

Differential form.

$$\int_a^b f'(x) dx = f(b) - f(a) \Rightarrow \int_{[a,b]} df = \int_{[a,b]} f = f(b) - f(a)$$

$$\int_M d\omega = \int_M \omega$$

Multilinear setup.

Define:

$$dx_i : \mathbb{R}^n \rightarrow \mathbb{R} \quad i = 1, \dots, n \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n, \quad dx_i(v) = v_i.$$

↪ dual space: $(\mathbb{R}^n)^* = \text{span}\{dx_i\}_{i=1, \dots, n}$

Now, define $I = (i_1, \dots, i_k)$

$$dx_I : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n}_k \rightarrow \mathbb{R} \quad dx_I(v_1, \dots, v_k) = \begin{vmatrix} dx_{i_1}(v_1) & \dots & dx_{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ dx_{i_k}(v_1) & \dots & dx_{i_k}(v_k) \end{vmatrix}$$

$$\text{Ex: } n=3, \quad I=(1,3), \quad v_1 = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$dx_{(1,3)} \left(\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right) = \begin{vmatrix} dx_1(v_1) & dx_1(v_3) \\ dx_3(v_1) & dx_3(v_3) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 11.$$

Property

if $i_j = i_k$ for $j \neq k \Rightarrow dx_I = 0$.

$\Lambda^k(\mathbb{R}^n)^*$: vector space from $(\mathbb{R}^n)^k$ to \mathbb{R} . $T \in \Lambda^k(\mathbb{R}^n)^* \Rightarrow T = \sum_I a_I dx_I$, $a_I = T(e_{i_1}, \dots, e_{i_k})$

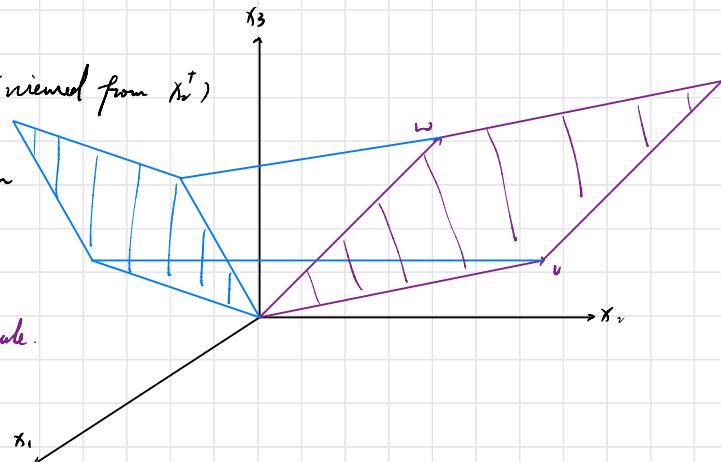
$$\dim(\Lambda^k(\mathbb{R}^n)^*) = \binom{n}{k} = C_n^k \quad \# \Lambda^k(\mathbb{R}^n)^* = \mathbb{R}.$$

the signed area (viewed from x_1^+)

of this parallelogram

in $d\chi_{(3,1)}(v, w)$

right hand rule.



Wedge product: generalization of cross product in \mathbb{R}^3

$$dx_I \wedge dx_J = d\chi_{(I,J)} \quad \text{Ex: } dx_{(1,2)} \wedge dx_{(2,3)} = d\chi_{(1,2,3,4)} = -d\chi_{(1,2,3,5)}$$

$$\text{Let } \omega = \sum a_I dx_I, \eta = \sum b_J dx_J, \quad \omega \wedge \eta = \sum (a_I b_J) dx_I \wedge dx_J = \sum (a_I b_J) d\chi_{(I,J)}$$

Ex:

$$\text{Let } \omega = a_1 dx_1 + a_2 dx_2, \eta = b_1 dx_1 + b_2 dx_2, \quad \omega \wedge \eta \in \Lambda^2(\mathbb{R}^2) = (\mathbb{R}^2)^*$$

$$\omega \wedge \eta = (a_1 dx_1 + a_2 dx_2) \wedge (b_1 dx_1 + b_2 dx_2)$$

$$= a_1 b_1 dx_1 \wedge dx_1 + a_1 b_2 dx_1 \wedge dx_2 + a_2 b_1 dx_2 \wedge dx_1 + a_2 b_2 dx_2 \wedge dx_2$$

$$= a_1 b_1 d\chi_{(1,2)} + a_2 b_1 d\chi_{(2,1)} = \frac{a_1 b_1 - a_2 b_1}{\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}} d\chi_{(1,2)} \in \Lambda^2(\mathbb{R}^2)$$

Proposition

$$1. \text{ bilinear } (\omega + \phi) \wedge \eta = \omega \wedge \eta + \phi \wedge \eta \text{ and } (c\omega) \wedge \eta = c(\omega \wedge \eta)$$

$$2. \text{ skew-commutative } \omega \wedge \eta = (-)^{kl} \eta \wedge \omega, \quad \omega \in \Lambda^k(\mathbb{R}^n)^*, \eta \in \Lambda^l(\mathbb{R}^n)^* \quad \left[\begin{matrix} i_1, \dots, i_k, j_1, \dots, j_l \\ k \downarrow \\ j_1, \dots, j_l, i_1, \dots, i_k \end{matrix} \right]$$

$$3. \text{ associative } (\omega \wedge \eta) \wedge \phi = \omega \wedge (\eta \wedge \phi)$$

Differential form on \mathbb{R}^n & the exterior derivative.

