VV186 RC Part IV

Differentiation

"Take everything Rigorously."

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Overview

In order to investigate a function's derivative, we should first take a close look of **Linear map**.

Definition : A linear map on $\mathbb{R}$ is a function given by :

$$L : \mathbb{R} \to \mathbb{R}, \qquad L(x) = \alpha x, \alpha \in \mathbb{R}$$

Clearly, such a function has lots of good properties, which made our discussion becomes easier.

In this perspective, we would like to *approximate* any functions which we are interested in by a linear map. And if such linear map exists, we say this function is *differentiable*.

Here comes the formal definition of differentiability.

Definition : Let $\Omega \subseteq \mathbb{R}$ be a set and $x \in \text{int}\Omega$. Moreover, Let $f : \Omega \to \mathbb{R}$ be a real function. Then we say f is **differentiable** if there exists a linear map L_x such that for all sufficiently small $h \in \mathbb{R}$,

$$f(x+h) = f(x) + L_x(h) + o(h) \quad as \ h \to 0$$

This linear map is **unique**, if it exists.

We call L_x "the derivative of f at x". If f is differentiable at all points of some open set $U \subseteq \Omega$, we say f is differentiable on U.

Derivative

Common misunderstandings:

$$L_x \text{ is a number for a fixed } x \in \Omega, \text{ because } L_x = \alpha.$$

L_x is **not a number**, but a **linear map**, or one can say "linear function", so it essentially is a *function*. $L_x \cdot h = \alpha \cdot h$ (for some α) doesn't mean $L_x = \alpha$.

To see this, one can consider a function given by

$$f(x) = 2x$$

,which doesn't mean $f = 2$.
In fact, the notation ($L_x = \alpha$) sometimes doesn't make sense, as you will see in the future, one can assign L_x by a more complicated linear function, then, such a multiplication will not make sense at all.

Derivative

Common misunderstandings:

The derivative of f at x is a line passing through $(x, f(x))$

Although it is usually a good idea to sketch something to help you to understand some mathematical concepts, but you always need to aware of the essential reason why such a graph make sense.

The derivative of f at x is a *function*, not a graph. We simply use the graph to illustrate our function sometimes, in this case($\mathbb{R}$), it will be a straight line, but in other case, it can be more complicated.

Derivative

Common misunderstandings:

For $f(x) = x^4$, $f'(x) = 4x^3$, so L_x may not be linear

You are confusing "derivative at a point" with "function that gives derivative". At certain point x, $4x^3$ is just a number in $\mathbb{R}$. Using our notation for L_x(or $f'(x)$), we can express L_x as

$$L_x(\cdot) = 4x^3(\cdot)$$

, the *variable* of L_x is not x, so L_x is **linear** for its input $(\cdot)$

Given a differentiable function $f : \Omega \to \mathbb{R}$, the function that gives a derivative can be denoted by $L : \Omega \to \mathbb{R},\ L(x) = L_x(\cdot)$.
It is a function that maps function to function.

Derivative

Common misunderstandings:

It doesn't make much sense to define differentiation on an open set.

The reason, or benefit to define differentiation on an open set U is because for any point $p \in U$, there is some interval(*neighborhood*) $(p-\epsilon, p+\epsilon) \subseteq U$. This ensures that we can define differentiation at this point p, so does *every point* in U.

However, if a set is not open, it may contain some boundary points. On these points our definition for differentiability fails. In fact, one may have to use one-sided differentiation to define it.

Rules of Differentiation

We not assume both f and g are differentiable functions, then:

- $(f+g)'(x) = f'(x) + g'(x)$
- $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- $(f \circ g)'(x) = f'(g(x))g'(x)$
- $(\frac{f}{g})'(x) = \frac{f'(xg(x) - f(x)g'(x))}{g^2(x)}$
- $f^{-1'}(y) = \frac{1}{f'(f^{-1}(y))}$
- $\lim\limits_{x \searrow b} \frac{f(x)}{g(x)} = \lim\limits_{x \searrow b} \frac{f'(x)}{g'(x)}$, if $\lim\limits_{x \searrow b} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ and $\lim\limits_{x \searrow b} \frac{f'(x)}{g'(x)}$ exists.

Application of Differentiation

We list some useful Results and Theorems.

1. If a real function is differentiable at x, then it is continuous at x.

2. *Hierarchy* of local smoothness.
    - Arbitrary function
    - Function continuous at x
    - Function differentiable at x
    - Function continuously differentiable at x
    - Function twice differentiable at x
    - ...

Result and Theorems.

3. Let f be a function and $(a, b) \subseteq \text{dom } f$ and open interval. If $x \in (a, b)$ is a maximum(or minimum) point of $f \subseteq (a, b)$ and if f is differentiable at x, then $f'(x) = 0$.

4. Let f be a function and $[a, b] \subseteq \text{dom } f$. Assume that f is differentiable on (a, b) and $f(a) = f(b)$. Then there is a number $x \in (a, b)$ such that $f'(x) = 0$.

