Lab Course
Scientific Computing
Worksheet 3

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Due December 15th, 2011

We study the solution of the differential equation

$$T_t = T_{xx} + T_{yy}, \tag{1}$$

on the square $]0,1[^2$, with Dirichlet boundary conditions given by

$$T(x,y,t) = 0, \qquad \forall (x,y) \in \partial]0,1[^2, \qquad t \in]0,\infty[, \tag{2}$$

and initial condition

$$T(x,y,0) = 1.0 \qquad \forall (x,y) \in]0,1[^2. \tag{3}$$

For the partial derivatives, we choose the following finite difference discretization:

$$T_{xx}|_{i,j} \approx \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{h_x^2}, \tag{4a}$$

$$T_{yy}|_{i,j} \approx \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{h_y^2}. \tag{4b}$$

a) When $t \to \infty$, the equation has to go to equilibrium (otherwise, the integral in time would tend to infinity), which means $T_t \to 0$. This means that $T_{xx} + T_{yy} = 0$, which is Laplace's equation. We know that Laplace's equation does not admit any local minima or maxima, and since the value of the solution at the boundary ∂ is zero, we must have $\lim_{t\to\infty} T(x,y,t) = 0$.

b, c) An explicit Euler method to solve Eq. (1) is implemented in the MATLAB file `expEuler.m` (with forward difference discretization in time), which is a function of N_x, N_y, δt, and the value of the temperature at the current time T_i. The plots were generated for the times $t = \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$. Figure 1 shows the plots for $N_x = N_y = 3, 7, 15, 31$ and $\delta t = \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024}, \frac{1}{2048}, \frac{1}{4096}$, when $t = \frac{1}{2}$.

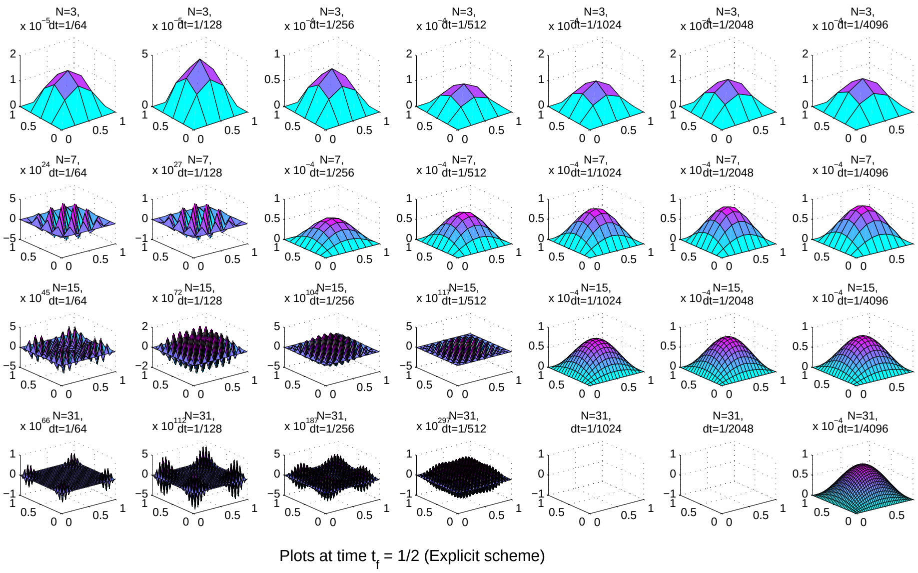

Figure 1: Plots for the different cases corresponding to the explicit Euler method. The plots corresponding to $N = 31$, $\delta t = \frac{1}{1024}, \frac{1}{2048}$ give `NaN` results which cannot be plotted.

