VV186 Mid 2 Big RC

Differentiation of Real Functions and their Properties

"Sometimes you need to admit that, practice really makes perfect."

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Overview

In order to investigate a function's derivative, we should first take a close look of **Linear map**.

Definition : A linear map on $\mathbb{R}$ is a function given by :

$$L : \mathbb{R} \to \mathbb{R}, \qquad L(x) = \alpha x, \alpha \in \mathbb{R}$$

We would like to *approximate* any functions which we are interested in by a linear map. And if such linear map exists, we say this function is *differentiable*.

Differentiation

Definition : Let $\Omega \subseteq \mathbb{R}$ be a set and $x \in \text{int}\Omega$. Moreover, Let $f : \Omega \to \mathbb{R}$ be a real function. Then we say f is **differentiable** if there exists a linear map L_x such that for all sufficiently small $h \in \mathbb{R}$,

$$f(x+h) = f(x) + L_x(h) + o(h) \quad as\ h \to 0$$

This linear map is **unique**, if it exists.

We call L_x "the derivative of f at x". If f is differentiable at all points of some open set $U \subseteq \Omega$, we say f is differentiable on U.

Derivative

There is an important thing that you should pay attention to.

If you want to use the definition to calculate the derivative of some function f, you **need** to show how the extra terms belong to $o(h)$.

Exercise: Use the definition to show

$$\frac{d}{dx}\sin x = \cos x$$

(Hint: Consider how to take advantage of $o(h)$)

Derivative

Does the following statement make sense?

L_x is a number for a fixed $x \in \Omega$, because $L_x = \alpha$.

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$$L_x \text{ is a number for a fixed } x \in \Omega, \text{ because } L_x = \alpha.$$

L_x is **not a number**, but a **linear map**, or one can say "linear function", so it essentially is a *function*. $L_x \cdot h = \alpha \cdot h$ (for some α) doesn't mean $L_x = \alpha$.

To see this, one can consider a function given by

$$f(x) = 2x$$

,which doesn't mean $f = 2$.

Derivative

Does the following statement make sense?

$$\text{For } f(x) = x^4,\ f'(x) = 4x^3, \text{ so } L_x \text{ may not be linear}$$

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You are confusing "derivative at a point" with "function that gives derivative". At certain point x, $4x^3$ is just a number in $\mathbb{R}$ (corresponds to the α for the definition of a linear map). Using our notation for L_x(or $f'(x)$), we can express L_x as

$$L_x(\cdot) = 4x^3(\cdot)$$

, the *variable* of L_x is not x, so L_x is **linear** for its input $(\cdot)$

Given a differentiable function $f : \Omega \to \mathbb{R}$, the function that gives a derivative can be denoted by $L : \Omega \to \mathbb{R},\ L(x) = L_x(\cdot)$.
It is a function that maps function to function.

Derivative

Exercise :

What are the following objects really are?

1. $\frac{d}{dx}$
2. $\frac{d}{dx}f = f'$
3. $f'(x)$

Consider further, what are their domain and range?

Rules of Differentiation

We not assume both f and g are differentiable functions, then:

- $(f+g)'(x) = f'(x) + g'(x)$
- $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
- $(f \circ g)'(x) = f'(g(x))g'(x)$
- $(\frac{f}{g})'(x) = \frac{f'(xg(x) - f(x)g'(x))}{g^2(x)}$
- $f^{-1'}(y) = \frac{1}{f'(f^{-1}(y))}$
- $\lim\limits_{x \searrow b} \frac{f(x)}{g(x)} = \lim\limits_{x \searrow b} \frac{f'(x)}{g'(x)}$, if $\lim\limits_{x \searrow b} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ and $\lim\limits_{x \searrow b} \frac{f'(x)}{g'(x)}$ exists.

Application of Differentiation

We now list some Results and Theorems that you should already be familiar with.

1. If a real function is differentiable at x, then it is continuous at x.

2. *Hierarchy* of local smoothness.
    - Arbitrary function
    - Function continuous at x
    - Function differentiable at x
    - Function continuously differentiable at x
    - Function twice differentiable at x
    - ...

Result and Theorems.

3. Let f be a function and $(a, b) \subseteq \text{dom } f$ and open interval. If $x \in (a, b)$ is a maximum(or minimum) point of $f \subseteq (a, b)$ and if f is differentiable at x, then $f'(x) = 0$.