A 0-form on \mathbb{R}^n is a smooth function. An n -form on \mathbb{R}^n is

$$\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$$

for a smooth $f(x)$.

A k -form on \mathbb{R}^n is

smooth
↓

$$\omega = \sum_{k\text{-tuples } I} f_I(x) dx_I = \sum_{1 \leq i_1 < i_2 < \dots < i_k} f_{i_1, i_2, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Reminder: dx_I gives a basis of $\Lambda^k(\mathbb{R}^n)^*$. $k > n$, the only k -form is 0.

* The set of k -form on \mathbb{R}^n is a vector space, denoted by $\Lambda^k(\mathbb{R}^n)$

key word: module of ring of smooth function

graded algebra

Proposition (summary)

Let $U \subset \mathbb{R}^n$ open. $\omega \in \Lambda^k(U)$, $\eta \in \Lambda^l(U)$, $\phi \in \Lambda^m(U)$.

1. If $k+l=m$, $\omega + \eta = \eta + \omega$; $(\omega + \eta) + \phi = \omega + (\eta + \phi)$

2. $\omega \wedge \eta = (-1)^{kl} (\eta \wedge \omega)$

3. $(\omega \wedge \eta) \wedge \phi = \omega \wedge (\eta \wedge \phi)$

4. If $k+l$, $(\omega + \eta) \wedge \phi = (\omega \wedge \phi) + (\eta \wedge \phi)$

$$\text{We want: } df(x) = Df(x) \Rightarrow df(x) = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j.$$

In particular, if f is the i^{th} coordinate function, $df = dx_i$; and $dx_i(v) = Dx_i(v) = v_i$, as we expected!

If $\omega = \sum_I f_I dx_I \in A^k(\mathbb{R}^n)$, we define:

$$d\omega = \sum_I df_I \wedge dx_I = \sum_I \sum_{j=1}^n \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I \wedge \cdots \wedge dx_{i_k}.$$

Ex:

$f: \mathbb{R} \rightarrow \mathbb{R}$ is smooth, $df = f'(x)dx$.

$$\omega = y dx - x dy \in A^1(\mathbb{R}), \quad d\omega = dy \wedge dx + dx \wedge dy = 0.$$

The operator d , called exterior derivative, has the following property:

Let $\omega \in A^k(V)$ and $\eta \in A^l(V)$, let f be a smooth function:

$$1. \text{ If } k=l, \quad d(\omega + \eta) = d\omega + d\eta$$

$$2. \quad d(f\omega) = df \wedge d\omega + f d\omega$$

$$3. \quad d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \quad (\text{consider } \omega = f dx_I, \eta = g dx_J, \text{ and use product rule}).$$

$$4. \quad d(d\omega) = 0$$

$$\text{If } (f): \text{ Let } \omega = f dx_I, \quad d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_I, \quad d(d\omega) = \sum_{i=0}^n \sum_{j=0}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I.$$

$$\text{Since } dx_i \wedge dx_j = - dx_j \wedge dx_i, \quad d(d\omega) = \sum_{i,j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right) dx_i \wedge dx_j \wedge dx_I = 0.$$

Pull Back

$$\int_a^b f(g(x)) g'(x) dx \xrightarrow{u:=g(x)} \int_{g(a)}^{g(b)} f(u) du. \quad (\text{Chain rule})$$

Def:

Let $V \subset \mathbb{R}^m$ open, $g: V \rightarrow \mathbb{R}^n$ smooth. If $w \in \Lambda^k(\mathbb{R}^n)$, we define

$g^* w \in \Lambda^k(V)$ (the pullback of w by g) as:

0-form: f

$$g^* f = f \circ g$$

1-form: if $g(u) = x$, then

$$g^* dx_i = dg_i = \sum_{j=1}^m \frac{\partial g_i}{\partial u_j} du_j$$

Then, let the pullback of a wedge product be the wedge product of the pullbacks:

$$g^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = dg_{i_1} \wedge \cdots \wedge dg_{i_k} = dg_I$$

Last, let the pullback of a sum to be the sum of pullbacks:

$$g^* \left(\sum_I f_I dx_I \right) = \sum_I (f_I \circ g) dg_I = \sum_I (f_I \circ g) dg_{i_1} \wedge \cdots \wedge dg_{i_k}$$

Ex:

(a) If $g: \mathbb{R} \rightarrow \mathbb{R}$, $g^*(f(x) dx) = f(g(u)) g'(u) du$

(b) Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$g(u) = \begin{pmatrix} u \cos v \\ u \sin v \end{pmatrix}$$

if $\omega = x dx + y dy$

$$g^* \omega = (u \cos v) (\cos v du - u \sin v dv) + (u \sin v) (\sin v du + u \cos v dv)$$

$$= u du$$

Exercise:

$$g^*(dx_I dy) = ? \quad (\text{ans: } u du \wedge dv)$$

Note that

$$g^* dx_I = \sum_{k \text{-tuples } j^k} \det \left(\frac{\partial g_I}{\partial u_j} \right) du_j \quad \text{i.e.,}$$

$$g^*(dx_{i_1} \wedge \dots \wedge dx_{i_k}) = \sum_{1 \leq j_1 < \dots < j_k \leq m} \begin{vmatrix} \frac{\partial g_{i_1}}{\partial u_{j_1}} & \dots & \frac{\partial g_{i_1}}{\partial u_{j_k}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{i_k}}{\partial u_{j_1}} & \dots & \frac{\partial g_{i_k}}{\partial u_{j_k}} \end{vmatrix} du_{j_1} \wedge \dots \wedge du_{j_k}$$

We need one more result for integration.

