

VV186 Mid 2 Big RC

Differentiation of Real Functions and their Properties

“Sometimes you need to admit that, practice really makes perfect.”

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1. Differentiation
2. Derivative
3. Rules of Differentiation
4. Application of Differentiation
5. Convexity and concavity
6. L'Hopital's Rule
7. Exercise

In order to investigate a function's derivative, we should first take a close look of **Linear map**.

Definition : A linear map on \mathbb{R} is a function given by :

$$L : \mathbb{R} \rightarrow \mathbb{R}, \quad L(x) = \alpha x, \alpha \in \mathbb{R}$$

We would like to *approximate* any functions which we are interested in by a linear map. And if such linear map exists, we say this function is *differentiable*.

Definition : Let $\Omega \subseteq \mathbb{R}$ be a set and $x \in \text{int}\Omega$. Moreover, Let $f : \Omega \rightarrow \mathbb{R}$ be a real function. Then we say f is **differentiable** if there exists a linear map L_x such that for all sufficiently small $h \in \mathbb{R}$,

$$f(x + h) = f(x) + L_x(h) + o(h) \quad \text{as } h \rightarrow 0$$

This linear map is **unique**, if it exists.

We call L_x "the derivative of f at x ". If f is differentiable at all points of some open set $U \subseteq \Omega$, we say f is differentiable on U .

There is an important thing that you should pay attention to.

If you want to use the definition to calculate the derivative of some function f , you **need** to show how the extra terms belong to $o(h)$.

Exercise: Use the definition to show

$$\frac{d}{dx} \sin x = \cos x$$

(Hint: Consider how to take advantage of $o(h)$)

Does the following statement make sense?

L_x is a number for a fixed $x \in \Omega$, because $L_x = \alpha$.

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L_x is **not a number**, but a **linear map**, or one can say "linear function", so it essentially is a *function*. $L_x \cdot h = \alpha \cdot h$ (for some α) doesn't mean $L_x = \alpha$.

To see this, one can consider a function given by

$$f(x) = 2x$$

,which doesn't mean $f = 2$.

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For $f(x) = x^4$, $f'(x) = 4x^3$, so L_x may not be linear

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You are confusing "derivative at a point" with "function that gives derivative". At certain point x , $4x^3$ is just a number in \mathbb{R} (corresponds to the α for the definition of a linear map). Using our notation for L_x (or $f'(x)$), we can express L_x as

$$L_x(\cdot) = 4x^3(\cdot)$$

, the *variable* of L_x is not x , so L_x is **linear** for its input (\cdot)

Given a differentiable function $f : \Omega \rightarrow \mathbb{R}$, the function that gives a derivative can be denoted by $L : \Omega \rightarrow \mathbb{R}$, $L(x) = L_x(\cdot)$.

It is a function that maps function to function.

Exercise :

What are the following objects really are?

1. $\frac{d}{dx}$

2. $\frac{d}{dx}f = f'$

3. $f'(x)$

Consider further, what are their domain and range?

We not assume both f and g are differentiable functions, then:

- ▶ $(f + g)'(x) = f'(x) + g'(x)$
- ▶ $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- ▶ $(f \circ g)'(x) = f'(g(x))g'(x)$
- ▶ $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$
- ▶ $f^{-1'}(y) = \frac{1}{f'(f^{-1}(y))}$
- ▶ $\lim_{x \searrow b} \frac{f(x)}{g(x)} = \lim_{x \searrow b} \frac{f'(x)}{g'(x)}$, if $\lim_{x \searrow b} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ and $\lim_{x \searrow b} \frac{f'(x)}{g'(x)}$ exists.

We now list some Results and Theorems that you should already be familiar with.

1. If a real function is differentiable at x , then it is continuous at x .
2. *Hierarchy* of local smoothness.
 - ▶ Arbitrary function
 - ▶ Function continuous at x
 - ▶ Function differentiable at x
 - ▶ Function continuously differentiable at x
 - ▶ Function twice differentiable at x
 - ▶ ...

Result and Theorems.

3. Let f be a function and $(a, b) \subseteq \text{dom } f$ and open interval. If $x \in (a, b)$ is a maximum(or minimum) point of $f \subseteq (a, b)$ and if f is differentiable at x , then $f'(x) = 0$.
4. Let f be a function and $[a, b] \subseteq \text{dom } f$. Assume that f is differentiable on (a, b) and $f(a) = f(b)$. Then there is a number $x \in (a, b)$ such that $f'(x) = 0$.

