Computational Physics
Problem Set

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1 Fractal Generation

Fractals are self-similar shapes, so they can be generated by repeatedly applying mappings to a set of points. The fractals in this problem set can all be created with linear iterated function systems (IFS). The transformations for these fractals can be described with affine transformations:

$$\mathbf{v}_{i+1,j} = T_j \mathbf{v}_{i,k} + \mathbf{c}_j \tag{1}$$

$$\mathbf{v}_{i,j} = \begin{pmatrix} x_{i,j} \\ y_{i,j} \end{pmatrix} \tag{2}$$

$$T_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \quad \mathbf{c}_j = \begin{pmatrix} e_j \\ f_j \end{pmatrix} \tag{3}$$

There are two ways to define the mappings: relative and absolute. Also, the fractals can be generated randomly with the absolute mappings.

1.1 Relative Mapping

This type of mapping is suitable for use with line-based fractals (i.e. the fractal consists of a single curve). In this method, the transformations are applied locally on each line; The origin of the coordinate system is set to on end of the line and the point on the other end of the line is transformed relative to that. Algorithm 1 is the implementation of this method.

1.2 Absolute Mapping

This type of mapping is suitable for non-line-based fractals (the fractal isn't only a single curve). In this method, the same transformations are applied to every point

Algorithm 1 Fractal Generation by Relative Mapping

function FRACTAL($x_{init}, y_{init}, T, \mathbf{c}, steps$) ▷ T and $\mathbf{c}$ represent the matrix and vector for every transformation

  $x \leftarrow x_{init}$

  $y \leftarrow y_{init}$

  for $step \leftarrow 1, steps$ **do**

    $i \leftarrow 1$

    while $i < N(x)$ **do** ▷ N(x) is the number of x coordinates (which is the same as the number of points)

      $\begin{pmatrix} x_{rel} \\ y_{rel} \end{pmatrix} \leftarrow \begin{pmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{pmatrix}$

      for all T_j and $\mathbf{c}_j$ **do**

        $\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} \leftarrow T_j \begin{pmatrix} x_{rel} \\ y_{rel} \end{pmatrix} + \mathbf{c}$

        insert x_{new} into x at index i

        insert y_{new} into y at index i

        $i \leftarrow i + 1$

      end for

    end while

  end for

  return x and y

end function

in each step to generate new points. The origin of the coordinate system is always set at the first point of the fractal. By repeatedly applying the transformations to the fractal in each step, the self-similar shape is created. Algorithm 2 is the implementation of this method.

Algorithm 2 Fractal Generation by Absolute Mapping

function FRACTAL($x_{init}, y_{init}, T, \mathbf{c}, steps$) ▷ T and $\mathbf{c}$ represent the matrix and vector for every transformation
  $x \leftarrow x_{init}$
  $y \leftarrow y_{init}$
  for $step \leftarrow 1, steps$ **do**
    define the x_{new} empty array
    define the y_{new} empty array
    for all $x_i \in x$ and $y_i \in y$ **do**
      for all T_j and $\mathbf{c}_j$ **do**
        $\begin{pmatrix} x_{gen} \\ y_{gen} \end{pmatrix} \leftarrow T_j \begin{pmatrix} x_{rel} \\ y_{rel} \end{pmatrix} + \mathbf{c}$
        append x_{gen} to x_{new}
        append y_{gen} to y_{new}
      end for
    end for
    $x \leftarrow x_{new}$
    $y \leftarrow y_{new}$
  end for
  return x and y
end function

1.3 Random Fractals

Some fractals can be generated by repeatedly applying random transformations from the fractal. If this is done with enough sample points, and enough steps so that the smalles unit of the shape is smaller than a pixel of the display, the fractal will be perfect. Algorithm 3 is the implementation of this method.

1.4 Representation of the fractals

The fractals are defined by functions $\{f_1, f_2, \ldots, f_n\}$. Each of the functions are

$$f_i(\mathbf{v}) = T_i\mathbf{v} + \mathbf{c}_i, \tag{4}$$

Algorithm 3 Fractal Generation by Absolute Mapping

function FRACTAL($range_x, range_y, T, \mathbf{c}, steps, samples$) ▷ T and $\mathbf{c}$ represent the matrix and vector for every transformation
    generate random x values in $range_x$ ($N(x) = samples$)
    generate random y values in $range_y$ ($N(y) = samples$)
    for all $x_i \in x$ and $y_i \in y$ **do**
        for $step \leftarrow 1, steps$ **do**
            choose random T_j and $\mathbf{c}_j$
            $\begin{pmatrix} x_i \\ y_i \end{pmatrix} \leftarrow T_j \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \mathbf{c}_j$
        end for
    end for
    return x and y
end function

where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} e \\ f \end{pmatrix}. \tag{5}$$

So we can represent each fractal by defining the values a, b, c, d, e.