Proposition:

Let $U \subset \mathbb{R}^m$ be open, and let $g: U \rightarrow \mathbb{R}^n$ be smooth.

If $\omega \in \Lambda^k(\mathbb{R}^n)$, then

$$g^*(d\omega) = d(g^*\omega)$$

Take it for granted, or ...

If:

$$\text{For } 0\text{-form, } d(g^*f) = d(f \cdot g) = \sum_{j=1}^m \left(\sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \cdot g \right) \frac{\partial g}{\partial u_j} \right) du_j$$

$$= \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \cdot g \right) \left(\sum_{j=1}^m \frac{\partial g}{\partial u_j} du_j \right) = \sum_{i=1}^n g^* \left(\frac{\partial f}{\partial x_i} \right) g^* dx_i$$

$$= g^* \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \right) = g^*(df)$$

Since the pullback of a wedge product is the wedge product of pullbacks, we have $g^*(dx_I) = dg_I$. And since d and

pullback is linear, it suffices to prove the case that

$$\omega = f dx_I.$$

Indeed. Since

$$g^*(d(f dx_I)) = g^*(df \wedge dx_I) = g^*(df) \wedge g^*(dx_I)$$

$$= g^*(df) \wedge dg_I = \underbrace{d((g^*f) dg_I)}_{d(g^*f) dg_I} = d(g^*f) \wedge dg_I$$

$$= d(g^*(f dx_I))$$

$$d(g^*f) dg_I = d(g^*f) \wedge dg_I + g^*f \wedge d(dg_I)$$

from o-form
 $= g^*(df) \wedge dg_I + 0$

Integration

smooth, integrable on any Ω

Given n -form $\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$ on a region $\Omega \subset \mathbb{R}^n$, we define

$$\int_{\Omega} \omega = \int_{\Omega} f dV$$

Change of Variables Theorem

Let $g: \mathbb{R}^n \setminus \Omega \rightarrow \mathbb{R}^n$ be smooth and one-to-one, with $\det(Dg) > 0$.

Then for any n -form $\omega = f dx_1 \wedge \cdots \wedge dx_n$ on $S = g(\Omega)$, we have

$$\int_S \omega = \int_{\Omega} g^* \omega.$$

Furthermore, if $\Omega \subset \mathbb{R}^k$, g 's derivative has rank k almost everywhere. We say $M = g(\Omega) \subset \mathbb{R}^n$ is a parametrized k -dimensional manifold. If $\omega \in \Lambda^k(\mathbb{R}^n)$, we define

$$\int_M \omega = \int_{\Omega} g^* \omega$$

Remark.

If there are two g_1 and g_2 with

$$\det D(g_1^{-1} \circ g_2) > 0 \quad (\text{same orientation})$$

then

$$\int_{\Omega_2} g_2^* \omega = \int_{\Omega_1} g_1^* \omega$$

\Rightarrow the integral of ω over M is well defined.

(We omit the proof here).

Line integrals and Green's Theorem

We begin with a 1-form $\omega = \sum F_i dx_i$ on \mathbb{R}^n and a parametrized curve C , given by $t \mapsto g: [a, b] \rightarrow \mathbb{R}^n$ with $g' \neq 0$. Then we define

$$\int_C \omega = \int_{[a, b]} g^* \omega = \int_a^b \sum_{i=1}^n F_i(g(t)) g'_i(t) dt$$

Now we define a vector field (vector valued function) $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$$

We then have

$$\int_C \omega = \int_a^b F(g(t)) \cdot g'(t) dt = \int_a^b F(g(t)) \cdot \frac{g'(t)}{\|g'(t)\|} \|g'(t)\| dt = \int_C F \cdot T ds$$

ds called element of arclength on C and T is the unit tangent vector.

Remark

Let \bar{C} be the curve with $\bar{h}: [a, b] \rightarrow \mathbb{R}^n$, $\bar{h}(u) = g(a+b-u)$. Then

$$\int_{[a, b]} \bar{h}^* \omega = \int_a^b F(\bar{h}(u)) \cdot \bar{h}'(u) du = \int_a^b F(g(a+b-u)) \cdot (-g'(a+b-u)) du$$

$$= - \int_a^b F(g(t)) \cdot g'(t) dt, \quad t = a+b-u$$

$$= - \int_{[a, b]} g^* \omega.$$

Ex.

Let C be the line segment from (-1) to (2) , and let $w = xy \, dz$.

$\int_C w = ?$ First, we find $g(t) = \begin{pmatrix} 1+t \\ -1+3t \\ rt \end{pmatrix}$, $0 \leq t \leq 1$ as a parametrization. Then

$$\int_C w = \int_{[0,1]} g^* w = \int_0^1 (1+t)(-1+3t)(r \, dt) = r \int_0^1 (1+3t+2t^2-1) \, dt = 2r.$$

The fundamental Theorem of calculus for line integral.

Proposition.

