CPSC 121 Midterm 2
Wednesday 14 June 2017

[4] 1. Oh, a Question
Prove $5n^5 + 3n^2 \in \mathcal{O}(n^5)$.

Solution: $3n^2 \leq 3n^2 \cdot n^3 = 3n^5$ for all $n \geq 1$
$\Rightarrow 5n^5 + 3n^2 \leq 5n^5 + 3n^5 = 8n^5$ for all $n \geq 1$

[4] 2. Oh No, a Question
Prove $3n^2 \notin \mathcal{O}(n)$.

Solution: Assume $3n^2 \in \mathcal{O}(n)$
$\Rightarrow \exists c \in \mathbb{N}, \exists t \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq t \rightarrow 3n^2 \leq cn$
$\Rightarrow \exists c \in \mathbb{N}, \exists t \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq t \rightarrow n \leq \frac{c}{3}$ ↯

[1] 3. How Many Moos?
The number of cows in field A is prime. The number of cows in field B is prime. Must the number of cows, taken together (that is, the number of cows in field A plus the number of cows in field B) be even? Justify your answer in no more than three sentences.

Solution: No. There could be 2 cows in one field and 3 in the other.

[1] 4. What Becomes of This?
Let $R = \emptyset$, $S = \emptyset$, and $L = \{cow, bunny, puppy\}$. What is $(R \cup S) \cup (L \backslash L)$?

Solution: $\emptyset$

[6] 5. Predicates Abound
Consider predicates $F(x)$, $G(x)$, and $H(x, y)$, on domain D (you can think of these three predicates and the domain in the abstract — they can have any meaning that you please). We know the following three statements are true:

- 1. $\exists a \in D$ s.t. $F(a)$
- 2. $\forall b \in D, \sim G(b) \vee H(b, b)$
- 3. $\forall c \in D, \forall d \in D, F(c) \rightarrow G(d) \wedge H(c, d)$

Using these known facts, prove $\exists e \in D$ s.t. $H(e, e)$. Explicitly state the rule used in each step of your proof, referring to previous line numbers in your proof as "inputs" to the rule.

Solution:

Line	Statement	Rule	On
4.	$x \in D \wedge F(x)$	Existential instantiation	1
5.	$\forall d \in D, F(x) \to G(d) \wedge H(x,d)$	Universal instantiation	3
6.	$F(x) \to G(x) \wedge H(x,x)$	Universal instantiation	5
7.	$G(x) \wedge H(x,x)$	Modus ponens	4, 6
8.	$H(x,x)$	Specialization	7
9.	$\exists e \in D, H(e,e)$	Direct proof	8

[12] 6. A Little Inductive Fun, With Lemmas

In this question, you will prove a statement in three stages. You are free to invoke the conclusion of an earlier part in a later part, even if you did not successfully prove the earlier part. That is, even if you did not solve Part "a", you are free to use the statement you are trying to prove in Part "a" in your answer to Part "b" or Part "c".

[4] a. Prove, via induction, that $2^n \geq 2$ for all $n \in \mathbb{Z}$ where $n \geq 1$.

Solution: Base case ($n = 1$): $2^1 = 2 \geq 2$
Inductive step: Assume $2^n \geq 2$ for some $n \geq 1$ (I.H.a)

$$\begin{aligned}
2^{n+1} &= 2^n + 2^n \\
&\geq 2 + 2 && \text{by I.H.a} \\
&\geq 2
\end{aligned}$$

[4] b. Next, prove, via induction, that $2^n \geq 2n + 1$ for all $n \in \mathbb{Z}$ where $n \geq 3$.

Solution: Base case ($n = 3$): $2^3 = 8 \geq 7 = 2 \cdot 3 + 1$
Inductive step: Assume $2^n \geq 2n + 1$ for some $n \geq 3$ (I.H.b)

$$\begin{aligned}
2^{n+1} &= 2^n + 2^n \\
&\geq 2^n + 2 && \text{by Part A} \\
&\geq 2n + 1 + 2 && \text{by I.H.b} \\
&= 2(n+1) + 1
\end{aligned}$$

[4] c. Finally, prove, via induction, that $2^n \geq n^2$ for all $n \in \mathbb{Z}$ where $n \geq 4$.

Solution: Base case ($n = 4$): $2^4 = 16 \geq 16 = 4^2$
Inductive step: Assume $2^n \geq n^2$ for some $n \geq 4$ (I.H.c)

$$\begin{aligned}
2^{n+1} &= 2^n + 2^n \\
&\geq 2^n + 2n + 1 && \text{by Part B} \\
&\geq n^2 + 2n + 1 && \text{by I.H.c} \\
&= (n+1)^2
\end{aligned}$$

[3] 7. Think Backwards About This One
Prove that, for all $n \in \mathbb{Z}$, if $n^2 + 4n - 3$ is even, then n is odd.

