上 海 交 通 大 学 试 卷

2020 – 2021 Academic Year (Fall Term)

Vv186 Honors Mathematics II
Second Midterm Exam – Solutions

Exercise 1

In the following exercises, mark the boxes corresponding to true statements with a cross (☒). In each case, it is possible that none of the statements are true or that more than one statement is true.

i) Suppose that $(V, +, \cdot)$ is s vector space and let $U, W \subset V$ be two subspaces. Then

☒ $U \cap W \neq \emptyset$.

☐ $V \setminus U$ is also a subspace of V.

☐ $U \cup W$ is a subspace of V.

☒ $U \cap W$ is a subspace of V.

ii) Let $(a, b) \subset \mathbb{R}$ be an open interval and denote by $C^1(a, b) \cap C([a, b])$ the vector space of those functions on (a, b) that are continuous on $[a, b]$ and continuously differentiable on (a, b). On this space a norm is defined by

☒ $\|f\| := \sup\limits_{x\in[a,b]} |f(x)|$

☐ $\|f\| := \sup\limits_{x\in[a,b]} |f'(x)|$

☒ $\|f\| := \sup\limits_{x\in[a,b]} |f(x)| + \sup\limits_{x\in[a,b]} |f'(x)|$

☒ $\|f\| := \sup\limits_{x\in[a,b]} \big(|f(x)| + |f'(x)|\big)$

iii) Let (a_n) be a sequence of real numbers such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges at least for $x \in [0, 1]$. Then $f(x)$

☒ $f(x)$ must converge for $x = -1/2$.

☐ $f(x)$ must converge for $x = -1$.

☒ $f(x)$ may or may not converge for $x = -1$.

☐ $f(x)$ never converges for $x = -2$.

iv) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and has a local minimum at $x = 0$. Then

☐ f is convex in a neigborhood of $x = 0$.

☐ $f''(0) > 0$.

☒ $f''(0) \geq 0$ and $f''(0) = 0$ is possible.

☐ $f'(x)$ is increasing in a neighborhood of $x = 0$.

(8 Marks); each correctly checked or unchecked box is worth 1/2 Mark

Exercise 2

Suppose that (f_n) is a sequence of increasing functions $f_n\colon [0,1] \to [0,1]$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad\qquad \text{for all } x \in [0,1].$$

Suppose that f is a continuous function. Show that the convergence is uniform.

(Note: the functions f_n are not assumed to be continuous.)
(5 Marks)

Solution 2

We fix $\varepsilon > 0$. Since f is continuous on the closed interval $[0,1]$, f is also uniformly continuous. Hence, we can find $N \in \mathbb{N} \setminus \{0\}$ such that $|x-y| < 1/N$ implies $|f(x) - f(y)| < \varepsilon$. **(1 Mark)**

We write

$$\begin{aligned}
\sup_{x\in[0,1]} |f_n(x) - f(x)| &= \max_{1\le k\le N} \sup_{x\in[\xi_{k-1},\xi_k]} |f_n(x) - f(x)| \\
&\le \underbrace{\max_{1\le k\le N} \sup_{x\in[\xi_{k-1},\xi_k]} |f_n(x) - f_n(\xi_{k-1})|}_{(1)} \\
&\quad + \underbrace{\max_{1\le k\le N} |f_n(\xi_{k-1}) - f(\xi_{k-1})|}_{(2)} \\
&\quad + \underbrace{\max_{1\le k\le N} \sup_{x\in[\xi_{k-1},\xi_k]} |f(\xi_{k-1}) - f(x)|}_{(3)}
\end{aligned}$$

(1 Mark) Term (3) will be smaller than ε by the uniform continuity of f. **(1/2 Mark)** Since each f_n is increasing, we may evaluate the supremum in Term (1), so we have

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \le \underbrace{\max_{1\le k\le N} |f_n(\xi_k) - f_n(\xi_{k-1})|}_{(1)} + \underbrace{\max_{1\le k\le N} |f_n(\xi_{k-1}) - f(\xi_{k-1})|}_{(2)} + \varepsilon$$

(1/2 Mark) For term (1) we write

$$\begin{aligned}
\max_{1\le k\le N} |f_n(\xi_k) - f_n(\xi_{k-1})| &\le \max_{1\le k\le N} |f_n(\xi_k) - f(\xi_k)| + \underbrace{\max_{1\le k\le N} |f(\xi_k) - f(\xi_{k-1})|}_{<\varepsilon} \\
&\quad + \max_{1\le k\le N} |f(\xi_{k-1}) - f_n(\xi_{k-1})|
\end{aligned}$$

(1 Mark) Since the sequence $f_n(x)$ converges to $f(x)$ for every $x \in [0,1]$, we can then find some $M \in n$ such that for all $n > M$ the remaining terms in (1) and (2) each become smaller than ε. **(1 Mark)** This completes the proof.

