Vv285 Honors Mathematics III

Summer 2020 First Midterm Exam

Questions and Solutions

Exercise 1.1
Suppose that V is a vector space over $\mathbb{F}$. Which of the following statements is true?

1. $\{x+y\colon x \in V,\ y \in V\} = V$
2. $\{x+y\colon x \in V,\ y \in V\} = V \times V$
3. $\{\lambda x\colon x \in V,\ \lambda \in \mathbb{F}\} = \mathbb{F} \times V$

Solution. Correct answer: 1.

Exercise 1.2
Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let U be a subspace of V. Then

1. $V \setminus U$ is always a subspace of V.
2. $V \setminus U$ is never a subspace of V.
3. $V \setminus U$ is possibly a subspace of V. That depends on the concrete choices of V and U.

Solution. Correct answer: 2.

Exercise 1.3
Which of the following spaces U is a subspace of $\mathbb{R}^n$?

1. $U = \{x \in \mathbb{R}^n\colon x_1 = x_2 = \cdots = x_n\}$.
2. $U = \{x \in \mathbb{R}^n\colon x_1^2 = x_2^2\}$.
3. $U = \{x \in \mathbb{R}^n\colon x_1 = 1\}$.

.

Solution. Correct answer: 1.

Exercise 1.4
Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The n-tuple $(v_1, \dots, v_n)$ of elements of V will be a basis if the following holds:

1. $\lambda_1 v_1 + \lambda_2 v_2 + \dots \lambda_n v_n = 0$ only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.
2. If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, then $\lambda_1 v_1 + \lambda_2 v_2 + \dots \lambda_n v_n = 0$.
3. $\lambda_1 v_1 + \lambda_2 v_2 + \dots \lambda_n v_n = 0$ for all $(\lambda_1, \dots, \lambda_n) \in \mathbb{F}^n$.

Solution. Correct answer: 1.

Exercise 1.5
Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\{v_1, v_2, v_3\}$ be a linearly independent set of vectors in V. Then

1. $\{v_1, v_2\}$ is always linearly dependent.
2. $\{v_1, v_2\}$ may or may not be linearly dependent, depending on the choice of $\{v_1, v_2, v_3\}$.
3. $\{v_1, v_2\}$ is always linearly independent.

Solution. Correct answer: 3.

Exercise 1.6
Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If one were to define

$$U_1 - U_2 := \left\{ z \in V \colon \underset{x \in U_1}{\exists} \ \underset{y \in U_2}{\exists} \ z = x - y \right\}$$

for subspaces U_1, U_2 of V, then one would have

1. $U_1 - U_2 = \{0\}$.
2. $(U_1 - U_2) + U_2 = U_1$.
3. $U_1 - U_2 = U_1 + U_2$.

.

Solution. Correct answer: 3.

Exercise 1.7
The vector space $V = \{0\}$

1. has the basis $\{0\}$.
2. has the basis $\emptyset$.
3. does not have a basis.

.

Solution. Correct answer: 3.

Exercise 1.8
Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Subspaces U_1, U_2 of V are said to be *transverse* (to each other) if $U_1 + U_2 = V$. One calls $\operatorname{codim} U := \dim V - \dim U$ the *codimension* of U in V. For transverse U_1, U_2, one has

1. $\dim U_1 + \dim U_2 = \dim(U_1 \cap U_2)$.
2. $\dim U_1 + \dim U_2 = \operatorname{codim}(U_1 \cap U_2)$.
3. $\operatorname{codim} U_1 + \operatorname{codim} U_2 = \operatorname{codim}(U_1 \cap U_2)$.

Solution. Correct answer: 3.

Exercise 1.9
Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and $U \subset V$ a subspace. Then the orthogonal complement of U is defined as

1. $U^\perp = \{u \in U \colon u \perp U\}$.
2. $U^\perp = \{x \in V \colon x \perp U\}$.
3. $U^\perp = \{x \in V \colon x \perp U, \ \|x\| = 1\}$.

Solution. Correct answer: 2.

Exercise 1.10
Let $V = \mathbb{R}^2$ with the standard inner product. Which of the following tuples of elements of V forms an orthonormal basis?

1. $\left\{ \binom{1}{-1}, \binom{-1}{-1} \right\}$.
2. $\left\{ \binom{0}{-1}, \binom{-1}{0} \right\}$.
3. $\left\{ \binom{1}{0}, \binom{1}{1}, \binom{0}{1} \right\}$.

Solution. Correct answer: 2.

Exercise 1.11
Which of the following maps $\langle \cdot, \cdot \rangle_i \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2, 3, 4$, are inner products (where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$)?

