Computational Physics
Problem Set 9

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1 RC Circuit: First Order Ordinary Differential Equation

In a RC circuit where a battery with voltage V_0 is connected to a capacitor and a resistor (kvl)

$$V_B + V_C + V_R = 0 \implies V_0 - \frac{q}{C} - IR = 0. \tag{1}$$

Here, the current charges the capacitor, so $I = \dot{q}$ and therefore

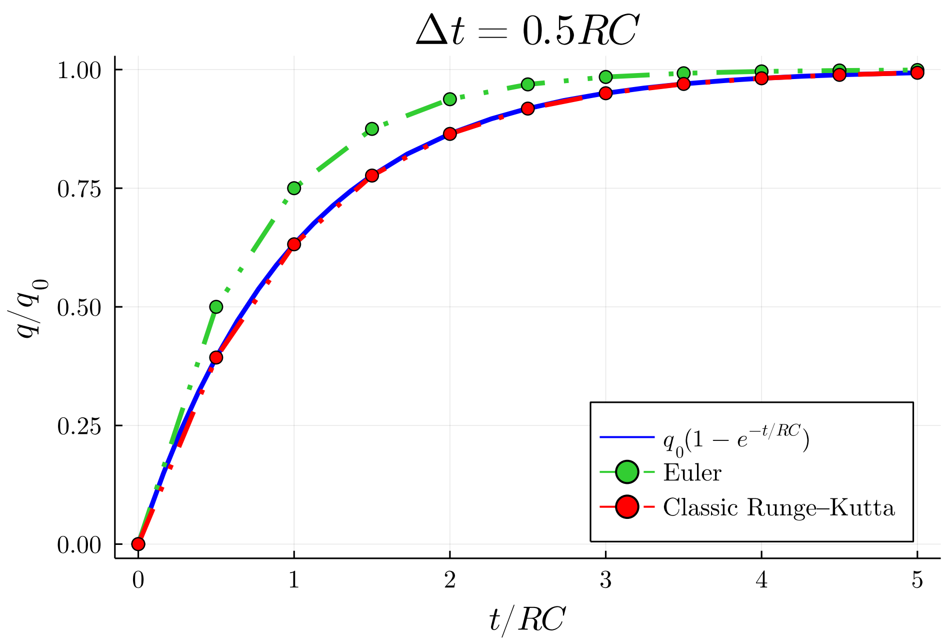
$$\dot{q} = \frac{V_0}{R} - \frac{q}{RC}. \tag{2}$$

