# Propositions, Sets, + truncation levels

Spaces that you can build


What is this hierarchy about? It concerns iterated identity types

$A \qquad x =_A y \qquad p =_{x =_A y} q \qquad \alpha =_{p =_{x=_A y} q} \beta \qquad - - -$

$\mathbb{N} \qquad 3 =_{\mathbb{N}} 4 \qquad p =_{2+2 =_{\mathbb{N}} 4} q \qquad -$

$S^1 \qquad base =_{S^1} base \qquad loop =_{base =_{S^1} base} refl \qquad \alpha =_{loop =_{S^1} loop} \beta$

## § Propositions

defn A type $A$ is contractible if it has a unique element.

$$\text{is-contr } A := \sum_{c:A} \prod_{x:A} c =_A x$$

defn A type $A$ is a proposition if all of its identity types are contractible.

$$\text{is-Prop } A := \prod_{x,y:A} \text{is-contr}(x =_A y).$$

**Ex** The unit type $\mathbb{1}$ is a proposition b/c its identity types are contractible.

**Ex** Any contractible type is a proposition.

**Ex** $\emptyset$ is a proposition.

$$\text{exfalso} : \text{is-prop}\, \emptyset := \prod_{x,y:\emptyset} \text{is-contr}\,(x =_{\emptyset} y)$$

**Prop** For a type $A$ the following are logically equivalent:

(i) $A$ is a proposition $\quad \prod_{x,y:A} \text{is-contr}\,(x=y)$

(ii) any two terms of type $A$ can be identified $\quad \prod_{x,y:A} x=y$

(iii) $A$ is contractible once it's inhabited $\quad A \to \text{is-contr}\, A$

(iv) the map $\text{const}_*: A \to \mathbb{1}$ is an embedding.

**Proof:**

(i) $\Rightarrow$ (ii)
$$\left(\prod_{x,y:A} \sum_{p:x=y} \prod_{q:x=y} p =_{x=y} q\right) \longrightarrow \left(\prod_{x,y:A} x=y\right)$$
$$\alpha \longmapsto \lambda x \lambda y.\, \text{pr}_1\, \alpha(x,y)$$

(ii) $\Rightarrow$ (iii)
$$\left(\prod_{x,y:A} x=y\right) \longrightarrow \left(A \to \sum_{c:A} \prod_{x:A} c=x\right)$$
$$p \longmapsto a \longmapsto (a, p(a))$$

(iii) $\Rightarrow$ (iv)
$$\left(A \to \text{is-contr}\, A\right) \longrightarrow \prod_{x,y:A} \text{is-equiv}\left(\text{ap}_{\text{const}_*} : (x =_A y) \to (* =_{\mathbb{1}} *)\right)$$
$$c$$
$$f : A \longrightarrow \prod_{x,y:A} \text{is-equiv}\left(\text{ap}_{\text{const}_*} : (x =_A y) \to (* =_{\mathbb{1}} *)\right)$$

$a \longmapsto$ $c(a) : \text{is-contr}\, A$ use this to conclude that $x =_A y$ are contractible and thus equivalent to $(* =_{\mathbb{1}} *)$

$$\left(A \to \text{is-contr}\, A\right) \longrightarrow \prod_{x,y:A} \text{is-equiv}\left(\text{ap}_{\text{const}_*} : (x =_A y) \to (* =_{\mathbb{1}} *)\right)$$

$c \longmapsto \lambda x, y : A,\ f(x, x, y)$

(iv) $\Rightarrow$ (i)

$$\left(\prod_{x,y:A} \text{is-equiv}\left(\text{ap}_{\text{const}_*} : (x =_A y) \to (* =_{\mathbb{1}} *)\right)\right) \longrightarrow \left(\prod_{x,y:A} \text{is-contr}(x = y)\right)$$

$\alpha \longmapsto$ $\alpha$ provides an equivalence $(x =_A y) \simeq (* =_{\mathbb{1}} *) \simeq \mathbb{1}$. So these identity types are contractible. $\boxed{}$

**Lem** If $A \simeq B$ then $(\text{isProp}\, A) \leftrightarrow (\text{isProp}\, B)$.

**Proof:** Suppose $e : A \simeq B$, write $e : A \to B$ for that equivalence.

We know $\prod_{x,y:A} \text{is-equiv}\left(\text{ap}_e : (x =_A y) \to (ex =_B ey)\right)$.