1. $\langle a, b \rangle_1 := 2a_1b_1 + a_1b_2 + a_2b_1 + 2a_2b_2$.
2. $\langle a, b \rangle_2 := a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$.
3. $\langle a, b \rangle_3 := a_1^2b_1 + a_2b_2^2$.

Solution. Correct answer: 1.

Exercise 1.12
Let U, V be two three-dimensional vector spaces with bases $\mathcal{B}_U = \{a_1, a_2, a_3\}$ and $\mathcal{B}_V = \{b_1, b_2, b_3\}$. Let $L \colon U \to V$ be a linear map such that $La_i = b_i$, $i = 1, 2, 3$. Then the matrix A representing this map is

1. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.
2. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
3. $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution. Correct answer: 2.

Exercise 1.13
Let $L \colon U \to V$ be a surjective linear map between two vector spaces with $\dim U = 5$ and $\dim V = 3$. Then

1. $\dim \ker L \geq 3$.
2. $\dim \ker = 0, 1$, or 2. Any of these cases is possible.
3. $\dim \ker = 2$.

Solution. Correct answer: 3.

Exercise 1.14
Let $A \in \operatorname{Mat}(n \times n; \mathbb{R})$. Then

1. If $\operatorname{rank} A = n$, then n is invertible, but there exist invertible matrices with $\operatorname{rank} A \neq n$.
2. If A is invertible, then $\operatorname{rank} A = n$, but there exist matrices with $\operatorname{rank} A = n$ that are not invertible.
3. A is invertible if and only if $\operatorname{rank} A = n$.

Solution. Correct answer: 3.

Exercise 1.15
Let $A \in \operatorname{Mat}(m \times n; \mathbb{R})$ with $m \leq n$. Then

1. $\operatorname{rank} A \leq m$.
2. $m \leq \operatorname{rank} A \leq n$.
3. $n \leq \operatorname{rank} A$.

Solution. Correct answer: 1.

Exercise 1.16
Let $A \in \mathrm{Mat}(n \times n; \mathbb{R})$. Then

1. $\det A = 0$ implies $\operatorname{rank} A = 0$.
2. $\det A = 0$ if and only if $\operatorname{rank} A \leq n - 1$.
3. $\det A = 0$ implies $\operatorname{rank} A = n$.

.

Solution. Correct answer: 2.

Exercise 1.17
Let $\lambda > 0$. Then $\det \begin{pmatrix} \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda \end{pmatrix} =$

1. 0.
2. λ.
3. λ^4.

.

Solution. Correct answer: 1.

Exercise 1.18
Let $A \in \mathrm{Mat}(n \times n; \mathbb{R})$ with $\det A = 0$. Then the system of equations $Ax = b$

1. only has a solution if $b = 0$.
2. is solvable for all $b \in \mathbb{R}^n$, but perhaps not uniquely for all b.
3. is solvable for some $b \in \mathbb{R}^n$, but even if the solution exists, it is never unique.

Solution. Correct answer: 3.

Exercise 1.19
Let $A \in \mathrm{Mat}(n \times n; \mathbb{R})$. Then the system of equations $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$ if

1. $\dim \ker A = 0$
2. $\dim \operatorname{ran} A = 0$
3. $\dim \ker A = n$

.

Solution. Correct answer: 1.

Exercise 1.20
Let $A \in \mathrm{Mat}(n \times n; \mathbb{R})$ and $b \in \mathbb{R}^n$. Suppose that the system of equations $Ax = b$ has two independent solutions. Then

1. $\operatorname{rank} A \leq n$, the case $\operatorname{rank} A = n$ is possible.
2. $\operatorname{rank} A \leq n - 1$, the case $\operatorname{rank} A = n - 1$ is possible.
3. $\operatorname{rank} A \leq n - 2$, the case $\operatorname{rank} A = n - 2$ is possible.

Solution. Correct answer: 2.

Exercise 2.1
Let $u_1, u_2 \in \mathbb{R}^3$ be given by

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

and define the subspace $U = \operatorname{span}\{u_1, u_2\} \subset \mathbb{R}^3$.

1. Find an orthonormal basis of U.
2. Find the orthogonal projection P_U onto U in matrix form, i.e., express P_U as a 3×3 matrix.

   Recall that P_U must satisfy $P_U^2 = P_U$ and $P_U^T = P_U$ (but you don't need to verify this for your answer).
3. Find $\ker P_U$.

Solution.