Defining $q_0 \coloneqq CV_0$ and assuming $q(0) = 0$

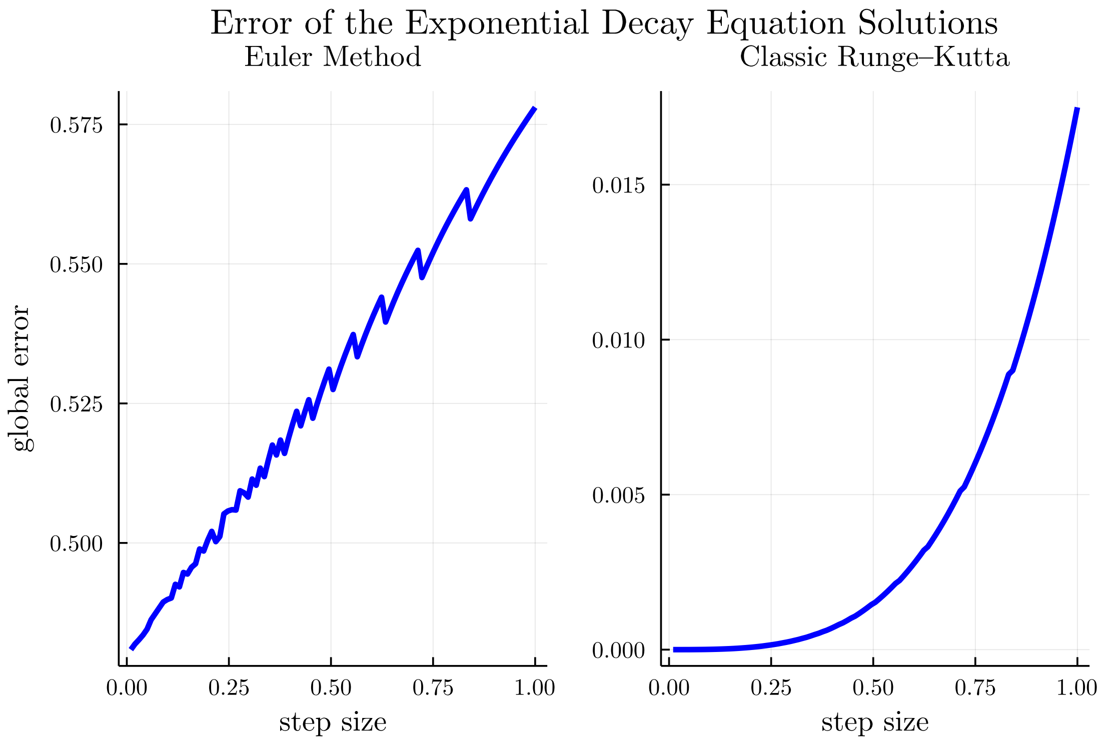
$$q(t) = q_0(1 - e^{-t/RC}). \tag{3}$$

Here, the Euler method and the classic Runge-Kutta methods are compared against the exact solution.

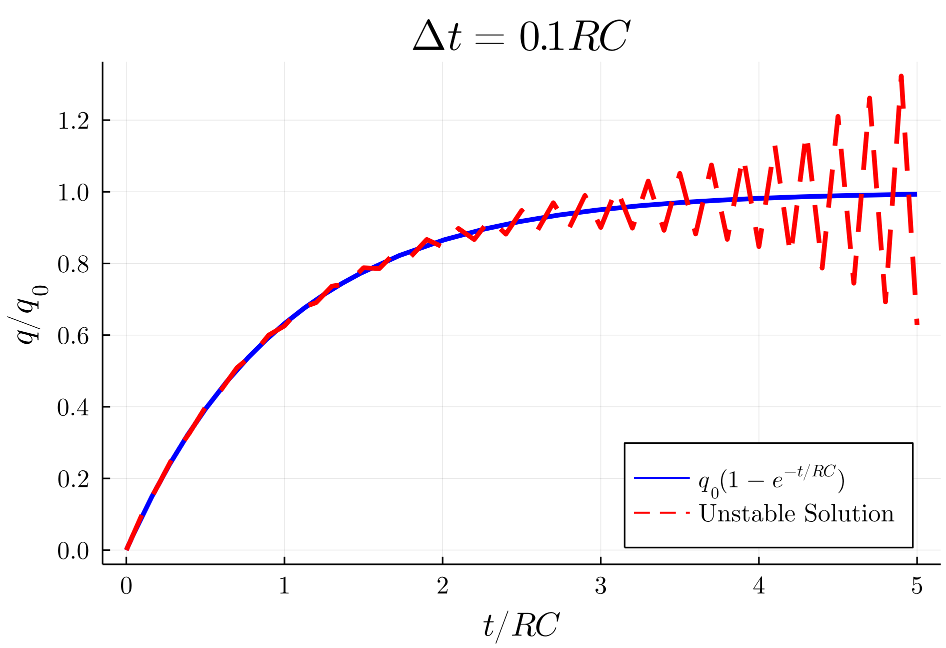
Also, in the last figure, it is shown that the method $x_{n+1} = x_{n-1} + 2\Delta t \dot{x}(t_n, x_n)$ is unstable.



