VV186 Mid 1 Big RC

Sequence II

"Find the problem, understand it, then explore more."

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Overview

1. Metric Space
2. Cauchy Sequence
3. Generalization of Convergence
4. Completeness
5. Construct Real Numbers
6. Exercise

Metric Space

We want to generalize the idea of *convergence*, and we want to define the most essential thing of convergence by ourselves, namely the **Length Function**.

What properties a usual length function should have?

1. Always positive.
2. Symmetric.
3. Followed *Triangle Inequality*.

Transform these into mathematical language. . .

Metric Space

A two variables functions

$$\rho(\cdot,\cdot) : M \times M \to \mathbb{R}$$

is called a metric if it satisfies:

1. $\forall x, y \in M,\ \rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$.
2. $\forall x, y \in M,\ \rho(x, y) = \rho(y, x)$.
3. $\forall x, y, z \in M,\ \rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

Examples

- $M = \mathbb{R}^n$, the usual metric is given by

$$\rho((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sqrt{\sum_{i=k}^{n} (x_i - y_i)^2}$$

  and this is so-called *Euclidean distance*.
- $M = \mathbb{N}, \rho(x, y) = \#\{a : a \in [min\{x, y\}, max\{x, y\}]\ \}$
- $M = \mathbb{R}, \rho(x, y) = 1$ if $x \neq y$; $\rho(x, y) = 0$ if $x = y$

Exercise :

Verify the above three functions are actually a metric.

Generalization of Convergence

Then, by replacing the usual matric $\rho(x, y) = |x - y|$ and choosing our universal set M, we get the natural definition for generalize convergence in metric space (M, ρ) for a sequence $(a_n) : \mathbb{N} \to M$, which is given by:

$$\lim_{n\to\infty} a_n = a \quad :\Leftrightarrow \quad \underset{\epsilon>0}{\forall} \ \underset{N\in\mathbb{N}}{\exists} \ \underset{n>\mathbb{N}}{\forall} a_n \in B_\epsilon(a)$$

where

$$B_\epsilon(a) = \{x \in M : \rho(x, a) < \epsilon\}, \quad \epsilon > 0, \quad a \in M.$$

Cauchy Sequence

- What is the definition of Cauchy Sequence?
- How to understand Cauchy Sequence?
- Why we want to introduce the idea of Cauchy Sequence?
- What new results can we explore from this new idea?

The fundamental reason why we want to introduce Cauchy Sequence is because we want to *further generalize* the idea of convergence.

Now, we are not only free to choose the metric we like, additionally, we remove the constraint which requires a sequence to converge to a *specific point*.

Cauchy Sequence

You should be familiar with the definition of a Cauchy sequence:

Definition. A sequence (a_n) in a metric space (M, ρ) is called a *Cauchy sequence* if

$$\underset{\epsilon>0}{\forall} \underset{N\in\mathbb{N}}{\exists} \underset{m,n>N}{\forall} \rho(a_m, a_n) < \epsilon$$

One can see that we define a Cauchy sequence without mentioned a *specific point* in our universal set M. This is the fundamental reason why Cauchy is important and interesting.

Cauchy Sequence

We list some important results and theorems for Cauchy Sequence.

- Every convergent sequence is a Cauchy sequence.
- Every Cauchy sequence in a metric space (M, ρ) is bounded.
- Every Cauchy sequence in $\mathbb{R}$ with the usual metric is convergent.

Exercise :

Give the outlines of the proof for these three theorem.

Completeness

The problem is, a Cauchy Sequence can simply not "*converge*" to anywhere anymore if we generalize the idea of convergence.

Then, after generalizing the idea of convergence, if **every** Cauchy sequence still converge, we say this metric space is *complete*.

We now take a look at some examples for a Cauchy sequence failed to converge.

