TECHNISCHE UNIVERSITÄT MÜNCHEN

Institut für Informatik

Informatik V
Chair for
Scientific Computing

Scientific Computing Lab

Ordinary Differential Equations

Explicit Discretization

Ordinary Differential Equations

- Equation with a function f(x) and derivatives of f(x)

$$f(x)+f'(x)=0$$

$$\sum_{i=0}^{\infty} a_i f^{(i)}(x)=0$$

- typical: development of a variable over time

Ordinary Differential Equations

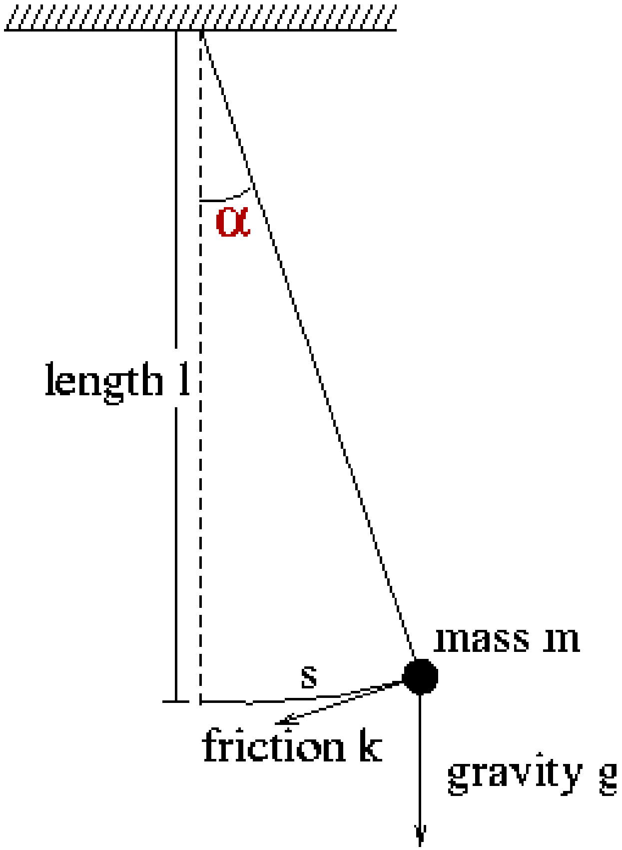
- Example: radioactive decay
  - Half-life: Period of time in which half of the atoms decay
  - Decay constant k: Describes rate of decay and thus half-life

$$\frac{dr(t)}{dt} = -k \cdot r(t)$$

Ordinary Differential Equations

- Example: pendulum

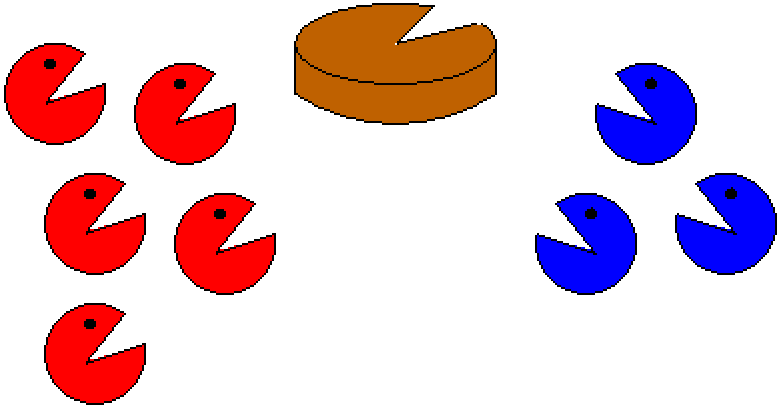
$$\frac{d^2\alpha}{dt^2} + \frac{k}{m} \cdot \frac{d\alpha}{dt} + \frac{g}{l} \cdot \sin(\alpha) = 0$$