$\Delta t = 0.5RC$
1.00
0.75
0.50
0.25
0.00
q/q_0
0
1
2
3
4
5
t/RC
$q_0(1 - e^{-t/RC})$
Euler
Classic Runge–Kutta



Error of the Exponential Decay Equation Solutions
Euler Method
Classic Runge–Kutta
global error
0.575
0.550
0.525
0.500
0.015
0.010
0.005
0.000
0.00
0.25
0.50
0.75
1.00
step size



$\Delta t = 0.1RC$
1.2
1.0
0.8
0.6
0.4
0.2
0.0
q/q_0
0
1
2
3
4
5
t/RC
$q_0(1 - e^{-t/RC})$
Unstable Solution

2 Simple Harmonic Oscillator: Second Order Ordinary Differential Equation

For the simple harmonic oscillator

$$\ddot{x} = -\omega x, \tag{4}$$

$$\text{for } v(0) = \dot{x}(0) = 0 \begin{cases} x(t) = x_0 \cos(\omega t) \\ v(t) = x_0 \omega \sin(\omega t) \end{cases} \tag{5}$$

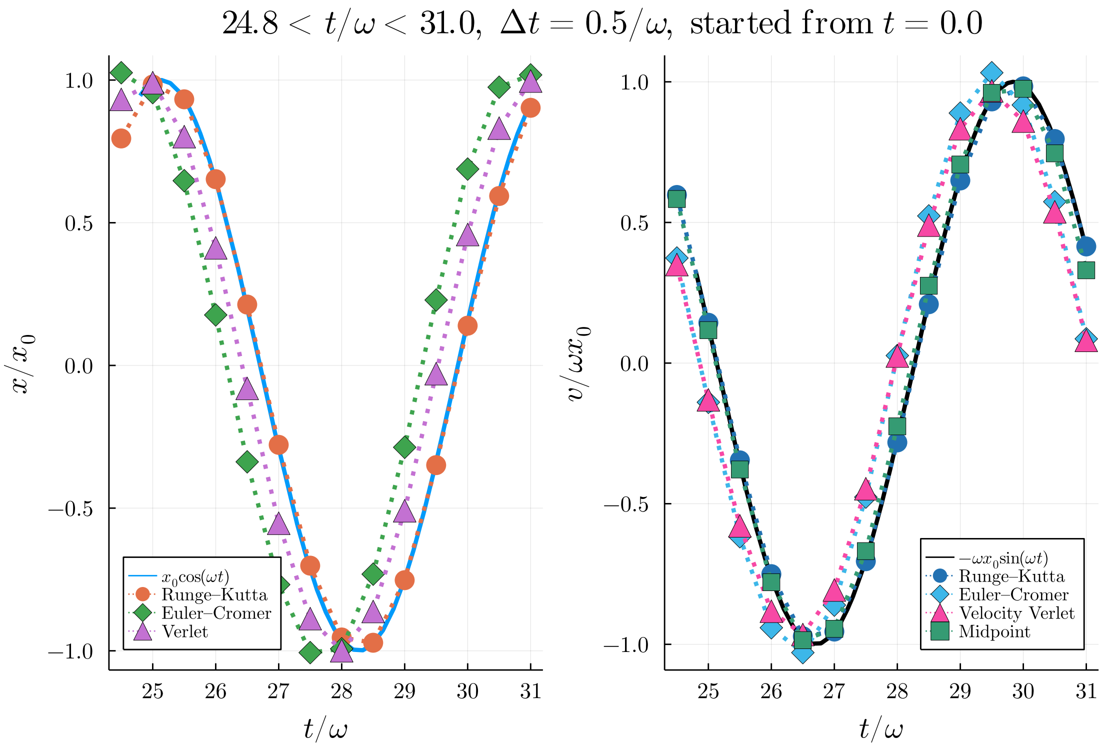
Here, 6 numerical methods are compared agains the exact solution:

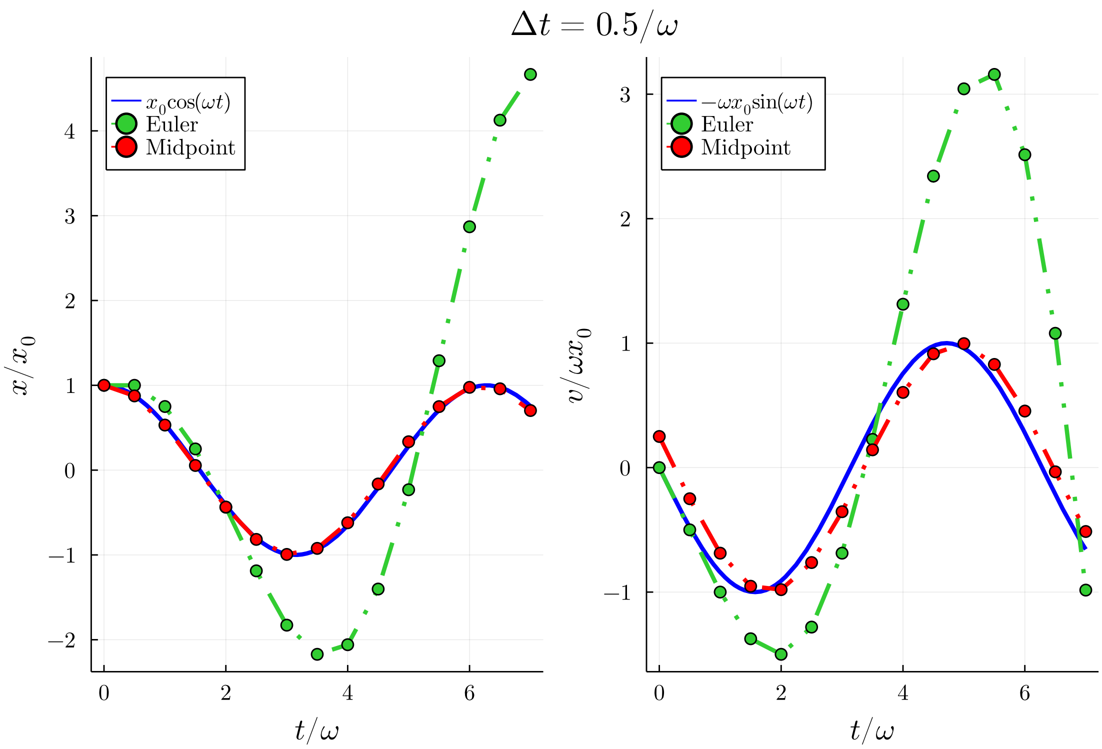
Method	Local Error	Global Error	Global Velocity ($\dot{x}(t)$) Error
Euler Method	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(\Delta t)$	$\mathcal{O}(\Delta t)$
Euler–Cromer Method (a.k.a. Semi-implicit Euler Method)	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(\Delta t)$	$\mathcal{O}(\Delta t)$
Midpoint Method (A Modified Euler Method)	$\mathcal{O}(\Delta t^3)$	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(\Delta t^2)$
Verlet Integration (a.k.a. Störmer–Verlet Method)	$\mathcal{O}(\Delta t^4)$	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(\Delta t)$
Velocity Verlet	$\mathcal{O}(\Delta t^4)$	$\mathcal{O}(\Delta t^2)$	$\mathcal{O}(\Delta t^2)$
Classic Runge–Kutta (a.k.a. RK4)	$\mathcal{O}(\Delta t^5)$	$\mathcal{O}(\Delta t^4)$	$\mathcal{O}(\Delta t^4)$

