# Computational Physics
# Problem Set 6

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## 1 Central Limit Theorem

The central limit theorem states that the sum of independent random variables tends towards a normal distribution. Specifically, if $X_1, X_2, \ldots, X_n$ are n random samples from a population with overall mean $\mu$ and finite variance $\sigma^2$, and if $\bar{X}_n$ is the sample mean, then the limiting form of the distribution

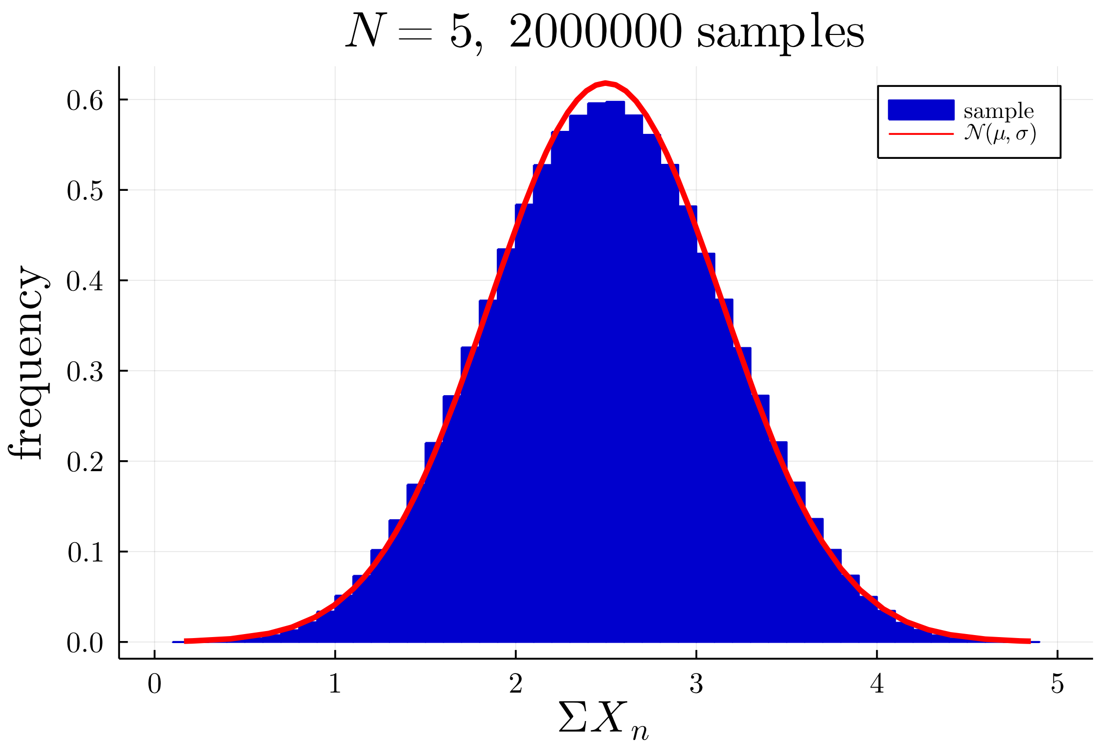
$$Z = \lim_{n\to\infty} \frac{\sqrt{n}}{\sigma}\left(\bar{X}_n - \mu\right) \tag{1}$$

