Low-Rank Matrix Approximation for Multi-Sensory Data Association

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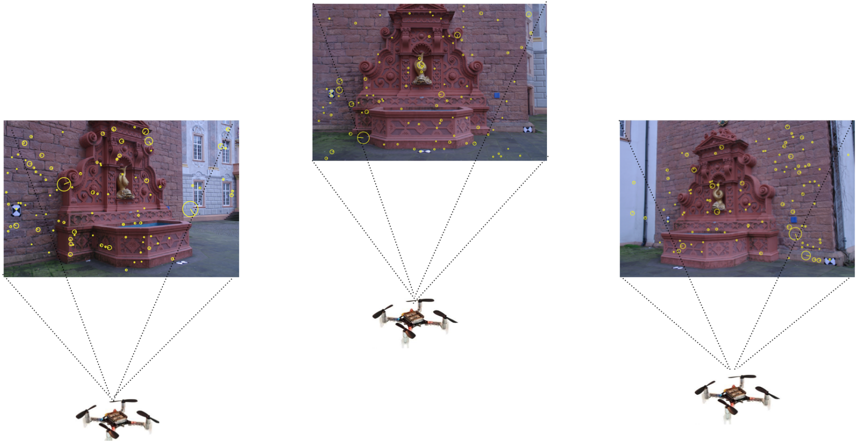
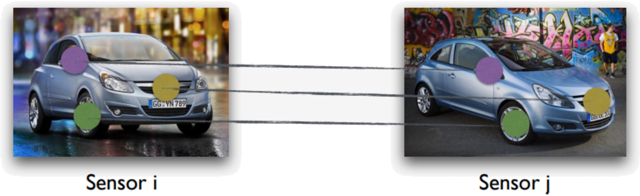
Collaborative data association

Problem: can we agree on what we see?

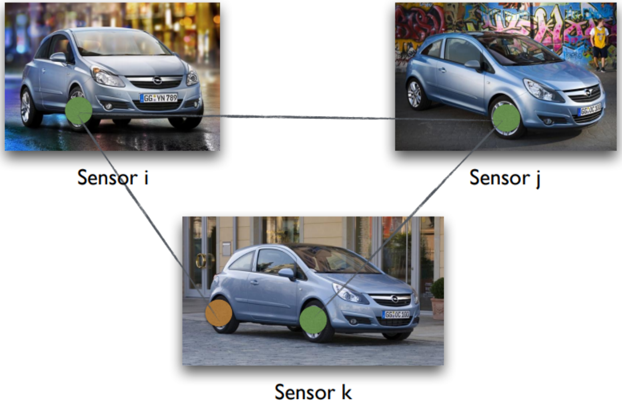


Two-way matching

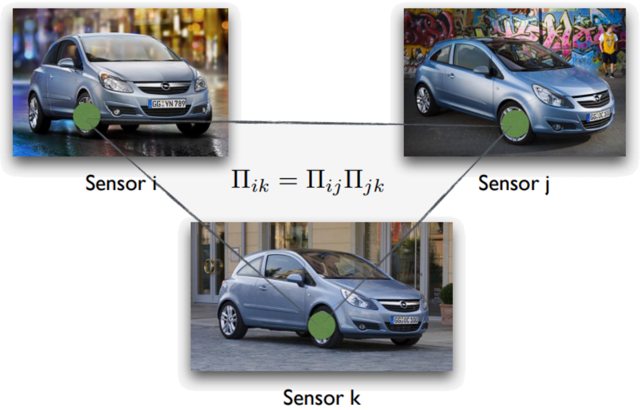


$$\Pi_{ij} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

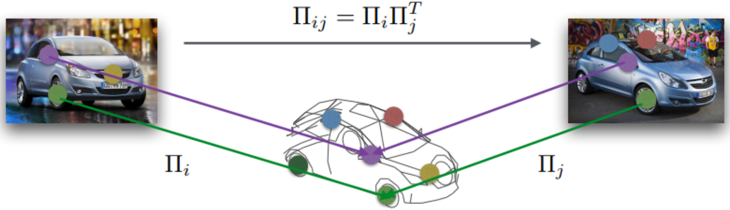
Multiway matching and inconsistencies



Cycle consistency

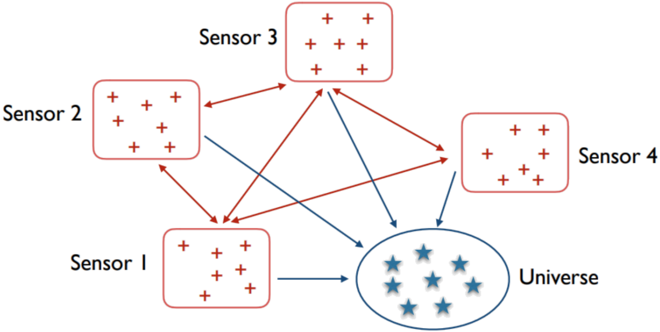


Equivalent condition for cycle consistency



$$\Pi_i = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \Pi_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem formulation