Suppose $\omega = df$ for $f \in C^1$. Then for any path (piecewise- C^1 manifold) C starting at A and ending at B , we have

$$\int_C \omega = f(B) - f(A)$$

or, when $F = \nabla f$, we have

$$\int_C F \cdot T ds = f(B) - f(A)$$

pf:

For C , we know it can be divided by

$$C = C_1 \cup C_2 \cup \dots \cup C_k, \quad C_i \in C^1, \quad j_i : [a_i, b_i] \rightarrow \mathbb{R}^n$$

Let $g_j(a_j) = A_j, g_j(b_j) = B_j$. We see $A_1 = A, B_k = B$.

Then we see it suffices to prove the result for only C_i since

$$\int_C \omega = \sum_{i=1}^k \int_{C_i} \omega = \sum_{i=1}^k (f(B_i) - f(A_i)) = f(B) - f(A)$$

if the Theorem holds for any C_i . Now we have

$$\begin{aligned} \int_{C_i} \omega &= \int_{a_i}^{b_i} g_i^* \omega = \int_{a_i}^{b_i} g_i^*(df) = \int_{a_i}^{b_i} d(g_i^* f) = \int_{a_i}^{b_i} d(f \circ g_i^*) \\ &= \int_{a_i}^{b_i} (f \circ g_i^*)'(t) dt = (f \circ g_i^*)(b_i) - (f \circ g_i^*)(a_i) = f(B) - f(A). \end{aligned}$$

Theorem

Let $\omega = \sum F_i dx_i$ be a 1-form (or let F be the vector field) on an open set $U \subset \mathbb{R}^n$. TFAE:

1. $\oint_C \omega = 0$ for every closed curve $C \subset U$.

2. $\int_A^B \omega$ is path-independent in U

3. $\omega = df$ (or $F = \nabla f$) for some potential function f on U .

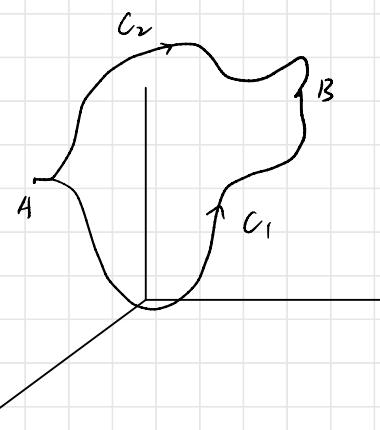
pf:

(1) \rightarrow (2)

If C_1, C_2 from $A \rightarrow B$, then $C = C_1 \cup C_2^-$ is a closed curve,

then

$$0 = \oint_C \omega = \int_{C_1} \omega - \int_{C_2} \omega \Rightarrow \int_{C_1} \omega = \int_{C_2} \omega$$



(2) \rightarrow (3)

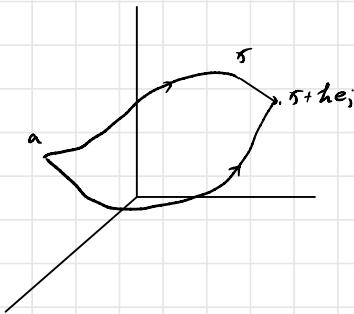
We assume any two points in V can be joined by a path. If not, one must repeat the argument on each connected "piece" of V .

Fix $a \in V$, and define $f: V \rightarrow \mathbb{R}$ by

$$f(x) = \int_a^x w \quad (\text{for any path from } a \text{ to } x)$$

By path-independence, f is well defined. Now we show

$$\frac{\partial f}{\partial x_i} \Big|_x = F_i(x)$$



As the graph suggests,

$$\frac{\partial f}{\partial x_i} \Big|_x = \lim_{h \rightarrow 0} \frac{1}{h} \left(f \begin{pmatrix} x_1 \\ \vdots \\ x_i + h \\ \vdots \\ x_n \end{pmatrix} - f \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} \right)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+he_i} w = \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h F_i(x + te_i) dt$$

$$= F_i(x)$$

(3) \rightarrow (1)

Clearly, since $\oint_C w = \oint_C df = 0$

Remark.

Given $w \in A^1(\mathbb{R}^n)$, by $d(dw) = 0$, a necessary condition for

$$w = df$$

for some function $f(w)$ exact is that $dw = 0$ (w closed).

\Rightarrow Not sufficient

\Rightarrow the topology of the region on which w is defined is relevant.

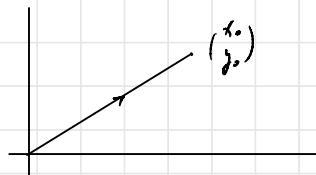
Finding a potential function

If $\int \omega$ is path-independent, then we can construct a potential function f , namely $df = \omega$. We show 3 ways by an example.

Eg.

$$\omega = (e^x + xy) dx + (x^2 + \cos y) dy$$

1. Let c from (x_0, y_0) to (\bar{x}_0, \bar{y}_0) with $g(t) = (tx_0, ty_0)$, $0 \leq t \leq 1$.



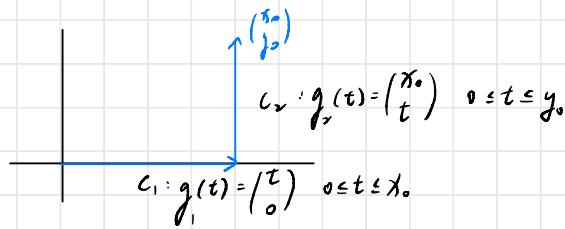
Then

$$\begin{aligned} f\left(\frac{\bar{x}_0}{\bar{y}_0}\right) &= \int_{x_0}^{\bar{x}_0} \omega = \int_{t=0,17} g^* \omega \\ &= \int_0^1 ((e^{tx_0} + tx_0 \cdot ty_0)x_0 + (t^2x_0^2 + \cos(ty_0))y_0) dt \\ &= (e^{\bar{x}_0} + \bar{x}_0 \bar{y}_0 + \sin \bar{y}_0) - 1. \end{aligned}$$

So we set $f(y) = e^x + x^2 y + \sin y - 1$. (Check $df = \omega$!)

