VV285 RC

Differential Calculus

Integral in $\mathbb{R}^n$

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Outline

At this moment, we have learned how to perform integral in finite dimensional vector space. To be more specific, we have investigated deeply into the volume and (hyper-)surface integral in $\mathbb{R}^n$. Additionally, powerful integral tools such as Fubini's theorem and substitution rule were introduced. Therefore, you are supposed to be able to figure out many complex integrals in high-dimensional space on your own! For example, the surface area and volume of a *hyperball* in $\mathbb{R}^5$. And that's something you will not be surprised if encountered during exams. In lectures, for most of time, we limited ourselves in $\mathbb{R}^2$ and $\mathbb{R}^3$. Thus, you should be particularly experienced to perform integral in both space.

Overview II

Trust me, you will reopen your VV285 lecture slides again and again in your future study, no matter what your major is! (Hopefully, some of you will reopen my RC slides) Here is a list of courses where you will find learning well in VV285 is so useful:

1. VV214 & 417: Linear Algebra
2. VV286: Honors Math IV
3. VV556 & 557: Methods of Applied Mathematics I&II
4. VP150/160 & VP250/260: (Honors) Physics I&II
5. VP390: Modern Physics
6. VE230 & VE330: Electromagnetics I&II
7. VM211: Introduction to Solid Mechanics
8. VM320 & 520: Fluid Mechanics & Advanced Fluid Mechanics
9. ...

Overview III

In summary, you need to be particularly experienced to perform any integral in $\mathbb{R}^2$&$\mathbb{R}^3$.i.e. Line integral/surface integral/volume integral on scalar/vector function, we haven't introduced some of which.

Cuboids

Let $a_k, b_k, k = 1, \ldots, n$ be pairs of numbers with $a_k < b_k$. Then the set $Q \subset \mathbb{R}^n$ given by

$$\begin{aligned} Q &= [a_1, b_1] \times \cdots \times [a_n, b_n] \\ &= \{x \in \mathbb{R}^n : x_k \in [a_k, b_k], k = 1, \ldots, n\} \end{aligned}$$

is called an *n-cuboid*. We define the volume of Q to be

$$|Q| := \prod_{k=1}^{n} (b_k - a_k).$$

We will denote the set of all n-cuboids by $\mathcal{Q}_n$.

Remark: Clearly, an n-cuboid is a compact set in $\mathbb{R}^n$.

Upper and Lower Volume

Let $\Omega \subset \mathbb{R}^n$ be a bounded non-empty set. We define the *outer* and *inner volume* of Ω by

$$\overline{V}(\Omega) := \inf \left\{ \sum_{k=0}^{r} |Q_k| : r \in \mathbb{N},\ Q_0, \ldots, Q_r \in \mathcal{Q}_n, \Omega \subset \bigcup_{k=1}^{r} Q_k \right\},$$

$$\underline{V}(\Omega) := \sup \left\{ \sum_{k=0}^{r} |Q_k| : r \in \mathbb{N},\ Q_0, \ldots, Q_r \in \mathcal{Q}_n, \Omega \supset \bigcup_{k=1}^{r} Q_k, \bigcap_{k=1}^{r} Q_k = \emptyset \right\}.$$

It is easy to see that $0 \le \underline{V}(\Omega) \le \overline{V}(\Omega)$.

Measure

We can now define Jordan Measurable based on outer and inner volume of a set.

Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then Ω is said to be *(Jordan) measurable* if either

(i) $\overline{V}(\Omega) = 0$ or

(ii) $\overline{V}(\Omega) = \underline{V}(\Omega)$.

In the first case, we say that Ω has (Jordan) *measure zero*, in the second case we say that

$$|\Omega| := \overline{V}(\Omega) = \underline{V}(\Omega)$$

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(How to prove the set $\mathbb{Q} \cap (0,1)$ has measure 0?)

