VV285 RC Final

Final Exam Exercises

Last but not least. . .

Pingbang Hu

University of Michigan-Shanghai Jiao Tong University Joint Institute

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Exercise 1

Calculate the directional derivative of the continuous function

$$f : \mathbb{R}^2 \to R, \qquad f(x, y) = \sqrt[3]{x^2 y}$$

at $(x, y) = (0, 0)$ in the direction of the vector $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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From the definition , we need to calculate

$$\frac{d}{dt} f(x_0 + th)\big|_{t=0}$$

where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Hence, we have

$$f(x_0 + th) = f(t \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1\sqrt{2} \end{pmatrix}) = \sqrt[3]{\frac{t^2}{2} \cdot \frac{t}{\sqrt{2}}} = \frac{t}{\sqrt{2}}$$

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so that

$$\frac{d}{dt} f(x_0 + th) = \frac{d}{dt}\frac{t}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

■

Exercise 2

Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.
Prove that the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = g(\|x\|)x$$

is conservative.

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The set $\mathbb{R}^3 \backslash \{0\}$ is simply connected , so it suffices to show that $\operatorname{rot} F = 0$. Now

$$|(\operatorname{rot} F)_i| = \left| \frac{\partial F_j}{\partial x_k} - \frac{\partial F_k}{\partial x_j} \right|$$

where (i, j, k) is any one of the permutation of $\{1, 2, 3\}$.

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We then have

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Hence, the rotation of F vanishes everywhere on $\mathbb{R}^3 \setminus \{0\}$. ∎

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Let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Find the flux of the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = \frac{x}{\|x\|^3}$$

through the surface

$$\mathcal{S} = \{x \in \mathbb{R}^3 : 4x_1^2 + 9x_2^2 + 6x_3^2 = 36\}$$

by using a suitable parametrization.

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where $0 < \phi < 2\pi$ and $0 < \theta < \pi$. We have chosen the outward-pointing (positively oriented) normal vector.

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$$\int_{\partial B_\epsilon(0)} \left\langle F, d\overrightarrow{A} \right\rangle = \int_0^{2\pi} \int_0^{\pi} \frac{1}{\|\varphi(\phi,\theta)^3\|} \left\langle \varphi(\phi,\theta), t_\phi \times t_\theta(\phi,\theta) \right\rangle \, d\theta \, d\phi$$

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$$\begin{aligned}\int_{\partial B_\epsilon(0)}\left\langle F, d\vec{A}\right\rangle &= \int_0^{2\pi}\int_0^{\pi}\underbrace{\frac{1}{\|\varphi(\phi,\theta)^3\|}}_{=1/\epsilon^3}\langle\varphi(\phi,\theta),t_\phi\times t_\theta(\phi,\theta)\rangle\,d\theta\,d\phi\\ &= \int_0^{2\pi}\int_0^{\pi}\frac{\epsilon^3}{\epsilon^3}\left\langle\begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix},\begin{pmatrix}\cos\phi\sin^2\theta\\ \sin\phi\sin^2\theta\\ \cos\theta\sin\theta\end{pmatrix}\right\rangle d\theta\,d\phi\\ &= \int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi\end{aligned}$$

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\int_{\partial B_\epsilon(0)} \left\langle F, d\vec{A} \right\rangle &= \int_0^{2\pi}\int_0^{\pi} \underbrace{\frac{1}{\|\varphi(\phi,\theta)^3\|}}_{=1/\epsilon^3} \langle \varphi(\phi,\theta), t_\phi \times t_\theta(\phi,\theta)\rangle \, d\theta\, d\phi \\
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\end{aligned}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set and $R \subset \Omega$ an admissible region. Let $u : \Omega \to \mathbb{R}$ be a twice continuously differentiable function such that

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \lambda u \text{ on } \mathrm{int} R \qquad \text{and} \qquad u|_{\partial R} = 0$$

for some $\lambda \in \mathbb{R}$. Suppose that $u \neq 0$, i.e., $u(x) \neq 0$ for some $x \in R$. Prove that $\lambda < 0$.

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$$\lambda < 0.$$

■