2.

Now, with two paths,



Then we have

$$\begin{aligned} f\left(\frac{x_0}{y_0}\right) &= \int_{C_1} w + \int_{C_2} w = \int_0^{x_0} e^t dt + \int_0^{y_0} (x_0^2 + \cos t) dt \\ &= e^{x_0} - 1 + x_0^2 y_0 + \sin y_0. \end{aligned}$$

Again, $f\left(\frac{x}{y}\right) = e^x - 1 + x^2 y + \sin y$.

3.

Only by antiderivating. We want $df = w$, which means

$$\frac{\partial f}{\partial x} = e^x + 2xy \quad \frac{\partial f}{\partial y} = x^2 + \cos y$$

Fixed y , integrating the first equation, we have

$$f = \int e^x + 2xy \, dx = e^x + x^2 y + \underbrace{h(y)}_{\text{constant}}$$

Then we have $\frac{\partial f}{\partial y} = x^2 + h'(y) = x^2 + \cos y \Rightarrow h'(y) = \sin y + C$

$$\Rightarrow f = e^x + x^2 y + \sin y + C.$$

Remark.

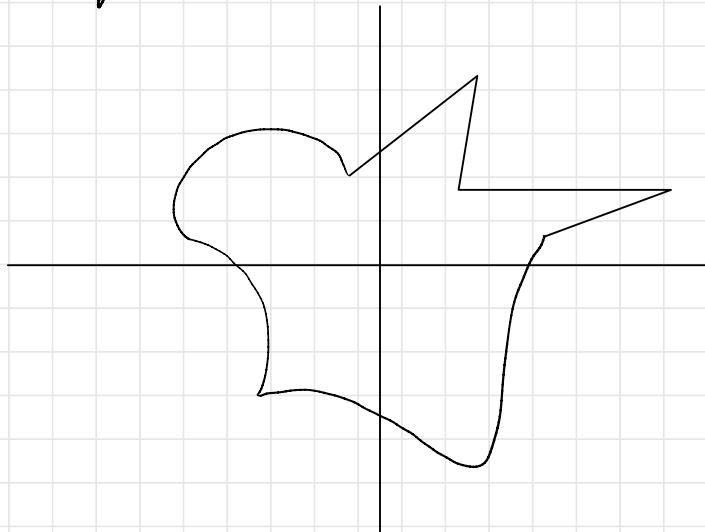
a. only needs a star region

b,c need a region which can reach every point along the axes.

Star-shaped region

Theorem:

Let $\omega \in \Lambda^1(\mathbb{R}^n)$ closed on a star-shaped region. Then ω is exact, namely $\exists f$ $df = \omega$. (No proof, no time !!)



Green's Theorem (\mathbb{R}^2)

Motivation:

We know if $w = df$, $\oint_C w = 0$ ($dw = d(df) = 0$).

Hence, we expect that the size of dw will affect $\oint_C w$.

Green's Theorem for a rectangle.

Let $R \subset \mathbb{R}^2$ be a rectangle, and let w be a 1-form on R .

Then

$$\int_{\partial R} w = \int_R dw. \quad \text{if } \partial R \text{ is } S$$

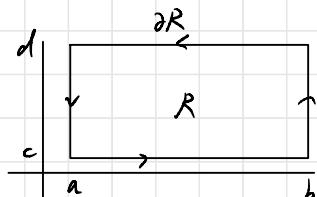
If. Let $R = [a, b] \times [c, d]$. Write $w = P dx + Q dy$. Then

$$dw = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

We then have:

$$\int_R dw = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\begin{aligned} (\text{Juhini}) &= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x} dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y} dy \right) dx \\ &= \int_c^d \left(Q(y) - Q(y) \right) dy - \int_a^b \left(P(b) - P(a) \right) dx \\ &= \int_a^b P(c) dx + \int_c^d Q(b) dy - \int_a^b P(d) dx - \int_c^d Q(a) dy \\ &= \int_{\partial R} w \end{aligned}$$



Corollary

If $S \subset \mathbb{R}^n$ is parametrized by a rectangle, and ω is a 1-form on S , then

$$\int_{\partial S} \omega = \int_S d\omega$$

H:

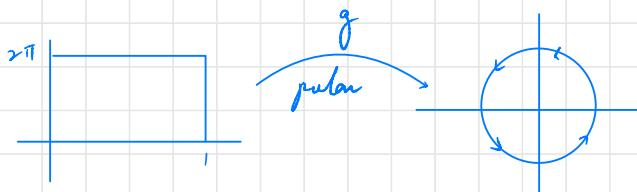
Let $g: R \rightarrow S \subset \mathbb{R}^n$ be a parametrization. Then

$$\int_{\partial S} \omega = \int_{\partial R} g^* \omega = \int_R d(g^* \omega) = \int_R g^*(d\omega) = \int_S d\omega.$$

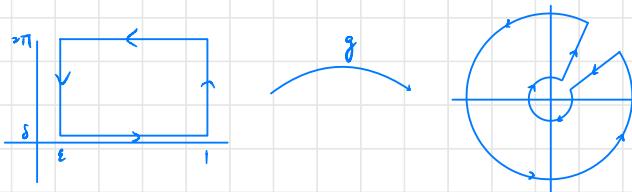
Ex.

