Counting

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1 The Basics of Counting: §6.1

1.1 Example: The bit string 0110

How many bit strings are there of length n that start and end with a 1?

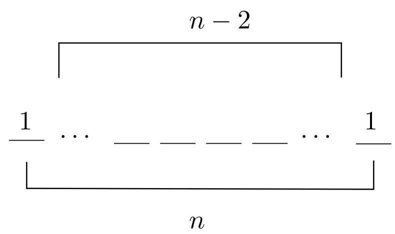

Figure 1: The Bit String

Since every slot adds two choices, and we already have two slots filled out of the n elements, we get that there are 2^{n-2} choices.

1.1.1 Examples

1. How many bit strings are there of 6 or less, not including the empty string?

Length	Number of Bit Strings That Are Exactly the Length
1	2
2	2^2
3	2^3
4	2^4
5	2^5
6	2^6

$$\begin{aligned}\text{Total} &= 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 \\ &= 2(1 + 2 + 2^2 + 2^3 + 2^4 + 2^5) \\ &= 2(\frac{1-2^6}{1-2}) = 2^7 - 2 \text{ (From definition of harmonic series)}\end{aligned}$$

2. How many positive integers between 100 and 999, inclusive, satisfy:

(a) Are divisible by 7, i.e. $n = 7k, k \in \mathbb{Z}$.
How many k's satisfy $100 \leq k \leq 999$.

$$7(14) < 100 < 7(15) < \cdots < 7(142) < 999$$

We want to count $15 \leq k \leq 142$. Thus, we can just count the numbers in the range. The general number of elements in a range from a to b can be represented as $b - a + 1$.

$$142 - 15 + 1 = 128 \text{ numbers between 100 and 999 are divisible by 7}$$

(b) Have the same three digits?
9, because we have nine choices for each digit, and they must be the same for every digit.

(c) Are divisible by 3 or 4?

$$A_3 = \{n : 3 \mid n \text{ and } 100 \leq n \leq 999\}$$
$$A_4 = \{n : 4 \mid n \text{ and } 100 \leq n \leq 999\}$$

Is $\#A_3 + \#A_4$ correct?
No, because that would double-count some numbers such as 12, which divide both 3 and 4. To get a correct answer, we need to remove where 3 and 4 both divide the number, i.e. $\#(A_3 \cap A_4)$.

$$100 \leq 3k \leq 999$$
$$34 \leq k \leq 333$$
$$(333 - 34) + 1 = 300$$

$$100 \leq 4k \leq 999$$
$$25 \leq k \leq 249$$
$$(249 - 25) + 1 = 225$$

$$100 \le 12k \le 999$$
$$9 \le k \le 83$$
$$(83 - 9) + 1 = 75$$

$$\#(A_3 \cup A_4) = 300 + 225 - 75 = \boxed{450}$$

(d) Groups: how many are not divisible by 4?

This is a case where it is easier to subtract the number of elements that divide 4 by the elements in the range.

$$\begin{aligned} \#A - \#A_4 &= \\ ((999 - 100) + 1) - \#A_4 &= \\ 900 - \#A_4 &= 900 - 225 = \boxed{675} \end{aligned}$$

(e) How many are divisible by 3 and 4?

$$\#A_3 \cap \#A_4 = \boxed{75} \text{ (calculated earlier)}$$

(f) Not divisible by 3 or 4?

$$\#A - \#(A_3 \cup A_4) = 900 - 450 = \boxed{450} \text{ (calculated earlier)}$$

3. How many ways can the photographer arrange the top 6 in the class?

(a) When the Valedictorian and Salutatorian are next to each other

We can treat the Val + Sal as "one person". If this is the case, we can just use the formula for permutations, and get $(6-1)! = 5!$ possible permutations. However, since we can switch the order of the two, we get two options within the one person, so our final number of permutations is $2 * 5!$.

(b) When the Val and Sal can't sit next to each other

In this case, the only way that this cannot be true is if the Val and Sal sit next to each other. Since we have 6! ways to arrange everyone, we can write this as:

$$6! - 2 * 5!$$

(c) Val must be to the left of Sal

If we put the Sal in spot 2, we have 4! other options for the 4 other open slots.

