Scientific Computing Lab

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Convergence of the Newton Iteration – Examples

Simple Root

Example: $G(x) = x^2 - 1$

Initial guess: $x_0 = 2$

Compute and sketch the first two Newton iterations for this example!

$$\begin{aligned} G'(x) &= 2x \\ x_1 &= x_0 - \frac{G(x_0)}{G'(x_0)} = 2 - \frac{3}{4} = \frac{5}{4} \\ x_2 &= x_1 - \frac{G(x_1)}{G'(x_1)} = \frac{5}{4} - \frac{\frac{25}{16} - 1}{\frac{5}{2}} = \frac{41}{40} \end{aligned}$$

Errors:

$$\begin{aligned} e_0 &= x_0 - x^* = 2 - 1 = 2 \\ e_1 &= x_1 - x^* = \frac{5}{4} - 1 = \frac{1}{4} \\ e_2 &= x_2 - x^* = \frac{41}{40} - 1 = \frac{1}{40}. \end{aligned}$$

Conclusion: Fast convergence (more than multiplication with a fixed factor per iteration) to the clostest root in this case

Divergence

Example: $G(x) = x^2 - 1$

Initial guess: $x_0 = 0$

Compute and sketch the first two Newton iterations for this example!

$$\begin{aligned} G'(x) &= 2x \\ x_1 &= x_0 - \frac{G(x_0)}{G'(x_0)} = 0 - \frac{-1}{0} = \infty. \end{aligned}$$

Conclusion: The Newton method does not always converge, not even for quadratic functions!!!

Oscillating Convergence

Example: $G(x) = \sin(x)$

Initial guess: $x_0 = \frac{\pi}{4}$

Compute and sketch the first two Newton iterations for this example!

$$\begin{aligned} G'(x) &= \cos(x) \\ x_1 &= x_0 - \frac{G(x_0)}{G'(x_0)} = \frac{\pi}{4} - \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{\pi}{4} - 1 \approx -0.2146 \\ x_2 &= x_1 - \frac{G(x_1)}{G'(x_1)} = \frac{\pi}{4} - 1 - \frac{\sin\left(\frac{\pi}{4} - 1\right)}{\cos\left(\frac{\pi}{4} - 1\right)} \approx +0.003456. \end{aligned}$$

Errors:

$$\begin{aligned} e_0 &= x_0 - x^* = \frac{\pi}{4} - 0 \approx= 0.7854 \\ e_1 &= x_1 - x^* \approx -0.2146 \\ e_2 &= x_2 - x^* \approx 0.003456. \end{aligned}$$

Conclusion: Fast convergence (more than multiplication with a fixed factor per iteration) to the clostest root in this case, but not montone as in the first example.

Multiple Roots

Example: $G(x) = x^2$

Initial guess: $x_0 = 1$

Compute and sketch the first two Newton iterations for this example!

$$\begin{aligned}
G'(x) &= 2x \\
x_1 &= x_0 - \frac{G(x_0)}{G'(x_0)} = 1 - \frac{1}{2} = \frac{1}{2} \\
x_2 &= x_1 - \frac{G(x_1)}{G'(x_1)} = \frac{1}{2} - \frac{\frac{1}{4}}{1} = \frac{1}{4}.
\end{aligned}$$

Errors:

$$\begin{aligned}
e_0 &= x_0 - x^* = 1 - 0 = 1 \\
e_1 &= x_1 - x^* = \frac{1}{2} \\
e_2 &= x_2 - x^* = \frac{1}{4}.
\end{aligned}$$

Conclusion: Slower convergence (multiplication with a fixed factor $1/2$ per iteration) to the clostest root in this case.

Several Roots

Example: $G(x) = \sin(x)$

Initial guess: $x_0 = \frac{\pi}{2} - 0.001$

Compute and sketch the first two Newton iterations for this example!

Closest root to x_0: $X^* = 0$.

$$\begin{aligned}
G'(x) &= \cos(x) \\
x_1 &= x_0 - \frac{G(x_0)}{G'(x_0)} = \frac{\pi}{2} - 0.001 - \frac{\sin\left(\frac{\pi}{2} - 0.001\right)}{\cos\left(\frac{\pi}{2} - 0.001\right)} \approx -998.4299 \\
x_2 &= x_1 - \frac{G(x_1)}{G'(x_1)} = -998.4299 - \frac{\sin(-998.4299)}{\cos(-998.4299)} \approx -999.1090 \\
x_3 &= x_2 - \frac{G(x_2)}{G'(x_2)} = -999.1090 - \frac{\sin(-999.1090)}{\cos(-999.1090)} \approx -999.0263
\end{aligned}$$

Conclusion: The Newton method converges in many but not in all cases to the root that is closest to the initial guess!!!