The fundamental theorem of identity types

§ What are identity types?

P.O.V. #1: Martin-Löf gave the following rules for identity types for any type A in any context.

- there is a type family: given $x:A$ $y:A$ then $x =_A y$ is a type.
- for any $x:A$ there is a term $\mathrm{refl}_x : x =_A x$
- for any family of types $P(x,y,p)$ depending on $x:A$ $y:A$ and $p : x =_A y$ to inhabit the type $\prod_{x:A} \prod_{y:A} \prod_{p:x=y} P(x,y,p)$ it suffices to inhabit the type $\prod_{z:A} P(z,z,\mathrm{refl}_z)$, i.e.,

$$\text{path-ind}: \left(\prod_{z:A} P(z,z,\mathrm{refl}_z)\right) \to \left(\prod_{x:A} \prod_{y:A} \prod_{p:x=_Ay} P(x,y,p)\right)$$

- for any $d : \prod_{z:A} P(z,z,\mathrm{refl}_z)$ then $\text{path-ind}(d, z, z, \mathrm{refl}_z) \doteq d(z)$ for any $z:A$

POV #2 For a particular type A in a context you might already know how to identify terms. A type A might be given an <u>observational</u> <u>equality</u> type family.

<u>Ex</u> Observational equality on the natural numbers defines a type family $\mathrm{Eq}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N} \to \mathcal{U}$ which may be defined by induction:

$\mathrm{Eq}_{\mathbb{N}}(0,0) := \mathbb{1}$ $\qquad\qquad$ $\mathrm{Eq}_{\mathbb{N}}(s(m),0) := \emptyset$

$\mathrm{Eq}_{\mathbb{N}}(0,s(n)) := \emptyset$ $\qquad\qquad$ $\mathrm{Eq}_{\mathbb{N}}(s(m),s(n)) := \mathrm{Eq}_{\mathbb{N}}(m,n)$

Ex Given types A and B we may define observational equality for $A+B$, this being a type family $\mathrm{Eq}^{+}_{A,B} : (A+B) \to (A+B) \to \mathcal{U}$

$\mathrm{Eq}^{+}_{A,B}(\mathrm{inl}(a), \mathrm{inl}(a')) :\equiv (a =_A a')$ $\qquad$ $\mathrm{Eq}^{+}_{A,B}(\mathrm{inl}(a), \mathrm{inr}(b)) :\equiv \emptyset$

$\mathrm{Eq}^{+}_{A,B}(\mathrm{inr}(b), \mathrm{inr}(b')) :\equiv (b =_B b')$ $\qquad$ $\mathrm{Eq}^{+}_{A,B}(\mathrm{inr}(b), \mathrm{inl}(a)) :\equiv \emptyset$

AIM: develop a technique to recognize when another type family is equivalent to an identity type family.

§ Families of equivalences

For any family of maps $f : \prod_{x:A} (B(x) \to C(x))$

there is a map $\mathrm{tot}(f) : \left(\sum_{x:A} B(x)\right) \to \left(\sum_{x:A} C(x)\right)$

defined by $\mathrm{tot}(f) :\equiv \lambda(x,y).\,(x, f(x,y))$

Theorem For any family of maps $f : \prod_{x:A} B(x) \to C(x)$ the following are logically equivalent:

(i) f is a <u>family of equivalences</u>: $f(x) : B(x) \to C(x)$ is an equivalence for each x

(ii) $\mathrm{tot}\,f$ is an equivalence.

$$\prod_{x:A} \text{is-equiv}(f(x)) \longleftrightarrow \text{is-equiv}(\mathrm{tot}(f))$$

proof: Recall equivalences are contractible maps, meaning maps whose fibers are contractible. So it suffices to show for each $x:A$ and each $c:C(x)$ that $\mathrm{fib}_{f(x)}\,c$ is contractible iff the fiber $\mathrm{fib}_{\mathrm{tot}(f)}(x,c)$ is contractible. It turns out these fibers are equivalent. $\square$

