

Simple Derivation

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$$\begin{aligned} f_X(x) &= \int_0^\infty f_{X|w}(x) f_w(w) dw \\ &\propto \int_0^\infty \frac{1}{\sqrt{w}} \exp\left(-\frac{x^2}{2w}\right) w^{-\frac{v}{2}-1} \exp\left(-\frac{v}{2w}\right) dw \\ &= \int_0^\infty w^{-\frac{v+1}{2}-1} \exp\left(-\frac{x^2+v}{2w}\right) dw \\ &= \Gamma\left(\frac{v+1}{2}\right) \left(\frac{x^2+v}{2}\right)^{-\frac{v+1}{2}} \end{aligned} \tag{1}$$

$$\begin{aligned} f_W(w|x) &= \frac{f_{X|w}(x) f_w(w)}{f_X(x)} \\ &= \frac{\frac{1}{\sqrt{w}} \exp\left(-\frac{x^2}{2w}\right) w^{-\frac{v}{2}-1} \exp\left(-\frac{v}{2w}\right)}{\Gamma\left(\frac{v+1}{2}\right) \left(\frac{x^2+v}{2}\right)^{-\frac{v+1}{2}}} \\ &\propto w^{-\frac{v+1}{2}-1} \exp\left(-\frac{x^2+v}{2w}\right) \end{aligned} \tag{2}$$

Thus we have the following result: $w|x$ is an inverse gamma distribution with parameters $\alpha = \frac{v+1}{2}$ and $\beta = \frac{x^2+v}{2}$. Now, assuming there exists a known patch of $X = \{x_1, x_2, \dots, x_n\}$,

$$\begin{aligned} f_W(w|X) &= f_w(w) \prod_{i=1}^n f_{X_i}(x_i|w) \\ &\propto w^{-\frac{v+n}{2}-1} \exp\left(-\frac{\sum_{i=1}^n x_i^2 + v}{2w}\right) \end{aligned} \tag{3}$$

$w|X$ is an inverse gamma distribution with parameters $\alpha = \frac{v+n}{2}$ and $\beta = \frac{\sum_{i=1}^n x_i^2 + v}{2}$.