VV285 RC Part VII

Vector Fields

Flux, Circulation and Fundamental Theorem of Calculus

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Overview

Flux, circulation, divergence, rotation, gradient,. . . The concepts we learned last week have practical significance in subject such as *fluid dynamics* and *electromagnetics*. It is important for us to be familiar with these concepts.

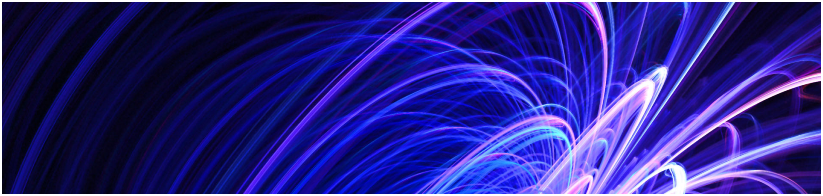


Figure: Electromagnetics

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field defined in a neighborhood of a hypersurface $\mathcal{S}$ with parametrization $\varphi : \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^{n-1}$. Then we define the *flux of F through $\mathcal{S}$* by

$$\int_{\mathcal{S}} F \, d\vec{A} := \int_{\mathcal{S}} \langle F , N \rangle \, dA \qquad \left(= \int_{\mathcal{S}} \langle F , d\vec{A} \rangle \right)$$
$$= \int_{\Omega} \langle F \circ \varphi(x) , N \circ \varphi(x) \rangle \sqrt{g(x)} \, dx_1 \ldots dx_{n-1}$$

And

$$\operatorname{div} F := \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

is called the *divergence* of F.

Remark: The total flux density at a point x is given by the divergence of the field at x.

In triangle calculus, the divergence of a vector field F can be expressed as

$$\operatorname{div} F = \langle \nabla , F \rangle = \left\langle \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} \right\rangle = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

In some other notations, one also uses $\nabla \cdot F$ to indicates the divergence of F. This type of notation is more common in a physical textbooks.

Find the flux of the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, $F(x, y, z) = (x, 1, z)$, along the surface $S = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leq 9, y = x^2 + z^2\right\}$ Assume y-axis to be negative orientation.

Let $\Omega \subset \mathbb{R}^n$ be an open set, $F : \Omega \to \mathbb{R}^n$ a continuously differentiable vector field and $\mathcal{C}^*$ an oriented closed curve in $\mathbb{R}^n$. Then

$$\int_{\mathcal{C}^*} \langle F , T \rangle \, ds$$

is called the (total) *circulation* of F along $\mathcal{C}$. Then the anti-symmetric, bilinear form

$$\mathit{rot} F|_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad \mathit{rot} F|_x(u, v) := \langle DF|_x u , v \rangle - \langle DF|_x v , u \rangle$$

is called the *rotation* (in mainland Europe) or *curl* (in anglo-saxon countries) of the vector field F at $x \in \mathbb{R}^n$.

Circulation and Rotation/Curl II

Let $\Omega \subset \mathbb{R}^2$ be open and $F : \Omega \to \mathbb{R}^2$ a continuously differentiable vector field. Then here exists a uniquely defined continuous **potential** function $\operatorname{rot} F : \Omega \to \mathbb{R}$ such that

$$\mathscr{rot}\, F|_x(u, v) = \operatorname{rot} F(x) \cdot \det(u, v).$$

Moreover,

$$\operatorname{rot} F(x) = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}.$$

Remark: $\operatorname{rot} F$ represents the circulation density in $\mathbb{R}^2$.

Circulation and Rotation/Curl III

Let $\Omega \subset \mathbb{R}^3$ be open and $F : \Omega \to \mathbb{R}^3$ a continuously differentiable vector field. Then here exists a uniquely defined continuous **vector** field $\operatorname{rot} F : \Omega \to \mathbb{R}^3$ such that

$$\cancel{\mathit{rot}}\, F|_x(u, v) = \langle \operatorname{rot} F(x)\,,\; u \times v \,\rangle = \det(\operatorname{rot} F(x), u, v).$$

Moreover,

$$\operatorname{rot} F(x) = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}.$$

Remark: $\operatorname{rot} F$ represents the circulation density in $\mathbb{R}^3$.

