Ex 1.

We first parametrize the surface,

$$\varphi : [0,3]\times[0,2\pi] \to \mathbb{R}^3 \qquad \Rightarrow \qquad F\circ\varphi(r,\theta) = (r\cos\theta,\ 1,\ r\sin\theta)$$

$$\varphi(r,\theta) = (r\cos\theta,\ r^2,\ r\sin\theta)$$

The tangent vectors are

$$\left.\begin{aligned} t_r(r,\theta) &= (\cos\theta,\ 2r,\ \sin\theta) \\ t_\theta(r,\theta) &= (-r\sin\theta,\ 0,\ r\cos\theta) \end{aligned}\right\} \quad n(r,\theta) = t_r \times t_\theta(r,\theta)$$

$$d\bar{A} = (2r^2\cos\theta,\ -r,\ 2r^2\sin\theta)\,dr\,d\theta \leftarrow$$ the vectorial surface element.

$$\int_S \langle F, d\bar{A}\rangle = \int_0^{2\pi}\int_0^3 (2r^3 - r)\,dr\,d\theta = 72\pi$$

Ex 2.

i) G is a potential field, because the region is simple-connected, and

$$\operatorname{rot} G = \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} = 2xe^{x^2} + \frac{1}{x} - \left(2xe^{x^2} + \frac{1}{x}\right) = 0$$

ii). $$\int \underline{G_x\,dx} = \int \left(2xye^{x^2} + \frac{y}{x}\right) dx = \underline{ye^{x^2} + y\ln x + C_1(y)}$$

$$\int \underline{G_y\,dy} = \int \left(e^{x^2} + \ln x + 6y\right) dy = \underline{ye^{x^2} + y\ln x + 3y^2 + C_2(x)}$$

$$\Rightarrow \quad g(x,y) = \underline{ye^{x^2} + y\ln x + 3y^2 + C}, \quad C \in \mathbb{R} \text{ is a constant.}$$

$$\frac{\partial}{\partial x} C = 0$$

Ex 3.

i) F is a potential field because $\mathbb{R}^3$ is simply-connected, and

$$\operatorname{rot} F = \begin{pmatrix} 0-0 \\ e^z - e^z \\ 0-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (\Rightarrow \text{irrotational})$$

ii)

$$\int F_x \, dx = xe^z + C_1(y,z)$$

$$\int F_y \, dy = y + C_2(x,z)$$

$$\int F_z \, dz = xe^z + C_3(x,y)$$

$\Rightarrow f(x,y,z) = xe^z + y + C$, where $C \in \mathbb{R}$ is a constant.

Ex 4

We apply Green's theorem,

$$|R| = \int_R 1\,dx = \frac{1}{2}\int_{\partial R} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} d\vec{s}$$

$$= \frac{1}{2}\int_0^{\pi} \left\langle \begin{pmatrix} -\sin t \\ \sin 2t \end{pmatrix}, \begin{pmatrix} 2\cos 2t \\ \cos t \end{pmatrix} \right\rangle dt$$

$$= \frac{1}{2}\int_0^{\pi} (\sin 2t \cos t - 2\sin t \cos 2t)\,dt$$

$$= \frac{1}{2}\int_0^{\pi} 2\sin^3 t\,dt$$

$$= \int_0^{\pi} \sin^3 t\,dt$$

$$= \frac{4}{3}$$

Ex 5.

If we define $F(x,y,z) = (y,z,x)$

we have
$$\int_{S^2} (xy + yz + zx)\, d\sigma = \int_{S^2} \langle F, N \rangle\, d\sigma \quad \left(= \int_{S^2} \langle F, d\vec{\sigma} \rangle\right)$$

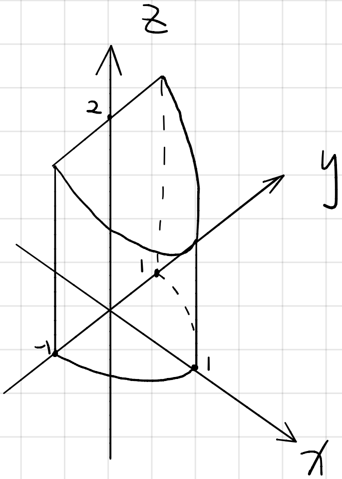
By Gauß's theorem,
$$\int_{S^2} \langle F, N \rangle\, d\sigma = \int_{B_1(0)} \operatorname{div} F \, dV. = 0$$

where $B_1(0)$ is the unit ball centered at 0, and

$\operatorname{div} F = 0 + 0 + 0 = 0$

Ex 6.

i). Sketch :



By Gauß's theorem,

ii). $$\int_{S^+} \langle F, N \rangle \, d\sigma = \int_U \operatorname{div} \vec{F}$$

$$= \int_U 3 \, dV$$

Using cylindrical coordinates,

$$\int_U 3 \, dV = 3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_0^{2 - r\cos\phi} r \, dz \, dr \, d\phi$$

$$= 3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 (2 - r\cos\phi) r \, dr \, d\phi$$

$$= 3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[r^2 - \frac{r^3}{3} \cos\phi \right]_0^1 d\phi$$

$$= 3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \frac{1}{3}\cos\phi\right) d\phi$$

$$= 3\pi - (\sin\phi)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 3\pi - 2$$

Ex 7.