D is a unit disk in \mathbb{R}^2 , ω is a smooth 1-form.

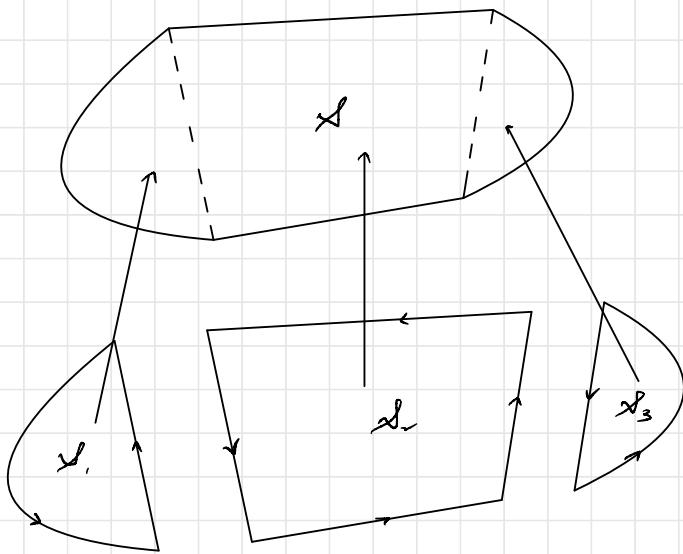
$$\int_{\partial D} \omega = \int_D d\omega ? \quad \checkmark$$



What about D/D_a , $a < 1$? \checkmark



Generally, for a \mathcal{X} which can be decomposed as a finite union of parametrized rectangles overlapping only along their edges.



$$\int_{\mathcal{X}} \omega = \sum_{i=1}^k \int_{\mathcal{X}_i} \omega = \sum_{i=1}^k \int_{\mathcal{X}_i} d\omega = \int_{\mathcal{X}} d\omega.$$

Surface integrals and flux

as expected, to define the integral of a 2-form over a parametrized surface Σ , we pullback and integrate.

When $\omega \in \Lambda^2(\mathbb{R}^n)$ and $\Sigma = g(V)$, we set

$$\int_{\Sigma} \omega = \int_V g^* \omega \quad (\text{if it exists})$$

We omit the most part in surface integral, since we have a stronger tool: metric tensor. However, in term of 2-form, we have something similar.

Def:

Σ be an oriented surface, its (oriented) area 2-form is the 2-form such that $\sigma(x)$ assigns to each pair of tangent vector at x the signed area of the parallelogram they span.

σ in \mathbb{R}^3

We claim that

$$\sigma = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy \quad N = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

is $\Sigma(\mathbb{C}\mathbb{R}^3)$'s area two form.

$$\star u, v \in T_x \Sigma, \sigma(u, v) = |N \cdot u \times v|$$

Meaning of integrating a 2-form.

1 form of $F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$, $\omega = F_1 dx_1 + F_2 dx_2 + F_3 dx_3$ along C : work.

2 form $\eta = F_1 dy_1 dz + F_2 dy_2 dx + F_3 dy_3$ over oriented \mathcal{S} ?

If u, v are tangent to \mathcal{S} , $\eta(u, v) = |F \cdot u \times v| = (F \cdot n) \times (\sigma(u, v))$.

$\Rightarrow \int_{\mathcal{S}} \eta$ represents the flux of F outward across \mathcal{S} , often written

as

$$\int_{\mathcal{S}} F \cdot n \, d\mathcal{S} \quad \text{non-oriented surface area.}$$

\Rightarrow Now, for flux, you can just find η_F and integrate it!

Ex:

$F = \begin{pmatrix} xy^2 \\ yz^2 \\ zx^2 \end{pmatrix}$, \mathcal{S} in a sphere with radius a center at origin.

Find the flux through \mathcal{S} .

$$\eta = xy^2 dy_1 dz + yz^2 dx_1 dy + zx^2 dx_2 dy$$

$$g: (0, \pi) \times (0, 2\pi) \rightarrow \mathbb{R}^3 \quad \gamma(\theta) = a \begin{pmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{pmatrix}$$

$$g^* dx = a \cos \phi \cos \theta d\phi - a \sin \phi \sin \theta d\theta \quad g^* dy = a \cos \phi \sin \theta d\phi + a \sin \phi \cos \theta d\theta \quad g^* dz = -a \sin \phi d\theta$$

$$\text{Ex: } g^*(x^* dy \wedge dz) = a^3 \sin^3 \phi \cos \theta \cos \phi (g^* dy \wedge g^* dz)$$

$$g^* dy \wedge g^* dz = a^3 \sin^3 \phi \cos \theta d\phi \wedge d\theta$$

$$g^*(y^* dz \wedge dx) = a^3 \sin^3 \phi \sin \theta \cos \theta (g^* dz \wedge g^* dx)$$

