VV285 RC Final

Final Exam Exercises

Last but not least...

Pingbang Hu

University of Michigan-Shanghai Jiao Tong University Joint Institute

July 30, 2021

Exercise 1

Calculate the directional derivative of the continuous function

$$f : \mathbb{R}^2 \to R, \qquad f(x, y) = \sqrt[3]{x^2 y}$$

at $(x, y) = (0, 0)$ in the direction of the vector $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Solution 1

Calculate the directional derivative of the continuous function

$$f : \mathbb{R}^2 \to R, \qquad f(x, y) = \sqrt[3]{x^2 y}$$

at $(x, y) = (0, 0)$ in the direction of the vector $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

From the definition

Solution 1

Calculate the directional derivative of the continuous function

$$f : \mathbb{R}^2 \to R, \qquad f(x,y) = \sqrt[3]{x^2 y}$$

at $(x,y) = (0,0)$ in the direction of the vector $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

From the definition , we need to calculate

$$\frac{d}{dt} f(x_0 + th)\big|_{t=0}$$

where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Calculate the directional derivative of the continuous function

$$f : \mathbb{R}^2 \to R, \qquad f(x, y) = \sqrt[3]{x^2 y}$$

at $(x, y) = (0, 0)$ in the direction of the vector $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

From the definition , we need to calculate

$$\frac{d}{dt}\, f(x_0 + th)\big|_{t=0}$$

where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Hence, we have

$$f(x_0 + th) = f(t \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1\sqrt{2} \end{pmatrix}) = \sqrt[3]{\frac{t^2}{2} \cdot \frac{t}{\sqrt{2}}} = \frac{t}{\sqrt{2}}$$

so that

Calculate the directional derivative of the continuous function

$$f : \mathbb{R}^2 \to R, \qquad f(x, y) = \sqrt[3]{x^2 y}$$

at $(x, y) = (0, 0)$ in the direction of the vector $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

From the definition , we need to calculate

$$\frac{d}{dt}\, f(x_0 + th)|_{t=0}$$

where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Hence, we have

$$f(x_0 + th) = f(t \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1\sqrt{2} \end{pmatrix}) = \sqrt[3]{\frac{t^2}{2} \cdot \frac{t}{\sqrt{2}}} = \frac{t}{\sqrt{2}}$$

so that

$$\frac{d}{dt} f(x_0 + th) = \frac{d}{dt} \frac{t}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

■

Exercise 2

Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.
Prove that the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = g(\|x\|)x$$

is conservative.

Solution 2

Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.
Prove that the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = g(\|x\|)x$$

is conservative.

The set $\mathbb{R}^3 \backslash \{0\}$ is simply connected

Solution 2

Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.
Prove that the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = g(\|x\|)x$$

is conservative.

The set $\mathbb{R}^3 \backslash \{0\}$ is simply connected , so it suffices to show that $\text{rot} F = 0$.

Solution 2

Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.
Prove that the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = g(\|x\|)x$$

is conservative.

The set $\mathbb{R}^3 \backslash \{0\}$ is simply connected , so it suffices to show that $\operatorname{rot} F = 0$. Now

$$|(\operatorname{rot} F)_i| = \left| \frac{\partial F_j}{\partial x_k} - \frac{\partial F_k}{\partial x_j} \right|$$

where (i, j, k) is any one of the permutation of $\{1, 2, 3\}$.

Solution 2

We then have

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial}{\partial x_k} \left(g(\|x\|)x\right)_j$$

Solution 2

We then have

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial}{\partial x_k}\left(g(\|x\|)x\right)_j = \frac{\partial}{\partial x_k} g(\|x\|)x_j$$

Solution 2

We then have

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial}{\partial x_k}\left(g(\|x\|)x\right)_j = \frac{\partial}{\partial x_k} g(\|x\|)x_j$$

$$= x_j g'(\|x\|)\frac{\partial}{\partial x_k}\sqrt{x_1^2 + x_2^2 + x_3^2}$$

We then have

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial}{\partial x_k}\left(g(\|x\|)x\right)_j = \frac{\partial}{\partial x_k} g(\|x\|)x_j$$

$$= x_j g'(\|x\|)\frac{\partial}{\partial x_k}\sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$= x_j g'(\|x\|)\frac{x_k}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

