Hereditary Structural Completeness of Weakly Transitive Logics

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Deductive systems

Let Fm be a set of propositional formulas.

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A deductive system is a relation $\vdash \subseteq \mathcal{P}(Fm) \times Fm$ such that

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- $\Gamma \vdash \phi$ implies that there is a finite set $\Delta \subseteq \Gamma$ such that $\Delta \vdash \phi$.
- for any substitution $\sigma : Fm \to Fm$, $\Gamma \vdash \phi$ implies $\sigma[\Gamma] \vdash \sigma(\phi)$.

Admissible and derivable rules

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In other words, the set of theorems of $\vdash$ is closed under $\Gamma \rhd \phi$, and adding $\Gamma \rhd \phi$ to $\vdash$ does not add new theorems.

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A rule $\Gamma \triangleright \psi$ is derivable in $\vdash$ if $\Gamma \vdash \phi$.

Structural completeness

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Definition

A deductive system $\vdash$ is structurally complete (SC) if every admissible rule in $\vdash$ is derivable in $\vdash$.

Extensions of a deductive system

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$$\Gamma \vdash' \phi \qquad \text{iff} \qquad \Gamma, \Delta \vdash \phi.$$

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Remark

If $\vdash'$ is an axiomatic extension of $\vdash$, we can always take Δ to be the theorems of $\vdash'$.

Hereditary structural completeness

Theorem (Olson, Raftery, Van Alten)

Let $\vdash$ be a deductive system, TFEA.

- *Every extension of $\vdash$ is SC.*
- *Every axiomatic extension of $\vdash$ is SC.*
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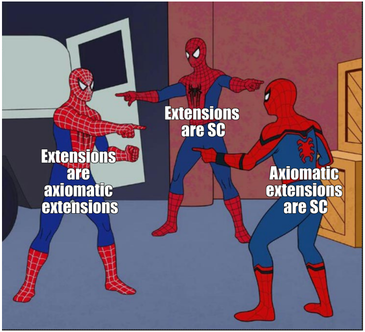

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Definition

A deductive system $\vdash$ is hereditarily structurally complete (HSC) if it validates any of the above.

Deductive systems from modal logics

Definition

Given a NML Λ, we define a deductive system $\vdash_\Lambda$ by $\Gamma \vdash_\Lambda \phi$ iff ϕ is derivable from Γ using

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Remark

Given a NML Λ, the axiomatic extension of $\vdash_\Lambda$ are those systems of the form $\vdash_{\Lambda'}$ where Λ' is a NML extending Λ.

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$$K + p \wedge \Box p \to \Box\Box p.$$

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$$x\,R\,y \quad \text{and} \quad y\,R\,z \qquad \text{implies} \qquad x\,R\,z \quad \text{or} \quad x = z$$

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A modal algebra is a wK4-algebra if it validates $a \wedge \Box a \leq \Box\Box a$ for all a.

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The master modality is definable as $\Box^{+} p := p \wedge \Box p$.

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Proposition (Blok, Pigozzi)

If Λ is a 1-transitive logic, then $\vdash_\Lambda$ has a deduction detachment theorem (DDT) witnessed by $\Box^+ p \to q$:

$$\Gamma, \phi \vdash_\Lambda \psi \qquad \textit{iff} \qquad \Gamma \vdash_\Lambda \Box^+\phi \to \psi.$$

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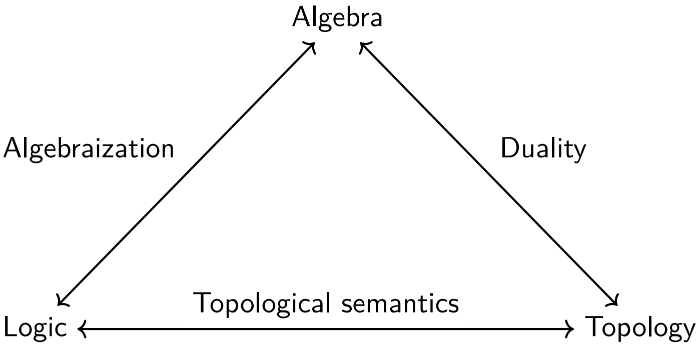
- SL, 2023: HSC of weakly transitive modal logics (extension of wK4)