2 Koch Snowflake

transformation	a	b	c	d	e	f
f_1	1/3	0	0	1/3	0	0
f_2	1/6	$-\sqrt{3}/6$	$\sqrt{3}/6$	1/6	1/3	0
f_3	1/6	$-\sqrt{3}/6$	$\sqrt{3}/6$	1/6	1/2	$\sqrt{3}/6$
f_4	1/3	0	0	1/3	2/3	0

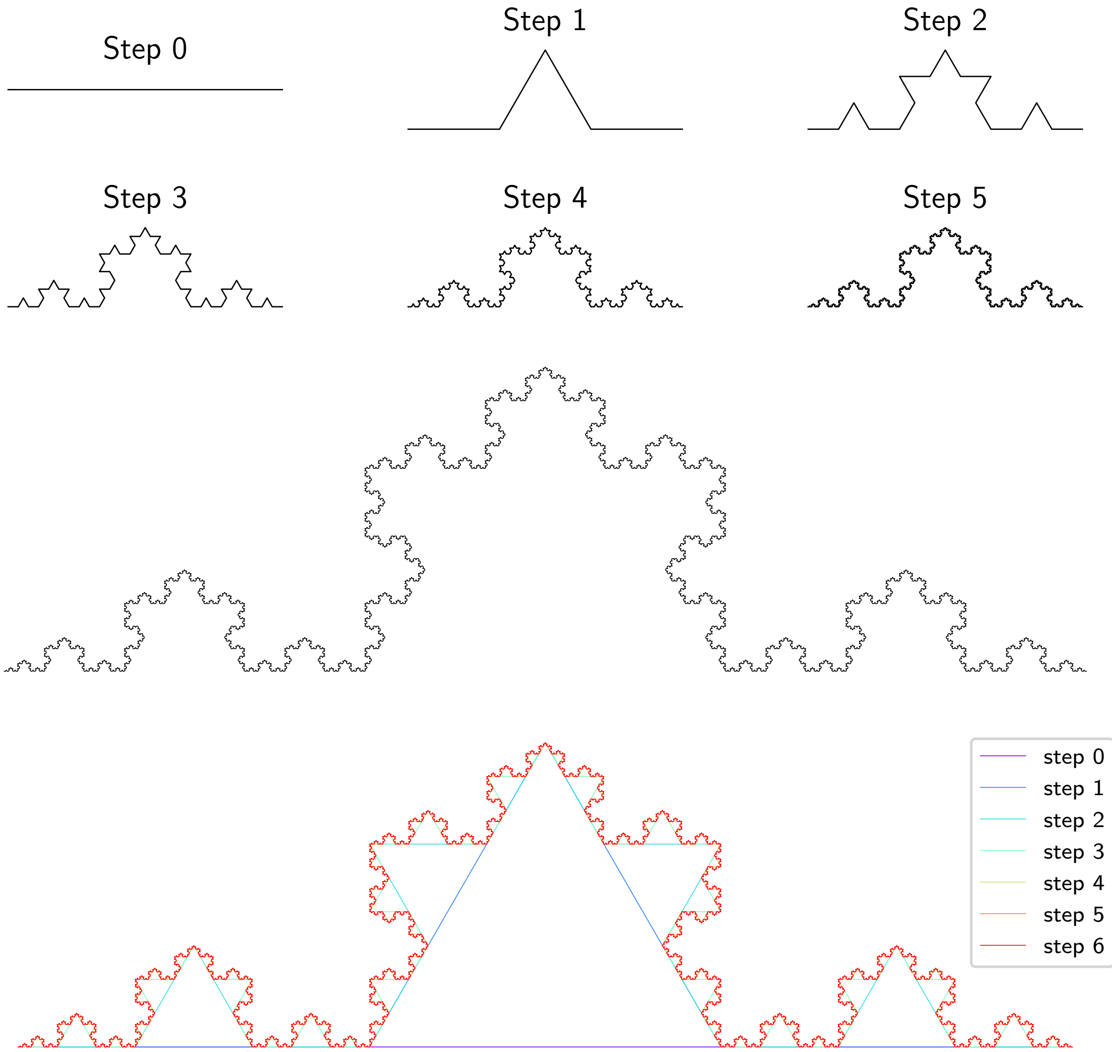


Figure 1: Koch Curve

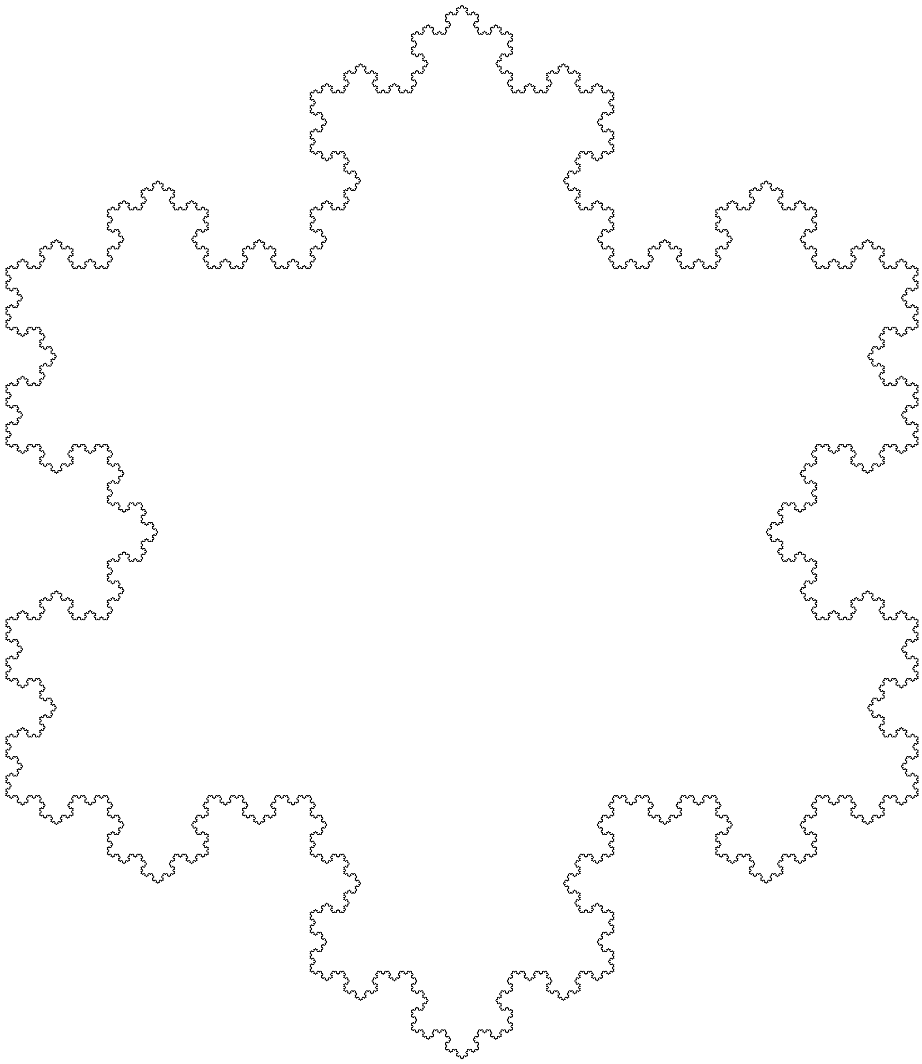