We then have

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial}{\partial x_k}\left(g(\|x\|)x\right)_j = \frac{\partial}{\partial x_k} g(\|x\|)x_j$$

$$= x_j g'(\|x\|)\frac{\partial}{\partial x_k}\sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$= x_j g'(\|x\|)\frac{x_k}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$= x_j x_k \frac{g'(\|x\|)}{\|x\|}$$

Solution 2

We then have

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial}{\partial x_k} \left(g(\|x\|)x\right)_j = \frac{\partial}{\partial x_k} g(\|x\|)x_j$$

$$= x_j g'(\|x\|) \frac{\partial}{\partial x_k} \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$= x_j g'(\|x\|) \frac{x_k}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$= x_j x_k \frac{g'(\|x\|)}{\|x\|} = \frac{\partial F_k}{\partial x_j}$$

Solution 2

We then have

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial}{\partial x_k}\left(g(\|x\|)x\right)_j = \frac{\partial}{\partial x_k} g(\|x\|)x_j$$

$$= x_j g'(\|x\|)\frac{\partial}{\partial x_k}\sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$= x_j g'(\|x\|)\frac{x_k}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$= x_j x_k \frac{g'(\|x\|)}{\|x\|} = \frac{\partial F_k}{\partial x_j}$$

Hence, the rotation of F vanishes everywhere on $\mathbb{R}^3 \setminus \{0\}$. ∎

Exercise 3

Let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Find the flux of the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = \frac{x}{\|x\|^3}$$

through the surface

$$\mathcal{S} = \{x \in \mathbb{R}^3 : 4x_1^2 + 9x_2^2 + 6x_3^2 = 36\}$$

by using a suitable parametrization.

Solution 3

Let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Find the flux of the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = \frac{x}{\|x\|^3}$$

through the surface

$$\mathcal{S} = \{x \in \mathbb{R}^3 : 4x_1^2 + 9x_2^2 + 6x_3^2 = 36\}$$

by using a suitable parametrization.

We note that

$$\frac{\partial F_i}{\partial x_i}$$

Solution 3

Let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Find the flux of the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = \frac{x}{\|x\|^3}$$

through the surface

$$\mathcal{S} = \{x \in \mathbb{R}^3 : 4x_1^2 + 9x_2^2 + 6x_3^2 = 36\}$$

by using a suitable parametrization.

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}$$

Solution 3

Let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Find the flux of the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = \frac{x}{\|x\|^3}$$

through the surface

$$\mathcal{S} = \{x \in \mathbb{R}^3 : 4x_1^2 + 9x_2^2 + 6x_3^2 = 36\}$$

by using a suitable parametrization.

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \frac{(x_1^2 + x_2^2 + x_3^2)^{3/2} - 3x_i^2 (x_1^2 + x_2^2 + x_3^2)^{1/2}}{(x_1^2 + x_2^2 + x_3^2)^3}$$

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{x_i}{(x_1^2+x_2^2+x_3^2)^{3/2}} = \frac{(x_1^2+x_2^2+x_3^2)^{3/2} - 3x_i^2(x_1^2+x_2^2+x_3^2)^{1/2}}{(x_1^2+x_2^2+x_3^2)^3}$$

Solution 3

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \frac{(x_1^2 + x_2^2 + x_3^2)^{3/2} - 3x_i^2(x_1^2 + x_2^2 + x_3^2)^{1/2}}{(x_1^2 + x_2^2 + x_3^2)^3}$$

so

$$\text{div} F(x) =$$

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \frac{(x_1^2 + x_2^2 + x_3^2)^{3/2} - 3x_i^2(x_1^2 + x_2^2 + x_3^2)^{1/2}}{(x_1^2 + x_2^2 + x_3^2)^3}$$