   Comment. We need the requirement that f is **differentiable everywhere** on (a, b). Otherwise, a counterexample can be:

$$[a, b] = [0, 2], \quad \begin{cases} f(x) = x & x \in [0, 1] \\ f(x) = 2 - x & x \in (1, 2] \end{cases}$$

Result and Theorems.

5. Let $[a, b] \subseteq \operatorname{dom} f$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a number $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

6. Let f be a real function and $x \in \operatorname{dom} f$ such that $f'(x) = 0$. If $f''(x) > 0$, then f has a local minimum at x, if $f''(x) < 0$, then f has a local maximum at x.

   Comment. The case in which $f''(x) = 0$ is more complicated, different conditions may occur.

   Example 1: $f'(x) = x^2$. Example 2: $f'(x) = x^3$.

   As you can see from example 2, f may not even have a local extremum if $f''(x) = 0$.

Result and Theorems.

7. Let f be a twice differentiable function on an open set $\Omega \subseteq \mathbb{R}$. If f has a local minimum at some point $a \in \Omega$, then $f''(a) \geq 0$.

Proof :

Suppose f has a local minimum at a. If $f''(a) < 0$, then f would also have a local maximum at a. Thus, f would be constant in some interval containing a. So $f''(a) = 0$. But this contradicts to our assumption.

Comment. An analogous statement is : If f has a local maximum at some point $a \in \Omega$, then $f''(a) \leq 0$.

Result and Theorems.

8. Let $a \in (0, \infty) \cup \{\infty\}$. Let $f : (-a, a) \to \mathbb{R}$ be a differentiable function. If f is odd, then its derivative is even; if f is even, then its derivative is odd.

Proof :

Suppose f is odd. Then

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = f'(x)$$

(Are there a more elegant way to proof it?)

To be honest, to this part, you may need more practice to really get familiar with all calculation tricks and procedure.

Without further saying, let do some exercise.

Exercise

1. Please calculate following functions' derivative.
(Suppose g' always exists and doesn't vanish)

   i. $f(x) = g(x \cdot g(a))$
   ii. $f(x) = g(x + g(x)) + \frac{1}{g(x)}$
   iii. $f(x) = g(x)(x - a)$
   iv. What is wrong with the following usage of L'Hopital's Rule?

$$\lim_{x \to 1} \frac{x^3 - x - 2}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{3x^2 - 1}{2x - 3} = \lim_{x \to 1} \frac{6x}{2} = 3$$

Exercise

2. Prove that if
$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

Then $a_0 + a_1 x + \cdots + a_n x^n = 0$ for some $x \in [0, 1]$

Exercise

3. This exercise aims to show that differentiation can also be used to prove sequential results. Recall the inequality

$$|a+b|^n \leq 2^{n-1}(|a|^n + |b|^n)$$

Now try to use differentiable function to prove it.

Exercise

4. Prove that if $f : I \to \mathbb{R}$ is a differentiable function on interval I, and f' is strictly increasing on I, then each tangent line intersects f only once.

Exercise

5. Suppose that f satisfies $f'' + f'g - f = 0$ for some function g. Prove that if f is 0 at two distinct points, then f is 0 on the interval between them.

Exercise

6. Let F be an increasing function on $J = (-K, K)$, continuous at $-K$ and K. Let G be another such increasing function on J such that

$$\underset{x \in J}{\forall} \quad \lim_{y \to x^-} F(y) \leq G(x) \leq \lim_{y \to x^+} F(y)$$

Then the set of points $x \in J$ where the derivative F' exists equals the set of points $x \in J$ where G' exists. Moreover, $F'(x) = G'(x)$ on these points.

Exercise

7. In this exercise, we would like to give a deeper investigation of Lipschitz condition. If a real function $T : \Omega \to \mathbb{R}$ satisfies

$$|T(x) - T(y)| \leq k \cdot |x - y|^{\alpha}$$

for any $x, y \in \Omega$, we say T satisfies "*Lipschitz condition of order α*".

1. Show that if $\alpha > 0$, then T is continuous.
2. Show that if $\alpha > 1$, then T is a constant function, i.e.,

$$\underset{C \in \mathbb{R}}{\exists}\ T(x) = C$$

Exercise

8. Suppose $f : [0, n], n \in \mathbb{N}$ is a continuous function, and is differentiable on $(0, n)$. Furthermore, assume that

$$f(0) + f(1) + \cdots + f(n-1) = n, \ \ f(n) = 1$$

Show that there must exist $c \in (0, n)$ such that $f'(c) = 0$.

Exercise

9. Let f, g be two differentiable functions with domain $[0, \infty)$. Prove that if

$$f(0) = g(0) \text{ and } f'(x) \geq g'(x) \text{ for all } x > 0$$

then

$$f(x) \geq g(x) \text{ on } [0, \infty)$$

10. Practical calculation shouldn't be ignored. Please calculate the derivatives of the following functions.

- $(2x + 5x^2)^6$
- $\frac{\sqrt{x}}{x+1}$
- $\sqrt[3]{\frac{3x^2+1}{x^2+1}}$

Have Fun and Learn Well!