Figure 2: Koch Snowflake

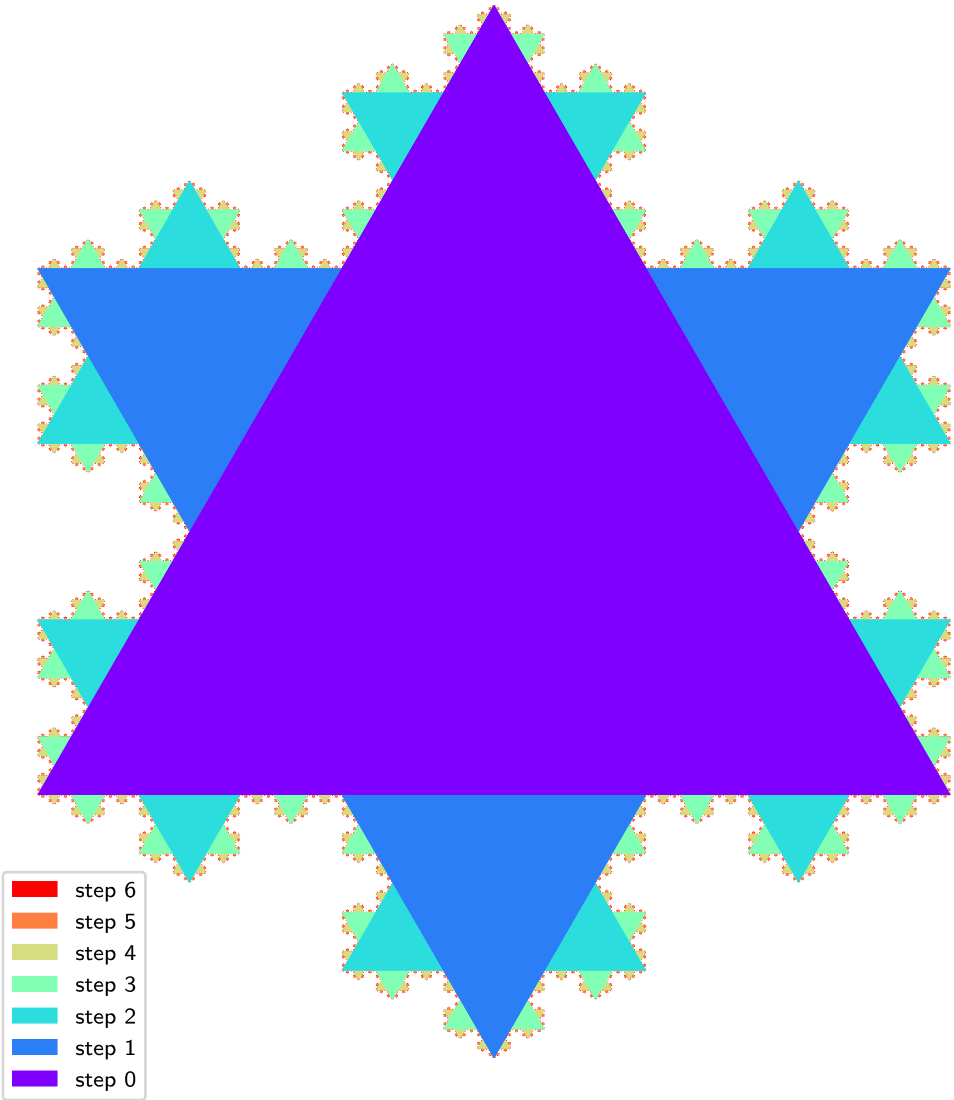
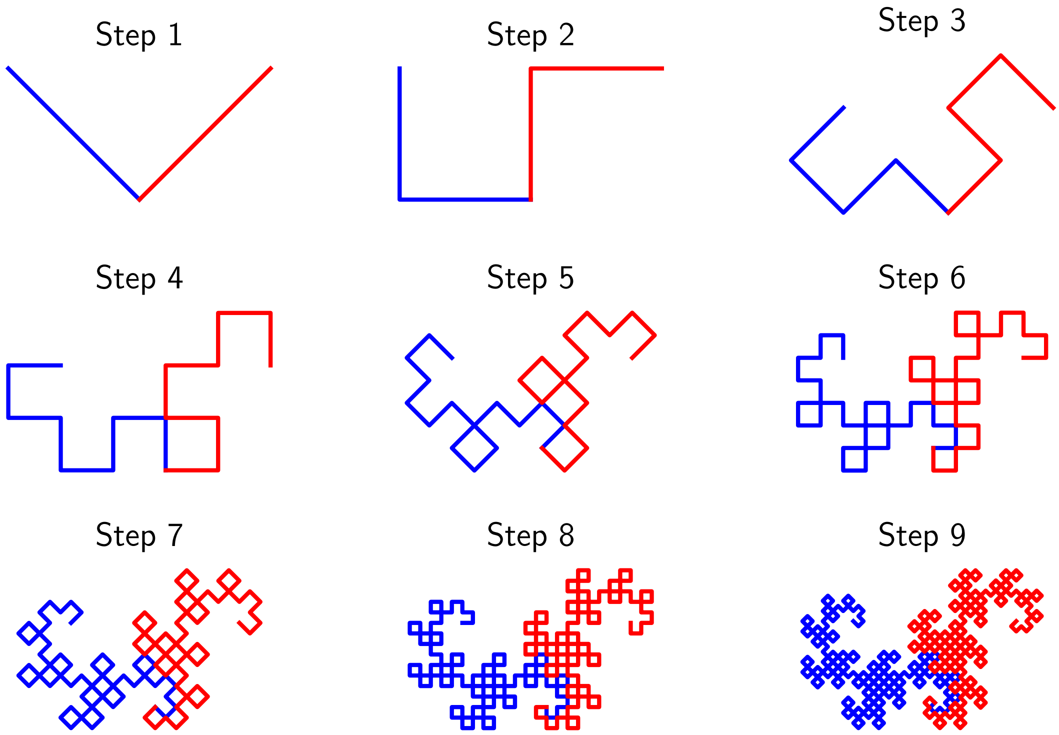
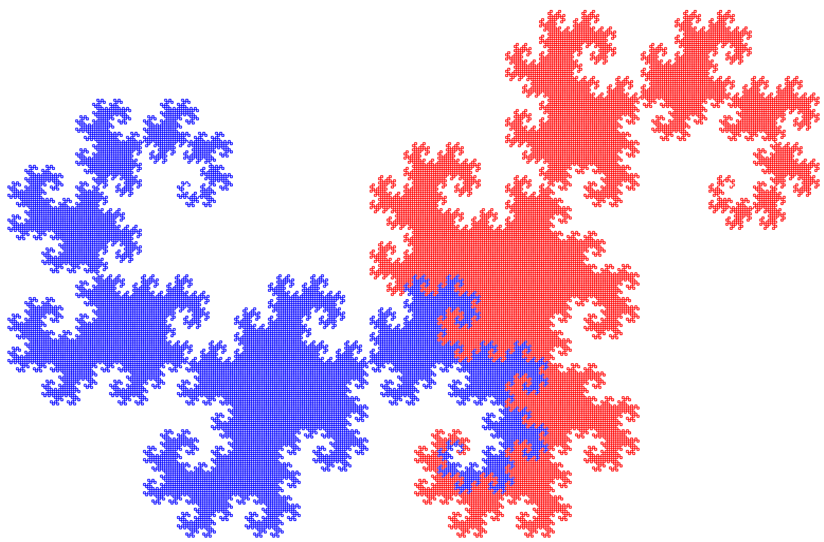