Comment. We need the requirement that f is **differentiable everywhere** on (a, b) . Otherwise, a counterexample can be:

$$[a, b] = [0, 2], \quad \begin{cases} f(x) = x & x \in [0, 1] \\ f(x) = 2 - x & x \in (1, 2] \end{cases}$$

Result and Theorems.

5. Let $[a, b] \subseteq \text{dom } f$ be a function that is continuous on $[a, b]$ and differentiable on (a, b) . Then there exists a number $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.
6. Let f be a real function and $x \in \text{dom } f$ such that $f'(x) = 0$. If $f''(x) > 0$, then f has a local minimum at x , if $f''(x) < 0$, then f has a local maximum at x .

Comment. The case in which $f''(x) = 0$ is more complicated, different conditions may occur.

Example 1: $f'(x) = x^2$. Example 2: $f'(x) = x^3$.

As you can see from example 2, f may not even have a local extremum if $f''(x) = 0$.

Result and Theorems.

7. Let f be a twice differentiable function on an open set $\Omega \subseteq \mathbb{R}$. If f has a local minimum at some point $a \in \Omega$, then $f''(a) \geq 0$.

Proof :

Suppose f has a local minimum at a . If $f''(a) < 0$, then f would also have a local maximum at a . Thus, f would be constant in some interval containing a . So $f''(a) = 0$. But this contradicts to our assumption.

Comment. An analogous statement is : If f has a local maximum at some point $a \in \Omega$, then $f''(a) \leq 0$.

Result and Theorems.

8. Let $a \in (0, \infty) \cup \{\infty\}$. Let $f : (-a, a) \rightarrow \mathbb{R}$ be a differentiable function. If f is odd, then its derivative is even; if f is even, then its derivative is odd.

Proof :

Suppose f is odd. Then

$$f'(-x) = \lim_{h \rightarrow 0} \frac{f(-x + h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f(x) - f(x - h)}{h} = f'(x)$$

(Are there a more elegant way to proof it?)

For further analysis of functions, we would introduce the concept of **Convexity** and **Concavity**.

The definition of these two concepts are as follows.

Let $\Omega \subseteq \mathbb{R}$ be any set and $I \subseteq \Omega$ an interval. A function $f : \Omega \rightarrow \mathbb{R}$ is called convex on I if for all

$$x, a, b \in I \text{ with } a < x < b, \frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}$$

A strictly convex function is a function that satisfies

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}. \quad (1)$$

We say a function f is concave if $-f$ is convex. We say a function f is strictly concave if $-f$ is strictly convex.

Comment.

We often use "—" (minus sign) to define a new definition from an existing one. The benefit is that these two definitions can be strongly related with each other.

Comment 2.

There is a quick way to memorize it... **Concave**...



As before, we list some Results and Theorems that you should already be familiar with.

1. Let $f : I \rightarrow \mathbb{R}$ be strictly convex on I and differentiable at $a, b \in I$. Then:

- i For any $h > 0$ ($h < 0$) such that $a + h \in I$, the graph of f over the interval $(a, a + h)$ lies below the secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$
- ii The graph of f over all I lies above the tangent line through the point $(a, f(a))$
- iii If $a < b$, then $f'(a) < f'(b)$

Results/Theorem & Comment

2. A function $f : I \rightarrow \mathbb{R}$ (I is an interval) is convex if and only if

$$\forall_{t \in (0,1)} \quad \forall_{x,y \in I} \quad \text{with } x < y, f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

3. Let I be an interval, $f : I \rightarrow \mathbb{R}$ differentiable and f' strictly increasing. If $a, b \in I$, $a < b$ and $f(a) = f(b)$, then

$$f(x) < f(a) = f(b) \text{ for all } x \in (a, b)$$

When calculating the limit for some function, you may bump into some cases including:

$$\text{i } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

$$\text{ii } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

in both cases, the right-hand side is the *pre-result* when you are trying to plug in the limit point into your function (in this case, $\frac{f(a)}{g(a)}$) and guess the result.

However, you might encounter above cases and then you have no idea what the limit is.

Fortunately, we have L'Hopital's Rule.