so

$$\mathrm{div} F(x) = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i}$$

Solution 3

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{x_i}{(x_1^2+x_2^2+x_3^2)^{3/2}} = \frac{(x_1^2+x_2^2+x_3^2)^{3/2} - 3x_i^2(x_1^2+x_2^2+x_3^2)^{1/2}}{(x_1^2+x_2^2+x_3^2)^3}$$

so

$$\mathrm{div}F(x) = \sum_{i=1}^{3}\frac{\partial F_i}{\partial x_i} = 0$$

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{x_i}{(x_1^2+x_2^2+x_3^2)^{3/2}} = \frac{(x_1^2+x_2^2+x_3^2)^{3/2} - 3x_i^2(x_1^2+x_2^2+x_3^2)^{1/2}}{(x_1^2+x_2^2+x_3^2)^3}$$

so

$$\operatorname{div} F(x) = \sum_{i=1}^{3}\frac{\partial F_i}{\partial x_i} = 0$$

for

Solution 3

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \frac{(x_1^2 + x_2^2 + x_3^2)^{3/2} - 3x_i^2(x_1^2 + x_2^2 + x_3^2)^{1/2}}{(x_1^2 + x_2^2 + x_3^2)^3}$$

so

$$\mathrm{div} F(x) = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i} = 0$$

for $x \neq 0$.

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \frac{(x_1^2 + x_2^2 + x_3^2)^{3/2} - 3x_i^2 (x_1^2 + x_2^2 + x_3^2)^{1/2}}{(x_1^2 + x_2^2 + x_3^2)^3}$$

so

$$\mathrm{div} F(x) = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i} = 0$$

for $x \neq 0$. Hence, it is sufficient to calculate the flux through the boundary of an

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \frac{(x_1^2 + x_2^2 + x_3^2)^{3/2} - 3x_i^2(x_1^2 + x_2^2 + x_3^2)^{1/2}}{(x_1^2 + x_2^2 + x_3^2)^3}$$

so

$$\mathrm{div} F(x) = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i} = 0$$

for $x \neq 0$. Hence, it is sufficient to calculate the flux through the boundary of an **arbitrary ball** $B_\epsilon(0)$ of radius $\epsilon > 0$ centered at the origin. We can parametrize $\partial B_\epsilon(0)$ by

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{x_i}{(x_1^2+x_2^2+x_3^2)^{3/2}} = \frac{(x_1^2+x_2^2+x_3^2)^{3/2} - 3x_i^2(x_1^2+x_2^2+x_3^2)^{1/2}}{(x_1^2+x_2^2+x_3^2)^3}$$

so

$$\mathrm{div}F(x) = \sum_{i=1}^{3}\frac{\partial F_i}{\partial x_i} = 0$$

for $x \neq 0$. Hence, it is sufficient to calculate the flux through the boundary of an **arbitrary ball** $B_\epsilon(0)$ of radius $\epsilon > 0$ centered at the origin. We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi,\theta) = \begin{pmatrix} \epsilon\cos\phi\sin\theta \\ \epsilon\sin\phi\sin\theta \\ \epsilon\cos\theta \end{pmatrix},$$

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{x_i}{(x_1^2+x_2^2+x_3^2)^{3/2}} = \frac{(x_1^2+x_2^2+x_3^2)^{3/2} - 3x_i^2(x_1^2+x_2^2+x_3^2)^{1/2}}{(x_1^2+x_2^2+x_3^2)^3}$$

so

$$\operatorname{div} F(x) = \sum_{i=1}^{3}\frac{\partial F_i}{\partial x_i} = 0$$

for $x \neq 0$. Hence, it is sufficient to calculate the flux through the boundary of an **arbitrary ball** $B_\epsilon(0)$ of radius $\epsilon > 0$ centered at the origin. We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi,\theta) = \begin{pmatrix} \epsilon\cos\phi\sin\theta \\ \epsilon\sin\phi\sin\theta \\ \epsilon\cos\theta \end{pmatrix}, \qquad t_\theta \times t_\phi = \begin{pmatrix} \epsilon^2\cos\phi\sin^2\theta \\ \epsilon^2\sin\phi\sin^2\theta \\ \epsilon^2\cos\theta\sin\theta \end{pmatrix}$$

