CPSC 121, Summer 2016: Final Exam

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This exam is open-book, and you may use any written or printed materials to assist you. You may not use any electronic devies, though (e.g., calculators, phones, tablets, etc.). Please do not take any copies of these questions from the examination room.

1. A Silly Situation (4 marks)
   Let c be a cow, and let p be a puppy. The cow says "moo", and the puppy says "woof". They are having a terribly silly argument about the nature of the universe, and what should be done with the farm's cat (neither has a great opinion of the cat). Because we have already used the variable c, we'll call the cat k (from the German for "cat", which is "Katze"). As the farmer, you want to create a machine that recognizes if there are an odd number of "moo"s produced in this argument — an argument that shall determine the fate of k.

   (a) This part of the question is worth 2 of the 4 marks
   Draw a DFA that accepts an input on the language $L = \{\text{moo}, \text{woof}\}$ (the cat gets no say in its fate) where there are an odd number of "moo"s.

   (b) This part of the question is worth 2 of the 4 marks
   Write a regular expression that recognizes the same input as the aforementioned DFA. To make your regular expression easier to read, please choose two symbols (letters or otherwise) to represent a "moo" and a "woof". Please define those symbols clearly.

2. As Simple As Possible (2 marks)
   Consider the following statement: "If p is a penguin, then p likes fish; but, if k is a kitten, then k likes tuna if and only if k also likes playing with toys with a feather on them". Assume the following are definitions are propositions: g means that p is a penguin; and t means that k likes toys with a feather on them. You may define any additional propositions that you please. Write the statement above, in quotation marks, using propositional logic. Use the fewest number of propositions possible.

3. Tasty Issues (1 mark)
   Assume that Ryan can buy hamburger at his corner store in packs of either 5 kg, 17 kg, or 22 kg. What is the largest amount of hamburger that he cannot buy? Justify your answer.

4. Real Life (1 mark)
   Give clear definitions of propositions a and b. Give a real-life example, and use it to *describe* the difference between $a \leftrightarrow b$ and $a \rightarrow b$.

5. This Function is Never Right (1 mark)
   Define a function that takes inputs b, r, and p. You can call the function anything that you like. Draw a truth table for this function, assuming that this function outputs a contradiction.

6. <u>More Fun Than Dominoes</u> (6 marks)
   You have a sheet of graph paper, containing squares. The squares are all equal size, and there are an equal number of squares in each row to the number of squares in each column.

   You also have three boxes. In one box, there is a single block, capable of covering one of the squares on your graph paper.

   The second box is a very magical box indeed. It contains an infinite number of "L"-shaped blocks, that can cover three of the squares on your paper in an "L" shape.

   The third box is also very magical. It contains an infinite number blocks that are straight lines, and can cover three squares in one row or in one column.

   Prove that if your graph paper has 2^n by 2^n squares on it, where $n \in \mathbb{Z} \geq 0$, you can cover it perfectly just using the contents of these boxes — that is, every square on your graph paper is covered by a single block, but none of the blocks you used are hanging off the edge of the graph paper.

7. <u>Dragons and Trolls, Again</u> (1 mark)
   Everyone in this world is a dragon or a troll. Dragons alway tell the truth, and trolls always lie. Alice says that she is a dragon, and Bob says that he is a troll. Is Alice a dragon, is Alice a troll, are both possible, or is neither possible? Justify your answer in no more than two sentences.

8. <u>Getting to the Base of the Matter</u> (2 marks)
   Express 334_7 in base-36. We recommend that you clearly show your steps. Use the same idea to express base-36 as how we expressed base-16 (hexadecimal) in class.

9. <u>Another Inference</u> (4 marks)
Determine whether the following argument is valid using *rules of inference* (that is, you may draw a truth table if it helps you to solve the problem, but all of the available marks are for reaching the correct answer via rules of inference). You must state what inference rule or logical equivalence you use at each step. Assume all uses of the word "or" are inclusive ors, not exclusive ors.

I have neither a cow nor an alpaca	(1)
If I have a donkey, then I cannot possibly have an elephant	(2)
I have an elephant, or I have an alpaca	(3)
If I don't have a pet fox but do have a bunny, then I also have a cow or a donkey	(4)
If I do not have an alpaca, then I must have a bunny	(5)

$\therefore$ I have a pet fox

10. <u>Oh the Problem Names</u> (2 marks)
Prove that, if $f(n) = 3n^5 + 2n^3 \cdot \log(n) + 40$, then $f \in \mathcal{O}(n^5)$.

11. Oh the Terrible, Terrible Problem Names (2 marks)
Assume that we've invented a sorting algorithm that is basically the most terrible sorting algorithm in existence. Well, not quite (we actually can do worse). But, our terrible algorithm, if given a list of n elements, takes $f(n) = n^{215} \cdot \log_2(n) \cdot \log_2(n) + 1$ operations to complete. Give an $\mathcal{O}$ bound for the runtime of this algorithm (i.e., how many operations it takes), and prove that it is a correct $\mathcal{O}$ bound.