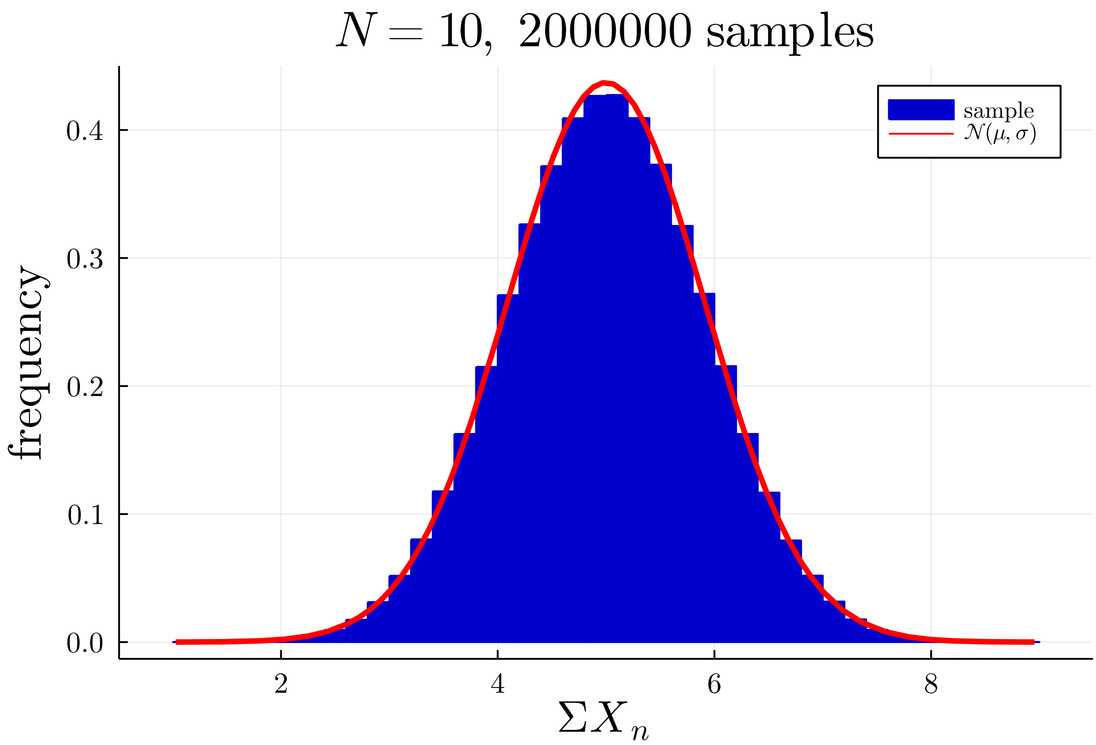
is a standard normal distribution, which is

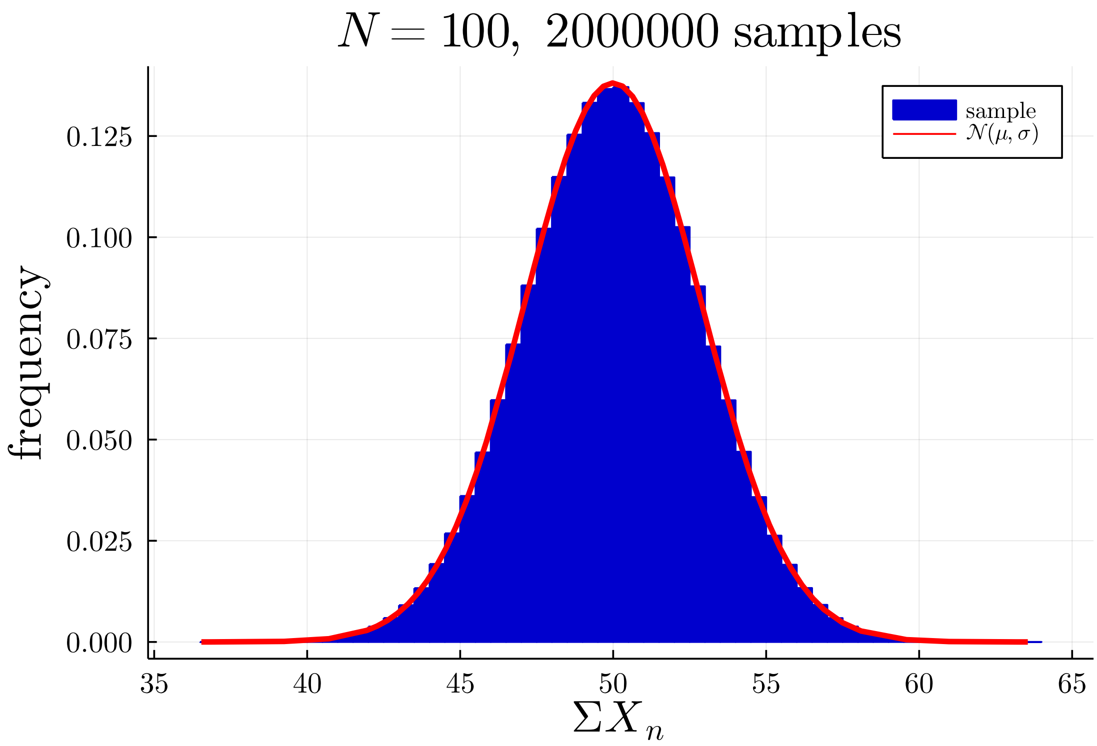
$$\mathcal{N}(0,1) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}. \tag{2}$$