Figure 3: Colored Koch Snowflake

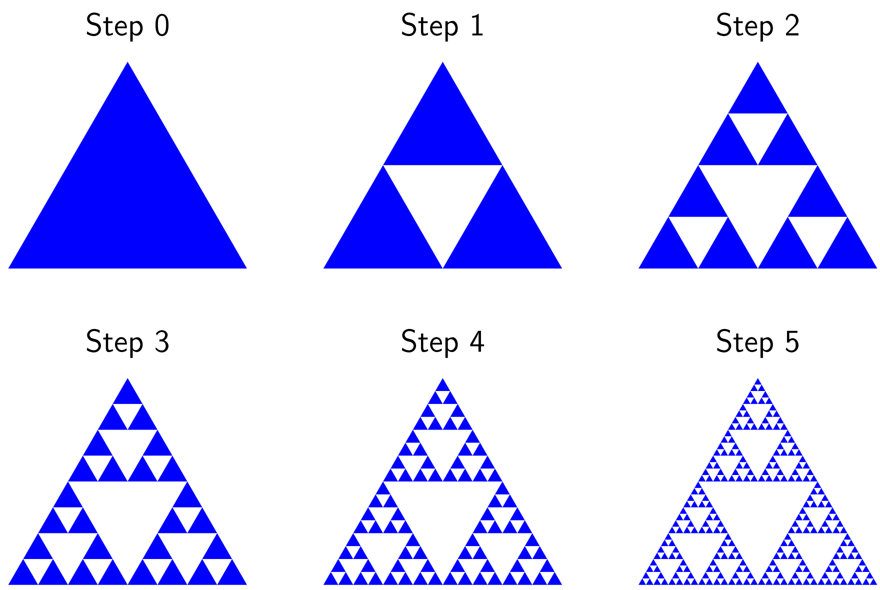
3 Heighway Dragon

transformation	a	b	c	d	e	f
f_1	1/2	1/2	− 1/2	1/2	0	0
f_2	− 1/2	1/2	− 1/2	− 1/2	1/2	− 1/2


Step 1
Step 2
Step 3
Step 4
Step 5
Step 6
Step 7
Step 8
Step 9



4 Sierpiński Triangle



transformation	a	b	c	d	e	f
f_1	$1/2$	0	0	$1/2$	0	0
f_2	$1/2$	0	0	$1/2$	$1/4$	$\sqrt{3}/4$
f_3	$1/2$	0	0	$1/2$	$1/2$	0