12. Input Selection (1 mark)
Consider a multiplexer with 18 inputs. How many selector lines, each one bit wide, are needed to choose an output? Answer the question in no more than one sentence, and then justify your answer with no more than one equation.

13. An Animal Parade (2 marks)
Assume we have animals walking across the stage, where the possible animals that could appear next are in the set of animals $S = \{\text{cow}, \text{kitten}, \text{puppy}\}$. Design a DFA that accepts if the parade of animals starts with at least one cow, then has at least three kittens, then ends with any number of puppies.

14. Let's Go With Meatballs this Time (2 marks)
Assume that I can purchase meatballs in packs of 6, 7, or 9. What is the largest integer number of meatballs that I cannot purchase by combining purchases of these three packet sizes?

15. Strong Induction to the Rescue (5 marks)
Prove that if $n \in \mathbb{Z}^+ \geq 1$, then $12 \mid (n^4 - n^2)$. Hint: you want six base cases.

16. Euclid of Alexandria (2 marks)
Use the Euclidean algorithm to determine $\gcd(99, 336)$.

17. A New Function (1 mark)
Is $f(x) = x^4 + 1$, with $f : \mathbb{Z}^+ \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ (for the purposes of this question) is defined to include 0, surjective? Prove your answer.

18. Coding Theory (1 mark)
The simplest way to transmit a message over a lossy communication channel in such a way that it will likely be received is to repeat each bit several times. If I have a message M comprising bits on the language $\{0, 1\}$, I could (for example) repeat each bit 5 times; or, I could repeat the entire message 5 times back-to-back. Why might I want to repeat each bit 5 times (that is, encode "0110" as "00000111111111100000"), instead of repeating the entire message 5 times (that is, encode "0110" as "01100110011001100110")?

19. A Multitude of Infinities (2 marks)
Prove that there are at least countably infinite different sizes of uncountably infinite sets.

20. They Have the Power (1 mark)
What is the power set of $\{cow, piggy, bear\}$?

21. Dividing Up the Animals (1 mark)
Create any correct partitioning of $\{cow, piggy, cow, bear, cow, puppy, bear\}$ into exactly three subsets, with no repeat elements among the partitions.

22. Some Inputs Again (1 mark)
Provide a propositional statement that is equivalent to the following function f, expressed in this truth table. Hint: use a disjunction of multiple conjunctions / negations — you may be able to devise a more efficient or more clever solution, but there are no more marks awarded for doing so.

a	b	c	d	$f(a,b,c,d) = ?$
0	0	0	0	1
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

23. A Negative Thought (1 mark)
Assume you have a four-bit two's complement (i.e., signed) binary number, 1101. What number does that represent in base-ten signed numbers?

24. An Infinitely Large Problem (4 marks)
Let $A \subseteq \mathbb{R}$ be equal to $(25.736, 29.744)$. Let $B \subseteq \mathbb{R}$ be equal to $[25.736, 29.744]$.

(a) This part of the question is worth 3 of the 4 marks
Prove that A is uncountably infinite in size. You are not allowed to assume that $\mathbb{R}$ is uncountably infinite in size, or otherwise call upon the proof we did in class.

(b) This part of the question is worth 1 of the 4 marks
Use the proof in the previous part of this question to prove that B is uncountably infinite in size.

25. A Circular Question (1 mark)
Assume $n \equiv 5 \mod 6$. What is the smallest non-negative number m such that $m \equiv (4n+3) \mod 4$?

26. Changing What Comes Next (1 mark)
Let f be a mapping from the positive (non-negative, non-zero) integers n to elements s_n of a sequence. Let $S = \langle s_i \rangle = \langle 1, 4, 9, 16, 25, \ldots \rangle$. Design a function f that outputs these five elements first, then outputs negative values as the next two elements. The sequence output should be infinite in length.

27. A Powerful Question (1 mark)
Is $\emptyset \subseteq \mathcal{P}(S)$, where S is an arbitrary set? Justify your answer in no more than two sentences.

28. A Predicated Thought (1 mark)
Let $C(x)$ mean that x makes cheese, and let $H(x)$ mean that x is happy. Write in predicate logic notation, defining any additional predicates that you feel you need and defining your domain, that if c is happy (for some cow c) then c produces more cheese than average.

29. The Prime Way of Doing the Problem (1 mark)
Solve $\gcd(25, 55)$ using the method of breaking both numbers into their prime factors (not using the Euclidean algorithm).

30. Divisibility and DFAs (6 marks)
Consider a DFA that operates on the language $L = \{0, 1\}$. It should accept if the number of "1"s is divisible by 2 or divisible by 3. Draw the DFA as a diagram, and represent the DFA in table format.

31. Text Searching with DFAs (3 marks)
Design a DFA, in diagram format, that operates on the language of lowercase letters. If should accept if and only if the string “cow” or the string “moo” appears somewhere in the input document.

32. Building Euclid’s Algorithm (2 marks)
Prove one of the Lemmas used to build Euclid’s algorithm. Namely, if $\gcd(a, b) = d$, then $\gcd(a/d, b/d) = 1$.

33. You’ve Seen This Question Coming All Term (1 mark)
Prove that there cannot exist a function that decides the halting problem.