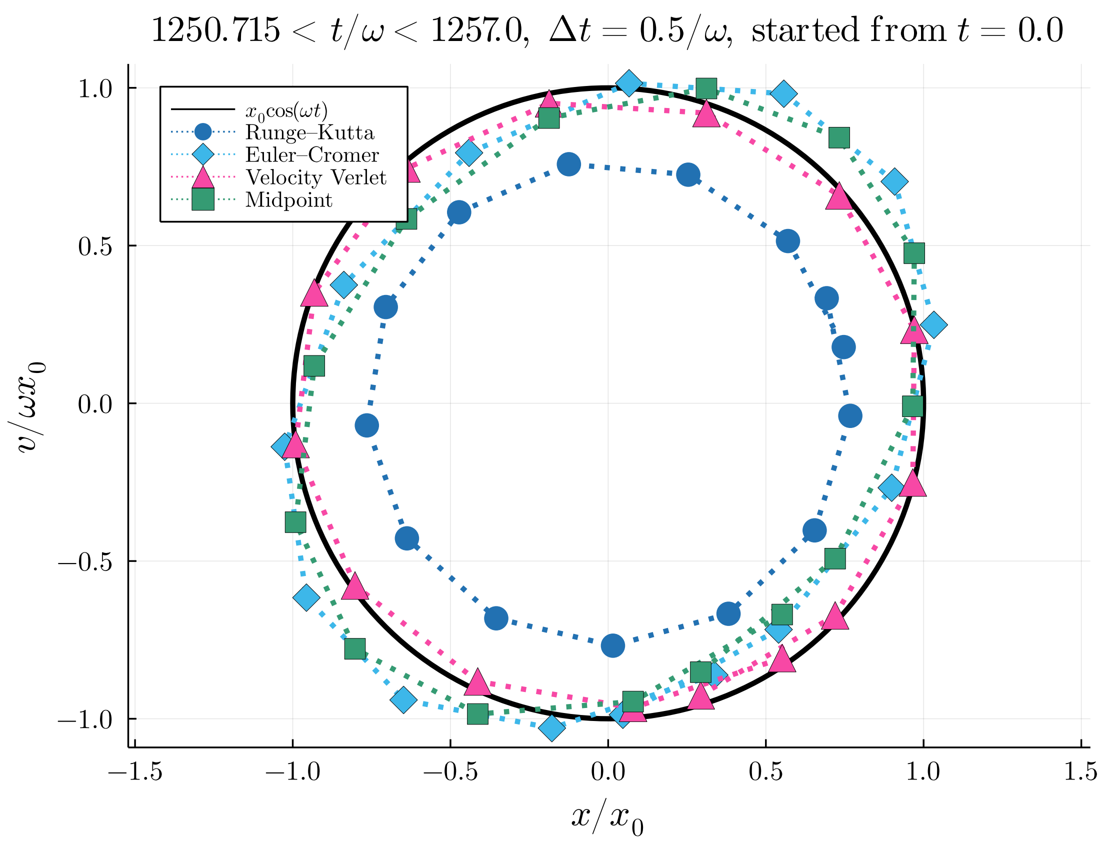
The classic Runge–Kutta method provides the most accurate solution, but it exhibits a severe energy drift in large time intervals (since it is not a sympletic integrator).

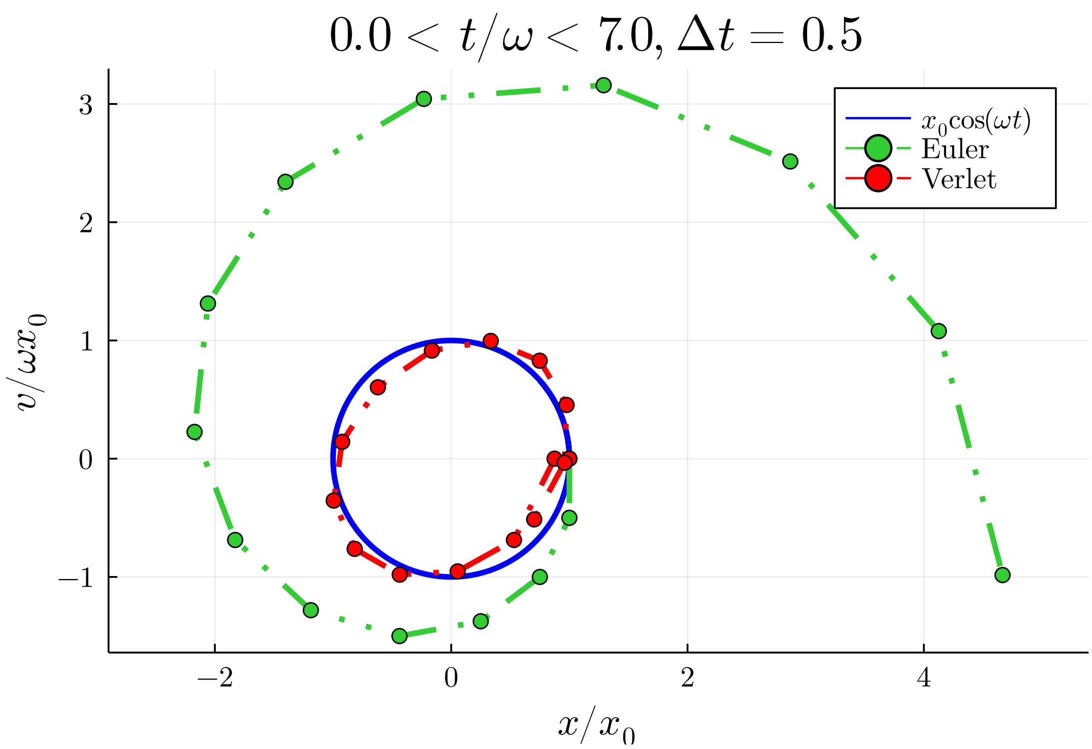
The velocity Verlet method performs the best in terms of conservation of energy. In terms of the displacement accuracy, it performs the same as the midpoint method and the regular verlet method.

