Differential form.

$$\int_a^b f'(x)\,dx = f(a) - f(b) \Rightarrow \int_{[a,b]} df = \int_{\partial[a,b]} f = f(b) - f(a)$$

$$\int_M d\omega = \int_{\partial M} \omega$$

Multilinear setup

Define:

$$dx_i : \mathbb{R}^n \longrightarrow \mathbb{R} \quad i = 1, \cdots, n \qquad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n, \quad dx_i(v) = v_i.$$

✿ dual space : $(\mathbb{R}^n)^* = \text{span}\{dx_i\}_{1 \le i \le n}$

Now, define $I = (i_1, \cdots, i_k)$

$$dx_I : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k} \longrightarrow \mathbb{R} \qquad dx_I(v_1, \cdots, v_k) = \begin{vmatrix} dx_{i_1}(v_1) & \cdots & dx_{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ dx_{i_k}(v_1) & \cdots & dx_{i_k}(v_k) \end{vmatrix}$$

Ex: $n = 3$, $I = (1, 3)$, $v_1 = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$ $v_2 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

$$dx_{(1,3)}\left(\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}\right) = \begin{vmatrix} dx_1(v_1) & dx_1(v_2) \\ dx_3(v_1) & dx_3(v_2) \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} = 11.$$

Property

If $i_j = i_k$ for $j \neq k \Rightarrow dx_I = 0$.

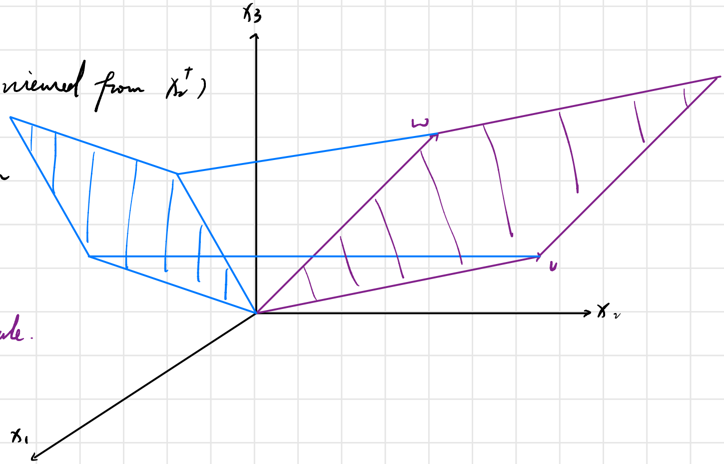
$\Lambda^k(\mathbb{R}^n)^*$: vector space from $(\mathbb{R}^n)^k$ to $\mathbb{R}$. $T \in \Lambda^k(\mathbb{R}^n)^* \Rightarrow T = \sum_{I'} a_I\, dx_I$, $a_I = T(e_{i_1}, \cdots, e_{i_k})$

$\dim(\Lambda^k(\mathbb{R}^n)^*) = \binom{n}{k} = C_n^k$. ✿ $\Lambda^0(\mathbb{R}^n)^* = \mathbb{R}$.

the signed area (viewed from x_3^+)

of this parallelogram

in $\underbrace{dx_{(3,1)}}_{\text{right hand rule.}}(v, w)$



Wedge product: generalization of cross product in $\mathbb{R}^3$

$dx_I \wedge dx_J = dx_{(I,J)}$ Ex: $dx_{(1,3)} \wedge dx_{(2,5)} = dx_{(1,3,2,5)} = -dx_{(1,2,3,5)}$

Let $\omega = \sum a_I dx_I$, $\eta = \sum b_J dx_J$, $\omega \wedge \eta := \sum (a_I b_J)\, dx_I \wedge dx_J = \sum (a_I b_J)\, dx_{(I,J)}$

Ex:

Let $\omega = a_1 dx_1 + a_2 dx_2$, $\eta = b_1 dx_1 + b_2 dx_2$ $\omega, \eta \in \Lambda^1(\mathbb{R}^2)^* = (\mathbb{R}^2)^*$

$$\omega \wedge \eta = (a_1 dx_1 + a_2 dx_2) \wedge (b_1 dx_1 + b_2 dx_2)$$

$$= a_1 b_1 \underline{dx_1 \wedge dx_1} + a_1 b_2\, dx_1 \wedge dx_2 + a_2 b_1\, dx_2 \wedge dx_1 + a_2 b_2 \underline{dx_2 \wedge dx_2}$$

$$= a_1 b_2\, dx_{(1,2)} + a_2 b_1\, dx_{(2,1)} = \underbrace{a_1 b_2 - a_2 b_1}_{\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}!}\, dx_{(1,2)} \in \Lambda^2(\mathbb{R}^2)^*$$

Proposition

1 bilinear $(\omega + \phi) \wedge \eta = \omega \wedge \eta + \phi \wedge \eta$ and $(c\omega) \wedge \eta = c(\omega \wedge \eta)$

2 skew-commutative $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$, $\omega \in \Lambda^k(\mathbb{R}^n)^*$, $\eta \in \Lambda^l(\mathbb{R}^n)^*$ $\begin{bmatrix} i_1, \dots, i_k, j_1, \dots, j_l \\ kl \downarrow \\ j_1, \dots, j_l, i_1, \dots, i_k \end{bmatrix}$

3. associative $(\omega \wedge \eta) \wedge \phi = \omega \wedge (\eta \wedge \phi)$

Differential form on $\mathbb{R}^n$ & the exterior derivative

A 0-form on $\mathbb{R}^n$ is a smooth function. An n-form on $\mathbb{R}^n$ is

$$\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n$$

for a smooth $f(x)$.

