Simple Derivation

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\begin{aligned}
f_X(x) &= \int_0^\infty f_{X|w}(x) f_w(w) dw \\
&\propto \int_0^\infty \frac{1}{\sqrt{w}} \exp\left(-\frac{x^2}{2w}\right) w^{-\frac{v}{2}-1} \exp\left(-\frac{v}{2w}\right) dw \\
&= \int_0^\infty w^{-\frac{v+1}{2}-1} \exp\left(-\frac{x^2+v}{2w}\right) dw \\
&= \Gamma\left(\frac{v+1}{2}\right)\left(\frac{x^2+v}{2}\right)^{-\frac{v+1}{2}}
\end{aligned} \tag{1}
$$

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\begin{aligned}
f_W(w|x) &= \frac{f_{X|w}(x) f_w(w)}{f_X(x)} \\
&= \frac{\frac{1}{\sqrt{w}} \exp\left(-\frac{x^2}{2w}\right) w^{-\frac{v}{2}-1} \exp\left(-\frac{v}{2w}\right)}{\Gamma\left(\frac{v+1}{2}\right)\left(\frac{x^2+v}{2}\right)^{-\frac{v+1}{2}}} \\
&\propto w^{-\frac{v+1}{2}-1} \exp\left(-\frac{x^2+v}{2w}\right)
\end{aligned} \tag{2}
$$

Thus we have the following result: $w|x$ is an inverse gamma distribution with parameters $\alpha = \frac{v+1}{2}$ and $\beta = \frac{x^2+v}{2}$. Now, assuming there exists a known patch of $X = \{x_1, x_2, ..., x_n\}$,

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\begin{aligned}
f_W(w|X) &= f_w(w) \prod_{i=1}^{n} f_x(x_i|w) \\
&\propto w^{-\frac{v+n}{2}-1} \exp\left(-\frac{\sum_{i=1}^{n} x_i^2 + v}{2w}\right)
\end{aligned} \tag{3}
$$

$w|X$ is an inverse gamma distribution with parameters $\alpha = \frac{v+n}{2}$ and $\beta = \frac{\sum_{i=1}^{n} x_i^2+v}{2}$.