1) $y_i = \beta_0 + \beta_1 x_i + \varepsilon$

$x^T = (0, \dots, 0, 1 \dots 1)$

n_0 zeroes and $100 - n_0$ ones.

$\mathrm{Var}[\hat{\beta}_0]$, $\mathrm{Var}[\hat{\beta}_1] = ?$

Group 0: $x_i = 0 \to$ mean $= \beta_0$

Group 1: $x_i = 1 \to$ mean $= \beta_0 + \beta_1$

$$\mathrm{Var}(\hat{\beta}_0) = \sigma^2 \cdot \frac{1}{n_0}$$

$$\mathrm{Var}(\hat{\beta}_1) = \sigma^2 \left(\frac{1}{n_0} + \frac{1}{100 - n_0} \right)$$

For $\hat{\beta}_0$,

To minimize variance, $\frac{1}{n_0}$ to be as small as possible.

$\therefore$ maximize n_0, when $n_0 = 99$

For $\hat{\beta}_1$, minimize $\frac{1}{n_0} + \frac{1}{100 - n_0}$

Set $n_0 = 50$

2)

$$s_i = \frac{r_i}{\sqrt{(I-P_x)_{ii}\ SS_{Res}/(n-p-1)}}$$

$$|s_i| \leq \sqrt{n-p-1}$$

Let $\vec{e_i}$ be the standard basis vector w/ 1 at the i^{th} position.

By Cauchy Schwarz inequality,

$$|\vec{e_i}\ (I-P)\vec{y}| \leq \sqrt{\vec{e_i}^T(I-P)\vec{e_i}} \cdot \sqrt{\vec{y}^T(I-P)\vec{y}}$$

$$= |r_i| \leq \sqrt{(I-P)_{ii}} \cdot \sqrt{SS_{Res}}$$

$$|s_i| = \frac{|r_i|}{\sqrt{(I-P)_{ii} \cdot SS_{Res}/n-p-1}} \leq \frac{\sqrt{(I-P)_{ii} \cdot SS_{Res}}}{\sqrt{(I-P) \cdot SS_{Res}/n-p-1}} = \sqrt{n-p-1}$$

$\therefore$ Since $|s_i| \leq \sqrt{n-p-1}$, residual is bounded whereas a true $t(n-p-1)$ is unbounded. Therefore s_i cannot follow a t-distribution.

3. 1) n obs. w/ $x_i \sim N(0, \sigma_1^2)$

m obs. w/ $x_i \sim N(0, \sigma_2^2)$

$$\frac{1}{n+m} \sum_{i=1}^{n+m} Var(x_i) = \frac{n\sigma_1^2 + m\sigma_2^2}{n+m}$$

Let $\delta = \frac{n}{n+m} \in [0, 1]$

then, $\delta \sigma_1^2 + (1-\delta)\sigma_2^2$

2) $\sigma_j^2 < E\left[\frac{SS_{Res}}{n+m+p-1}\right]$, $\sigma_1^2 < \sigma_2^2$

$$\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{\sigma_1^2} + \frac{\sum_{i=n+1}^{n+m} (x_i - \bar{x})^2}{\sigma_2^2} \sim \chi^2(n+m+p-1)$$

But if $\sigma_2^2 >> \sigma_1^2$, the outlier inflates the residuals

$\therefore$ $SS_{Res} = \sum (x_i - \bar{x})^2$ is larger than

expected under σ_j^2

$$\therefore \sigma_1^2 < E\left[\frac{SS_{Res}}{n+m+p-1}\right] < \sigma_2^2$$