A k-form on $\mathbb{R}^n$ is

smooth

$$\omega = \sum_{k\text{-tuples } I} f_I(x)\, dx_I = \sum_{i_1 < \cdots < i_k} f_I\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Remind: dx_I gives a basis of $\Lambda^k(\mathbb{R}^n)^*$. $k > n$, the only k-form is 0.

⊕ The set of k-form on $\mathbb{R}^n$ is a vector space, denoted by $A^k(\mathbb{R}^n)$

Key word: module of ring of smooth function

graded algebra

Proposition (summary)

Let $U \subset \mathbb{R}^n$ open. $\omega \in A^k(U)$, $\eta \in A^l(U)$, $\phi \in A^m(U)$.

1. If $k = l = m$, $\omega + \eta = \eta + \omega$; $(\omega + \eta) + \phi = \omega + (\eta + \phi)$
2. $\omega \wedge \eta = (-1)^{kl}(\eta \wedge \omega)$
3. $(\omega \wedge \eta) \wedge \phi = \omega \wedge (\eta \wedge \phi)$
4. If $k = l$, $(\omega + \eta) \wedge \phi = (\omega \wedge \phi) + (\eta \wedge \phi)$

We want: $df(x) = Df(x) \Rightarrow df(x) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j$.

In particular, if f is the i^{th} coordinate function, $df = dx_i$ and $dx_i(V) = Dx_i(V) = V_i$, as we expected!

If $\omega = \sum_I df_I \wedge dx_I \in A^k(\mathbb{R}^n)$, we define:

$$d\omega = \sum_I df_I \wedge dx_I = \sum_I \sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

Ex:

$f: \mathbb{R} \to \mathbb{R}$ is smooth, $df = f'(x)dx$.

$\omega = y\,dx \wedge x\,dy \in A^1(\mathbb{R}^2)$, $d\omega = dy \wedge dx + dx \wedge dy = 0$.

The operator d, called exterior derivative, has the following property:

Let $\omega \in A^k(U)$ and $\eta \in A^l(U)$, let f be a smooth function:

1. If $k = l$, $d(\omega + \eta) = d\omega + d\eta$
2. $d(f\omega) = df \wedge d\omega + f d\omega$
3. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ (consider $\omega = f dx_I$, $\eta = g dx_J$, and use product rule).
4. $d(d\omega) = 0$

pf (4): Let $\omega = f dx_I$, $d\omega = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$, $d(d\omega) = \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I$.

Since $dx_i \wedge dx_j = -dx_j \wedge dx_i$, $d(d\omega) = \sum_{i<j} \underbrace{\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right)}_{= 0} dx_i \wedge dx_j \wedge dx_I = 0$.

Pull Back

$$\int_a^b f(g(x))\, g'(x)\, dx \overset{u:=g(x)}{\underset{du=g'(x)dx}{\Rightarrow}} \int_{g(a)}^{g(b)} f(u)\, du. \quad \text{(Chain rule)}$$

Def:

Let $U \in \mathbb{R}^m$ open, $g: U \to \mathbb{R}^n$ smooth. If $\omega \in A^k(\mathbb{R}^n)$, we define $g^*\omega \in A^k(U)$ (the pullback of ω by g) as:

0-form: f

$$g^* f = f \circ g$$

1-form: if $g(u) = x$, then

$$g^* dx_i = dg_i = \sum_{j=1}^{m} \frac{\partial g_i}{\partial u_j} du_j$$

Then, let the pullback of a wedge product be the wedge product of the pullbacks:

$$g^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = dg_{i_1} \wedge \cdots \wedge dg_{i_k} = dg_I$$

Last, let the pullback of a sum to be the sum of pullbacks:

$$g^*\left(\sum_I f_I\, dx_I\right) = \sum_I (f_I \circ g)\, dg_I = \sum_I (f_I \circ g)\, dg_{i_1} \wedge \cdots \wedge dg_{i_k}$$

Ex:

(a) If $g: \mathbb{R} \to \mathbb{R}$, $g^*(f(x)\, dx) = f(g(u))\, g'(u)\, du$

(b) Let $g: \mathbb{R}^2 \to \mathbb{R}^2$

$$g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u\cos v \\ u\sin v \end{pmatrix}$$

if $\omega = x\, dx + y\, dy$

$$g^*\omega = (u\cos v)(\cos v\,du - u\sin v\,dv) + (u\sin v)(\sin v\,du + u\cos v\,dv)$$
$$= u\,du$$

exercise:

$g^*(dx \wedge dy) = ?$ (ans: $u\,du \wedge dv$)

Note that

$$g^* dx_I = \sum_{k\text{-tuples } J^{<}} \det\left(\frac{\partial g_I}{\partial u_J}\right) du_J \quad \text{i.e.,}$$

$$g^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{1 \le j_1 < \cdots < j_k \le m} \begin{vmatrix} \frac{\partial g_{i_1}}{\partial u_{j_1}} & \cdots & \frac{\partial g_{i_1}}{\partial u_{j_k}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{i_k}}{\partial u_{j_1}} & \cdots & \frac{\partial g_{i_k}}{\partial u_{j_k}} \end{vmatrix} du_{j_1} \wedge \cdots \wedge du_{j_k}$$

We need one more result for integration.