Solution 3

We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \frac{(x_1^2 + x_2^2 + x_3^2)^{3/2} - 3x_i^2(x_1^2 + x_2^2 + x_3^2)^{1/2}}{(x_1^2 + x_2^2 + x_3^2)^3}$$

so

$$\mathrm{div} F(x) = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i} = 0$$

for $x \neq 0$. Hence, it is sufficient to calculate the flux through the boundary of an **arbitrary ball** $B_\epsilon(0)$ of radius $\epsilon > 0$ centered at the origin. We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi, \theta) = \begin{pmatrix} \epsilon \cos\phi \sin\theta \\ \epsilon \sin\phi \sin\theta \\ \epsilon \cos\theta \end{pmatrix}, \qquad t_\theta \times t_\phi = \begin{pmatrix} \epsilon^2 \cos\phi \sin^2\theta \\ \epsilon^2 \sin\phi \sin^2\theta \\ \epsilon^2 \cos\theta \sin\theta \end{pmatrix}$$

where $0 < \phi < 2\pi$ and $0 < \theta < \pi$. We have chosen the outward-pointing (positively oriented) normal vector.

Solution 3

We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi,\theta) = \begin{pmatrix} \epsilon\cos\phi\sin\theta \\ \epsilon\sin\phi\sin\theta \\ \epsilon\cos\theta \end{pmatrix}, \qquad t_\theta \times t_\phi = \begin{pmatrix} \epsilon^2\cos\phi\sin^2\theta \\ \epsilon^2\sin\phi\sin^2\theta \\ \epsilon^2\cos\theta\sin\theta \end{pmatrix}$$

where $0 < \phi < 2\pi$ and $0 < \theta < \pi$. We have chosen the outward-pointing (positively oriented) normal vector.

We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi,\theta) = \begin{pmatrix} \epsilon\cos\phi\sin\theta \\ \epsilon\sin\phi\sin\theta \\ \epsilon\cos\theta \end{pmatrix}, \qquad t_\theta \times t_\phi = \begin{pmatrix} \epsilon^2\cos\phi\sin^2\theta \\ \epsilon^2\sin\phi\sin^2\theta \\ \epsilon^2\cos\theta\sin\theta \end{pmatrix}$$

where $0 < \phi < 2\pi$ and $0 < \theta < \pi$. We have chosen the outward-pointing (positively oriented) normal vector. Then the flux through $\mathcal{S}$ is

Solution 3

We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi,\theta)=\begin{pmatrix}\epsilon\cos\phi\sin\theta\\ \epsilon\sin\phi\sin\theta\\ \epsilon\cos\theta\end{pmatrix},\qquad t_\theta\times t_\phi=\begin{pmatrix}\epsilon^2\cos\phi\sin^2\theta\\ \epsilon^2\sin\phi\sin^2\theta\\ \epsilon^2\cos\theta\sin\theta\end{pmatrix}$$

where $0<\phi<2\pi$ and $0<\theta<\pi$. We have chosen the outward-pointing (positively oriented) normal vector. Then the flux through $\mathcal{S}$ is

$$\int_{\partial B_\epsilon(0)}\left\langle F, d\overrightarrow{A}\right\rangle$$

Solution 3

We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi,\theta)=\begin{pmatrix}\epsilon\cos\phi\sin\theta\\ \epsilon\sin\phi\sin\theta\\ \epsilon\cos\theta\end{pmatrix},\qquad t_\theta\times t_\phi=\begin{pmatrix}\epsilon^2\cos\phi\sin^2\theta\\ \epsilon^2\sin\phi\sin^2\theta\\ \epsilon^2\cos\theta\sin\theta\end{pmatrix}$$

where $0<\phi<2\pi$ and $0<\theta<\pi$. We have chosen the outward-pointing (positively oriented) normal vector. Then the flux through $\mathcal{S}$ is

$$\int_{\partial B_\epsilon(0)}\left\langle F, d\overrightarrow{A}\right\rangle=\int_0^{2\pi}\int_0^{\pi}\frac{1}{\|\varphi(\phi,\theta)^3\|}\left\langle\varphi(\phi,\theta),t_\phi\times t_\theta(\phi,\theta)\right\rangle\,d\theta\,d\phi$$

