Numerical Methods - Lecture 1

University Leipzig

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For $n+1$ data points $(x_i, f(x_i))$, $i = 0, 1, \ldots, n$ with pairwise distinct x_i there is exactly one interpolating polynomial $P(x)$ of degree $\leq n$ which is given by

$$\begin{gathered} P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \\ \ldots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \ldots (x - x_{n-1}) \end{gathered}$$

where the divided differences are given by

$$f[x_i] = f(x_i) f[x_i, \ldots, x_{i+k}] = \frac{f[x_i, \ldots, x_{i+k-1}] - f[x_{i+1}, \ldots, x_{ik}]}{x_i - x_{i+k}}$$

Now we calculate the interpolating polynomials in the newton base, for previous example

i	x_i	y_i
1	2	1/2
2	2.5	2/5
3	4	1/4

$$f[x_0, x_1, x_2] = \frac{-0.1 - (-0.2)}{4 - 2} = 0.05$$

$$f[x_0, x_2] = \frac{0.4 - 0.5}{2.5 - 2} = -0.2 \qquad f[x_1, x_2] = \frac{0.25 - 0.4}{4 - 2.5} = -0.1$$

$$f[x_0] = f(x_0) = 0.5 \qquad f[x_1] = f(x_1) = 0.4 \qquad f[x_2] = f(x_2) = 0.25$$

$$P(x) = 0.5 - 0.2(x - 2) + 0.05(x - 2)(x - 2.5)$$

0.1 Error of polynomial interpolation

Theorem. Let $f : [a, b] \to \mathbb{R}$ be at least $(n + 1)$ times continuously differentiable and $P(x)$ be the interpolating polynomial of f in the nodes $x_0, \dots, x_n \in [a, b]$ of degree $\leq n$. Then for every $x \in [a, b]$ there is a $\xi = \xi(x) \in [a, b]$ such that

$$f(x) - P(x) = \underbrace{(x - x_0)(x - x_1) \dots (x - x_n)}_{:= W_{n+1}(x)} \frac{f^{n+1}(\xi)}{(n + 1)!}$$

Proof. Let $F(x) := f(x) - p(x) - k \cdot W_{n+1}(x)$ and select some $\hat{x} \in [a, b], \hat{x} \neq x_i$. We choose K such that $F(\hat{x}) = 0$. This is always possible since $W_{n+1}(x) \neq 0$. $F(x)$ has therefore $n + 2$ roots. This means that $F'(x)$ has $n+1$ roots, $F''(x)$ has n roots etc., until $F^{n+1}(x) = f^{n+1}(x) - k(n + 1)!$ has one root, which we call $\xi = \xi(\hat{x})$

$$\begin{aligned} 0 &= f^{n+1}(\xi) - k \cdot (n + 1)! \\ K &= \frac{f^{n+1}(\xi)}{(n + 1)!} \end{aligned}$$

□

Remark. In order to estimate the error, one needs to know the boundaries of the $(n+1)$th derivative on $[a, b]$

0.2 Chebyshev/Tschebyscheff-polynomials

If we have the freedom to choose the nodes x_i, then this can be used to minimize $|W_{n+1}(x)|$ and thus the error of the interpolating polynomial. We constrict ourselves to $[a, b] = [-1, 1]$. If $[a, b] \neq [-1, 1]$ then the transformation

$$\begin{aligned} [-1, 1] &\to [a, b] \\ x &\to \frac{a + b}{2} + x\frac{b - a}{2} = y \end{aligned}$$

with the inverse transformation

$$\begin{aligned} [a, b] &\to [-1, 1] \\ y \to \frac{2y}{b - a} &- \frac{a + b}{b - a} = x \end{aligned}$$

does not change the properties of the interpolation and approximation. The Chebyshev polynomials are recursively defined by

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_n + 1 &= 2x \cdot T_n(x) - T_{n-1}(x) \end{aligned}$$

This implies $T_n(x) = \cos(n \cdot \arccos(x))$

Proof.

