CFD Lab Course

The Lattice Boltzmann Method

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Review: The story so far

So far: Navier–Stokes

- Compute pressure and velocity of the fluid
- Nonlinear equations
- Equation system to be solved in each timestep

However:

- Only valid in the continuum regime, that is for

$$\mathrm{Kn} := \frac{l_{mfp}}{L_c} << 1$$

  Kn Knudsen number, l_{mfp} mean free path of fluid molecules, L_c characteristic length of flow system
- Not that trivial to bring Navier–Stokes to supercomputers

Lattice What???

Lattice Boltzmann Method:

Derived from Boltzmann equation ($\rightarrow$ microscopic description)

- Describes fluid by means of statistics
  $\rightarrow$ Find probability $f(\vec{x}, \vec{v}, t)$ that fluid molecules are in a small surrounding of $\vec{x}$ at time t with a certain velocity $\vec{v}$
- Macroscopic quantities are obtained by integration over the velocity space:

$$\begin{aligned} \rho(x,t) &= \int_{-\infty}^{\infty} f(\vec{x}, \vec{v}, t) \mathrm{dv} \\ \rho(x,t)\vec{u} &= \int_{-\infty}^{\infty} f(\vec{x}, \vec{v}, t)\vec{v} \ \mathrm{dv} \end{aligned} \tag{1}$$

  with fluid density ρ and fluid velocity $\vec{u}$
- Assumption “Ideal gas” $\Rightarrow$ Pressure $p = c_s^2 \rho$, c_s speed of sound

Lattice What???

Problems:

- Probabilities live in a $2D+1$-dimensional space
  $\rightarrow$ "Curse of dimensionality"
- Integration over infinite domains can be quite hard

How can we solve these issues?

Welcome to the dark ages: Castle invasion

Let's find a simple model for a castle invasion:
Assume Q bowmen on a castle tower, defending the castle.
Assume, we have 'fair" battle, i.e. there are only Q invaders

taken from http://www.robinhood2010.de/DE/

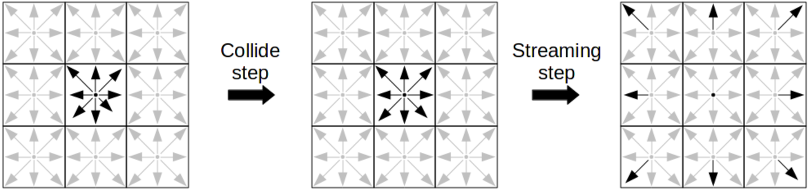
wp-content/uploads/2010/08/robin-hood_disney.jpg

- Each bowman can shoot arrows only into the direction of one enemy (and vice versa)
- If a bowman does not have any more arrows, he gets new arrows from the other bowmen
- The arrows coming from the enemies are collected and re-used by the bowmen

Dark ages $\leftrightarrow$ Lattice Boltzmann

Q Bowmen on a tower	**Probability densities** f_i ($i = 1, ..., Q$) in a (cubic) lattice cell
Q directions for the arrows	Q **lattice velocities** $\vec{c}_i$ for fluid molecules
Re-distribution of arrows	**Collision** process between populations f_i within a cell
Incoming arrows from Q attackers	Molecules **streamed** from Q neighboured cells

Collision and Streaming



1 collision + 1 streaming = 1 timestep

Reduce the effort...

Problems:

- Probabilities live in a $2D+1$-dimensional space $\rightarrow$ “Curse of dimensionality”
- Integration over infinite domains can be quite hard

Our discrete formulation...