1. Taking the vectors $v_1 = u_1 + u_2$ and $v_2 = u_1 - u_2$, **(1 Mark)** we see immediately that an orthonormal basis is given by

   $$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

   (2 Marks)
2. We have, for $x \in \mathbb{R}^3$,

   $$\begin{aligned} P_U x &= \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 \\ &= \frac{x_1 + x_3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} x_1 + x_3 \\ 2x_2 \\ x_1 + x_3 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

   so

   $$P_U = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

   (3 Marks)
3. The kernel of an orthogonal projection is orthgonal to its range, so by inspection of e_1 and de_2 we see that

   $$\ker P_U = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

   (2 Marks)

Exercise 2.2
Let $u_1, u_2 \in \mathbb{R}^3$ be given by

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and define the subspace $U = \operatorname{span}\{u_1, u_2\} \subset \mathbb{R}^3$.

1. Find an orthonormal basis of U.
2. Find the orthogonal projection P_U onto U in matrix form, i.e., express P_U as a 3×3 matrix.

   Recall that P_U must satisfy $P_U^2 = P_U$ and $P_U^T = P_U$ (but you don't need to verify this for your answer).
3. Find $\ker P_U$.

Solution.

1. Taking the vectors $v_1 = u_1 - u_2$ and $v_2 = u_2$, **(1 Mark)** we see immediately that an orthonormal basis is given by

$$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

   (2 Marks)

2. We have, for $x \in \mathbb{R}^3$,

$$\begin{aligned} P_U x &= \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 \\ &= \frac{x_1 + x_3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} x_1 + x_3 \\ 2x_2 \\ x_1 + x_3 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

   so

$$P_U = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

   (3 Marks)

3. The kernel of an orthogonal projection is orthgonal to its range, so by inspection of e_1 and de_2 we see that

$$\ker P_U = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

   (2 Marks)

Exercise 2.3
Let $u_1, u_2 \in \mathbb{R}^3$ be given by

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and define the subspace $U = \operatorname{span}\{u_1, u_2\} \subset \mathbb{R}^3$.

1. Find an orthonormal basis of U.
2. Find the orthogonal projection P_U onto U in matrix form, i.e., express P_U as a 3×3 matrix.

   Recall that P_U must satisfy $P_U^2 = P_U$ and $P_U^T = P_U$ (but you don't need to verify this for your answer).
3. Find $\ker P_U$.

Solution.

1. The two vectors are already orthogonal, **(1 Mark)** so we have an orthonormal basis given by

$$e_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \qquad e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

   (2 Marks)

2. We have, for $x \in \mathbb{R}^3$,

$$\begin{aligned} P_U x &= \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 \\ &= \frac{x_1 - x_2 - x_3}{3} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \frac{x_1 + x_3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{6} \begin{pmatrix} 5x_1 - 2x_2 + x_3 \\ -2x_1 + 2x_2 + 2x_3 \\ -5x_1 + 2x_2 + 5x_3 \end{pmatrix} \\ &= \begin{pmatrix} 5/6 & -1/3 & 1/6 \\ -1/3 & 1/3 & 1/3 \\ 1/6 & 1/3 & 5/6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned}$$

   so

$$P_U = \begin{pmatrix} 5/6 & -1/3 & 1/6 \\ -1/3 & 1/3 & 1/3 \\ 1/6 & 1/3 & 5/6 \end{pmatrix}.$$

   (3 Marks)

3. The kernel of an orthogonal projection is orthgonal to its range, so by inspection of e_1 and de_2 we see that

$$\ker P_U = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}.$$

   (2 Marks)

Exercise 3.1
Consider the endormorphism

$$L\colon \operatorname{Mat}(2\times 2;\mathbb{R}) \to \operatorname{Mat}(2\times 2;\mathbb{R}), \qquad\qquad L\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}-a_{12} & a_{12}+a_{22} \\ a_{11}+a_{22} & a_{21}+a_{22} \end{pmatrix}.$$

1. Using the ordered basis $\mathcal{B} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$ find the matrix $\varPhi_{\mathcal{B}}^{\mathcal{B}} \in \operatorname{Mat}(4\times 4;\mathbb{R})$ representing this linear map.
2. Using $\varPhi_{\mathcal{B}}^{\mathcal{B}}$, determine L^{-1} if it exists. If not, find $\ker L$ and $\operatorname{ran} L$.

Solution.