$$
\begin{aligned}
T_{n+1} &= 2x \cdot \cos(n \cdot \arccos(x)) - \cos((n-1)\arccos(x)) \\
&= 2\cos(\arccos(x)) \cdot \cos(n \cdot \arccos(x)) - \cos((n-1))\underbrace{\arccos((x))}_{\varphi} \\
&= \cos((n+1)\varphi) + \cos((n-1)\varphi) - \cos((n-1)\varphi) = \cos((n+1)\varphi)
\end{aligned}
$$

□

Properties of T_n:

- the roots of T_n are $\cos(\frac{2k+1}{2k} \cdot \pi)$ $(k = 0, \dots, n-1)$
- $T_n\left(\cos(\frac{k \cdot \pi}{n})\right) = (-1)^k$
- $|T_n(x)| \leq 1$ for $x \in [-1, 1]$

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1
\end{aligned}
$$

Theorem. From all possible $(x_0, \dots, x_n)^T \in [-1,1]^n \subset \mathbb{R}^n$ the value $\max_{x \in [-1,1]} |w_{n+1}(x)|$ becomes minimal if the nodes x_i are the roots of the $(n+1)$st Chebyshev polynomial T_{n+1}

$$
\begin{aligned}
x_i &= \cos\left(\frac{2i+1}{2n+2}\pi\right), i = 0, \dots, n \\
\text{Then } w_{n+1}(x) &= \frac{1}{2^n} \cdot T_{n+1}(x) \\
\text{and } \max_{x \in [-1,1]} |x_{n+1}(x)| &= \frac{1}{2^n}
\end{aligned}
$$

Proof.

Lemma. Let $q(x) = 2^{n-1}x^n + \dots$ be a polynomial unequal to the nth Chebyshev polynomial T_n. Then:

$$
\max_{x \in [-1,1]} |q(x)| > 1 = \max_{x \in [-1,1]} |T_n(x)|
$$

We assume $|q(x)| \leq 1$ for all $x \in [-1,1]$, $T_n(1) = 1$, $T_n\left(\cos\frac{\pi}{n}\right) = -1$. We consider $T_n(x) - q_n(x)$ at $[\cos\frac{\pi}{n}, 1]$. $q(x)$ and $T_n(x)$ have the same highest order coefficient (2^{n-1}). Therefore $T_n(x) - q(x)$ is of degree $n-1$

$$
\begin{aligned}
&|q(x)| \leq 1 \\
&\to T_n(1) - q(1) \geq 0 \\
&\to T_n\left(\cos\left(\frac{\pi}{n}\right)\right) - q\left(\cos\left(\frac{\pi}{n}\right)\right) \leq 0
\end{aligned}
$$

$T_n(x) - q_(x)$ has at least one root on $[\cos\frac{\pi}{n}, 1]$. In the same way, we can show that $T_n(x) - q(x)$ has at least one root on $[\cos\frac{2\pi}{n}, \cos\frac{\pi}{n}]$ and on $[\cos\frac{3\pi}{n}, \cos\frac{2\pi}{n}]$ and on ...and on $[-1, \cos\left(\frac{n-1\pi}{n}\right)]$. Therefore $T_n(x) - q(x)$ has n roots in $[-1, 1]$. If the root is situated on the boundary of two sub intervals, then it's a double root, since $T_n(x)$ and $q(x)$ extremal. However $T_n(x) - q(x)$ is only of degree $\leq n-1$

$$\begin{aligned} T_n(x) - q(x) = 0 \text{ (!) contradicition to assumption } T_n(x) \neq q(x) \\ \rightarrow \max_{x\in[-1,1]} |w_{n+1}(x)| = \frac{1}{2^n} \underbrace{\max_{x\in[-1,1]} |\underbrace{2^n w_{n+1}(x)}_{T_{n+1}(x)}|}_{\geq 1 \text{ with Lemma}} \\ \max_{x\in[-1,1]} |w_{n+1}| \geq \frac{1}{2^n} \text{ with equality if } w_{n+1}(x) = \frac{1}{2^n} \cdot T_{n+1}(x) \end{aligned}$$