Exercise 3

i) Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}$$

ii) For which $a \in \mathbb{R}$ does the following series converge?

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right) \right)^a$$

(5 Marks)

Solution 3

i) Applying the ratio test,

$$\frac{(2n+2)!(3n+3)!}{(n+1)!(4n+4)!} \frac{n!(4n)!}{(2n)!(3n)!} = \frac{(2n+2)(2n+1)(3n+3)(3n+2)(3n+1)}{(n+1)(4n+4)(4n+3)(4n+2)(4n+1)} \to \frac{108}{256} < 1$$

as $n \to \infty$, so the series converges. **(2 Marks)**

ii) For $a \leq 0$, the sequence of the summands does not converge to zero as $n \to \infty$, so the series does not converge.**(1 Mark)**

We may write

$$\sin\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{6n^3}(1 + o(1))$$

as $n \to \infty$. Hence,

$$\frac{1}{n} - \sin\left(\frac{1}{n}\right) = \frac{1}{6n^3}(1 + o(1))$$

and for large enough n we have

$$\frac{1}{12n^3} < \frac{1}{n} - \sin\left(\frac{1}{n}\right) < \frac{1}{3n^3}$$

(1 Mark) Then

$$\frac{1}{12^a n^{3a}} < \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right) \right)^a < \frac{1}{3^a n^{3a}}$$

for large enough n and the series converges for $a > 1/3$ by the comparison test. **(1 Mark)**

Exercise 4
Let $f\colon [0,1] \to \mathbb{R}$ be continuous on $[0,1]$ and twice differentiable on $(0,1)$. Suppose that f satisfies

$$f''(x) + f'(x) - f(x) = 0 \qquad\qquad \text{for } x \in (0,1)$$

and $f(0) = f(1) = 0$. Show that $f(x) = 0$ for all $x \in [0,1]$.
(4 Marks)

Solution 4
Since f is continuous on $[0,1]$, f will assume a maximum value in this interval. **(1 Mark)** Suppose there exists some $\xi \in (0,1)$ such that $f(\xi) > 0$ is this maximum value. Then

$$f(\xi) > 0, \qquad\qquad f'(\xi) = 0, \qquad\qquad f''(\xi) \leq 0.$$

(1 Mark) But this contradicts the differential equation. **(1 Mark)** Hence, f can not assume a strictly positive maximum. Similarly, f can not assume a strictly negative minimum, because then $-f$ would have a strictly positive maximum, which leads to the same contradiction. **(1 Mark)** Hence, $f(x) = 0$ for all $x \in [0,1]$.

Exercise 5
Let $f\colon [0,2\pi] \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{1 + e^{\pi - x}\sin(x)}.$$

i) For which x is $f'(x) = 0$? Derive the solution to the equation $\sin(x) = \cos(x)$, $x \in [0,2\pi]$.

ii) Where is f increasing? Where is f decreasing?

iii) Find the local extrema of f.

iv) What can you say about the convexity and concavity of f?

v) Sketch the graph of f, clearly indicating any significant features of the graph.

(8 Marks)

Solution 5

i) A simple calculation shows that

$$f'(x) = -\frac{e^{\pi - x}}{(e^{\pi - x}\sin(x) + 1)^2}\big(\cos(x) - \sin(x)\big)$$

Bakemono

so that $f'(x) = 0$ when $\cos(x) = \sin(x)$. **(1/2 Mark)** To solve this equation, we note that if x satisfies $\cos(x) = \sin(x)$, then

$$1 = \cos^2(x) + \sin^2(x) = \cos(x)\sin(x) + \sin(x)\cos(x) = \sin(2x)$$

so that $x = \pi/4$ is one solution. **(1/2 Mark)** Since $\cos(x) = \sin(x)$ is equivalent to $\tan(x) = 1$ and the tangent function is injective on $(-\pi/2, \pi/2)$ and has period π, we find that all solutions are given by $x = \pi/4 + n\pi$, $n \in \mathbb{Z}$. **(1/2 Mark)** Hence we find that $x = \pi/4$ and $x = 5\pi/4$ are the critical points for f on $[0,2\pi]$.

