Written in 7 days

-1 0 1 2

$\sqrt{2} = \{q \in \mathbb{Q} \mid q^2 < 2 \text{ or } q < 2\}$.
$\mathbb{Q}$ embeds in $\mathbb{R}$ via $q \mapsto \{r \in \mathbb{Q} \mid r < q\}$, completion
Requirements are not crucial
$r = \emptyset \to -\infty$, $r = \mathbb{Q} \to +\infty$
r contains a greatest element $\to$ non unique repr.

Opposite approach
Ordinals introduced by Cantor as equivalence class of well-ordered sets
For von Neumann, each ordinal is the well-ordered set of all smaller ordinals
In other words, an ordinal is a set of previously constructed ordinals
We start from nothing!!

Write def on board
Let's mix it up!
$\geq$ is the native connective, $=$ follows.
This is a well-founded definition!
This defines a real closed field

$\leq$: We want $x^L < x < x^R$,
if $y \leq x$ and $x^R \leq y$
then $x \geq x^R$!

$x - x^L, y - y^L > 0$
$\Rightarrow (x - x^L)(y - y^L) > 0$
$xy > xy^L + x^L y - x^L y^L$

mult: because well. $\{x^L y, x y^L \mid x^R y, x y^R\}$ might not contain enough info.
It defines $+$.

$\{\mid\} = 0$ is it a number? $0+0=0$ $0 \cdot 0 = 0$
$0 + x = x$ $0 \cdot x = 0$
$\{0 \mid\} = 1$ is it a number? $1 \cdot 1 = 1$
$\{\mid 0\} = -1$ ——— $-1 = -(1)$ $-1 \leq 0 \leq 1$
$\{0 \mid 0\}$ not a number

$x = \{-1 \mid 2\}$? $x \geq 0$ unless $2 \leq 0$,
$x \leq 0$ unless $0 \leq -1$.

$y \ngeq x \to \{y, x^L \mid x^R\} = x$
$x' \geq x$ unless $\underbrace{x'^R \leq x}_{\text{no, } x'^R \text{ is an } x^R}$ or $\underbrace{x' \leq x^L}_{\text{no, } x^L \text{ is an } x'^L}$
$x' \leq x$ unless $x'^L \geq x$ or $x' \geq x^R$
$y \geq x$ no, x^R is an x'^R

If we keep going this way, we obtain dyadic rationals

Day ω: $\omega = \{0, 1, 2, \dots \mid\} = \{2, 4, 8, \dots\}$ & $-\omega$
$\{0 \mid 1, \frac{1}{2}, \frac{1}{4}, \dots\} = \frac{1}{\omega}$
Other real numbers: $\frac{1}{3} = \{\frac{1}{4}, \frac{1}{4} + \frac{1}{16}, \dots \mid \frac{1}{2}, \frac{1}{2} - \frac{1}{8}, \dots\}$
$\sqrt{2}, e, \pi$, as cuts of dyadic rationals
Dyadic rationals are recreated on day ω:
$\frac{3}{8} = \{\text{dyadic} < \frac{3}{8} \mid \text{dyadic} > \frac{3}{8}\}$
von Neumann's ordinals also embed there as
$0 = \{\mid\}$, $1 = \{0 \mid\}$, $2 = \{0, 1 \mid\}$, ..., $\alpha = \{\beta < \alpha \mid\}$

More numbers:
$\{0, 1, 2, \dots \mid \omega\} = \omega - 1$, $\omega - 2, \dots$
$\{0, 1, 2, \dots \mid \omega, \omega - 1, \omega - 2, \dots\} = \frac{\omega}{2}, \frac{\omega}{4}, \dots$
$\{0, 1, 2, \dots \mid \omega, \frac{\omega}{2}, \frac{\omega}{4}, \dots\} = \sqrt{\omega}$
$\{0 \mid \frac{1}{\omega}\} = \frac{1}{2\omega}$, $\{\frac{1}{\omega} \mid 1, \frac{1}{2}, \frac{1}{4}, \dots\} = \frac{2}{\omega}$,
$\{0 \mid \frac{1}{\omega}, \frac{1}{2\omega}, \frac{1}{4\omega}, \dots\} = \frac{1}{\omega^2}$
$\sqrt[3]{\omega}, \omega^{1/\omega}, \omega + \pi, \frac{1}{\omega + 1}, \sqrt{\omega - 1}$

I scammed you! Are operations well-defined?
That is, do they yield a number and are they independent on the repr.
Numbers can be generalised to games.
Allows us to write expressions without having to immediately check that they are numbers.
We will study games later in this talk.
Identity = they are the same representation.

