Differential Geometry: The Frenet-Serret Frame and Formulas

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The Frenet-Serret Frame and Formulas were named after the two mathematicians that independently discovered them: Jean Frederic Frenet in 1847, and Joseph Alfred Serret in 1851 although it is worth noting that at that time, mathematics lacked the vector notation and linear algebra currently used to write these formulas today. The Frenet-Serret Formulas are given by:

$$\frac{d\vec{\mathbf{T}}}{ds} = \kappa\vec{\mathbf{N}},$$

$$\frac{d\vec{\mathbf{N}}}{ds} = -\kappa\vec{\mathbf{T}} + \tau\vec{\mathbf{B}},$$

$$\frac{d\vec{\mathbf{B}}}{ds} = -\tau\vec{\mathbf{N}},$$

where $\frac{d}{ds}$is the derivative with respect to an arc length s, κ is the curvature, τ is the torsion of the curve and the unit vectors, $\{\vec{\mathbf{T}}, \vec{\mathbf{N}}, \vec{\mathbf{B}}\}$ correspond to the Unit Tangent Vector ($\vec{\mathbf{T}}$), Unit Normal Vector ($\vec{\mathbf{N}}$), and Unit Binormal vector ($\vec{\mathbf{B}}$) that together, form an orthonormal basis spanning $\mathbf{R^3}$. These vectors collectively are known as The Frenet-Serret Frame.

The Frenet-Serret formulas describe the kinematic properties of a particle moving along a continuous, differential curve in three-dimensional Euclidean space. Specifically, these formulas are used to describe the derivatives of the unit vectors in the Frenet-Serret Frame in terms of each other.

Let $\vec{p}(t)$ represent a position vector of a particle as a function of time in Euclidean space. So long as $\vec{p}(t)$ has non-zero curvature, then there exists an arclength, namely this function is defined as

$$\mathbf{r} = \int_a^b \|\vec{p}(t)\| dt$$

In order for the Frenet-Serret Frame to be defined, the quantity $\mathbf{r}$ is used to give the curve $\vec{p}(t)$ a natural parametrization by its arc length, thus let $\mathbf{s}$(t) denote the parameterized arc length curve:

$$\mathbf{s}(t) = \int_0^t \|\vec{r}'(\sigma)\| d\sigma$$

We see here that $\mathbf{s}$(t) represents the arc length at which the particle has moved along the curve in time t.
With the use of this, it is possible to solve for t as a function of s, and thus to write

$$\mathbf{r}(\mathrm{s}) = \mathbf{r}(t(\mathrm{s}))$$

thus we see that the curve is parametrized in a preferred manner by its arc length and now The Frenet-Serret Frame is able to be defined as follows:

$$\vec{\mathbf{T}} = \frac{\frac{d\mathbf{r}}{ds}}{\|\frac{d\mathbf{r}}{ds}\|}$$

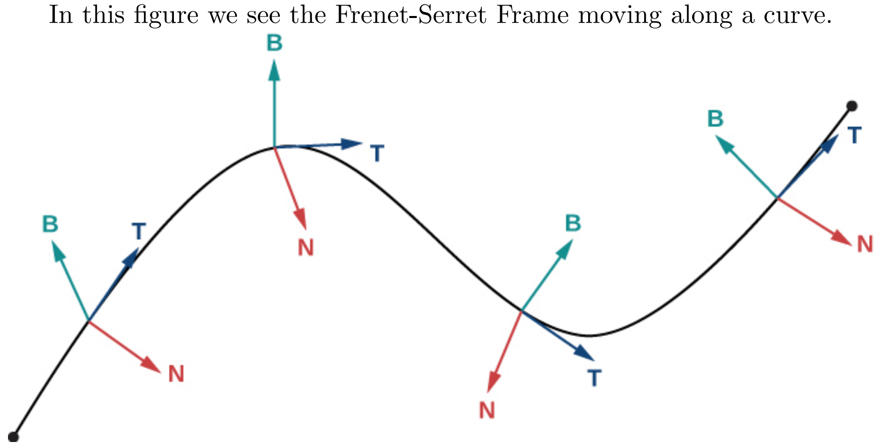
where $\vec{\mathbf{T}}$ always points in the direction of motion,