Examples:

1. The space $([0,1),\rho)$, where $\rho(x,y)=|x-y|$.
2. The space $(\mathbb{Q},\rho)$, where $\rho(x,y)=|x-y|$.

We take a close look for the second example.
For a metric space $(\mathbb{Q},\rho)$ given by example 2, consider a sequence given by

$$(a_n)_{n\in\mathbb{N}} := (1,\ 1.4,\ 1.41,\ 1.414,\ \dots)$$

is a Cauchy sequence but *failed to converge in $\mathbb{Q}$* if we choose the following terms appropriately and let (a_n) *converges to $\sqrt{2}$ in $\mathbb{R}$* eventually. Clearly, (a_n) is not a convergent sequence in $(\mathbb{Q},\rho)$.

From here, it seems we somehow find a good way to *construct* Real Numbers.

Construct Real Numbers

The main idea to **complete** $\mathbb{Q}$ is to consider the following

Every Cauchy sequence can *represent* a number in $\mathbb{R}$.

For example, from the last slide, we can view the sequence

$$(a_n)_{n\in\mathbb{N}} = (1,\ 1.4,\ 1.41,\ 1.414,\ \dots)$$

as $\sqrt{2}$ by defining a *(equivalence) class* for each number in $\mathbb{R}$, and in this case, we define the sequence (a_n) as $\sqrt{2}$.

Then, from now on, the number defined through the Cauchy sequence which is given by

$$0.9, 0.99, 0.999, 0.9999, 0.99999$$

is equal to 1, namely $0.999999_{\dots} = 1$.

More about Cauchy Sequence

Since we do not make any constraint to our universal set because we are using simple set theory in default, we can collect any element in our universal set and define a **reasonable** metric to form a metric space.

Consider follows:

Define the universal set as

$$S := \{a_n : a_n \text{ is a sequence.}\}$$

Then, we can define a subset of S as

$$M := \{a^{(k)} \in S : a^{(k)} = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, 0, 0, 0, \dots)\}$$

clearly, $M \subset S$.

More about Cauchy Sequence

Then, we want to construct a metric space with the universal set M, which means we should construct a reasonable metric to measure the distance *between two sequence*.

Consider follows:

Define $\rho(\ a^{(n)},\ a^{(m)}\ )$ as

$$\rho(\ a^{(n)},\ a^{(m)}\ ) = \sup_{k\in\mathbb{N}} |(a^{(n)} - a^{(m)})_k|$$

For example,

$$a^{(3)} = (1, \frac{1}{2}, \frac{1}{3}, 0, 0, 0, \dots)$$

Then we will have

$$a_1^{(3)} = 1, \qquad a_2^{(3)} = \frac{1}{2}, \qquad a_3^{(3)} = \frac{1}{3}, \qquad a_4^{(3)} = 0, \dots$$

More about Cauchy Sequence

Furthermore, we define the subtraction between two sequence as *term by term* subtraction.
Without losing the generality, we suppose $m > n$, then we have

$$\begin{aligned}\rho(a^{(n)}, a^{(m)}) &= sup_{k\in\mathbb{N}^*}|(a^{(n)} - a^{(m)})_k| \\ &= \sup_{k\in\mathbb{N}^*}|(0, \dots, 0, \frac{1}{n+1}, \dots, \frac{1}{m}, \dots)_k| \\ &= \frac{1}{n+1}\end{aligned}$$

Clearly, as n, m getting bigger and bigger, $\rho(a^{(n)}, a^{(m)}) \to 0$.
Now, let us consider the sequence defined as

$$(A_n)_{n\in\mathbb{N}^*} := (a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)}, \dots)$$

More about Cauchy Sequence

Obviously, (A_n) is a Cauchy sequence since if n, m is big enough, $\rho(a^{(n)}, a^{(m)})$ will be small enough as shown in the last slide.