A simply connected innovational vector field is a potential field.

Circulation and Rotation/Curl IV

In triangle calculus, the rotation of a vector field F can be formally written as

$$\operatorname{rot} F = \nabla \times F(x) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{pmatrix},$$

where e_1, e_2, e_3 are the standard basis vectors in $\mathbb{R}^3$.

Remark: The triangle notation is a good mnemonics to help us memorize these formulas. (Why?)

Exercise 2 & 3: Circulation & Potential

Let $G(x, y) = \left(2xye^{x^2} + \frac{y}{x}, e^{x^2} + \ln x + 6y\right)$ be a vector field in $\mathbb{R}^2$, $G : \{(x, y) : x > 0, y > 0\} \to \mathbb{R}^2$.

- Is G a potential field?
- If so, find the form of its potential function. Otherwise, prove it's not conservative.

Let $F(x, y, z) = (e^z, 1, xe^z)$ be a vector field in $\mathbb{R}^3$.

- Is F a potential field?
- If so, find the form of its potential function. Otherwise, prove it's not conservative.

Green's Theorem

Let $R \subset \mathbb{R}^2$ be a bounded, simple region and $\Omega \supset R$ an open set containing R. Let $F : \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field. Then

$$\int_{\partial R^*} F \, d\vec{s} = \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \, dx$$

where ∂R^* denotes the boundary curve of R with positive (counter-clockwise) orientation.

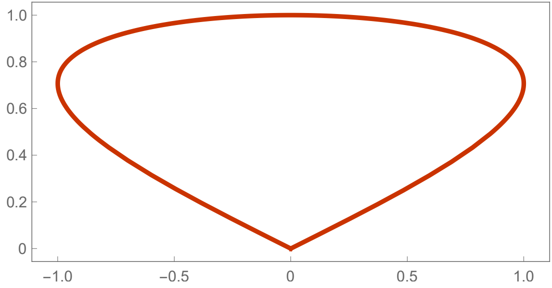
Remark: Green's theorem is basically a particular case of Stokes's theorem. A clever use of it is to calculate the area of a simple region. For $F(x_1, x_2) = (-x_2, x_1)$ we obtain

$$|R| = \int_R 1 \, dx = \frac{1}{2} \int_{\partial R} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} d\vec{s}.$$

For a bounded simple region, the value along the boundary contains the information of the area.

Find the area of region R that is bounded by the curve $\mathcal{C}$, whose parametrization is

$$\gamma(t) = \begin{pmatrix} \sin 2t \\ \sin t \end{pmatrix}, \quad t \in [0, \pi].$$



Gauß's Theorem

Let $R \subset \mathbb{R}^n$ be an admissible region and $F : \overline{R} \to \mathbb{R}^n$ a continuously differentiable vector field. Then

$$\int_R \operatorname{div} F \, dx = \int_{\partial R^*} F \, d\vec{A}.$$

Exercise 5: Gauß's Theorem

Calculate the integral

$$\int_{S^2} (xy + yz + zx) d\sigma$$

where $S^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$ is the unit sphere.

Let $F : \mathbb{R}^3 \to \mathbb{R}^3, F(x, y, z) = (x, y, z)$ and

$$U = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, x + z \leq 2, z \geq 0, x \geq 0\right\}$$

Denote by $S = \partial U$ the surface of U and by N the outward-pointing unit normal vector on S. Let S^+ denote the surface S with positive orientation.

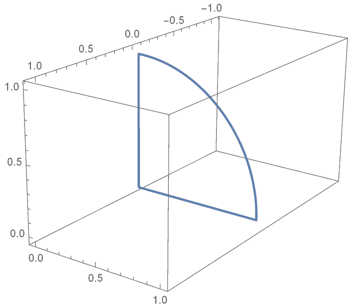
- Sketch U.
- Calculate the flux $\int_{S^+} \langle F, N \rangle d\sigma$

Let $\Omega \subset \mathbb{R}^3$ be an open set, $\mathcal{S} \subset \Omega$ a parametrized, admissible surface in $\mathbb{R}^3$ with boundary $\partial\mathcal{S}$ and let $F : \Omega \to \mathbb{R}^3$ be a continuously differentiable vector field. Then

$$\int_{\partial\mathcal{S}^*} F \, d\vec{s} = \int_{\mathcal{S}^*} \operatorname{rot} F \, d\vec{A}$$

where the orientations of the boundary curve $\partial\mathcal{S}^*$ and the surface $\mathcal{S}^*$ are chosen according to *right hand law*.