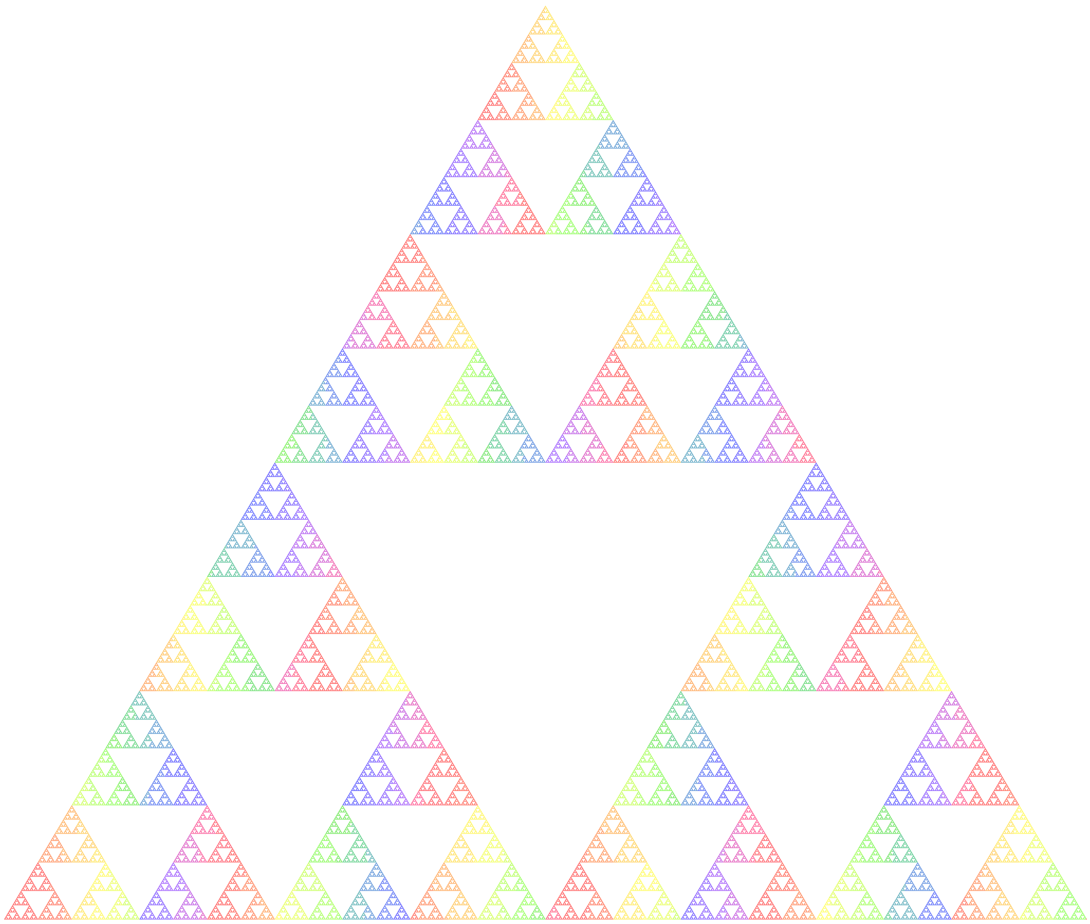


Figure 4: Step 10 Sierpiński triangle

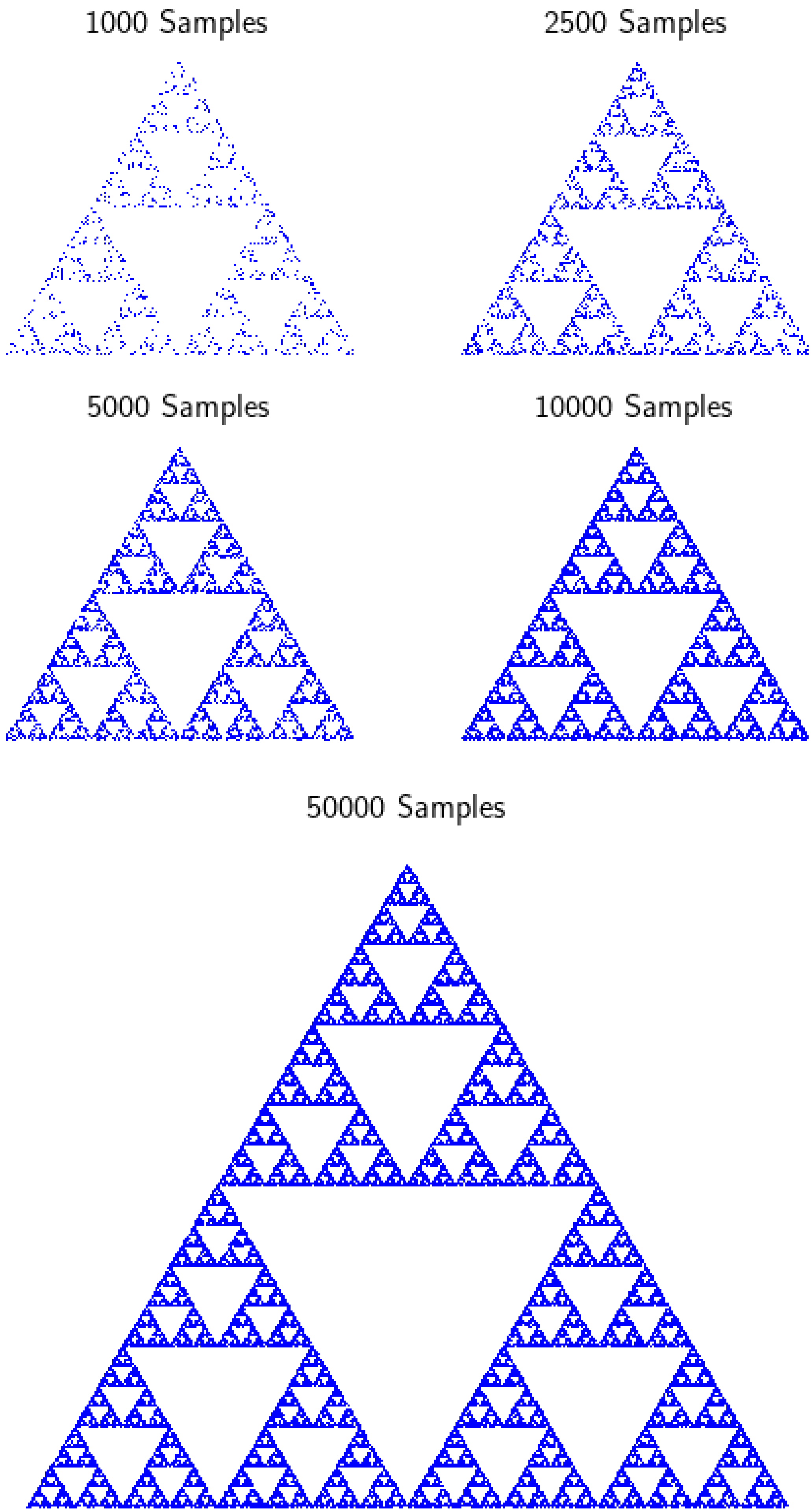
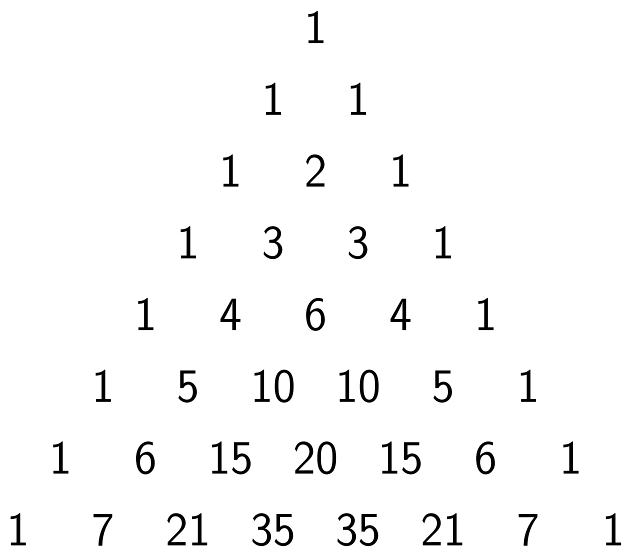