Proposition:

Let $U \subset \mathbb{R}^m$ be open, and let $g: U \to \mathbb{R}^n$ be smooth.

If $\omega \in A^k(\mathbb{R}^n)$, then

$$g^*(d\omega) = d(g^*\omega)$$

take it for granted, or ...

pf:

For 0-form, $d(g^*f) = d(f \circ g) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i} \circ g\right)\frac{\partial g_i}{\partial u_j}\right) du_j$

$$= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i} \circ g\right)\left(\sum_{j=1}^{m}\frac{\partial g_i}{\partial u_j}\,du_j\right) = \sum_{i=1}^{n} g^*\left(\frac{\partial f}{\partial x_i}\right) g^* dx_i$$

$$= g^*\left(\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\,dx_i\right) = g^*(df).$$

Since the pullback of a wedge product is the wedge product of pullback, we have $g^*(dx_I) = dg_I$. And since d and pullback is linear, it suffices to prove the case that

$$\omega = f\, dx_I.$$

Indeed. Since

$$g^*(d(f\, dx_I)) = g^*(df \wedge dx_I) = g^*(df) \wedge g^*(dx_I)$$

$$= g^*(df) \wedge dg_I = d((g^*f)\, dg_I)$$

$d((g^*f)\, dg_I) = d(g^*f) \wedge dg_I + g^*f \wedge d(dg_I)$

form $\downarrow$ 0-form

$= g^*(df) \wedge dg_I + 0$

$$= d(g^*(f\, dx_I))$$

Integration

smooth, integrable on any Ω

Given n-form $\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n$ on a region $\Omega \subset \mathbb{R}^n$,

we define

$$\int_\Omega \omega = \int_\Omega f \, dV$$

Change of Variables Theorem

Let $g : \mathbb{R}^n \supset \Omega \longrightarrow \mathbb{R}^n$ be smooth and one-to-one, with $\det(Dg) > 0$.

Then for any n-form $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ on $S = g(\Omega)$, we have

$$\int_S \omega = \int_\Omega g^* \omega .$$

Furthermore, if $\Omega \subset \mathbb{R}^k$, g's derivative has rank k almost everywhere. We say $M = g(\Omega) \subset \mathbb{R}^n$ is a parametrized k-dimensional manifold. If $\omega \in A^k(\mathbb{R}^n)$, we define

$$\int_M \omega = \int_\Omega g^* \omega$$

Remark.

If there are two g_1 and g_2 with

$$\det D(g_2^{-1} \circ g_1) > 0 \quad \text{(same orientation)}$$

then

$$\int_{\Omega_2} g_2^* \omega = \int_{\Omega_1} g_1^* \omega$$

$\Rightarrow$ the integral of ω over M is well defined.

(We omit the proof here).

Line integrals and Green's Theorem

We begin with a 1-form $\omega = \sum F_i \, dx_i$ on $\mathbb{R}^n$ and a parametrized curve C, given by $C' \Rightarrow g : [a,b] \longrightarrow \mathbb{R}^n$ with $g' \neq 0$. Then we define

$$\int_C \omega = \int_{[a,b]} g^* \omega = \int_a^b \sum_{i=1}^n F_i(g(t)) \, g_i'(t) \, dt$$

Now we define a vector field (vector valued function) $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$$

We then have

$$\int_C \omega = \int_a^b F(g(t)) \cdot g'(t) \, dt = \int_a^b F(g(t)) \cdot \frac{g'(t)}{\|g'(t)\|} \|g'(t)\| \, dt = \int_C F \cdot T \, ds$$

ds called element of arclength on C and T is the unit tangent vector.

Remark

Let C^- be the curve with $h : [a,b] \longrightarrow \mathbb{R}^n$, $h(u) = g(a+b-u)$. Then

$$\int_{[a,b]} h^* \omega = \int_a^b F(h(u)) \cdot h'(u) \, du = \int_a^b F(g(a+b-u)) \cdot (-g'(a+b-u)) \, du$$

$$= -\int_a^b F(g(t)) \cdot g'(t) \, dt \quad , \quad t = a+b-u$$

$$= -\int_{[a,b]} g^* \omega .$$

Ex.

Let C be the line segment from $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ to $\begin{pmatrix}2\\2\\2\end{pmatrix}$, and let $\omega = xy\,dz$.

$\int_C \omega = ?$ First, we find $g(t) = \begin{pmatrix}1+t\\-1+3t\\2t\end{pmatrix}$, $0 \le t \le 1$ as a parametrization. Then

$$\int_C \omega = \int_{[0,1]} g^* \omega = \int_0^1 (1+t)(-1+3t)(2\,dt) = 2\int_0^1 (3t^2+2t-1)\,dt = 2.$$

The fundamental theorem of calculus for line integral.

Proposition.

Suppose $\omega = df$ for $f \in \mathcal{C}^1$. Then for any path (piecewise-$\mathcal{C}^1$ manifold) C starting at A and ending at B, we have

$$\int_C \omega = f(B) - f(A)$$

or, when $F = \nabla f$, we have

$$\int_C F \cdot T\, ds = f(B) - f(A)$$

pf:

For C, we know it can be divided by

$$C = C_1 \cup C_2 \cup \cdots \cup C_k, \quad C_i \in \mathcal{C}^1, \quad g_i : [a_i, b_i] \longrightarrow \mathbb{R}^n$$

Let $g_j(a_j) = A_j$, $g_j(b_j) = B_j$. We see $A_1 = A$, $B_j = A_{j+1}$, $B_k = B$.

