TECHNISCHE UNIVERSITÄT MÜNCHEN

Institut für Informatik

Informatik V
Chair for
Scientific Computing

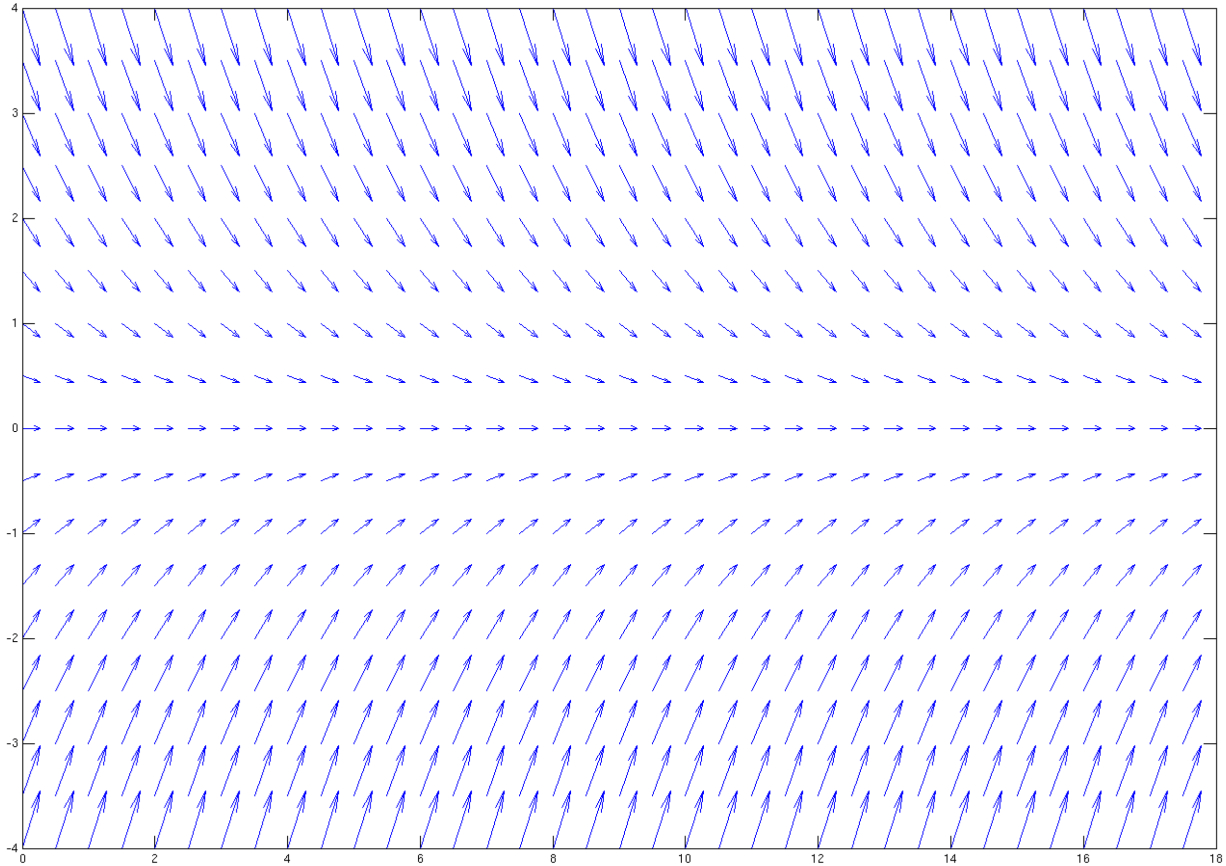
Scientific Computing Lab

Ordinary Differential Equations

Implicit Discretization