$\mathbb{Q} \cap (0, 1)$ has measure 0

The trick is that we first noting the fact that $Q \cap (0, 1)$ is *countable*. Then, for every $q_i \in \mathbb{Q}$, where q_i denoting the $i - th$ rational number, we let the interval with the length

$$\frac{\epsilon}{2^{i+1}}$$

cover q_i, namely

$$q_i \in (q_i - \frac{\epsilon}{2^{i+2}}, q_i + \frac{\epsilon}{2^{i+2}})$$

where $\epsilon > 0$ is fixed first. Then since we know that

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \quad \Rightarrow \quad \sum_{i=2}^{\infty} \frac{1}{2^i} = \frac{1}{2}$$

$\mathbb{Q} \cap (0,1)$ has measure 0

So the total length of the cover is

$$\sum_{i=\epsilon}^{\infty} \frac{1}{2^{i+1}} = \sum_{i=2}^{\infty} \frac{\epsilon}{2^i} = \frac{\epsilon}{2} < \epsilon$$

Hence we show that we can let the cover as small as we want, so $Q \cap (0,1)$ is a Jordan Measured zero set.

Failure of Step Functions

In $\mathbb{R}$, we use a sequence of step functions that converges uniformly to some regulated function f to define the integral. However, this method fails in $\mathbb{R}^n$. The reason is that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ may not be approximated uniformly by step functions.

Actually, the way we have defined the Riemann integral is not quite the way it is done in the literature; our integral is more properly called a *Darboux integral*. However, the definitions of the Riemann and Darboux integral are fully equivalent. There is no difference between a Darboux-integrable and a Riemann-integrable function, and the two integrals coincide.

It is ok even though we cannot use the step functions to approximate uniformly f. We can still utilize the step functions in the following ways:

Riemann and Darboux Integral II

We will now formulate the definition of the Riemann integral for functions of several variables with real values.

3.3.12. Definition. Let $Q \subset \mathbb{R}^n$ be an n-cuboid and f a bounded real function on Q. let $\mathscr{U}_f$ denote the set of all step functions u on Q such that $u \geq f$ and $\mathscr{L}_f$ the set of all step functions v on Q such that $v \leq f$. The function f is then said to be ***(Darboux)-integrable*** if

$$\sup_{v \in \mathscr{L}_f} \int_Q v = \inf_{u \in \mathscr{U}_f} \int_Q u.$$

In this case, the ***(Darboux) integral of f over Q***, $\int_Q f$, is defined to be this common value.

3.3.13. Theorem. A bounded function $f\colon Q \to \mathbb{R}$ is Riemann-integrable if and only if for every $\varepsilon > 0$ there exist step functions u_ε and v_ε such that $v_\varepsilon \leq f \leq u_\varepsilon$ and

$$\int_Q u_\varepsilon - \int_Q v_\varepsilon \leq \varepsilon.$$

Integration over Jordan-Measurable Sets

Let $\Omega \subset \mathbb{R}^n$ be a bounded Jordan-measurable set and let $f : \Omega \to \mathbb{R}$ be continuous a.e. Then f is integrable on Ω.

That's what we learned in the course. However, we had a hidden precondition here: The function f needs to be bounded on its domain. And in this whole section, we consider only bounded functions. So, to be more precise, the statement should be:

Let $\Omega \subset \mathbb{R}^n$ be a bounded Jordan-measurable set and let $f : \Omega \to \mathbb{R}$ be (bounded) and (continuous a.e.). Then f is integrable on Ω.

Unbounded functions can still be integrable

Despite the fact that we only consider bounded functions in this section, we should notice that an unbounded function can still have integrals. (Improper Riemann Integrals)

Question: m and n are two integers. Judge whether the following integrals exist or not. $\|x\|$ denotes the Euclidean norm.

$$\int_{\mathbb{R}^n} \frac{1}{\|x\|^m} dx$$

$(m, n) = (1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)$ are the multiple-choice question of final exam in last year! We will talk about this question after learning the substitution rule.