□

0.3 hermite interpolation

If in addition to the values of f also the values of the derivative f' are known, we can construct the hermite interpolating polynomial. Every node $x_i \in \{x_0, \ldots, x_n\}$ corresponds to 2 conditions which implies that the interpolating polynom is of degree $2n+1$

Theorem. Let $f \in C^1([a,b])$ and $x_0, \ldots, x_n \in [a,b]$ pairwise distinct. Then the only polynomial of degree $2n+1$ that equals f and f' in $x_0, , x_n$ is given by

$$\begin{aligned} \mathcal{H}_{2n+1}(x) &= \sum_{j=0}^{n} f(x_j) \cdot \mathcal{H}_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \cdot \widehat{\mathcal{H}}_{n,j}(x) \\ \text{where } \mathcal{H}_{n,j}(x) &= [1 - 2(x - x_j) \cdot L'_{n,j}(x_j)] \cdot L_{n,j}(x) \\ \text{and } \widehat{\mathcal{H}}_{n,j}(x) &= (x - x_j) \cdot L_{n,j}(x)^2 \end{aligned}$$

Here $L_{n,j}(x)$ are the Lagrange polynomials. It's easy to see that $\mathcal{H}_{n,j}(x_i) = \delta_{ij}$ $(L_{n,j}(x_i) = \delta_{ij})$ and $\widehat{\mathcal{H}}_{n,j}(x_i) = 0 \ \forall x_i$

$$\begin{aligned} \mathcal{H}'_{n,j}(x) &= 2L'_{n,j}(x_j) \cdot L_{n,j}(x)^2 + 2[1 - 2(x - x_j)L'_{n,j}(x_j)] \cdot L'_{n,j}(x) \cdot L_{n,j}(x) \\ \mathcal{H}'_{n,j}(x_j) &= -2L'_{n,j}(x_j) + 2L'_{n,j}(x_j) = 0 \\ H_{n,j}(x_i)|_{i\neq j} &= 0 \end{aligned}$$

$$\begin{aligned} \widehat{\mathcal{H}}'_{n,j}(x) &= L_{n,j}(x)^2 + 2(x - x_j)L'_{n,j}(x)L_{n,j}(x) \\ \widehat{\mathcal{H}}'_{n,j}(x_j) &= 1 \\ \widehat{\mathcal{H}}'_{n,j}(x_i)|_{i\neq j} &= 0 \end{aligned}$$

In summary we obtain

$$\mathcal{H}_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) \cdot \mathcal{H}_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\widehat{\mathcal{H}}_{n,j}(x)$$

$$\begin{aligned} \mathcal{H}_{n,j}(x_i) &= \delta_{ij} \qquad & \mathcal{H}'_{n,j}(x_i) &= 0 \\ \widehat{\mathcal{H}}_{n,j}(x_i) &= 0 \qquad & \widehat{\mathcal{H}}'_{n,j}(x_i) &= \delta_{ij} \end{aligned}$$

However, the construction by means of Lagrange polynomials is computationally expensive. Again, divided differences are useful. From the remainder of polynomial interpolation it follows:

Lemma. Let $f \in C^n([a,b])$ and $X_0, \dots, x_n$ be pairwise distinct in $[a,b]$. Then there is a $\xi \in [a,b]$ such that

$$f[x_0, \dots, x_n] = \frac{f^{(n)}(x)}{n!} \ (*)$$

Proof. If $n+1$ nodes $x_0, \dots, x_n$ and the values $f(x_i), f'(x_i)$ $(i = 0, \dots, n)$ are given, we define a sequence [TODO] □

0.4 Spline interpolation

Although an interpolating polynomial can always be found, this is often not a very good approximation, especially in the case of large n. Fluctuation in a small region can lead to great changes throughout the entire interval. Piecewise interpolation can be preferable.

The simplest version of piecewise interpolation would employ polynomials of degree 1. This has the big disadvantage that the resulting curve is not smooth/differentiable. Better choice: Piecewise hermite interpolation with polynomials of degree 3. A cubic spline interpolating function is a function s with the properties