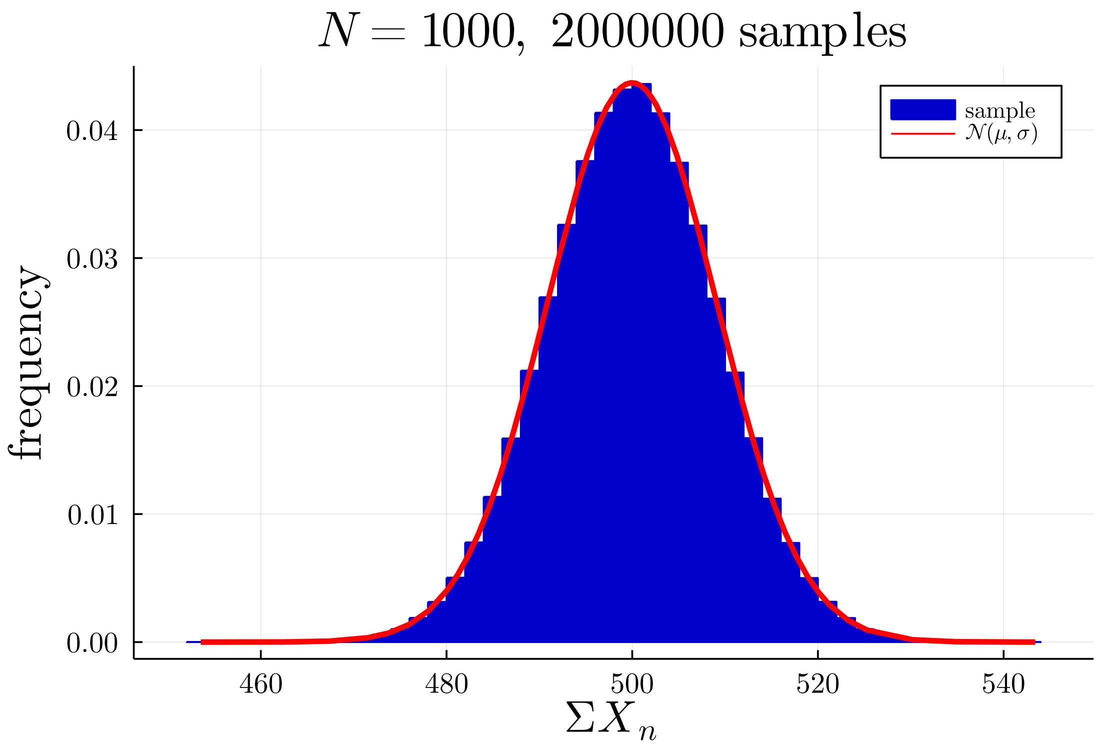
In the following figures, I generated random numbers between 0 and 1 and then compared the results (sample sums) to a perfect normal distribution with the same mean and variance as the generated sample. As you can see, the distribution of the sample is very close to a normal distribution in all cases, and as $N$ grows, the distribution becomes more similar to a normal distribution.

Note that random walk generates a normal distribution exactly because of this theorem; The final position of the walker is the sum of a random sample from the numbers $-1$ and 1. Each $X_n$ corresponds to a $-1$ or 1 and the probability of each step is independent from other steps.


