Inner product

i) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$

$\mathbb{R}$?

ii) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

[linear in 2nd entry]

iii) $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$

iv) $\langle u, v \rangle = \overline{\langle v, u \rangle}$

① Euclidean inner product $\sum x_i y_i$ ($\mathbb{R}$)

② Weighted " $\sum w_i x_i y_i$

$\Downarrow$

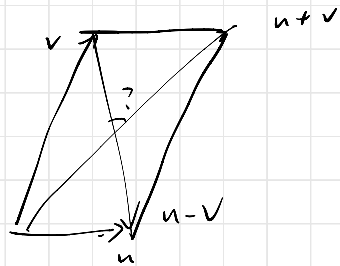
induced norm. $\sqrt{\langle \cdot, \cdot \rangle}$

Ex: pf: $\langle u, v \rangle = \dfrac{\|u+v\|^2 - \|u-v\|^2}{4}$, and geometry interpretation.

$$\frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 = \frac{1}{4}\langle u+v, u+v\rangle - \frac{1}{4}\langle u-v, u-v\rangle$$

$$= \frac{1}{4}\left(\langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle\right) - \frac{1}{4}\left(\langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle\right)$$

$$= \frac{1}{2}\langle u,v\rangle + \frac{1}{2}\langle v,u\rangle = \frac{1}{2}\langle u,v\rangle + \frac{1}{2}\langle u,v\rangle = \langle u,v\rangle$$



Ex: $x \in \mathbb{F}^n$, $\|x\|_\infty = \max_n \{|x|_n\}$

(1) $\frac{\|x\|_2}{\sqrt{n}} \le \|x\|_\infty \le \|x\|_2$

$$\|x\|_2^2 = \sum |x_i|^2 \le n\,|x_k|^2 = n \cdot \|x\|_\infty = (\sqrt{n}\,\|x\|_\infty)^2$$

$$\Rightarrow \|x\|_2 \le \sqrt{n}\,\|x\|_\infty \Rightarrow \frac{\|x\|_2}{\sqrt{n}} \le \|x\|_\infty$$

and $\|x\|_\infty = (|x_k|^2)^{1/2} \le \left(\sum |x_i|^2\right)^{1/2} = \|x\|_2$ $\blacksquare$

(2) $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2$

$\|x\|_2 = \left(\sum |x_i|^2\right)^{1/2} \le \sum |x_i| = \|x\|_1$. Let $u = \begin{bmatrix} |x_1| \\ \vdots \\ |x_n| \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

By Cauchy-Schwarz inequality, $|\langle u, v\rangle| \le \|u\|_2 \|v\|_2$

$$\Rightarrow \left|\sum |x_i|\right| \le \left(\sum |x_i|^2\right)^{1/2} \cdot n^{1/2} \Rightarrow \|x\|_1 \le \|x\|_2 \sqrt{n}.$$

(3) $\frac{\|x\|_1}{\sqrt{n}} \le \|x\|_\infty \le \|x\|_1$

$$\|x\|_1 = \sum |x_i| \le n \cdot |x_k| = n\,\|x\|_\infty$$

$\Rightarrow \frac{1}{n}\|x\|_1 \le \|x\|_\infty$. and $\|x\|_\infty = |x_k| \le \sum |x_i| = \|x\|_1$

Ex:

Find one weighted Euclidean inner product s.t.

$v_1 = (1, 0, \cdots, 0)$,

$v_2 = (0, \sqrt{2}, \cdots, 0)$

$\vdots$

$v_n = (0, \cdots, \sqrt{n})$

form an orthonormal basis

$1 = \langle v_i, v_i \rangle = w_i (\sqrt{i})(\sqrt{i}) = i w_i \Rightarrow w_i = \frac{1}{i}$

Why inner product? ① convergence

Ex: Consider $\|t\|_\infty = \sup_{i \in \mathbb{N}} |t_i|$ incomplete normed space.