Here is the theorem.

Let f and g be real functions such that the $b \in \overline{\text{dom}f \cap \text{dom}g}$ and $\lim_{x \searrow b} f(x) = \lim_{x \searrow b} g(x) = 0$. Suppose further that f and g are defined and differentiable on $(b, b + \delta)$ and $g'(x) \neq 0$ on it. Moreover, if the limit $\lim_{x \searrow b} \frac{f'(x)}{g'(x)} =: L$ exists, then

$$\lim_{x \searrow b} \frac{f(x)}{g(x)} = \lim_{x \searrow b} \frac{f'(x)}{g'(x)} = L$$

Comments. This result doesn't require dealing with whole neighborhood, but instead, *half-neighborhood*.

As $b \rightarrow \infty$, we expect a similar method will hold, which is shown in next slide.

Let f and g be real functions such that the interval $(C, \infty) \subseteq \text{dom}f \cap \text{dom}g$ and $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = 0$. Suppose further that f and g are defined and differentiable on (C, ∞) and $g'(x) \neq 0$ on it. Moreover, if the limit $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} =: L$ exists, then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} = L$$

Actually, we have one more variations of L'Hopital's Rule.

Let f and g be real functions such that the interval $(C, \infty) \subseteq \text{dom}f \cap \text{dom}g$ and $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty$. Suppose further that f and g are defined and differentiable on (C, ∞) and $g'(x) \neq 0$ on it. Moreover, if the limit $\lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} =: L$ exists, then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} = L$$

The only difference is that $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty$.

There are something you should know for this section.

1. Be familiar with the definition of L_x
2. Know the difference between $\frac{d}{dx}$, $\frac{df}{dx}$, $f'(x)$, or further, $\frac{d^2}{dx^2}$ and so on. . .
3. Get familiar with the basic calculation process for calculating a function's derivative.
4. Know what are **Rolle's theorem**, **Mean Value theorem**, and also **Cauchy Mean Value Theorem**.
5. Know when you can apply **L'Hopital's Rule** when you perform limit calculations.

1. Please calculate following functions' derivative.

(Suppose g' always exists and doesn't vanish)

i. $f(x) = g(x/g(a))$

ii. $f(x) = g(x + g(x)) + g(x + a)$

iii. $f(x) = ax \cdot g(x)$

iv. What is wrong with the following usage of L'Hopital's Rule?

$$\lim_{x \rightarrow 1} \frac{x^3 - x - 2}{x^2 - 3x + 2} = \lim_{x \rightarrow 1} \frac{3x^2 - 1}{2x - 3} = \lim_{x \rightarrow 1} \frac{6x}{2} = 3$$

2. Prove that if

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

Then $a_0 + a_1x + \cdots + a_nx^n = 0$ for some $x \in [0, 1]$

3. Practical calculation shouldn't be ignored. Please calculate the derivatives of the following functions.

► $(2x + 5x^2)^6$

► $\frac{\sqrt{x}}{x+1}$

► $\sqrt[3]{\frac{3x^2+1}{x^2+1}}$

► x^x

4. Remember the problem that asks you to calculate

$$\lim_{x \rightarrow 1} \frac{\sqrt{x+3} - \sqrt{3x+1}}{x-1} ?$$

Now please use L'Hopital's rule to solve it.

5. Let f be a continuous convex real function on $[a, b]$. Show that f either has one local minimum or infinitely many local minimums on $[a, b]$.

Exercise*

6. Suppose $f : [0, n]$, $n \in \mathbb{N}$ is a continuous function, and is differentiable on $(0, n)$. Furthermore, assume that

$$f(0) + f(1) + \cdots + f(n-1) = n, \quad f(n) = 1$$

Show that there must exist $c \in (0, n)$ such that $f'(c) = 0$.

Exercise*

7. Let f, g be two differentiable functions with domain $[0, \infty)$.
Prove that if

$$f(0) = g(0) \text{ and } f'(x) \geq g'(x) \text{ for all } x > 0$$

then

$$f(x) \geq g(x) \text{ on } [0, \infty)$$

Exercise*



8. Suppose that f satisfies $f'' + f'g - f = 0$ for some function g . Prove that if f is 0 at two distinct points, then f is 0 on the interval between them.

Have Fun and Learn Well!