Lemma For any family of maps $f : \prod_{x:A} B(x) \to C(x)$ and any term $t : \sum_{x:A} C(x)$ there is an equivalence

$$\mathrm{fib}_{\mathrm{tot}(f)}\, t \simeq \mathrm{fib}_{f(\mathrm{pr}_1 t)}\, \mathrm{pr}_2 t .$$

proof: By a lot of induction. $\square$

§ The fundamental theorem

The fundamental theorem characterizes one-sided identity types

$$\frac{\Gamma \vdash a : A}{\Gamma, x : A \vdash a =_A x \ \mathrm{type}} \qquad \frac{\Gamma \vdash a : A \quad \Gamma, x : A, p : a = x \vdash P(x,p)\ \mathrm{type}}{\Gamma \vdash \text{path-ind}_a : P(a, \mathrm{refl}_a) \to \prod_{x:A} \prod_{p : a = x} P(x,p)}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathrm{refl}_a : a =_A a} \qquad \frac{\Gamma \vdash a : A \quad \Gamma, x : A, p : a = x \vdash P(x,p)\ \mathrm{type}}{\Gamma \vdash \text{path-ind}_a(q, a, \mathrm{refl}_a) \doteq q : P(a, \mathrm{refl}_a)}$$

The identity type family has the form $\lambda x.\, a = x : A \to \mathcal{U}$ and has a special term $\mathrm{refl}_a : a =_A a$.

We'll consider arbitrary type families with a similar form $E : A \to \mathcal{U}$ with $r : E(a)$ in a context with a type A and $a : A$.

This theorem will allow us to recognize a family of equivalences

$$\text{path-ind}_a(r) : \prod_{x:A} (a=x) \to E(x) \quad \text{defined by } (a, \text{refl}(a)) \mapsto r.$$

<u>Defn</u> Given a type A and term a, a type family $E : A \to \mathcal{U}$ and $r : E(a)$ is a <u>unary identity system</u> if for all $P(x,e)$ depending on $x : A$ and $e : E(x)$ the function

$$\text{ev}_{(a,r)} : \prod_{x:A} \prod_{E(x)} P(x,e) \longrightarrow P(a,r) \quad \text{has a section.}$$

This section gives a version of the computation rule up to identification.

<u>Fundamental Theorem of Identity Types</u> For any type A, term $a : A$, $E : A \to \mathcal{U}$ and $r : E(a)$. For the family of maps

$$\text{path-ind}_a(r) : \prod_{x:A} (a=x) \to E(x) \qquad (a, \text{refl}(a)) \mapsto r.$$

The following are logically equivalent:

(i) $\text{path-ind}_a(r)$ is a family of equivalences

(ii) $\sum_{x:A} E(x)$ is contractible onto (a,r)

(iii) E is an identity system.

<u>proof</u>: By the theorem about families of equivalences (i) holds iff the map on total spaces $\left(\sum_{x:A} a=x\right) \to \left(\sum_{x:A} E(x)\right)$ is an equivalence. But the domain is contractible so this is an equivalence iff $\sum_{x:A} E(x)$ is contractible.

For (ii) $\Leftrightarrow$ (iii) consider for $P : \sum_{x:A} E(x) \to \mathcal{U}$

$$\begin{tikzcd}
\prod_{t:\sum_{x:A} E(x)} P(t) \arrow[rr, "\text{ev-pair}"] \arrow[dr, "\text{ev-pt}_{(a,r)}"'] & & \prod_{x:A} \prod_{y:E(x)} P(x,y) \arrow[ll, dashed, bend left] \arrow[dl, "\text{ev}_{(a,r)}"] \\
& P(a,r) &
\end{tikzcd}$$

By Σ-induction the top map has a section. Thus the left map has a section iff the right map does. (iii) asserts that the right map has a section while the left map has a section iff $\sum_{x:A} E(x)$ satisfies singleton induction. This holds iff $\sum_{x:A} E(x)$ is contractible. $\square$

§ Equality on $\mathbb{N}$.

$Eq_{\mathbb{N}}(0,0) := \mathbb{1}$ $\qquad$ $Eq_{\mathbb{N}}(s(m),0) := \emptyset$

$Eq_{\mathbb{N}}(0,s(n)) := \emptyset$ $\qquad$ $Eq_{\mathbb{N}}(s(m),s(n)) := Eq_{\mathbb{N}}(m,n)$

Note we have a term $\text{refl-eq} : \prod_{n:\mathbb{N}} Eq_{\mathbb{N}}(n,n)$, by induction

$\text{refl-eq}(0) := \star : \mathbb{1}$

$\text{refl-eq}(s(n)) := \text{refl-eq}(n) : \underset{=}{Eq_{\mathbb{N}}(s(n),s(n))}\ Eq_{\mathbb{N}}(n,n)$

Theorem The canonical map

$\text{path-ind}(\text{refl-eq}) : \prod_{m:\mathbb{N}} \prod_{n:\mathbb{N}} (m=n) \to Eq_{\mathbb{N}}(m,n)$ is an equivalence.

proof: By the fundamental theorem it suffices to show for all $m:\mathbb{N}$ that $\sum_{n:\mathbb{N}} Eq_{\mathbb{N}}(m,n)$ is contractible onto $(m, \text{refl-eq}(m))$.