$$\vec{\mathbf{N}} = \frac{\frac{d\mathbf{T}}{ds}}{\|\frac{d\mathbf{T}}{ds}\|}$$

where $\vec{\mathbf{N}}$ is orthogonal to $\vec{\mathbf{T}}$,

$$\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$$

where $\vec{\mathbf{B}}$ is the cross product of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ thus making $\vec{\mathbf{B}}$ orthogonal to both vectors.
In this figure we see the Frenet-Serret Frame moving along a curve.



The Frenet-Serret Frame and Formulas can be very insightful for understanding curves in space. An example of such a curve is known as a helix. A helix can be characterized by the height 2πh and radius r of a single turn. Consider the curve paramaterized by its arclength:

$\vec{\mathbf{r}}(s) = \Bigg\langle acos(s), asin(s), as\Bigg\rangle$, where a $\in$R

Then the Unit Tangent Vector to $\vec{\mathbf{r}}(s)$ is,

$$\vec{\mathbf{T}} = \frac{\frac{d\mathbf{r}}{ds}}{\|\frac{d\mathbf{r}}{ds}\|} = \frac{\Bigg\langle -asin(s), acos(s), a\Bigg\rangle}{\sqrt{a^2sin^2(s) + a^2cos^2(s) + a^2}}$$

Recall the trigonometry identity $sin^2(s) + cos^2(s) = 1$, thus

$$\vec{\mathbf{T}} = \frac{\Bigg\langle -asin(s), acos(s), a\Bigg\rangle}{\sqrt{2}a}$$

Simplifying we see that,

$$\vec{\mathbf{T}} = \Bigg\langle -\frac{sin(s)}{\sqrt{2}}, \frac{cos(s)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Bigg\rangle$$

Now to find the Unit Normal Vector, we know that

$$\vec{\mathbf{N}} = \frac{\frac{d\mathbf{T}}{ds}}{\|\frac{d\mathbf{T}}{ds}\|} = \frac{\Bigg\langle -\frac{cos(s)}{\sqrt{2}}, -\frac{sin(s)}{\sqrt{2}}, 0\Bigg\rangle}{\sqrt{\frac{cos^2(s)}{2} + \frac{sin^2(s)}{2}}}$$

by using the same trigonometric identity, we see that,

$$\vec{\mathbf{N}} = \frac{\Bigg\langle -\frac{cos(s)}{\sqrt{2}}, -\frac{sin(s)}{\sqrt{2}}, 0\Bigg\rangle}{\frac{1}{\sqrt{2}}}$$

now we simply have,

$$\vec{\mathbf{N}} = \Bigg\langle -cos(s), -sin(s), 0\Bigg\rangle$$

To complete the last element in the Frenet-Serret Frame, we need to find the Binormal Vector $\vec{\mathbf{B}}$ which is the vector cross product of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$, hence

$$\vec{\mathbf{B}} = \vec{\mathbf{T}} \times \vec{\mathbf{N}}$$

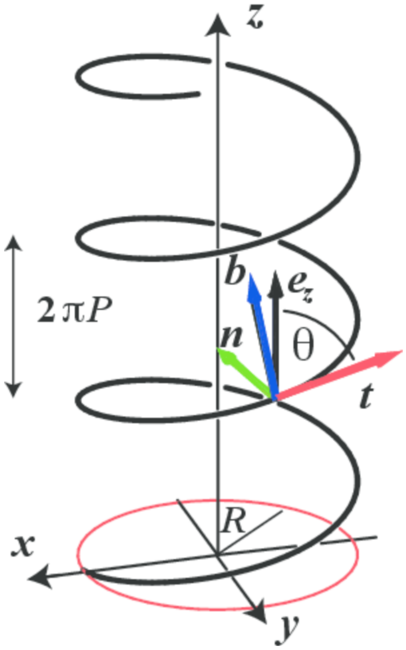
By solving using the cross product we see that,

