

VV186 RC Part III

Convergence

“Find the problem, understand it, then explore more.”

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1. Review
2. Metric Space
3. Cauchy Sequence
4. Generalization of Convergence
5. Completeness
6. Construct Real Numbers
7. Real Functions
8. Exercise
9. More about Cauchy Sequence

Let first look at the Exercise 7 from last time.

Let (x_n) be a bounded real sequence. Then define

$$a_n := \sup_{m \geq n} (x_m), \quad b_n := \inf_{m \geq n} (x_m)$$

1. Prove that (a_n) is decreasing, while (b_n) is increasing.
2. Since both $(a_n), (b_n)$ are monotonic and bounded, they are convergent. We denote $\underline{\lim} x_n = \lim b_n$; $\overline{\lim} x_n = \lim a_n$. Show that:

$$\underline{\lim} y_n + \underline{\lim} z_n \leq \underline{\lim} (y_n + z_n) \leq \overline{\lim} y_n + \underline{\lim} z_n \leq \overline{\lim} y_n + \overline{\lim} z_n$$

This is a title!



Any Questions?

- ▶ What is the definition of a metric?
- ▶ Why we want to introduce the idea of Metric Space?
- ▶ What new results can we explore from this new idea?

We want to generalize the idea of *convergence*, we want to define the most essential thing of convergence by ourselves, namely the **Length Function**.

What properties a usual length function should have?

1. Always positive.
2. Symmetric.
3. Followed *Triangle Inequality*.

Transform these into mathematical language...

A two variables functions

$$\rho(\cdot, \cdot) : M \times M \rightarrow \mathbb{R}$$

is called a metric if it satisfies:

1. $\forall x, y \in M, \rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$.
2. $\forall x, y \in M, \rho(x, y) = \rho(y, x)$.
3. $\forall x, y, z \in M, \rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

- ▶ $M = \mathbb{R}^n$, the usual metric is given by

$$\rho((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

and this is so-called *Euclidean distance*.

- ▶ $M = \mathbb{N}$, $\rho(x, y) = \#\{a : a \in [\min\{x, y\}, \max\{x, y\}]\}$
- ▶ $M = \mathbb{R}$, $\rho(x, y) = 1$ if $x \neq y$; $\rho(x, y) = 0$ if $x = y$

A simple question arise, is any metric **well-defined**?

Then, by replacing the usual metric $\rho(x, y) = |x - y|$ and choosing our universal set M , we get the natural definition for generalize convergence in metric space (M, ρ) for a sequence $(a_n) : \mathbb{N} \rightarrow M$, which is given by:

$$\lim_{n \rightarrow \infty} a_n = a \quad :\Leftrightarrow \quad \forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{n > N} a_n \in B_{\epsilon}(a)$$

where

$$B_{\epsilon}(a) = \{x \in M : \rho(x, a) < \epsilon\}, \quad \epsilon > 0, \quad a \in M.$$

- ▶ What is the definition of Cauchy Sequence?
- ▶ How to understand Cauchy Sequence?
- ▶ Why we want to introduce the idea of Cauchy Sequence?
- ▶ What new results can we explore from this new idea?

The fundamental reason why we want to introduce Cauchy Sequence is because we want to *further generalize* the idea of convergence.

Now, we are not only free to choose the metric we like, additionally, we remove the constraint which requires a sequence to converge to a *specific point*.

We list some important results and theorems for Cauchy Sequence.

- ▶ Every convergent sequence is a Cauchy sequence.
- ▶ Every Cauchy sequence in a metric space (M, ρ) is bounded.
- ▶ Every Cauchy sequence in \mathbb{R} with the usual metric is convergent.

We now take a deep breath, and look back what we have introduced.
Where does all these new definitions lead us to?

The problem is, a Cauchy Sequence can simply not "*converge*" to anywhere anymore if we generalize the idea of convergence.

Then, after generalizing the idea of convergence, if **every** Cauchy sequence still converge, we say this metric space is *complete*.

We now take a look at some examples for a Cauchy sequence failed to converge.

For a metric space (\mathbb{Q}, ρ) , where $\rho(x, y)$ is analogous to our usual metric. Then, a sequence

$$(a_n)_{n \in \mathbb{N}} := 1, 1.4, 1.41, 1.414, \dots$$

is a Cauchy sequence but failed to converge in \mathbb{Q} if we choose the following terms appropriately and let it *converge to $\sqrt{2}$ in \mathbb{R}* eventually.

From here, it seems we somehow find a good way to *construct* Real Numbers.

The main idea to **complete** \mathbb{Q} is to consider the following

Every Cauchy sequence can *represent* a number in \mathbb{R} .

For example, from the last slide, we can view the sequence

$$(a_n)_{n \in \mathbb{N}} = 1, 1.4, 1.41, 1.414, \dots$$

as $\sqrt{2}$ by defining a (*equivalence*) *class* for each number in \mathbb{R} , and in this case, we define the sequence (a_n) as $\sqrt{2}$.

Then, from now on, the number defined through the Cauchy sequence which is given by

$$0.9, 0.99, 0.999, 0.9999, 0.99999$$

is equal to 1, namely $0.999999\dots = 1$.

We already know how to describe a function, so we list some *elementary functions* and some categories of functions which you will encounter a lot in the future.

- ▶ Polynomial Function : $f(x) = \sum_{k=0}^n c_k x^k$, $n \in \mathbb{N}$. $\forall_{1 \leq i \leq n} c_i \in \mathbb{R}$
- ▶ Power Function : $f(x) = x^n$, $n \in \mathbb{N}$
- ▶ Rational Function : $f(x) = \frac{p(x)}{q(x)}$, $p, q \in \mathbb{P}(\mathbb{R})$
- ▶ Periodic Function : $\forall_{x \in \mathbb{R}} f(x + T) = f(x)$, where T is *period*.
- ▶ Piecewise Function

(What is the domain of each kind of function?)

Then, we can do some basic manipulation to a function, including:

1. Translation in different directions
2. Multiplying a function's argument
3. Multiplying a function's values

1. Suppose the function which is given by

$$f : [2, +\infty) \rightarrow \mathbb{R}$$

and satisfies

$$x^2 f(x) - f(1-x) = x$$

Please calculate the value of

$$\lim_{x \rightarrow \infty} \frac{f(x)}{3}$$

(Can we further expand the domain of f ?)

2. Please construct a metric that makes \mathbb{R} incomplete with regard to this metric. You can use the method given in Horst's slides.

More About Cauchy Sequence



Let us take a look at some tricky examples of Cauchy sequence.

In class demonstration!

Have Fun and Learn Well!