- ... lives in a $D+1$-dimensional space, having Q unknowns per lattice cell
- ... can easily (and locally!) handle the integration procedure:

$$\rho = \sum_{i=1}^{Q} f_i$$

$$\rho\vec{u} = \sum_{i=1}^{Q} f_i \vec{c}_i$$

Collision

- Interaction of local populations f_i inside each lattice cell
- Denote the post-collision state by f_i^*
- Assume, our system is close to an equilibrium state
- If f_i^{eq} represents the equilibrium state of the fluid system, the molecular collisions should drive our system even closer towards this equilibrium

BGK model (Bhatnagar – Gross – Krook):

$$\boxed{f_i^*(x,t) = f_i(x,t) - \frac{1}{\tau}\left(f_i(x,t) - f_i^{eq}\right)}$$

with relaxation time $\tau \in (0.5, 2)$

The equilibrium distribution

- Can be derived from the Maxwell-Boltzmann distribution
- Equilibrium state only depends on macroscopic quantities $(\rho, \vec{u})$

$$\boxed{f_i^{eq}(\rho, \vec{u}) = w_i \rho \left(1 + \frac{\vec{c}_i \cdot \vec{u}}{c_s^2} + \frac{\left(\vec{c}_i \cdot \vec{u}\right)^2}{2c_s^4} - \frac{\vec{u} \cdot \vec{u}}{2c_s^2} \right)}$$

with $c_s := \frac{1}{\sqrt{3}}$ speed of sound on the lattice

The Lattice Boltzmann Algorithm

Given: $f_i(x, t_{start})$ everywhere in our computational domain

`for` $(t = t_{start}\,;\; t < t_{end}\,;\; t = t + dt)$ `; do`

`Compute density and velocity inside a fluid cell:`

$$\rho = \sum_i f_i, \qquad \rho\vec{u} = \sum_i f_i \vec{c}_i$$

`Local collision:` $f_i^*(x, t) = f_i(x, t) - \frac{1}{\tau}\left(f_i(x, t) - f_i^{eq}(\rho, \vec{u})\right)$

`Streaming to/ from neighbours:` $f_i(x + \vec{c}_i dt, t + dt) = f_i^*(x, t)$

`done`

$$\boxed{f_i(x + c_i dt, t + dt) = f_i(x, t) - \frac{1}{\tau}\left(f_i - f_i^{eq}\right)}$$

Remarks

- **Explicit method**, no equation system that needs to be solved!
- Asymptotic theory: In the continuum limit (Kn$\rightarrow$0):

  Lattice Boltzmann $\rightarrow$ (incompr.) Navier-Stokes

- Lattice Boltzmann: Simulation of **weakly compressible** flows
  $\rightarrow \quad \rho \approx 1.0, \quad \|\vec{u}\| << c_s$

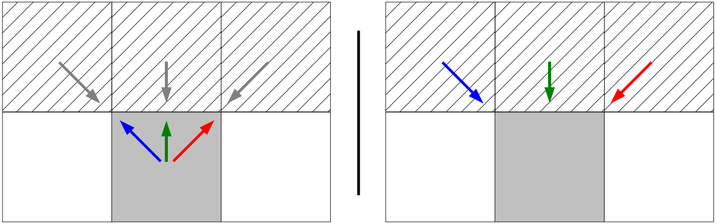
Boundary Conditions: Back to the dark ages

Bowman vs. Bowman

What should a bowman do when there's no enemy?

Boundary conditions - No-slip wall

- No-slip wall $\Leftrightarrow \vec{u} = 0$ at wall
- In “Dark Ages”-words:
  If an arrow (loaded with particles) leaves the domain
  $\Leftrightarrow$ An arrow with the same amount of particles needs to enter the domain



- Write $f_i^*(x,t)$ to $f_{inv(i)}^*(x + \vec{c}_i dt, t)$
- Similar modeling of moving walls (see worksheet)

Worksheet 2: 3D Cavity flow

- Program a 3D cavity based on the D3Q19 Lattice Boltzmann model ($\rightarrow$ 3D, 19 lattice velocities)
- Simulate cavities at different Reynolds numbers

  $\rightarrow$ kinematic viscosity ν is defined by $\nu = c_s^2 \left(\tau - \frac{1}{2}\right)$

  $\rightarrow$ viscosity limited to the range $\left(0, \frac{1}{2}\right)$

  $\rightarrow$ velocity needs to be (much) smaller than the speed of sound