The Euler method can be unstable under certain conditions. Here, one example of this instability is evident.

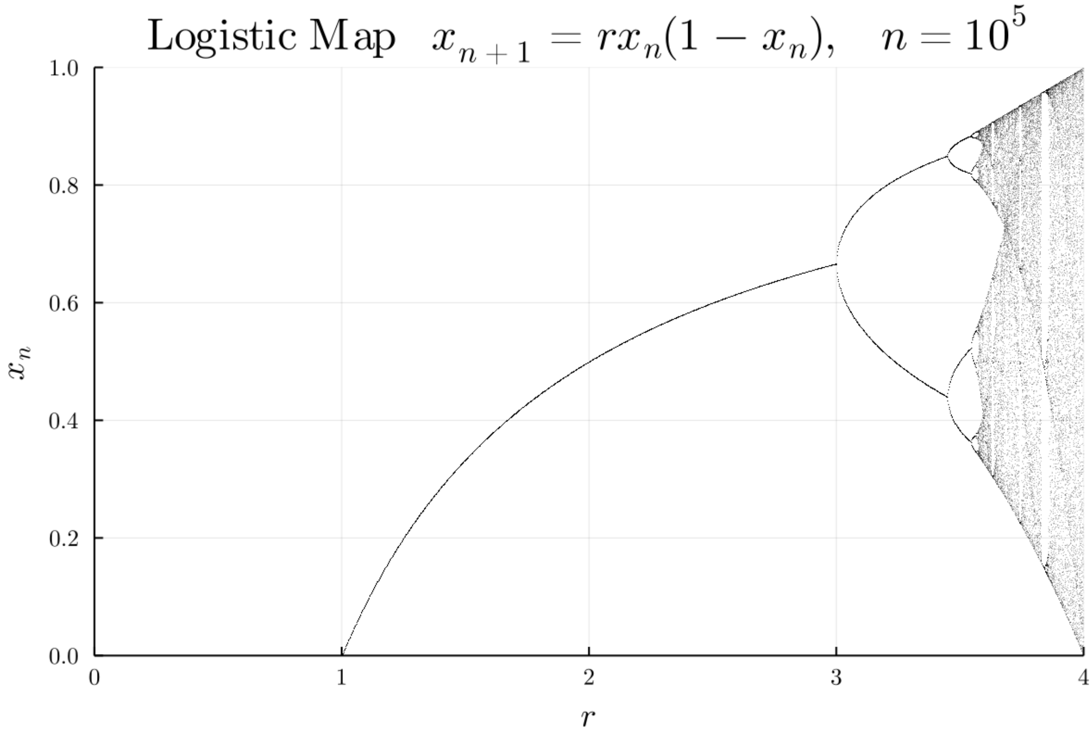

24.8 < t/ω < 31.0, Δt = 0.5/ω, started from t = 0.0
x/x0
t/ω
x0cos(ωt)
Runge–Kutta
Euler–Cromer
Verlet
v/ωx0
−ωx0sin(ωt)
Runge–Kutta
Euler–Cromer
Velocity Verlet
Midpoint