Then we see it suffices to prove the result for only C_i since

$$\int_C \omega = \sum_{i=1}^{k} \int_{C_i} \omega = \sum_{i=1}^{k} (f(B_i) - f(A_i)) = f(B) - f(A)$$

if the theorem holds for any C_i. Now we have

(only depend on t)

$$\int_{C_i} \omega = \int_{a_i}^{b_i} g_i^* \omega = \int_{a_i}^{b_i} g_i^*(df) = \int_{a_i}^{b_i} d(g^* f) = \int_{a_i}^{b_i} d(f \circ g^*)$$

$$= \int_{a_i}^{b_i} (f \circ g^*)'(t)\, dt = (f \circ g^*)(b_i) - (f \circ g^*)(a_i) = f(B) - f(A).$$

Theorem

Let $\omega = \sum F_i dx_i$ be a 1-form (or let F be the vector field) on an open set $U \subset \mathbb{R}^n$. TFAE:

1. $\oint_C \omega = 0$ for every closed curve $C \subset U$.

2. $\int_A^B \omega$ is path-independent in U

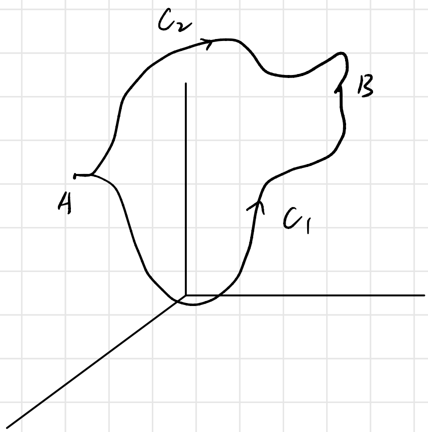
3. $\omega = df$ (or $F = \nabla f$) for some potential function f on U.

pf:

$(1) \to (2)$

If C_1, C_2 from $A \to B$, then $C := C_1 \cup C_2^-$ is a closed curve, then

$$0 = \oint_C \omega = \int_{C_1} \omega - \int_{C_2} \omega \Rightarrow \int_{C_1} \omega = \int_{C_2} \omega$$



$(2) \longrightarrow (3)$

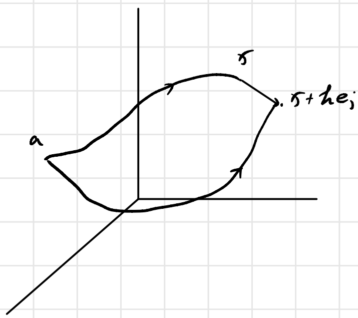
☆ We assume any two points in U can be joined by a path. If not, one must repeat the argument on each connected "piece" of U

Fixed $a \in U$, and define $f: U \to \mathbb{R}$ by

$$f(x) = \int_a^x \omega \quad \text{(for any path from } a \text{ to } x\text{)}$$

By path-independence, f is well defined. Now we show

$$\left.\frac{\partial f}{\partial x_j}\right|_x = F_j(x)$$



As the graph suggests,

$$\left.\frac{\partial f}{\partial x_j}\right|_x = \lim_{h \to 0} \frac{1}{h}\left(f\begin{pmatrix} x_1 \\ \vdots \\ x_j + h \\ \vdots \\ x_n \end{pmatrix} - f\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}\right)$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+he_j} \omega = \lim_{h \to 0} \frac{1}{h} \int_0^h F_j(x + te_j)\,dt$$

$$= F_j(x)$$

$(3) \longrightarrow (1)$

Clearly, since $\oint_C w = \oint_C df = 0$

Remark.

Given $w \in A^1(\mathbb{R}^n)$, by $d(dw) = 0$, a necessary condition for

$$w = df$$

for some function f (w exact) is that $dw = 0$ (w closed).

$\Rightarrow$ Not sufficient

$\Rightarrow$ the topology of the region on which w is defined is relevant.

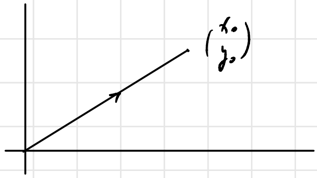
Finding a potential function

If $\int \omega$ is path-independent, then we can construct a potential function f, namely $df = \omega$. We show 3 ways by an example.

Ex.

$$\omega = (e^x + 2xy)\,dx + (x^2 + \cos y)\,dy$$

1\. Let C from $\begin{pmatrix}0\\0\end{pmatrix} \longrightarrow \begin{pmatrix}x_0\\y_0\end{pmatrix}$ with $g(t) = \begin{pmatrix}tx_0\\ty_0\end{pmatrix}$, $0 \le t \le 1$.



Then

$$f\begin{pmatrix}x_0\\y_0\end{pmatrix} = \int_0^{\bar{x}_0} \omega = \int_{[0,1]} g^* \omega$$

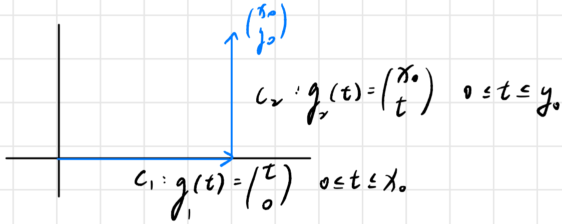
$$= \int_0^1 \left((e^{tx_0} + 2tx_0 ty_0)x_0 + (t^2 x_0^2 + \cos(ty_0))y_0 \right) dt$$

$$= (e^{x_0} + x_0^2 y_0 + \sin y_0) - 1.$$

So we set $f\begin{pmatrix}x\\y\end{pmatrix} = e^x + x^2 y + \sin y - 1$. (Check $df = \omega$!)

