$$H(p) = -\sum_{i=1}^{K} p_i \log(p_i)$$

Probability:

$$P(X = x_i) = p_i, for\ i = 1,2,\dots K$$

Constriant:

$$\sum_{i=1}^{K} p_i = 1 \rightarrow \sum_{i=1}^{K} p_i - 1 = 0$$

Lagrange multipliers:

$$\mathcal{L} = H(p) + \lambda\left(\sum_{i=1}^{K} p_i - 1\right)$$

$$= -\sum_{i=1}^{K} p_i \log(p_i) + \lambda\left(\sum_{i=1}^{K} p_i - 1\right)$$

To find the highest entropy, set $\dfrac{\partial \mathcal{L}}{\partial p_i} = 0$

$$\frac{\partial \mathcal{L}}{\partial p_i} = \frac{\partial\left(-\sum_{i=1}^{K} p_i \log(p_i) + \lambda(\sum_{i=1}^{K} p_i - 1)\right)}{\partial p_i} = -\log(p_i) - 1 + \lambda = 0$$

$$\log(p_i) = \lambda - 1$$

$$p^* = e^{\lambda-1}$$

The categorical distribution has the highest entropy when $p^* = e^{\lambda-1}$