$t^{(1)} = (1, 0, \cdots, 0, \cdots),$

$t^{(2)} = (1, \frac{1}{2}, 0, \cdots, 0, \cdots)$

$t^{(3)} = (1, \frac{1}{2}, \frac{1}{3}, 0, \cdots, 0, \cdots)$

consider $i > j$. $\Rightarrow$

$$\|t^i - t^j\|_\infty = \|(0, 0, \cdots, \frac{1}{j+1}, \cdots, \frac{1}{i}, 0, \cdots)\|_\infty$$

$$= \frac{1}{j+1} \longrightarrow 0 \text{ as } j \to \infty$$

$\Rightarrow$ cauchy sequence in M_0 (follow by 0)

But in $\mathbb{R}^{\mathbb{N}}$ $(\mathbb{R}^{\omega})$, t converges to

$(1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots) \notin M_0$

$\Rightarrow$ not complete

Why inner product? ② Complete normed V MUST have orthonormal basis. (Next)

Prove is base on Hausdorff maxmal principle (Zorn's Lemma) and ... "Topology"

Brief introduction of Topology:

Topolgical space : (S, T).

$T = \{ X \mid X \subseteq S$ is open set$\}$ (collection)

$$\begin{cases} 1. \ \emptyset \in T \\ 2. \ \bigcup t \in T \\ 3. \ \bigcap_{\text{finite}} t \in T \end{cases}$$

Basis: $\mathcal{B}$

$$\begin{cases} 1. \ \forall s \in S \quad \exists b \in \mathcal{B} \quad s \in b \\ 2. \ b_1, b_2 \in \mathcal{B} \Rightarrow \exists b_3 \in b_1 \cap b_2 \quad b_3 \in \mathcal{B} \end{cases}$$

Now, Topology induced by $\|\cdot\|$

Def: $\mathcal{B} = \{ B_\varepsilon(x) \mid \forall x \ \forall \varepsilon \}$

(for any $\dim V$)

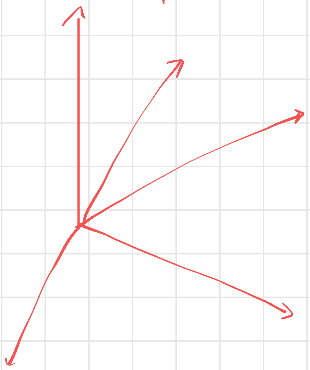
Importance: Continuity: $f^{-1}(\text{open})$ is open

Compactness

⋮

Separated axiom.

Gram-Schmidt



choose one as reference

↓

$v_2 - \text{proj}_{v_1} v_2 \perp v_1$

Projection P_V

i) P is linear

ii) $P(v) = v \quad \forall v \in V$

iii) $\mathrm{Im}(P) = V$

iv) $P^2 = P$.

Best Approx.

$\|v - P_W(v)\| \le \|v - w\|, \ \forall w \in W$

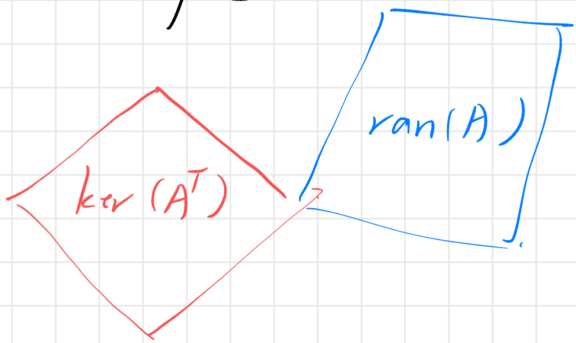
$$\ker(P_W) = W^{\perp}$$

$$v \in \ker(P) \Leftrightarrow P(v) = 0 \Leftrightarrow \text{proj}_W v = 0 \Leftrightarrow v \perp W$$

Important Theorem

both span whole space

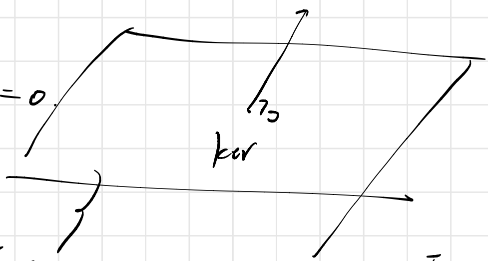
$\iff$



Show by $\perp$

Geometric: $L = \text{span}\left\{\begin{pmatrix}a\\b\\c\end{pmatrix}\right\}$

$ax+by+cz=0$



$\begin{pmatrix}a\\b\\c\end{pmatrix}^T \begin{pmatrix}x\\y\\z\end{pmatrix} = 0 \Rightarrow \ker\left(\begin{pmatrix}a\\b\\c\end{pmatrix}^T\right) = L^\perp$