Let $u_1, u_2 \in \mathbb{R}^3$ as $u_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $u_2 = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ and define $U = \text{span}\{u_i\}$

1. Find an orthonormal basis of U.
2. Find the orthogonal projection P_U onto U in matrix form.
3. Find $\ker P_U$.

1. Method 1: Gram-Schmidt:

Let $v_1' := u_1$, $v_1 := \dfrac{v_1'}{\|v_1'\|} = \dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$v_2' := u_2 - \langle u_2, v_1\rangle v_1 = \begin{pmatrix}1\\-1\\1\end{pmatrix} - \left\langle \begin{pmatrix}1\\-1\\1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right\rangle \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\-1\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\-4\\2\end{pmatrix}$$

$$v_2 = \frac{v_2'}{\|v_2'\|} = \frac{1}{\sqrt{1^2+2^2+1^2}}\begin{pmatrix}1\\-2\\1\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

Method 2.

Let $v_1' = u_1 + u_2$, $v_2' = u_1 - u_2 \Rightarrow v_1' = \begin{pmatrix}2\\0\\2\end{pmatrix}$, $v_2' = \begin{pmatrix}0\\2\\0\end{pmatrix}$

Normalize: $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$.

2. $\forall x \in \mathbb{R}^3$ $P_U x = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2$, here $e_1 = v_1$, $e_2 = v_2$.

Let $v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \Rightarrow P_U x = \left\langle x, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\right\rangle \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} + \left\langle x, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\rangle \begin{pmatrix}0\\1\\0\end{pmatrix}$

$$= \frac{1}{2}(x_1 + x_3)\begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}0\\x_2\\0\end{pmatrix} = \begin{pmatrix}\frac{x_1+x_3}{2}\\ x_2 \\ \frac{x_1+x_3}{2}\end{pmatrix}$$

read out: $P_U = \begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 1 & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}$.

3.

$$\ker P_U = \{ x \mid P_U x = \bar{0} \}$$

$$\Rightarrow \forall x \in \ker P_U \quad \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_3}{2} \\ x_2 \\ \frac{x_1+x_3}{2} \end{pmatrix} = 0 \Rightarrow x_2 = 0,\ x_1 = -x_3$$

$$\Rightarrow x = \begin{pmatrix} x_1 \\ 0 \\ -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \ker P_U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Consider the endomorphism:

$$\mathcal{L} : Mat(2\times 2;\mathbb{R}) \longrightarrow Mat(2\times 2;\mathbb{R}), \quad \mathcal{L}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}-a_{12} & a_{12}+a_{22} \\ a_{11}+a_{22} & a_{21}+a_{22} \end{pmatrix}$$

1. $\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$, find $\Phi^B_B \in Mat(4\times 4;\mathbb{R})$ representing this linear map.

2. Using Φ^B_B, determine $\mathcal{L}^{-1}$ if it exists. If not, find $\ker L$ and $\operatorname{ran} L$.

1. Using $\mathcal{B}$, we have $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}$

$$\begin{tikzcd} Mat(2\times 2) \arrow[r, "L"] \arrow[d, "\varphi_B"] & Mat(2\times 2) \arrow[d, "\varphi_B"] \\ R^4 \arrow[r, "\Phi^B_B"] & R^4 \end{tikzcd}$$

$$\Rightarrow \Phi^B_B \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}-a_{12} \\ a_{12}+a_{22} \\ a_{11}+a_{22} \\ a_{21}+a_{22} \end{pmatrix} \Rightarrow \Phi^B_B = \begin{array}{c} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \begin{matrix} \;1 & \;2 & \;3 & \;4 \end{matrix} \end{array}$$

2. Notice that $\underset{4}{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}} = \underset{1}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}} + \underset{2}{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}} + \underset{3}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \Rightarrow \operatorname{rank} \Phi^B_B < 4 \Rightarrow \Phi^B_B$ is not invertible

$\Rightarrow \mathcal{L}^{-1}$ does not exists. We can check that

$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is linearly independent

$$\Rightarrow \operatorname{ran} \Phi^B_B = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\Rightarrow \operatorname{ran} \mathcal{L} = \left\{ A \in Mat(2\times 2;\mathbb{R}) \,\middle|\, A = \begin{pmatrix} \alpha-\beta & \beta \\ \alpha & \gamma \end{pmatrix}, \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

