Permutations and Combinations

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1 Permutations

Within a set of n *distinct* elements (no duplicates), a **permutation** is one way of arranging all the elements in the set

For example, one permutation of the set $\{a, b, c, d\}$ is $\{a, b, c, d\}$.

1.1 Example: How many ways are there to fill a basketball team?

Say we have 5 slots on a basketball team, how many ways are there to fill the slots?

Going through this by hand, we can take these steps:

1. Start with 5 slots open, so we have five options for the first slot. Choose one
2. We now have 4 options for the second slot, choose one of those
3. We have 3 options left, choose
4. We have 2 options left, choose
5. We have 1 option left, choose

Mathematically, we can represent this as:

$$5 \text{ choices} * 4 \text{ choices} * \cdots * 1 \text{ choice} = 120 \text{ different combinations}$$

1.2 Definition of the Number of Permutations

In a set of n distinct elements, there are $n!$ ways to order the set, or there are $n!$ permutations of the set. This is written as $P(n, n)$.

The first n represents the length of the original set, and the second length represents the size of the subsets that we are choosing. Since we are finding permutations of the original set, this length is the same size as the original set.

1.3 The Number of Permutations on Smaller Subsets

However, if we want to know the number of permutations of n for a set of smaller subsets, say of size r, we can use a more general formula. An arrangement of a subset of n is called an **r-permutation**, and is defined as:

$$P(n, r) = n(n-1) * \cdots * (n-r+1) = \frac{n!}{(n-r)!}$$

1.3.1 Examples

1. There are 10 people that participate in a race, and we need to know the number of combinations if we give out a first, second, and third place medal. All the others do not get medals.

   If we start at the beginning, 10 people can get the gold medal. Once we have chosen one, there are 9 people left to get the silver medal. After that, we have 8 people that can get the bronze medal. After that, not more people get a medal. This leaves us with:

   $$10 * 9 * 8 = \boxed{720} \text{ potential combinations of medals}$$

   Since we do not give everyone a medal, we only want the factorial to a certain number of elements. This means that we can instead use:

   $$10 * 9 * 8 = 720 = \frac{10!}{7!}$$

   We can even find our formula for r-permutations in this:

   $$\frac{10!}{7!} = \frac{10!}{(10-3)!}$$

   Where 10 represents the number of elements in the set, and 3 represents the number of elements chosen in the set (3 medals).

2 Combinations

A **combination** is a set of r elements from a set of n elements, without caring about ordering

2.1 Subsets of Permutations and Combinations

How many subsets are there of size 2 in the set $\{a, b, c, d\}$?

$$\# \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\} = \boxed{6}.$$

There are 6 subsets, because with combinations, $\{a, b\} = \{b, a\}$.

With ordering, treating the problem as a permutation, the permutation formula gives us $P(4, 2) = 12$ combinations.

But, since we don't care about ordering, we have 2! ways to order every pair.

$$\frac{P(4,2)}{2!} = \frac{12}{2*1} = \boxed{6}.$$

Which is the same answer that we got previously.

2.2 The General form of Getting the Number of Unordered Subsets in a Set

$$C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$$

This is also denoted as $\binom{n}{r}$ and read as "n choose r".

2.3 Examples

1. With a deck of 52 cards, how many standard-size poker hands are there (a hand is 5 cards)

   First, we can see that this will be a combination, since the order of cards do not matter once they are in a hand. All hands with cards in shuffled order are the same.

   $$\binom{52}{5} = \frac{52!}{(52-5)!5!} = 2,598,960, \text{ or "52 choose 5"}.$$

   Therefore, there are 2,598,960 potential poker hands in a deck of 52 cards.

2. How many bit strings of length 10 contain:

   (a) Exactly 4 1's

   Does order matter? No, because the ones can be in any spot on the number. Therefore we can use the formula for combinations.

   $$C(10,4) = \binom{10}{4} = \frac{10!}{4!6!} = 210.$$

   (b) An equal number of zeroes and ones

   Since we have a length of 10, there must be exactly 5 ones and 5 zeroes. This again can be in any order, so we use the rule for combinations:

   $$C(10,5) = \binom{10}{5} = \frac{10!}{5!5!} = 252.$$

   (c) Groups: At most four ones

   Since $\binom{10}{4}$ is exactly 4 ones

   $$\begin{aligned} \binom{10}{4} + \binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0} &= \\ \frac{10!}{4!6!} + \frac{10!}{3!7!} + \frac{10!}{2!8!} + \frac{10!}{1!9!} + \frac{10!}{0!10!} &= \\ \frac{10!}{17280} + \frac{10!}{30240} + \frac{10!}{80640} + \frac{10!}{362880} + \frac{10!}{10!} &= \\ 210 + 120 + 45 + 10 + 1 &= \boxed{386}. \end{aligned}$$