4. Let f be a function and $[a, b] \subseteq \text{dom } f$. Assume that f is differentiable on (a, b) and $f(a) = f(b)$. Then there is a number $x \in (a, b)$ such that $f'(x) = 0$.

   Comment. We need the requirement that f is **differentiable everywhere** on (a, b). Otherwise, a counterexample can be:

$$[a, b] = [0, 2], \quad \begin{cases} f(x) = x & x \in [0, 1] \\ f(x) = 2 - x & x \in (1, 2] \end{cases}$$

Result and Theorems.

5. Let $[a, b] \subseteq \operatorname{dom} f$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a number $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.

6. Let f be a real function and $x \in \operatorname{dom} f$ such that $f'(x) = 0$. If $f''(x) > 0$, then f has a local minimum at x, if $f''(x) < 0$, then f has a local maximum at x.

   Comment. The case in which $f''(x) = 0$ is more complicated, different conditions may occur.

   Example 1: $f'(x) = x^2$. Example 2: $f'(x) = x^3$.

   As you can see from example 2, f may not even have a local extremum if $f''(x) = 0$.

Result and Theorems.

7. Let f be a twice differentiable function on an open set $\Omega \subseteq \mathbb{R}$. If f has a local minimum at some point $a \in \Omega$, then $f''(a) \geq 0$.

Proof :

Suppose f has a local minimum at a. If $f''(a) < 0$, then f would also have a local maximum at a. Thus, f would be constant in some interval containing a. So $f''(a) = 0$. But this contradicts to our assumption.

Comment. An analogous statement is : If f has a local maximum at some point $a \in \Omega$, then $f''(a) \leq 0$.

Result and Theorems.

8. Let $a \in (0, \infty) \cup \{\infty\}$. Let $f : (-a, a) \to \mathbb{R}$ be a differentiable function. If f is odd, then its derivative is even; if f is even, then its derivative is odd.

Proof :

Suppose f is odd. Then

$$f'(-x) = \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = f'(x)$$

(Are there a more elegant way to proof it?)

For further analysis of functions, we would introduce the concept of **Convexity** and **Concavity**.

The definition of these two concepts are as follows.

Let $\Omega \subseteq \mathbb{R}$ be any set and $I \subseteq \Omega$ an interval. A function $f : \Omega \to \mathbb{R}$ is called convex on I if for all

$$x, a, b \in I \text{ with } a < x < b, \frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}$$

A strictly convex function is a function that satisfies

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}. \tag{1}$$

We say a function f is concave if $-f$ is convex. We say a function f is strictly concave if $-f$ is strictly convex.

Convexity and Concavity

Comment.

We often use "$-$"(minus sign) to define a new definition from an existing one. The benefit is that these two definitions can be strongly related with each other.

Comment 2.

There is a quick way to memorize it... Con**cave**...

Convexity and Concavity

As before, we list some Results and Theorems that you should already be familiar with.

1. Let $f : I \to \mathbb{R}$ be strictly convex on I and differentiable at $a, b \in I$. Then:
   - i For any $h > 0 (h < 0)$ such that $a + h \in I$, the graph of f over the interval $(a, a + h)$ lies below the secant line through the points $(a, f(a))$ and $(a + h, f(a + h))$
   - ii The graph of f over all I lies above the tangent line through the point $(a, f(a))$
   - iii If $a < b$, then $f'(a) < f'(b)$

Results/Theorem & Comment

2. A function $f : I \to \mathbb{R}$(I is an interval) is convex if and only if

$$\underset{t\in(0,1)}{\forall}\ \underset{x,y\in I}{\forall} \text{ with } x < y, f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

3. Let I be an interval, $f : I \to \mathbb{R}$ differentiable and f' strictly increasing. If $a, b \in I$, $a < b$ and $f(a) = f(b)$, then

$$f(x) < f(a) = f(b) \text{ for all } x \in (a, b)$$

L'Hopital's Rule

When calculating the limit for some function, you may bump into some cases including:

i $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$

ii $\lim\limits_{x \to a} \frac{(f(x)}{g(x)} = \frac{0}{0}$

in both cases, the right-hand side is the *pre-result* when you are trying to plug in the limit point into your function (in this case, $\frac{f(a)}{g(a)}$) and guess the result.

