$$H(p) = -\sum_{i=1}^{K} p_i \log(p_i)$$

$Probability$:

$P(X = x_i) = p_i, for\ i = 1,2, \ldots K$

$Constriant$:

$$\sum_{i=1}^{K} p_i = 1 \rightarrow \sum_{i=1}^{K} p_i - 1 = 0$$

$Lagrange\ multipliers$:

$$\mathcal{L} = H(p) + \lambda\left(\sum_{i=1}^{K} p_i - 1\right)$$

$$= -\sum_{i=1}^{K} p_i \log(p_i) + \lambda\left(\sum_{i=1}^{K} p_i - 1\right)$$

$To\ find\ the\ highest\ entropy, set\ \dfrac{\partial \mathcal{L}}{\partial p_i} = 0$

$$\frac{\partial \mathcal{L}}{\partial p_i} = \frac{\partial\left(-\sum_{i=1}^{K} p_i \log(p_i) + \lambda(\sum_{i=1}^{K} p_i - 1)\right)}{\partial p_i} = -\log(p_i) - 1 + \lambda = 0$$

$\log(p_i) = \lambda - 1$

$Thus, p_1 = p_2 = \cdots = p_i = \dfrac{1}{K}$

$Therefore, when\ p^* = \dfrac{1}{K}, the\ categorical\ distribution\ has\ the\ highest\ entropy,$

$H(p^*) = logK.$