Fubini's Theorem

Let Q_1 be an n_1-cuboid and Q_2 an n_2-cuboid so that $Q := Q_1 \times Q_2 \subset \mathbb{R}^{n_1+n_2}$ is an (n_1+n_2)-cuboid. Assume that $f : Q \to \mathbb{R}$ is integrable on Q and that for every $x \in Q_1$ the integral

$$g(x) = \int_{Q_2} f(x, \cdot)$$

exists. Then

$$\int_Q f = \int_{Q_1 \times Q_2} f = \int_{Q_1} g = \int_{Q_1} \left(\int_{Q_2} f \right).$$

Remark: This is a very powerful tool. So that we can divide and conquer a integral in $\mathbb{R}^n$.

Ordinate Region

For $x \in \mathbb{R}^n$ we define

$$\hat{x}^{(k)} := (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-1}$$

as the vector x with the kth component omitted.

A subset $U \subset \mathbb{R}^n$ is said to be an *ordinate region (with respect to x_k)* if there exists a measurable set $\Omega \subset \mathbb{R}^{n-1}$ and continuous, almost everywhere differentiable functions $\varphi_1, \varphi_2 : \Omega \to \mathbb{R}$, such that

$$U = \{x \in \mathbb{R}^n : x \in \Omega, \varphi_1(\hat{x}^{(k)}) \le x_k \le \varphi_2(\hat{x}^{(k)})\}.$$

If U is an ordinate region with respect to each $x_k, k = 1, \dots, n$, it is said to be a *simple region*.

Remark: Any ordinate region is measurable.

Exercise

Please find the value of following integrals:

$$\int_0^1 \int_x^1 \sin\left(y^2\right) dydx$$

$$\int_0^1 \int_x^1 e^{y^2} dydx$$

Exercise

The primitives of e^{y^2} and $\sin(y^2)$ are both non-elementary functions, so it's hard for us to integrate them in the original sequence. So we want to change the order of variables. The region can be express by two inequalities:

$$0 \leq x \leq 1, x \leq y \leq 1$$

which is equivalent to

$$0 \leq y \leq 1, \quad 0 \leq x \leq y$$

Therefore

$$\int_0^1 \int_x^1 \sin\left(y^2\right) dydx = \int_0^1 \int_0^y \sin\left(y^2\right) dxdy$$
$$= \int_0^1 y \sin\left(y^2\right) dy$$
$$= -\left.\frac{\cos\left(y^2\right)}{2}\right|_0^1$$
$$= \frac{1-\cos 1}{2}$$

$$\int_0^1 \int_x^1 e^{y^2} dydx = \int_0^1 \int_0^y e^{y^2} dxdy$$
$$= \int_0^1 y e^{y^2} dy = \left.\frac{e^{y^2}}{2}\right|_0^1 = \frac{e-1}{2}$$

Substitution Rule

Here is another powerful tool named substitution rule. Let $\Omega \subset \mathbb{R}^n$ be open and $g : \Omega \to \mathbb{R}^n$ injective and continuously differentiable. Suppose that $\det J_g(y) \neq 0$ for all $y \in \Omega$. Let K be a compact measurable subset of Ω. The $g(K)$ is compact and measurable and if $f : g(K) \to \mathbb{R}$ is integrable, then

$$\int_{g(K)} f(x)\, dx = \int_K f(g(y)) \cdot |\det J_g(y)|\, dy.$$

Applications I

(i) Polar coordinates in $\mathbb{R}^2$ are defined by a map

$$\Phi : (0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \setminus \{0\}, \qquad (r, \phi) \mapsto (x, y)$$

where

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Note that this map is bijective and even C^∞ in the interior of its domain. An alternative (but rarely used) version of polar coordinates would map $x = r\sin\phi, y = r\cos\phi$. This simply corresponds to a different geometrical interpretation of the angle ϕ. In any case,

$$|\det J_\Phi(r, \phi)| = \left|\det \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}\right| = r$$

Applications II

(ii) Cylindrical coordinates in $\mathbb{R}^3$ are given through a map

$$\Phi : (0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3 \setminus \{0\}, \qquad (r, \phi, \zeta) \mapsto (x, y, z)$$

defined by

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = \zeta$$

In this case

$$|\det J_\Phi(r, \phi, \zeta)| = \left| \det \begin{pmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = r$$