Ordinary Differential Equations

- Example: population growth
- populations P and Q

competition



$$\frac{dp(t)}{dt} = (2 - p - q) \cdot p$$

$$\frac{dq(t)}{dt} = (2 - p - q) \cdot q$$

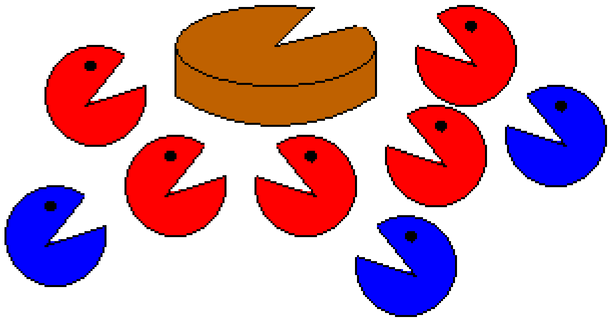
Ordinary Differential Equations

Example: population growth

populations P and Q – predator-prey

$$\frac{dp(t)}{dt} = (2 - p + q) \cdot p$$

$$\frac{dq(t)}{dt} = (2 - 10\mathrm{p} - q) \cdot q$$



Solving
Ordinary Differential Equations

- Solving = finding f(x)
- Two ways to solve the ODE:
  - ~~Analytically~~
  - Numerically

→ Needs discretization (timestepping)

Solving
Ordinary Differential Equations

- Your turn: Try to compute f(t)

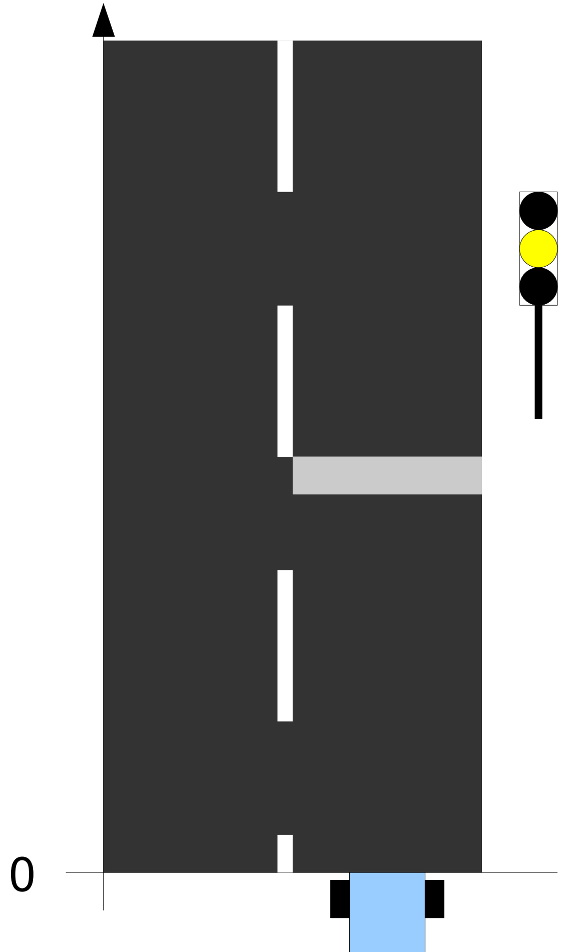
Time t	0	5	10	15	20	25	30	35	40	45
f'(t)	5	1	0	-1	0	0	0	0	2	5

- Time in seconds, f'(t) in m/s
- Any ideas what f(t) describes?

Solving Ordinary Differential Equations

- Your turn: Try to compute f(t)

t = 0



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Solving
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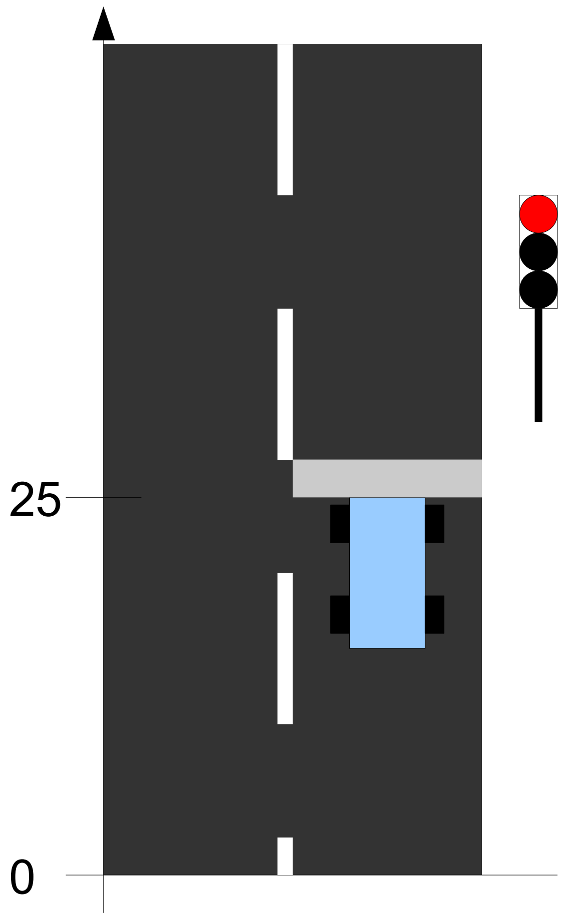
t = 5