Input: noisy pairwise associations $\{\widetilde{\Pi}_{ij}\}_{\{i,j\}\in\mathcal{E}}$

Output: labeling of observations $\{\Pi_i\}_{i\in\mathcal{V}}$

Such that $\widetilde{\Pi}_{ij} \approx \Pi_i \Pi_j^T, \quad \forall \{i,j\} \in \mathcal{E}$

Cycle consistency as low-rank matrix factorization

Pairwise associations $\{\widetilde{\Pi}_{ij}\}_{\{i,j\}\in\mathcal{E}}$ are cycle consistent iff

$$\widetilde{\mathbf{\Pi}} \doteq \begin{bmatrix} I & \widetilde{\Pi}_{12} & \cdots & \widetilde{\Pi}_{1n} \\ \widetilde{\Pi}_{21} & I & \cdots & \widetilde{\Pi}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\Pi}_{n1} & \widetilde{\Pi}_{n2} & \cdots & I \end{bmatrix} = \begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \vdots \\ \Pi_n \end{bmatrix} \begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \vdots \\ \Pi_n \end{bmatrix}^T$$

for some (partial) permutation matrices $\Pi_1, \Pi_2, \ldots, \Pi_n$.

Solve for $\Pi_1, \Pi_2, \ldots, \Pi_n$ instead of $\{\Pi_{ij}\}_{\{i,j\}\in\mathcal{E}}$

Related work and contributions

- Convex relaxations and semidenite programming [Huang '12, Huang '13, Chen '14, Zhou '15].
- Spectral relaxations and spectral clustering [Pachauri '13, Maset '17, Fathian '19].
- Density-based clustering approaches [Tron '17].
- Decentralized optimization [Leonardos '18, '19].
- Nonnegative matrix factorization [Bernard '19, Birdal '19] (concurrent works).

Proposed approach

- Formulation as a combinatorial optimization problem:

$$\begin{aligned} \underset{\{\Pi_i\}_{i\in\mathcal{V}}}{\text{minimize}} \quad & \phi(\Pi) \doteq \sum_{\{i,j\}\in\mathcal{E}} \underbrace{\|\widetilde{\Pi}_{ij} - \Pi_i \Pi_j^T\|_F^2}_{\text{cycle consistency}} \\ \text{subject to} \quad & \Pi_i \in \{0,1\}^{n_i \times K},\ \Pi_i \Pi_i^T = I_{n_i},\ \forall i \in \mathcal{V} \end{aligned}$$

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- First approach: solution with ADMM without relaxation.

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- First approach: solution with ADMM without relaxation.

- Second approach: relax from permutation to stochastic matrices, and use a trust-region method on the manifold of stochastic matrices.

First approach

ADMM standard form (separable objective and linear equality constraints):

$$\begin{aligned} \underset{\Pi, Z}{\text{minimize}} \quad & \phi(\Pi) + \sum_{i \in \mathcal{V}} I_{\mathcal{P}_{n_i \times K}}(Z_i) \\ \text{subject to} \quad & \Pi_i - Z_i = 0, \quad i \in \mathcal{V}. \end{aligned}$$

Indicator function:

$$I_{\mathcal{P}_{n_i \times K}}(Z_i) = \begin{cases} 0, & Z_i \in \mathcal{P}_{n_i, K} \\ +\infty, & Z_i \notin \mathcal{P}_{n_i, K}. \end{cases}$$

First numerical algorithm

Algorithm 1 MatchADMM

Input: Noisy correspondences $\{\widetilde{\Pi}_{ij}\}_{\{i,j\}\in\mathcal{E}}$, universe size K, parameters $\alpha > 1$, ρ^0

Ouput: Permutations $\{\Pi_i\}_{i\in\mathcal{V}}$, consistent correspondences $\{\Pi_{ij} \doteq \Pi_i\Pi_j^T\}_{\{i,j\}\in\mathcal{E}}$

repeat

$$\Pi^{k+1} := \underset{\Pi}{\text{argmin}} \ \left\{\phi(\Pi) + (\rho^k/2)\|\Pi - Z^k + U^k\|_F^2\right\}$$

$$Z_i^{k+1} := \text{Proj}_{\mathcal{P}_{n_i,K}}\left(\Pi_i^{k+1} + U_i^k\right)$$

$$U^{k+1} := (1/\alpha)(U^k + \Pi^{k+1} - Z^{k+1})$$

$$\rho^{k+1} := \alpha\rho^k$$

until Convergence

Second approach

- Basic ideas:
  - relax from permutation to stochastic matrices,
  - use approximate 2nd order methods to efficiently solve the relaxation.