Solution 3

We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi,\theta) = \begin{pmatrix} \epsilon\cos\phi\sin\theta \\ \epsilon\sin\phi\sin\theta \\ \epsilon\cos\theta \end{pmatrix}, \qquad t_\theta \times t_\phi = \begin{pmatrix} \epsilon^2\cos\phi\sin^2\theta \\ \epsilon^2\sin\phi\sin^2\theta \\ \epsilon^2\cos\theta\sin\theta \end{pmatrix}$$

where $0 < \phi < 2\pi$ and $0 < \theta < \pi$. We have chosen the outward-pointing (positively oriented) normal vector. Then the flux through $\mathcal{S}$ is

$$\int_{\partial B_\epsilon(0)} \left\langle F, d\vec{A} \right\rangle = \int_0^{2\pi}\int_0^{\pi} \underbrace{\frac{1}{\|\varphi(\phi,\theta)^3\|}}_{=1/\epsilon^3} \langle \varphi(\phi,\theta), t_\phi \times t_\theta(\phi,\theta)\rangle \, d\theta\, d\phi$$

We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi,\theta) = \begin{pmatrix} \epsilon\cos\phi\sin\theta \\ \epsilon\sin\phi\sin\theta \\ \epsilon\cos\theta \end{pmatrix}, \qquad t_\theta \times t_\phi = \begin{pmatrix} \epsilon^2\cos\phi\sin^2\theta \\ \epsilon^2\sin\phi\sin^2\theta \\ \epsilon^2\cos\theta\sin\theta \end{pmatrix}$$

where $0 < \phi < 2\pi$ and $0 < \theta < \pi$. We have chosen the outward-pointing (positively oriented) normal vector. Then the flux through $\mathcal{S}$ is

$$\int_{\partial B_\epsilon(0)} \left\langle F, d\overrightarrow{A} \right\rangle = \int_0^{2\pi}\int_0^{\pi} \underbrace{\frac{1}{\|\varphi(\phi,\theta)^3\|}}_{=1/\epsilon^3} \langle \varphi(\phi,\theta), t_\phi \times t_\theta(\phi,\theta)\rangle \, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\epsilon^3}{\epsilon^3} \left\langle \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\phi\sin^2\theta \\ \sin\phi\sin^2\theta \\ \cos\theta\sin\theta \end{pmatrix} \right\rangle d\theta\, d\phi$$

We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi,\theta)=\begin{pmatrix}\epsilon\cos\phi\sin\theta\\ \epsilon\sin\phi\sin\theta\\ \epsilon\cos\theta\end{pmatrix},\qquad t_\theta\times t_\phi=\begin{pmatrix}\epsilon^2\cos\phi\sin^2\theta\\ \epsilon^2\sin\phi\sin^2\theta\\ \epsilon^2\cos\theta\sin\theta\end{pmatrix}$$

where $0<\phi<2\pi$ and $0<\theta<\pi$. We have chosen the outward-pointing (positively oriented) normal vector. Then the flux through $\mathcal{S}$ is

$$\begin{aligned}\int_{\partial B_\epsilon(0)}\left\langle F,d\overrightarrow{A}\right\rangle &=\int_0^{2\pi}\int_0^{\pi}\underbrace{\frac{1}{\|\varphi(\phi,\theta)^3\|}}_{=1/\epsilon^3}\langle\varphi(\phi,\theta),t_\phi\times t_\theta(\phi,\theta)\rangle\,d\theta\,d\phi\\ &=\int_0^{2\pi}\int_0^{\pi}\frac{\epsilon^3}{\epsilon^3}\left\langle\begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix},\begin{pmatrix}\cos\phi\sin^2\theta\\ \sin\phi\sin^2\theta\\ \cos\theta\sin\theta\end{pmatrix}\right\rangle d\theta\,d\phi\\ &=\int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi\end{aligned}$$