If $\vec{a} \in \ker \Phi^B_B \Rightarrow \begin{pmatrix} a_{11}-a_{12} \\ a_{12}+a_{22} \\ a_{11}+a_{22} \\ a_{21}+a_{22} \end{pmatrix} = \vec{0} \Rightarrow \begin{cases} a_{11} = a_{12} \\ a_{12} = -a_{22} \\ a_{11} = -a_{22} \\ a_{21} = -a_{22} \end{cases} \Rightarrow a_{11} = a_{12} = a_{21} = -a_{22}$

$$\Rightarrow \ker L = \left\{ A \in Mat(2\times 2;\mathbb{R}) \,\middle|\, A = \begin{pmatrix} \lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}, \lambda \in \mathbb{R} \right\}$$

Let $a, b \in \mathbb{R}^n \setminus \{0\}$ be non-zero vector with $a \perp b$. Let

$$\mathcal{S} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad \mathcal{S}x = x + \langle b, x \rangle a.$$

1. Verify that $\mathcal{S}$ is a linear map.

2. Express $\mathcal{S}$ as a matrix in terms of a and b.

3. Show that $\det \mathcal{S} = 1$.

1. $\forall \alpha, \beta \in \mathbb{R} \quad \forall x, y \in \mathbb{R}^n$

$$\mathcal{S}(\alpha x + \beta y) = (\alpha x + \beta y) + \langle b, (\alpha x + \beta y) \rangle a = \alpha x + \langle b, \alpha x \rangle a + \beta y + \langle b, \beta y \rangle a$$

$$= \alpha (x + \langle b, x \rangle a) + \beta (y + \langle b, y \rangle a) = \alpha \mathcal{S}(x) + \beta \mathcal{S}(y).$$

2.

Notice that $\langle x, y \rangle = x^T y$ if $x, y \in \mathbb{R}^n$

$$\Rightarrow \mathcal{S}x = x + \langle b, x \rangle a = x + (b^T x) a = x + a (b^T x) = x + (ab^T) x = (I + ab^T) x$$

$$\Rightarrow \mathcal{S} = I + ab^T.$$

3.

Since $a \perp b \Rightarrow a, b$ are linearly independent. By basis extension theorem, we can obtain an orthonormal basis:

$$\mathcal{B} = \left\{ \frac{a}{\|a\|}, \frac{b}{\|b\|}, e_1, \cdots, e_{n-2} \right\}$$

$$=: \{ \tilde{a}, \tilde{b}, e_1, \cdots, e_{n-2} \}$$

$\star$ $\mathcal{B}$ is a basis $\Rightarrow \mathcal{B}^{-1}$ exists

Notice that $\mathcal{S}B = \mathcal{S}(\tilde{a}, \tilde{b}, e_1, \cdots, e_{n-2})$

$$= (\mathcal{S}\tilde{a}, \mathcal{S}\tilde{b}, \mathcal{S}e_1, \cdots, \mathcal{S}e_{n-2})$$

$$= (\breve{a} + \underbrace{\langle b, \breve{a}\rangle}_{0} a, \breve{b} + \langle b, \breve{b}\rangle a, e_1 + \underbrace{\langle b, e_1\rangle}_{0} e_1, \cdots, e_{n-2} + \underbrace{\langle b, e_{n-2}\rangle}_{0} e_{n-2})$$

$$= (\breve{a}, \breve{b} + |b|a, e_1, \cdots, e_{n-2})$$

$\Rightarrow \det \mathcal{S}B = \det(\tilde{a}, \tilde{b} + |b|a, e_1, \cdots, e_{n-2})$

$$= \det(\tilde{a}, \tilde{b}, e_1, \cdots, e_{n-2}) + |b| \det(\underbrace{\breve{a}, a}_{0}, e_1, \cdots, e_{n-2})$$

$$= \det B$$

$\Rightarrow \det \mathcal{S} = \det \mathcal{S}BB^{-1} = \det \mathcal{S}B \det B^{-1} = \det B \det B^{-1} = \dfrac{\det B}{\det B} = 1$ $\ast$

$$\det\begin{pmatrix} 0 & & & 1 \\ \vdots & & \iddots & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix}_n$$

$$= (-1)^{n+1} \cdot \det\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}_{n-1}$$

$$= (-1)^{n+1} \cdot (-1)^{n} \cdot \dots \cdot (-1)^{2} \cdot \det(1)$$

$$= (-1)^{\frac{(n+1+2)n}{2}} = (-1)^{\frac{n^2+3n}{2}}$$

if $n = 2k \Rightarrow \det = (-1)^{2k^2+3k} = (-1)^{2k^2+2k+k} = (-1)^{k}$

if $n = 2k+1 \Rightarrow \det = (-1)^{\frac{4k^2+4k+1+6k+3}{2}} = (-1)^{2k^2+5k+2}$

$$= (-1)^{2k^2+4k+2+k} = (-1)^{k}.$$