$$L = \varphi_m^{-1} \circ A_{n\times m} \circ \varphi_U$$

$$\begin{tikzcd}
U \arrow[r, "L"] \arrow[d, "\varphi_U"] & V \arrow[d, "\varphi_m"] \\
\mathbb{R}^n \arrow[r, "A_{n\times n}"] & \mathbb{R}^m
\end{tikzcd}$$

$\{v_1, v_2, \cdots, v_m\}$

$\{u_1, \cdots, u_n\}$

$\varphi_U(u_i) = e_i \qquad \varphi_V(v_i) = e_i$

Ex: $B_1\{1-2t+t^2,\ 3-5t+4t^2,\ 2t+3t^2\} \longrightarrow B_2\{1, t, t^2\}$

$$\begin{tikzcd}
B_1 \arrow[r, "C_{B_1\to B_2}"] \arrow[d, "\varphi_1"] & B_2 \arrow[d, "\varphi_2"] \\
\mathbb{R}^3 \arrow[r, "I"] & \mathbb{R}^3
\end{tikzcd}$$

$$\varphi_1 = \begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{pmatrix}^{-1}$$

$$\varphi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$\Rightarrow C_{B_1\to B_2} = \varphi_2^{-1} \circ I \circ \varphi_1 = I \circ I \circ \begin{pmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{pmatrix}$$

Let $A_{m \times n}$ $m \leq n$ s.t. $Ax = b \in \mathbb{R}^m$, $b \neq 0$ has no sol, but $\exists x$ s.t. $Ax = d \in \mathbb{R}^n$, $d \neq 0$.

pf: first, since $m \leq n$, $\Rightarrow$ rank $\leq m$.

since $Ax = b$ has no sol, then

$1 \leq \operatorname{rank} A \leq m-1$.

$$\left(A \mid b \right) \sim \left(\begin{array}{cccc|c} 1 & & & & \vdots \\ & 1 & & & \vdots \\ & & \ddots & & \\ 0 & 0 & 0 & 0 & \neq 0 \end{array} \right)$$

$\dim(\ker A)$ is the dim. of the solution set.

$n = \dim \ker A + \dim \operatorname{Im} A$.

$n - 1 \geq \dim \ker A \geq n - (m-1)$ #

2.

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ t & 0 & 1 & t+1 \\ 1 & t-1 & 1 & t+1 \\ t & 0 & t & t \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & t & 1 & 1 \\ 0 & 0 & 0 & t-1 \\ 0 & 0 & t-1 & -1 \end{pmatrix}$$

$$t=1 \Rightarrow A \sim \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad r=3.$$

$$t=0 \Rightarrow A \sim \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \quad r=3$$

$t \neq 0, -1 \Rightarrow r=4. \Rightarrow$ invertible

$P_k(\mathbb{F})$

1. $\dim(P_k(\mathbb{F})) = k+1$. $\mathcal{B} = \{1, t, \cdots, t^k\}$

2. $W = \{p \in P_k : p(0) + p(1) = 0\}$ is a subspace.

i) $0_{P_k(\mathbb{F})} = 0_W \in W$. $0(0) + 0(0) = 0 + 0 = 0$.

ii)

$\alpha w_1 + \beta w_2 \in W$ . . .

3.

$W = \{p = \sum a_k t^k : p(0) + p(1) = 0\}$

$\Rightarrow \sum a_k 1^k = a_0 + \sum a_k = 0$

$\Rightarrow a_0 = -\frac{1}{2}\sum_1 a_k$

$\Rightarrow W = \{p = a_1(t - \frac{1}{2}) + a_2(t^2 - \frac{1}{2}) + \cdots + a_k(t^k - \frac{1}{2})\}$

$\{t - \frac{1}{2}, t^2 - \frac{1}{2}, \ldots, t^k - \frac{1}{2}\}$ span W

⋮

Let $A = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$. define $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$

$T(B) = AB - BA$. $\mathcal{B}_{M_2(\mathbb{R})} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

1. $T_A = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & -2 \\ 0 & 2 & 0 & 0 \end{pmatrix}$

2. $\ker T = ?$ $\left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

   $\operatorname{Im} T = ?$ $\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix} \right\}$

3. find V s.t. $V \oplus \ker T = M_2(\mathbb{R})$

   $V = \operatorname{span} \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

8.

1.
$$T(f_1) + T(f_2) = T(f_1 + f_2)$$

since $f_1 + f_2 \big|_{[1,2]} = f_1 \big|_{[1,2]} + f_2 \big|_{[1,2]}$.

$$T(\alpha f_1) = \alpha T(f_1)$$

since $\alpha\left(f_1 \big|_{[1,2]}\right) = \alpha f_1 \big|_{[1,2]}$.

2.

$$S: C[1,2] \to C[0,3]$$

$\forall f \in C[1,2]$, $S(f) = f'$, $\begin{cases} f, & [1,2] \\ 0, & \text{otherwise.} \end{cases}$

3. T surjective : $\forall f \in C[1,2]$ $\exists f' \in C[0,3]$ s.t. $T(f') = f$. (just let $f' = S(f)$.)

$\mathrm{Im}\, ST = \mathrm{Im}\, S$: $\mathrm{Im}(ST) \overset{\forall f \in C[0,3]}{=} ST(f) = S(f|_{[1,2]}) = f'$, $\begin{cases} f, & (1,2] \\ 0, & \text{otherwise} \end{cases}$ $= \mathrm{Im}\, S$.

S injective : $S(f_1) = S(f_2) \Rightarrow f_1 = f_2$. since

If $f_1 \neq f_2$, $S(f_1) = f_1' = \begin{cases} f_1, & (1,2] \\ 0, & \text{otherwise} \end{cases} \neq \begin{cases} f_2, & (1,2] \\ 0, & \text{otherwise} \end{cases} = f_2' = S(f_2)$. $\times$

$\mathrm{Ker}\, ST = \mathrm{Ker}\, T$: $\mathrm{Ker}\, ST = \{f : ST(f) = 0\}$.

$$= \{f : S(f|_{[1,2]}) = 0\}$$

$$= f' \begin{cases} 0, & [1,2] \\ f, & \text{otherwise} \end{cases}$$

$$= \{f : f|_{[1,2]} = 0\}.$$

$$= \{f : T(f) = 0\}$$

4. $\mathrm{Im}\, S \oplus \mathrm{Ker}\, T = C[0,3]$

since $\mathrm{Im}\, S \cap \mathrm{Ker}\, T = \{0\}$ (obviously, from 3).

And we can see $C[0,3] = \mathrm{Im}\, S + \mathrm{Ker}\, T$

(obviously, also from 3

$\mathrm{Im}\, S = f',\ \begin{cases} f, & [1,2] \\ 0, & \text{otherwise} \end{cases}$ $\quad \mathrm{Ker}\, T = f' \begin{cases} 0, & [1,2] \\ f, & \text{otherwise} \end{cases}$)

Then from 3, we know $\begin{cases} \mathrm{Im}\, S = \mathrm{Im}\, ST \\ \mathrm{Ker}\, ST = \mathrm{Ker}\, T \end{cases}$

$\Rightarrow C[0,3] = \mathrm{Im}(ST) \oplus \mathrm{Ker}(ST).$