Proof: Since f is continuous on a closed interval $[a, b]$, f attains its minimum on it, so f has at least one local minimum. Then we would like to show that f cannot have finitely many local minimums: suppose $\{f(x_1), f(x_2), f(x_3), \ldots, f(x_n)\}$ are all of f's local minimums, and $n \geq 2$. Between two adjacent points $f(x_i), f(x_{i+1})$, since f is continuous, f attains a maximum, denoted by $f(y_i)$, on $[x_i, x_{i+1}]$. Furthermore, $f(y) > max\,\{f(x_i), f(x_{i+1})\}$. It follows that

$$\frac{f(x_i) - f(y)}{x_i - y} > 0 > \frac{f(x_{i+1}) - f(y)}{x_{i+1} - y}$$

So f is not convex, leading to a contradiction. Third we need to show that f can have infinitely many local minimums: the example is $f: [-2,2], f(x) = \begin{cases} -x, -2 \leq x < -1 \\ 1, -1 \leq x \leq 1 \\ x, \quad 1 < x \leq 2 \end{cases}$.

Proof: If $f(0) = f(1) = \cdots = f(n-1)$, then $f(i) = 1, i \in \{0,1,2, \ldots n\}$. This means for any interval $[i, i+1], i \in \{0,1,2, \ldots, n-1\}$, there is some $c_i \in (i, i+1)$ with $f'(c_i) = 0$. If "f(i)"s are not all equal, then there is some $f(i) > 1$ and some $gf(j) < 1$. Without losing generality we assume that $i < j$. By Mean Value Theorem of continuous function, there is some point $x \in (i, j)$ such that $f(x) = 1$. Thus, on the interval (x, n), there is some $c \in (x, n)$ such that $f'(c) = 0$.

Proof: $f(0) \geq g(0)$ is trivially true. When $x > 0$, construct a function $T: [0, +\infty) \rightarrow \mathbb{R}$, $T(x) = f(x) - g(x)$. Since both f and g are differentiable, we know that T is differentiable. Given $x > 0$, by the Mean value Theorem, there is some $y \in (0, x)$ such that $T'(y) = \frac{T(x)-T(0)}{x-0}$. Since $T(0) = 0$ and $T'(y) = f'(y) - g'(y) \geq 0$, $\frac{T(x)}{x} \geq 0$. But this means $T(x) \geq 0$, or $f(x) \geq g(x)$.

Proof: Let the two points where f attains zero be a and b, namely, $f(a) = f(b) = 0$. Let $I = (a, b), \bar{I} = [a, b]$. Suppose the maximum of f on $\bar{I}$, denoted by $f(x_0)$ is strictly greater than zero. Then $x_0 \in I$ and $f'(x_0) = 0$. Since $f(x_0) > 0$, $f''(x_0) = -f'(x_0)g(x_0) + f(x_0) = f(x_0) > 0$. But this means there exists some $\varepsilon > 0$ such that on the neighborhood $(x_0 - \varepsilon, x_0 + \varepsilon)$, $f(x_0)$ is a strict local minimum. It follows that $f(x_0)$ is not the maximum point of f, leading to a contradiction. Suppose the maximum of f on $\bar{I}$, denoted by $f(x_0)$ is strictly smaller than zero. A similar proof shows that this assumption fails. So $\max_{x \in \bar{I}} f(x) = \min_{x \in \bar{I}} f(x) = 0$. It follows that f is 0 on the interval between $f(a)$ and $f(b)$.