$N = 5,\ 2000000$ samples
frequency
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0
1
2
3
4
5
$\Sigma X_n$
sample
$\mathcal{N}(\mu, \sigma)$


$N = 10,\ 2000000$ samples
frequency
0.0
0.1
0.2
0.3
0.4
2
4
6
8
$\Sigma X_n$
sample
$\mathcal{N}(\mu, \sigma)$


$N = 100,\ 2000000$ samples
sample
$\mathcal{N}(\mu, \sigma)$
frequency
0.125
0.100
0.075
0.050
0.025
0.000
35
40
45
50
55
60
65
$\Sigma X_n$


$N = 1000,\ 2000000$ samples
sample
$\mathcal{N}(\mu, \sigma)$
frequency
0.04
0.03
0.02
0.01
0.00
460
480
500
520
540
$\Sigma X_n$

## 2 Generating a Gaussian Distribution with an RNG[1]

The probability density function of a random number generator is constant in the range $[0, 1]$ and zero elsewhere. If the probability density function of a random variable $x$ is given by the function $f(x)$, then with a change of variables $y$ can be chosen as a function of $x$ such that the probability density function of $y$ is the arbitrary normalized function $g(y)$; The probability of associated $x$ and $y$ is equal, so

$$f(x)\,dx = g(y)\,dy \implies \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{y} g(y)\,dy\,. \tag{3}$$

If $x$ is generated by an RNG, then

$$x = \int_{-\infty}^{y} g(y)\,dy = G(y) \tag{4}$$

since $G(y)$ is monotonically increasing ($g(y) > 0$, because it is a probability density function), it is invertible and

$$y = G^{-1}(x). \tag{5}$$

For a gaussian (normal) distribution

$$g(y) = \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}. \tag{6}$$

In this form, $G(y)$ cannot be expressed in elementary functions. to solve this issue, we can generate two random numbers $y_1$ and $y_2$ from $x$ instead of one:

$$g(y_1, y_2)\,dy_1\,dy_2 = g(y_1)g(y_2)\,dy_1\,dy_2 = \frac{e^{-\frac{y_1^2+y_2^2}{2\sigma^2}}}{2\pi\sigma^2}\,dy_1\,dy_2 \tag{7}$$

With a change of coordinates to polar coordinates

$$\begin{cases} y_1 = \rho\sin\theta \\ y_2 = \rho\cos\theta \end{cases} \tag{8}$$

$$g(y_1, y_2)\,dy_1\,dy_2 = g(\rho,\theta)\rho\,d\rho\,d\theta = \frac{e^{-\frac{\rho^2}{2\sigma^2}}}{2\pi\sigma^2}\rho\,d\rho\,d\theta \tag{9}$$

[1] Random Number Generator

For normalized distributions in terms of $\rho$ and $\theta$, we can use

$$\begin{cases} g_1(\rho) = \dfrac{e^{-\frac{\rho^2}{2\sigma^2}}}{\sigma^2}\rho & (10) \\ g_2(\theta) = \dfrac{1}{2\pi} & (11) \end{cases}$$

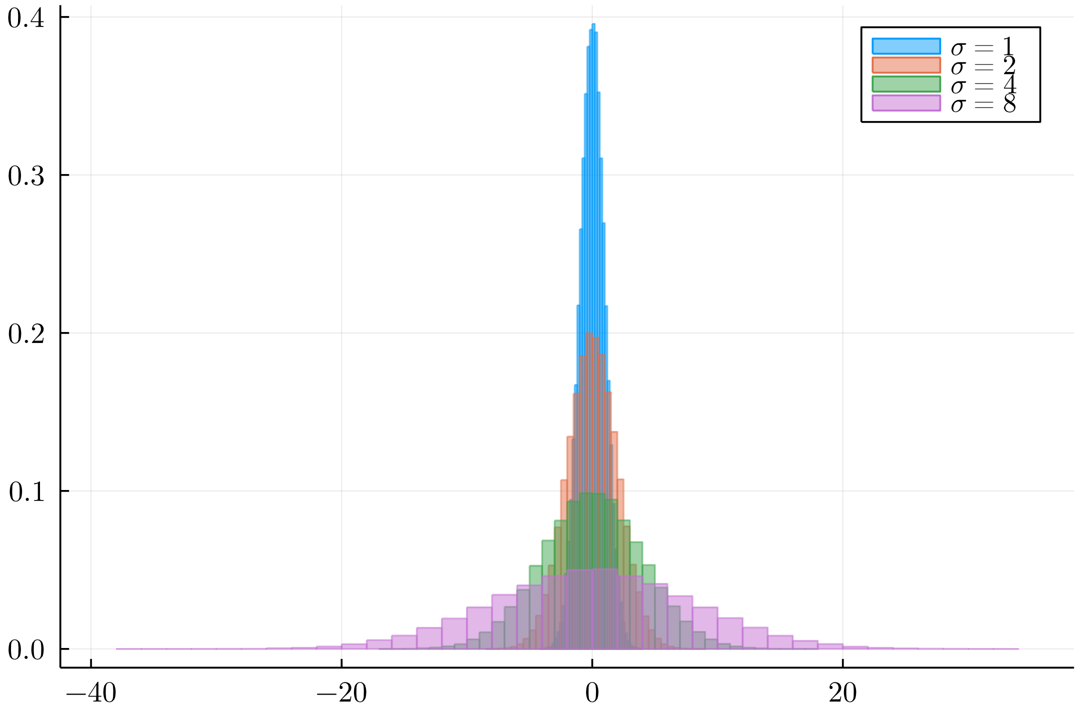
Now, two random numbers with gaussian distribution $y_1$ and $y_2$ can be generated from two random numbers form RNG $x_1$ and $x_2$:

