$$f(A+H) = (A+H)^3$$

$f(A)$ $Df|_A H$

$$= A^3 + AHA + A^2H + HA^2 + o(H)$$

operator norm

to prove this

$$Df|_A H = AHA + A^2H + HA^2$$

$$Df : A \mapsto Df|_A$$

$$D^2f \Rightarrow \quad Df|_{A+J} = Df|_A + D^2f|_A J + o(J)$$

$$Df|_{A+J} H = (A+J)H(A+J) + (A+J)^2 H + H(A+J)^2$$

$$= AHA + A^2H + HA^2 + AHJ + JHA + AJH + JAH + HAJ + HJA$$

$$+ o(J)$$

$$\Rightarrow D^2f|_A(H,J) = AHJ + JHA + AJH + JAH + HAJ + HJA$$

Using chain rule,

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

$$Dg|_A H = \mathrm{tr}(AHA + A^2H + HA^2) = 3\,\mathrm{tr}\,A^2H$$

$$D^2g|_A(H,J) = 3\,\mathrm{tr}(AJH) + 3\,\mathrm{tr}(JAH)$$

$g(x,y) = x^3 - y^2 - xy + 1$

$$\frac{\partial g}{\partial x} = 3x^2 - y, \quad \frac{\partial g}{\partial y} = -2y - x$$

Let $3x^2 - y = -2y - x = 0$

$\Rightarrow$ we have critical points $(0,0)$ and $\left(-\frac{1}{6}, \frac{1}{12}\right)$

Find its Hessian,

$$\text{Hess } g(x) = \begin{pmatrix} 6x & -1 \\ -1 & -2 \end{pmatrix} \qquad \Delta(x,y) = -12x - 1$$

At $(0,0)$ $\Delta(x,y) = -1 \Rightarrow$ indefinite $\Rightarrow (0,0)$ saddle point

At $\left(-\frac{1}{6}, \frac{1}{12}\right)$ $\Delta(x,y) = 1 \Rightarrow$ negative definite $\Rightarrow \left(-\frac{1}{6}, \frac{1}{12}\right)$ local maxima

$g\left(-\frac{1}{6}, \frac{1}{12}\right) = \frac{433}{432}$ is a local maxima, and no global minimum & maximum

$f(x,y) = x^3 + 2y^2 + 1$ $\qquad \frac{\partial f}{\partial x} = 3x^2,\ \frac{\partial f}{\partial y} = 4y \Rightarrow (0,0)$ critical

$\mathrm{Hess} f(x) = \begin{pmatrix} 6x & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow$ no info at $(0,0)$

$f(0,0) = 1$

Check boundary: $x^2+y^2=9 \Rightarrow 2y^2 = 18-2x^2$

$\Rightarrow f(x,y) = x^3+2y^2+1 = x^3-2x^2+19,\ \underline{x \in [-3,3]}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \varphi(x)$

$\varphi'(x) = 3x^2-4x$, (critical: $x=0, \frac{4}{3}$)

We also check boundary points $(x=\pm 3)$

In summary

$$\begin{cases} f(0,\pm 3) = 19 \\ f(\frac{4}{3}, \pm\frac{\sqrt{65}}{3}) = \frac{481}{27} \\ f(-3,0) = -26 \\ f(3,0) = 28 \end{cases}$$

Global maximum: $f(3,0) = 28$

Global minimum: $f(-3,0) = -26$

$DT|_{(x,y,z)} = (50 \quad 0 \quad 50)$

$Dg|_{(x,y,z)} = (2x \quad 2y \quad 2z)$

$Dh|_{(x,y,z)} = (1 \quad 1 \quad 1)$

let $\xi = (x,y,z)$

and $g(x,y,z) = x^2+y^2+z^2-1$

$h(x,y,z) = x+y+z-1$

$\begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix}$ has a 2X2 submatrix whose det $\neq 0$ for all $\xi \in \mathbb{R}^3$.

otherwise, $x=y=z$ for all $x,y,z \in \mathbb{R}$, ↯

$$\begin{cases} DT|_\xi = \lambda Dg|_\xi + \mu Dh|_\xi \\ g(\xi) = 0 \\ h(\xi) = 0 \end{cases} \Rightarrow \begin{cases} 50 = 2\lambda x + \mu & ① \\ 0 = 2\lambda y + \mu & ② \\ 50 = 2\lambda z + \mu & ③ \\ x^2+y^2+z^2 = 1 & ④ \\ x+y+z = 1 & ⑤ \end{cases}$$

①③ $\Rightarrow x = z$

3 $\Rightarrow y = 1-2z$

$\Rightarrow x^2+(1-2x)^2+x^2=1$

2 $\Rightarrow x = 0, \frac{2}{3}$

$(0,1,0)$

$(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$

Respectively, $T(0,1,0) = 80 \to \min$

$T(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}) = \frac{440}{3} \to \max$

We first define optical path length $\Delta := n \cdot S$ 光程.

where n is refractive index, S is distance.

(Conclusion: $\Delta = c \cdot t$) $\frac{c_0}{n}$

light speed

$$\Delta = n_1 l_1 + n_2 l_2 = n_1 \frac{a}{\cos\alpha} + n_2 \frac{b}{\cos\beta} =: \Delta(\alpha, \beta)$$

Constraint: $a\tan\alpha + b\tan\beta = d$

Apply Lagrangian multiplier rule,

$Dg|_{(\alpha,\beta)}$

$$D\Delta|_{(\alpha,\beta)} \left(n_1 \frac{-\sin\alpha}{\cos^2\alpha} a,\ n_2 \frac{-\sin\beta}{\cos^2\beta} b\right) + \lambda\left(\frac{a}{\cos^2\alpha}, \frac{b}{\cos^2\beta}\right) = 0$$

$$\Rightarrow \begin{array}{l} \lambda = n_1 \sin\alpha \\ \lambda = n_2 \sin\beta \end{array} \Rightarrow n_1 \sin\alpha = n_2 \sin\beta.$$

$\square$