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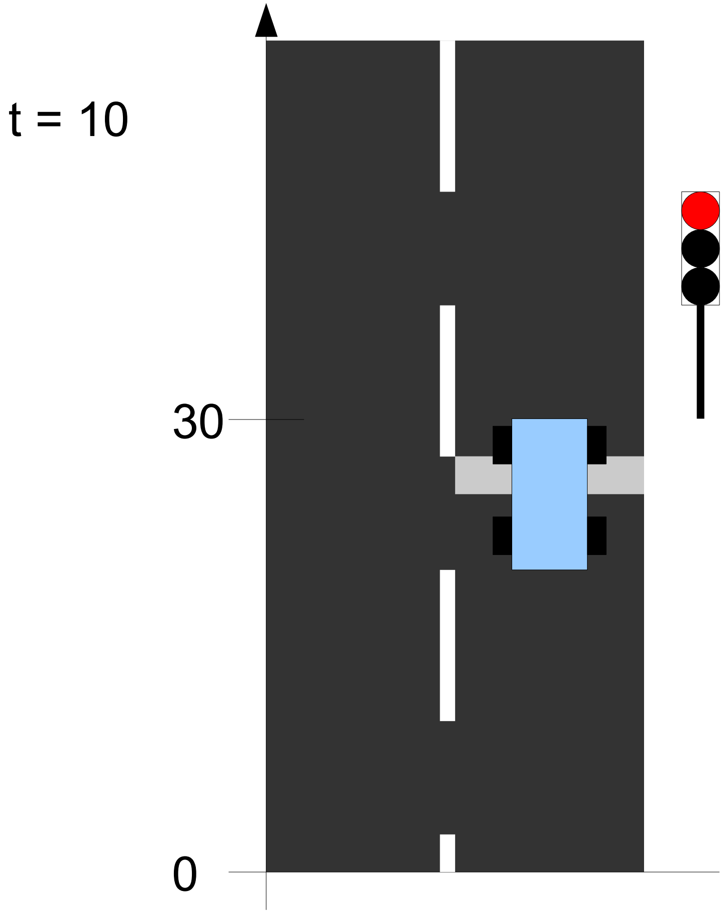

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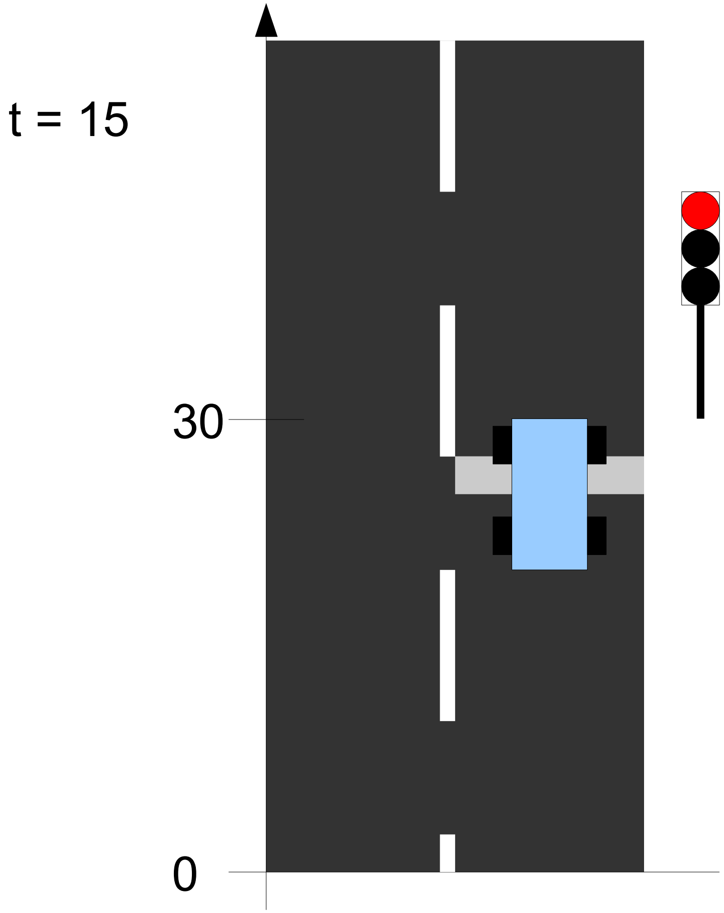