Second approach

- Basic ideas:
  - relax from permutation to stochastic matrices,
  - use approximate 2nd order methods to efficiently solve the relaxation.
- Non-convex relaxation (convex domain, non-convex objective):

$$\begin{aligned} \underset{\Pi}{\text{minimize}} \quad & \phi_\lambda(\Pi) \doteq \sum_{\{i,j\}\in\mathcal{E}} \underbrace{\|\widetilde{\Pi}_{ij} - \Pi_i\Pi_j^T\|_F^2}_{\text{cycle consistency}} + \lambda \sum_{i\in\mathcal{V}} \underbrace{\|I - \Pi_i\Pi_i^T\|_F^2}_{\text{regularizer}} \\ \text{subject to} \quad & \Pi_i\mathbf{1} = \mathbf{1},\ \Pi_i \geq 0, \quad i \in \mathcal{V}. \end{aligned}$$

The multinomial manifold and trust-region methods

- Definition:

$$\mathcal{M}_K^N = \{X \in \mathbb{R}^{N \times K} : X > 0, X\mathbf{1} = \mathbf{1}\}.$$

- Riemannian metric (Fisher information metric):

$$g_X(U, V) = \sum\nolimits_{i,j} (U)_{ij}(V)_{ij}/(X)_{ij}$$

- Optimization problem at hand of the form:

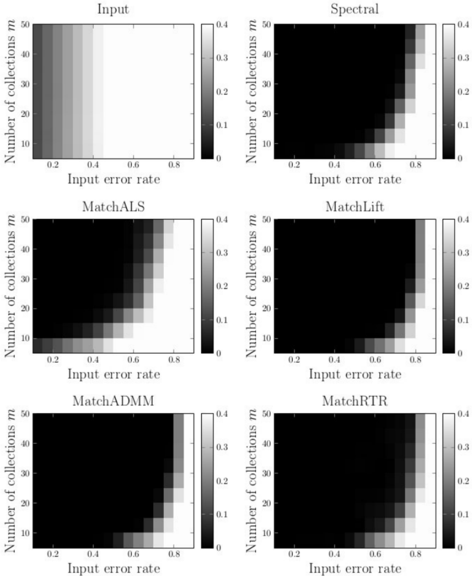
$$\underset{X \in \mathcal{M}_K^N}{\text{minimize}} \quad \phi_\lambda(X)$$

- Trust-region subproblem at iteration k:

$$\begin{aligned} \underset{U \in T_{X^k}\mathcal{M}_K^N}{\text{minimize}} \quad & \phi_\lambda(X^k) + g_{X^k}(\operatorname{grad}\phi_\lambda(X^k), U) + (1/2) g_{X^k}(\operatorname{Hess}\phi_\lambda(X^k)[U], U) \\ \text{subject to} \quad & \|U\|^2 \le r^2 \end{aligned}$$

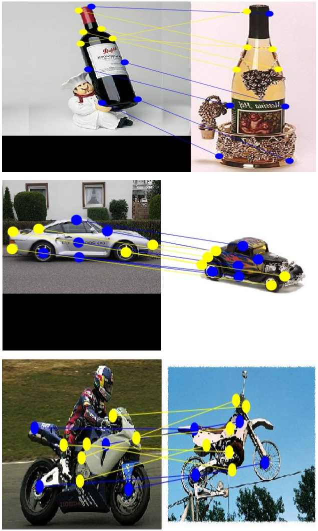
- Solution with Manopt [Boumal et al. '14].

Experiments with synthetic data

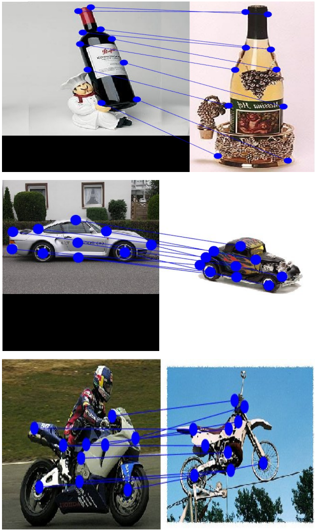


Experiments with WILLOW object class datasets

Input Matches



Optimized Matches



Conclusions and future work

- Conclusions:
  - Proposed two low-rank approximation approaches for multiway matching based on cycle consistency.
  - State of the art results in a multi-image feature matching setting, and robustness to outliers.
- Future work:
  - Conditions for exact recovery guarantees.
  - Are there undesirable local minima?