Instability

- Vector field of ODE $f'(x) = -k \cdot f(x)$



Instability

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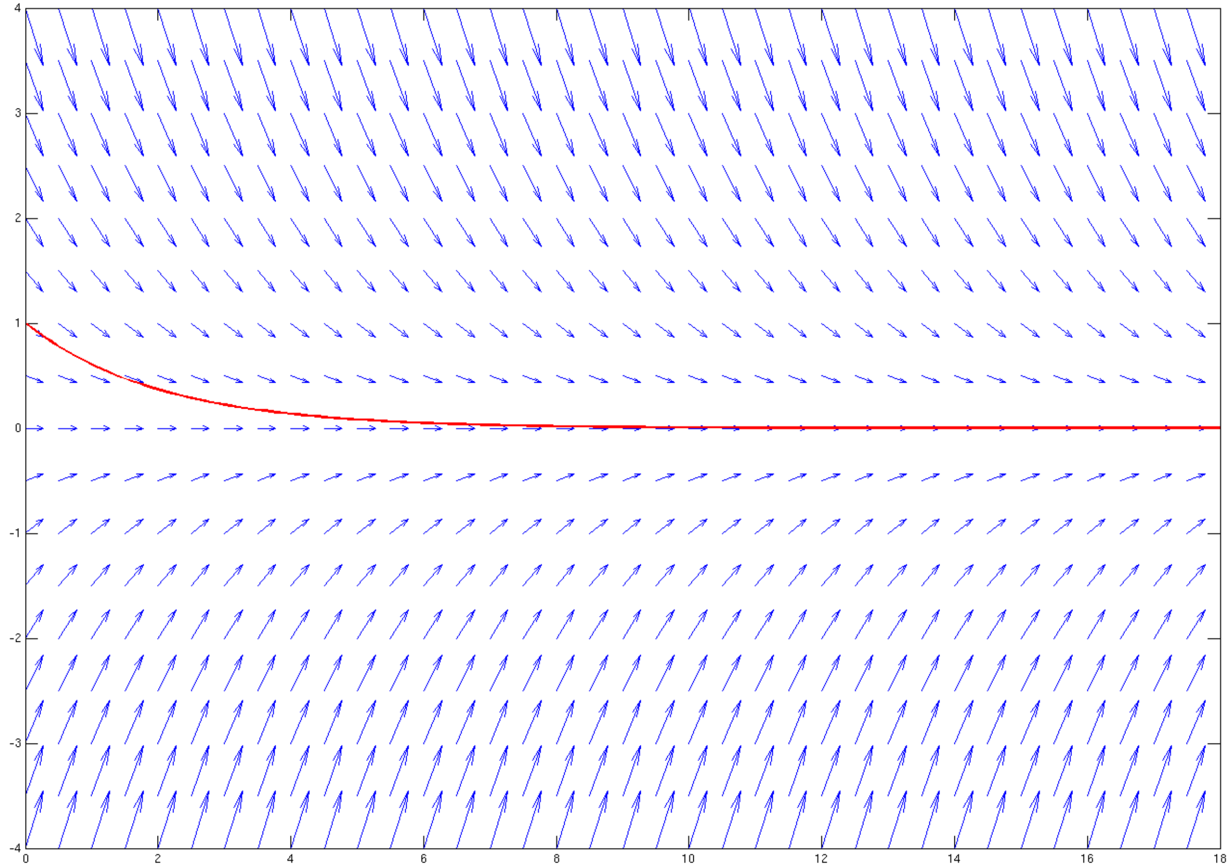
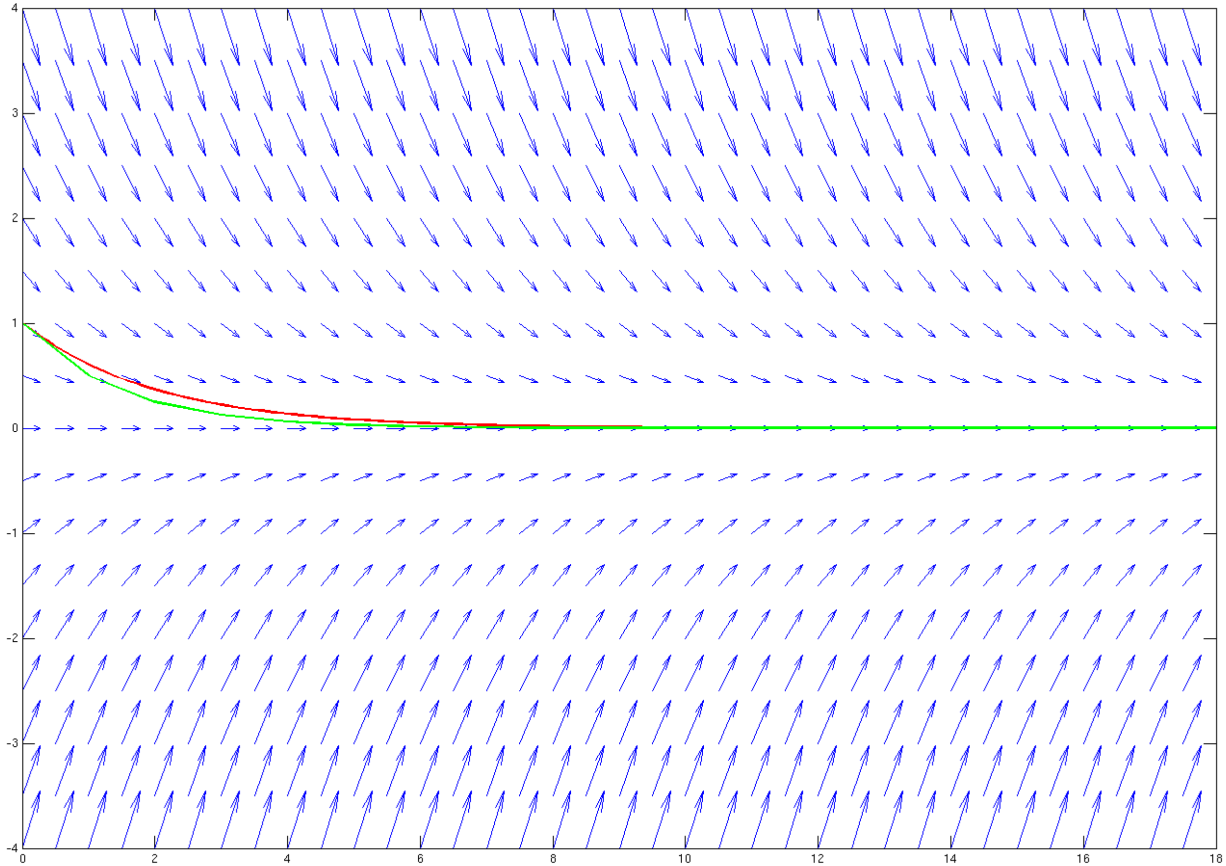