$$g^* dz \wedge g^* dx = a^3 \sin^3 \phi \sin \theta d\phi \wedge d\theta$$

$$g^*(z^* dx \wedge dy) = a^3 \cos \phi \sin^3 \phi \sin \theta (g^* dx \wedge g^* dy)$$

$$g^* dx \wedge g^* dy = a^5 (\sin \phi \cos \phi \cos^2 \theta + \sin \phi \cos \phi \sin^2 \theta) d\phi \wedge d\theta = a^5 \sin \phi \cos \phi d\phi d\theta$$

$$g^* \gamma = a^5 (\sin \phi \cos \theta \cos \phi / (\sin^2 \phi \cos \theta))$$

$$+ \sin^3 \phi \sin \theta \cos^2 \theta (\sin^2 \phi \sin \theta)$$

$$+ \cos \phi \sin^3 \phi \sin^2 \theta (\sin \phi \cos \phi) d\phi \wedge d\theta$$

$$= a^5 (\sin^3 \phi \cos^2 \phi + \sin^5 \phi \cos^2 \theta \sin^2 \theta) d\phi \wedge d\theta$$

Then

$$\int_S \gamma = \int_{(0, \pi) \times (0, 2\pi)} g^* \gamma = \frac{4}{5} \pi a^5.$$

Remark: $\sigma(u, v)$ only gives us 2-dim manifold surface area, metric tensor can be generalized to n-dim manifold!

Stokes Theorem

Integrating over a general compact, oriented k -dim manifold

We know how to integrate a k -form over a k -dim manifold by
pulling back.

Instead of dealing with the whole manifold, we now chop this
global problem into local ones.

Def:

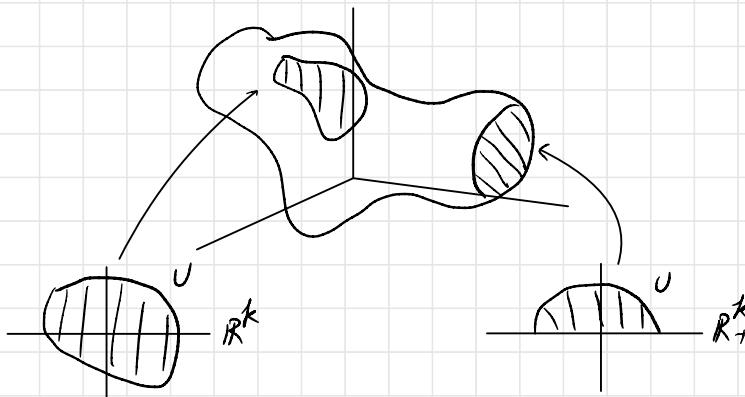
$M \subset \mathbb{R}^n$ is called a k -dim manifold with boundary if

$\forall p \in M \exists V \subset \mathbb{R}^n$ V is open and $p \in V$

and a parametrization $g: U \rightarrow \mathbb{R}^n$ so that

i) $g(U) = V \cap M$ and

ii) U is an open subset either of \mathbb{R}^k or $\mathbb{R}_+^k = \{u \in \mathbb{R}^k : u_k \geq 0\}$



We say $p \in \partial M$ if $p = g(u)$ for some $u \in \partial \mathbb{R}_+^k = \{u \in \mathbb{R}^k : u_k = 0\}$

$g(U)$ is sometimes called a coordinate chart on M .

Def:

$g: U \rightarrow \mathbb{R}^n, \left\{ \frac{\partial g}{\partial u_i} \right\}_i$ is a positive basis of $T_{g(x)}$.

Let $M \subset \mathbb{R}^n$ be an [↑] oriented k -dim manifold with boundary.

Its (oriented) volume form is the k -form σ such that $\sigma(x)$ assigns to each k -tuple of tangent vector at x the signed volume of the parallelepiped they span.

To define integration,

Theorem:

Let $M \subset \mathbb{R}^n$ be a compact k -dim manifold with boundary. Then there are smooth real-valued functions ρ_1, \dots, ρ_n on M so that

i) $0 \leq \rho_i \leq 1$ for all i

ii) each ρ_i is 0 outside some coordinate ball

iii) $\sum_{i=1}^n \rho_i = 1$. \downarrow
image of some ball under some parametrization

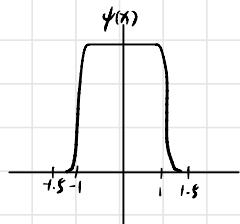
We omit the proof. Just purely construction!

Remark.

Such ρ_i are called a partition of unity.

Then, let $M \subset \mathbb{R}^n$ be a compact, oriented k -dim manifold (with piecewise smooth boundary). Let $\omega \in \Lambda^k(M)$. Let $\{\rho_i\}$ be a partition of unity, and let g_i be the corresponding parametrizations (orientation preserving). Now we set

$$\int_M \omega = \int_M \left(\sum_{i=1}^N \rho_i \omega \right) = \sum_{i=1}^N \int_{B_i(o)} g_i^*(\rho_i \omega) = \sum_{i=1}^N \int_{B_i(o)} \psi g_i^* \omega$$



technical detail from the above theorem

Stokes' Theorem

Let M be a compact, oriented k -dim manifold with boundary, and let ω be a smooth $(k-1)$ form on M . Then

$$\int_{\partial M} \omega = \int_M d\omega$$

where ∂M endowed with the boundary orientation.

Remark.

