4. Let $A, B \in Mat(n\times n; \mathbb{R})$, prove that $I-AB$ is inv. $\Leftrightarrow$ $I-BA$ is inv.

1.

Sps. $I-AB$ not inv., then $\ker(I-AB) \supsetneq \{0\} \Leftrightarrow \exists\, u\neq 0 \ (I-AB)u = 0$.

But then $0 = B(I-AB)u = (B-BAB)u = (I-BA)Bu \Rightarrow Bu \in \ker(I-BA)$.

Furthermore, since $(I-AB)u = 0 \Leftrightarrow u = ABu \neq 0 \Leftrightarrow Bu \neq 0 \Rightarrow \ker(I-BA) \supsetneq \{0\}$

$\Rightarrow I-BA$ is not inv. $\Rightarrow$ proved the converse.

2.

Sps $I-AB$ is inv. Let $C := (I-AB)^{-1}$, then

$$\begin{aligned} I &= I - BA + BA \\ &= I - BA + BC(I-AB)A \\ &= I - BA + BC(A - ABA) \\ &= I - BA + BCA(I-BA) \\ &= (I+BCA)(I-BA) \end{aligned}$$

$\Rightarrow (I-BA)^{-1} = (I+BCA) \Rightarrow I-BA$ is inv.

3.

$I-AB \Leftrightarrow I-BA$

$$\det(I-AB) = \det\begin{pmatrix} I & B \\ A & I \end{pmatrix} = \det\begin{pmatrix} I & B \\ A & I \end{pmatrix}\begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} = \det\begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}\begin{pmatrix} I & B \\ A & I \end{pmatrix}$$

$$= \det\begin{pmatrix} I-BA & 0 \\ A & I \end{pmatrix} = \det(I-BA) \ \text{※}$$

Topology.

Open ball

$$B_\varepsilon(a) := \{ x \in V : \|x-a\| < \varepsilon \}$$

An open set is a set such that $\forall_{v \in V} \exists_{\varepsilon>0} B_\varepsilon(v) \subset V$

$(x^{(m)})_{m\in\mathbb{N}} = (x_1^{(m)}, x_2^{(m)}, \dots, x_n^{(m)})$. Sps $\exists_{C>0}\ |x_k^{(m)}| < C$

$\Rightarrow \exists (x^{(m_j)})_{j\in\mathbb{N}} \to y$. ✩ There is a stopping point.

Interior point / boundary point / exterior point

$e(M) = \operatorname{int}(V \setminus M)$

M is open iff $M = \operatorname{int} M$

M is closed iff $\partial M \subset M$

M is closed iff $M = \overline{M}$

$\overline{M} := M \cup \partial M$ $\left(\{ x \in V : \exists_{(x_n)_{n\in\mathbb{N}}} x_n \in M \text{ and } x_n \to x \} \right.$

Ex: $A = \{(x,y) : 0 < x < 1,\ \ln x < y < 0\}$

1 open? Yes

2 $\overline{A}$? $\{(x,y) : 0 < x \le 1,\ \ln x \le y \le 0\} \cup \{(0,y) : y \le 0\}$

3. $\operatorname{int} A$, ∂A, $\operatorname{ext} A$?

$\operatorname{int} A = A$

$\partial A = \{(0,y) : y \le 0\} \cup \{(x,0) : 0 < x < 1\} \cup \{(x, \ln x) : 0 < x \le 1\}$

$\operatorname{ext} A = \mathbb{R}^2 / (\operatorname{int} A \cup \partial A)$

4 $\partial A \cap A$? $\emptyset$

Continuity $(U, \|\cdot\|_1), (V, \|\cdot\|_2)$. $f : U \to V$. We say f is cont. at a if

$\forall_{\varepsilon > 0}\ \exists_{\delta > 0}\ \forall_{x \in U}\ \|x - a\|_1 < \delta \Rightarrow \|f(x) - f(a)\|_2 < \varepsilon$

Ex:

$f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,0) = 0$ and $f(x,y) = (1 - \cos\frac{x^2}{y})\sqrt{x^2+y^2}$ for $y \ne 0$.

Show f is cont. at $(0,0)$ by def.

$\forall x, y \in \mathbb{R}$ $|f(x,y)| = |1 - \cos\frac{x^2}{y}|\sqrt{x^2+y^2} \le 2\sqrt{x^2+y^2} \longrightarrow 0.$

Ex:

Calculate $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$ if exists.

Let $y = \pm x^2 \Rightarrow \begin{cases} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \\ \frac{-x^4}{x^4 + x^4} = -\frac{1}{2} \end{cases}$?

Image / pre-image

$f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$? F

$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$? T

$f(A_1 \cap A_2) = f(\phi) = V.$

$(f(A_1) = a) \cap (f(A_2) = a) = a$ ξ.

Claim: Det is cont.

Take away: View Det is in form of polynomial.

Formally: 1 define a norm for det

2. Use sequence to prove its cont.

Tricky: Is Det a uniformly continuous function?

Compact: K is compact if $\forall$ sequence in K has a convergent with limit contained in K.

$n = \infty$: compact $\Rightarrow$ closed & bounded

$n < \infty$ compact $\Leftrightarrow$ closed & bounded.

f is cont. $\Rightarrow$ 1. $f(K)$ is compact

2. f has T

3. f is uniformly cont.

Compact is [,]'s generalization.

Uniformly cont: $f: \Omega \to V$

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in \Omega \ \|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$$

θ Det is not u. cont

$$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x, y \in \Omega \ \|x - y\| < \delta \wedge \|f(x) - f(y)\| \geq \varepsilon$$

$x := A_{n+1}$ $\quad A_n := \operatorname{diag}(\sqrt{n}, \sqrt{n}, 1, \cdots, 1)$

$y := A_n$

Let $\varepsilon := 1$, $\|\det A_{n+1} - \det A_n\| = 1$

$\|\cdot\|$ operator norm.

$$\|A_{n+1} - A_n\| = (\sqrt{n+1} - \sqrt{n}) = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \delta$$

Derivative

$\Omega \subset X$ open $f: \Omega \to V$ is differentiable at $x \in \Omega$

if $\overset{\exists}{L} \in \mathcal{L}(X, V)$ such that $f(x+h) = f(x) + L_x h + o(h)$ as $h \to 0$

where $L_x = Df|_x = df|_x$

$f(x) := x$

$x + h = x + 1 \cdot h + o \Rightarrow L_x \overset{\circledast}{=} 1 = \mathbb{1} = x.$

Ex $\circledast$: Identify domain and range $f: X \to V$

1. D $\quad C^1(\Omega, V) \longrightarrow C(\Omega, \mathcal{L}(X, V))$ $\quad f \mapsto Df$ linear

2. Df $\quad \Omega \longrightarrow \mathcal{L}(X, V)$ $\quad x \mapsto Df|_x$ non-linear

3. $Df|_x$ $\quad X \longrightarrow V$ $\quad y \mapsto Df|_x y$ linear

4. D^2 $\quad C^2(\Omega, V) \longrightarrow C(\Omega, \mathcal{L}(X, \mathcal{L}(X, V)))$ $\quad f \mapsto D^2 f$ linear

5. $D^2 f$ $\quad \Omega \longrightarrow \mathcal{L}(X, \mathcal{L}(X, V))$ $\quad x \mapsto D^2 f|_x$ non-linear

$\vdots$