1. Using the given basis,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}.$$

Therefore, the matrix $\varPhi_{\mathcal{B}}^{\mathcal{B}}$ satisfies

$$\varPhi_{\mathcal{B}}^{\mathcal{B}} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}-a_{12} \\ a_{12}+a_{22} \\ a_{11}+a_{22} \\ a_{21}+a_{22} \end{pmatrix}$$

so that

$$\varPhi_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(2 Marks)

2. The last column of $\varPhi_{\mathcal{B}}^{\mathcal{B}}$ is the sum of the first three columns. Therefore, A is not invertible and L^{-1} does not exist. We also have

$$\operatorname{ran} \varPhi_{\mathcal{B}}^{\mathcal{B}} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

so that

$$\operatorname{ran} L = \left\{ A \in \operatorname{Mat}(2\times 2;\mathbb{R})\colon A = \begin{pmatrix} \alpha-\beta & \beta \\ \alpha & \gamma \end{pmatrix},\ \alpha,\beta,\gamma\in\mathbb{R} \right\}$$

(2 Marks) We further see that

$$\varPhi_{\mathcal{B}}^{\mathcal{B}} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}-a_{12} \\ a_{12}+a_{22} \\ a_{11}+a_{22} \\ a_{21}+a_{22} \end{pmatrix} = 0$$

if and only if

$$a_{11} = a_{12} = -a_{22} = a_{21}$$

so that

$$\ker L = \left\{ A \in \operatorname{Mat}(2\times 2;\mathbb{R})\colon A = \begin{pmatrix} \lambda & \lambda \\ \lambda & -\lambda \end{pmatrix},\ \lambda\in\mathbb{R} \right\}.$$

(2 Marks)

Exercise 3.2
Consider the endormorphism

$$L\colon \operatorname{Mat}(2\times 2;\mathbb{R}) \to \operatorname{Mat}(2\times 2;\mathbb{R}), \qquad L\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}-a_{22} & a_{12}+a_{22} \\ a_{11}+a_{12} & a_{21}+a_{12} \end{pmatrix}.$$

1. Using the ordered basis $\mathcal{B} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$ find the matrix $\Phi_{\mathcal{B}}^{\mathcal{B}} \in \operatorname{Mat}(4\times 4;\mathbb{R})$ representing this linear map.
2. Using $\Phi_{\mathcal{B}}^{\mathcal{B}}$, determine L^{-1} if it exists. If not, find $\ker L$ and $\operatorname{ran} L$.

Solution.

1. Using the given basis,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}.$$

Therefore, the matrix $\Phi_{\mathcal{B}}^{\mathcal{B}}$ satisfies

$$\Phi_{\mathcal{B}}^{\mathcal{B}} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}-a_{22} \\ a_{12}+a_{22} \\ a_{11}+a_{12} \\ a_{21}+a_{12} \end{pmatrix}$$

so that

$$\Phi_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

(2 Marks)

2. The second column of $\Phi_{\mathcal{B}}^{\mathcal{B}}$ is the sum of the other three columns. Therefore, A is not invertible and L^{-1} does not exist. We also have

$$\operatorname{ran} \Phi_{\mathcal{B}}^{\mathcal{B}} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

so that

$$\operatorname{ran} L = \left\{ A \in \operatorname{Mat}(2\times 2;\mathbb{R})\colon A = \begin{pmatrix} \alpha & \beta \\ \alpha+\beta & \beta+\gamma \end{pmatrix},\ \alpha,\beta,\gamma \in \mathbb{R} \right\}$$

(2 Marks) We further see that

$$\Phi_{\mathcal{B}}^{\mathcal{B}} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}-a_{22} \\ a_{12}+a_{22} \\ a_{11}+a_{12} \\ a_{21}+a_{12} \end{pmatrix} = 0$$

if and only if

$$a_{11} = a_{22} = -a_{12} = a_{21}$$

so that

$$\ker L = \left\{ A \in \operatorname{Mat}(2\times 2;\mathbb{R})\colon A = \begin{pmatrix} \lambda & -\lambda \\ \lambda & \lambda \end{pmatrix},\ \lambda \in \mathbb{R} \right\}.$$

(2 Marks)

Exercise 3.3
Consider the endormorphism

$$L\colon \operatorname{Mat}(2\times 2;\mathbb{R}) \to \operatorname{Mat}(2\times 2;\mathbb{R}), \qquad L\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{12}-a_{22} & a_{11}+a_{22} \\ a_{11}+a_{12} & a_{21}+a_{11} \end{pmatrix}.$$

1. Using the ordered basis $\mathcal{B} = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$ find the matrix $\Phi^{\mathcal{B}}_{\mathcal{B}} \in \operatorname{Mat}(4\times 4;\mathbb{R})$ representing this linear map.
2. Using $\Phi^{\mathcal{B}}_{\mathcal{B}}$, determine L^{-1} if it exists. If not, find $\ker L$ and $\operatorname{ran} L$.