We parametrize the surface, $\varphi : [0,1]\times[0,\frac{\pi}{2}] \to S$

$$\varphi(r,\theta) = \begin{pmatrix} 0 \\ r\cos\theta \\ r\sin\theta \end{pmatrix},$$

$$\operatorname{rot} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

The tangent vector: $t_r(r,\theta) = \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}$, $t_\theta(r,\theta) = \begin{pmatrix} 0 \\ -r\sin\theta \\ r\cos\theta \end{pmatrix}$

normal vector: $t_r \times t_\theta(r,\theta) = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \xrightarrow[\text{by orientation}]{\text{change sgn}} \begin{pmatrix} -r \\ 0 \\ 0 \end{pmatrix}$

Apply Stokes theorem,

$$\int_{\partial S^*} \langle F, d\vec{l} \rangle = \int_S \langle \operatorname{rot} F, d\vec{A} \rangle = \int_0^1 \int_0^{\frac{\pi}{2}} \left\langle \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -r \\ 0 \\ 0 \end{pmatrix} \right\rangle d\theta\, dr$$

$$= \frac{\pi}{4}$$

Ex8.

We parametrize the surface : $\varphi\ [0,1]\times[0,2\pi]\to S$

$$\varphi(r,\theta) = (r\cos\theta, r\sin\theta, 1)$$

$$\operatorname{rot} F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x - \frac{y^3}{3} & \cos y + \frac{x^3}{3} & xyz \end{vmatrix} = \begin{pmatrix} xz \\ -yz \\ x^2+y^2 \end{pmatrix} = \boxed{\begin{pmatrix} r\cos\theta \\ -r\sin\theta \\ r^2 \end{pmatrix}}$$

The tangent vector: $t_r(r,\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$, $t_\theta(r,\theta) = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix}$

normal vector: $\begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$ (verify direction is correct)

Apply Stokes's theorem,

$$\int_{\partial S^*} \langle F, d\bar{l}\rangle = \int_S \langle \operatorname{rot} F, d\bar{A}\rangle = \int_0^1\int_0^{2\pi} \left\langle \begin{pmatrix} r\cos\theta \\ -r\sin\theta \\ r^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \right\rangle d\theta\, dr$$

$$= 2\pi\int_0^1 r^3\, dr = \frac{\pi}{2}$$

Ap1.

Using Gauß's theorem,

$$\underline{\oint_{\partial\Omega} E\cdot dS} = \int_\Omega \underline{\nabla\cdot E}\, dV = \int_\Omega \frac{\rho}{\varepsilon_0}\, dV = \underline{\frac{1}{\varepsilon_0}\int \rho\, dV}$$

$$\oint_{\partial\Omega} B\cdot dS = \int_\Omega \nabla\cdot B\, dV = \underline{\int_\Omega 0\cdot dV} = 0$$

$$\oint_{\partial\Sigma} E\cdot dl = \int_\Sigma \underline{\nabla\times E}\cdot dS = \int_\Sigma -\frac{\partial B}{\partial t}\cdot dS = -\frac{d}{dt}\int_\Sigma B\cdot dS$$

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$$\oint_{\partial\Sigma} \underline{B\cdot dl} = \int_\Sigma \underline{\nabla\times B}\cdot dS = \int_\Sigma \mu_0\left(J+\varepsilon_0\frac{\partial E}{\partial t}\right)\cdot dS$$

$$= \mu_0\left(\int_\Sigma J\cdot dS + \varepsilon_0\frac{d}{dt}\int_\Sigma E\cdot dS\right)$$

Ex 9.

$$B = \nabla \times A \Rightarrow$$

$$\underline{\nabla \cdot (\nabla \times A) = 0} \Leftrightarrow \underline{\nabla \cdot B = 0}$$

$$\underline{E = -\nabla \phi - \frac{\partial A}{\partial t}} \Rightarrow$$

$$\underline{\nabla \times E} = \nabla \times \left(-\nabla \phi - \frac{\partial A}{\partial t}\right)$$

$$= -\underline{\nabla \times \nabla \phi} - \frac{\partial}{\partial t} \underline{\nabla \times A}$$

$$= 0 - \frac{\partial}{\partial t} B = \underline{-\frac{\partial B}{\partial t}}$$

...... (Learn more in VE230!)

Ap 2. i) By Green's first identity,

$$\int_\Omega \langle \nabla u, \nabla u \rangle \, dx = \underline{-\int_\Omega u \cdot \Delta u \, dx} + \underline{\int_{\partial\Omega^*} u \frac{\partial u}{\partial n} \, dA}$$

$\to 0$ $\to 0$

Since $\Delta u = 0$, and $\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$, we have

$$\underline{\int_\Omega \langle \nabla u, \nabla u \rangle dx = 0} \Rightarrow \underline{\int_\Omega \|\nabla u\|^2 dx} = 0$$

Therefore, $\nabla u(x) = 0$ a.e. on Ω.

Since $u \in C^2(\Omega) \Rightarrow \underline{\nabla u(x) = 0}$ for all $x \in \Omega$

In summary, u is constant in Ω.

ii). 1. The flow is irrotational (Because we model it using a potential field. $v = \nabla u$)

2. There are no sources/sinks, and the fluid is incompressible. (Because $\Delta u = \operatorname{div} v = 0$)

3. Here u represents the potential of the velocity $v = \nabla u$ of the fluid flow.

4. The boundary condition $\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$ indicates $\left.V_n\right|_{\partial\Omega} = 0$. So that no fluid flows through Ω's boundary $\Rightarrow$ "solid-wall" boundary

5. u is constant $\Rightarrow v = 0$. The fluid is stationary in Ω.

☆ This means fluid moving in a closed container cannot be both irrotational & incompressible. Otherwise it would be stationary.