Let $\mathcal{C}$ be the curve illustrated below.

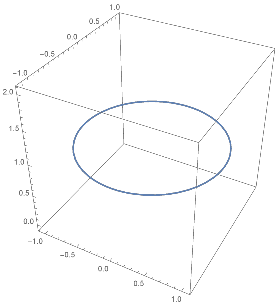


For $F(x, y, z) = (y, z, x)$, compute the circulation along $\mathcal{C}$ using Stokes's theorem.

Let

$$F(x,y,z) = \left(\sin x - \frac{y^3}{3}, \cos y + \frac{x^3}{3}, xyz\right).$$

Calculate the circulation of F along $\mathcal{C}$, where $\mathcal{C}$ is the oriented unit circle in the plane $z = 1$. (The orientation is counter-clockwise if viewing from positive z-axis).



Application 1: Electromagnetics

Integral Equations

$$\oint_{\partial\Omega} E \cdot \mathrm{d}S = \frac{1}{\varepsilon_0}\int_{\Omega} \rho \mathrm{d}V$$

$$\oint_{\partial\Omega} B \cdot \mathrm{d}S = 0$$

$$\oint_{\partial\Sigma} E \cdot \mathrm{d}\ell = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma} B \cdot \mathrm{d}S$$

$$\oint_{\partial\Sigma} B \cdot d\ell = \mu_0 \left(\int_{\Sigma} J \cdot \mathrm{d}S + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Sigma} E \cdot \mathrm{d}S\right)$$

Differential Equations

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 \left(J + \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

The famous equations above are called the *Maxwell equations*. They together describe fundamental behaviors of electromagnetic fields. Can you find the relations between the integral equations and differential equations?

Exercise 9: *Vector Calculus Identity

As we have discussed last week, physicists really love using triangle notations because this good notation makes life easier. See those common equalities in vector calculus, and think about what they indicate/why they hold/why the triangle notations are user-friendly:

- $\nabla \times (\nabla f) = 0$
- $\nabla \cdot (\nabla \times F) = 0$
- $\nabla \times \left(\dfrac{F}{\phi}\right) = \dfrac{\phi \nabla \times F - \nabla \phi \times F}{\phi^2}$

Remark: In electromagnetics, we define *magnetic vector potential* A to be a vector field such that

$$B = \nabla \times A$$

and *electric potential* ϕ to be scalar field such that

$$E = -\nabla \phi - \frac{\partial A}{\partial t}.$$

How does such a definition correspond with Maxwell equations?

Green's Identity

Let $R \subset \mathbb{R}^n$ be an admissible region and $u, v : \overline{R} \to \mathbb{R}$ be twice continuously differentiable potential functions. Then

$$\int_R \langle \nabla u \, , \, \nabla v \rangle \, dx = -\int_R u \cdot \Delta v \, dx + \int_{\partial R^*} u \frac{\partial v}{\partial n} \, dA \tag{1}$$

and

$$\int_R (u \cdot \Delta v - v \cdot \Delta u) \, dx = \int_{\partial R^*} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dA. \tag{2}$$

(1) is commonly called *Green's first identity* and (2) *Green's second identity*.

The properties of the Laplace operator Δ acting on smooth functions defined on some set $\Omega \subset \mathbb{R}^n$ are important in the theory of differential equations. Green's identities are your friend here:

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set and $u : \Omega \to \mathbb{R}$ a twice differentiable function such that

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \qquad \textit{and} \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$$

1. Prove that u is constant, i.e., there exists some $c \in \mathbb{R}$ such that $u(x) = c$ for all $x \in \Omega$.
2. Interpret the statement of this exercise physically in terms of fluid flow (potential flow) in a container.

Enjoy the following!