Section 4.4:

20. Suppose a is an integer. If a mod 7 = 4, what is 5a mod 7? In other words, if division of a by 7 gives a remainder of 4, what is the remainder when 5a is divided by 7?

We have: $a \bmod 7 = 4$

For some integer r $\Rightarrow$ $a = 7r + 4$ (def. modulo)

$\Rightarrow$ $5a = 5(7r+4)$

$\Rightarrow$ $5a = 35r + 20$

$\Rightarrow$ $5a = 35r + 14 + 6$

$\Rightarrow$ $5a = 7\underbrace{(5r+2)}_{\text{an integer}} + 6$

and $0 \le 6 < 7$

$\therefore$ $5a \bmod 7 = 6$

23. Prove that for all integers n, if n mod 5 = 3 then n^2 mod 5 = 4.

Given some integer n such that $n \bmod 5 = 3$

For some integer r $\Rightarrow$ $n = 5r + 3$

$\Rightarrow$ $n^2 = (5r+3)^2$

$\Rightarrow$ $n^2 = 25r^2 + 30r + 9$

$\Rightarrow$ $n^2 = 25r^2 + 30r + 5 + 4$

$\Rightarrow$ $n^2 = 5\underbrace{(5r^2 + 6r + 1)}_{\text{an integer}} + 4$

and $0 \le 4 < 5$

$\therefore$ $\boxed{n^2 \bmod 5 = 4}$

brata

In 31–33, you may use the properties listed in Example 4.2.3.

32. Given any integers a, b, and c, if a – b is even and b – c is even, what can you say about the parity of 2a – (b + c)? Prove your answer.

Let a, b, and c be some integers such a-b is even and b-c is even.

We have: $2a - (b+c) = a + a - b - c$

$= (a-b) + \boxed{(a-c)}$

(a-b) is even and (a-c) is even and sum of two even number is even

$\therefore$ $\boxed{2a - (b+c) \text{ is even}}$

a-b is even
b-c is even
(a-b+b-c) is even
(a-c) is even

Prove the statement

39. The sum of any four consecutive integers has the form 4k + 2 for some integer k.

Let n be some integer

$\Rightarrow$ n, n+1, n+2, n+3 are 4 consecutive integers

We have: $n + (n+1) + (n+2) + (n+3)$

$= 4n + 1 + 2 + 3$

$= 4n + 6$

$= 4n + 4 + 2$

$= 4\underbrace{(n+1)}_{\text{an integer}} + 2$

Let n+1 = k

$\Rightarrow n + (n+1) + (n+2) + (n+3) = 4.k + 2$

$\therefore$ The sum of any four consecutive integers has the form 4k+2 forsome Integer k

Def. floor: Given any real number x, the floor of x, $\lfloor x \rfloor$, is defined as $\lfloor x \rfloor = n$ s.t $n \le x < n+1$

Section 4.5:

Prove the statement

26. For all real numbers x, if $x - \lfloor x \rfloor < 1/2$ then $\lfloor 2x \rfloor = 2\lfloor x \rfloor$.

Let x be some real number s.t $x - \lfloor x \rfloor < \frac{1}{2}$

We have: $x - \lfloor x \rfloor < \frac{1}{2}$

$\Leftrightarrow 2x - 2\lfloor x \rfloor < 1$

$\Leftrightarrow 2x < 2\lfloor x \rfloor + 1$ (1)

We have: $\lfloor x \rfloor \le x$ (def. floor)

$\Leftrightarrow 2\lfloor x \rfloor \le 2x$ (2)

(1) (2) $\Rightarrow 2\lfloor x \rfloor \le 2x < 2\lfloor x \rfloor + 1$

$\Rightarrow \lfloor 2x \rfloor = 2\lfloor x \rfloor$

$\therefore$ For all real numbers x, if $x - \lfloor x \rfloor < \frac{1}{2}$ then $\lfloor 2x \rfloor = 2\lfloor x \rfloor$

Section 4.6:

Prove each statement in 10–17 by contradiction.

10. The square root of any irrational number is irrational.

Negation:

$\exists$ irrational number X, s.t. $\sqrt{X}$ is rational

by definition of rational numbers

$\sqrt{X} = \frac{a}{b}$ $(a, b \in \mathbb{Z}, b \neq 0)$

$X = \frac{a^2}{b^2}$

$\because a^2, b^2 \in \mathbb{Z}, b^2 \neq 0$

$\therefore X$ is rational, contradiction

Prove each of the statements in 23–29 in two ways: (a) by contraposition and (b) by contradiction.

26. For all integers a, b, and c, if $a \nmid bc$ then $a \nmid b$. (Recall that the symbol $\nmid$ means "does not divide.")

a) contraposition:

$\forall a, b, c \ (a, b, c \in \mathbb{Z})$, if $a \mid b$, then $a \mid bc$

According to the definition of divisibility,

$b = ak_1 \ (k_1 \in \mathbb{Z})$

$bc = ak_1 \cdot c = a(k_1 \cdot c) \ (k_1 \cdot c \in \mathbb{Z})$

$\therefore a \mid bc$

b) contradiction:

$\exists a, b, c \ (a, b, c \in \mathbb{Z})$. s.t. $a \nmid bc$ and $a \mid b$

According to the definition of divisibility,

$\because a \mid b$

$\therefore b = ak_1 \ (k_1 \in \mathbb{Z})$

$bc = a(k_1 \cdot c) \ (k_1 c \in \mathbb{Z})$

$\therefore a \mid bc$, contradiction to $a \nmid bc$

Section 4.7:

Determine which statements in 3–13 are true and which are false. Prove those that are true and disprove those that are false.

