上 海 交 通 大 学 试 卷

2020 – 2021 Academic Year (Fall Term)

Vv186 Honors Mathematics II
Final Exam – Solutions

Exercise 1

In the following exercises, mark the boxes corresponding to true statements with a cross (☒). In each case, it is possible that none of the statements are true or that more than one statement is true.

i) Suppose that (f_n) is a sequence of bounded, absolutely integrable functions $f_n\colon \mathbb{R} \to \mathbb{R}$. Then, as $n \to \infty$,

- ☐ $\sup\limits_{x\in\mathbb{R}}|f_n(x)| \to 0$ implies $\int_{-\infty}^{\infty}|f_n(x)|\,dx \to 0$;
- ☐ $\int_{-\infty}^{\infty}|f_n(x)|\,dx \to 0$ implies $\sup\limits_{x\in\mathbb{R}}|f_n(x)| \to 0$;
- ☐ $f_n(x) \to 0$ for all $x \in \mathbb{R}$ implies $\int_{-\infty}^{\infty}|f_n(x)|\,dx \to 0$;
- ☐ $\int_{-\infty}^{\infty}|f_n(x)|\,dx \to 0$ implies $f_n(x) \to 0$ for all $x \in \mathbb{R}$.

ii) If $f\colon [a, b] \to \mathbb{R}$ is regulated, then f is

- ☒ bounded;
- ☐ continuous;
- ☐ differentiable;
- ☐ Darboux-integrable.

iii) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is such that $\int_{-\infty}^{\infty} f(x)\,dx < \infty$. Then

- ☐ f is bounded;
- ☐ $\lim\limits_{x\to\infty} f(x) = 0$;
- ☒ $\int_a^b f(x)\,dx$ exists for any $a, b \in \mathbb{R}$;
- ☒ $\lim\limits_{N\to\infty} \int_{-N}^{N} f(x)\,dx = \lim\limits_{N\to\infty} \int_{-N}^{2N} f(x)\,dx$.

iv) Let $f\colon [\alpha, \beta] \to \mathbb{R}$ and $g\colon [a, b] \to [\alpha, \beta]$ be continuous with g bijective and continuously differentiable on (a, b). Then a correct form of the substitution rule is

- ☒ $\int_\alpha^\beta f(x)\,dx = \int_a^b f(g(x)) \cdot |g'(x)|\,dx$.
- ☐ $\int_{g(a)}^{g(b)} f(g(x)) \cdot g'(x)\,dx = \int_\alpha^\beta f(x)\,dx$.
- ☒ $\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(x)\,dx$.
- ☐ $\int f(x)g'(x)\,dx = \int f(g(x))\,dx$.

(8 Marks); each correctly checked or unchecked box is worth 1/2 Mark

Exercise 2
Explain in two to four English sentences what the regulated integral is. Your explanation should include

- for which functions the regulated integral is defined and
- how the regulated integral is calculated.

You should NOT use any formulas. You should use only words and possibly symbols (e.g., "a function f") if needed.
(4 Marks)

Solution 2
The regulated integral is defined for functions on a closed interval that can be approximated uniformly by step functions. **(1 Mark)** The integral gives the (signed) area between the graph of the function and the abscissa. **(1 Mark)** It is calculated for a given function f by finding a sequence of step functions to approximate f uniformly, determining the aforementioned area for these step functions, and then taking the limit of these areas. **(2 Marks)**

Exercise 3
Show that

$$\int_0^1 x^n \ln(x)^n \, dx = -\frac{n!}{[-(n+1)]^{n+1}} \qquad \text{for } n \in \mathbb{N}.$$

Next, using that $x^x = e^{x \ln(x)}$, show that

$$\int_0^1 x^x \, dx = -\sum_{n=1}^{\infty} (-n)^{-n}.$$

(5 Marks)

Solution 3

i) First, we substitute $y = \ln(x)$, $dy = \frac{1}{x} \, dx$, we have

$$\int_0^1 \frac{x^n \ln(x)^n}{n!} \, dx = \frac{1}{n!} \int_{-\infty}^0 e^{(n+1)y} y^n \, dy$$

(1 Mark) Setting $z = -(n+1)y$, $dz = -(n+1)\, dy$, this becomes

$$\int_0^1 \frac{x^n \ln(x)^n}{n!} \, dx = -\frac{1}{n!} \frac{1}{(-1)^{n+1}(n+1)^{n+1}} \underbrace{\int_0^\infty e^{-z} z^n \, dz}_{=\Gamma(n+1)=n!} = -[-(n+1)]^{-(n+1)}$$