Method


Algebra
Algebraization
Duality
Topological semantics
Logic
Topology

Algebraisable logics

Definition

A deductive system $\vdash$ is algebraisable if there exist

- a quasi-variety K,
- a set of equations $\tau(x)$,
- a set of formulas $\Delta(x, y)$,

such that for all set of equation Θ, all equation $\varepsilon \approx \delta$, all set of formulas Γ and all formula ϕ, we have

- $\Gamma \vdash \phi$ iff $\tau[\Gamma] \models_K \tau(\phi)$,
- $\Theta \models_K \varepsilon \approx \delta$ iff $\Delta[\Theta] \vdash \Delta(\varepsilon, \delta)$,
- $\phi \vdash \Delta[\tau(\phi)]$ and $\Delta[\tau(\phi)] \vdash \phi$,
- $\varepsilon \approx \delta \models_K \tau[\Delta(\varepsilon, \delta)]$ and $\tau[\Delta(\varepsilon, \delta)] \models_K \varepsilon \approx \delta$.

The quasi-variety K is the equivalent algebraic semantics (EAS) of $\vdash$. It is unique when it exists.

Rules and quasi-equations

Let $\vdash$ be a deductive system with quasi-variety K as its EAS.

Remark

A rule $\Gamma \rhd \phi$ corresponds to a quasi-equation $\bigwedge \tau[\Gamma] \to \tau(\phi)$. Conversely, a quasi-equation $\bigwedge \Theta \to \varepsilon \approx \delta$ corresponds to a rule $\Delta[\Theta] \rhd \Delta(\varepsilon, \delta)$

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Proposition

A rule is derivable in $\vdash$ iff the corresponding quasi-equation is valid in K

Structural completeness

Let $\vdash$ be a deductive system with quasi-variety K as its EAS.

Corollary

$\vdash$ is SC iff every quasi-equation which is valid in $F_K(\omega)$ is valid in K.

Theorem (Prucnal, Wroński)

$\vdash$ is SC iff $K = \mathbb{Q}(F_K(\omega))$.

Extensions and subquasi-varieties

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Theorem (Blok, Pigozzi)

The lattice of axiomatic extensions of $\vdash$ is dually isomorphic to the lattice of relative subvarieties of K.

Hereditary structural completeness

Let $\vdash$ be a deductive system with variety K as its EAS.

Corollary

$\vdash$ *is HSC iff every subquasi-variety of* K *is a variety.*

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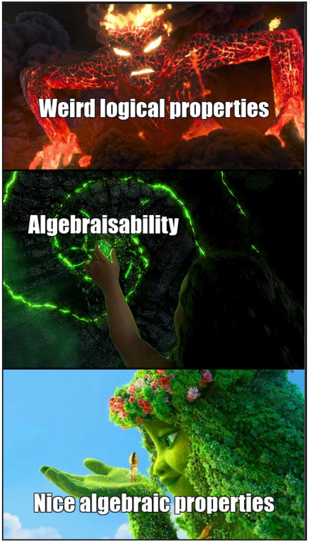
Definition

A variety is primitive if all of its subquasi-varieties are varieties.

Dictionary

Logic	Algebra
Deductive system	EAS (variety)
Rules	Quasi-equations
Admissible	Valid in $F_K(\omega)$
Derivable	Valid in K
SC	$K = \mathbb{Q}(F_K(\omega))$
Extensions	Subquasi-variety
Axiomatic extensions	Subvarieties
HSC	Primitive



Wrapping up

The following problems are equivalent:

- Characterising the 1-transitive modal logics Λ such that $\vdash_\Lambda$ is HSC.
- Characterising the axiomatic extensions of $\vdash_{\mathsf{wK4}}$ which are HSC.
- Characterising the primitive varieties of wK4-algebras.