3. $6 - 7\sqrt{2}$ is irrational.

Proof by contradiction

negation: $6 - 7\sqrt{2}$ is rational

According to the definition of rational numbers

$$6 - 7\sqrt{2} = \frac{a}{b} \quad (a, b \in \mathbb{Z}, b \neq 0)$$

$$-7\sqrt{2} = \frac{a}{b} - 6$$

$$\sqrt{2} = \frac{a-6b}{b} \times -\frac{1}{7}$$

$$= -\frac{a-6b}{7b}$$

$\because a-6b,\ 7b \in \mathbb{Z},\ 7b \neq 0$

$\therefore \sqrt{2}$ is rational,

contradiction to theorem $\sqrt{2}$ is irrational

30. The following "proof" that every integer is rational is incorrect. Find the mistake.

"**Proof (by contradiction):** Suppose not. Suppose every integer is irrational. Then the integer 1 is irrational. But $1 = 1/1$, which is rational. This is a contradiction. *[Hence the supposition is false and the theorem is true.]*"

Negation:

there exists one integer which is irrational

negation form incorrect

16. a. Use proof by contradiction to show that for any integer n, it is impossible for n to equal both $3q_1 + r_1$ and $3q_2 + r_2$, where q_1, q_2, r_1, and r_2, are integers, $0 \le r_1 < 3$, $0 \le r_2 < 3$, and $r_1 \ne r_2$.

b. Use proof by contradiction, the quotient-remainder theorem, division into cases, and the result of part (a) to prove that for all integers n, if n^2 is divisible by 3 then n is divisible by 3.

c. Prove that $\sqrt{3}$ is irrational.

a. proof by contradiction

negation: there ~~is at least~~ exists integers, q_1, r_1, q_2, r_2

s.t. $3q_1 + r_1 = 3q_2 + r_2$ $(0 \le r_1 < 3, 0 \le r_2 < 3, r_1 \ne r_2)$

$3q_1 + r_1 = 3q_2 + r_2$

$3(q_1 - q_2) = r_2 - r_1$

$\therefore r_2 - r_1 = -2, -1, 1$ or 2

$\because q_1 - q_2 \in \mathbb{Z}$, $r_2 - r_1 \in \mathbb{Z}$

$3 \mid r_2 - r_1$ contradiction

b. negation:

$\exists$ integer n, s.t. $3 \mid n^2$ and $3 \nmid n$

$\therefore n^2 = 3k$, $(k \in \mathbb{Z})$

$\because 3 \nmid n$

According to the definition of quotient-remainder theorm, $n = 3k_1 + 1$ or $3k_1 + 2$ $(k_1 \in \mathbb{Z})$

case ①: $n = 3k_1 + 1$ $(k \in \mathbb{Z})$

$n^2 = 9k_1^2 + 6k_1 + 1$

$= 3(3k_1^2 + 2k_1) + 1$

$\therefore 3 \nmid n^2$, contradiction

case 2: $n = 3k_1 + 2$ $(k \in \mathbb{Z})$

$n^2 = 9k_1^2 + 12k_1 + 4$

$= 3(3k_1^2 + 4k_1 \cancel{+5}) + \cancel{4}\ 1$

$= 3(3k_1^2 + 4k_1 + 1) + 1$

$\therefore 3 \nmid n^2$, contradiction

c. proof by contradiction

negation: $\sqrt{3}$ is rational

$\therefore \sqrt{3} = \frac{a}{b}$ $(a, b \in \mathbb{Z}, b \neq 0)$, a, b have no common factor

$a^2 = 3b^2$

by definition of divisibity. $3 \mid a^2$

$\because$ from part (b)

$\therefore 3 \mid a$

$\therefore a = 3k$ $(k \in \mathbb{Z})$

$\therefore a^2 = 9k^2$

$\therefore 9k^2 = 3b^2$

$b^2 = 3k^2$ $\therefore 3 \mid b^2$ contradiction

$\therefore 3 \mid b$

For exercises 32–35 note that to show there is a unique object with a certain property, show that (1) there is an object with the property and (2) if objects A and B have the property, then A = B.

34. Prove that there is at most one real number a with the property that a + r = r for all real numbers r. (Such a number is called an additive identity.)

direct proof

for real numbers a_1, a_2, s.t.

$$\begin{cases} a_1 + r = r & ① \\ a_2 + r = r & ② \end{cases}$$

by ②, $a_1 + a_2 + r = a_2 + r$

$a_1 + a_2 = a_2$

by ① $a_2 + a_1 + r = a_1 + r$

$a_2 + a_1 = a_1$

$a_2 = a_1 + a_2 = a_2 + a_1 = a_1$

$\therefore a_2 = a_1$

$\therefore$ there is at most one real number s.t. $\hat{a}$

$a + r = r$