If $B$ is proposition then $\prod_{x,y:A} \text{is-contr}\left(ex =_B ey\right)$.

Thus $(x =_A y) \simeq (ex =_B ey)$ is contractible as well.

This proves $\text{isProp}\, B \to \text{isProp}\, A$. For the converse use $(A \simeq B) \to (B \simeq A)$. $\boxed{}$

**Prop** If $P$ and $Q$ are propositions then

$$(P \simeq Q) \longleftrightarrow (P \leftrightarrow Q).$$

Proof: For any types $(P \simeq Q) \to (P \leftrightarrow Q)$.

If $P$ and $Q$ are propositions and $f : P \to Q$, $g : Q \to P$ then to prove that $P \simeq Q$ I require homotopies

$$f \circ g \sim id_Q := \prod_{x:Q} f(g(x)) = x$$

$$g \circ f \sim id_P := \prod_{y:P} g(f(y)) = y$$

Any pair of terms in a proposition can be identified. ∎

§ Sets

defn A type $A$ is a set if its identity types are propositions

$$\text{is-Set}(A) := \prod_{x,y:A} \text{is-prop}(x =_A y)$$

Ex For $m, n : \mathbb{N}$ $(m =_{\mathbb{N}} n) \simeq Eq_{\mathbb{N}}(m,n)$

By induction $\prod_{m,n:\mathbb{N}} \text{is-prop}(Eq_{\mathbb{N}}(m,n))$

By the invariance of propositions under equivalence

$\prod_{m,n} \text{is-prop}(m = n)$. Thus $\mathbb{N}$ is a set.

Theorem For a type $A$ the following are logically equivalent

(i) $A$ is a set $\quad \text{is-Set}\, A := \prod_{x,y:A} \prod_{p,q:x=y} p = q$

(ii) $A$ satisfies axiom K

$$\text{axiom-K}(A) := \prod_{x:A} \prod_{p:x=x} p = \text{refl}_x .$$

Proof: (i) $\Rightarrow$ (ii) ✓

(ii) $\Rightarrow$ (i). $k: \prod_{x:A} \prod_{p:x=x} p = \mathrm{refl}_x$

Suppose $x,y:A$, $p,q: x=y$. Want to show $p=q$.

$$p \overset{\text{r-unit}}{=} p\cdot \mathrm{refl}_y \overset{\text{inv}}{=} p\cdot(q^{-1}\cdot q) \overset{\text{assoc}}{=} (p\cdot q^{-1})\cdot q \overset{k(x,\, p\cdot q^{-1})}{=} \mathrm{refl}_x \cdot q \overset{\text{l-unit}}{=} q. \quad \blacksquare$$

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## § General truncation levels

So far

$\text{is-trunc}_{-2} A :=$ is-contr $A := \sum_{c:A} \prod_{x:A} c=x$ ← $-2$-types

$\text{is-trunc}_{-1} A :=$ is-prop $A := \prod_{x,y:A}$ is-contr$(x=y)$ ← $-1$-types

$\text{is-trunc}_{0} A :=$ is-set $A := \prod_{x,y:A}$ is-prop$(x=y)$ ← $0$-types

is-1-type $A := \prod_{x,y:A}$ is-set$(x=y)$ ← $1$-types

defn. Let $\mathbb{T}$, the type of truncation levels, be the type freely generated by $-2_{\mathbb{T}} : \mathbb{T}$ and $\mathrm{succ}_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$.

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There is an inclusion $i: \mathbb{N} \to \mathbb{T}$ defined by

$$i(0_{\mathbb{N}}) := \mathrm{succ}_{\mathbb{T}}(\mathrm{succ}_{\mathbb{T}}(-2_{\mathbb{T}}))$$

$$i(\mathrm{succ}_{\mathbb{N}}(n)) := \mathrm{succ}_{\mathbb{T}}(i(n)).$$

This lets us abuse notation and write

$$-2, -1, 0, 1, 2, \ldots, k, k+1 \quad : \mathbb{T}.$$

Defn is-trunc : $\mathbb{Z} \to \mathcal{U} \to \mathcal{U}$ by induction

$$\text{is-trunc}_{-2} A := \text{is-contr}(A)$$

$$\text{is-trunc}_{k+1} A := \prod_{x,y:A} \text{is-trunc}_k (x =_A y).$$

When $\text{is-trunc}_k A$ holds say that $A$ is a <u>k-type</u>.