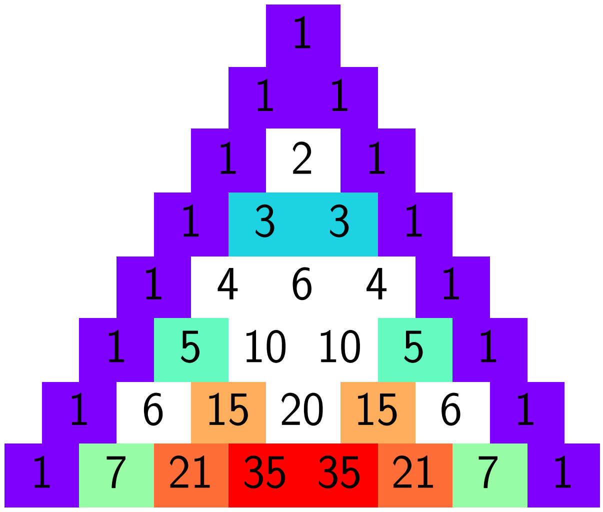


Figure 5: Randomly Generating the Sierpiński Triangle

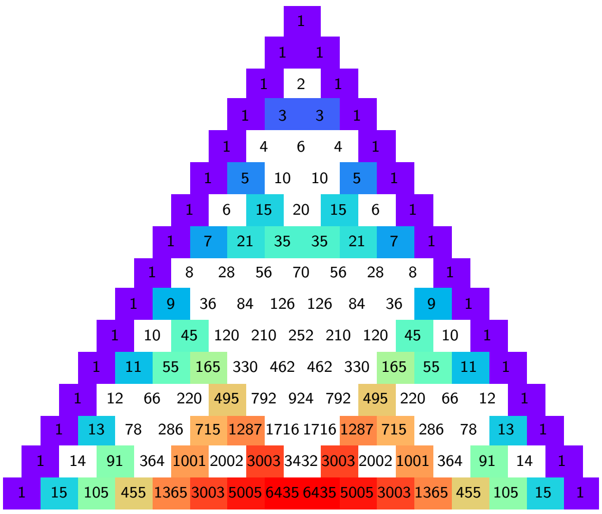
4.1 The Sierpiński Triangle Inside Pascal's Triangle