(iii) Spherical coordinates in $\mathbb{R}^3$ are often defined through a map

$$\Phi : (0,\infty) \times [0,2\pi) \times (0,\pi) \to \mathbb{R}^3 \setminus \{0\}, \quad (r,\phi,\theta) \mapsto (x,y,z),$$
$$x = r\cos\phi\sin\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\theta.$$

Of course, there is a certain freedom in defining θ and ϕ, so there are alternative formulations. The modulus of the determinant of the Jacobian is given by

$$|\det J_\Phi(r,\phi,\theta)| = \left|\det \begin{pmatrix} \cos\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\theta & 0 & -r\sin\theta \end{pmatrix}\right|$$
$$= r^2\sin\theta$$

Applications IV

(iv) In $\mathbb{R}^n$, we can define spherical coordinates by

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\ &\vdots \\ x_{n-1} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\cos\theta_{n-1} \\ x_n &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\sin\theta_{n-1} \end{aligned}$$

with $r > 0$ and $0 < \theta_k < \pi, k = 1, \ldots, n-2, 0 < \theta_{n-1} < 2\pi$. Here,

$$|\det J_\Phi(r,\theta_1,\ldots,\theta_{n-1})| = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\ldots\sin\theta_{n-2}.$$

Exercise

The volume of a Jordan-measurable measurable set $\Omega \subset \mathbb{R}^n$ is given by

$$|\Omega| = \int_{\Omega} 1.$$

Question: Calculate the volume of a 4-dimensional unit ball B^4.

Improper Integrals

Just as for integrals of a single variable, we can treat improper Riemann integrals of functions $f : \mathbb{R}^n \to \mathbb{R}$ over measurable sets $\Omega \subset \mathbb{R}^n$. These occur if either

1. f is unbounded or
2. Ω is unbounded.

In either case, one considers the improper integral as a suitable limit of "proper" integrals; if the limit exists, so does the improper integral.

Exercise

Now let's review this questions:
m and n are two integers. Judge whether the following integrals exist or not. $\|x\|$ denotes the Euclidean norm.

$$\int_{\mathbb{R}^n} \frac{1}{\|x\|^m} dx$$

$(m, n) =$

1. (1,2),
2. (1,3),
3. (2,2),
4. (2,3),
5. (3,2),
6. (3,3).

Exercise

With the substitution rule, we can use the spherical coordinates in $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} \frac{1}{\|x\|^m} dx = \int_K \frac{r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \ldots \sin\theta_{n-2}}{r^m} dr d\theta_1 d\theta_2 \cdots d\theta_{n-1}$$
$$= \left(\int_0^{\pi} \sin^{n-2}\theta_1 d\theta_1 \right) \cdots \left(\int_0^{2\pi} d\theta_{n-1} \right) \left(\int_0^{\infty} r^{n-m-1} dr \right)$$

We can just consider whether $\int_0^\infty r^{n-m-1} dr$ exists For all $k \in \mathbb{Z}$, the targeted value of $\int_0^\infty r^k dr$ is ∞, so all of the integrals do not exist.

Exercise

What if we replace $\mathbb{R}^n$ by B^n?

If the integral

$$\int_{B^n} \frac{1}{\|x\|^m} dx$$

in $\mathbb{R}^n$ exists, what relationship does n and m have?

Exercise

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$$\int_{B^n} \frac{1}{\|x\|^m} dx$$

in $\mathbb{R}^n$ exists, what relationship does n and m have?

$(n \geq m + 1)$

Parametrized Surface

A *smooth parametrized m-surface in* $\mathbb{R}^n$ is a subset $\mathcal{S} \subset \mathbb{R}^n$ together with a locally bijective, continuously differentiable map (parametrization)

$$\varphi : \Omega \to \mathcal{S}, \qquad \Omega \subset \mathbb{R}^m,$$

such that

$$\operatorname{rank} D\varphi|_x = m$$

for almost every $x \in \Omega$. If $m = n - 1$, $\mathcal{S}$ is said to be a *parametrized hypersurface*.