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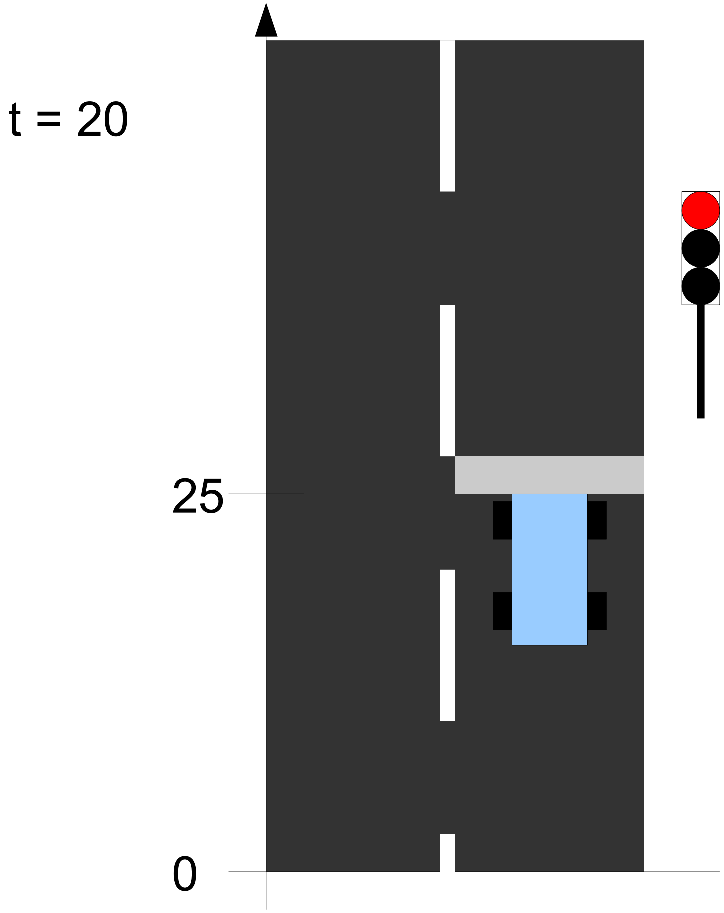

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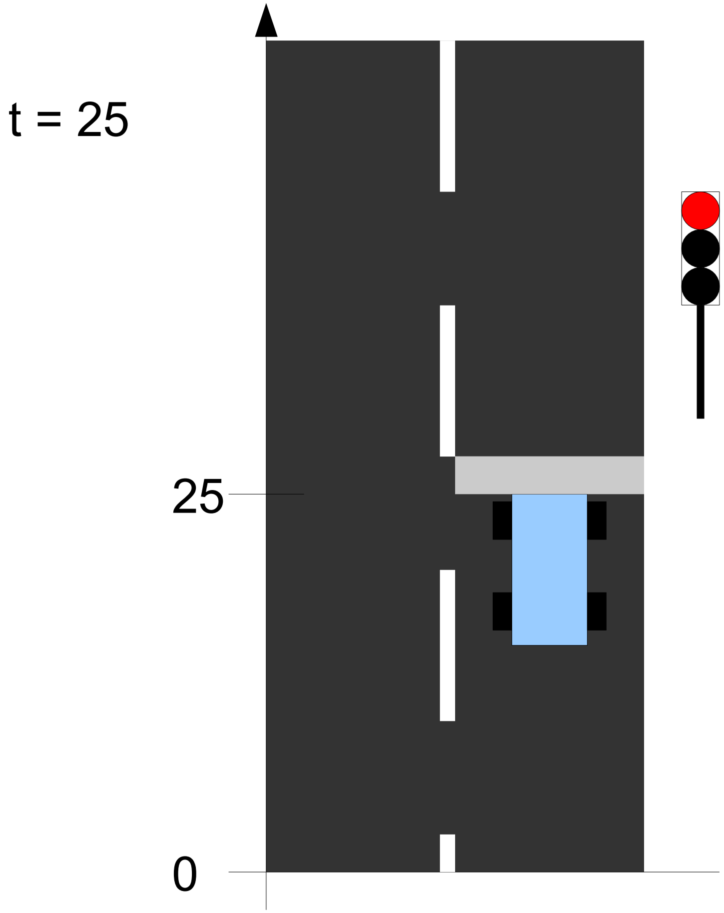

