VV186 RC Part II Solution

From Numbers to Sequences – Exercises

"Practice makes Perfect..."

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Exercises-I

1. When learning the axioms of rational number, one student found that the operation of subsets of a non-empty set X is somewhat similar to that of rational number:

If we regard $\cup$ as $+$, $\cap$ as $\cdot$, then the equation

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

is just the distributivity law. Help him check whether P1 – P9 also hold for such operations.

Solution-I

We check by enumerating P1 – P9.

1. $A \cup (B \cup C) = (A \cup B) \cup C$
2. $A \cup \emptyset = \emptyset \cup A = A$
3. $A \cup B = B \cup A$
4. $A^c \cup A = A \cup A^c = X$
5. $A \cap (B \cap C) = (A \cap B) \cap C$
6. $A \cap X = X \cap A = A$
7. $A \cap B = B \cap A$
8. $A \cap \emptyset = \emptyset \cap A = \emptyset$
9. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

As we can see, P4 and P8 does not hold, since we can't find the inverse element for $\cup$ and $\cap$, so the answer of this exercise is no. $\square$

Exercises-I

2.

1. Prove that for $a_i \in \mathbb{Q}, i \in \mathbb{N}^*$, and n is a natural number,

$$|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$$

2. Prove that $|a - c| \leq |a - b| + |c - b|$

2 – 1

Proof :

$$
\begin{aligned}
|\sum_{i=1}^{n} a_i| &= |a_1 + a_2 + \cdots + a_n| = |(a_1 + \cdots + a_{n-1}) + a_n| \\
&\leq |a_1 + \cdots + a_{n-1}| + |a_n| \leq \ldots \\
&. \\
&. \\
&. \\
&\leq |a_1| + |a_2| + \cdots + |a_n| = \sum_{i=1}^{n} |a_i|
\end{aligned}
$$

$\square$

Solution-I

2 – 2

Proof :

$$\begin{aligned} |a-c| &= |a+b-b-c| \\ &= |(a-b)+(b-c)| \\ &\le |a-b|+|b-c| \\ &= |a-b|+|c-b| \end{aligned}$$

$\square$

Exercises-I

3. We define $\mathbb{R}$ as a set containing rational numbers and the limits of rational numbers. Let $E \subseteq \mathbb{R}$ a non-empty subset of that is bounded above in $\mathbb{R}$. Now we don't assume P13 is an axiom. We want to prove P13 with P1 – P12 and other facts that we know:

- Let's call the set of form $(a, +\infty)$ or $[a, +\infty)$ *up interval*. Prove that the set of E's upper bound, denoted by U, is an *up interval*.
- (**Theorem 2.2**) E has a least upper bound.

3 – 1

Proof :

We first prove that U is an interval by contradiction. Suppose U is not an interval, then we can see

$$\exists\ m \notin U\ \begin{cases} U \cap (-\infty, m) \neq \emptyset \\ U \cap (m, \infty) \neq \emptyset \end{cases}$$

From here, we have two useful conditions as follows.

1. Since $m \notin U$, $\underset{x \in E}{\exists}\ x > m$.
2. Since $(-\infty, m) \cap U \neq \emptyset$, $\exists\ y \in (-\infty, m) \cap U \Rightarrow \begin{cases} y < m \\ y \in U \end{cases}$

Finally, we get $y < m < x$, but this is a contradiction since y is an upper bound but $y < x \in E$.

3 – 1

Proof (*continue*) :

Then, we know U is an interval now. Because U is clearly bounded below by some element in E, let say a, then U should have this form:

$$U = (a, b) \ or \ U = [a, b)$$

for some b. And because if $b \in U$, then clearly $b + c \in U$ for $c > 0$, we know that

$$U = (a, \infty) \ or \ U = [a, \infty)$$

so U is an up interval. □

3 – 2

Proof :

We consider two cases.

First, if $U = [a, \infty)$, then clearly, U has a minimum a, and which is the least upper bound of E.

Second, if $U = (a, \infty)$, then we know

$$\begin{cases} \underset{x \in E}{\exists} \ x > a \Rightarrow \ \underset{\epsilon > 0}{\exists} \ x = a + 2\epsilon \\ \underset{\epsilon > 0}{\forall} \ a + \epsilon \in U \ \Rightarrow \ a + \epsilon > x \end{cases}$$

We simply fixed the ϵ we found in equation 1, and then plug in to the equation 2, we get

$$a + \epsilon > x = a + 2\epsilon \ \Rightarrow \ \epsilon < 0$$

Which contradicts to our chosen condition of ϵ, so we know the second case $U = (a, \infty)$ can not be the case. $\square$

Exercises-I*

4. Let A be bounded set in $\mathbb{R}$ (which means that the total set is $\mathbb{R}$), for any $\epsilon > 0$, there is an element x in A such that $|x - supA| < \epsilon$.