Solution 3

We can parametrize $\partial B_\epsilon(0)$ by

$$\varphi(\phi,\theta)=\begin{pmatrix}\epsilon\cos\phi\sin\theta\\ \epsilon\sin\phi\sin\theta\\ \epsilon\cos\theta\end{pmatrix},\qquad t_\theta\times t_\phi=\begin{pmatrix}\epsilon^2\cos\phi\sin^2\theta\\ \epsilon^2\sin\phi\sin^2\theta\\ \epsilon^2\cos\theta\sin\theta\end{pmatrix}$$

where $0<\phi<2\pi$ and $0<\theta<\pi$. We have chosen the outward-pointing (positively oriented) normal vector. Then the flux through $\mathcal{S}$ is

$$\int_{\partial B_\epsilon(0)}\left\langle F, d\vec{A}\right\rangle=\int_0^{2\pi}\int_0^{\pi}\underbrace{\frac{1}{\|\varphi(\phi,\theta)^3\|}}_{=1/\epsilon^3}\langle\varphi(\phi,\theta),t_\phi\times t_\theta(\phi,\theta)\rangle\;d\theta\,d\phi$$

$$=\int_0^{2\pi}\int_0^{\pi}\frac{\epsilon^3}{\epsilon^3}\left\langle\begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix},\begin{pmatrix}\cos\phi\sin^2\theta\\ \sin\phi\sin^2\theta\\ \cos\theta\sin\theta\end{pmatrix}\right\rangle\,d\theta\,d\phi$$

$$=\int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi=4\pi$$

■

Exercise 4

Let $\Omega \subset \mathbb{R}^n$ be an open set and $R \subset \Omega$ an admissible region. Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function such that

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \lambda u \text{ on } \operatorname{int} R \qquad \text{and} \qquad u|_{\partial R} = 0$$

for some $\lambda \in \mathbb{R}$. Suppose that $u \neq 0$, i.e., $u(x) \neq 0$ for some $x \in R$. Prove that $\lambda < 0$.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $R \subset \Omega$ an admissible region. Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function such that

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \lambda u \text{ on int}R \qquad \text{and} \qquad u|_{\partial R} = 0$$

for some $\lambda \in \mathbb{R}$. Suppose that $u \neq 0$, i.e., $u(x) \neq 0$ for some $x \in R$. Prove that $\lambda < 0$.

We multiply both side of the equation with u

$$u\Delta u \quad = \quad \lambda u^2 \quad .$$

Solution 4

Let $\Omega \subset \mathbb{R}^n$ be an open set and $R \subset \Omega$ an admissible region. Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function such that

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \lambda u \text{ on } \operatorname{int} R \qquad \text{and} \qquad u|_{\partial R} = 0$$

for some $\lambda \in \mathbb{R}$. Suppose that $u \neq 0$, i.e., $u(x) \neq 0$ for some $x \in R$. Prove that $\lambda < 0$.

We multiply both side of the equation with u and integrate over Ω to obtain

$$\int_\Omega u \Delta u \, dx = \int_\Omega \lambda u^2 \, dx.$$

Solution 4

Let $\Omega \subset \mathbb{R}^n$ be an open set and $R \subset \Omega$ an admissible region. Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function such that

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \lambda u \text{ on int} R \qquad \text{and} \qquad u|_{\partial R} = 0$$

for some $\lambda \in \mathbb{R}$. Suppose that $u \neq 0$, i.e., $u(x) \neq 0$ for some $x \in R$. Prove that $\lambda < 0$.

We multiply both side of the equation with u and integrate over Ω to obtain

$$\int_\Omega u \Delta u \, dx = \int_\Omega \lambda u^2 \, dx.$$

Since u is not identically zero and u is continuous, the integral on the right is non-zero and we can divide

Solution 4

Let $\Omega \subset \mathbb{R}^n$ be an open set and $R \subset \Omega$ an admissible region. Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function such that

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \lambda u \text{ on int} R \qquad \text{and} \qquad u|_{\partial R} = 0$$

for some $\lambda \in \mathbb{R}$. Suppose that $u \neq 0$, i.e., $u(x) \neq 0$ for some $x \in R$. Prove that $\lambda < 0$.