$\leq$ is an order, antisymmetry by definition.
GAMES form a group!
Numbers are a subgroup.

Advantage of this approach: No need to reprove the same statement for $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.
But maybe a bit artificial and less natural.

We have already defined ordinals, but there is another surprising definition.
Normal forms also generalise to all numbers.

From now on, no proof because too technical.
More of a survey

Integers = omnific integer.
Waring's problem: Write numbers as sums of square

We consider combinatorial games. For example chess.

Let's look at those games. What are the winning strats?
0 is the endgame. Winning strategy: don't start!
1 is always winning for Left.
-1 ——— Right.
$*$ is a fuzzy game. Winning strategy: start!

$G \leq 0$ R wins if L starts	$G \mid\!\triangleright 0$ L wins if L starts	
$G = 0$ 2nd wins	$G > 0$ L wins	$G \geq 0$ L wins if R starts
$G < 0$ R wins	$G \parallel 0$ 1st wins	$G \triangleleft\!\mid 0$ R wins if R starts

Proof Goal: Every game G is either ≤ 0 or $\mid\!\triangleright 0$, and either $G \geq 0$ or $\triangleleft\!\mid 0$.
If $G^L \geq 0$ for some G^L, then $G \mid\!\triangleright 0$.
Otherwise every $G^L \triangleleft\!\mid 0$, and $G \leq 0$.

Negation: The legal moves of L in the reversed game are the legal moves of R in the initial game. But after L moves, the roles are still reversed, thus giving a recursive definition.
Sum: A legal move for L is either
- picking G and moving to some $G^L \to G^L + H$
- picking H and moving to some $H^L \to G + H^L$

How sums happen: Two parts of the board may become independent from each other.
Game of dominos: Left vertical, Right horizontal.
$= 0$, $= 1$, $= -2$
$= \{-1, 0 \mid 1\} = \frac{1}{2}$
In a sense L is half a move ahead
$= *$, $= \{1 \mid -1\} \parallel 0$
$= \frac{1}{2} + 1 - 2 = \frac{-1}{2}$, R is half a move ahead
$1 + 1$:
$* + *$: First player to move ends up in $* + 0$ or $0 + *$, which is winning for 2nd

It is illegal to move in 0.
$G - G$: 2nd player mirrors strategy of the first
$G + H \geq 0$: Right starts, and L always replies on the same board using his winning strat.
$G + H > 0$: If L starts, do so in $G > 0$.
$G + H = G$: Only play in H if replying to a move in H.
$G + H = G + K$: $G + K = G + K + H - K = G + H$

At this point, you may be wondering whether the two orders are the same.
$G \geq 0$ unless $G^R \leq 0$ for some G^R
L wins G if R starts unless R wins G^R if L starts, for some G^R

And so on! This is completely justified

$\{1 \mid -1\} < a$ for all $a > 1$, $\{1 \mid -1\} > b$ for all $b < -1$,
$\{1 \mid -1\} \parallel c$ for all $c \in [-1, 1]$.
$\ngeq$ $-1 \leq c$
$\nleq$ $1 \geq c$
$\{2 \mid -2\}$ < than any number > 2, > than any number < -2 and $\parallel$ against any $[-2, 2]$.
It seems that we can learn something about a game by comparing it to all numbers.
We can define Leftmost and Rightmost values st any number outside the interval is comparable and any number inside is fuzzy.
We can also define a mean value of a game.
I'm not going in all the details, but let me give a brief overview.
Unfortunately, a large part of this argument is inapplicable to infinite games.