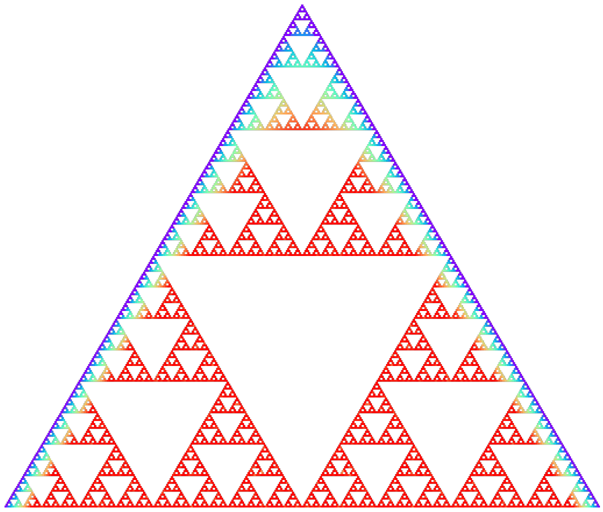
(a) Pascal's Triangle



(b) Connecting the odd numbers generates the Sierpiński triangle



(c) The Sierpiński triangle inside a larger Pascal's triangle



(d) Painted on Pascal's triangle of size 256

5 Barnsley Fern

To create this fractal, we need to apply random functions with different probabilities.

transformation	a	b	c	d	e	f	probability
f_1	0	0	0	0.16	0	0	1%
f_2	0.85	0.04	-0.04	0.85	0	1.6	85%
f_3	0.20	-0.26	0.23	0.22	0	1.6	7%
f_4	-0.15	0.28	-0.26	0.24	0	0.44	7%

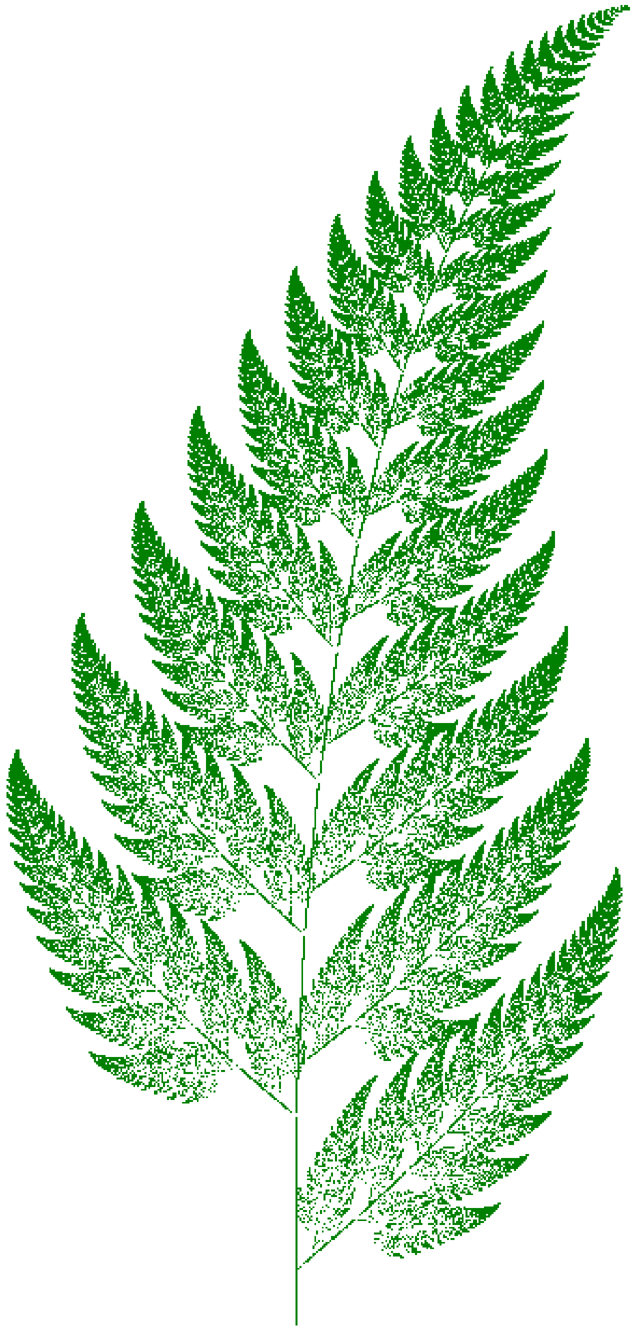


Figure 7: Barnsley fern with 200000 random samples and 25 transformation steps