(2 Marks)

ii) Using the series expansion of the exponential,

$$\int_0^1 x^x \, dx = \int_0^1 e^{x \ln(x)} \, dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^n \ln(x)^n}{n!} \, dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^n \ln(x)^n}{n!} \, dx$$

(1 Mark) With the previous result we obtain

$$\int_0^1 x^{-x}\,dx = \sum_{n=0}^{\infty}\int_0^1 \frac{(-x)^n \ln(x)^n}{n!}\,dx = -\sum_{n=0}^{\infty}[-(n+1)]^{-(n+1)} = -\sum_{n=1}^{\infty}(-n)^{-n}.$$

(1 Mark)

Exercise 4
Show that the improper integral

$$\int_0^{\infty} \sin(x^2)\,dx$$

converges.
(4 Marks)

Solution 4

- **Solution 1:** We substitute $y = x^2$, $dy = 2x\,dx$ to write the integral as

$$\int_0^{\infty} \frac{\sin(y)}{\sqrt{y}}\,dy.$$

Fix $R > 0$ and let $R < x < y$ and consider

$$I(x,y) := \int_x^y \frac{\sin t}{\sqrt{t}}\,dt, \tag{1}$$

which we can rewrite as

$$I(x,y) = \int_x^{x+\pi} \frac{\sin t}{\sqrt{t}}\,dt - \int_y^{y+\pi} \frac{\sin t}{\sqrt{t}}\,dt + \int_{x+\pi}^{y+\pi} \frac{\sin \tau}{\sqrt{\tau}}\,d\tau.$$

Substituting $t = \tau - \pi$ in the last integral, we obtain

$$I(x,y) = \int_x^{x+\pi} \frac{\sin t}{\sqrt{t}}\,dt - \int_y^{y+\pi} \frac{\sin t}{\sqrt{t}}\,dt - \int_x^{y} \frac{\sin t}{\sqrt{t+\pi}}\,dt. \tag{2}$$

Adding (1) to (2), we obtain

$$2I(x,y) = \int_x^{x+\pi} \frac{\sin t}{\sqrt{t}}\,dt - \int_y^{y+\pi} \frac{\sin t}{\sqrt{t}}\,dt + \int_x^{y} \frac{(\sqrt{t+\pi}-\sqrt{t})\sin t}{\sqrt{t(t+\pi)}}\,d\tau. \tag{3}$$

Hence,

$$2|I(x,y)| \le \frac{2\pi}{\sqrt{x}} + \int_x^y \frac{dt}{2t^{3/2}} < \frac{2\pi}{\sqrt{x}} + \int_x^{\infty} \frac{dt}{2t^{3/2}} = \frac{2\pi}{\sqrt{x}} + \frac{1}{\sqrt{x}}.$$

By choosing R large enough this quantity can be made as small as desired, independent of x and y. Therefore, by the Cauchy property for improper integrals, the integral converges.

- **Solution 2:** We substitute $y = x^2$, $dy = 2x\,dx$ to write the integral as
$$\int_0^\infty \frac{\sin(y)}{\sqrt{y}}\,dy.$$
Set
$$I(R) = \int_0^R \frac{\sin(y)}{\sqrt{y}}\,dy.$$
Then
$$I(n\pi) = \int_0^x \frac{\sin(y)}{\sqrt{y}}\,dy = \sum_{k=1}^{n}(-1)^k a_k$$
where
$$a_k = \int_{(k-1)\pi}^{k\pi} \frac{|\sin(y)|}{\sqrt{y}}\,dy$$
Since
$$\frac{2}{k\pi} < \int_{(k-1)\pi}^{k\pi} \frac{|\sin(y)|}{\sqrt{y}}\,dy\frac{2}{(k-1)\pi}$$
we see that $a_{k+1} < a_k$ and $a_k \to 0$ as $k \to \infty$. Hence, the sequence $I(n\pi)$ converges as $n \to \infty$ by the :Leibniz theorem. Furthermore, if n_R is the largest integer less than R/π, i.e., $n_R = \lfloor R/\pi \rfloor$, then
$$|I(R) - I(n_R\pi)| = \int_{n_R\pi}^{R} \frac{|\sin(y)|}{\sqrt{y}}\,dy < \int_{n_R\pi}^{(n_R+1)\pi} \frac{|\sin(y)|}{\sqrt{y}}\,dy \le \frac{2}{\sqrt{n_R\pi}}$$
This shows that $\lim\limits_{R\to\infty} I(R)$ exists and so the integral converges.