$$\vec{\mathbf{B}} = \Bigg\langle \frac{cos(s)}{\sqrt{2}}(0) - \frac{1}{\sqrt{2}}(-sin(s)), \frac{1}{\sqrt{2}}(-cos(s)) - (-\frac{sin(s)}{\sqrt{2}}(0)), -\frac{sin(s)}{\sqrt{2}}(-sin(s)) - \frac{cos(s)}{\sqrt{2}}(-cos(s))\Bigg\rangle$$

Simplifying we get,

$$\vec{\mathbf{B}} = \left\langle \frac{sin(s)}{\sqrt{2}}, \frac{-cos(s)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle,$$

thus completing the Frenet-Serret Frame of unit vectors that together, form an orthonormal basis spanning $\mathbf{R^3}$ along the helix curve.



Here we can see the Frenet-Serret Frame moving along the helix curve. It is clear to see all three vectors forming an orthonormal basis as well as noting the $\vec{\mathbf{T}}$ is pointing in the direction of motion, and $\vec{\mathbf{N}}$ is pointing in the direction of the central axis.

Going further, the Frenet-Serret Formulas are used to describe the derivatives of the unit vectors that make up the Frenet-Serret Frame as a way to show kinematic properties of a particle moving along a differentiable curve. In this helix curve example, we see that the Frenet-Serret Formulas can be described as,

$$\frac{d\vec{\mathbf{T}}}{ds} = \kappa\vec{\mathbf{N}} = \kappa\left\langle -cos(s), -sin(s), 0 \right\rangle = \left\langle -\kappa cos(s), -\kappa sin(s), 0 \right\rangle,$$

For some helix curvature κ.

Similarly, we can substitute each Frenet-Serret Frame unit vector to find all

Frenet-Serret formulas for the helix curve:

$$\frac{d\vec{\mathbf{N}}}{ds} = -\kappa\vec{\mathbf{T}} + \tau\vec{\mathbf{B}}$$

substituting we get,

$$\frac{d\vec{\mathbf{N}}}{ds} = \left\langle \frac{\kappa sin(s)}{\sqrt{2}}, -\frac{\kappa cos(s)}{\sqrt{2}}, -\frac{\kappa}{\sqrt{2}} \right\rangle + \left\langle \frac{\tau sin(s)}{\sqrt{2}}, -\frac{\tau cos(s)}{\sqrt{2}}, \frac{\tau}{\sqrt{2}} \right\rangle$$

where κ, τ are the curvature and torsion of the helix curve.
And finally we have,

$$\frac{d\vec{\mathbf{B}}}{ds} = -\tau\vec{\mathbf{N}} = -\tau\left\langle -cos(s), -sin(s), 0 \right\rangle$$

Distributing we are left with,

$$\frac{d\vec{\mathbf{B}}}{ds} = \left\langle \tau cos(s), \tau sin(s), 0 \right\rangle.$$

These Frenet-Serret formulas are very useful when looking at kinematic properties because each derivative is a multiple of one of Frenet-Serret Frame vectors, therefore resulting in the derivative of the Tangent and Binormal Vectors to point in the direction of its multiple. It is also worth noting that κ, or the curvature describes the speed of rotation of the frame while τ describes the torsion of a space curve where torsion measures the failure of a curve to be planar. The results of these properties sheds light to the study of spacial curvature.

References

[1]: Wikipedia: Frenet-Serret Formulas and Frame. Used information from this site as a basis for this paper.
[2]: The Frenet-Serret Equations lecture, Differential Geometry Lecture by NJ Wildberger (11/17/13) found on YouTube.com
[3]: Found both images from researchgate.net