VV186 RC Part IV

Integration

"When learning integration, integrate all knowledge you have. . . "

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Overview

Motivation

The motivation for integration is simple:

- Find area under curves.
- Find signed area for real life application.
- Do different kinds of *transformation*.
- Solving equations.

Step Function

- Partition
- Properties of step functions
  1. If φ is a step function on $[a, b]$, then $\forall \alpha \in \mathbb{R}$, $\alpha\varphi$ is also a step function on $[a, b]$.
  2. If φ, Ψ are two step functions on $[a, b]$, then $\varphi + \Psi$ is also a step function on $[a, b]$.
  3. The set consisting of step functions on $[a, b]$ where $a, b \in \mathbb{R}$ and $a < b$ is a **vector space**.(Why?)

Step Function

Properties of step function integral

- Given a step function $\varphi : [a, b] \to \mathbb{R}$, its integral exists and doesn't depend on the choice of partition.
- Let $T : Step([a, b]) \to \mathbb{R}$ be a function (functional) and $T(\varphi) = \int_a^b \varphi$, then T is a linear function that maps non-negative step functions to non-negative values.
- Let $\varphi \in [a, b]$, then $|\int_a^b \varphi| \leq \int_a^b |\varphi|$

(T is sometimes called "positive linear functional")

Regulated Integral

Let $f \in \text{Reg}([a,b])$ and (φ_n) a sequence in $\text{Step}([a,b])$ converging uniformly to f. Then the regulated integral of f, defined by

$$\int_a^b f := \lim \int_a^b \varphi_n$$

exists and does not depend on the choice of (φ_n).

Darboux Integral

Let $[a, b] \subseteq \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Let u_f denote the set of all step functions u on $[a, b]$ such that $u \geq f$ and denote l_f the set of all step functions l on $[a, b]$ such that $f \geq l$. The function f is then said to be Darboux-integrable if

$$\underline{\mathbb{I}}(f) = \sup_{l \in l_f} \int_a^b l = \inf_{u \in u_f} \int_a^b u = \overline{\mathbb{I}}(f)$$

We denote the integral of f by $\mathbb{I}(f)$ to distinguish with the lower step functions l.

Riemann Integral

Let$[a, b] \subseteq \mathbb{R}$ be a closed interval and f a bounded real function on $[a, b]$. Then f is Riemann-integrable with integral $\int_a^b f \in \mathbb{R}$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition on $[a, b]$ with mesh size $m(P) < \delta$,

$$|\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) - \int_a^b f| < \epsilon$$

Comment. It is also a usual way to define Riemann integral by Darboux integral. In fact, Riemann integral is usually *just the Darboux integral*.

Results/Theorem & Comment

The following are some results / Theorems & comments for integrals. They apply to all the three kinds of integrals that we learn in Vv186.

1. The integral is a linear map that maps non-negative functions to non-negative values.

2. Let f, g be two regulated functions on $[a, b]$. Moreover, if $f \leq g$, then $\int_a^b f \leq \int_a^b g$.
   Comment. This can be proved by taking the limit of the step functions.

Results/Theorem & Comment

3. The integral of f doesn't change if f changes its value on a **finite** set and is still integrable.

4. The integral of f doesn't change if f changes its value on a **countable** set and is still integrable. If we take the usual understanding of "length of interval" to integrate.
   Comment. It is important that f is *integrable*.
   Comment. This set is usually called *measured zero set*.

5. The Riemann integral, Darboux integral and the regulated integral of f exist and coincide if f is regulated.

Exercise

1. This exercise gives an example of using step function sequence to find the integral of a function. Calculate

$$\int_0^1 x^4$$

Exercise

2.

- Check whether the function J given by

$$J : [0,1] \to \mathbb{R}, \qquad J(x) = \begin{cases} 1 & x = \frac{1}{n} \\ 0 & otherwise \end{cases}$$

  is regulated.

- Check whether the function J we defined in i) is Riemann integrable.

3. Please judge whether the following statements are true or false.
   - $f(x) = x^3 + e^x$ on $[-1, 1]$ is regulated.
   - Since $f(x) = x^2$ is regulated on each $[-n, n]$, where $n \in \mathbb{N}$, f is regulated on $\mathbb{R}$, because we can let $n \to \infty$.
   - A continuous function is piecewise continuous.
   - Let f, g be two real-valued functions defined on $[0, 1]$. Furthermore, assume $f - g = x$, then the equation

     $$\int_0^1 f - \int_0^1 g = \frac{1}{2}$$

     holds.
   - Let $f \in \text{Reg}([0, 1])$, let g be a real-valued function. The $f \circ g$ is regulated.

Exercise

4. Now let's prove the assertion on Slide 543. Let f be a bounded real function on $[a, b]$ and f is Darboux integrable. Please show that it is Riemann integrable on $[a, b]$.

Exercise

5. Let f be a piecewise continuous function on $[a, b]$. Prove that f is regulated, i.e., for any $\epsilon > 0$, there exists a step function φ such that

$$\sup_{x \in [a,b]} |f(x) - \varphi(x)| < \epsilon$$

Exercise

6. This exercise aims at showing two more characteristics of $\text{Reg}([a, b])$. Let $f, g \in \text{Reg}([a, b])$,

1. Show that the $f \cdot g \in \text{Reg}([a, b])$
2. Show that $f^2 \in \text{Reg}([a, b])$
3. Suppose $f \geq p > 0$ for some $p \in \mathbb{R}$. Show that $\frac{1}{f} \in \text{Reg}([a, b])$.

Exercise

7. Give an example such that $\int_a^b |f|$ is Riemann integrable but $\int_a^b f$ is not Riemann integrable. Is the converse true?

Exercise

8. Let $f : [a, b] \to \mathbb{R}$, f is monotonic on $[a, b]$. Prove that f is regulated. Is f integrable?

Exercise*

9. Given $\epsilon > 0$. For any regulated function $f : [0, 1] \to \mathbb{R}$. There exists a continuous function $g : [0, 1] \to \mathbb{R}$ such that

$$|\int_0^1 f - \int_0^1 g| < \epsilon \qquad \text{and} \qquad |f(x) - g(x)| < \epsilon \text{ on } [0, 1]$$

except for a finite number of intervals with total length less than ϵ.

Any Question?

Have Fun and Learn Well!