2.

Now, with two paths,



Then we have

$$f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \int_{C_1} \omega + \int_{C_2} \omega = \int_0^{x_0} e^t \, dt + \int_0^{y_0} (x_0^2 + \cos t)\, dt$$

$$= e^{x_0} - 1 + x_0^2 y_0 + \sin y_0$$

Again, $f\begin{pmatrix} x \\ y \end{pmatrix} = e^x - 1 + x^2 y + \sin y$.

3.

Only by antidifferentiating. We want $df = \omega$, which means

$$\frac{\partial f}{\partial x} = e^x + 2xy \qquad \frac{\partial f}{\partial y} = x^2 + \cos y$$

Fixed y, integrating the first equation, we have

$$f = \int e^x + 2xy \, dx = e^x + x^2 y + \underbrace{h(y)}_{\text{constant}}$$

Then we have $\dfrac{\partial f}{\partial y} = x^2 + h'(y) = x^2 + \cos y \Rightarrow h(y) = \sin y + C$

$$\Rightarrow f = e^x + x^2 y + \sin y + C.$$

Remark.

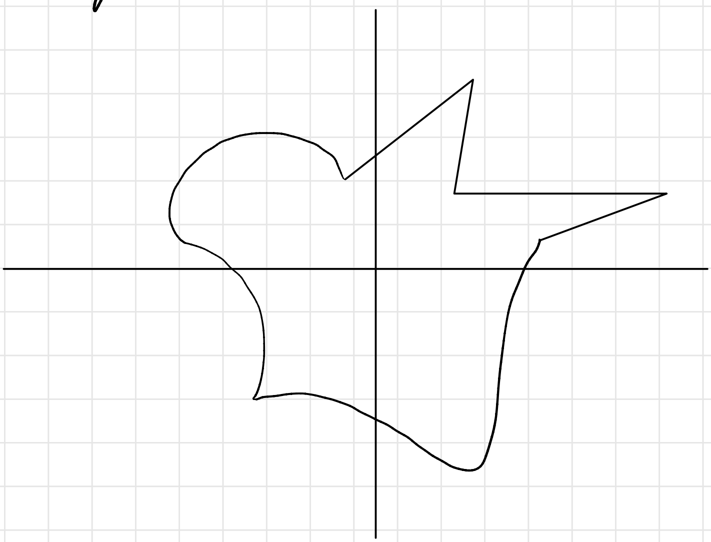
a. only needs a star region

b, c need a region which can reach every point along the axes.

Star-shaped region

Theorem:

Let $\omega \in A^1(\mathbb{R}^n)$ closed on a star-shaped region. Then ω is exact, namely $\exists f$ $df = \omega$. (No proof, no time :( )



Green's Theorem ($\mathbb{R}^2$)

Motivation:

We know if $\omega = df$, $\oint_C \omega = 0$ ($d\omega = d(df) = 0$).

Hence, we expect that the sign of $d\omega$ will affect $\oint_C \omega$.

Green's Theorem for a rectangle.

Let $R \subset \mathbb{R}^2$ be a rectangle, and let ω be a 1-form on R. Then

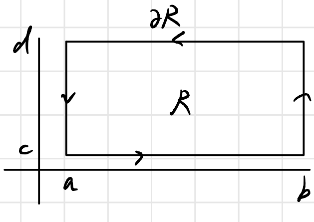
$$\int_{\partial R} \omega = \int_R d\omega. \qquad \oint \partial R \text{ is } \circlearrowleft$$

pf. Let $R = [a,b] \times [c,d]$. Write $\omega = P\,dx + Q\,dy$. Then

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

We then have:

$$\int_R d\omega = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



(Fubini) $$= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x}\, dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y}\, dy \right) dx$$

$$= \int_c^d \left(Q\begin{pmatrix} b \\ y \end{pmatrix} - Q\begin{pmatrix} a \\ y \end{pmatrix} \right) dy - \int_a^b \left(P\begin{pmatrix} d \\ x \end{pmatrix} - P\begin{pmatrix} c \\ x \end{pmatrix} \right) dx$$

$$= \int_a^b P\begin{pmatrix} x \\ c \end{pmatrix} dx + \int_c^d Q\begin{pmatrix} b \\ y \end{pmatrix} dy - \int_a^b P\begin{pmatrix} x \\ d \end{pmatrix} dx - \int_c^d Q\begin{pmatrix} a \\ y \end{pmatrix} dy$$

$$= \int_{\partial R} \omega.$$

Corollary

If $S \subset \mathbb{R}^2$ is parametrized by a rectangle, and ω is a 1-form on S, then

$$\int_{\partial S} \omega = \int_S d\omega$$

Pf:

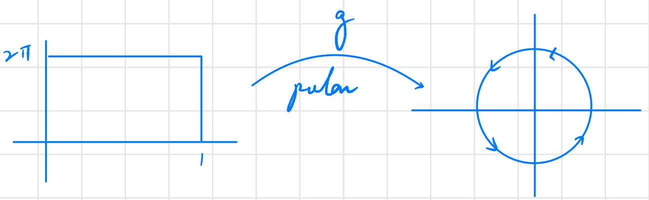
Let $g : R \to S \subset \mathbb{R}^2$ be a parametrization. Then

$$\int_{\partial S} \omega = \int_{\partial R} g^* \omega = \int_R d(g^*\omega) = \int_R g^*(d\omega) = \int_S d\omega .$$

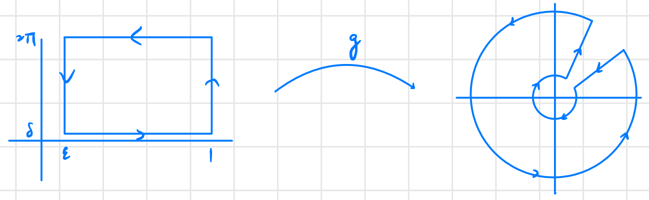
Ex.

D is a unit disk in $\mathbb{R}^2$, ω is a smooth 1-form.

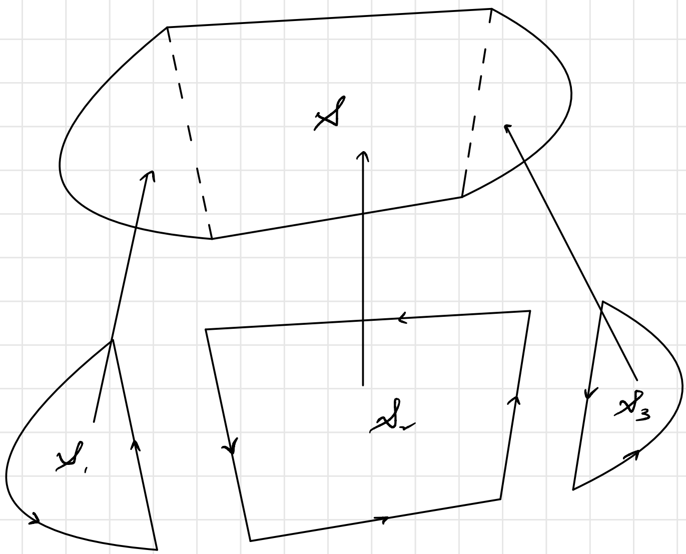
$$\int_{\partial D} \omega = \int_D d\omega \ ? \qquad \checkmark$$



What about D/D_a, $a < 1$? $\checkmark$



Generally, for a S which can be decomposed as a finite union of parametrized rectangles overlapping only along their edges.



$$\int_{\partial S} \omega = \sum_{i=1}^{k} \int_{\partial S_i} \omega = \sum_{i=1}^{k} \int_{S_i} d\omega = \int_{S} d\omega.$$

Surface integrals and flux

As expected, to define the integral of a 2-form over a parametrized surface S, we pullback and integrate:

When $\omega \in A^2(\mathbb{R}^n)$ and $S = g(U)$, we set

$$\int_S \omega = \int_U g^* \omega \quad \text{(if it exists)}$$

※ We omit the most part in surface integral, since we have a stronger tool: metric tensor. However, in terms of 2-form, we have something similar.

Def:

S be an oriented surface, its (oriented) area 2-form is the 2-form such that $\sigma(x)$ assigns to each pair of tangent vector at x the signed area of the parallelogram they span.

σ in $\mathbb{R}^3$

We claim that

$$\sigma = n_1\, dy \wedge dz + n_2\, dz \wedge dx + n_3\, dx \wedge dy \qquad N = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

is $S (\subset \mathbb{R}^3)$'s area two form.

※ $u, v \in T_S$, $\sigma(u, v) = \begin{vmatrix} N & u & v \end{vmatrix}$

Meaning of integrating a 2-form.

1 form of $F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$, $\omega = F_1\,dx_1 + F_2\,dx_2 + F_3\,dx_3$ along C : work.

2 form $\eta = F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$ over oriented S ?

If u, v are tangent to S, $\eta(u,v) = \begin{vmatrix} F & u & v \end{vmatrix} = (F \cdot n) \times (\sigma(u,v))$.

$\Rightarrow \int_S \eta$ represents the flux of F outward across S, often written

as

$$\int_S F \cdot n \, \underline{dS}$$ non-oriented surface area.

$\Rightarrow$ Now, for flux, you can just find η_F and integrate it!

Ex:

$F = \begin{pmatrix} xz^2 \\ yx^2 \\ zy^2 \end{pmatrix}$, S in a sphere with radius a center at origin.

Find the flux through S.

$$\eta = xz^2\,dy \wedge dz + yx^2\,dz \wedge dx + zy^2\,dx \wedge dy$$

$$g : (0,\pi) \times (0, 2\pi) \longrightarrow \mathbb{R}^3 \qquad g\binom{\phi}{\theta} = a \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

$$g^*\eta = a^5\left(\sin\phi\cos\theta\cos^2\phi\,(\sin^2\phi\cos\theta) + \sin^3\phi\sin\theta\cos^2\theta\,(\sin^2\phi\sin\theta) + \cos\phi\sin^2\theta\,(\sin\phi\cos\phi)\right) d\phi \wedge d\theta$$

$$= a^5\left(\sin^3\phi\cos^2\phi + \sin^5\phi\cos^2\theta\sin^2\theta\right) d\phi \wedge d\theta$$

Then

$$\int_S \eta = \int_{(0,\pi)\times(0,2\pi)} g^* \eta = \frac{4}{5}\pi a^5 .$$

Remark: $\sigma(u,v)$ only gives us 2-dim manifold surface area,

metric tensor can be generalized to n-dim manifold!