ii) Since $\cos(0) > \sin(0)$, we see that $f'(0) < 0$ for $x \in (0, \pi/4)$, so f is decreasing in this interval. Similarly, f is increasing in $(\pi/4, 5\pi/4)$ and decreasing in $(5\pi/4, 2\pi]$. **(1 Mark)**

iii) From the monotonicity of f by ii) above) it follows that $x = \pi/4$ and $x = 2\pi$ are a minimum points and $x = 5\pi/4$ and $x = 0$ are local maximum points. **(1 Mark)** We have

$$f(0) = 1, \qquad f(\pi/4) = \frac{\sqrt{2}}{\sqrt{2} + e^{3\pi/4}}, \qquad f(5\pi/4) = \frac{\sqrt{2}}{\sqrt{2} - e^{-\pi/4}}, \qquad f(2\pi) = 1.$$

(1/2 Mark)

iv) The second derivative of f is given by

$$f''(x) = \frac{2e^{x-\pi}}{\left(e^{x-\pi} + \sin(x)\right)^3}\Big(\big(e^{x-\pi} - \sin(x)\big)\cos(x) + 1\Big)$$

(1/2 Mark) and $f''(x) = 0$ when

$$g(x) := \big(e^{x-\pi} - \sin(x)\big)\cos(x) + 1 = 0.$$

One solution is found by inspection: $x = \pi$. **(1/2 Mark)**

Furthermore, for $x \in (0, \pi)$ we have $\sin(x) > 0$, $0 < e^{x-\pi} < 1$, and $\cos(x) > -1$ so there are no points where $f'' = 0$ in that interval.

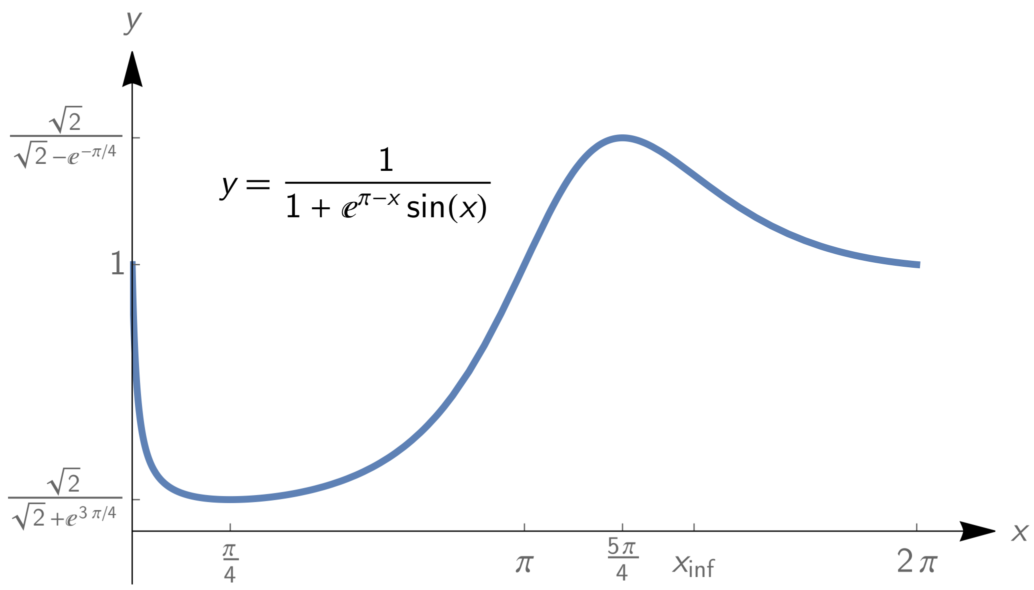
Since $\sin(x) < 0$, $\cos(x > 0$ and $e^{\pi - x} > 1$ for $3\pi/2 < x < 2\pi$, we see that $f''(x) > 0$ and that there are no inflection points for $x > 3\pi/2$.

Since f has a maximum at $x = 5\pi/4$, it follows that $f''(5\pi/4) < 0$ ($f''(5\pi/4)$ does not equal zero, as can be seen directly). Since $f''(3\pi/2) = 1 > 0$, it follows that there is at least one further inflection point x_{inf} in the interval $(5\pi/4, 3\pi/2]$.

Since the exponential, sine and cosine functions are monotonic in the interval $(\pi, 3\pi/2)$, so is g and hence there will be no other zeroes of f'' in this interval.

We conclude that f is convex in $(0, \pi)$, concave in $(\pi, x_{\text{inf}}$ and convex in $(x_{\text{inf}}, 2\pi)$, where $x_{\text{inf}} \in (5\pi/4, 3\pi/2]$. **(1 Mark)**

v) The sketch should look as follows:



Marks for the sketch:

- **(1/2 Mark)** for labeling the axes x and y and the curve $y = f(x)$ or similarly.
- **(1/2 Mark)** for labeling the minimum and maximum points and values on the axes. (The inflection points do not have to be indicated.)
- **(1 Mark)** for getting the general shape of the curve correct and having an adequate size for the sketch.