If we put the Sal in spot 3, we still have 4 open slots, so 4! permutations, but we also have two different places for the Val, so we get $2 * 4!$ possible permutations.

Continuing this on for Sal in spot 3, we get 3 possible spaces for the val, plus 4! permutations for the other slots.

This continues on until $5 * 4!$.

We get:

$$4! + 2 * 4! + 3 * 4! + 4 * 4! + 5 * 4! =$$
$$(1 + 2 + 3 + 4 + 5)4! = 360$$

2 The Pigeonhole Principle: §6.2

The rules of the pigeonhole principle state that every pigeon must be in a hole.

2.1 Introductory Example

For example, say you have 4 pigeons and 3 pigeon holes. The pigeonhole principle states that at least one hole must contain more than one pigeon.

2.2 A Simple General Representation of the Pigeonhole Principle

If we have m objects and n spots, and $m > n$, then at least one must contain more than one object.

2.3 Examples

1. Let $f : A \to B$ be a function, A and B are finite sets. If $\#A > \#B$ ($\#A$ is the number of items in A), then f is not injective.

   Proof:

   Proof. Let $\#A = m, \#B = n$ and assume $m > n$.

   Write $A = \{a_1, \dots, a_n, a_{n+1}, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$.

   Apply f to $\{a_1, \dots, a_n\}$.

   If $f(ai) = f(aj)$ for some $i \neq j, 1 \leq i, j \leq n$ we have f is not injective.

   If $f(ai) \neq f(aj)$ for all $1 \leq i, j \leq n$ with $i = j$, then $\{b_1, \dots, b_n\} = \{f(a_1), \dots, f(a_n)\}$.

   We have $f(a_{n+1}) \in B$, so $f(a_{n+1}) = bi$, for some i, thus f is not injective. □

2. There is a jar with 10 red balls and 10 green balls

   Groups:

   (a) How many do you have to pick to guarantee 3 of the same color?

   You would need to pick at least 5, because in the worst case, you would pick two of the same color each, and then the last one that you would pick would make three of the same color either way.

   (b) How many do you need to guarantee three green balls?

   You would need to pick at most 13 balls. Since you could plausibly choose all 10 red balls, and after that, there are only green balls left, so you are forced to choose three.

2.3.1 Aside: How many people have to be in a room to guarantee that one pair will share a birthday?

Answer: Since there are 366 possible birthdays (365 days + February 29th on a leap year), we will need at most $366 + 1 = \boxed{367}$ people.

2.4 The Generalized Pigeonhole Problem

If N objects are placed in K boxes, there is at least one box containing at least $\lceil \frac{N}{K} \rceil$ objects.

2.4.1 The Definition of The Ceiling Function $\lceil x \rceil$

Definition: $x \in \mathbb{R}$, $\lceil x \rceil$ is the integer n so that $n - 1 < x \leq n$.

2.5 Framing the Original Pigeonhole Problem Within the More General Method

Recall that in the original problem we had $k+1$ objects, or pigeons, and k boxes, or pigeonholes.

$$\implies \left\lceil \frac{k+1}{k} \right\rceil = \left\lceil 1 + \frac{1}{k} \right\rceil = 2$$

Thus, this says that at least one pigeonhole must have at least two pigeons.

2.6 Applications and Examples

1. Suppose every student in Math 110 is a Freshman (frosh), a Sophomore (soph), or a Junior (jr). There are 25 total students.

   (a) Show that there are at least 9 frost, at least 9 soph, or at least 9 juniors in the class.

   $$\left\lceil \frac{25}{3} \right\rceil = 9$$

   Because:

   $$\frac{25}{3} = 8.\bar{3}$$

   Which we round up to 9 because of the ceiling function.

   This means that at least one of the categories (in this case frosh, soph, or jrs) must have at least 9 elements (students).

   (b) There are at least 3 frosh, at least 19 soph, or at least 5 jrs

   Proof. If not at least 3 frost, there must be at most 2 frosh.

   If not at least 19 soph, at most 18 soph.

   If not at least 5 jrs, at most 4 jrs.

   We have at most $3 + 19 + 5 = 24$ students. Since we have at most 25 students, we have a contradiction. □