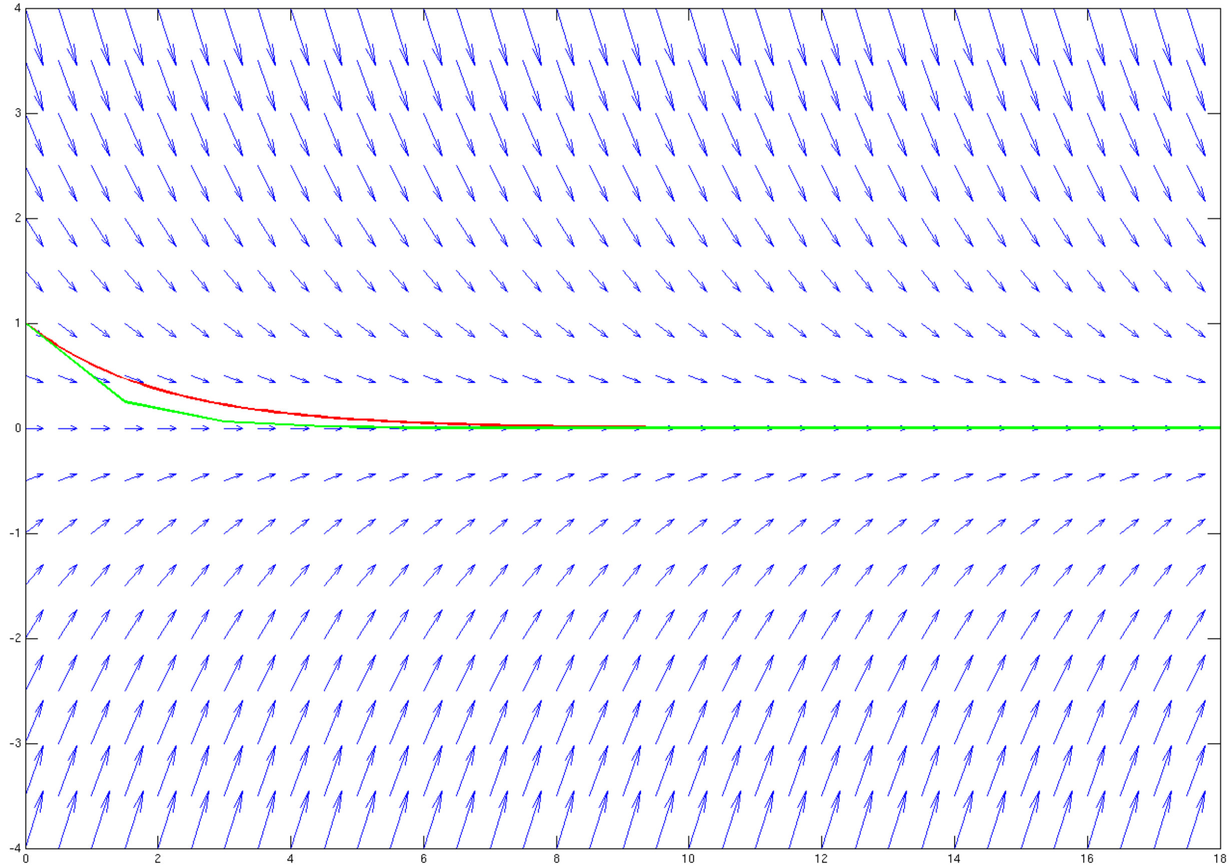
Instability

- Exlicit Euler, time step size: 1



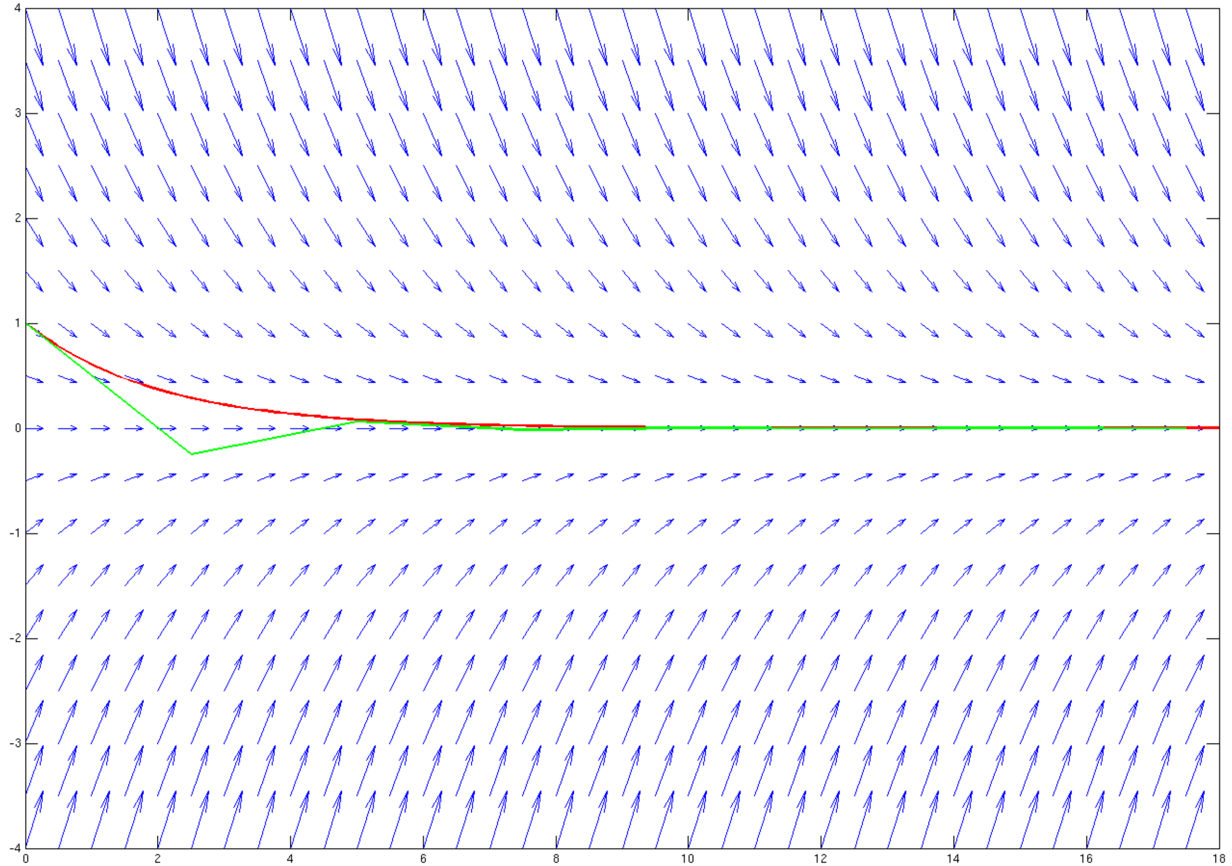
Instability

- Exlicit Euler, time step size: 1.5



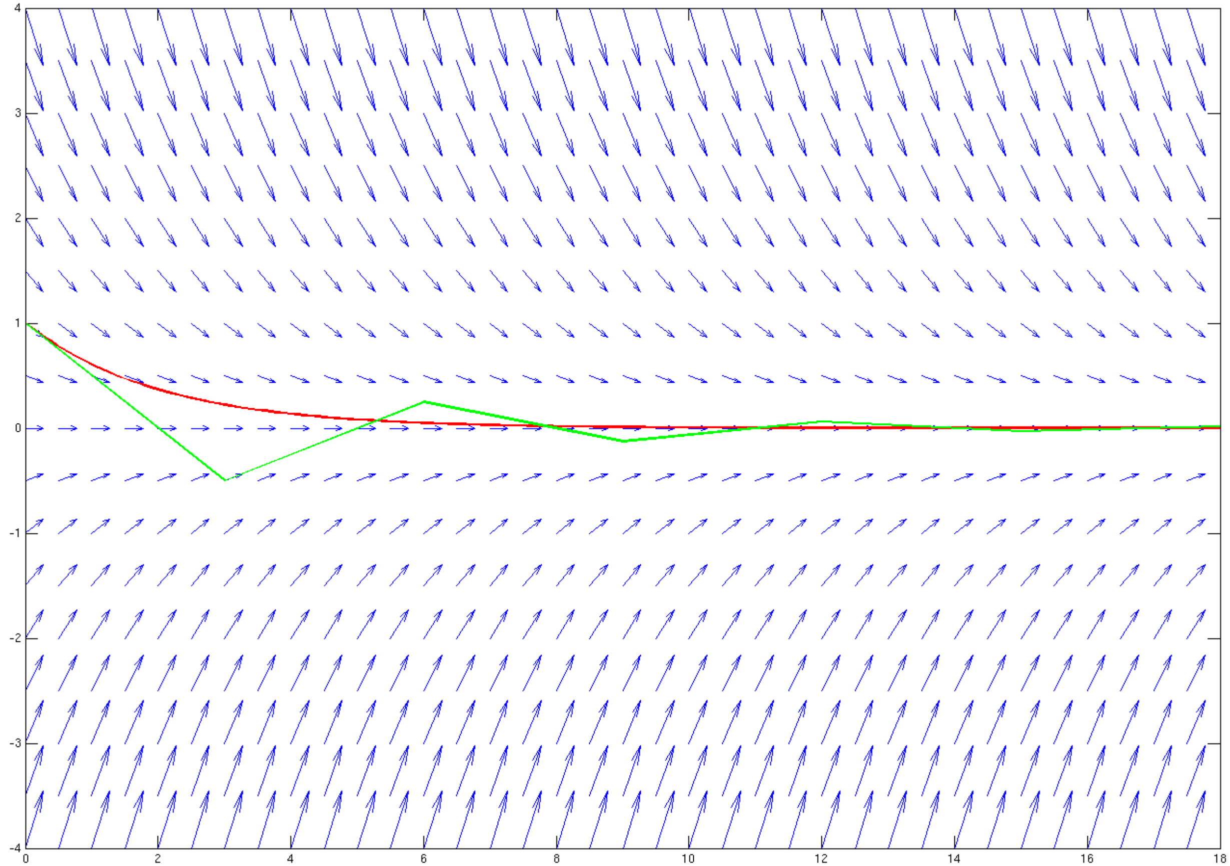
Instability

- Exlicit Euler, time step size: 2.5



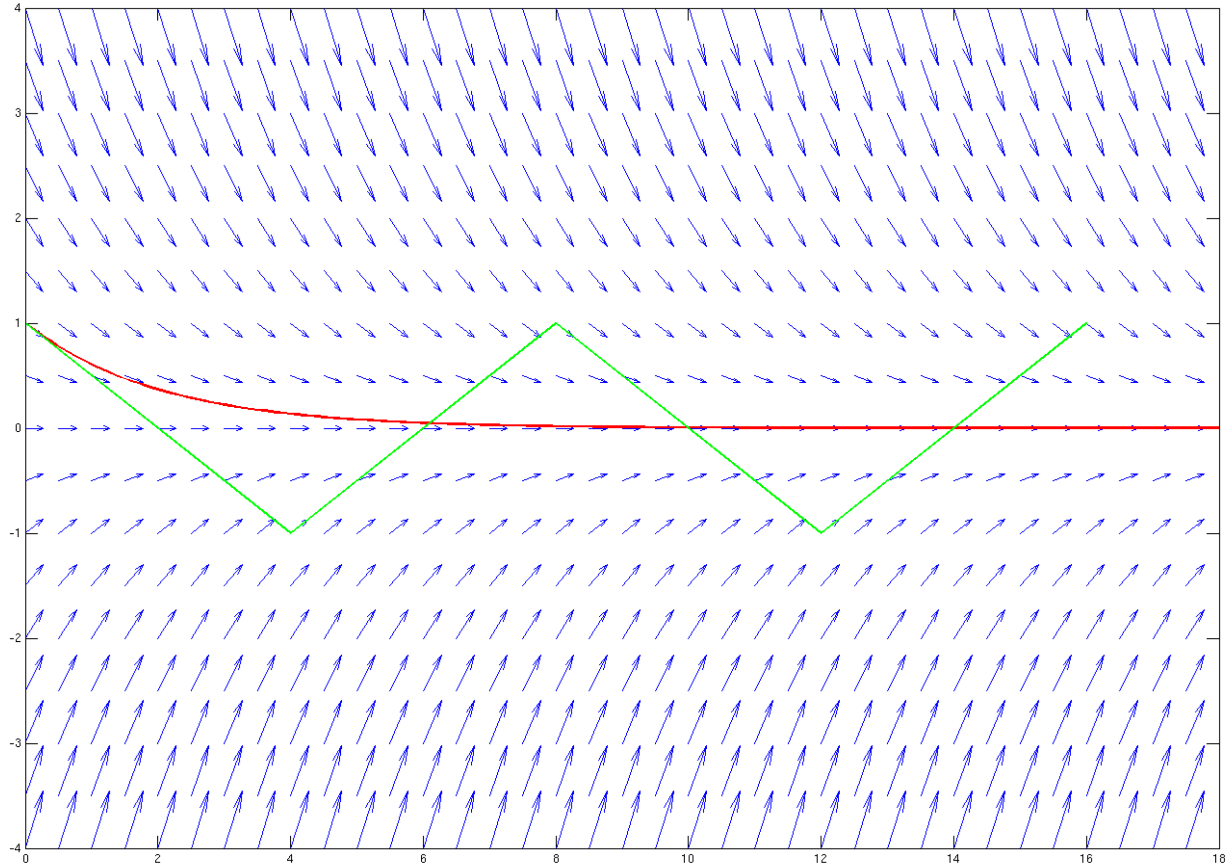