$$\begin{cases} G_1(\rho) = \displaystyle\int_0^{\rho} \frac{e^{-\frac{\rho^2}{2\sigma^2}}}{\sigma^2}\rho\, d\rho = 1 - e^{-\frac{\rho^2}{2\sigma^2}} & (12) \\ G_2(\theta) = \displaystyle\int_0^{\theta} \frac{1}{2\pi}\, d\theta = \frac{\theta}{2\pi} & (13) \end{cases}$$

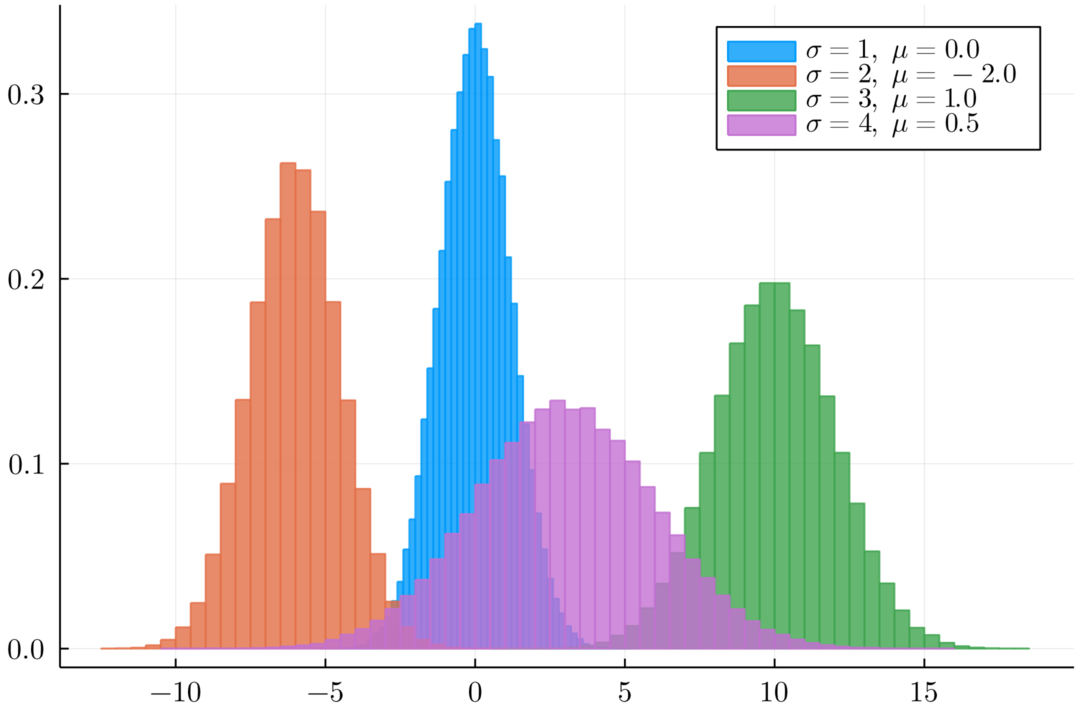
$$\begin{cases} \rho = G_1^{-1}(x_1) = \sigma\sqrt{2\ln\left(\dfrac{1}{1-x_1}\right)} & (14) \\ \theta = G_2^{-1}(x_2) = 2\pi x_2 & (15) \end{cases}$$

$$\boxed{\begin{cases} y_1 = \sigma\sin(2\pi x_2)\sqrt{2\ln\left(\dfrac{1}{1-x_1}\right)} \\ y_2 = \sigma\cos(2\pi x_2)\sqrt{2\ln\left(\dfrac{1}{1-x_1}\right)} \end{cases}} \quad \begin{matrix} (16) \\ (17) \end{matrix}$$

