1) $y = \beta_1 x_1 + \dots \beta_p x_p + \varepsilon$

$$\hat{\beta}_\lambda^R = (X^T X + \lambda I_n)^{-1} X^T Y$$

$$E[\hat{\beta}_\lambda^R] = E[(X^T X + \lambda I)^{-1} X^T Y]$$

$$= (X^T X + \lambda I)^{-1} X^T . E[Y]$$

$$= (X^T X + \lambda I)^{-1} X^T . X\beta$$

$$E[\hat{\beta}_\lambda^R] - \beta = (X^T X + \lambda I)^{-1} X^T X \beta - \beta$$

$$= [(X^T X + \lambda I)^{-1} X^T X - I]\beta$$

$$= [(X^T X + \lambda I)^{-1} (X^T X + \lambda I - \lambda I) - I]\beta$$

$$= [I - \lambda (X^T X + \lambda I)^{-1} - I]\beta$$

$$= -\lambda (X^T X + \lambda I)^{-1} \beta$$

2)

$$\pi(x_1, \dots x_p) = \frac{1}{1+e^{-(\beta_0 + \beta_1 x_1 + \dots \beta_p x_p)}} = \frac{1}{1+e^{-z}}, \text{ where}$$

$$z = \beta_0 + \beta^T x$$

to show: $\nabla \pi = \pi(1-\pi)\beta$

$$\beta^T = (\beta_1, \dots \beta_p)$$

$$\frac{d\pi}{dz} = \pi(1-\pi)$$

$$\frac{dz}{\partial x_j} = \frac{d\pi}{\partial z} \cdot \frac{\partial z}{\partial x_j} = \pi(1-\pi)\beta_j$$

$$\therefore \quad \nabla \pi = \begin{bmatrix} \frac{\partial \pi}{\partial x_1} \\ \vdots \\ \vdots \\ \frac{\partial \pi}{\partial x_n} \end{bmatrix} = \pi(1-\pi) \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \pi(1-\pi)\beta$$

3) $E[y] = a + bx = \lambda_i \qquad Var[y_i] = \lambda_i$

$$f(y) = \frac{e^{-(a+bx)}(a+bx)^y}{y!}$$

log likelihood: $\sum_{i=1}^{n} \left[-(a+bx_i) + y_i \cdot \log(a+bx_i) - \underbrace{\log(y_i!)}_{\to 0} \right]$

$$\frac{\partial \ell}{\partial a} = \sum_{i=1}^{n} \left[-1 + \frac{y_i}{a+bx_i} \right] = 0$$

$$\therefore \sum \frac{y_i}{a+bx_i} = n$$

$$\frac{1}{n} \cdot \frac{\sum y_i}{\hat{a}+\hat{b}x_i} = 1$$

$$\frac{\partial \ell}{\partial b} = \sum_{i=1}^{n} \left[-x_i + \frac{x_i y_i}{a+bx_i} \right]$$

$$= \sum_{i=1}^{n} \frac{x_i y_i}{a+bx_i} = \sum_{i=1}^{n} x_i$$

$$\frac{1}{n} \cdot \sum_{i=1}^{n} \left[\frac{x_i y_i}{\hat{a}+\hat{b}x_i} \right] = \frac{\sum x_i}{n} = \bar{x}$$

with $a = 0$

$$\ell(b) = \sum_{i=1}^{n} \left[-bx_i + y_i \cdot \log(bx_i) \right]$$

$$\frac{\partial \ell}{\partial b} = \sum -x_i + \frac{y_i}{b} = 0$$

$$\hat{b} = \frac{\sum y_i}{\sum x_i}$$

$\mathrm{Var}[\hat{b}]$, let $S_y = \sum y_i$

since $y_i \sim \mathrm{Poisson}(b x_i)$.

$S_y \sim \mathrm{Poisson}(b \sum x_i)$

$$\hat{b} = \frac{S_y}{\sum x_i}$$

$E[\hat{b}] = b$.

$$\mathrm{Var}[\hat{b}] = \frac{1}{(\sum x_i)^2} \cdot \mathrm{Var}(S_y) = \frac{1}{(\sum x_i)^2} \cdot b(\sum x_i)$$

$$= \frac{b}{\sum x_i}$$

HW6 Q1

$$y = \beta_1 x_1 + \cdots \beta_p x_p + \varepsilon$$

$$\hat{\beta}_\lambda^R = (X^T X + \lambda I)^{-1} X^T Y$$

$$E[\hat{\beta}_\lambda^R] = E[(X^T X + \lambda I)^{-1} X^T Y]$$

$$E[\hat{\beta}_\lambda^R] = -\lambda (X^T X + \lambda I_n)^{-1} \beta + \beta.$$

$$= (X^T X + \lambda I_n)^{-1} X^T X \beta - \beta$$

$$= (X^T X + \lambda I_n)^{-1} X^T X \beta - (X^T X + \lambda I_n)^{-1} (X^T X + \lambda I_n) \beta$$

$$= (X^T X + \lambda I_n)^{-1} \beta \left[\cancel{X^T X} - (\cancel{X^T X} + \lambda I_n) \right]$$

$$= (X^T X + \lambda I_n)^{-1} \beta \left[-\lambda I_n \right].$$

$$= -\lambda (X^T X + \lambda I_n)^{-1} \beta.$$

2)

$$\pi(x_1, \ldots x_p) = \frac{1}{1+e^{-x_i^T\beta}} = \frac{1}{1+e^{-z}}$$

$$\nabla \pi = \pi(1-\pi)\beta.$$

$$\frac{d\pi}{dz} = \frac{(1+e^{-z}).0 - (1)(0+e^{-z}.-1)}{(1+e^{-z})^2}$$

$$= \frac{e^{-z}}{(1+e^{-z})^2} = \frac{1}{1+e^{-z}} \cdot \frac{e^{-z}}{1+e^{-z}}$$

$$= \pi(1-\pi)$$

$$\frac{d\pi}{dx} = \frac{d\pi}{dz} \cdot \frac{dz}{dx} = \pi(1-\pi).\beta$$

HW6 Q3

$$L = \prod_{i=1}^{n} \frac{e^{-a-bx_i}(a+bx_i)^{y_i}}{y_i!}$$

$$\ell = \log\left[\prod_{i=1}^{n} \frac{e^{-a-bx_i}(a+bx_i)^{y_i}}{y_i!}\right]$$

$$= \sum_{i=1}^{n} \log\left(e^{-a-bx_i}\right) + \log\left((a+bx_i)^{y_i}\right) - \log(y_i!)$$

$$= \sum_{i=1}^{n} -a-bx_i + y_i \log(a+bx_i) - \log(y_i!)$$

I/P - N

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$$\hat{a} = \frac{\partial \ell}{\partial a} = \sum -1 + y_i \frac{1}{a+bx_i} \cdot (1+0) - 0$$

$$= \sum 1 + \frac{y_i}{a+bx_i}$$

$$0 = -n + \sum \frac{y_i}{a+bx_i}$$

$$1 = \frac{1}{n} \sum \frac{y_i}{a+bx_i}$$

$$\frac{\partial \ell}{\partial b} = \sum_{i=1}^{n} (-a - bx_i) + y_i \log(a + bx_i) - \log(y_i!)$$

$$= -x_i + \frac{y_i}{a+bx_i} \cdot x_i - 0$$

$$= -x_i + \frac{x_i y_i}{a+bx_i} = \sum x = \sum \frac{x_i y_i}{a+bx_i}$$

$$= \bar{x} = \frac{1}{n} \frac{\sum x_i y_i}{a+bx}$$