Instability

- Exlicit Euler, timestep-size: 3.0



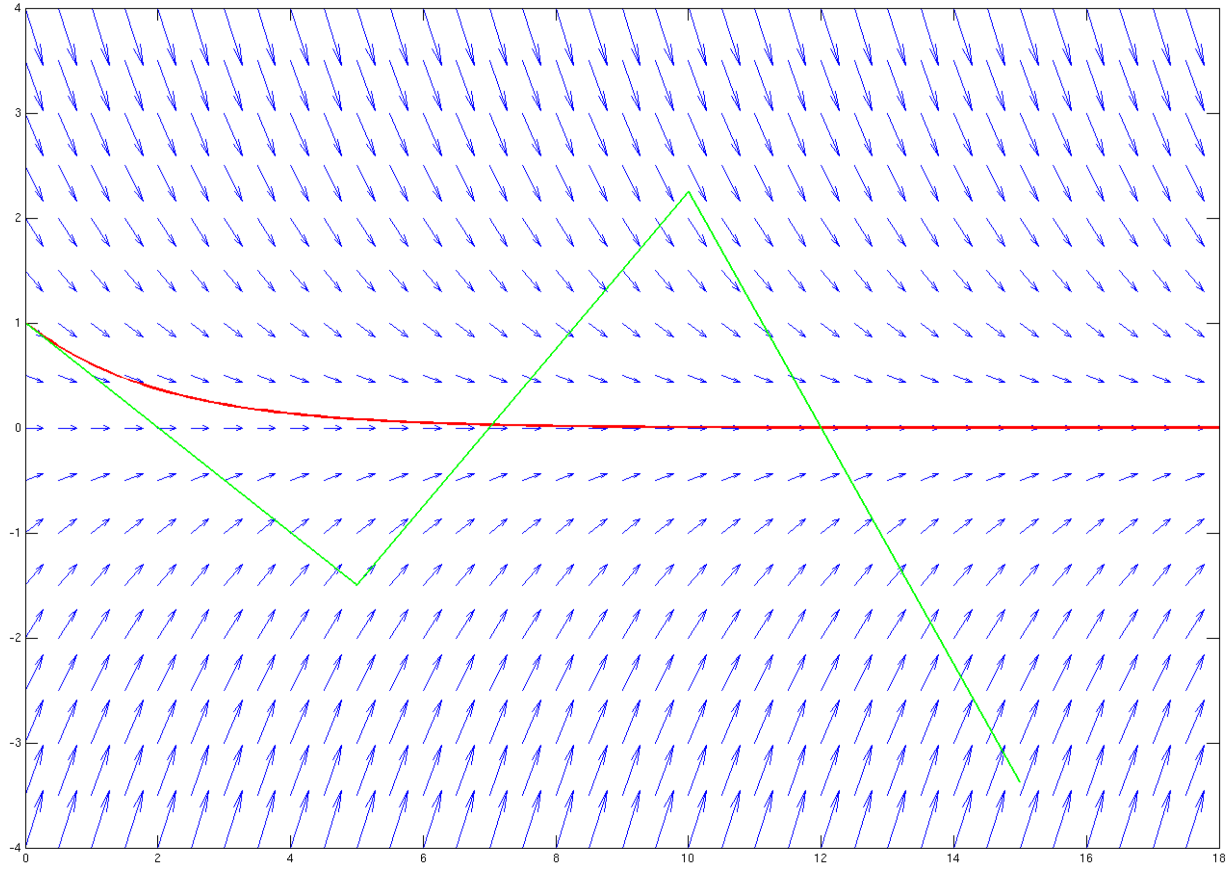
Instability

- Exlicit Euler, time step size: 4.0



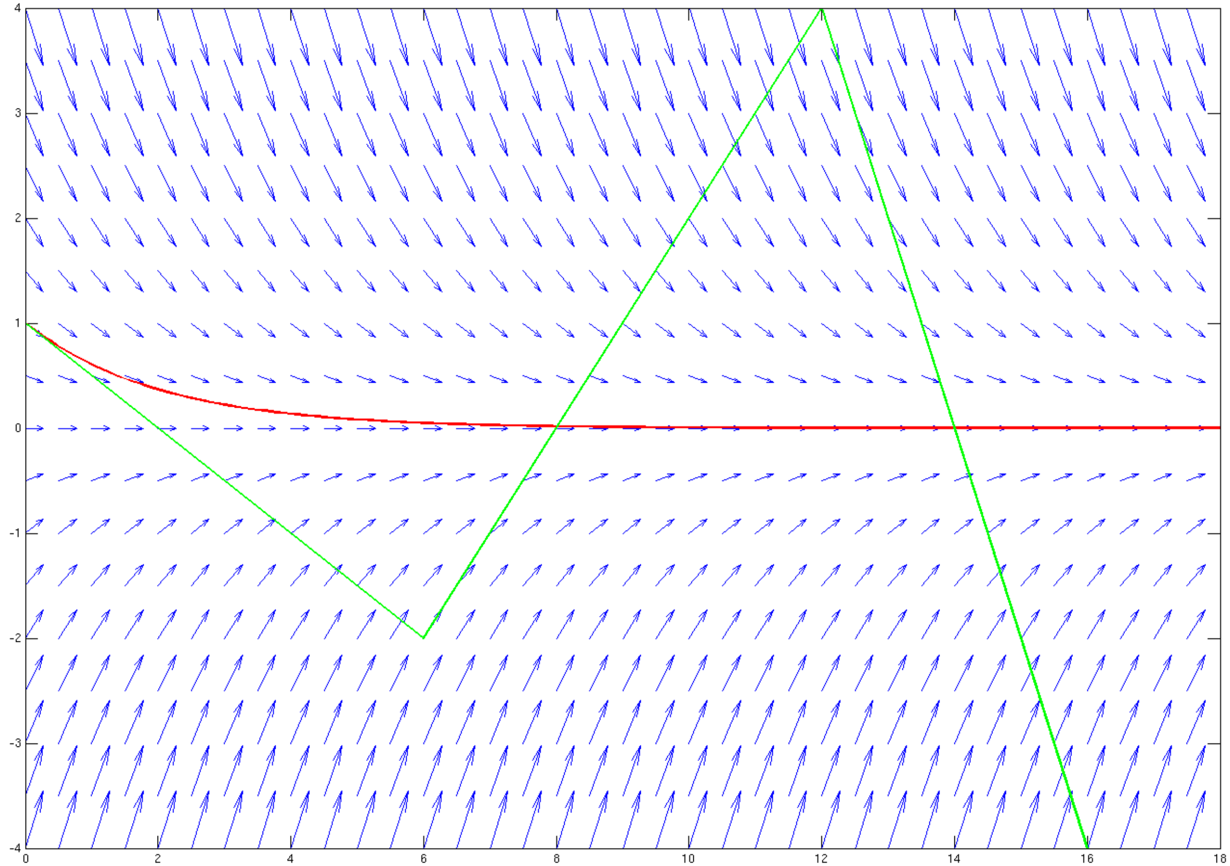
Instability

- Exlicit Euler, time step size: 5.0



Instability

- Exlicit Euler, time step size: 6.0

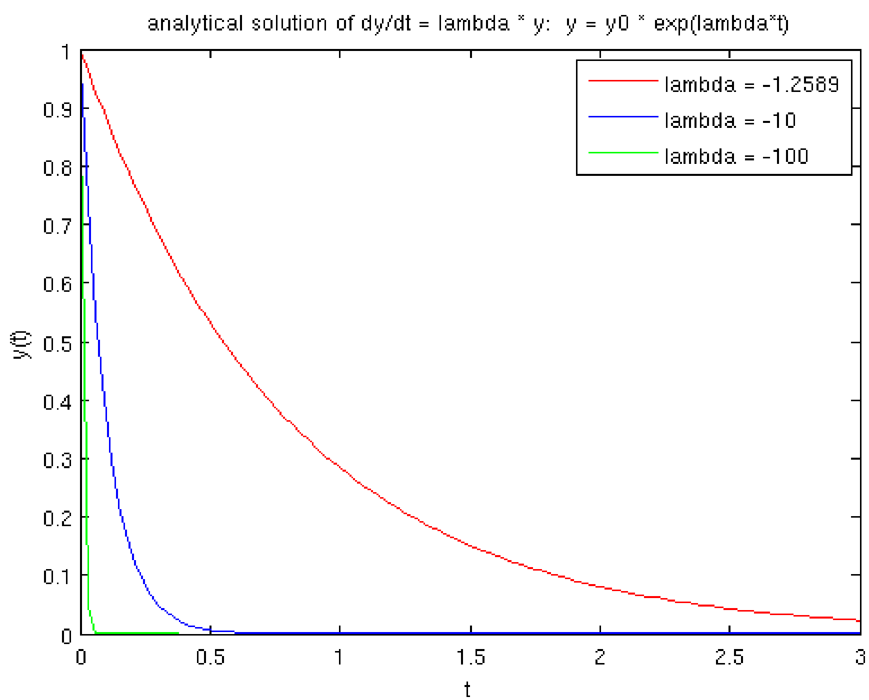