4

Proof :

We proceed our proof by contradiction. Suppose this is not true, namely there is an $\epsilon > 0$ such that for all $x \in A$ we have

$$|x - \sup A| \geq \epsilon$$

But then we know, $\sup A - \frac{1}{2}\epsilon$ is an upper bound of A, which leads to a contradiction. $\square$

5. A sequence is defined as

$$(S_n)_{n\in\mathbb{N}},\ \ S_1=\sqrt{2},\ \ S_2=\sqrt{2\sqrt{2}},\ \ S_3=\sqrt{2\sqrt{2\sqrt{2}}}$$

Please calculate the limit of (S_n) as $n\to\infty$, if it exists.

5.

Since we don't know whether the limit of (S_n) exist or not, we must first prove the existence, then do the following calculation.

We proceed as follows. First, we observe that (S_n) is bounded since

$$0 \leq S_n = \underbrace{\sqrt{2\sqrt{2\sqrt{2\ldots\sqrt{2}}}}}_{n \ \ square \ \ roots} \leq \underbrace{\sqrt{2\sqrt{2\sqrt{2\ldots 2}}}}_{n \ \ square \ \ roots} = 2$$

We get the right-hand side by replacing the last $\sqrt{2}$ by 2 from the left-hand side.

We conclude that the sequence (S_n) is bounded. Our goal is to prove (S_n) is convergent, so we now only need to prove (S_n) is monotonic, then we can deduce the convergence from our already known lemma.

Solution-I

($continue$)

Since $S_{n+1} = \sqrt{2S_n}$ for $n \geq 1$. When $n \geq 1$,

$$\frac{S_{n+1}}{S_n} = \frac{\sqrt{2Sn}}{Sn} = \frac{\sqrt{2}}{\sqrt{S_n}} > \frac{\sqrt{2}}{\sqrt{2}} = 1$$

It is easy to see that (S_n) is strictly increasing, which means it is monotonic, hence convergent.

To calculate the limit, we use the equation

$$\lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} S_n =: L$$

Which leads to $\sqrt{2L} = L$. By solving this equation and the fact that $S_n \geq S_1 = 1$, so $L = 2$, which mean the limit is 2. $\square$

Exercise-II

6. Let $(a_n), (b_n)$ be two real sequences. Furthermore, assume that $a_n < b_n$ for all n, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, $\lim(a_n - b_n) = 0$. Prove that there is an unique $m \in [a_n, b_n]$ for all n, such that

$$\lim a_n = \lim b_n = m$$

Solution-II

6.

Proof :

We first show that $(a_n), (b_n)$ are convergent. Notice that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ means that (a_n) is increasing, while (b_n) is decreasing. Furthermore, both (a_n) and (b_n) are bounded by $[a_0, b_0]$. Therefore, (a_n) and (b_n) are convergent.

Next we show that $\lim a_n = \lim b_n$. Suppose $\lim b_n = L$, where L is a unique number since a sequence has precisely one limit. Then we know

$$\lim a_n = \lim[(a_n - b_n) + b_n] = \lim(a_n - b_n) + \lim b_n = 0 + L = L$$

Moreover, since $\begin{cases} L \geq a_n \\ L \leq b_n \end{cases}$, we know that $L \in [a_n, b_n]$. $\square$

Exercise-II

7. Let (x_n) be a bounded real sequence. Then define

$$a_n := \sup_{m\geq n}(x_m), \quad b_n := \inf_{m\geq n}(x_m)$$

1. Prove that (a_n) is decreasing, while (b_n) is increasing.
2. Since both $(a_n), (b_n)$ are monotonic and bounded, they are convergent. We denote $\underline{\lim} x_n = \lim b_n$; $\overline{\lim} x_n = \lim a_n$. Show that:

$$\underline{\lim} y_n + \underline{\lim} z_n \leq \underline{\lim}(y_n + z_n) \leq \overline{\lim} y_n + \underline{\lim} z_n \leq \overline{\lim} y_n + \overline{\lim} z_n$$

Solution-II

7 – 1

Proof :

This follows from the fact that for two bounded sets A, B in $\mathbb{R}$ with $A \supseteq B$, the supremum of A is no less than the supremum of B. Similarly, for two bounded sets A, B in $\mathbb{R}$ with $A \supseteq B$, the infimum of A is no greater than the infimum of B.

7 – 2

Proof :

We precede from the left-hand side. Start from the first inequality:

$$\underline{\lim} y_n + \underline{\lim} z_n \le \underline{\lim}(y_n + z_n)$$

Fix $\epsilon > 0$, there is an $M \in \mathbb{N}$ such that $\underset{n>M}{\forall}\ y_n \ge \underline{\lim} y_n - \frac{\epsilon}{4}$, and $z_n \ge \underline{\lim} z_n - \frac{\epsilon}{4}$. So we have $\underline{\lim} y_n + \underline{\lim} z_n - \frac{\epsilon}{2} \le (y_n + z_n)$ for all such n. Next, we claim that there is always a subsequence of $(y_n + z_n)$ such that it converges to $\underline{\lim}(y_n + z_n)$.