Prop If $A$ is a k-type then $A$ is also a k+1 type.

proof: By induction on $k : \mathbb{Z}$.

Base case $\underset{\text{is-contr} A}{\underset{\shortparallel}{\text{is-trunc}_{-2} A}} \to \underset{\text{is-prop} A}{\underset{\shortparallel}{\text{is-trunc}_{-1} A}}$ ✓

Inductive step: want to prove

$$\text{is-trunc}_{k+1} A \to \text{is-trunc}_{k+2} A$$

assuming IH: $\text{is-trunc}_k A \to \text{is-trunc}_{k+1} A$.

$$\underset{\shortparallel}{\text{is-trunc}_{k+1} A} \to \underset{\shortparallel}{\text{is-trunc}_{k+2} A}$$

$$\prod_{x,y:A} \text{is-trunc}_k (x=y) \longrightarrow \prod_{x,y:A} \text{is-trunc}_{k+1} (x=y)$$

This follows from the inductive hypothesis! ∎

Cor If $A$ is a k-type then its identity types are k-types.

Lemma If $A \simeq B$ then $\text{is-trunc}_k A \leftrightarrow \text{is-trunc}_k B$.

proof: As before we'll use $e : A \simeq B$, $e : A \to B$ to prove is-trunc$_k$ B $\to$ is-trunc$_k$ A, by induction on $k : \mathbb{T}$.

Base case is the invariance of contractibility under equivalence.

For the induction step, $e : A \to B$ provides an equivalence
$ap_e : (x =_A y) \to (ex =_B ey)$. If B is (k+1)-truncated

its identity types are k-truncated. By the inductive hypothesis

it follows that $(x =_A y)$ are k-truncated, which is what we want!

Exercise If $f : A \to B$ is an embedding then if B is a
k+1 type so is A. (for $k : \mathbb{T}$).

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In addition to contractible types we have contractible maps
aka equivalences:

Theorem For $A \xrightarrow{f} B$ the following are logically equivalent
(i) f is an equivalence
(ii) for all $b : B$ $fib_f(b)$ is contractible.

This generalizes to higher levels.

§ Subtypes

defn A type family B over A is a subtype if for
all $x : A$ the type $B(x)$ is a proposition.

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Why are these called subtypes? Consider
$$pr_1 : \sum_{x : A} B(x) \longrightarrow A$$

We'll show that $B$ is a subtype iff this is an embedding.

<u>Ex</u> $A \simeq B := \sum_{f:A\to B} \text{is-equiv}(f)$.

$\text{pr}_1 : (A \simeq B) \longrightarrow (A \to B)$ modulo something to be discussed next time.

<u>Theorem</u> For $f : A \to B$ the following are logically equivalent

(i) $f$ is an embedding

(ii) for all $b : B$, $\text{fib}_f(b)$ is a proposition.

<u>Proof:</u> By the fundamental theorem of identity types

$f$ is an embedding $\longleftrightarrow \prod_{x,y:A} \text{is-equiv}\left((x =_A y) \xrightarrow{\text{ap}_f} (fx =_B fy)\right)$

$\longleftrightarrow \sum_{x:A} f(x) =_B f(y)$ is contractible.

$=: \text{fib}_f(f(y))$.

If $b : B$ and $p : f(y) = b$ then transport along $p$ gives an equivalence

$$\text{fib}_f(f(y)) \simeq \text{fib}_f(b).$$

$$\text{fib}_f(b) \to \text{is-contr}(\text{fib}_f(b)) \quad : \qquad \square$$

<u>Cor</u> For any $B : A \to \mathcal{U}$ the following are logically equivalent

(i) $\text{pr}_1 : \sum_{x:A} B(x) \to A$ is an embedding.

(ii) $B(x)$ is a proposition for $x : A$.

Proof: $B(x) \simeq \text{fib}_{\text{pr}_1}(x)$.