Instability

- stiff equations
  - instabilities if dt does not obey restrictions
  - „all explicit SSM with stable dt provide very small local errors”
  - examples: damped mech. systems, parabolic PDE, chem. reaction kinetics
- remedy: (special) implicit methods

Stiff Equations

- Example: Dahlquist's test equation: $y' = \lambda \cdot y$
  - stable anal. solution
  - $\Delta t \leq c / \lambda$ also in steady phase



transient phase steady phase

Δt limited by stability & accuracy

Δt limited by stability only!

Implicit Methods

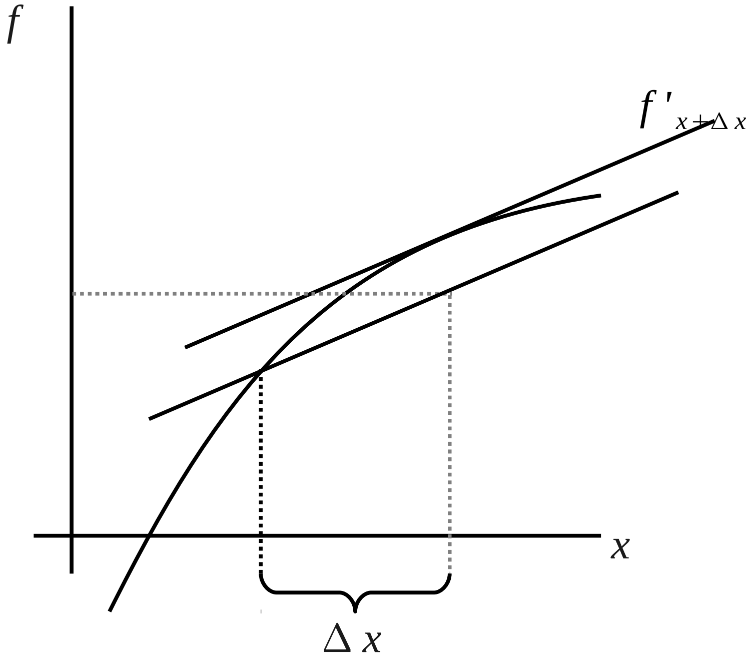
- implicit Euler
- second order Adams-Moulton