It is straightforward to identify which cases give stable solutions. These results are seen in the following table (crosses indicate stable solutions):

$N_x = N_y$	$\delta t = \frac{1}{64}$	$\delta t = \frac{1}{128}$	$\delta t = \frac{1}{256}$	$\delta t = \frac{1}{512}$	$\delta t = \frac{1}{1024}$	$\delta t = \frac{1}{2048}$	$\delta t = \frac{1}{4096}$
3	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
7			$\times$	$\times$	$\times$	$\times$	$\times$
15					$\times$	$\times$	$\times$
31							$\times$

Table 1: Stable solutions for the explicit Euler method.

d, e) An implicit Euler step was implemented to solve Eq. (1) in the MATLAB file `impEuler.m`. It is a function of N_x, N_y, δt, and the value of the temperature at the current time T_i. A Gauss-Seidel solver is used to obtain the solutions of the systems of equations, with tolerance of 10^{-4} for the residual norm. The scheme we chose is the following:

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\delta t} = \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{h_x^2} + \frac{T_{i,j-1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j+1}^{n+1}}{h_y^2}. \tag{5}$$

Fig. shows the resulting plots for $N_x = N_y = 3, 7, 15, 31$ at times $t = \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$ using $\delta t = \frac{1}{64}$. We can see that all the solutions are stable.

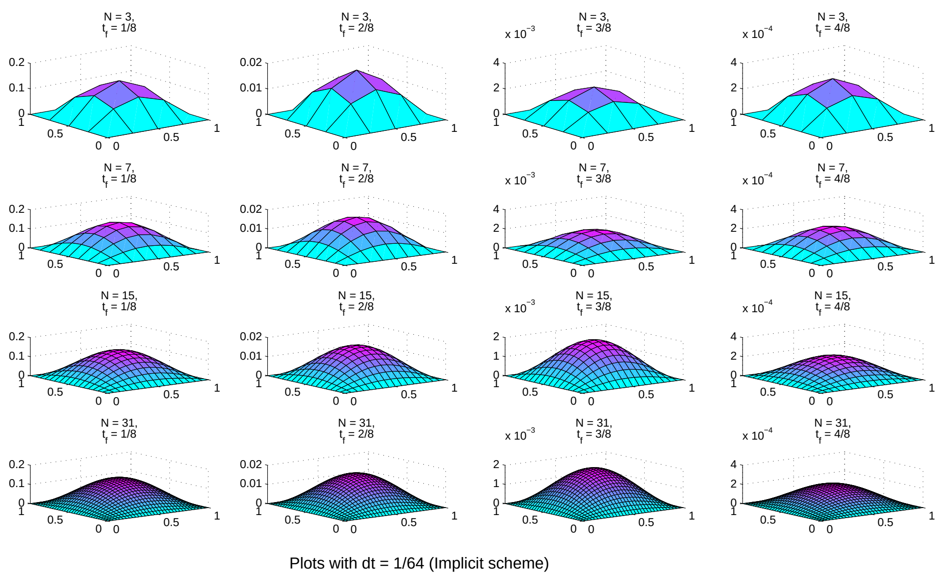

Figure 2: Plots for the different cases corresponding to the explicit Euler method. We observe that all solutions are stable.

Questions

1) From table 1, we observe that, when the spatial step $h = \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^4}$, the maximum timesteps we require are $\delta t = \frac{1}{2^8}, \frac{1}{2^{10}}$, and $\frac{1}{2^{12}}$, correspondingly, from which we can deduce

$$\delta t \leq \frac{h^2}{4}. \tag{6}$$

2) It is easy to verify that the local truncation error for the explicit difference scheme is of order $O(h^2 + \delta t)$. It is desirable to have methods with balanced orders

in time and space, but the explicit schemes are limited by stability constraints such as (6). If we wanted to have a balanced local truncation error, for instance, $e = O(h^2 + \delta t^2)$, condition (6) still requires δt to be much smaller than h, and thus we gain nothing in terms of the error.

3) The implicit method we chose has the same truncation error as the explicit scheme, namely, $e = O(h^2 + \delta t)$, so it is not a good choice if we wish to have a balanced accuracy in time and space.

4) From the theory of numerical analysis, we know that the implicit scheme is unconditionally stable, so the order of the time discretization can be effectively increased without constraints of the type (6). The Richardson scheme or the Crank-Nicolson scheme achieve this and are a common choice for solving equations similar to ours.