Define the contracting homotopy

$$\gamma(m) : \prod_{n:\mathbb{N}} \prod_{e:\mathrm{Eq}(m,n)} (m, \text{refl-eq}(m)) = (n, e)$$

By induction on m and n. Base case

$\gamma(0,0,\star) := \text{refl}$. For $m=0$ and $s(n)$ or vice versa use ex-falso

For the final case

$$\gamma(s(m), s(n), e) : (s(m), \text{refl-eq}(s(m))) = (s(n), e)$$

use a map

$$\lambda(n,e).(n+1,e) : \sum_{n:\mathbb{N}} \mathrm{Eq}_{\mathbb{N}}(m,n) \longrightarrow \sum_{k:\mathbb{N}} \mathrm{Eq}(m+1,k)$$

This carries $(m, \text{refl-eq}(m))$ to $(m+1, \text{refl-eq}(m+1))$

So we apply the map to the identification (m,n,e) to get the identification we need. ∎

§ Disjointness of coproducts.

$$\mathrm{Eq}^+_{A,B}(\mathrm{inl}(a), \mathrm{inl}(a')) := (a =_A a') \qquad \mathrm{Eq}^+_{A,B}(\mathrm{inl}(a), \mathrm{inr}(b)) := \emptyset$$

$$\mathrm{Eq}^+_{A,B}(\mathrm{inr}(b), \mathrm{inr}(b')) := (b =_B b') \qquad \mathrm{Eq}^+_{A,B}(\mathrm{inr}(b), \mathrm{inl}(a)) := \emptyset$$

Theorem For $a, a' : A$ and $b, b' : B$ there are equivalences

$$\left(\mathrm{inl}(a) =_{A+B} \mathrm{inl}(a')\right) \simeq (a =_A a') \qquad \left(\mathrm{inl}(a) =_{A+B} \mathrm{inr}(b)\right) \simeq \emptyset$$

$$\left(\mathrm{inr}(b) =_{A+B} \mathrm{inr}(b')\right) \simeq (b =_B b') \qquad \left(\mathrm{inr}(b) =_{A+B} \mathrm{inl}(a)\right) \simeq \emptyset$$

Define $\text{refl-eq} : \prod_{s:A+B} \mathrm{Eq}^+_{A,B}(s,s)$ by induction

$\mathrm{refl\text{-}eq}(\mathrm{inl}(a)) := \mathrm{refl}_a \qquad \mathrm{refl\text{-}eq}(\mathrm{inr}(b)) := \mathrm{refl}_b.$

Prop For any $s : A+B$, $\sum_{t:A+B} \mathrm{Eq}_{A+B}(s,t)$ is contractible onto $(s, \mathrm{refl\text{-}eq}(s))$.

<u>proof:</u> By induction suffices to assume s is $\mathrm{inl}(a)$ or $\mathrm{inr}(b)$.

$$\left(\sum_{t:A+B} \mathrm{Eq}_{A+B}(\mathrm{inl}(a),t)\right) \simeq \left(\sum_{x:A} \mathrm{Eq}_{AB}(\mathrm{inl}(a), \mathrm{inl}(x))\right) + \left(\sum_{y:B} \mathrm{Eq}_{A+B}(\mathrm{inl}(a), \mathrm{inr}(b))\right)$$

$$\simeq \left(\sum_{x:A} a = x\right) + \sum_{y:B} \emptyset$$

$$\simeq \sum_{x:A} a = x \simeq \mathbb{1}$$

$\square$

§ Embeddings

<u>Defn</u> A map $f : A \to B$ is an <u>embedding</u> if

$$\mathrm{ap}_f : \prod_{x:A} \prod_{y:A} (x =_A y) \to (fx =_B fy)$$ is a family of equivalences.

<u>Theorem</u> Equivalences are embeddings.

<u>proof:</u> Suppose $e : A \to B$ is an equivalence. Want to show

$$\mathrm{ap}_e : \prod_{x,y:A} (x =_A y) \to (ex =_B ey)$$ is a family of equivalences.

By the fundamental theorem it suffices to show that

$\sum_{y:A} ex =_B ey$ is contractible. This is equivalent to

$$\left(\sum_{y:A} (ex =_B ey)\right) \simeq \sum_{y:A} (ey = ex) =: \mathrm{fib}_e(ex).$$

Since e is an equivalence its fibers are contractible. $\square$