Tangent Spaces of Surface

Let $\mathcal{S} \subset \mathbb{R}^n$ be a parametrized m-surface with parametrization $\varphi : \Omega \to \mathcal{S}$. Then

$$t_k(p) = \frac{\partial}{\partial x_k} \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} \Bigg|_{x=\varphi^{-1}(p)}, \qquad k = 1, \ldots, m.$$

is called the *kth tangent vector of $\mathcal{S}$ at $p \in \mathcal{S}$* and

$$T_p\mathcal{S} := \operatorname{ran} D\varphi|_x = \operatorname{span}\{t_1(p), \ldots, t_m(p)\}$$

is called the *tangent space* to $\mathcal{S}$ at p. The vector field

$$t_k : \mathcal{S} \to \mathbb{R}^n, \qquad p \mapsto t_k(p)$$

is called the *kth tangent vector field* on $\mathcal{S}$.

Exercise

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere and $p = (a, b, c) \in S^2$. Show that the tangent plane at p is given by

$$T_pS^2 = \left\{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 1\right\}$$

We parametrize the unit sphere by

$$\Phi(\phi, \theta) = \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix}$$

The tangent vectors at the point p are

$$t_\phi(p) = \frac{\partial}{\partial\phi} \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix}\Bigg|_p = \begin{pmatrix} -\sin\phi \sin\theta \\ \cos\phi \sin\theta \\ 0 \end{pmatrix}\Bigg|_p$$

$$t_\theta(p) = \frac{\partial}{\partial\theta} \begin{pmatrix} \cos\phi \sin\theta \\ \sin\phi \sin\theta \\ \cos\theta \end{pmatrix}\Bigg|_p = \begin{pmatrix} \cos\phi \cos\theta \\ \sin\phi \cos\theta \\ -\sin\theta \end{pmatrix}\Bigg|_p$$

The point $q = (x, y, z) \in T_p S^2$ if and only if $q - p \in \text{span}\{t_\phi(p), t_\theta(p)\}$, so

$$\det \begin{pmatrix} x-a & -\sin\phi\sin\theta & \cos\phi\cos\theta \\ y-b & \cos\phi\sin\theta & \sin\phi\cos\theta \\ z-c & 0 & -\sin\theta \end{pmatrix}\Bigg|_p = 0$$

Evaluating the determinant, we obtain

$$(x-a)\left(-\sin^2\theta\cos\phi\right)+(y-b)\left(-\sin^2\theta\sin\phi\right)+(z-c)(-\sin\theta\cos\theta) = 0$$

Dividing by $\sin\theta$ and inserting the parametrization, we obtain

$$(x-a)x + (y-b)y + (z-c)z = 0$$

Using $x^2 + y^2 + z^2 = 1$, we obtain $ax + by + cz = 1$.

Solution 2

It is known that the tangent vectors to the sphere at p are orthogonal to p, so $q \in T_pS^2$ if and only if $\langle q - p, p \rangle = 0$. This is equivalent to

$$\langle q, p \rangle = \langle p, p \rangle = 1$$

which also gives $ax + by + cz = 1$

Normal Vector to Hypersurface

Let $\mathcal{S} \subset \mathbb{R}^n$ be a hypersurface. Then a unit vector that is orthogonal to all tangent vectors to $\mathcal{S}$ at p is called a *unit normal vector to $\mathcal{S}$ at p* and denoted by $N(p)$. The vector field

$$N : \mathcal{S} \to \mathbb{R}^n, \qquad\qquad p \mapsto N(p)$$

is called the *normal vector field* on $\mathcal{S}$.

(i) A hypersurface that is the boundary of a measurable set $\Omega \subset \mathbb{R}^n$ with non-zero measure is said to be a *closed surface*.

(ii) A closed hypersurface is said to have *positive orientation* if the normal vector field is chosen so that the normal vectors point outwards from Ω. (Important. We will discuss flux and divergence of a vector field later.)

Have Fun
And
Learn Well!

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