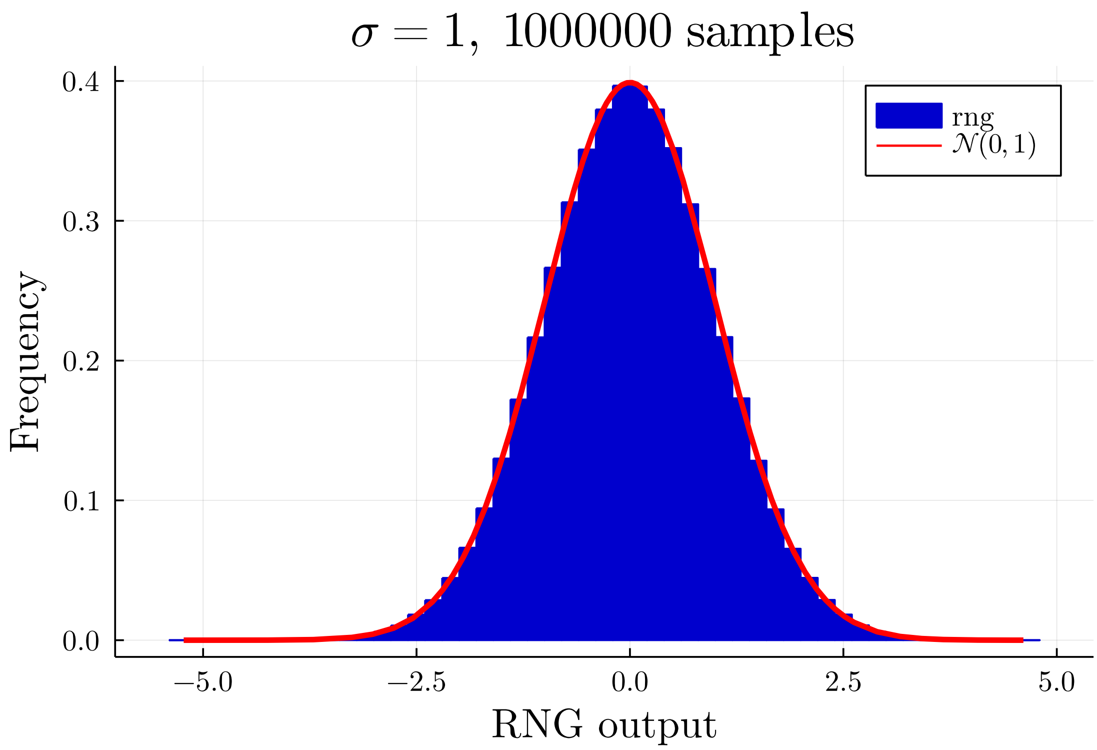
To create a normal distribution with mean $\mu$ and variance $\sigma^2$ it suffices to add $\mu$ to $y_1$ and $y_2$.



$\sigma = 1$
$\sigma = 2$
$\sigma = 4$
$\sigma = 8$
0.4
0.3
0.2
0.1
0.0
−40
−20
0
20

$\sigma = 1,\ \mu = 0.0$
$\sigma = 2,\ \mu = -2.0$
$\sigma = 3,\ \mu = 1.0$
$\sigma = 4,\ \mu = 0.5$
0.3
0.2
0.1
0.0
−10
−5
0
5
10
15



$\sigma = 1,\ 1000000$ samples
0.4
0.3
0.2
0.1
0.0
Frequency
−5.0
−2.5
0.0
2.5
5.0
RNG output
rng
$\mathcal{N}(0,1)$