Solution: Proof by contrapositive.

Assume n is even. So write $n = 2k$, for some $k \in \mathbb{Z}$.

$$\begin{aligned} \Rightarrow n^2 + 4n - 3 &= (2k)^2 + 4(2k) - 3 \\ &= 4k^2 + 8k - 3 \\ &= 2(2k^2 + 4k - 1) - 1 \end{aligned}$$

Since $k \in \mathbb{Z}$, we know $2k^2 + 4k - 1 \in \mathbb{Z}$. So, we can write

$$n^2 + 4n - 3 = 2j - 1$$

for some $j \in \mathbb{Z}$, meaning it is odd.

[1] 8. Puppies Are Silly
Assume p is some puppy in the set of all puppies, P. Assume that p likes chasing its own tail. Let $C(x)$ mean that x likes chasing its own tail. Do we know that $\forall\ z \in P, C(z)$? Justify your answer in no more than three sentences.

Solution: No. We only know $\exists z \in P, C(z)$. There could be other puppies that do not like chasing their own tails.

[1] 9. Powerful Power Sets
What is $\mathcal{P}\big(\{\emptyset\}\big)$?

Solution: $\{\emptyset, \{\emptyset\}\}$

[2] 10. What a Nice Compliment
Assume you have an 8-bit signed (i.e., 2's-complement) number. The number stored is 0xE9. What value, in base-10, does this number represent.

Solution: The represented value is -23.

0xE9 represents 1110 1001. Flipping the bits and adding one yields 0001 0111, or positive 23.

[4] 11. A Bijective Function
Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^5 + 8$. Prove that f is bijective.

Solution: First, prove injectivity: assume $f(x) = f(y)$ for some $x, y \in \mathbb{R}$
$\Rightarrow x^5 + 8 = y^5 + 8$
$\Rightarrow x^5 = y^5$
$\Rightarrow x = y$

Next, prove surjectivity: let $y \in \mathbb{R}$.
Consider $x = \sqrt[5]{y-8}$. Note $x \in \mathbb{R}$, because $y \in \mathbb{R}$.
$f(x) = \left(\sqrt[5]{y-8}\right)^5 + 8 = y - 8 + 8 = y$

[2] 12. <u>A Tricky Conversion</u>
Convert 320_7 into base-12.

Solution: First, convert into base-10:

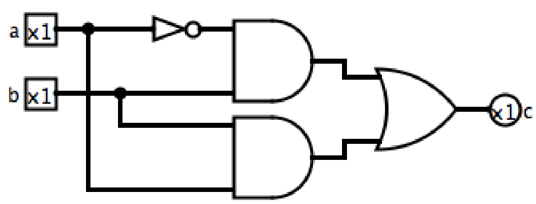
$3 \cdot 7^2 + 2 \cdot 7^1 + 0 \cdot 7^0 = 161_{10}$

Next, convert into base-12:

$$\begin{aligned} 161 - 1 \cdot 12^2 &= 17 \\ 17 - 1 \cdot 12^1 &= 5 \\ 5 - 5 \cdot 12^0 &= 0 \end{aligned}$$

Therefore, the value is 115_{12}.

[4] 13. <u>A Subtle Glitch</u>
Consider the following circuit. Denote the inputs to the circuit as a and b, and the output of the circuit as c.



[2] a. The circuit has a glitch in it, such that its output is not stable. That is, there exist inputs a_1 and b_1 that produce output c', and there exists other inputs a_2 and b_2 that also produce output c'.

However, each gate takes time to change its output after its input changes. Because of this delay, when you change the inputs to the circuit from a_1 and b_1, to a_2 and b_2, the circuit may briefly output the opposite of c'.

Identify inputs a_1, b_1, a_2, and b_2 that trigger this glitch.

Solution: $a_1 = b_1 = b_2 =$ true, and $a_2 =$ false

[2] b. Assume each gate takes 5 ns to operate (that is, to change its output when its input changes). For what period of time will the circuit be outputting the incorrect output? Justify your answer by describing the timeline.

(Denote the moment when inputs a_1 and b_1 change to a_2 and b_2 as $t = 0$ ns.)

Solution: The glitch lasts from 10 ns – 15 ns.

- At $t = 0$ ns, the circuit (i.e., the OR gate) is outputting true.
- At $t = 5$ ns, the NOT gate starts outputting true, and the lower AND gate false.
- At $t = 10$ ns, the top AND gate starts outputting true, but the OR gate begins outputting false.
- At $t = 15$ ns, the OR gate starts outputting true, resolving the glitch.