We multiply both side of the equation with u and integrate over Ω to obtain

$$\int_\Omega u \Delta u \, dx = \int_\Omega \lambda u^2 \, dx.$$

Since u is not identically zero and u is continuous, the integral on the right is non-zero and we can divide, yielding

$$\lambda = \frac{\int_\Omega u \Delta u \, dx}{\int_\Omega u^2 \, dx}.$$

Applying Green's first identity

Solution 4

Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$

Solution 4

Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$, we obtain

$$\lambda = \frac{-\int_\Omega (\nabla u)^2 \, dx}{\int_\Omega u^2 \, dx}.$$

Solution 4

Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$, we obtain

$$\lambda = \frac{-\int_\Omega (\nabla u)^2 \, dx}{\int_\Omega u^2 \, dx}.$$

Since both integrands on the right are greater or equal to zero, we see that $\lambda \leq 0$.

Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$, we obtain

$$\lambda = \frac{-\int_\Omega (\nabla u)^2\, dx}{\int_\Omega u^2\, dx}.$$

Since both integrands on the right are greater or equal to zero, we see that $\lambda \leq 0$.
Furthermore, $\lambda = 0$ only if

Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$, we obtain

$$\lambda = \frac{-\int_\Omega (\nabla u)^2 \, dx}{\int_\Omega u^2 \, dx}.$$

Since both integrands on the right are greater or equal to zero, we see that $\lambda \leq 0$.
Furthermore, $\lambda = 0$ only if

$$\int_\Omega (\nabla u)^2 \, dx = 0$$

which implies $\nabla u(x) = 0$ for all x, i.e.

Solution 4

Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$, we obtain

$$\lambda = \frac{-\int_\Omega (\nabla u)^2 \, dx}{\int_\Omega u^2 \, dx}.$$

Since both integrands on the right are greater or equal to zero, we see that $\lambda \leq 0$. Furthermore, $\lambda = 0$ only if

$$\int_\Omega (\nabla u)^2 \, dx = 0$$

which implies $\nabla u(x) = 0$ for all x, i.e., u is constant.

Solution 4

Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$, we obtain

$$\lambda = \frac{-\int_\Omega (\nabla u)^2 \, dx}{\int_\Omega u^2 \, dx}.$$

Since both integrands on the right are greater or equal to zero, we see that $\lambda \le 0$.
Furthermore, $\lambda = 0$ only if

$$\int_\Omega (\nabla u)^2 \, dx = 0$$

which implies $\nabla u(x) = 0$ for all x, i.e., u is constant. Since $u|_{\partial\Omega} = 0$

Solution 4

Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$, we obtain

$$\lambda = \frac{-\int_\Omega (\nabla u)^2 \, dx}{\int_\Omega u^2 \, dx}.$$

Since both integrands on the right are greater or equal to zero, we see that $\lambda \leq 0$.
Furthermore, $\lambda = 0$ only if

$$\int_\Omega (\nabla u)^2 \, dx = 0$$

which implies $\nabla u(x) = 0$ for all x, i.e., u is constant. Since $u|_{\partial\Omega} = 0$, this would mean $u(x) = 0$ for all x, which we have excluded.

Solution 4

Applying Green's first identity and using the fact that $u|_{\partial\Omega} = 0$, we obtain

$$\lambda = \frac{-\int_\Omega (\nabla u)^2 \, dx}{\int_\Omega u^2 \, dx}.$$

Since both integrands on the right are greater or equal to zero, we see that $\lambda \leq 0$.
Furthermore, $\lambda = 0$ only if

$$\int_\Omega (\nabla u)^2 \, dx = 0$$

which implies $\nabla u(x) = 0$ for all x, i.e., u is constant. Since $u|_{\partial\Omega} = 0$, this would mean $u(x) = 0$ for all x, which we have excluded. Therefore, $\lambda = 0$ is impossible and we conclude that

$$\lambda < 0.$$

■