  - $$f_{x+\Delta x} = F\left(f_{x+\Delta x}, x, \Delta x\right)$$

Implicit Methods

- implicit Euler (first-order method)

$$f_{x+\Delta x} = f_x + \Delta x \cdot f'_{x+\Delta x}$$



with

$$f'_{x+\Delta x} = f'\left(f_{x+\Delta x}, x+\Delta x\right)$$

Implicit Methods

- Adams Moulton (second-order method)

$$f_{x+\Delta x} = f_x + \frac{\Delta x}{2} \cdot \left(f'_x + f'_{x+\Delta x} \right)$$

with

$$f'_x = f'\left(f_x, x \right)$$

$$f'_{x+\Delta x} = f'\left(f_{x+\Delta x}, x + \Delta x \right)$$

Newton Method

- Implicit method may result in complex expressions:

  ODE

$$f'(t) = -\log(f(t))$$

  implicit Euler:

$$f_{t+\Delta t} = f_t + \Delta t \cdot f'_{t+\Delta t}$$

$$f_{t+\Delta t} = f_{-\Delta} t \cdot \log(f_{t+\Delta t})$$

Newton Method

- Implicit method may result in complex expressions:

  implicit Euler:

  $$f_{t+\Delta t} = f_t - \Delta t \cdot \log\left(f_{t+\Delta t}\right)$$

  needs to be solved:

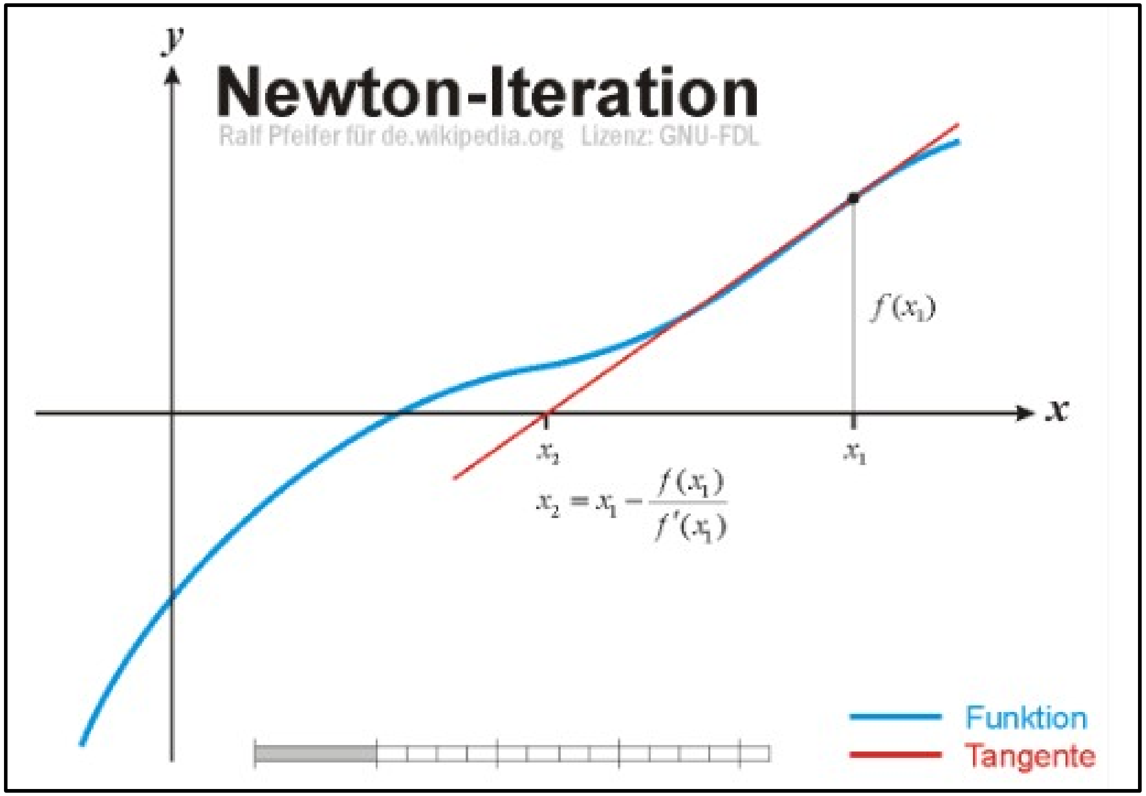
  $$f_{t+\Delta t} + \Delta t \cdot \log\left(f_{t+\Delta t}\right) - f_t = 0$$

  Can (only) be solved numerically!

