VV285 RC Final

Final Exercise

Last but not least...

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July 30, 2021

Exercise 1

Calculate the directional derivative of the continuous function

$$f : \mathbb{R}^2 \to R, \qquad f(x, y) = \sqrt[3]{x^2 y}$$

at $(x, y) = (0, 0)$ in the direction of the vector $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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where $x_0 = 0$ and $h = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Hence, we have

$$f(x_0 + th) = f(t \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1\sqrt{2} \end{pmatrix}) = \sqrt[3]{\frac{t^2}{2} \cdot \frac{t}{\sqrt{2}}} = \frac{t}{\sqrt{2}}$$

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so that

$$\frac{d}{dt} f(x_0 + th) = \frac{d}{dt} \frac{t}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Exercise 2

Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.
Prove that the vector field

$$F : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3, \qquad F(x) = g(\|x\|)x$$

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$$|(\operatorname{rot} F)_i| = \left| \frac{\partial F_j}{\partial x_k} - \frac{\partial F_k}{\partial x_j} \right|$$

where (i, j, k) is any one of the permutation of $\{1, 2, 3\}$.

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We then have

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial}{\partial x_k} \left(g(\|x\|)x\right)_j = \frac{\partial}{\partial x_k} g(\|x\|)x_j$$

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Hence, the rotation of F vanishes everywhere on $\mathbb{R}^3 \setminus \{0\}$. ■

Exercise 3

Exercise 4

Exercise 5