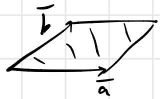
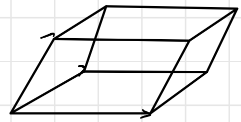
Det.

defined geometrically:  2d, 3d 

we want:

① multilinear

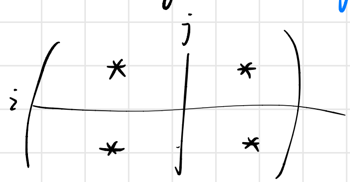
② alternating $\Rightarrow$ induced a unique map

③ normed. $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$

1. Induced raw form: $\sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_j a_{\pi(j)j}$

2. Laplace Expansion: $\sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$

$(-1)^{ij} \det A_{ij}$: $(i,j)^{th}$ minor of A $=: c_{ij}$



$\Rightarrow \mathrm{Cof}\, A = (c_{ij})$

adjugate of A : $A^{*} := (\mathrm{Cof}\, A)^{T}$

$A^{-1} = \frac{A^{*}}{\det A}$

Fredholm Alternative

$$\begin{cases} \det A = 0 , & \exists\, x \neq \vec{0},\ x \in \ker A \quad Ax = 0 \\ \det A \neq 0 , & \forall\, b \in \mathbb{R}^n \quad Ax = b,\ \exists\, x = A^{-1} b \text{ since } A^{-1} \text{ exists} = \frac{A^{\#}}{\det A}. \end{cases}$$

Cramer's Rule

$$Ax = b. \quad \det A \neq 0 \Rightarrow x_i = \frac{1}{\det A} \det(a_1, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n)$$

$$\det(AB) = \det A \det B.$$

proof idea: induce a function from A has the three properties of det $\Leftrightarrow f = \det \Rightarrow f(B) = \det B$ ...

another way: use E. $\begin{cases} i \to j \\ i \leftrightarrow j \\ ki \end{cases}$

$\Downarrow$

$$\det A^{-1} = \frac{1}{\det A}.$$

Let $A = \begin{pmatrix} \lambda-5 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 2 \\ 3 & 1 & \lambda-1 & 3 \\ 2 & 4 & 3 & 5 \end{pmatrix}$, find λ s.t. A^{-1} exists.

$$(\lambda-5)\cdot\left[\lambda(\lambda-1)5 + 12 + 6 - 8(\lambda-1) - 5 - 9\lambda\right]$$

$$= (\lambda-5)(\lambda-3)(5\lambda-7)$$

Proof that if A is skew-symmetric and n is odd, then A is not invertible.

$$A^T = -A \Rightarrow \det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$$
$$= -\det A$$

$$\Rightarrow \det A = 0$$ ※

Assume $A = (a_1 \quad a_2 \quad a_3)$, $B = \begin{pmatrix} 2a_2^T \\ a_1^T + a_2^T + a_3^T \\ a_3^T \end{pmatrix}$

If $\det A = 2$, find $\det AB^{-1}$.

Sol:

We find that $\det A = \det A^T$

$$= \det \begin{pmatrix} a_1^T \\ a_2^T \\ a_3^T \end{pmatrix}$$

$$= \det \begin{pmatrix} a_1^T + a_2^T + a_3^T \\ a_2^T \\ a_3^T \end{pmatrix}$$

$$= \det \begin{pmatrix} a_1^T + a_2^T + a_3^T \\ 2a_2^T \\ a_3^T \end{pmatrix} \cdot \frac{1}{2}$$

$$= \det \begin{pmatrix} 2a_2^T \\ a_1^T + a_2^T + a_3^T \\ a_3^T \end{pmatrix} \cdot \frac{1}{2} \cdot -1 = \det B \cdot \frac{1}{2} \cdot -1$$

$\Rightarrow \det B = \det A \cdot 2 \cdot -1 = 2 \cdot 2 \cdot -1 = -4.$

$\det AB^{-1} = \det A \det B^{-1} = \frac{2}{-4} = -\frac{1}{2}.$

If A is invertible, then $A+B$ and $I+BA^{-1}$ are both (not) invertible.

$$A+B=(I+BA^{-1})A$$

$$\begin{aligned} A+B \text{ inv.} \Leftrightarrow \det A+B \neq 0 &\Leftrightarrow \det((I+BA^{-1})A) \neq 0 \\ &\Leftrightarrow \det(I+BA^{-1}) \det A \neq 0 \\ &\Leftrightarrow \det I+BA^{-1} \neq 0 \\ &\Leftrightarrow I+BA^{-1} \text{ inv.} \end{aligned}$$

Wronskian

$C^{(n-1)}[a,b]$ ($n-1$ times diff.), $C^{n-1}[a,b]$ is a vector space. If $f_1, \cdots, f_n \in C^{(n-1)}[a,b]$, define

$$W(x) := W[f_1, f_2, \cdots, f_n](x) = \det \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}$$

Theorem: If $\exists\, x_0 \in [a,b]$ s.t. $W[f_1, \cdots, f_n](x) \neq 0$, then $f_1, \cdots, f_n$ are linearly independent.

Assume not. Namely existe α_i not all $0 \Rightarrow \sum \alpha_i f_i = 0$

$\Rightarrow$ do $n-1$ times diff.

$$\begin{cases} \alpha_1 f_1(x) + \cdots + \alpha_n f_n(x) = 0 \\ \alpha_1 f_1'(x) + \cdots + \alpha_n f_n'(x) = 0 \\ \vdots \qquad \ddots \qquad \vdots \\ \alpha_1 f_n^{(n-1)}(x) + \cdots + \alpha_n f_n^{(n-1)}(x) = 0 \end{cases} \Rightarrow \underbrace{\begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix}}_{= F} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

$\Rightarrow F(x)\,\bar{\alpha} = \bar{0}$ has non zero solution for every $x \in [a,b]$

$\Rightarrow F(x)$ is not inv. $\forall x \in [a,b]$

$\Rightarrow \det(F(x)) = 0$, $\forall x \in [a,b]$

$\Rightarrow W(x) = 0$, $\forall x \in [a,b]$ $\lightning$ $\Rightarrow$ $f_1, \cdots, f_n$ are linearly independent. #

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$

$$= \langle u, u\rangle + \langle v, v\rangle + \langle u, v\rangle + \langle v, u\rangle$$

$$= \|u\|^2 + \|v\|^2$$

iff $\langle u, v\rangle + \langle v, u\rangle = 0.$

$$\Rightarrow \langle u, v\rangle + \overline{\langle u, v\rangle} = 0.$$

$$\Rightarrow 2\mathrm{Re}(\langle u, v\rangle) = 0$$

$$\|P\| = \sup_{v} \frac{\|Pv\|}{\|v\|} = \sup_{\substack{\|v\|=1 \\ v}} \|Pv\|$$

iff $v \in U \subset V$

$$\Rightarrow \|Pv\| = 1.$$