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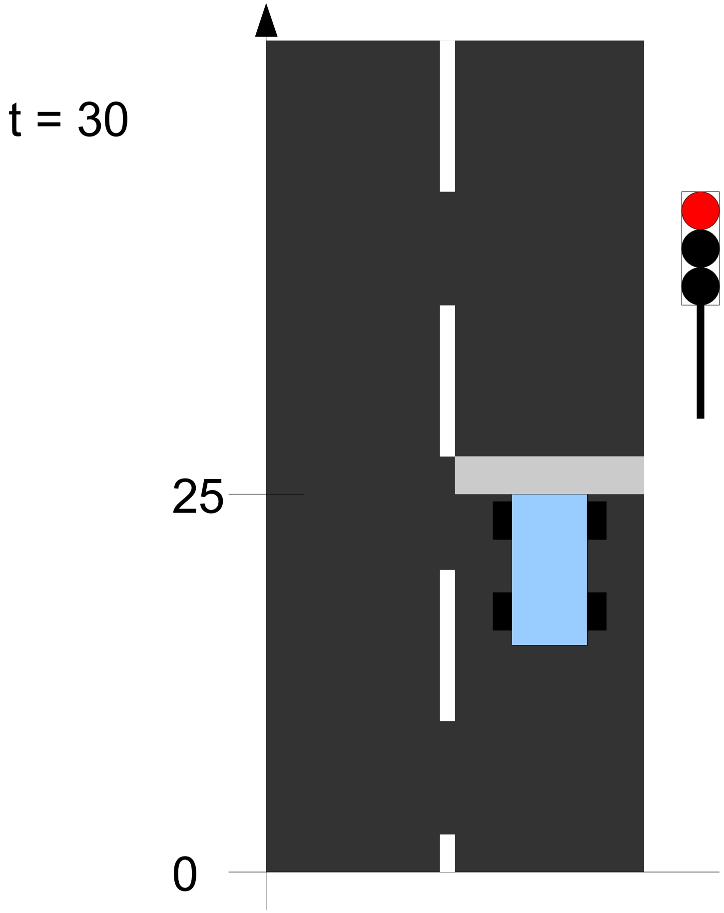

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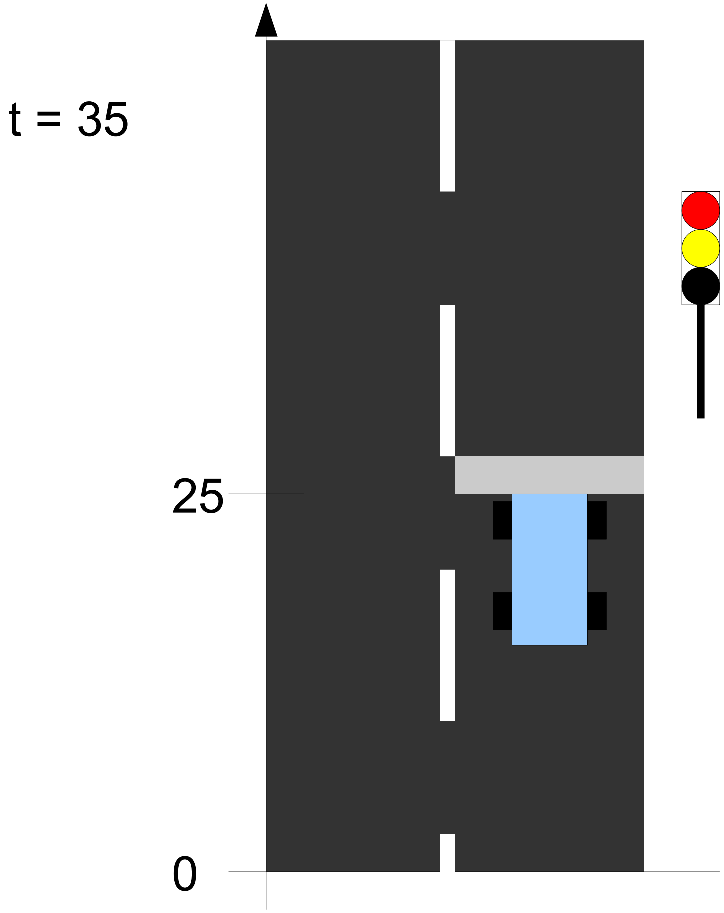