Stoke's Theorem

Integrating over a general compact, oriented k-dim manifold

We know how to integrate a k-form over a k-dim manifold by pulling back.

Instead of dealing with the whole manifold, we now chop this global problem into local ones.

Def:

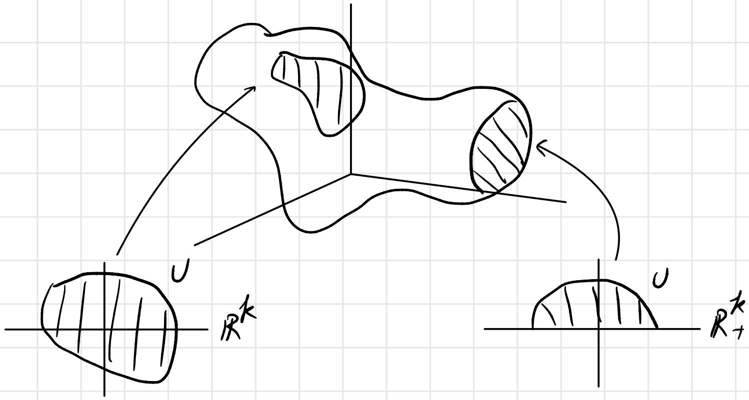
$M \subset \mathbb{R}^n$ is called a k-dim manifold with boundary if

$$\overset{\forall}{p \in M} \quad \overset{\exists}{W \subset \mathbb{R}^n} \quad W \text{ is open and } p \in W$$

and a parametrization $g: U \longrightarrow \mathbb{R}^n$ so that

i $\quad g(U) = V = W \cap M \quad$ and

ii $\quad U$ is an open subset either of $\mathbb{R}^k$ or $\mathbb{R}^k_+ = \{u \in \mathbb{R}^k : u_k \geq 0\}$



We say $p \in \partial M$ if $p = g(u)$ for some $u \in \partial \mathbb{R}^k_+ = \{u \in \mathbb{R}^k : u_k = 0\}$

$g(U)$ is sometimes called a coordinate chart on M.

Def: $g: U \to \mathbb{R}^n$, $\{\frac{\partial g}{\partial u_i}\}_i$ is a positive basis of T_xM.

Let $M \subset \mathbb{R}^n$ be an oriented k-dim manifold with boundary. Its (oriented) volume form is the k-form σ such that $\sigma(x)$ assigns to each k-tuple of tangent vector at x the signed volume of the parallelepiped they span.

To define integration,

Theorem:

Let $M \subset \mathbb{R}^n$ be a compact k-dim manifold with boundary. Then there are smooth real-valued functions $\rho_1, \cdots, \rho_N$ on M so that

i $0 \le \rho_i \le 1$ for all i

ii each ρ_i is 0 outside some coordinate ball

image of some ball under some parametrization

iii $\sum_{i=1}^{N} \rho_i = 1$.

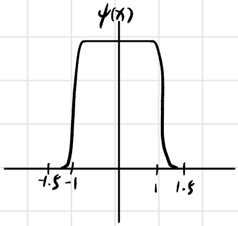
We omit the proof. Just purely construction!

Remark.

Such ρ_i are called a partition of unity.

Then, let $M \subset \mathbb{R}^n$ be a compact, oriented k-dim manifold (with piecewise-smooth boundary). Let $\omega \in A^k(M)$. Let $\{\rho_i\}$ be a partition of unity, and let g_i be the corresponding parametrizations (orientation preserving). Now we set

$$\int_M \omega = \int_M \left(\sum_{j=1}^{N} \rho_j \omega\right) = \underline{\sum_{j=1}^{N} \int_{B_2(0)} g_j^*(\rho_j \omega) = \sum_{j=1}^{N} \int_{B_2(0)} \psi\, g_j^* \omega}$$



technical detail from the above Theorem.

Stoke's Theorem

Let M be a compact, oriented k-dim manifold with boundary, and let ω be a smooth $(k-1)$ form on M. Then

$$\int_{\partial M} \omega = \int_M d\omega$$

where ∂M endowed with the boundary orientation.

Remark.

FTC, FTCL, GT are all special cases of this.

pf:

Both side of the equation are linear in ω, we can then use the partition of unity reduces the case that $\omega = 0$ outside a compact subset of a single coordinate chart $g: U \to \mathbb{R}^n$ (U is open in $\mathbb{R}^k$ or in $\mathbb{R}^k_+$) Then we have

$$\int_M d\omega = \int_{g(U)} d\omega = \int_U g^*(d\omega) = \int_U d(g^*\omega)$$

Now, $g^*\omega \in A^{k-1}(U)$, can be written as

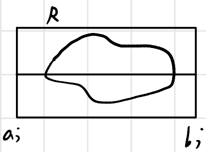
omit

$$g^*\omega = \sum_{i=1}^{k} f_i(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$$

Then,

$$\begin{aligned} d(g^*\omega) &= \sum_{i=1}^{k} \frac{\partial f_i(x)}{\partial x_i}\, dx_i \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k \\ &= \sum_{i=1}^{k} \frac{\partial f_i(x)}{\partial x_i} (-1)^{i-1}\, dx_1 \wedge \cdots \wedge dx_n \end{aligned}$$

Case I — U is open in $\mathbb{R}^k$



$\Rightarrow \omega = 0$ at ∂M, so we only need to show $\int_M d\omega = \int_U d(g^*\omega) = 0$.