Newton Method

- Numerical approach to find roots of functions

$$G(x) = 0$$

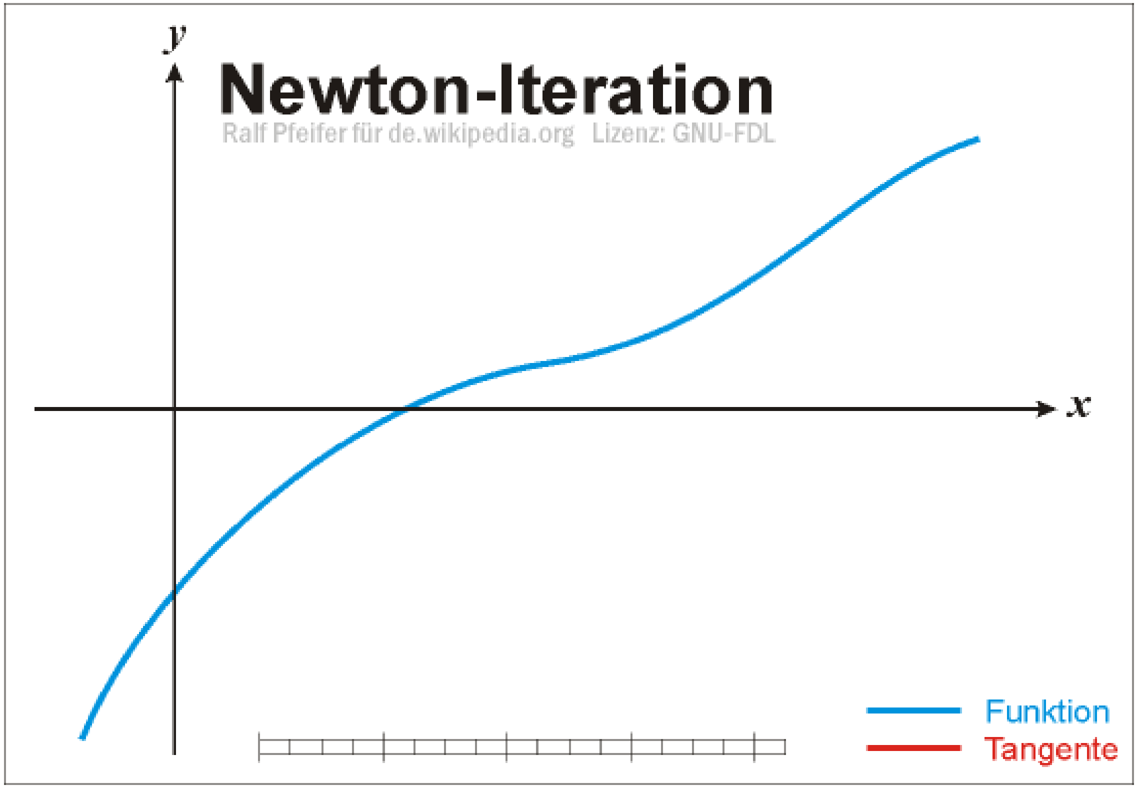


Ralf Pfeifer – http://upload.wikimedia.org/wikipedia/commons/e/e0/NewtonIteration_Ani.gif

Newton Method

- Numerical approach to find roots of functions

$$G(x) = 0$$



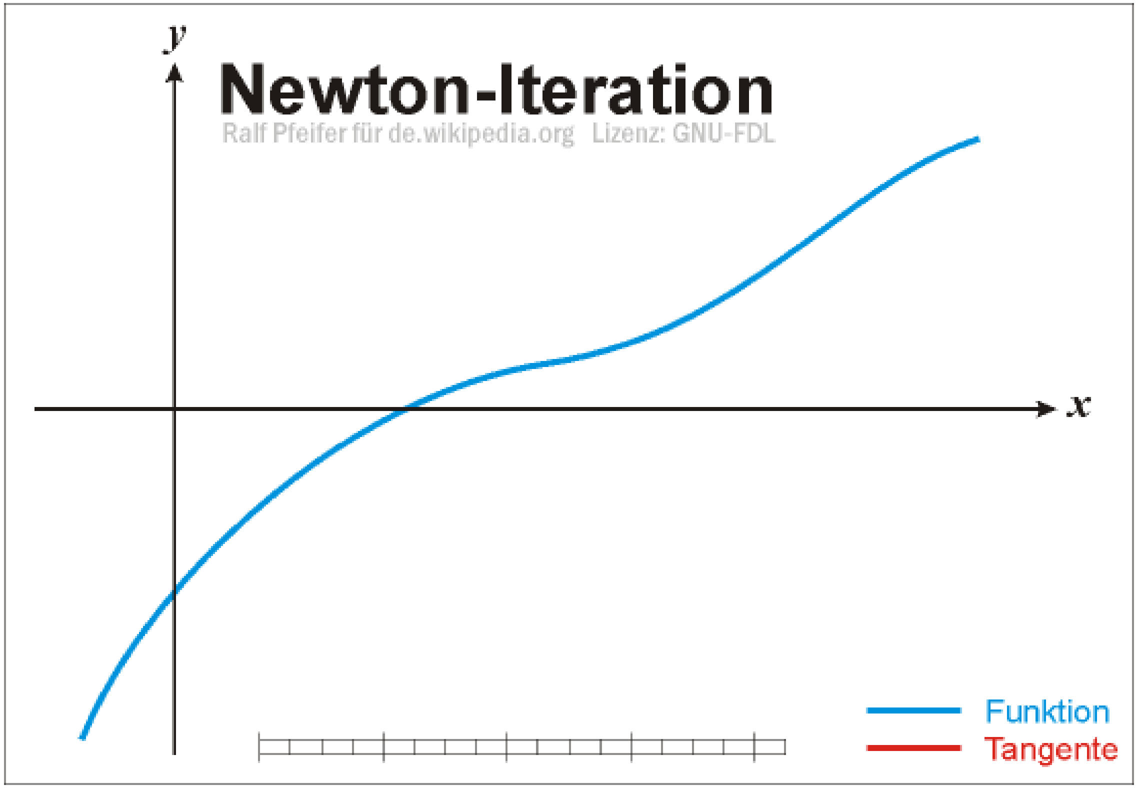
Ralf Pfeifer – http://upload.wikimedia.org/wikipedia/commons/e/e0/NewtonIteration_Ani.gif

Newton Method

- Numerical approach to find roots of functions

$$G(x) = 0$$

$$x_{n+1} = x_n - \frac{G(x_n)}{G'(x_n)}$$



Ralf Pfeifer – http://upload.wikimedia.org/wikipedia/commons/e/e0/NewtonIteration_Ani.gif

Newton Method

- Numerical approach to find roots of functions
- In our case we get

$$G\left(f_{x+\Delta x}\right)=0$$

- Thus, we need $G'\left(f_{x+\Delta x}\right)$

Newton Method

- Initial example: $f_{t+\Delta t} = f_t - \Delta t \cdot \log\left(f_{t+\Delta t}\right)$

$$G\left(f_{t+\Delta t}\right) = f_{t+\Delta t} + \Delta t \cdot \log\left(f_{t+\Delta t}\right) - f_t$$

$$G'\left(f_{t+\Delta t}\right) = f'_{t+\Delta t} + \frac{\Delta t}{f_{t+\Delta t}}$$

$$G'\left(f_{t+\Delta t}\right) = \log\left(f_{t+\Delta t}\right) + \frac{\Delta t}{f_{t+\Delta t}}$$

Newton Method

- Convergence examples of the Newton iteration (cf. separate file)

Explicit versus Implicit

- explicit:
  - cheap time steps
  - many time steps
- implicit:
  - expensive/impossible time steps
  - less time steps