quad \epsilon > 0, \quad a \in M.$$

Cauchy Sequence

- What is the definition of Cauchy Sequence?
- How to understand Cauchy Sequence?
- Why we want to introduce the idea of Cauchy Sequence?
- What new results can we explore from this new idea?

Cauchy Sequence

The fundamental reason why we want to introduce Cauchy Sequence is because we want to *further generalize* the idea of convergence.

Now, we are not only free to choose the metric we like, additionally, we remove the constraint which requires a sequence to converge to a *specific point*.

Cauchy Sequence

We list some important results and theorems for Cauchy Sequence.

- Every convergent sequence is a Cauchy sequence.
- Every Cauchy sequence in a metric space (M, ρ) is bounded.
- Every Cauchy sequence in $\mathbb{R}$ with the usual metric is convergent.

Digest...

We now take a deep breath, and look back what we have introduced. Where does all these new definitions lead us to?

Completeness

The problem is, a Cauchy Sequence can simply not "*converge*" to anywhere anymore if we generalize the idea of convergence.

Then, after generalizing the idea of convergence, if **every** Cauchy sequence still converge, we say this metric space is *complete*.

We now take a look at some examples for a Cauchy sequence failed to converge.

Completeness

For a metric space $(\mathbb{Q}, \rho)$, where $\rho(x, y)$ is analogous to our usual metric. Then, a sequence

$$(a_n)_{n\in\mathbb{N}} := 1, 1.4, 1.41, 1.414, \ldots$$

is a Cauchy sequence but failed to converge in $\mathbb{Q}$ if we choose the following terms appropriately and let it *converge to $\sqrt{2}$ in $\mathbb{R}$* eventually.

From here, it seems we somehow find a good way to *construct* Real Numbers.

Construct Real Numbers

The main idea to **complete** $\mathbb{Q}$ is to consider the following

Every Cauchy sequence can *represent* a number in $\mathbb{R}$.

For example, from the last slide, we can view the sequence

$$(a_n)_{n\in\mathbb{N}} = 1, 1.4, 1.41, 1.414, \ldots$$

as $\sqrt{2}$ by defining a *(equivalence) class* for each number in $\mathbb{R}$, and in this case, we define the sequence (a_n) as $\sqrt{2}$.

Then, from now on, the number defined through the Cauchy sequence which is given by

$$0.9, 0.99, 0.999, 0.9999, 0.99999$$

is equal to 1, namely $0.999999_{\ldots} = 1$.

Real Functions

We already know how to describe a function, so we list some *elementary functions* and some categories of functions which you will encounter a lot in the future.

- Polynomial Function : $f(x) = \sum_{k=0}^{n} c_k x^k,\ n \in \mathbb{N}.\ \underset{1\leq i\leq n}{\forall} c_i \in \mathbb{R}$
- Power Function : $f(x) = x^n,\ n \in \mathbb{N}$
- Rational Function : $f(x) = \frac{p(x)}{q(x)},\ p, q \in \mathbb{P}(\mathbb{R})$
- Periodic Function : $\underset{x\in\mathbb{R}}{\forall} f(x+T) = f(x)$, where T is *period*.
- Piecewise Function

(What is the domain of each kind of function?)

Real Functions

Then, we can do some basic manipulation to a function, including:

1. Translation in different directions
2. Multiplying a function's argument
3. Multiplying a function's values

Exercise

1. Suppose the function which is given by

$$f : [2, +\infty) \to \mathbb{R}$$

and satisfies

$$x^2 f(x) - f(1 - x) = x$$

Please calculate the value of

$$\lim_{x \to \infty} \frac{f(x)}{3}$$

(Can we further expand the domain of f?)

Exercise

2. Please construct a metric that makes $\mathbb{R}$ incomplete with regard to this metric. You can use the method given in Horst's slides.

More About Cauchy Sequence

Let us take a look at some tricky examples of Cauchy sequence.

In class demonstration!

Have Fun and Learn Well!