However, you might encounter above cases and then you have no idea what the limit is.

Fortunately, we have L'Hopital's Rule.

Here is the theorem.

Let f and g be real functions such that the $b \in \overline{domf \cap domg}$ and $\lim\limits_{x \searrow b} f(x) = \lim\limits_{x \searrow b} g(x) = 0$. Suppose further that f and g are defined and differentiable on $(b, b+\delta)$ and $g'(x) \neq 0$ on it. Moreover, if the limit $\lim\limits_{x \searrow b} \frac{f'(x)}{g'(x)} =: L$ exists, then

$$\lim_{x \searrow b} \frac{f(x)}{g(x)} = \lim_{x \searrow b} \frac{f'(x)}{g'(x)} = L$$

Comments. This result doesn't require dealing with whole neighborhood, but instead, *half-neighborhood*.

As $b \to \infty$, we expect a similar method will hold, which is shown in next slide.

L'Hopital's Rule

Let f and g be real functions such that the interval $(C, \infty) \subseteq domf \cap domg$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$. Suppose further that f and g are defined and differentiable on (C, ∞) and $g'(x) \neq 0$ on it. Moreover, if the limit $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} =: L$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$$

Actually, we have one more variations of L'Hopital's Rule.

L'Hopital's Rule

Let f and g be real functions such that the interval $(C, \infty) \subseteq dom f \cap dom g$ and $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$. Suppose further that f and g are defined and differentiable on (C, ∞) and $g'(x) \neq 0$ on it. Moreover, if the limit $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} =: L$ exists, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$$

The only difference is that $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$.

Brief Summary

There are something you should know for this section.

1. Be familiar with the definition of L_x
2. Know the difference between $\frac{d}{dx}$, $\frac{df}{dx}$, $f'(x)$, or further, $\frac{d^2}{dx^2}$ and so on. . .
3. Get familiar with the basic calculation process for calculating a function's derivative.
4. Know what are **Rolle's theorem**, **Mean Value theorem**, and also **Cauchy Mean Value Theorem**.
5. Know when you can apply **L'Hopital's Rule** when you perform limit calculations.

Exercise

1. Please calculate following functions' derivative.
(Suppose g' always exists and doesn't vanish)

   i. $f(x) = g(x/g(a))$
   ii. $f(x) = g(x + g(x)) + g(x + a)$
   iii. $f(x) = ax \cdot g(x)$
   iv. What is wrong with the following usage of L'Hopital's Rule?

$$\lim_{x \to 1} \frac{x^3 - x - 2}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{3x^2 - 1}{2x - 3} = \lim_{x \to 1} \frac{6x}{2} = 3$$

Exercise

2. Prove that if
$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0$$

Then $a_0 + a_1 x + \cdots + a_n x^n = 0$ for some $x \in [0, 1]$

Exercise

3. Practical calculation shouldn't be ignored. Please calculate the derivatives of the following functions.

- $(2x+5x^2)^6$
- $\frac{\sqrt{x}}{x+1}$
- $\sqrt[3]{\frac{3x^2+1}{x^2+1}}$
- x^x

Exercise

4. Remember the problem that asks you to calculate

$$\lim_{x\to 1} \frac{\sqrt{x+3}-\sqrt{3x+1}}{x-1} ?$$

Now please use L'Hopital's rule to solve it.

Exercise

5. Let f be a continuous convex real function on $[a, b]$. Show that f either has one local minimum or infinitely many local minimums on $[a, b]$.

Exercise*

6. Suppose $f : [0, n], n \in \mathbb{N}$ is a continuous function, and is differentiable on $(0, n)$. Furthermore, assume that

$$f(0) + f(1) + \cdots + f(n-1) = n, \;\; f(n) = 1$$

Show that there must exist $c \in (0, n)$ such that $f'(c) = 0$.

Exercise*

7. Let f, g be two differentiable functions with domain $[0, \infty)$. Prove that if

$$f(0) = g(0) \text{ and } f'(x) \geq g'(x) \text{ for all } x > 0$$

then

$$f(x) \geq g(x) \text{ on } [0, \infty)$$

Exercise*

8. Suppose that f satisfies $f'' + f'g - f = 0$ for some function g. Prove that if f is 0 at two distinct points, then f is 0 on the interval between them.

End

Have Fun and Learn Well!