VV285 RC Final

Final Exercise

Last but not least...

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Exercise 1

Calculate the directional derivative of the continuous function

$$f : \mathbb{R}^2 \to R, \qquad f(x, y) = \sqrt[3]{x^2 y}$$

at $(x, y) = (0, 0)$ in the direction of the vector $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Hence, we have

$$f(x_0 + th) = f(t \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1\sqrt{2} \end{pmatrix}) = \sqrt[3]{\frac{t^2}{2} \cdot \frac{t}{\sqrt{2}}} = \frac{t}{\sqrt{2}}$$

so that

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so that

$$\frac{d}{dt} f(x_0 + th) = \frac{d}{dt}\frac{t}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Exercise 2

Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.
Prove that the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = g(\|x\|)x$$

is conservative.

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$$|(\mathrm{rot} F)_i| = \left| \frac{\partial F_j}{\partial x_k} - \frac{\partial F_k}{\partial x_j} \right|$$

where (i, j, k) is any one of the permutation of $\{1, 2, 3\}$.

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Hence, the rotation of F vanishes everywhere on $\mathbb{R}^3 \setminus \{0\}$. ■

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Let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Find the flux of the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = \frac{x}{\|x\|^3}$$

through the surface

$$\mathcal{S} = \{x \in \mathbb{R}^3 : 4x_1^2 + 9x_2^2 + 6x_3^2 = 36\}$$

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We note that

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial}{\partial x_i} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} = \frac{(x_1^2 + x_2^2 + x_3^2)^{3/2} - 3x_i^2 (x_1^2 + x_2^2 + x_3^2)^{1/2}}{(x_1^2 + x_2^2 + x_3^2)^3}$$

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$$\varphi(\phi, \theta) = \begin{pmatrix} \epsilon \cos\phi \sin\theta \\ \epsilon \sin\phi \sin\theta \\ \epsilon \cos\theta \end{pmatrix}, \qquad t_\theta \times t_\phi = \begin{pmatrix} \epsilon^2 \cos\phi \sin^2\theta \\ \epsilon^2 \sin\phi \sin^2\theta \\ \epsilon^2 \cos\theta \sin\theta \end{pmatrix}$$

where $0 < \phi < 2\pi$ and $0 < \theta < \pi$.

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where $0<\phi<2\pi$ and $0<\theta<\pi$. We have chosen the outward-pointing (positively oriented) normal vector. Then the flux through $\mathcal{S}$ is

$$\int_{\partial B_\epsilon(0)}\left\langle F, d\overrightarrow{A}\right\rangle=\int_0^{2\pi}\int_0^{\pi}\frac{1}{\|\varphi(\phi,\theta)^3\|}\left\langle\varphi(\phi,\theta), t_\phi\times t_\theta(\phi,\theta)\right\rangle\, d\theta\, d\phi$$

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$$\int_{\partial B_\epsilon(0)} \left\langle F, d\overrightarrow{A} \right\rangle = \int_0^{2\pi} \int_0^{\pi} \underbrace{\frac{1}{\|\varphi(\phi,\theta)^3\|}}_{=1/\epsilon^3} \langle \varphi(\phi,\theta), t_\phi \times t_\theta(\phi,\theta) \rangle \; d\theta \, d\phi$$

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$$=\int_0^{2\pi}\int_0^{\pi}\frac{\epsilon^3}{\epsilon^3}\left\langle\begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix},\begin{pmatrix}\cos\phi\sin^2\theta\\ \sin\phi\sin^2\theta\\ \cos\theta\sin\theta\end{pmatrix}\right\rangle\,d\theta\,d\phi$$

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where $0 < \phi < 2\pi$ and $0 < \theta < \pi$. We have chosen the outward-pointing (positively oriented) normal vector. Then the flux through $\mathcal{S}$ is

$$\int_{\partial B_\epsilon(0)} \left\langle F, d\overrightarrow{A} \right\rangle = \int_0^{2\pi}\int_0^{\pi} \underbrace{\frac{1}{\|\varphi(\phi,\theta)^3\|}}_{=1/\epsilon^3} \langle \varphi(\phi,\theta), t_\phi \times t_\theta(\phi,\theta)\rangle \, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\epsilon^3}{\epsilon^3} \left\langle \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}, \begin{pmatrix} \cos\phi\sin^2\theta \\ \sin\phi\sin^2\theta \\ \cos\theta\sin\theta \end{pmatrix} \right\rangle d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta\, d\phi$$

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$$\begin{aligned}\int_{\partial B_\epsilon(0)}\left\langle F, d\vec{A}\right\rangle &= \int_0^{2\pi}\int_0^{\pi}\underbrace{\frac{1}{\|\varphi(\phi,\theta)^3\|}}_{=1/\epsilon^3}\langle\varphi(\phi,\theta), t_\phi\times t_\theta(\phi,\theta)\rangle\,d\theta\,d\phi\\ &= \int_0^{2\pi}\int_0^{\pi}\frac{\epsilon^3}{\epsilon^3}\left\langle\begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix},\begin{pmatrix}\cos\phi\sin^2\theta\\ \sin\phi\sin^2\theta\\ \cos\theta\sin\theta\end{pmatrix}\right\rangle d\theta\,d\phi\\ &= \int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi = 4\pi\end{aligned}$$

■

Exercise 4

Exercise 5