- **Solution 3:** We consider simply $\int_1^\infty \sin(x^2)\,dx$ and substitute $y = x^2$, $dy = 2x\,dx$ to write the integral as
$$\int_1^\infty \frac{\sin(y)}{\sqrt{y}}\,dy.$$
Then, integrating by parts,
$$\begin{aligned} I(R) &:= \int_1^R \frac{\sin(y)}{\sqrt{y}}\,dy \\ &= -\frac{\cos(y)}{\sqrt{y}}\bigg|_1^R - \int_1^R \frac{\cos(y)}{2y^{3/2}}\,dy \\ &= \cos(1) - \frac{\cos(R)}{\sqrt{R}} - \int_1^R \frac{\cos(y)}{2y^{3/2}}\,dy \end{aligned}$$
Since $\int_1^\infty y^{-3/2}\,dy$ converges, the limit $\lim\limits_{R\to\infty} I(R)$ exists and the integral converges.

Exercise 5
Find a series expansion centered at $x = 0$ for the function

$$f \colon \mathbb{R} \to \mathbb{R}, \qquad\qquad f(x) = e^x \cos(x).$$

(4 Marks)

Solution 5
We have

$$\begin{aligned}
e^x \cos(x) &= \frac{1}{2}(e^{ix} + e^{-ix})e^x = \frac{1}{2}(e^{(1+i)x} + e^{(1-i)x} \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(1+i)^n x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(1-i)^n x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(1+i)^n + (1-i)^n}{2} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} (\mathrm{Re}(1+i)^n) \frac{x^n}{n!}
\end{aligned}$$

Since

$$(1+i)^n = (\sqrt{2}e^{i\pi/4})^n = 2^{n/2} e^{i\pi n/4}$$

we obtain

$$e^x \cos(x) = \sum_{n=0}^{\infty} 2^{n/2} \cos(n\pi/4) \frac{x^n}{n!}$$

Exercise 6

i) Show that on $C([a,b])$, the space of continuous functions on the interval $[a,b]$, a norm can be defined by

$$\|f\|_1 := \int_a^b |f(x)|\, dx.$$

ii) Let (f_n) be a sequence of functions in $C([a,b])$ and let $f \in C([a,b])$. Prove or disprove:

$$\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0 \qquad \Leftrightarrow \qquad \|f_n - f\|_1 \xrightarrow{n\to\infty} 0$$

where $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$.

(5 Marks)

Solution 6

i) We need to check the following properties:

- $\|f\|_1 \geq 0$ follows from $|f| \geq 0$ and the positivity of the integral; **(1/2 Mark)**
- $\|\lambda \cdot f\|_1 = \int_a^b |\lambda \cdot f(x)|\, dx = |\lambda| \cdot \int_a^b |f(x)|\, dx = |\lambda| \cdot \|f\|_1$; **(1/2 Mark)**

- $\|f+g\|_1 = \int_a^b |f(x)+g(x)|\,dx \leq \int_a^b (|f(x)|+|g(x)|)\,dx = \int_a^b |f(x)|\,dx + \int_a^b |g(x)|\,dx = \|f\|_1 + \|g\|_1$; **(1/2 Mark)**
- If $f = 0$, then $f(x) = 0$ for all $x \in [a,b]$ and hence $\|f\|_1 = 0$. If $\|f\|_1 = 0$, then $f(x) = 0$ for all $x \in [a,b]$ as follows: Suppose $f(x_0) > 0$ for some $x_0 \in (a,b)$. Then $f(x) > 0$ on $(x_0 - \varepsilon, x_0 + \varepsilon)$ for some $\varepsilon > 0$ because f is continuous. This leads to a contradiction because

$$\|f\|_1 = \int_a^b |f(x)|\,dx \geq \int_{x_0-\varepsilon}^{x_0+\varepsilon} |f(x)|\,dx > 0.$$

(1 Mark) The same argument works if $f(x_0) < 0$ or $x_0 = a$ or $x_0 = b$. **(1/2 Mark)**

ii) The sequence (f_n) defined by the functions $f_n \colon [0,1] \to \mathbb{R}$, $f_n(x) = e^{-nx}$ has the property that

$$\|f_n - 0\|_1 = \int_0^1 |f_n(x)|\,dx = \int_0^1 e^{-nx}\,dx = \frac{1}{n}(1 - e^{-n}) \xrightarrow{n\to\infty} 0,$$

so (f_n) converges to the constant function $f(x) = 0$ as $n \to \infty$ in the $\|\cdot\|_1$ norm. However,

$$\|f_n - f\|_\infty = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} e^{-nx} \geq e^{-n\cdot 0} = 1,$$

so the sequence doesn't converge uniformly. The equivalence is false. **(2 Marks)**