Solution.

1. Using the given basis,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}.$$

Therefore, the matrix $\Phi^{\mathcal{B}}_{\mathcal{B}}$ satisfies

$$\Phi^{\mathcal{B}}_{\mathcal{B}} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{12}-a_{22} \\ a_{11}+a_{22} \\ a_{11}+a_{12} \\ a_{21}+a_{11} \end{pmatrix}$$

so that

$$\Phi^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(2 Marks)

2. The fourth column of $\Phi^{\mathcal{B}}_{\mathcal{B}}$ is the difference of the first and the second columns. Therefore, A is not invertible and L^{-1} does not exist. We also have

$$\operatorname{ran} \Phi^{\mathcal{B}}_{\mathcal{B}} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

so that

$$\operatorname{ran} L = \left\{ A \in \operatorname{Mat}(2\times 2;\mathbb{R})\colon A = \begin{pmatrix} \alpha & \beta \\ \alpha+\beta & \gamma \end{pmatrix},\ \alpha,\beta,\gamma \in \mathbb{R} \right\}$$

(2 Marks) We further see that

$$\Phi^{\mathcal{B}}_{\mathcal{B}} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{12}-a_{22} \\ a_{11}+a_{22} \\ a_{11}+a_{12} \\ a_{21}+a_{11} \end{pmatrix} = 0$$

if and only if

$$a_{12} = a_{22} = -a_{11} = a_{21}$$

so that

$$\ker L = \left\{ A \in \operatorname{Mat}(2\times 2;\mathbb{R})\colon A = \begin{pmatrix} -\lambda & \lambda \\ \lambda & \lambda \end{pmatrix},\ \lambda \in \mathbb{R} \right\}.$$

(2 Marks)

Exercise 4.1
Let $a, b \in \mathbb{R}^n \setminus \{0\}$ be non-zero vectors with $a \perp b$. Define the map

$$S\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad Sx = x + \langle b, x\rangle a$$

1. Verify that S is a linear map.
2. Express S as a matrix in terms of a and b.
3. Show that $\det S = 1$.

Solution.

1. For $\lambda, \mu \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$ we calculate

$$\begin{aligned} S(\lambda x + \mu y) &= \lambda x + \mu y + \langle b, (\lambda x + \mu y)\rangle a \\ &= \lambda x + \mu y + \lambda\langle b, x\rangle a + \mu\langle b, y\rangle a \\ &= \lambda Sx + \mu Sy \end{aligned}$$

   which shows that S is linear. **(1 Mark)**

2. We have
$$Sx = x + \langle b, x\rangle a = x + a\langle b, x\rangle = x + a(b^T x) = x + (ab^T)x$$
   so $S = \mathbb{1} + ab^T$. **(1 Mark)**

3. We start by taking a and b and choosing another $n-2$ vectors to obtain a basis of $\mathbb{R}^n$. After applying Gram-Schmidt orthonormalization, we define the matrix whose columns consist of the orthonormal basis,

$$U = (\widetilde{a}, \widetilde{b}, e_1, \ldots, e_{n-2})$$

   where $\widetilde{a} = \frac{1}{|a|}a$, $\widetilde{b} = \frac{1}{|b|}b$ are the normalized vectors, $|a|^2 = \langle a, a\rangle$.
   Since the columns of U are orthonormal, $U^{-1} = U^T$ exists.
   Since b is orthogonal to all columns of U except the second, we have

$$\begin{aligned} SU &= S(\widetilde{a}, \widetilde{b}, e_1, \ldots, e_{n-2}) \\ &= (S\widetilde{a}, S\widetilde{b}, Se_1, \ldots, Se_{n-2}) \\ &= (\widetilde{a}, \widetilde{b} + |b|a, e_1, \ldots, e_{n-2}) \end{aligned}$$

   and so

$$\begin{aligned} \det(SU) &= \det(\widetilde{a}, \widetilde{b} + |b|a, e_1, \ldots, e_{n-2}) \\ &= \det(\widetilde{a}, \widetilde{b}, e_1, \ldots, e_{n-2}) + \underbrace{\det(\widetilde{a}, |b|a, e_1, \ldots, e_{n-2})}_{=0} \\ &= \det U. \end{aligned}$$

   Finally,

$$\det S = \det(SUU^{-1}) = \det(SU)\det U^{-1} = \det U \det U^{-1} = \det(UU^{-1}) = \det \mathbb{1} = 1.$$

   (4 Marks)