Since $g^*\omega$ is smooth and 0 outside of a compact subset of U, we can choose a rectangle $R \supsetneq U$ and extend f_i to $\bar{f}_i$ by setting

$$\bar{f}_i = \begin{cases} f_i & x \in U \\ 0 & x \notin U \end{cases}$$

Finally, we integrate over $R = [a_1, b_1] \times \cdots \times [a_k, b_k]$:

$$\int_U d(g^* \omega) = \int_R \sum_{i=1}^{k} \left((-1)^{i-1} \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_k$$

$$= \sum_{i=1}^{k} (-1)^{i-1} \int_R \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k$$

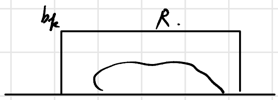
$$= \sum_{i=1}^{k} (-1)^{i-1} \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} \left(\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_k$$

$$= \sum_{i=1}^{k} (-1)^{i-1} \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} \left(f_i \begin{pmatrix} x_1 \\ \vdots \\ b_i \\ \vdots \\ x_k \end{pmatrix} - f_i \begin{pmatrix} x_1 \\ \vdots \\ a_i \\ \vdots \\ x_k \end{pmatrix} \right) dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_k$$

$$= 0$$

since $f_i = 0$ everywhere on ∂R.

Case II — U is open in $\mathbb{R}^k_+$



In this case, we again extend f_i to functions on $R \subsetneq \mathbb{R}^k_+$ by letting them be 0 outside of U. Then, $R = [a_1, b_1] \times \cdots \times [a_{k-1}, b_{k-1}] \times [0, b_k]$, so we have

$$\int_U d(g^* \omega) = \int_R \sum_{i=1}^{k} \left((-1)^{i-1} \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_k$$

$$= \sum_{i=1}^{k} (-1)^{i-1} \int_R \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k$$

$$= \sum_{i=1}^{k} (-1)^{i-1} \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} \left(\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_k$$

$$= \sum_{i=1}^{k} (-1)^{i-1} \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} \left(f_i\begin{pmatrix} x_1 \\ \vdots \\ b_i \\ \vdots \\ x_k \end{pmatrix} - f_i\begin{pmatrix} x_1 \\ \vdots \\ a_i \\ \vdots \\ x_k \end{pmatrix} \right) dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_k$$

$$= (-1)^{k-1} \int_{a_{k-1}}^{b_{k-1}} \cdots \int_{a_1}^{b_1} \left(f_k\begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ b_k \end{pmatrix} - f_k\begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ 0 \end{pmatrix} \right) dx_1 \cdots dx_{k-1}$$

$$= (-1)^{k} \int_{U \cap \partial \mathbb{R}^k_+} f_k\begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ 0 \end{pmatrix} dx_1 \cdots dx_{k-1}$$

$$= \int_{U \cap \partial \mathbb{R}^k_+} g^* \omega = \int_{\partial M} \omega \, . \quad \divideontimes$$

Analogy

We already see that for any function f, $df \in A^1(\mathbb{R}^n)$ is of.
Now we want to give the traditional intepretation of the exterior derivative to 1-form and 2-form.

Given a 1-form

$$\omega = F_1 dx_1 + F_2 dx_2 + F_3 dx_3 \in A^1(\mathbb{R}^3).$$

we have

$$d\omega = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1 \wedge dx_2$$

Correspondingly, given the vector field

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \quad \text{we set} \quad \underset{(\text{rot})}{\text{curl}}\, F = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}$$

Then we have our classical Stoke's Theorem:

$$\int_{\partial S} \underbrace{F \cdot T\, ds}_{\omega} = \int_S \underbrace{\text{curl}\, F \cdot n\, dS}_{d\omega}$$

Now, given the 2-form

$$\omega = F_1\, dx_2 \wedge dx_3 + F_2\, dx_3 \wedge dx_1 + F_3\, dx_1 \wedge dx_2 \in A^2(\mathbb{R}^3),$$

we then have

$$d\omega = \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3$$

Correspondingly, given the vector field

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \quad \text{we set} \quad \operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

Then we have the classical Gauss's Theorem

$$\int_{\partial\Omega} F \cdot n \, dS = \int_{\Omega} \operatorname{div} F \, dV$$

Summary

$$
\begin{tikzcd}[row sep=large]
\underline{\text{differential form}} & \underline{\text{field}} \\
\text{0-form} \arrow[d, "d"] \arrow[dd, bend right=60, "d^2=0"'] & \text{function (scalar field)} \arrow[d, "\text{grad}"] \arrow[dd, bend left=60, "\text{curl}(\text{grad})=0"] \\
\text{1-form} \arrow[d, "d"] \arrow[dd, bend right=60, "d^2=0"'] & \text{vector field (work)} \arrow[d, "\text{curl}"] \arrow[dd, bend left=60, "\text{div}(\text{curl})=0"] \\
\text{2-form} \arrow[d, "d"] & \text{vector field (flux)} \arrow[d, "\text{div}"] \\
\text{3-form} & \text{function (scalar field)}
\end{tikzcd}
$$