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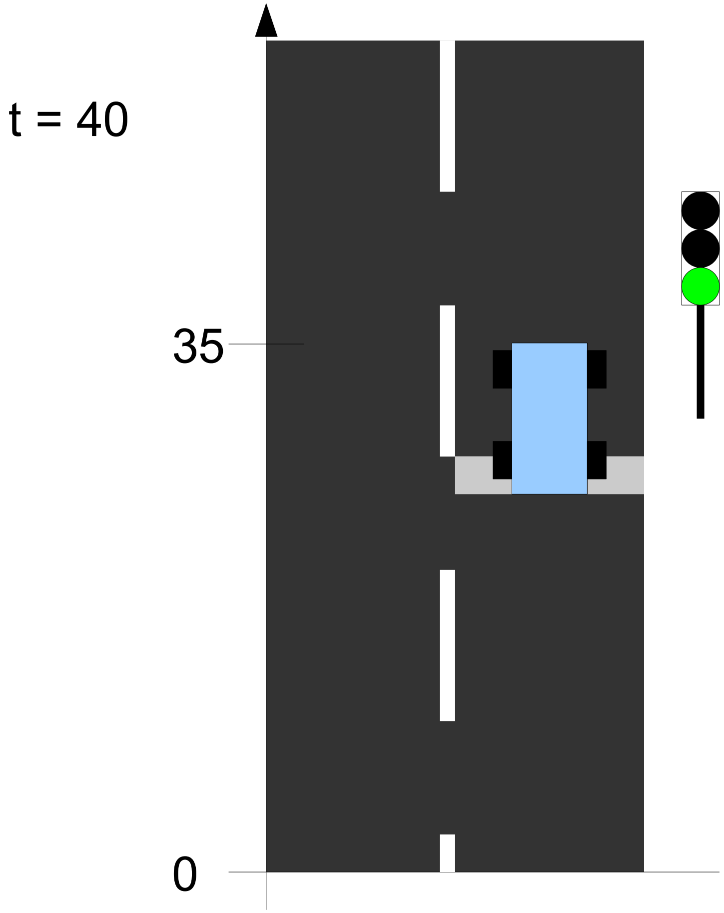

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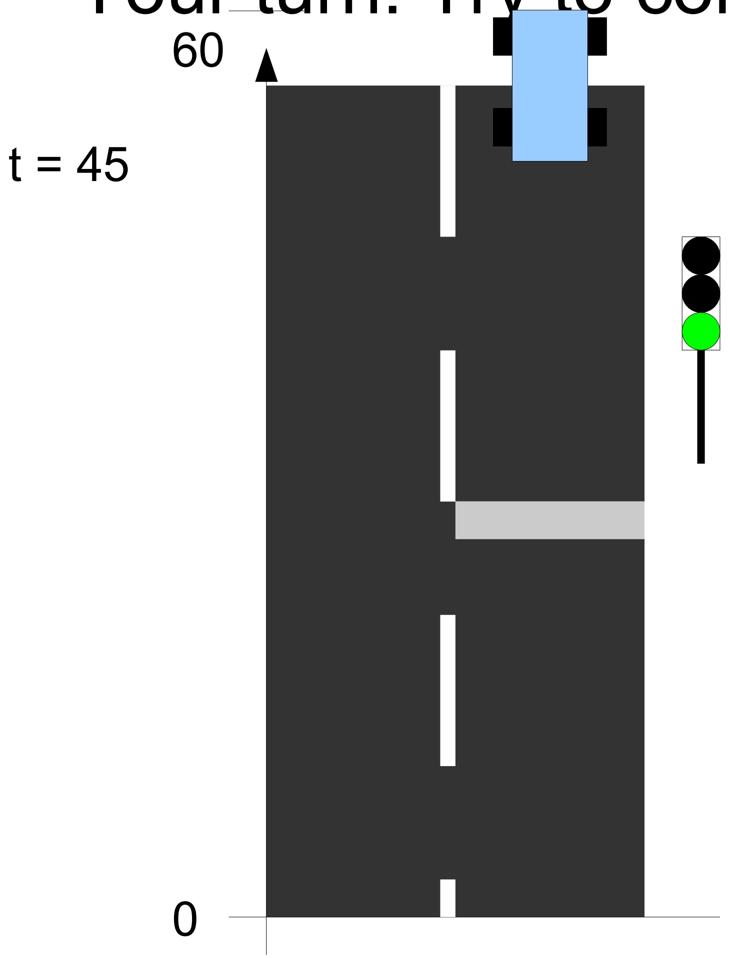