But let us take a step back to look at what A_n actually is. We list some terms for A_n below.

$$\begin{aligned} A_1 &= a^{(1)} = (1, 0, 0, 0, \dots) \\ A_2 &= a^{(2)} = (1, \frac{1}{2}, 0, 0, \dots) \\ A_3 &= a^{(3)} = (1, \frac{1}{2}, \frac{1}{3}, 0, \dots) \end{aligned}$$

The limit of this sequence clearly is

$$A := (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$$

More about Cauchy Sequence

Now, we see A_n's limit is A and A_n is a Cauchy sequence in our metric space (M, ρ). But the question is:

$$\text{Does } A \in M?$$

The answer is simply **no**.

Since in our set M, all elements (sequences) will have infinity many zeros in their last terms (tail).

But A has no zeros appeared in any entry of it, so clearly $A \notin M$, which means (A_n) does **not** converge in the metric space (M, ρ).

Hence, (M, ρ) can **not** be complete.

More about Cauchy Sequence

After digesting all the contents above, I think you really understand the concept of Cauchy sequence, metric space and the idea of *generalizing convergence*.

So now, let us do some exercises.

1. Prove or disprove the following statement:

   A Cauchy sequence in a metric space may have at most one accumulation point.

(This exercise is directly taken from last year's midterm!)

Solution

Since I would like to give you guy some taste for how you do a complete proof in an actual exam, so here is the solution.

Solution. The statement is true. Suppose that a Cauchy sequence (x_n) is given in a metric space (M, ρ) and that x and y are two accumulation points in M.(**1 Mark**) Choose $\epsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that $\rho(x_n, x_m) < \frac{\epsilon}{4}$ for all $n, m > N$.(**1 Mark**) Furthermore, there exist $n_0, m_0 > N$ such that $\rho(x.x_{n_0}) < \frac{\epsilon}{4}$ and $\rho(y, x_{m_0}) < \frac{\epsilon}{4}$.(**1 Mark**) Hence,

$$\begin{aligned} \rho(x, y) &\leq \rho(x, x_{n_0}) + \rho(x_{n_0}, y) \\ &\leq \rho(x, x_{n_0}) + \rho(x_{n_0}, x_{m_0}) + \rho(x_{m_0}, x_{n_0}) + \rho(x_{m_0}, y) \\ &< \epsilon \end{aligned}$$

Exercise

2. Please construct a metric that makes $\mathbb{R}$ incomplete with regard to this metric. You can use the method given in Horst's slides or one in mine.

Exercise

3. A sequence is defined as

$$(S_n)_{n\in\mathbb{N}}, S_1 = \sqrt{2}, S_2 = \sqrt{2\sqrt{2}}, S_3 = \sqrt{2\sqrt{2\sqrt{2}}}$$

Please calculate the limit of (S_n) as $n \to \infty$, if it exists.

4. Prove that $\lim \sqrt[n]{n} = 1$.

5. Let (x_n) be a bounded real sequence. Then define

$$a_n := \sup_{m\geq n}(x_m), \quad b_n := \inf_{m\geq n}(x_m)$$

1. Prove that (a_n) is decreasing, while (b_n) is increasing.
2. Since both $(a_n), (b_n)$ are monotonic and bounded, they are convergent. We denote $\underline{\lim} x_n = \lim b_n$; $\overline{\lim} x_n = \lim a_n$. Show that:

$$\underline{\lim} y_n + \underline{\lim} z_n \leq \underline{\lim}(y_n + z_n) \leq \overline{\lim} y_n + \underline{\lim} z_n \leq \overline{\lim} y_n + \overline{\lim} z_n$$

6. Let (a_n) be a sequence such that

$$a_n = \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

Calculate the limit of (a_n).

Solution

Since all the exercise is taken from my RC-II slides, and I have made all the solutions for them, you can refer to the solution of RC-II in canvas.

Some of them are quite easy, and some of them aren't. Just take them easy. If you can't figure some exercises out in the first time, that's *totally normal*.

Just think about the structure of the solution, and then take a look of the solution to get some ideas.

Hope you all get a good grades!

Have Fun and Learn Well!