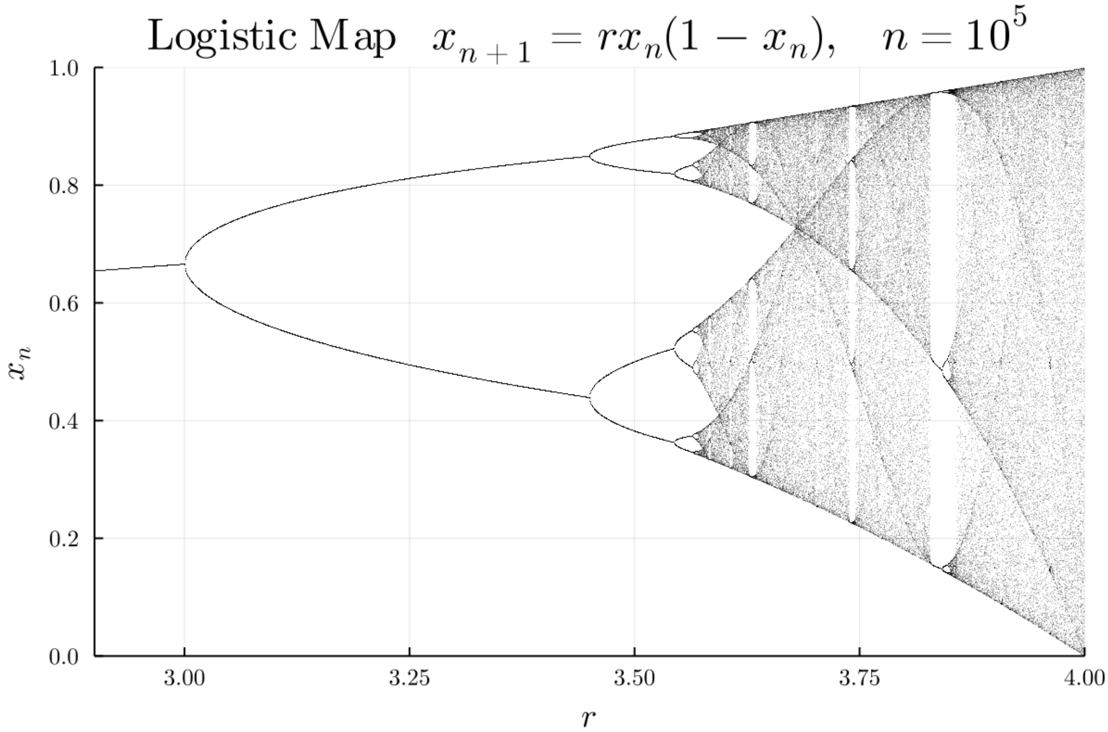

Δt = 0.5/ω
x/x0
t/ω
x0cos(ωt)
Euler
Midpoint
v/ωx0
−ωx0sin(ωt)
Euler
Midpoint


$1250.715 < t/\omega < 1257.0,\ \Delta t = 0.5/\omega,$ started from $t = 0.0$
$x_0\cos(\omega t)$
Runge–Kutta
Euler–Cromer
Velocity Verlet
Midpoint
$v/\omega x_0$
x/x_0


$0.0 < t/\omega < 7.0, \Delta t = 0.5$
$x_0\cos(\omega t)$
Euler
Verlet
$v/\omega x_0$
x/x_0

3 Logistic Map: Chaos





$$\text{Feigenbaum constants} \begin{cases} \text{first constant: } \delta = \dfrac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} \approx 4.669 \\ \text{second constant: } \alpha = \dfrac{\text{width of a tine}}{\text{width of the next tine}} \approx 2.509 \end{cases}$$

δ is calculated with $n = 5$ and α is calculated with a branch of a carefully chosen 8 point tine. δ is accurate up to the third decimal place and α is accurate up to the second decimal place (the error is much larger for α, as tine shapes are varied).

I also made an animation (named `bifurcation.gif`) showing how the bifurcation diagram changes in each step of applying the logistic map.