FTC, FTCL, GT are all special cases of this.

pf:

Both side of the equation are linear in ω , we can then use the partition of unity reduce the case that $\omega = 0$ outside a compact subset of a single coordinate chart $g: U \rightarrow \mathbb{R}^n$ (U is open in \mathbb{R}^k or in \mathbb{R}^k_+) Then we have

$$\int_M d\omega = \int_{g(U)} d\omega = \int_U g^*(d\omega) = \int_U d(g^*\omega)$$

Now, $g^*\omega \in A^{k-1}(U)$, can be written as

$$g^*\omega = \sum_{i=1}^k f_i(x) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k$$

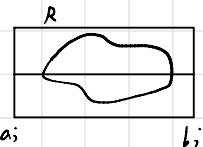
omit

Then,

$$\begin{aligned} d(g^*\omega) &= \sum_{i=1}^k \frac{\partial f_i}{\partial x_i} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k \\ &= \sum_{i=1}^k \frac{\partial f_i}{\partial x_i} (-1)^{i-1} dx_1 \wedge \cdots \wedge dx_n \end{aligned}$$

Case I — U is open in \mathbb{R}^k

$$\Rightarrow \omega = 0 \text{ at } \partial M, \text{ so we only need to show } \int_M d\omega = \int_U d(g^*\omega) = 0.$$



Since $g^*\omega$ is smooth and 0 outside of a compact subset of U , we can choose a rectangle $R \supseteq U$ and extend f_i to \bar{f}_i by setting

$$\bar{f}_i = \begin{cases} f_i & x \in U \\ 0 & x \notin U \end{cases}$$

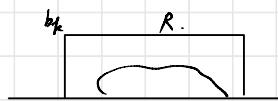
Finally, we integrate over $R = [a_1, b_1] \times \cdots \times [a_k, b_k]$:

$$\begin{aligned}
 \int_U d(g^* \omega) &= \int_R \sum_{i=1}^k (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k \\
 &= \sum_{i=1}^k (-1)^{i-1} \int_R \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k \\
 &= \sum_{i=1}^k (-1)^{i-1} \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} \left(\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_k \\
 &= \sum_{i=1}^k (-1)^{i-1} \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} \left(f_i \begin{pmatrix} x_1 \\ \vdots \\ b_i \\ \vdots \\ x_k \end{pmatrix} - f_i \begin{pmatrix} x_1 \\ \vdots \\ a_i \\ \vdots \\ x_k \end{pmatrix} \right) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_k \\
 &= 0
 \end{aligned}$$

since $f_i = 0$ everywhere on ∂R .

Case II — U is open in \mathbb{R}_+^k

In this case, we again extend f_i to functions on



$R \subsetneq \mathbb{R}_+^k$ by letting them be 0 outside of U . Then, $R = [a_1, b_1] \times \cdots \times [a_k, b_k] \times [0, b_k]$,

so we have

$$\begin{aligned}
 \int_U d(g^* \omega) &= \int_R \sum_{i=1}^k (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k \\
 &= \sum_{i=1}^k (-1)^{i-1} \int_R \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k \\
 &= \sum_{i=1}^k (-1)^{i-1} \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} \left(\int_{a_i}^{b_i} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_k
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^k (-1)^{i-1} \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} \left(f_i \begin{pmatrix} x_1 \\ \vdots \\ b_i \\ \vdots \\ x_k \end{pmatrix} - f_i \begin{pmatrix} x_1 \\ \vdots \\ a_i \\ \vdots \\ x_k \end{pmatrix} \right) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_k \\
&= (-1)^{k-1} \int_{a_{k-1}}^{b_{k-1}} \cdots \int_{a_1}^{b_1} \left(f_k \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ b_k \end{pmatrix} - f_k \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ a_k \end{pmatrix} \right) dx_1 \cdots dx_{k-1} \\
&= (-1)^k \int_{U \cap \partial R^k_+} f_k \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ 0 \end{pmatrix} dx_1 \cdots dx_{k-1} \\
&= \int_{U \cap \partial R^k_+} g^* \omega = \int_M \omega - *
\end{aligned}$$

Analogy

We already see that for any function f , $df \in A^1(\mathbb{R}^n)$ is ωf .
Now we want to give the traditional interpretation of the exterior derivative to 1-form and 2-form.

Given a 1-form

$$\omega = F_1 dx_1 + F_2 dx_2 + F_3 dx_3 \in A^1(\mathbb{R}^3),$$

we have

$$d\omega = \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \right) dx_2 \wedge dx_3 + \left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \right) dx_3 \wedge dx_1 + \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dx_1 \wedge dx_2$$

Correspondingly, given the vector field

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \quad \text{we set} \quad \text{curl } F = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}$$

(not)

Then we have our classical Stokes' Theorem:

$$\int_{\partial S} \frac{F \cdot T \, ds}{\omega} = \int_S \frac{\text{curl } F \cdot n \, dS}{\omega}$$

Now, given the 2-form

$$\omega = F_1 dx_2 \wedge dx_3 + F_2 dx_3 \wedge dx_1 + F_3 dx_1 \wedge dx_2 \in \Lambda^2(\mathbb{R}^3),$$

we then have

$$d\omega = \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3$$

Correspondingly, given the vector field

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \quad \text{we set } \operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

Then we have the classical Gauss's Theorem

$$\int_{\partial\Omega} F \cdot n \, dS = \int_{\Omega} \operatorname{div} F \, dV$$

Summary

