

VV285 RC Final

Final Exam Exercises

Last but not least...

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Exercise 1

Calculate the directional derivative of the continuous function

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = \sqrt[3]{x^2 y}$$

at $(x, y) = (0, 0)$ in the direction of the vector $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Hence, we have

$$f(x_0 + th) = f\left(t \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right) = \sqrt[3]{\frac{t^2}{2} \cdot \frac{t}{\sqrt{2}}} = \frac{t}{\sqrt{2}}$$

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so that

$$\frac{d}{dt} f(x_0 + th) = \frac{d}{dt} \frac{t}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$



Exercise 2

Let $g : (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function and let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Prove that the vector field

$$F : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}^3, \quad F(x) = g(\|x\|)x$$

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$$|(\text{rot} F)_i| = \left| \frac{\partial F_j}{\partial x_k} - \frac{\partial F_k}{\partial x_j} \right|$$

where (i, j, k) is any one of the permutation of $\{1, 2, 3\}$.

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We then have

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Hence, the rotation of F vanishes everywhere on $\mathbb{R}^3 \setminus \{0\}$.



Exercise 3

Let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Find the flux of the vector field

$$F : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}^3, \quad F(x) = \frac{x}{\|x\|^3}$$

through the surface

$$\mathcal{S} = \{x \in \mathbb{R}^3 : 4x_1^2 + 9x_2^2 + 6x_3^2 = 36\}$$

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$$\varphi(\phi, \theta) = \begin{pmatrix} \epsilon \cos \phi \sin \theta \\ \epsilon \sin \phi \sin \theta \\ \epsilon \cos \theta \end{pmatrix}, \quad t_\theta \times t_\phi = \begin{pmatrix} \epsilon^2 \cos \phi \sin^2 \theta \\ \epsilon^2 \sin \phi \sin^2 \theta \\ \epsilon^2 \cos \theta \sin \theta \end{pmatrix}$$

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$$\int_{\partial B_\epsilon(0)} \langle F, d\vec{A} \rangle = \int_0^{2\pi} \int_0^\pi \frac{1}{\|\varphi(\phi, \theta)\|^3} \langle \varphi(\phi, \theta), t_\phi \times t_\theta(\phi, \theta) \rangle d\theta d\phi$$

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Exercise 4

Let $\Omega \subset \mathbb{R}^n$ be an open set and $R \subset \Omega$ an admissible region. Let $u : \Omega \rightarrow \mathbb{R}$ be a twice continuously differentiable function such that

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \lambda u \text{ on } \text{int}R \quad \text{and} \quad u|_{\partial R} = 0$$

for some $\lambda \in \mathbb{R}$. Suppose that $u \neq 0$, i.e., $u(x) \neq 0$ for some $x \in R$. Prove that $\lambda < 0$.

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We multiply both side of the equation with u

$$u\Delta u = \lambda u^2$$

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We multiply both side of the equation with u and integrate over Ω to obtain

$$\int_{\Omega} u \Delta u \, dx = \lambda \int_{\Omega} u^2 \, dx.$$

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We multiply both side of the equation with u and integrate over Ω to obtain

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Since u is not identically zero and u is continuous, the integral on the right is non-zero and we can divide

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Since u is not identically zero and u is continuous, the integral on the right is non-zero and we can divide, yielding

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Applying Green's first identity

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Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$, we obtain

$$\lambda = \frac{-\int_{\Omega} (\nabla u)^2 \, dx}{\int_{\Omega} u^2 \, dx}.$$

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which implies $\nabla u(x) = 0$ for all x , i.e.

Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$, we obtain

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Since both integrands on the right are greater or equal to zero, we see that $\lambda \leq 0$. Furthermore, $\lambda = 0$ only if

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$$\lambda < 0.$$