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Explicit Discretization

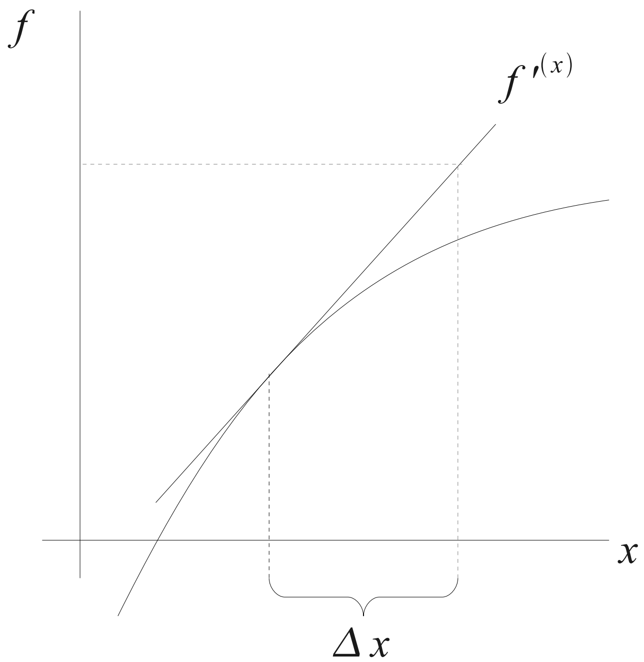
- explicit Euler
- method of Heun
- Runge-Kutta

➢ $$f^{(x+\Delta x)} = F\left(f^{(x)}, x, \Delta x\right)$$

Explicit Discretization

- explicit Euler

$$f^{(x+\Delta x)} = f^{(x)} + \Delta x \cdot f'^{(x)}$$



with

$$f'^{(x)} = f'\left(f^{(x)}, x\right)$$

Explicit Discretization

- method of Heun

$$f^{(x+\Delta x)} = f^{(x)} + \Delta x \cdot \frac{1}{2}\left(f'^{(x)} + \tilde{f}'^{(x+\Delta x)}\right)$$

with

$$\tilde{f}'^{(x+\Delta x)} = f'\left(f^{(x)} + \Delta x \cdot f'^{(x)}, x + \Delta x\right)$$

Explicit Discretization

- Runge-Kutta

$$f^{(x+\Delta x)} = f^{(x)} + \Delta x \cdot \frac{1}{6}(f'_1 + 2\mathrm{f}'_2 + 2\mathrm{f}'_3 + f'_4)$$

with

$$f'_1 = f'(f(x), x)$$

$$f'_2 = f'(f(x) + \frac{\Delta x}{2} \cdot f'_1, x + \frac{\Delta x}{2})$$

$$f'_3 = f'(f(x) + \frac{\Delta x}{2} \cdot f'_2, x + \frac{\Delta x}{2})$$

$$f'_4 = f'(f(x) + \Delta x \cdot f'_3, x + \Delta x)$$

What is Efficiency?

- number of operations?
- runtime?
- accuracy?

What is Efficiency?

- number of operations?
- runtime?
- accuracy?

  ➢ relation accuracy/cost !!!!!

Accuracy

- definition of convergence:

$$\left\| u_{exact} - u_{approx} \right\| = O\left(dt^{p}\right)$$

- experimental computation?