Fix arbitrary $t > 0$, $\underset{M\in\mathbb{N}}{\forall}\ \underset{n_k\ge M}{\exists}$ such that

$$\begin{cases} |(y_{n_k} + z_{n_k}) - \inf_{m\ge M}(y_m + z_m)| < \frac{1}{2}t \\ |(y_{n_k} + z_{n_k}) - \underline{\lim}(y_n + z_n)| < \frac{1}{2}t \end{cases}$$

Solution-II

7 – 2

Proof (*continue*) :

We know $|(y_{n_k} + z_{n_k}) - \underline{\lim}(y_n - z_n)| < t$. This means for large enough n_k (at least $n_k > M$) we make sure that $(y_{n_k} + z_{n_k}) \le \underline{\lim}(y_n + z_n) + \frac{\epsilon}{2}$ and also

$$\underline{\lim} y_n + \underline{\lim} z_n - \frac{\epsilon}{2} \le (y_{n_k} + z_{n_k}) \le \underline{\lim}(y_n + z_n) + \frac{\epsilon}{2}$$

$$\Rightarrow \underline{\lim} y_n + \underline{\lim} z_n \le \underline{\lim}(y_n + z_n) + \epsilon$$

Since this is true for all $\epsilon > 0$, we conclude that

$$\underline{\lim} y_n + \underline{\lim} z_n \le \underline{\lim}(y_n + z_n)$$

Solution-II

7 – 2

Proof (*continue*) :

Now we prove the second inequality.

$$\underline{\lim}(y_n + z_n) \leq \overline{\lim} y_n + \underline{\lim} z_n$$

From the fact that

$$-\underline{\lim}(-x_n) = \overline{\lim} x_n$$

This equality holds because $-\inf_{m \geq n}(-x_m) = \sup_{m \geq n} x_m$. Now, $\underline{\lim}(y_n + z_n) + \underline{\lim}(-y_n) \leq \underline{\lim}(y_n + z_n + (-y_n)) = \underline{\lim} z_n$ by what we have proved above. Thus,

$$\underline{\lim}(y_n + z_n) \leq -\underline{\lim}(-y_n) + \underline{\lim} z_n = \overline{\lim} y_n + \underline{\lim} z_n$$

Solution-II

7 – 2

Proof (continue) :

Finally, we now prove the last inequality.

$$\overline{\lim} y_n + \underline{\lim} z_n \leq \overline{\lim} y_n + \overline{\lim} z_n$$

Since $a_n := \sup_{m\geq n} x_m \geq \inf_{m\geq n} x_m =: b_n$, we have

$$\underline{\lim} z_n \leq \overline{\lim} z_n$$

Therefore,

$$\overline{\lim} y_n + \underline{\lim} z_n \leq \overline{\lim} y_n + \overline{\lim} z_n$$

$\square$

8. Let (a_n) be a sequence such that

$$a_n = \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

Calculate the limit of (a_n).

Solution-II*

8.

We first consider two sequence $(b_n)_{n\geq 1}, (c_n)_{n\geq 1}$ given by

$$\begin{cases} b_n := \frac{1}{\sqrt{n^2}} + \cdots + \frac{1}{\sqrt{n^2}} \\ c_n := \frac{1}{\sqrt{N^2+n}} + \cdots + \frac{1}{\sqrt{N^2+n}} \end{cases}$$

Clearly, $\begin{cases} (b_n) \leq (a_n) \\ (c_n) \geq (a_n) \end{cases}$ for all $n \geq 1$.

Then one can easily find out the limit for both (b_n) and (c_n), which can be calculated as

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} = \lim_{n\to\infty} \frac{n}{n} = \lim_{n\to\infty} 1 = 1$$

8.(*continue*)

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$$

By Squeeze Theorem, we know

$$\lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} c_n$$

, namely

$$1 \leq \lim_{n\to\infty} a_n \leq 1$$

We conclude that $\lim_{n\to\infty} a_n = 1$ $\square$

9. Prove that $\lim \sqrt[n]{n} = 1$.

Solution-II*

9.

We consider the case for $n \geq 2$. Since $\sqrt[n]{n} > 1$ for all $n \geq 2$, we can choose a non-negative sequence, let say $(b_n)_{n \geq 2}$ such that

$$1 + b_n = \sqrt[n]{n}$$

From the fact that

$$(1 + b_n)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot b_n^{n-k}$$

and another fact that $(1 + b_n)^n = n$, we get the following inequality

$$n = (1 + b_n)^n \leq \binom{n}{2} b_n^2 = \frac{n(n-1)}{2} b_n^2$$

9.(*continue*)

After some algebraic works, one can see

$$b_n \leq \sqrt{\frac{2}{n-1}}$$

But this means

$$0 \leq \sqrt[n]{n} - 1 \leq \sqrt{\frac{2}{n-1}}$$

By taking the limit, we have

$$0 \leq \lim_{n\to\infty} (\sqrt[n]{n} - 1) \leq \lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$$

Solution-II*

9.(*continue*)

Finally, by Squeeze theorem, we get

$$\lim_{n\to\infty}(\sqrt[n]{n}-1)=\lim_{n\to\infty}\sqrt[n]{n}-1=0$$

We conclude

$$\lim_{n\to\infty}\sqrt[n]{n}=1$$

$\square$

Have Fun and Learn Well!